A quantum optics inspired approach to phase space distributions of quantum states on surfaces.

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Overview of of the talk

- What are eigenfunctions?
- Why are they important?
- What did I do?

What are eigenfunctions?

I will focus on eigenfunctions for two broad families of linear differential operators

- Differential operators of the form $-\frac{d^2}{dx^2} + V(x)$ on $\mathbf{R}$
- The Laplace Operator Δ_g on a compact manifold(surface) without boundary.

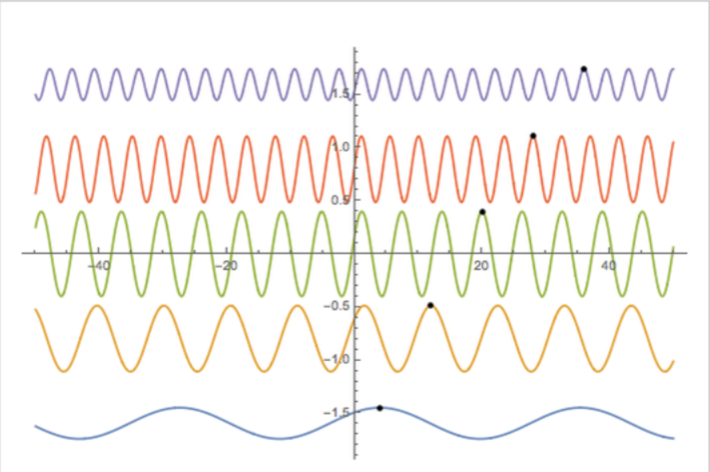
Eigenfunctions on $\mathbf{R}$ - Pictures and Examples

$$\left(-\frac{d^2}{d^2x} + V(x)\right)\varphi_\lambda = \lambda\varphi_\lambda$$

Eigenfunctions on $\mathbb{R}$ - Pictures and Examples

Example: $V = 0$

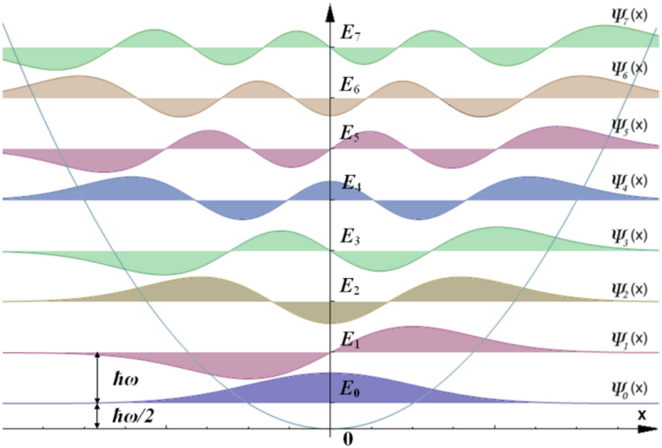
$$\left(-\frac{d^2}{d^2x}\right)\varphi_\lambda = \lambda\varphi_\lambda, \quad \varphi_\lambda = e^{i\sqrt{\lambda}x}, \quad \lambda \geq 0$$



Eigenfunctions on $\mathbb{R}$ - Pictures and Examples

Example: $V = x^2$

$$\left(-\frac{d^2}{d^2x} + x^2\right)\varphi_{\lambda_j} = \lambda_j \varphi_{\lambda_j}, \quad 0 = \lambda_0 <= \lambda_1 <=, \ldots \to \infty$$



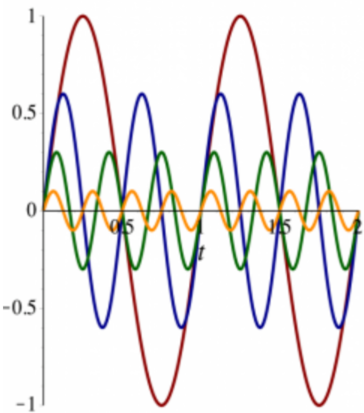
Laplace Eigenfunctions on manifolds - Pictures and Examples

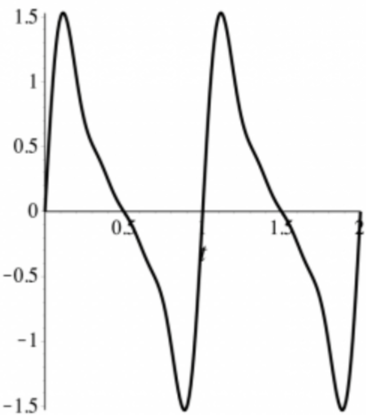
$$\Delta_g \varphi_{\lambda_j} = \lambda_j \varphi_{\lambda_j}, \quad 0 = \lambda_0 <= \lambda_1 <=, ... \to \infty$$

Laplace Eigenfunctions on manifolds - Pictures and Examples

Many of you may already know what these are if you are familiar with Fourier Series on the Circle S^1

$$\frac{d^2}{d^2\theta}\varphi_{n^2}(\theta) = n^2\varphi_{n^2}, \quad \varphi_{n^2}(\theta) = e^{in\theta}$$

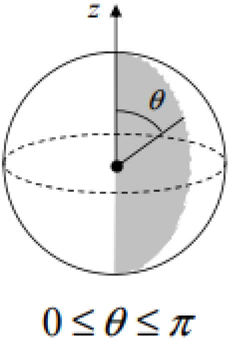


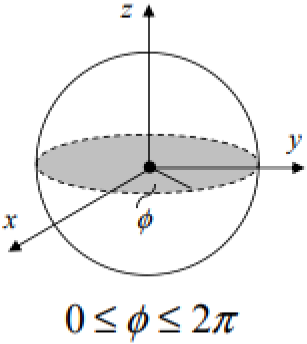


Laplace Eigenfunctions on manifolds - Pictures and Examples

On a sphere this equation takes the form

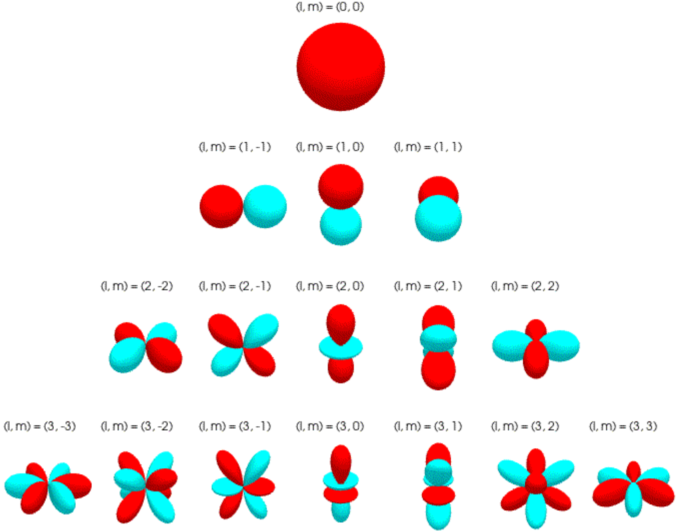
$$\left(\frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial \varphi^2} \right) Y_{\lambda_j} = \lambda_j Y_{\lambda_j}$$





Laplace Eigenfunctions on manifolds - Pictures and Examples

Pictures of Y_λ, Red = Positive, Blue = Negative



Why should we care about eigenfunctions?

Signal Processing

- Laplace eigenfunctions form an orthonormal basis for all functions on a surface. Waveforms are often decomposed into eigenfunctions.

Geometry

- Geometric information such as the volume and curvature of the surface is encoded in the distribution of eigenvalues and behavior of eigenfunctions.

Machine Learning

- Manifold learning seeks to "learn" the underlying shape of a point cloud. By estimating eigenfunctions and understanding their relation to geometry we can uncover structure in data.

Why should we care about eigenfunctions?

Quantum Mechanics

- $|\varphi_\lambda(x)|^2$ is a probability distribution of the position of a particle with energy λ

(Only on $\mathbf{R}$!) It's Fourier Transform

$$\hat{\varphi}(p) = \int_{\mathbf{R}} e^{-ix \cdot p} \varphi(x) dx$$

- $|\hat{\varphi}(p)|^2$ is a probability density describing the momentum of a particle of energy λ

My work - an overview

Focus: Overcoming the lack of a Fourier Transform on general surfaces in order to introduce joint position and momentum distributions for Laplace eigenfunctions and study their properties

Project 1

- Mathematical justification for a theory of position and momentum distributions for Laplace eigenfunctions inspired by quantum optics on $\mathbf{R}$.

Project 2

- Reflections of underlying geometry in these distributions and quantifying their rate of growth as $\lambda \to \infty$.

Joint Distributions of position and momentum

- Measuring position and momentum distributions separately is often costly and impractical.
- Desire: An **experimentally accessible** probability distribution P(x,p) such that

$$\int P(x,p)dx = |\hat{\varphi}(p)|^2, \quad \int P(x,p)dp = |\varphi(x)|^2$$

Joint Distributions of position and momentum

Wigner Quasi-Probability Distribution

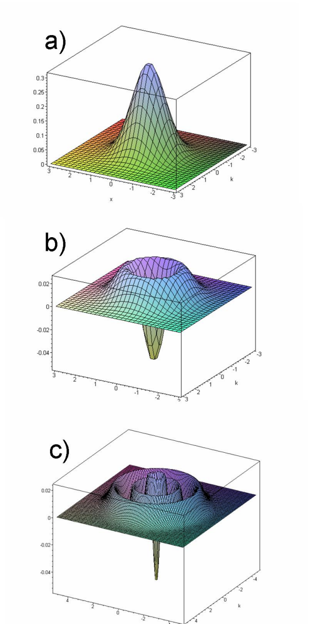
$$W(x,p) = \int_{\mathbf{R}} \overline{\varphi}(x+y)\varphi(x-y)e^{2ipy}dy$$

- Marginals

$$\int W(x,p)dx = |\hat{\varphi}(p)|^2, \quad \int W(x,p)dp = |\varphi(x)|^2$$

- Except for some very special states (that we will get to) there are regions where $W < 0$

Joint Distributions of position and momentum

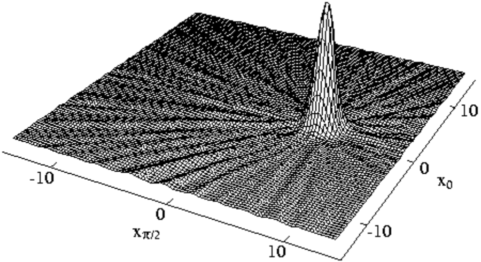


Joint Distributions of position and momentum

Lets introduce another type of phase space distribution that has no negative regions! Fix a position x, and a momentum p **coherent state** centered at (x,p)

$$C_{x,p}(y) = e^{ip\cdot(y-x)}e^{-\frac{|y-x|^2}{2}}$$

- The Wigner function of the coherent state is a bivariate Gaussian (Normal) distribution centered at (x,p)



Joint Distributions of position and momentum

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$$C_{x,p}(y) = e^{ip\cdot(y-x)}e^{-\frac{|y-x|^2}{2}}$$

- The same way that the Fourier Transform decomposes a function into plane waves, the Coherent State transform decomposes a function φ into Coherent States

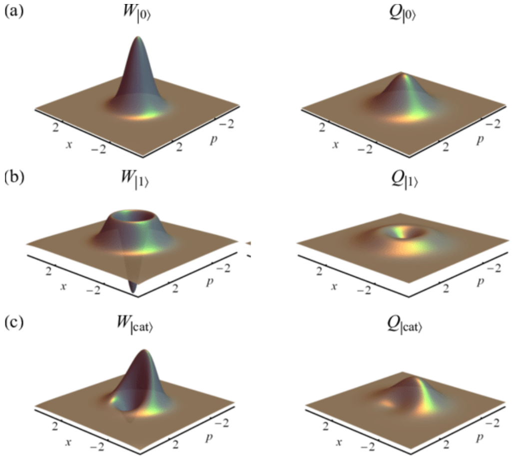
$$\varphi_H(x,p) = \int C_{x,p}(y)\varphi(y)dy$$

- These $|\varphi_H|^2$ are called **Husimi Quasi-Probability Distributions**

Joint Distributions of position and momentum

Husimi Distributions are regularizations of Wigner functions!

$$|\varphi_H|^2 = W_\varphi * G(z), \quad G(z) = e^{-|z|^2}$$



Joint Distributions of position and momentum

- Husimi Distributions are experimentally accessible quantities that has been realized in experiments in quantum optics.
- Positivity allows for a theory of entropy of phase space distributions (Wehrl Entropy)
- Used in the study of quantum effects in superconductors and in nucleon tomography

Constructing a complex phase space on a surface

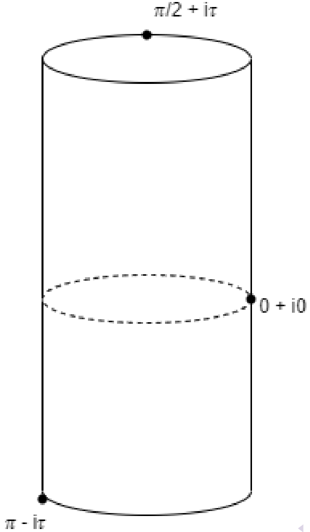
I will show you how to introduce Husimi Distributions of surface Laplace eigenfunctions in two steps

- Constructing a position and momentum space for the surface on which these distributions live
- Extending Laplace eigenfunctions to this phase space

We will carry out this construction for the circle, but the construction for any shape is similar.

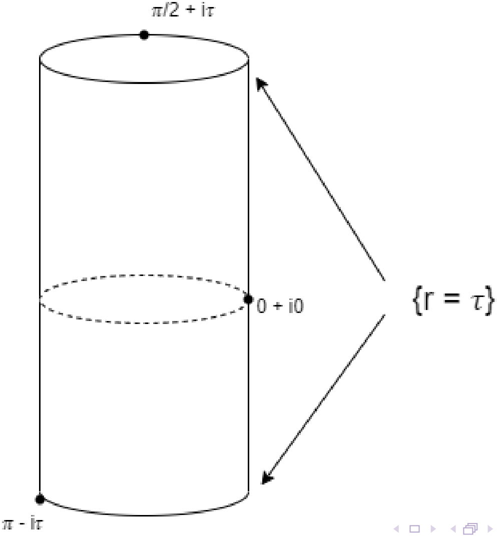
Constructing a complex phase space on a surface

We start with a circle, parameterized by an angle, θ, and simply add an imaginary component $i\tau$ to parameterize points on complex space



Constructing a complex phase space on a surface

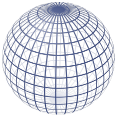
$r(\theta + i\tau) = |\tau|$ is a radius function describing how far into the cylinder we go.



Constructing a complex phase space on a surface

Consider the 2 dimensional sphere with the defining function

$$x_1^2 + x_2^2 + x_3^2 = 1$$



- Complex phase space is given by "Complexifying" the defining function to

$$(x_1 + iy_1)^2 + (x_2 + iy_2)^2 + (x_3 + iy_3)^2 = 1$$

Phase space distribution of eigenfunctions

Phase space distributions for Laplace eigenfunctions defined by "continuing" the function to complex phase space.

For a circle

$$e^{ik\theta} \to e^{ik(\theta+i\eta)} = e^{-k\eta}e^{ik\theta}$$

To associate phase space distributions to ANY function on S^1 all we do is "continue" its Fourier series to the complex phase space

$$f(\theta) = \sum_{-\infty}^{\infty} c_k e^{ik\theta} \to f_H(\theta + i\eta) = \sum_{-\infty}^{\infty} c_k e^{ik(\theta+i\eta)}$$

Phase space distribution of eigenfunctions

Phase space distributions for Laplace eigenfunctions defined by "continuing" the function to complex phase space.

For a general surface, we analytically continue φ_λ

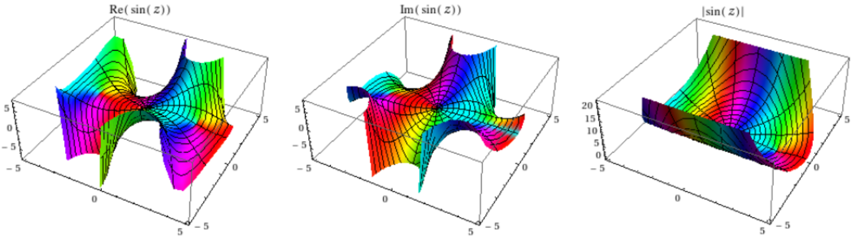
$$\varphi_\lambda(x) \to \varphi_{H,\lambda}(x+iy)$$

This allows us to analytically continue any function into complex phase space

$$f(x) = \sum_{\lambda_j} c_{\lambda_j} \varphi_{\lambda_j}(x) \to f(x+iy) = \sum c_{\lambda_j} \varphi_{H,\lambda_j}(x+iy)$$

Phase space distribution of eigenfunctions

A cross section of a Husimi Distribution for a spherical eigenfunction



A result on Husimi Distributions

- Individual Husimi Distributions on a general surface are too hard to study.
- We study weighted sums of Husimi Distributions instead.
- χ places most of the "weight" on φ_λ

$$\Pi_{\chi,\lambda}(u,v) = \sum_{\lambda_j < \lambda} \chi(\lambda - \lambda_j)\varphi_{\lambda_j,H}(u)\overline{\varphi}_{\lambda_j,H}(v)$$

A result on Husimi Distributions

At "frequency scale" on $r = \tau$ this weighted sum has the following formula

Chang–R. '21: Scaling asymptotics

$$\Pi_{\chi,\lambda}\left(\frac{u}{\sqrt{\lambda}}, \frac{v}{\sqrt{\lambda}}\right) = \frac{C}{\tau}\left(\frac{\lambda}{\tau}\right) e^{\frac{1}{\tau}\left(-\frac{|u|^2}{2} - \frac{|v|^2}{2} + v \cdot \overline{u}\right)}$$

$+$ lower order terms

Same form as sums of Husimi distributions of eigenfunctions of $-\frac{d^2}{dx^2} + x^2$

$$\sum_{\lambda} \varphi_{\lambda,H}(u)\overline{\varphi}_{\lambda,H}(v) = e^{\left(-\frac{|u|^2}{2} - \frac{|v|^2}{2} + v \cdot \overline{u}\right)}$$

End of Project 1

Questions?

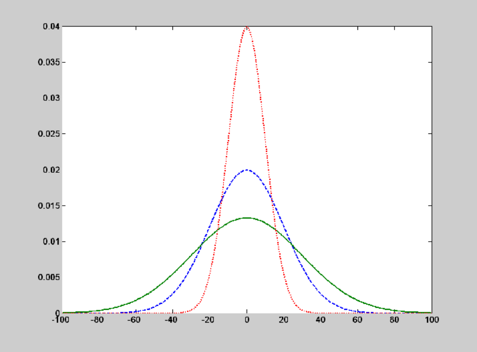
Project 2 - Concentration of Husimi Distributions

- We will focus on the question: when do eigenfunctions peak, and why?
- I will explain what I mean by "peaking" and how it relates to straight line trajectories on a surface

Eigenfunction peaking

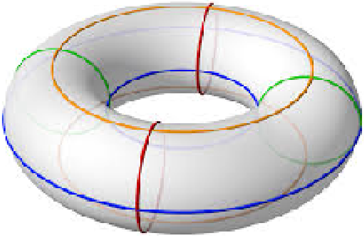
- Given a point x on a surface, how fast does $\varphi_\lambda(x)$ grow as a function of λ

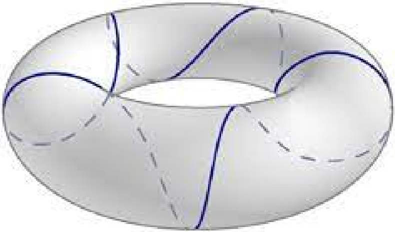
$$|\varphi_\lambda(x)| \leq f(\lambda), \quad \text{What is f?}$$



Straight Line Trajectories (geodesics)

- Given a point on a surface and a direction we can trace out a straight line trajectory (known as a geodesic)





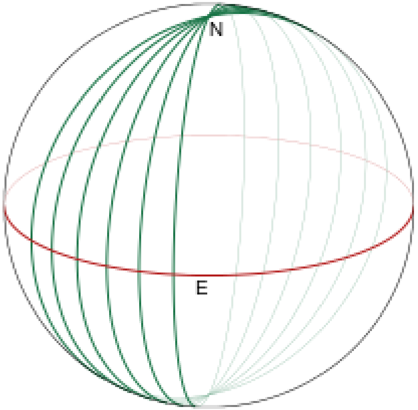
Straight Line Trajectories (geodesics)

I would like to focus on two types of geodesics both of which are found on a sphere

- Periodic Geodesics
- Stable Geodesics

Straight Line Trajectories (geodesics)

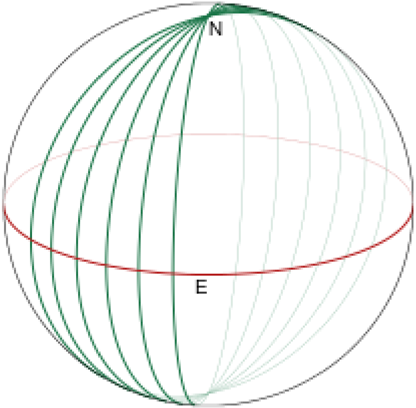
Periodic geodesics return back to the point they started from maintaining the same direction



Every geodesic starting at the north pole is periodic

Straight Line Trajectories (geodesics)

Stable geodesics when perturbed remain close to the original geodesic



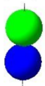
Slight perturbations of a great circle remains close

Peaking of eigenfunctions

- Foundational theorem, proved by Hörmander and true on ANY surface of dimension n

$$\sup_x |\varphi_\lambda(x)| \leq C\lambda^{\frac{n-1}{2}}$$

Equality occurs for eigenfunctions peaking at the poles which have many periodic geodesics!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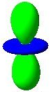
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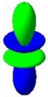
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Peaking of eigenfunctions

There are results on the growth of moments of φ_λ as well due to Sogge

$$\left(\int_S |\varphi_\lambda(x)|^q dx\right)^{\frac{1}{q}} \leq \begin{cases} C\lambda^{\frac{n-1}{2}(\frac{1}{2}-\frac{1}{q})}, & 2 \leq q \leq c_n \\ C\lambda^{n(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}}, & c_n \leq q \end{cases}$$

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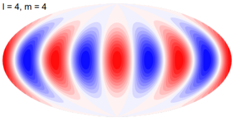
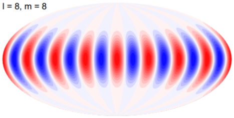
Equality for this small moment regime occurs for beam eigenfunctions along the stable equator





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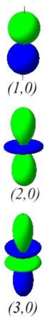
Equality for the large moment regime occurs for zonal harmonics which occur at points with many periodic geodesics!



My Result - Peaking of norms of Husimi Distributions

Chang-R '22: Peaking of Phase Space Norms

For any surface of dimension n on the boundary of its phase space

$$\left(\int |\varphi_{H,\lambda}(x)|^q dx\right)^{\frac{1}{q}} \leq C\lambda^{(n-1)(\frac{1}{2}-\frac{1}{q})} \qquad (2 \leq q \leq \infty).$$

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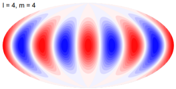
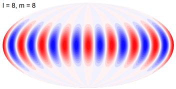
Equality is achieved for phase space distributions associated to beam eigenfunctions along stable geodesics!





End of Project 2

Why is there only one regime?

To be continued...