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Advanced Research Project

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A Topology for Two-Person Games and Evolutionary Dynamics of Iterated Play

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Abstract

The Prisoner's Dilemma has been the focus of much research in the field of Game theory with countless papers analysing its characteristics and applications. Its ability to model strategic conflict however, is not unique. This infamous dilemma is part of a broad class of games, most of which often go unexplored. In this paper, we introduce this class of two-person non-cooperative non-zero-sum games known as 2×2 *strict ordinal games*. Exploring the space of games and constructing a coherent topology, we aim to identify the similarities and differences between the various strategic situations of conflict and cooperation they represent. Following from the interest in the iterated Prisoner's Dilemma, we develop a Markov model for simulating iterated play for a broader range of games. Through the analysis of these simulations, we look to assess the correlation between successful strategy selection and the relationships between 2×2 games. We also present an alternative approach to strategy selection through the more recent study of evolutionary game theory. Analysing the replicator dynamics of two-population systems, we look to develop a toolkit for the analysis of asymmetric games.

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Signed : Abrar A. Phoplunker Date : 8^{th} June 2021

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Chapter 1

Introduction

In this project, we will investigate a range of two-person non-cooperative non-zero-sum games known as ***2×2 strict ordinal games***. As the simplest form of two-person game, these provide a good foundation for modelling strategic conflict. Due to their diversity, these games have much potential for real-world application, the most common example of this being the Prisoner's Dilemma. Despite extensive work having been done on a smaller set of these games and the topology they ascribe, they have been mostly overlooked in the context of iterated play.

1.1 Prisoner's Dilemma

The most prevalent of the larger set of two-person games, the Prisoner's Dilemma describes a situation where the strategy for common good is directly opposed to that of personal gain. A common analogy for the problem is as follows:

The police are separately interrogating two prisoners. They have sufficient evidence to convict both of a minor crime, sentencing them to 3 years in prison. They do however, suspect that the pair are guilty of a major crime for which there is insufficient evidence. In order to convict the prisoners, the police offer each of them the same deal. The prisoners are given the option to confess and implicate the other of the major crime and in doing so, earning a reduced sentence of 1 year whilst the other serves 10. If however, both prisoners implicate each other, they will both serve a sentence of 6 years. We can describe this scenario using the table below.

		Prisoner 2	
		Confess	Not Confess
Prisoner 1	Not Confess	1: 10 years 2: 1 year	1: 3 years 2: 3 years
	Confess	1: 6 years 2: 6 years	1: 1 year 2: 10 years

Table 1.1: Prisoner's Dilemma payoff table

From a utilitarian perspective, it is clear that the optimal outcome is achieved through both prisoners staying silent, receiving a sum total of 6 years. The dilemma arises from noticing the ability both prisoners have to unilaterally reduce their own sentence by confessing. This however, results in the worst possible outcome of a sum total of 12 years.

In this project, we will be considering ***normal-form games*** as shown in Table 1.1.

Definition 1.1. *A* ***normal-form game*** *is defined by* (N, A, g) *with the following properties:*

- $N = 1, 2, \ldots, n$ *is a finite set of* ***players***
- A_{S_i} *is the set of* ***pure strategies*** a_i *of player* i*;* $A = A_{S_1} \times \ldots \times A_{S_n} \ni a = (a_1, \ldots, a_n)$
- $g_i : A \to \mathbb{R}$ *is the* ***payoff function*** *of player* i*;* $g = (g_1, ..., g_n)$*.*

Here the payoff for player $i \in N$ is given $g_i(a_1, \ldots, a_n)$ where $a = (a_1, \ldots, a_n) \in A$ is played. This is a ***finite game*** if A is finite. In Chapter 2, we will go on to introduce some key ideas for solving the general N-player game.

For two-player games, as in the Prisoner's Dilemma, $A_{S_1} = A_S = \{a_1, a_2\}$ and $A_{S_2} = B_S = \{b_1, b_2\}$. These games can be expressed as a bi-matrix of payoffs. Looking at *strict ordinal games*, the value of the payoffs aims to provide a preference order and so we can reassign those for higher values to be good. In Table 1.2, we use the values $0, 1, 3, 5$ for payoffs in a reformulation of our original problem.

		Player 2	
		b_2	b_1
Player 1	a_1	(0, 5)	(3, 3)
	a_2	(1, 1)	(5, 0)

Table 1.2: Prisoner's Dilemma bi-matrix

Here a_1, b_1 correspond to cooperating (Not Confessing), C, with a_2, b_2 being the strategy of defecting, D. Comparing the payoffs for each strategy against a fixed opponent, we see that,

$$g_1(a_1, b_1) = 3 < g_1(a_2, b_1) = 5,$$
$$g_1(a_1, b_2) = 0 < g_1(a_2, b_2) = 1.$$

Hence, for Player 1, a_2 (defecting) is the best strategy. The same can be seen for Player 2, resulting in both players getting a payoff of 1 by playing a_2 and b_2. Calculating the optimal strategy for single play of this game is fairly straightforward, however what happens if they were to play repeatedly? Is Defecting Punished? Will they learn to Cooperate?

1.2 Axelrod and the Iterated Prisoner's Dilemma

In 1980, professor of Political Science Robert Axelrod held a tournament inviting strategies from participants all over the world for the Iterated Prisoner's Dilemma (IPD). This round-robin tournament pitted strategies against each other, playing repeated games of the Prisoner's Dilemma. Despite a large number of strategies being submitted, with various lengths and complexity, one of the simplest strategies, namely Tit-For-Tat (TFT), performed surprisingly well. Later, we aim to simulate our own versions of Axelrod's tournament, analysing the dynamics of strategy selection in the Prisoner's Dilemma and other 2×2 games.

1.3 Other popular games

Prisoner's Dilemma is but one possible arrangement of these payoffs and we will investigate games with alternative payoff structures later on. Some other games, which have been of particular interest in game theory research, are detailed below:

Example 1.1 (**Chicken**)
Another game which garnered much attention due to its ready applicability to conflict scenarios, Chicken, was popularised by Bertrand Russell. Russell (2009) develops the idea of Chicken as a metaphor for nuclear stalemate. Due to the direct conflict nature of the problem and the tension it gave rise to, it also gained notoriety in popular culture, a notable example being in the movie *Rebel Without a Cause (1995).* Here, the problem is depicted as a game consisting of two boys simultaneously driving their cars towards a cliff and jumping out at the last possible moment with the first to jump out being the "Chicken". Thus, we get the following payoff structure, prioritising not being "Chicken" and the worst possible outcome arising from both driving off the cliff or nuclear apocalypse in Russell's metaphor.

		Player 2	
		D	C
Player 1	C	(1, 5)	(3, 3)
	D	(0, 0)	(5, 1)

Table 1.3: Chicken bi-matrix

Example 1.2 (**Battle**)
Also known as *Bach or Stravinsky* (Osborne and Rubinstein, 1994), Battle presents a coordination game where the players choosing opposite strategies is in both of their best interests. A source of conflict however, is a battle to determine which of the two will obtain the greater payoff with the other receiving their second best payoff. A further point to note in the table below is the additional loss suffered from both players choosing to defect in an attempt to get the higher payoff.

		Player 2	
		D	C
Player 1	C	(3, 5)	(1, 1)
	D	(0, 0)	(5, 3)

Table 1.4: Battle bi-matrix

Example 1.3 (**Stag Hunt**)
Stag Hunt introduces a shift in priorities by providing each player with the option to reduce their own payoff but in doing so, reduce their opponent's further. Rousseau (2014) describes an analogy for this game with two hunters each given the following choice: pursue a stag which a lone hunter cannot catch or abandon the other in order to catch a rabbit for themselves. Here, a player may look to forego the greater payoff in order to obtain a payoff greater than their opponent.

		Player 2	
		D	C
Player 1	C	(0, 3)	(5, 5)
	D	(1, 1)	(3, 0)

Table 1.5: Stag Hunt bi-matrix

1.4 Overview

In the following chapters, we will explore a range of 2×2 games, highlighted key features of these games, how to solve them and the effects of repeated play.

In the next chapter, we will cover some of the basic theory of N-person non-cooperative games. We will introduce solution concepts and derive equations for payoff sets of 2×2 *strict ordinal games* going on to prove that there are only 78 unique of this type.

Chapter 3 will take a closer look at a smaller group of these games, Symmetric games, finding the relationships between them and the resulting geometry. These form an index for identifying and classifying all 78 *strict ordinal games*.

In Chapter 4, we will explore the Asymmetric games, categorizing them using inherent properties i.e. payoff sets, equilibria, etc. We will use this as to identify groups of similar games, building a coherent taxonomy through the categorization of these games.

Chapter 5 will introduce the strategies that will be considered when analysing the outcomes of repeated play for all *strict ordinal games*. Reproducing the tournaments run by Axelrod we will analyse the success of TFT against other strategies. By developing our own Markov model for tournaments of memory-one strategies, we will simulate iterated play across the broader set of all 2×2 games.

In Chapter 6, we will see more recent steps taken to analyse strategies through evolutionary game theory. Here we will simulate, visualise and analyse the replicator dynamics of strategies in two-person games.

Chapter 2

Two-Person Non-Cooperative Games

In this chapter we will be introducing the two-person, non-cooperative game. Though during this project we will mostly be encountering the most fundamental 2×2 game, we will begin by stating a few theorems and definitions for an arbitrary number, N, of players.

2.1 An Introduction to Non-Cooperative Game Theory

The theory of non-cooperative games explores games between individual players working independently for self-interest. The inability to communicate and bargain, unlike in cooperative games, means that any coalitions/alliances formed must be self-enforcing and so the theory aims to predict the actions of individuals as oppose to what coalitions will form.

2.1.1 Solution Concepts for Non-Cooperative Games

A simple start to solving games can be provided by eliminating dominated pure strategies.

Definition 2.1. *A strategy $a_i \in A_i$ is* ***strictly dominated*** *by a_i' if,*

$$g_i(a_i, a_{-i}) \leqslant g_i(a_i', a_{-i}), \forall a_{-i} \in A_{-i}$$

A more complete set of strategies can be provided by mixed strategies.

Definition 2.2. *A* ***mixed strategy*** *for i is given by a probability distribution over pure strategies A_{s_i}. For finite set $A_{s_i} = a_{i_1}, \dots, a_{i_m}$, a randomised mixed strategy α_i is given by probability distribution, say $p_1, \dots, p_m$ such that $p_i \geqslant 0$ and $\sum p_i = 1$. The randomised strategy is given, $\alpha_i = (p_1, \dots, p_m)$ or as a combination of pure strategies, $\alpha_i = p_1 a_{i_1} + \dots + p_m a_{i_m}$ with $\alpha_i \in \Delta_i$, the set of all mixed strategies for player i.*

From this we can see that ***pure strategies*** are just special cases of ***mixed strategies***, where $p_i = 1$ and $p_j = 0$, $\forall j \neq i$ for $i \in 1, \dots, m$. A simple solution for single play games is finding ***equilibrium points***.

Definition 2.3. *Let α^* be an N-tuple of mixed strategies and $E_i(\alpha^*)$ be the expected payoff of player i for mixed strategy profile α^*. If $\forall i, \forall \alpha_i \in \Delta_i$ we have*

$$E_i(\alpha^* || \alpha_i) \leq E_i(\alpha^*)$$

*then α^* is an **equilibrium point**. Here $E_i(\alpha^*||\alpha_i)$ is the expected payoff to i when α_i^* is replaced by α_i.*

In other words, no player can benefit from changing strategy if the others stay the same. The existence of these equilibrium points was first established by John Nash in 1950 and so were named ***Nash equilibrium* (NE)**. Nash et al. (1950) proved the existence of such equilibria in any finite, N-person game, where each player has a finite number of pure strategies. With the focus of this paper being on non-cooperative games, we state the following theorem and lemma from White (2018).

Theorem 2.1. *Every finite N-person non-cooperative game has at least one **Nash equilibrium**.*

Lemma 2.2. *For any mixed strategy profile α^*,*

$$\sup_{\alpha_i \in \Delta_i} E_i(\alpha^*||\alpha_i) = \sup_{a \in A_{S_i}} E_i(\alpha^*||a_i).$$

Using Lemma 2.2 from White (2018), we see that it suffices to check only pure strategies in Definition 2.3. This provides the simpler necessary and sufficient conditions for an N-tuple of (mixed) strategies to be a Nash equilibrium.

Theorem 2.3. *A necessary and sufficient condition for α^* to be **Nash equilibrium** is that, $\forall i$ and every pure strategy $a_i \in A_{S_i}$, we have,*

$$E_i(\alpha^*||a_i) \leq E_i(\alpha^*),$$

where $E_i(\alpha^*||a_i)$ is the expected payoff to i when α_i^* is replaced by pure strategy a_i.

2.2 Two-Person Games

Though the ideas demonstrated in 2.1 are consistent for games of any size, the determination of equilibrium points becomes increasingly complicated with the number of players and pure strategies. Our discussion will be restricted to the 2×2 case expressed as in Table 2.1.

		Player 2	
		b_2	b_1
Player 1	a_1	(a, a')	(b, b')
	a_2	(c, c')	(d, d')

Table 2.1: General 2×2 bi-matrix

2.2.1 Finding Equilibrium Pairs

The pure Nash equilibria of a 2×2 game can be found trivially by comparing each player's payoff. With Player 1 deciding the row and Player 2 the column, we see that the relevant comparisons are a to c and b to d for Player 1 and a' to b' and c' to d' for Player 2. With each comparison providing the preferred pure strategy, pure equilibria are found as fixed points of trajectories going to the higher payoff in each comparison.

Mixed Equilibrium Pairs

For the mixed strategies $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ for players 1 and 2, we have the expected payoffs are functions of p and q given,

$$E_1(p,q) = ap(1-q) + bpq + c(1-p)(1-q) + d(1-p)q$$
$$E_2(p,q) = a'p(1-q) + b'pq + c'(1-p)(1-q) + d'(1-p)q.$$

Observing E_1, we see that for any q, if $a(1-q)+bq \neq c(1-q)+dq$ then the optimal payoff can be obtained by Player 1 by selecting a pure strategy a_i i.e. $p=1$ if $a(1-q)+bq > c(1-q)+dq$, $p=0$ otherwise. Thus, in order to find the equilibrium pair, we find the strategy pair $((p^*, 1-p^*), (q^*, 1-q^*))$ such that,

$$a(1-q^*) + bq^* = c(1-q^*) + dq^*$$
$$a'p^* + c'(1-p^*) = b'p^* + d'(1-p^*).$$

Strategies $(p^*, 1-p^*)$ and $(q^*, 1-q^*)$ that satisfy the corresponding equation above are called **equalizer strategies** (ES) for Player 1 in Player 2's payoffs and vice-versa. Solving the pair of equations, we find the mixed equilibrium $((p^*, 1-p^*), (q^*, 1-q^*))$ where,

$$p^* = \frac{d'-c'}{a'-b'+d'-c'} \text{ and } q^* = \frac{a-c}{a-b+d-c}.$$

With the range of possible mixed strategies giving rise to a set of possible payoffs, we look to obtain geometric insight on the structure of this set.

2.2.2 Payoff Sets

Consider mixed strategies α, β for Players 1 and 2 in Prisoner's Dilemma game given in Table 1.2.

$$\alpha = pa_1 + (1-p)a_2$$
$$\beta = qb_1 + (1-q)b_2$$

with $\{a_1, a_2\} = A$ (Player 1's pure strategies) and $\{b_1, b_2\} = B$ (Player 2's pure strategies). It follows that the expected payoff pair $\in \mathbb{R}^2$ is,

$$pq(3,3) + (1-p)q(5,0) + p(1-q)(0,5) + (1-p)(1-q)(1,1)$$

we can rearrange this to get,

$$p(q(3,3) + (1-q)(0,5)) + (1-p)(q(5,0) + (1-q)(1,1))$$

with p $\in [0,1]$, it is clear that the expected payoff pair lies on the line connecting $(q(3,3)+(1-q)(0,5))$ to $(q(5,0)+(1-q)(1,1))$. The payoff set for the game is given by the union of these lines for q $\in [0,1]$.

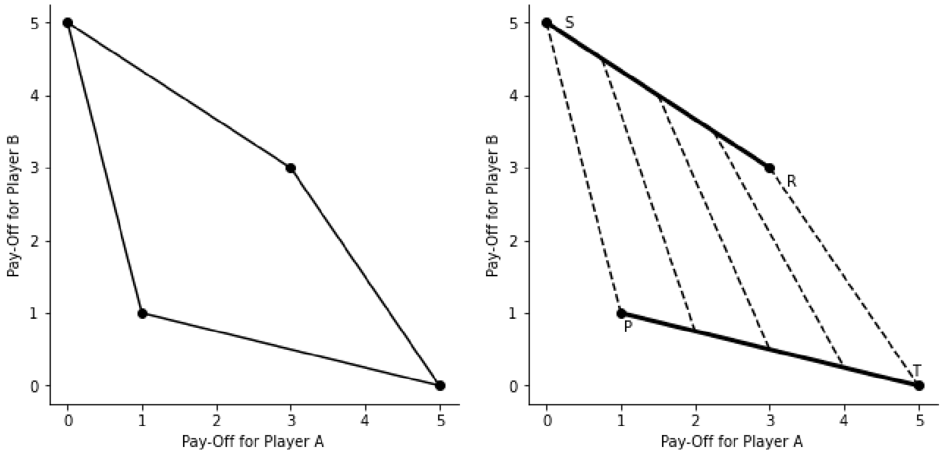


Figure 2.1: Construction of the payoff set for Prisoner's Dilemma

In the case of Prisoner's dilemma, we can see that the payout set generated by these lines is the convex hull of pure strategy payoffs as it would be in a cooperative game. The payoff sets however, aren't always the convex hull of payoff pairs. One example of this is given by Battle, where one boundary of the payoff set is given by a parabola between points $(3, 5)$ and $(5, 3)$.

Parabola from the envelope of non-parallel lines

Here we look to derive the parametric form of this parabolic boundary of the payoff set. In order to do this, we first consider a simpler version of the problem

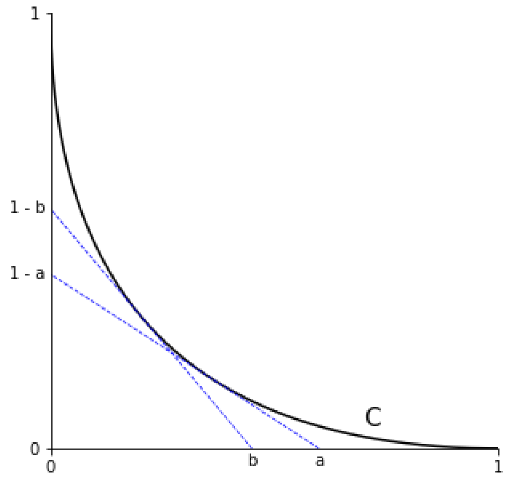


Figure 2.2: Parabola of a simple envelope

Let l_a be a line connecting point $(a, 0)$ to $(0, 1-a)$ for $a \in [0, 1]$. The family of lines l_a is called an *envelope* and form a set of tangent lines to the curve C. In order to find the point on l_a that is tangent to C we can consider another line l_b. The point on C can be found by the intersection of l_a and l_b as $b \to a$.

$$\lim_{b \to a}(ab, (1-a)(1-b)) = (a^2, (1-a)^2)$$

It can be seen from this that $\sqrt{x} + \sqrt{y} = 1$ and some simple manipulation gives us,

$$x^2 + y^2 - 2xy - 2x - 2x + 1 = 0$$

ensuring that the resulting curve is in fact a parabola.

Now we look to change the axis interval on which this envelope is constructed. We start by defining general linear functions, X and Y such that l_a now connects points $(X(a), 0)$ and $(0, Y(a))$.

$$X(a) = x_d a + x_0$$
$$Y(a) = y_d a + y_0$$

where $x_0 = X(0)$, $y_0 = Y(0)$, $x_d = X(1) - X(0)$ and $y_d = Y(1) - Y(0)$.

The intersect of lines l_a and l_b as $b \to a$, can now be found to be,

$$x = \frac{(x_d a + x_0)^2 y_d}{x_0 y_d - y_0 x_d}, \quad y = -\frac{(y_d a + y_0)^2 x_d}{x_0 y_d - y_0 x_d} \tag{2.1}$$

Further manipulation allows us to put this in standard form,

$$x^2(y_d)^2 + y^2(x_d)^2 + (2x_d y_d)xy - 2(x_0 y_d - y_0 x_d)(xy_d - yx_d) + (x_0 y_d - y_0 x_d)^2 = 0, \tag{2.2}$$

which evidently, is a parabola.

In order to find the envelope of two non-parallel lines not on the axis however, we require an affine transformation, $\mathcal{A}$, mapping intervals, $[x_0, x_d]$ and $[y_0, y_d]$, of the x and y axes onto the lines.

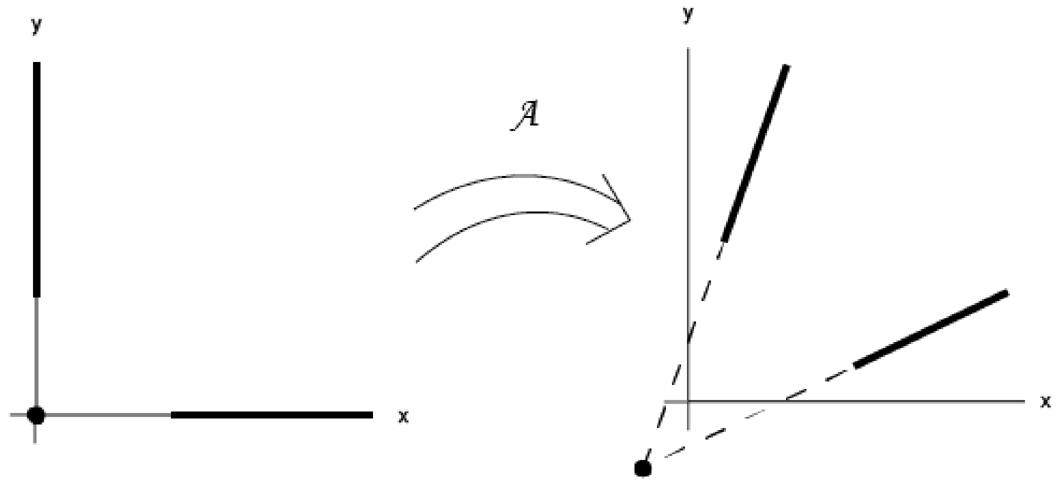


Figure 2.3: Affine transformation mapping axes segments onto non-parallel lines

The form of this transformation is given by,

$$\mathcal{A} : (x, y) \longmapsto (x + y + x_c,\ m_1 x + m_2 y + y_c) \tag{2.3}$$

with (x_c, y_c) being the intercept of the lines and m_1, m_2 being their respective gradients. Finally, we obtain the equation for the parabola by applying the transformation $\mathcal{A}$ to the equation for the parabola (2.2) or its parametric form (2.1).

Example 2.1
In the case of Battle, the above method gives us the Affine transformation:

$$\mathcal{A} : (x, y) \longmapsto (x + y + \frac{5}{7},\ 2x + \frac{3}{5}y + \frac{3}{7}).$$

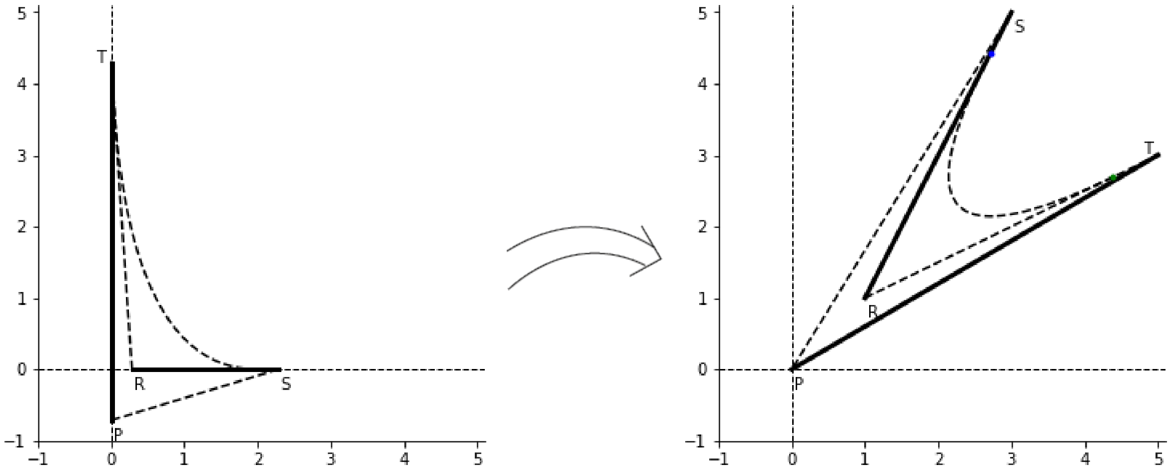


Figure 2.4: Affine transformation for Battle

As the points R, S, T and P are known, substitution into $\mathcal{A}$ results in a pair of simultaneous equations for x and y for each point, i.e. for $R = (R_x, R_y)$ we have $x + y + 5/7 = R_x$ and $2x + 3y/5 + 3/7 = R_y$. Solving these we obtain the required values x_0, x_d, y_0 and y_d. In this example, we find that the transformation $\mathcal{A}$ maps the interval $[\frac{2}{7}, \frac{16}{7}]$ of the x-axis onto line RS and $[\frac{-5}{7}, \frac{30}{7}]$ of the y-axis onto PT and the parabola in parametric form is given,

$$(7a^2 - 8a + \frac{31}{7},\ 7a^2 - 4a + \frac{19}{7}) \qquad a \in [0, 1]$$

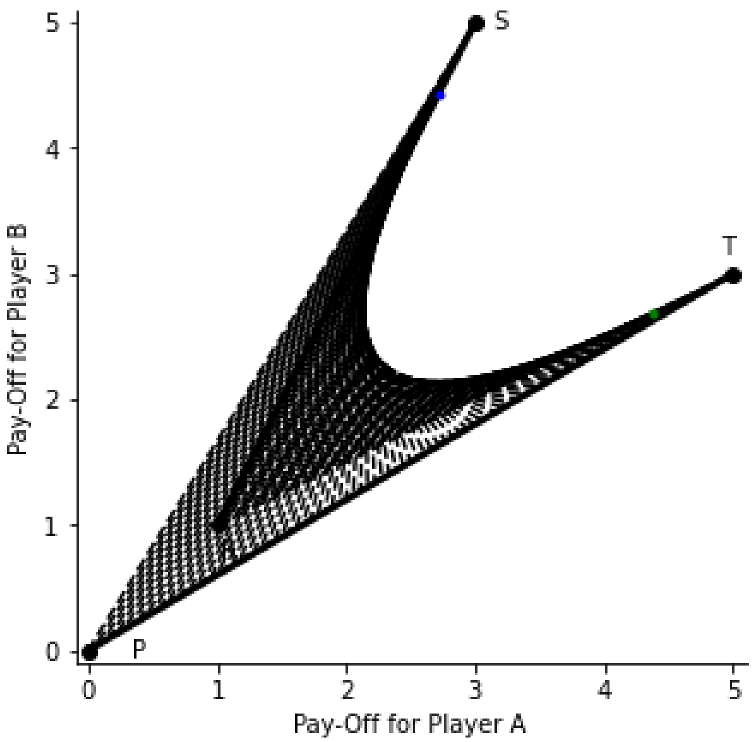


Figure 2.5: Battle Payoff Set

One thing that can be read directly from the payoff region is the relation of *dominance* between payoffs. The definition follows:

Definition 2.4. *Let (x, y) and (x', y') be two payoff pairs. Then (x, y) dominates (x', y') if,*

$$x \geq x' \text{ and } y \geq y'.$$

Thus, a point within the payoff region above and to the right of another point is dominant. Further we say, an equilibrium pair is said to be ***Pareto optimal*** if it is not dominated by any other payoff pair.

Returning to the game Battle, we identify two pure Nash equilibria $(5, 3)$ and $(3, 5)$, both of which are Pareto optimal. An additional mixed equilibrium pair, $((5/7, 2/7), (5/7, 2/7))$, can be found using the method detailed in 2.2.1, with the corresponding payoff pair $(15/7, 15/7)$. From the parametric form (2.3), we find that the highest symmetric payoff, given by the intercept of the parabola and $y = x$, is given by $a = 3/7$. As this gives the payoff pair $(16/7, 16/7)$, we see that the mixed equilibrium gives a ***Pareto-inferior*** payoff.

2.2.3 Finding 2×2 games

Here we aim to find the number of unique 2×2 games. We will however, be restricting our view to games with a clear preference order of payoffs for each player, using the payoff values {0, 1, 3, 5} as used in Axelrod (1980a).

Consider the general strict ordinal 2×2 game below, where the four payoffs for each player are distinct.

		Player 2	
		b_2 (D)	b_1 (C)
Player 1	a_1 (C)	(S, t)	(R, r)
	a_2 (D)	(P, p)	(T, s)

Table 2.2: General Bi-Matrix game

In the case of the Prisoner's Dilemma, we see that the payoff preference is as follows:

$$T > R > P > S \qquad\qquad t > r > p > s$$

To start finding the other games, notice that there are $4! = 24$ possible preference orders for each player's payoffs. The set of total 2×2 games is given by the product of the set of each player's orderings. With each player having 24 orderings, we get a total of $24^2 = 576$ total games though many of these games can be found to be equivalent. Note that relabelling (swapping) each player's strategies as well as swapping the players themselves will provide us with a game that is strategically identical to the one we started with.

Let X be the game bi-matrix,

$$X = \begin{bmatrix} (S, t) & (R, r) \\ (P, p) & (T, s) \end{bmatrix} = (A, B)$$

with the corresponding individual payoff matrices,

$$A = \begin{bmatrix} S & R \\ P & T \end{bmatrix} \text{ and } B = \begin{bmatrix} t & r \\ p & s \end{bmatrix}.$$

Define the row, column and player swap operators,

$$\hat{r} : \hat{r}X = \begin{bmatrix} (P,p) & (T,s) \\ (S,t) & (R,r) \end{bmatrix}$$

$$\hat{c} : \hat{c}X = \begin{bmatrix} (R,r) & (S,t) \\ (T,s) & (P,p) \end{bmatrix}$$

$$\hat{p} : \hat{p}X = \begin{bmatrix} (s,T) & (r,R) \\ (p,P) & (t,S) \end{bmatrix}$$

$$\hat{r}\hat{r} = \hat{c}\hat{c} = \hat{p}\hat{p} = \hat{e} \tag{2.4}$$

Noting that $\hat{c} = \hat{p}\hat{r}\hat{p}$ we can see that the group of all operators can be generated by $\hat{r}$ and $\hat{p}$. Thus, from (2.4), we can conclude that all possible transformations can be expressed as alternating sequences of $\hat{r}$ and $\hat{p}$. Here we look to determine the number of unique transformations provided by these sequences.

1. $\hat{e}$, identity operator
2. $\hat{r}$
3. $\hat{p}\hat{r}$ $(= \hat{c}\hat{p})$
4. $\hat{r}\hat{p}\hat{r}$ $(= \hat{c}\hat{r}\hat{p})$
5. $\hat{p}\hat{r}\hat{p}\hat{r}$ $(= \hat{c}\hat{r})$
$$\hat{p}\hat{r}\hat{p}\hat{r}X = \begin{bmatrix} (T,s) & (P,p) \\ (R,r) & (S,t) \end{bmatrix} = \hat{r}\hat{p}\hat{r}\hat{p}X$$
6. $\hat{r}\hat{p}\hat{r}\hat{p}\hat{r} = \hat{r}(\hat{r}\hat{p}\hat{r}\hat{p}) = \hat{p}\hat{r}\hat{p}$ $(= \hat{c})$
7. $\hat{p}\hat{r}\hat{p}\hat{r}\hat{p}\hat{r} = \hat{p}(\hat{p}\hat{r}\hat{p}) = \hat{r}\hat{p}$
8. $\hat{r}\hat{p}\hat{r}\hat{p}\hat{r}\hat{p}\hat{r} = \hat{r}(\hat{r}\hat{p}) = \hat{p}$

with $\hat{p}\hat{r}\hat{p}\hat{r}\hat{p}\hat{r}\hat{p}\hat{r} = \hat{e}$, any continuation of the sequence would just cycle through the above list.

Figure 2.6 below, provides as visualisation of the connections between these transformations. Using this representation, along with the arguments described above, we can see that each game can form a set of equivalent games given row, column and player symmetry.

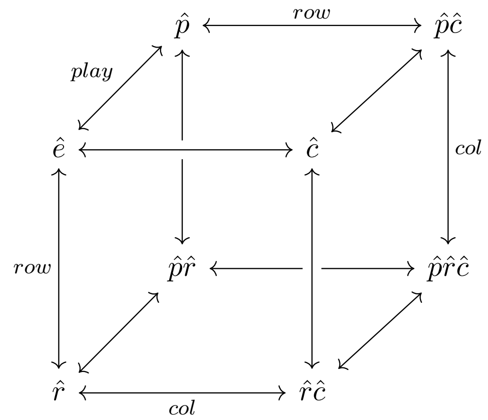


Figure 2.6: Graph of transformations connecting equivalent games

In order to find the number of unique games in the 576, we can use the following result.

Lemma 2.4 (Burnside's Lemma). *For finite group G that acts on set X, let X/G be set of orbits of X. Then, Burnside's lemma states that,*

$$|X/G| = \frac{1}{|G|} \sum_{\hat{g} \in G} |X^{\hat{g}}|$$

where $|X^g|$ is the subset of X that is fixed by $\hat{g}$.

In our case, G is defined by the 8 transformations seen above. To apply Burnside's we must find the number of elements in X, the set of 576 games, fixed by $\hat{g} \in G$. The following table lists the transformations along with their respective conditions for a game to be fixed and the number of games that meet said conditions.

$\hat{g}$	$\hat{g}X$	Conditions for $X = \hat{g}X$	$\lvert X^{\hat{g}}\rvert$
$\hat{e}$	$(S,t)\ (R,r)$ $(P,p)\ (T,s)$	-	576
$\hat{r}$	$(P,p)\ (T,s)$ $(S,t)\ (R,r)$	$S = P$	0
$\hat{p}\hat{r}$	$(r,R)\ (s,T)$ $(t,S)\ (p,P)$	$S = r = T$	0
$\hat{r}\hat{p}\hat{r}$	$(t,S)\ (p,P)$ $(r,R)\ (s,T)$	$(R,S,T,P) = (p,t,s,r)$	24*
$\hat{p}\hat{r}\hat{p}\hat{r}$	$(T,s)\ (P,p)$ $(R,r)\ (S,t)$	$S = T$	0
$\hat{p}\hat{r}\hat{p}$	$(R,r)\ (S,t)$ $(T,s)\ (P,p)$	$S = P$	0
$\hat{r}\hat{p}$	$(p,P)\ (t,S)$ $(s,T)\ (r,R)$	$S = p = T$	0
$\hat{p}$	$(s,T)\ (r,R)$ $(p,P)\ (t,S)$	$(R,S,T,P) = (r,s,t,p)$	24*

Table 2.3: Conditions for a game to be fixed by a transformation $\hat{g}$ and the magnitude of the subset of fixed games $X^{\hat{g}}$

Most cases above follow from noting that R, S, T, P are distinct with the result for cases marked * coming from seeing that there are only 4 free variables with the second player's payoff dependent on the first's, hence we have 4! possible games that fulfill the required condition.

Applying Burnside's with $|G| = 8$ and $|X^g|$ calculated above, we get the result,

$$\frac{1}{8}\ (576 + 0 + 0 + 24 + 0 + 0 + 0 + 24) = 78$$

In the following chapters we will go on to look at these 78 games in greater detail, looking to layout a comprehensive way to categorize them.

2.3 Iterated Prisoner's Dilemma

Returning to Prisoner's Dilemma, from the solution concepts detailed above, we can see that the one-shot version of the game would encourage each player to Defect with that being the dominant strategy, resulting in a Pareto-inferior Nash equilibrium payoff of 1. Clearly mutual cooperation provides a better (Pareto optimal) result for both players, this is where iterated play can change our perspective of what is a "good" strategy.

In iterated play, the players have an opportunity to adapt and "signal" an intention to cooperate without a need to explicitly confer. An opponent responding to this signal could lead to some sort of agreement between players for achieving the most mutually beneficial result. In chapter 5, we will go on to simulate the iterated variant of the game and observe how it affects the choice of strategy.

Chapter 3

Symmetric Games

Returning to the transformations listed in 2.2.3, consider the set of equivalent games for a given game X. We can notice that for X that is fixed by $\hat{p}$ i.e. $\hat{p}X = X$, we have the following:

$$\begin{aligned} \hat{r}\hat{p}\hat{r}(\hat{r}X) &= \hat{r}\hat{p}X = \hat{r}X \\ \hat{r}\hat{p}\hat{r}(\hat{p}\hat{r}X) &= \hat{r}\hat{r}\hat{p}\hat{r}\hat{p}X = \hat{p}\hat{r}X \\ \hat{p}(\hat{r}\hat{p}\hat{r}X) &= \hat{r}\hat{p}\hat{r}\hat{p}X = \hat{r}\hat{p}\hat{r}X \end{aligned}$$

And so, we see that for any game that is fixed by $\hat{p}$, the set of equivalent games consists of four games, two of which are fixed by $\hat{p}$ and two fixed by $\hat{r}\hat{p}\hat{r}$. We call a game that is fixed by $\hat{p}$ or $\hat{r}\hat{p}\hat{r}$, **symmetric**. Since these games form sets of four equivalent games, two of which are fixed by $\hat{p}$, we can use $|X^{\hat{p}}|$ from 2.3 to show that there are exactly 12 unique games of this type.

In order to standardise our analysis, we will, in each case, consider the $\hat{p}$ fixed equivalent of each of these games. Further, we will adopt a Cartesian layout of payoffs for each player with the greatest payoff for player 1 being in the right column and for player 2 in the top row. For a symmetric game with $(R, S, T, P) = (r, s, t, p)$, this condition can be expressed as,

$$\max(R, S, T, P) = R \text{ or } T \tag{3.1}$$

with the standard form of a game given,

$$X = \begin{bmatrix} (S,T) & (R,R) \\ (P,P) & (T,S) \end{bmatrix}. \tag{3.2}$$

From this requirement, we yield the 12 possible permutations of R,S,T and P. Each permutation defines a family of games, listed below with their names, as given in Goforth and Robinson (2004).

$$\begin{array}{cccc} \text{Chicken, Ch} & \text{Battle, Ba} & \text{Hero, Hr} & \text{Compromise, Cm} \\ T > R > S > P & T > S > R > P & T > S > P > R & T > P > S > R \\ \text{Deadlock, Dl} & \text{Prisoner's Dilemma, Pd} & \text{Stag Hunt, Sh} & \text{Assurance, As} \\ T > P > R > S & T > R > P > S & R > T > P > S & R > P > T > S \\ \text{Coordination, Co} & \text{Peace, Pc} & \text{Harmony, Ha} & \text{Concord, Nc} \\ R > P > S > T & R > S > P > T & R > S > T > P & R > T > S > P \end{array} \tag{3.3}$$

We will use a simplified notation, e.g. TRPS, to refer to the payoff preference ordering for a player, in this case, $T > R > P > S$ and * to represent the identical game with choices relabelled by swapping $S \leftrightarrow T$ and $R \leftrightarrow P$, i.e. *Pd can be represented as SPRT.

3.1 Geometry of Symmetric Games

We begin analysing the relationship between these games by viewing the effect of swapping adjacent payoffs. Using the connection from swapping adjacent payoffs, we can start mapping the games as triangles with neighbours obtained by the three adjacent payoff swaps. We show this below for the Prisoner's Dilemma in Figure 3.1.

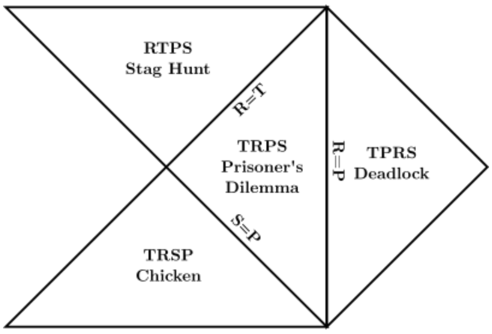


Figure 3.1: Prisoner's Dilemma and its adjacent swap neighbours

Here we see the game, Stag Hunt, can be obtained from the Prisoner's Dilemma by swapping the two highest payoffs R,T. Similarly, we obtain the games referred to as Chicken (by swapping S and P) and Deadlock (by swapping R and P). This gives us the neighbouring symmetric games of Prisoner's Dilemma. Repeating this for other games, we can map the 12 standard symmetric games onto a cube with each face divided into four triangular regions as shown below. The 24 regions generated here are the 24 $\hat{p}$ invariant games we found in Table 2.3.

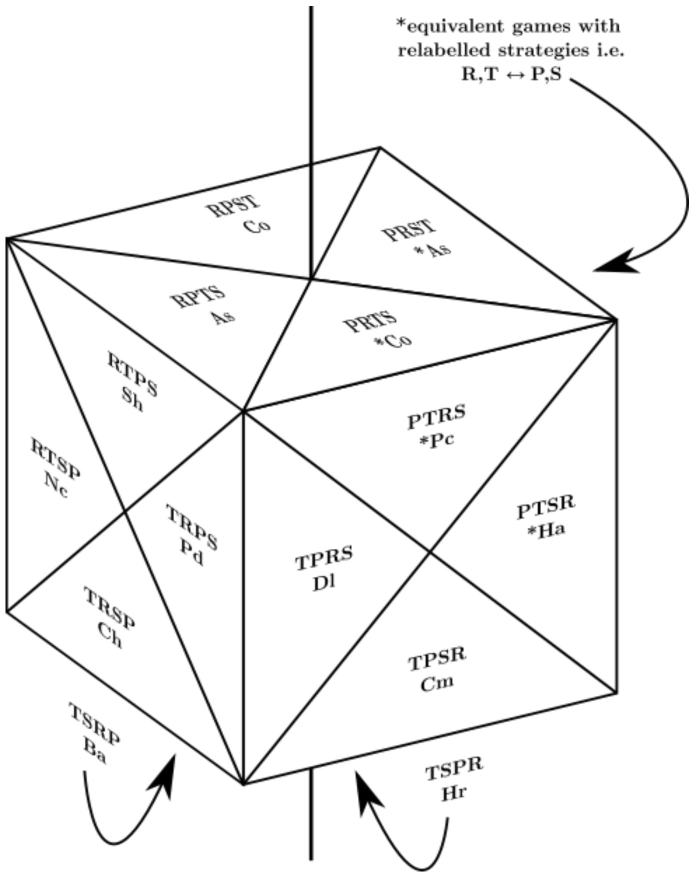


Figure 3.2: Cube map of 24 symmetric games

Thus, the surface of the cube maps the space of all symmetric 2×2 games. Here, the games on the lines separating triangular segments correspond to games with only 3 distinct payoffs and the vertices of the cube as well as points on faces where four triangular regions meet correspond to games with only 2 distinct payoffs. Though there is much to be explored in games with tied payoffs, we will restrict our view here to strict ordinal games with 4 distinct payoffs.

Noting, as we did before, that the set of 24 consists of 12 unique games, given in (3.3), along with their relabelled versions, it suffices to consider only half the net of the cube when looking to analyse games. Below we see a net for the half-cube mapping the space of all unique symmetric games with the 12 families of games with distinct payoffs as seen in (3.3) that follow our Cartesian standardisation.

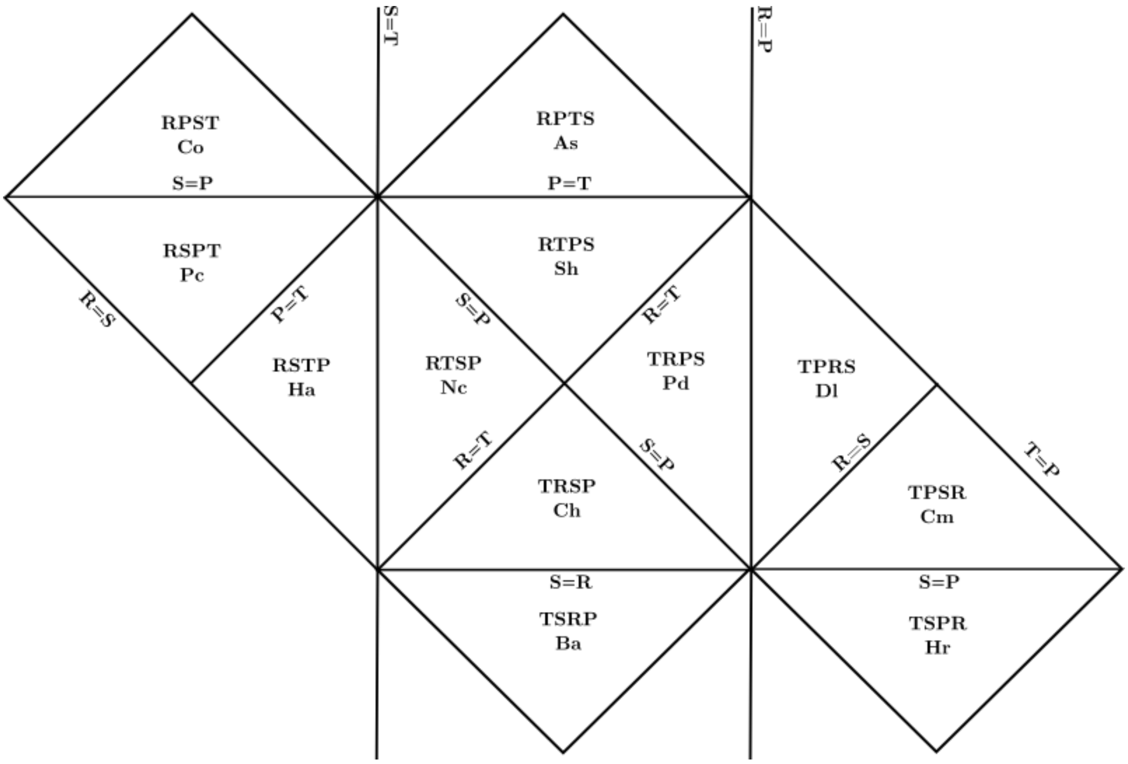


Figure 3.3: Net of half-cube mapping standard form symmetric games

An alternative net which maps the space of symmetric games onto a closed polyhedron is obtained by mapping each family of games to an equilateral triangle. The net for the resulting "flying octahedron" (Goforth and Robinson, 2004) is provided in Appendix A.1.

3.2 Classification of symmetric games

We now look to find characteristics for these games that can enable us to build a natural ordering and identify some underlying structure for the described geometry.

3.2.1 Nash Equilibria

We can begin by observing a key property of finite games, the Nash equilibria. Here we see that for all games where the largest payoff is given by R, there exists a pure Nash equilibrium at (R,R), hence when aiming to maximise one's score there is no conflict between the players. Further, for the subset of these with $S < P$ (Co, As, Sh), a second NE is present at (P,P) which is Pareto dominated by that at (R,R). We note further that, by the Oddness theorem

(Wilson, 1971), which states that in any finite game, the total number of NE must be odd, there must exist a third NE of mixed type. For the subset with $S > P$, we see that the strategy C dominates D, thus the NE at (R,R) is the only equilibrium and each player can guarantee getting a score greater than or equal to the other by choosing to cooperate (C). Recognising this lack-of-conflict Brams (1993) refers to these as **boring games**.

For games where $\max(R, S, T, P) = T$, named **conflict games**, we see that (R,R) is no longer a NE. We consider again, two cases as before. If $S > P$ (Ch, Ba, Hr) then there exist 2 pure NE, (S,T) and (T,S) as well as a third mixed NE. With the payoffs to each player being distinct in this case, we can determine the **fairness** of the equilibrium by the difference between S and T. For the last case, with $S < P$, containing Pd, we see that the strategy D dominates C, resulting in a NE at (P,P).

3.2.2 Effective play in symmetric games

An alternate approach to classifying these games, proposed by Goforth and Robinson (2012), allows us to categorize symmetric games using the maximum payoff, finding the best mutual outcome for a non-zero sum game in terms of total joint payoffs.

We begin by observing the traditional Prisoner's Dilemma, the requirements for which are given by the inequalities, $T > R > P > S$ and $2R > T + S$. This creates the "dilemma" with mutual cooperation yielding the higher payoff sum $2R$. For now however, we will forgo the second of these inequalities, considering the larger set of Prisoner's Dilemma games given by the triangular region seen above.

Noticing that adding a common value and multiplying by a positive constant preserves the relationship between payoffs, we can begin by normalising. We do this by subtracting $\min(R, S, T, P)$ and dividing by $\max(R, S, T, P) - \min(R, S, T, P)$, which, in the case of Prisoner's Dilemma, gives the normalised payoffs,

$$\left(\frac{R-S}{T-S}, 0, 1, \frac{P-S}{T-S}\right). \tag{3.4}$$

With only two variables, given by the normalised R and P, we can plot the set of games on the xy-plane. With $(x, y) = (P, R)$, for $x < y$ we have a Prisoner's Dilemma game and swapping the middle payoffs of Prisoner's Dilemma in (3.3) gives the family of Deadlock games in the region $x > y$. On the plot below, we also highlight the highest total payoff possible for each game, with the region given by the lines $x < 0.5$ and $x > y$ being Prisoner's Dilemma games that do not meet the second inequality of the traditional problem.

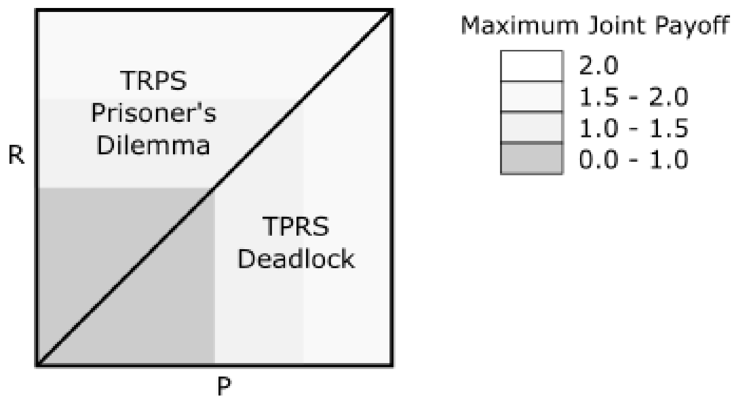


Figure 3.4: Pd-Dl middle swap plot indicating maximum joint payoff regions

Similarly, we can obtain plots for the other games, forming pairs that are connected to each other by swapping the middle payoffs. Below we illustrate the middle swap xy plots for other sets of games where $T = \max\{R, S, T, P\}$.

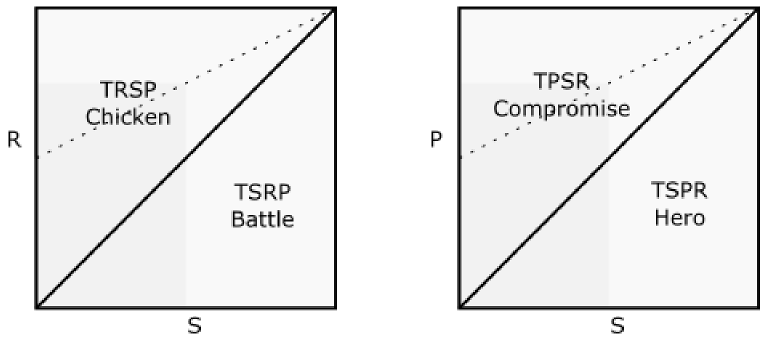


Figure 3.5: Ch-Ba and Cm-Hr middle swap plots for maximum joint payoff

We can say a strategy **effective** if it is able to obtain this optimal value for the joint payoff. Noting that for games where R is the greatest payoffs, the maximum joint payoff is $2R$, we can combine the plots to view the payoff regions on the net from Figure 3.3.

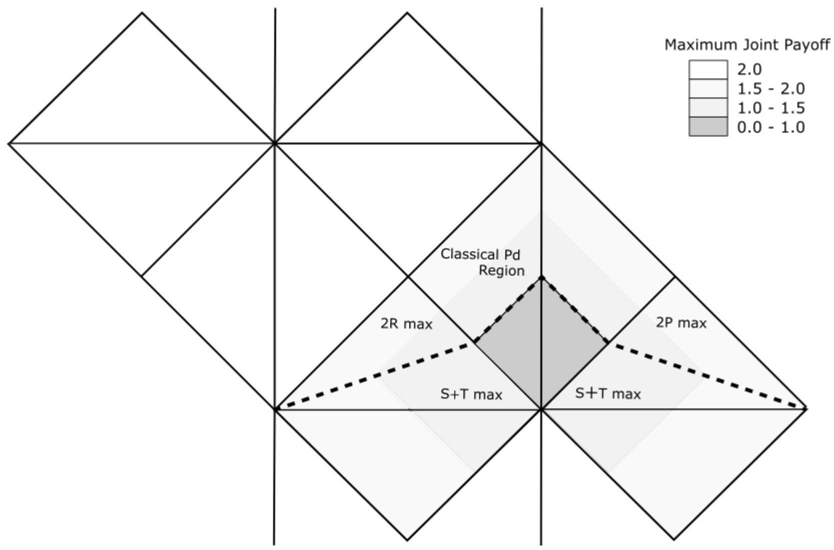


Figure 3.6: Half-cube net of maximum joint payoffs

Here, we see the coordination line above which coordination (choosing the same pure strategy) is favoured between players in order to achieve the optimal joint payoff given by $\max\{2R, 2P\}$. For games below the line, the most effective outcome is S + T and so, a fair optimal joint payoff is impossible in single-play and although players can't devise a cooperative plan, a sort of agreement can be reached through interactive behaviour in iterated play. This view allows us to find our first distinction between games, namely those in which coordination yields the optimal outcome R and those in which the optimal outcome is not fair in single play due to it being achieved by a asymmetric payoff S, T.

Earlier literature refers to the games we previously identified as boring games by the name **coordination games** due their regions lying entirely above the coordination line. Goforth and Robinson refer to Nc specifically as boring due to its apparent triviality when considering Nash equilibrium aiming to maximize individual score. By varying this approach slightly, looking instead to prioritize out-performing an opponent, we introduce a source of conflict in the three coordination games on the right of line S=T. Imposing this new incentive however, deviates from the scope of this paper since defining payoffs by the difference in scores would result in a zero-sum game. Returning to the non-zero sum case, we see this property commonly described in Stag Hunt, where a player can choose not to receive the symmetric payoff, switching to an option where they can instead aim to get a higher payoff than their opponent. Put concisely, the standard form of these games are given by the inequalities $R > T > S$. We call this group, containing Nc, As and Sh as **stag hunt games**.

For the other three games with a optimal joint payoff of 2R, Co, Pc and Ha, we see that with $R > S > T$, players have no similar incentive to move away from outcome R. We can refer to these as **no-conflict games**.

The remaining six games all contain subsets of games that have unfair optimal joint payoffs. Of the six, we see that only Hr and Ba lie completely under the coordination line so players aim to select different pure strategies in order to achieve most effective play. However, with the payoff pair being distinct values S and T, a source of conflict exists in their attempt to get the higher payoff.

An idea of effective strategy in these games can be suggested as one which is able to achieve this maximum value across all the symmetric games. Goforth and Robinson (2012) lays out the ideas described above, plotting the success of various strategies competing in symmetric games.

As we saw in the previous chapter, earlier research by Axelrod and Hamilton in the iterated version of the Prisoner's Dilemma helped highlight the benefits of altruism. With TFT receiving a lot of attention due to its success in Axelrod's tournaments, we may look to assess its success.

In an approach similar to that seen in Goforth and Robinson (2012), we will consider a set of simple strategies with which to run simulations using the python Axelrod library. Here we will consider a smaller set of strategies, described below:

- AllC : Always cooperates. (Yellow)
- AllD : Always defects. (Purple)
- CD Alt : Initial cooperation followed by alternating pure strategies.
- DC Alt : Initial defection followed by alternating pure strategies. (Teal)
- TFT : Initial cooperation followed by copying the opponent's previous move. (Green)

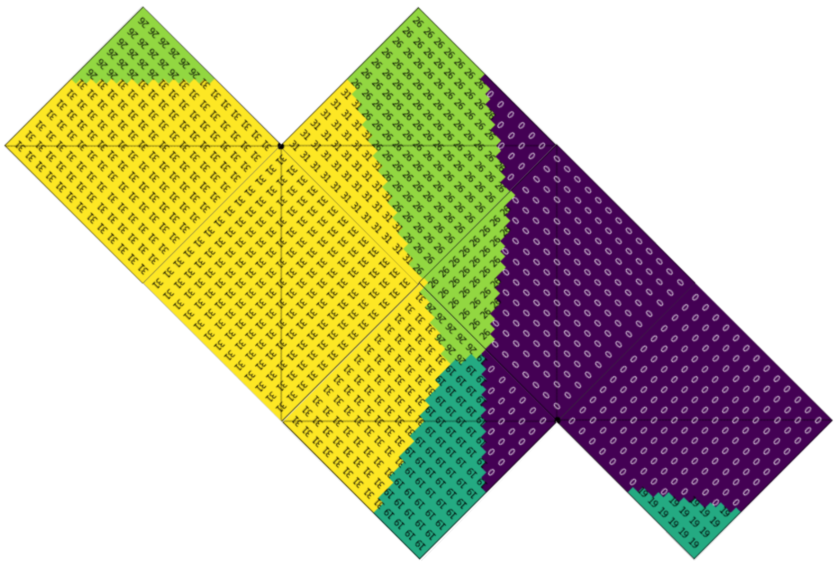


Figure 3.7: Winning strategies in iterated play on the space of symmetric games

Figure 3.7 divides the axes of each xy plot in intervals of 0.04, simulating iterated play for each of the resulting 625 games. It then displays the highest scoring strategy via the corresponding colour and its 5-bit code representation which we will discuss further in Chapter 5.

Here we see the success of the cooperative strategies (AllC, TFT) in games with R as optimal payoff as expected, however for Sh and As games with sufficiently large T, AllD wins. Noticing both of these as stag hunt games, this may be due to AllD's ability to exploit naive strategies such as AllC, reducing the joint payoff whilst still obtaining a payoff close to that of the optimal payoff. The success of TFT in this region can been seen through its resistance to such play, as well as the ability to obtain optimal payoff through games against AllC and itself.

In conflict games, with optimal joint payoff $S + T$, the alternating strategy DC Alt sees some success due to its ability to perfectly split the joint payoff with TFT and CD Alt. It does however, perform poorly against itself, being unable to obtain the best payoff, resulting in a fall off as either R or T decrease, regions where respectively, AllC and AllD prevail. Similarly, for regions with R sufficiently low, close to the vertex where Pd, Dl and Cm meet, the motivation to cooperate disappears resulting in clearer wins for AllD. We can however, recognise the Prisoner's Dilemma region here, in connection to Axelrod's results showing the region where TFT wins. In Figure 3.7 this region is approximately given by the additional constraint $R \geq 0.5 + 0.75P$.

We will go on to consider other characteristic properties of symmetric games when considering them in the set of all 2×2 games. However before doing so, we look to find a natural ordering with which to interpret the taxonomy of all 78 distinct games.

3.2.3 Ordering symmetric games

The swapping connection between games, provides a useful way to measure their similarity. It is clear that swapping the lowest payoffs would make the least change to a game, with middle payoff swaps being second-least we can try to find a natural ordering for these games. Through this process we find a path moving across the cube in Figure 3.2 creating a list of the 12 games.

$$\begin{tikzcd}
Pd \arrow[r, leftrightarrow, "mid"] \arrow[d, leftrightarrow, "high"] & Dl \arrow[r, leftrightarrow, bend right, "low"'] & Cm \arrow[r, leftrightarrow, "mid"] & Hr \arrow[r, leftrightarrow, bend right, "low"'] & Ba \arrow[r, leftrightarrow, "mid"] & Ch \arrow[r, leftrightarrow, bend right, "low"'] \arrow[d, leftrightarrow, "high"] & Pd \\
Sh \arrow[r, leftrightarrow, "mid"] & As \arrow[r, leftrightarrow, bend right, "low"'] & Co \arrow[r, leftrightarrow, "mid"] & Pc \arrow[r, leftrightarrow, bend right, "low"'] & Ha \arrow[r, leftrightarrow, "mid"] & Nc \arrow[r, leftrightarrow, bend right, "low"'] & Sh
\end{tikzcd}$$

Figure 3.8: Graph of mid, low and high swap connections

Here we took the classical Pd as a starting point finding neighbouring symmetric game. Using mid and low swaps, we obtained a loop of six games, the same set seen earlier with $\max(R, S, T, P) = T$ all centered around a single vertex in Figure 3.2. Exhausting the games that can be reached through low and mid swaps, we move on to Pd's high swapped neighbour Sh, low and mid neighbours of which also form a loop, this time using the remaining five games. And so, in order to highlight the similarities of neighbouring games, we can construct an ordering for the symmetric games as shown.

$$\left|\ \text{Nc} \quad \text{Ha} \quad \text{Pc} \quad \text{Co} \quad \text{As} \quad \text{Sh}\ \right|\ \text{Pd} \quad \text{Dl} \quad \text{Cm} \quad \text{Hr} \quad \text{Ba} \quad \text{Ch}\ \Big| \tag{3.5}$$

In the following chapter, we can use this ordering as a foundation on which to structure the space of all 78 2×2 games.

Chapter 4

Asymmetric Games

Though the symmetric games provide a broad range of games through the diversity of payoff structures present, they limit our view to situations where both parties have the same perspective. Here we look to use the symmetric games as a basis to derive the larger set of asymmetric games.

4.1 Constructing Asymmetric Games

Previously, we considered the larger set of payoffs, varying the middle pair for normalised minimum and maximum payoffs; we now restrict our view to the standard Axelrod payoffs to allow for comparison with Axelrod's results across our simulations in Chapter 5. It is important to note that in analysing these games, preference order is of greater importance and so, in other literature, it is common to encounter games described using this ordering as 1, 2, 3, 4. Using the Axelrod payoffs, we maintain this clear order whilst providing clearer distinction between the games we are considering and those of zero-sum.

To begin building the space of asymmetric games, we deconstruct the bi-matrix payoffs of the symmetric games into their Player 1 and Player 2 counterparts. Thus, we list possible individual payoff matrices, $A = \begin{smallmatrix}\mathrm{S} & \mathrm{R}\\ \mathrm{P} & \mathrm{T}\end{smallmatrix}$ & $B = \begin{smallmatrix}\mathrm{T} & \mathrm{R}\\ \mathrm{P} & \mathrm{S}\end{smallmatrix}$.

$$\begin{array}{lcccccccccccc} & \text{Nc} & \text{Ha} & \text{Pc} & \text{Co} & \text{As} & \text{Sh} & \text{Pd} & \text{Dl} & \text{Cm} & \text{Hr} & \text{Ba} & \text{Ch} \\ \text{Player 1, } A\text{:} & \begin{smallmatrix}1&5\\0&3\end{smallmatrix} & \begin{smallmatrix}3&5\\0&1\end{smallmatrix} & \begin{smallmatrix}3&5\\1&0\end{smallmatrix} & \begin{smallmatrix}1&5\\3&0\end{smallmatrix} & \begin{smallmatrix}0&5\\3&1\end{smallmatrix} & \begin{smallmatrix}0&5\\1&3\end{smallmatrix} & \begin{smallmatrix}0&3\\1&5\end{smallmatrix} & \begin{smallmatrix}0&1\\3&5\end{smallmatrix} & \begin{smallmatrix}1&0\\3&5\end{smallmatrix} & \begin{smallmatrix}3&0\\1&5\end{smallmatrix} & \begin{smallmatrix}3&1\\0&5\end{smallmatrix} & \begin{smallmatrix}1&3\\0&5\end{smallmatrix} \\ \text{Player 2, } B\text{:} & \begin{smallmatrix}3&5\\0&1\end{smallmatrix} & \begin{smallmatrix}1&5\\0&3\end{smallmatrix} & \begin{smallmatrix}0&5\\1&3\end{smallmatrix} & \begin{smallmatrix}0&5\\3&1\end{smallmatrix} & \begin{smallmatrix}1&5\\3&0\end{smallmatrix} & \begin{smallmatrix}3&5\\1&0\end{smallmatrix} & \begin{smallmatrix}5&3\\1&0\end{smallmatrix} & \begin{smallmatrix}5&1\\3&0\end{smallmatrix} & \begin{smallmatrix}5&0\\3&1\end{smallmatrix} & \begin{smallmatrix}5&0\\1&3\end{smallmatrix} & \begin{smallmatrix}5&1\\0&3\end{smallmatrix} & \begin{smallmatrix}5&3\\0&1\end{smallmatrix} \end{array} \tag{4.1}$$

Notice here that symmetric games of form $X = (A, B)$ are defined by the property $A = B^T$, where B^T is the transpose of B along the secondary diagonal.

With there now being 12 distinct payoff arrangements for each player, we can build the larger set of games generated by the combinations of these. We use the ordering (3.5) established in 3.2.3 as the two axes representing each player's payoff structure.

Contrary to our result from Chapter 2, below we see a table of the resulting 144 games expressed in bi-matrix form, with Player 1's payoffs indicated in red and Player 2's in blue. However, moving across the line of symmetric games (outlined in bold), on the secondary diagonal, we see that each asymmetric game is represented here as well as its player swapped equivalent, i.e. (Ch, Ba) and (Ba, Ch). Excluding these, we can count the remaining games to arrive at our previous calculation of 78 unique games.

	Nc	Ha	Pc	Co	As	Sh	Pd	Dl	Cm	Hr	Ba	Ch
Ch	1 3 3 5 0 0 5 1	1 1 3 5 0 0 5 3	1 0 3 5 0 1 5 3	1 0 3 5 0 3 5 1	1 1 3 5 0 3 5 0	1 3 3 5 0 1 5 0	1 5 3 3 0 1 5 0	1 5 3 1 0 3 5 0	1 5 3 0 0 3 5 1	1 5 3 0 0 1 5 3	1 5 3 1 0 0 5 3	1 5 3 3 0 0 5 1
Ba	3 3 1 5 0 0 5 1	3 1 1 5 0 0 5 3	3 0 1 5 0 1 5 3	3 0 1 5 0 3 5 1	3 1 1 5 0 3 5 0	3 3 1 5 0 1 5 0	3 5 1 3 0 1 5 0	3 5 1 1 0 3 5 0	3 5 1 0 0 3 5 1	3 5 1 0 0 1 5 3	3 5 1 1 0 0 5 3	3 5 1 3 0 0 5 1
Hr	3 3 0 5 1 0 5 1	3 1 0 5 1 0 5 3	3 0 0 5 1 1 5 3	3 0 0 5 1 3 5 1	3 1 0 5 1 3 5 0	3 3 0 5 1 1 5 0	3 5 0 3 1 1 5 0	3 5 0 1 1 3 5 0	3 5 0 0 1 3 5 1	3 5 0 0 1 1 5 3	3 5 0 1 1 0 5 3	3 5 0 3 1 0 5 1
Cm	1 3 0 5 3 0 5 1	1 1 0 5 3 0 5 3	1 0 0 5 3 1 5 3	1 0 0 5 3 3 5 1	1 1 0 5 3 3 5 0	1 3 0 5 3 1 5 0	1 5 0 3 3 1 5 0	1 5 0 1 3 3 5 0	1 5 0 0 3 3 5 1	1 5 0 0 3 1 5 3	1 5 0 1 3 0 5 3	1 5 0 3 3 0 5 1
Dl	0 3 1 5 3 0 5 1	0 1 1 5 3 0 5 3	0 0 1 5 3 1 5 3	0 0 1 5 3 3 5 1	0 1 1 5 3 3 5 0	0 3 1 5 3 1 5 0	0 5 1 3 3 1 5 0	0 5 1 1 3 3 5 0	0 5 1 0 3 3 5 1	0 5 1 0 3 1 5 3	0 5 1 1 3 0 5 3	0 5 1 3 3 0 5 1
Pd	0 3 3 5 1 0 5 1	0 1 3 5 1 0 5 3	0 0 3 5 1 1 5 3	0 0 3 5 1 3 5 1	0 1 3 5 1 3 5 0	0 3 3 5 1 1 5 0	0 5 3 3 1 1 5 0	0 5 3 1 1 3 5 0	0 5 3 0 1 3 5 1	0 5 3 0 1 1 5 3	0 5 3 1 1 0 5 3	0 5 3 3 1 0 5 1
Sh	0 3 5 5 1 0 3 1	0 1 5 5 1 0 3 3	0 0 5 5 1 1 3 3	0 0 5 5 1 3 3 1	0 1 5 5 1 3 3 0	0 3 5 5 1 1 3 0	0 5 5 3 1 1 3 0	0 5 5 1 1 3 3 0	0 5 5 0 1 3 3 1	0 5 5 0 1 1 3 3	0 5 5 1 1 0 3 3	0 5 5 3 1 0 3 1
As	0 3 5 5 3 0 1 1	0 1 5 5 3 0 1 3	0 0 5 5 3 1 1 3	0 0 5 5 3 3 1 1	0 1 5 5 3 3 1 0	0 3 5 5 3 1 1 0	0 5 5 3 3 1 1 0	0 5 5 1 3 3 1 0	0 5 5 0 3 3 1 1	0 5 5 0 3 1 1 3	0 5 5 1 3 0 1 3	0 5 5 3 3 0 1 1
Co	1 3 5 5 3 0 0 1	1 1 5 5 3 0 0 3	1 0 5 5 3 1 0 3	1 0 5 5 3 3 0 1	1 1 5 5 3 3 0 0	1 3 5 5 3 1 0 0	1 5 5 3 3 1 0 0	1 5 5 1 3 3 0 0	1 5 5 0 3 3 0 1	1 5 5 0 3 1 0 3	1 5 5 1 3 0 0 3	1 5 5 3 3 0 0 1
Pc	3 3 5 5 1 0 0 1	3 1 5 5 1 0 0 3	3 0 5 5 1 1 0 3	3 0 5 5 1 3 0 1	3 1 5 5 1 3 0 0	3 3 5 5 1 1 0 0	3 5 5 3 1 1 0 0	3 5 5 1 1 3 0 0	3 5 5 0 1 3 0 1	3 5 5 0 1 1 0 3	3 5 5 1 1 0 0 3	3 5 5 3 1 0 0 1
Ha	3 3 5 5 0 0 1 1	3 1 5 5 0 0 1 3	3 0 5 5 0 1 1 3	3 0 5 5 0 3 1 1	3 1 5 5 0 3 1 0	3 3 5 5 0 1 1 0	3 5 5 3 0 1 1 0	3 5 5 1 0 3 1 0	3 5 5 0 0 3 1 1	3 5 5 0 0 1 1 3	3 5 5 1 0 0 1 3	3 5 5 3 0 0 1 1
Nc	1 3 5 5 0 0 3 1	1 1 5 5 0 0 3 3	1 0 5 5 0 1 3 3	1 0 5 5 0 3 3 1	1 1 5 5 0 3 3 0	1 3 5 5 0 1 3 0	1 5 5 3 0 1 3 0	1 5 5 1 0 3 3 0	1 5 5 0 0 3 3 1	1 5 5 0 0 1 3 3	1 5 5 1 0 0 3 3	1 5 5 3 0 0 3 1

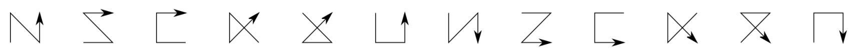
Figure 4.1: Table of 2×2 strict ordinal games with symmetric games on the secondary diagonal

Thus, we obtain a taxonomy and nomenclature for identifying and naming 2×2 games, first proposed by Goforth and Robinson (2004). Though a more extensive exploration of this space in presented in their book, *The Topology of the 2×2 Games*, in the following section we look to summarise a few key results to help illustrate the relationships present amongst the 2×2 games.

4.2 A topology for the 2×2 games

We begin by returning to the individual payoff structures of the symmetric games, this time using arrows to show the preference ordering.

$$\begin{array}{cccccccccccc} \text{Nc} & \text{Ha} & \text{Pc} & \text{Co} & \text{As} & \text{Sh} & \text{Pd} & \text{Dl} & \text{Cm} & \text{Hr} & \text{Ba} & \text{Ch} \\ \begin{smallmatrix}1&5\\0&3\end{smallmatrix} & \begin{smallmatrix}3&5\\0&1\end{smallmatrix} & \begin{smallmatrix}3&5\\1&0\end{smallmatrix} & \begin{smallmatrix}1&5\\3&0\end{smallmatrix} & \begin{smallmatrix}0&5\\3&1\end{smallmatrix} & \begin{smallmatrix}0&5\\1&3\end{smallmatrix} & \begin{smallmatrix}0&3\\1&5\end{smallmatrix} & \begin{smallmatrix}0&1\\3&5\end{smallmatrix} & \begin{smallmatrix}1&0\\3&5\end{smallmatrix} & \begin{smallmatrix}3&0\\1&5\end{smallmatrix} & \begin{smallmatrix}3&1\\0&5\end{smallmatrix} & \begin{smallmatrix}1&3\\0&5\end{smallmatrix} \end{array} \tag{4.2}$$



Here we can notice that the latter six are just horizontal reflections of the first. With a similar result for player 2's payoffs, Goforth and Robinson propose a numbering scheme based of three indices r, c and l, where r and c are the respective payoff patterns for each player numbered,

$$\begin{array}{cccccc} \text{Pd} & \text{Dl} & \text{Cm} & \text{Hr} & \text{Ba} & \text{Ch} \\ \mathbf{1} & \mathbf{6} & \mathbf{5} & \mathbf{4} & \mathbf{3} & \mathbf{2} \end{array} \tag{4.3}$$

The relative orientation of the payoff structure is then given by $l \in \{1, 2, 3, 4\}$ where $l = 1$ corresponds to Player 1's highest payoff being, and so the arrow pointing to, the bottom-right and Player 2's, the top-left. Using this system, we can express the space of games using a three coordinate system where Prisoner's Dilemma is given g_{111}.

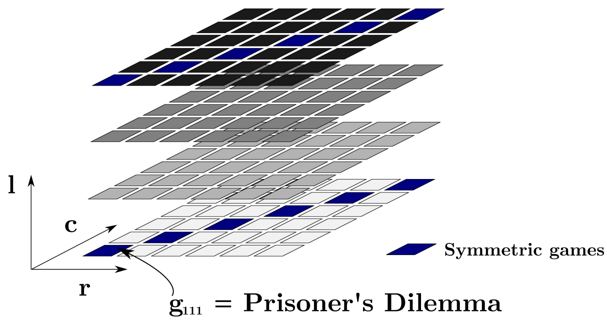


Figure 4.2: Index representation of games using row, column and layer

With the numbering being arbitrary, Goforth and Robinson choose to center the indexing around Prisoner's Dilemma due to its prevalence in game theory research. Using this representation of games, we see that the 144 games can be visualised as four layers of 36 games as shown in Figure 4.2, with a r, c, l index for each game.

4.2.1 Structuring layers

We now look find relationships between games within the layers given by Goforth and Robinson's index.

The payoff matrix is the defining characteristic of a game and in recognising this, we can say that games with similar payoff matrices are themselves, more similar. We then look to transform an initial game the least possible amount to obtain the game closest related to it. This was our motivation behind the structure produced in the previous chapter, where we transformed the payoff matrices of both players. Intuitively, changing only one player's payoffs would create a game more similar to the initial one. Following these ideas, we consider Layer 1, which contains the infamous Prisoner's Dilemma.

	Pd	Dl	Cm	Hr	Ba	Ch
Ch	1 5 3 3 0 1 5 0	1 5 3 1 0 3 5 0	1 5 3 0 0 3 5 1	1 5 3 0 0 1 5 3	1 5 3 1 0 0 5 3	1 5 3 3 0 0 5 1
Ba	3 5 1 3 0 1 5 0	3 5 1 1 0 3 5 0	3 5 1 0 0 3 5 1	3 5 1 0 0 1 5 3	3 5 1 1 0 0 5 3	3 5 1 3 0 0 5 1
Hr	3 5 0 3 1 1 5 0	3 5 0 1 1 3 5 0	3 5 0 0 1 3 5 1	3 5 0 0 1 1 5 3	3 5 0 1 1 0 5 3	3 5 0 3 1 0 5 1
Cm	1 5 0 3 3 1 5 0	1 5 0 1 3 3 5 0	1 5 0 0 3 3 5 1	1 5 0 0 3 1 5 3	1 5 0 1 3 0 5 3	1 5 0 3 3 0 5 1
Dl	0 5 1 3 3 1 5 0	0 5 1 1 3 3 5 0	0 5 1 0 3 3 5 1	0 5 1 0 3 1 5 3	0 5 1 1 3 0 5 3	0 5 1 3 3 0 5 1
Pd	0 5 3 3 1 1 5 0	0 5 3 1 1 3 5 0	0 5 3 0 1 3 5 1	0 5 3 0 1 1 5 3	0 5 3 1 1 0 5 3	0 5 3 3 1 0 5 1

Figure 4.3: Table of games on Layer 1

Low Swaps

Taking Ba, we can create a a set of four games consisting of combinations of Ba and Hr, each connected by swapping the lowest two payoffs. Figure 4.4 below, shows the described tile of four games.

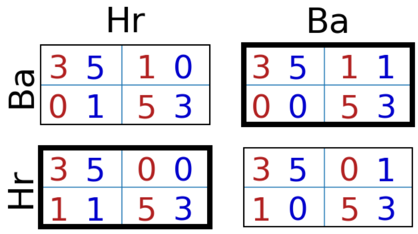


Figure 4.4: Low swap tile of Ba-Hr

Similarly, we can split the entire layer into two by two tiles, connected by low swaps. The low swap tiles wrap around the edges of the layer, top to bottom and left to right, with Pd-Ch forming a tile on the four corners.

Middle Swaps

Continuing from the arguments made above, we can claim that the second least significant change we can make to the payoff matrices is intuitively swapping the middle payoffs. Once again, due to our initial symmetric ordering from 3.2.3, these form two by two tiles intersecting four adjacent low swap tiles. The middle swap tile for Ch-Ba is shown below.

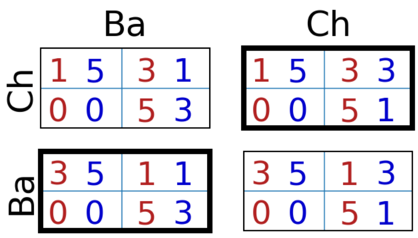


Figure 4.5: Mid swap tile of Ch-Ba

By noting the connections made across the layer, using the group of low and mid swap operations along with the identity operator, namely, $\{identity, low_A, low_B, mid_A, mid_B\}$, we can conclude, in the topology proposed by Goforth and Robinson, that each layer forms a torus. Here the mid swap operations form the connections between low swaps tiles which wrap across the layer, top to bottom and left to right. We can confirm this using the **Euler-Poincare** characteristic.

$$\text{Euler Number} = V + F - E \tag{4.4}$$

Viewing each game on the layer as a vertex with the swap connections being edges, we have $V = 36$, $E = 72$. Each vertex is surrounded by four faces and each face has four edges, hence $F = 36$. Using these, we get a Euler number of 0, proving that each layer can be mapped onto a torus.

4.2.2 Connecting layers

Visualising the connection between the four layers proves to be a trickier task but it can be seen that swapping the two highest payoffs moves to a subsequent game on another layer, connecting the four toroidal layers with the group of operations, $\{identity$, low_A, low_B, mid_A, mid_B, $high_A$, $high_B\}$, the six payoff swap operations and the identity operator. The method detailed above excludes the *high* swap operators utilising a proper subgroup of operations {*identity*, low_A, low_B, mid_A, mid_B}, to generate partitions of the entire space of 2×2 games, constructing four layers.

In order to establish the topological link across layers, Goforth and Robinson (2004) utilize an alternative subgroup {*identity*, low_A, low_B, $high_A$, $high_B$}, excluding mid swaps. Goforth and Robinson provide a much more mathematically rigorous description for the topological connections made by high swaps using the partition generated by this subgroup. They find that the space of 144 games, formed by connecting the 4 toroidal layers, can be mapped onto a torus with 37 holes using topological features of pipes and hot-spots in their book *The Topology of 2×2 games*.

4.3 Classifying Games and Equilibria

Following the intuitive arguments from 3.2 and connections described in the previous section, we can begin to identify similarities between games. Here we borrow the table of 2×2 games from Bruns (2015) which follows the topology described earlier. Note however, that the payoffs used in Figure 4.6 demonstrate a priority structure, with the numbers representing each player's preference order (with 4 - highest preference) as oppose to the payoff values we use in this project.

The analysis of equilibria across the set of all games, consists of both the type of equilibria present in each game as well as the number of pure equilibria. We can begin by noting a classification for types of equilibria, looking to highlight previously addressed concepts of effectiveness and fairness in payoffs, described as in the payoff families table in Figure 4.6.

Using the payoff structures and ordering of the symmetric games in (4.2), we can observe another feature exhibited by the layers in this topology. Splitting each layer into four quadrants, we see that these differ by the existence of dominant pure strategies. With the structures of Nc, Ha, Pc, Pd, Dl and Cm all having dominant strategies, we see that within each layer, games in the bottom-left quadrant have dominant strategies for both players. The top-right quadrant has no dominant strategies whilst the remaining two quadrants each have dominant strategies for one of the two players. Figure 4.6 indicates this property using the pair of arrows at the end of each quadrant with both arrows pointing the same way representing the presence of a dominant strategy. Thus, the ordering established in 3.2 gives rise to further patterns when considering the number of NE present. With the existence of dominant strategies, we fix either a column, row or both in the payoff table. As the payoffs in the unfixed axis are distinct, we then obtain a single NE. Due to the absence of dominant strategies, the top-right quadrant of each layer deviates from this, with the said quadrant in Layer 1 (as shown with Layer 3) having 3 NE and Layers 2 and 4 containing **cyclic** games, with no pure NE.

4.3.1 Layer 3: Win-win Games

By initially observing Layer 3, we notice that games constructed from combinations of boring games all have a **win-win** type equilibria obtained from mutual cooperation (Bruns, 2020). With layers being connected by high swaps we note that all equilibria of this type are confined to Layer 3.

We can further note, in combinations of Sh, As and Co, there exists a second equilibrium due to P being greater than S for both players, resulting in a (P, P) equilibrium. Bruns proposes an alternative categorization for stag games to that presented in 3.2.2, instead recognising the existence of two pure NE, one win-win and the other Pareto-inferior, as a definition for stag games. We will compare these definitions later, referring to the latter as **Bruns' stag games**.

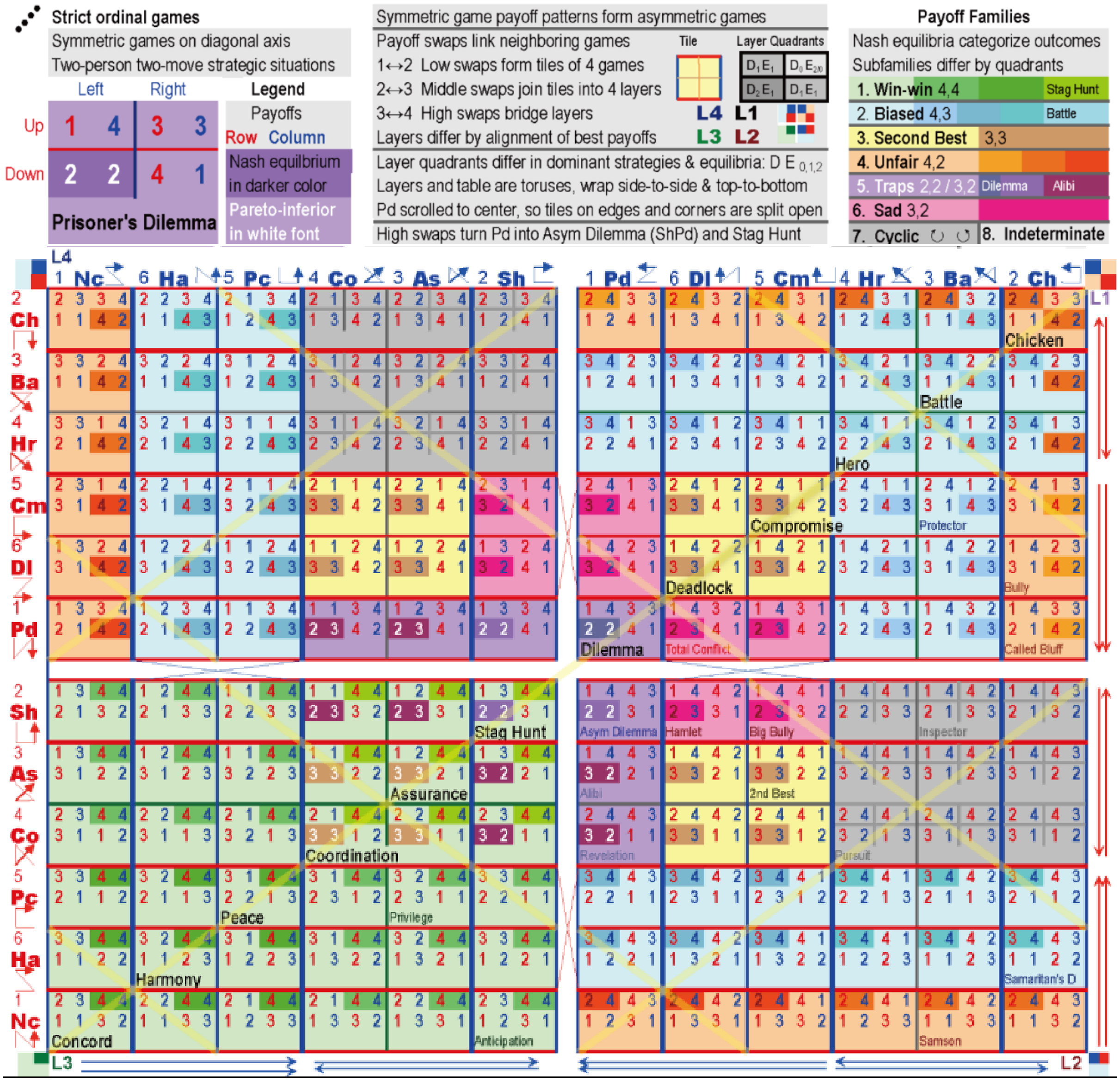


Figure 4.6: Table of all 2×2 strict ordinal games from Bruns (2012), categorizing the Nash equilibria using the Payoff Families table above

The types of the Pareto-inferior equilibria present in Bruns' stag games, indicated on Figure 4.6 in white font, differs with combinations of just Co and As generating a **second-best** equilibria and combinations consisting of Sh being of type **traps**. It should be noted here that, although we have confined our classification to pure Nash equilibria, by Wilson (1971), there exists a third mixed equilibrium in all games with two pure equilibria in order to satisfy the Oddness theorem.

With the absence of a win-win equilibrium, more competitively "exciting" games exist in Layers 1, 2 and 4. Noting that the latter contains games that are the player swap equivalent of those in Layer 2, we need only to consider the spectrum of equilibria in Layers 1 and 2.

4.3.2 Layer 1: Conflict Games

On Layer 1, we encounter the set of **conflict games** (generated by combinations of Pd, Dl, Cm, Hr, Ba and Ch). With the the preferred payoff for each player being off the symmetric diagonal i.e. $T = max\{R, S, T, P\}$, these games exhibit an inherent conflict with players aiming to maximise their own payoffs.

As oppose to equilibria of win-win type, combinations of Hr and Ba each contain a **biased** equilibrium, which provides the second-highest possible joint payoff but conflict arises as the payoffs aren't equal. A more severe version of this is seen in the **unfair** equilibrium present in combinations of Ch.

Here, we can observe what makes the Prisoner's Dilemma unique amongst the symmetric games. With defecting being the dominant strategy for both players, (P, P) is the single Nash equilibrium. Due to its payoff structure, we thus find Pd to be the only symmetric game with a single Pareto-inferior NE. In the set of all games, however, this property is also seen in combinations of Pd with Bruns' stag games in Layers 2 and 4.

4.3.3 Layer 2 (and 4)

In Layer 2, we see that the games constructed from Nc, favour the opposing player via an Unfair equilibrium with the Nc player receiving their third best payoff whilst the opposing player gets the best. This row (column in Layer 4) thus consists of games that are inherently unfair with the Nc player's optimal strategy being to Cooperate, leaving the opponent to choose between R, T for themselves with $T > R$. A similarly result can be seen in games of Pc and Ha, with a better yet still Biased equilibrium. There do however, exist games that closer resemble the Chicken scenario discussed earlier. The game titled **Samson** by Brams (1993) is an example of this, with the Nc player being able to exploit the opponent by threatening the mutually least preferred payoff. This strategy would force the opponent to cooperate for mutual gain, with the Nc player obtaining a payoff greater than that of the Nash equilibrium.

Returning to games with similar equilibrium characteristics to Pd, we now look at combinations of Pd with Bruns' stag games. Here the Nash equilibrium forms a mutually unfavourable outcome with neither player having independent incentive to switch to the strategy which would yield the maximum joint outcome. Due to their structural similarity to the Prisoner's Dilemma, these games have received further recognition being named the Asymmetric Dilemma, Alibi and Revelation (Brams, 2011). These games present an asymmetric version of Pd which, in the original analogy, can be represented as one of the Prisoner's having an alibi creating the resulting asymmetric structure. Robinson and Goforth point out that although there has been an enormous amount of research devoted to Prisoner's Dilemma, there is almost no research done on the strategically similar asymmetric versions, which could allow for the modelling of a wider range of scenarios.

4.4 Other representations of 2×2 games

Though in this chapter we focused primarily on the topology described by Goforth and Robinson (2004), it should be noted that there have been alternative approaches to categorizing these games.

4.4.1 Payoff sets

The nomenclature we adopted here is similar to that of Rapoport and Guyer (1966). A key difference however was the approach they took regarding a standardised normal form for games. Rapoport opted to describe games using the payoff sets. This was a resolution to the problem of defining a standard form for equivalent games, the payoff sets of which would be identical. Below we use the method established in 2.2.2 to plot the corresponding payoff sets of all 144 games.

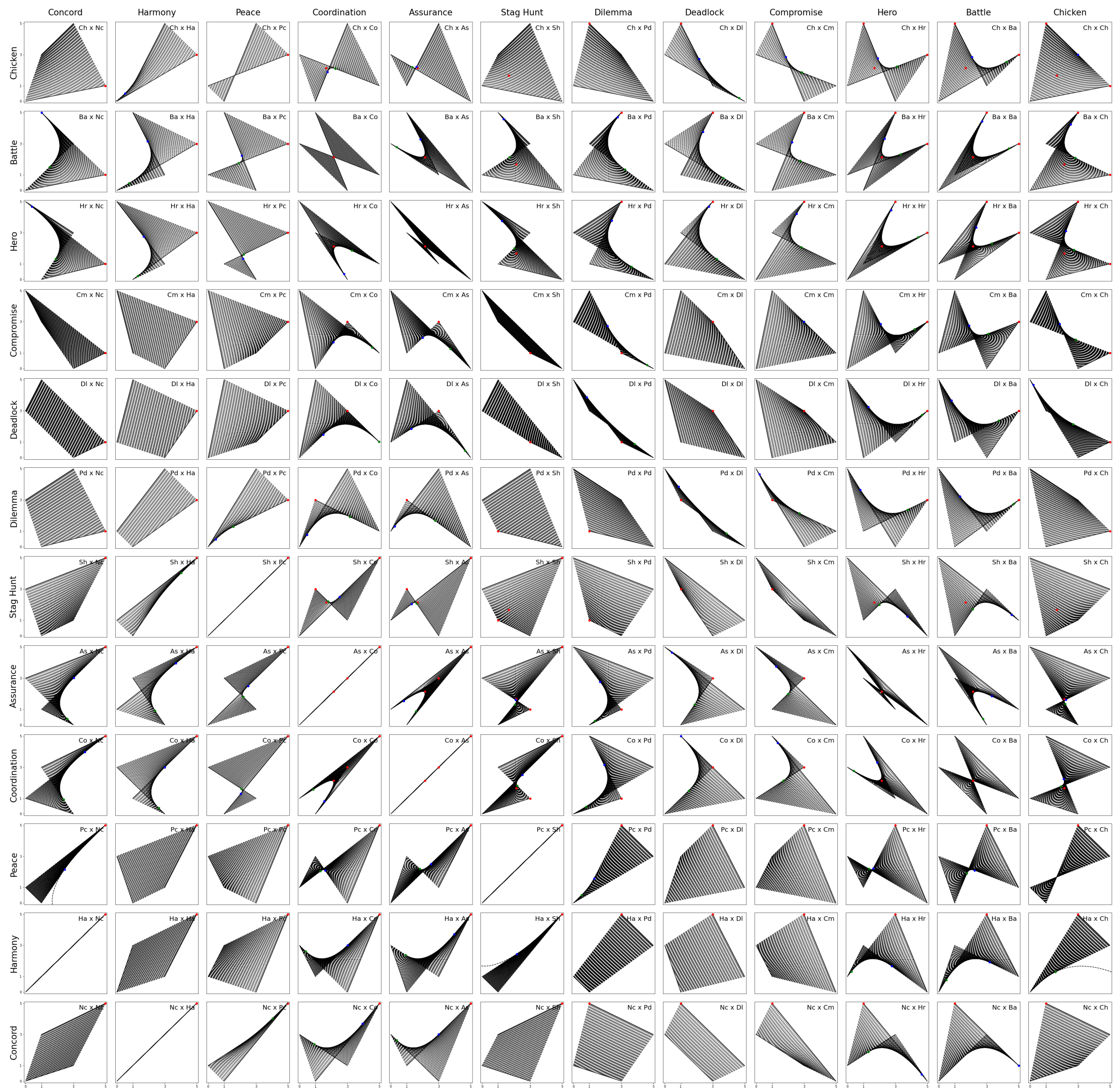


Figure 4.7: Table of 2×2 strict ordinal games using payoff sets

4.4.2 Fraser and Kilgour (1988)

An alternative to the 3-digit index (l, r, c) proposed by Goforth and Robinson is seen in Fraser and Kilgour's algorithm that generates a unique key number for distinct games. An advantage Fraser and Kilgour's method has is the inclusion of non-strict ordinal games allowing for ties in payoffs. Their paper, *Non-strict ordinal 2×2 games*, provides a comprehensive computer-assisted proof of the existence of 726 unique games and a following paper, *Taxonomy of All Ordinal 2×2 Games*, proposes an index for all 726 2×2 games. Though we excluded tied payoffs in this project with the index convention not accounting for them, Bruns (2010) develops the explored topology, integrating games with ties into the table seen in Figure 4.6.

4.4.3 Brams (1993)

Brams provides yet another indexing of games in his *Theory of Moves*. Numbers however, were not assigned to boring and stag hunt games, those with win–win outcomes, since they were not of interest for that analysis. The numbers assigned to games are shown in the figure below.

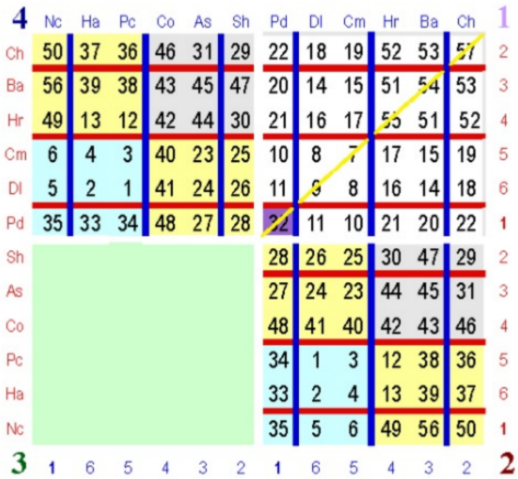


Figure 4.8: Numbers for 2×2 games from Brams (1993)

4.5 Summary

The Robinson-Goforth topology we explored, reveals a natural order in the payoff space of 2×2 games, visualised here in a four-layered "periodic table" format that elegantly organizes the diversity of 2×2 games. Using the symmetric games as coordinates, we explored an efficient nomenclature for the naming and classification of games. Using the figure from Bruns (2015), we utilised solution concepts discussed in Chapter 2 to analyse and categorize groups of games. The resulting taxonomy enables a clearer understanding of the range of conflict scenarios that can be modelled through the use of 2×2 games.

A key aim in the remainder of this project will be to assess the validity of classical solution concepts and the relationships drawn from this topology when considering evolving players and strategy selection in iterated play.

Chapter 5

Simulating Iterated Play

Our analysis of the Prisoner's Dilemma and other games, thus far, has been correct when considering only a single round of play however, a pivotal branch of research regarding 2×2 games has been the idea of iterated play.

In iterated games (*supergames*, Morris (1994)) the two players play a number of repetitions (either a fixed number or of probabilistic length) of the original game. With both players having knowledge of past play and the ability to alter their strategies (either changing the chosen pure strategy or the probabilities in a mixed strategy), the games maintain their non-cooperative nature, but a form of communication emerges.

5.1 Axelrod's Tournaments

Professor of political science, Robert Axelrod organised a tournament in order to study the effectiveness of various strategies in the IPD. He invited collaborators from various fields to submit strategies to participate in a round-robin tournament, where each strategy faced every other strategy as well as a copy of itself. The tournament was run as 200 rounds of the classic Prisoner's Dilemma with the payoffs shown below, and the entire tournament was repeated fives times to ensure reproducibility. Though the number of rounds was fixed, the participants were not informed of this value. This was in order to emulate a tournament of unknown length. For a tournament of known length, a cooperating player may have incentive to defect on the last round as this cannot be punished. This however, would recursively lead to defecting on every round as a method of preemptive punishment for the opportunistic defection on the last round. The problem presented by this, is resolved by modelling the simulation as a tournament of length unknown to participants.

Axelrod's first tournament proved to be a revolution for game theory, showing the success of altruistic strategies in a game where the one-shot variant clearly favoured otherwise. The success of TFT in Axelrod's findings led to further research on similar strategies with Axelrod and Hamilton (1981) exploring the implications on the evolution of cooperation in populations. In *The Calculus of Selfishness*, Sigmund explores the dependence of TFT's success on the competing population as well as exploring a more successful variant in Generous-TFT.

The tournament consisted 14 strategies as well as *Random* which had a 50% chance of cooperating each round. The results of our recreation of this tournament, as seen in Phoplunker (2020), are shown below.

Name	Author	Avg. Score	Wins	Rank*
SteinAndRapoport	Stein and Anatol Rapoport	482.8	10	6
Grofman	Bernard Grofman	473.3	1	4
Shubik	Martin Shubik	467.8	3	5
TitForTat	Anatol Rapoport	467.1	0	1
Nydegger	Rudy Nydegger	459.4	0	3
TidemanAndChieruzzi	T. Tideman and P. Chieruzz	460.0	11	2
Grudger	James W Friedman	448.6	5	7
Davis	Morton Davis	447.6	4	8
Graaskamp	Jim Graaskamp	428.4	4	9
RevisedDowning	Leslie Downing	411.9	7	10
Feld	Scott Feld	358.9	10	11
Joss	Johann Joss	330.1	11	12
Tullock	Gordon Tullock	333.8	8	13
UnnamedStrategy	Unknown	313.9	3	14
Random	Unknown	310.3	3	15

Table 5.1: Results of recreated Axelrod's 1^{st} tournament from Phoplunker (2020) (*reported rank from original tournament)

Recreating the tournament, we find results that deviate from those reported by Axelrod (1980a,b). From the original ranks shown on the below table, we see that the first six we found are different to to those reported in Axelrod's paper. By repeating the tournament to mitigate stochastic effects of the strategies involved, we conclude that disparity between our simulations and the original tournament may lie in something other than the probabilities at play. Looking at the descriptions for the participants given in the Axelrod library and in the original paper, we see that there may have been some issues in implementation, in particular the strategy submitted by Jim Graaskamp. The strategy is described in Axelrod (1980a) as follows:

> *This rule plays tit for tat for 50 moves, defects on move 51, and then plays 5 more moves of tit for tat. A check is then made to see if the player seems to be RANDOM, in which case it defects from then on. A check is also made to see if the other is TIT FOR TAT, ANALOGY (a program from the preliminary tournament), and its own twin, in which case it plays tit for tat. Otherwise it randomly defects every 5 to 15 moves, hoping that enough trust has been built up so that the other player will not notice these defections.*

With the preliminary tournament having not been as clearly documented, the difference in findings could be attributed to an inaccurate implementation of the strategy by Graaskamp, highlighting the dependence of the results on the set of participating strategies.

Another factor to consider may be alternative incentives in tournaments and what qualifies success for a strategy. Table 5.1 shows two possible metrics for success: the average total score (across repetitions of the tournament) and the number of wins for each strategy. In the latter, a strategy is said to have won against another strategy if in a match, consisting of the set number of rounds of play, the first strategy has accumulated more points than its opponent. Notice that these two definitions for success greatly differ with TFT having not won a single game. We could note that using number of wins as a metric reduces each match to a zero-sum game. Further, there is no incentive for cooperation as mutual benefit is futile unless one can ensure obtaining more points than their opponent, hence we could map all payoffs to either multiples of $(1, -1)$ or $(0, 0)$ corresponding to the bias of the payoff.

A key difference here from our previous considerations, is the preference order becoming dependent on the opponent's payoffs structure. As in the ideas explored with stag games in

3.1, using the number of wins metric prioritises outperforming the opponent over maximising one's score, which, by definition, becomes an iterated zero-sum game. Thus, we choose to use the average score as a metric for success in our simulations.

5.2 Markov Model

In our analysis of iterated play, we can look to simulate the iterations using a Markov model, viewing the iterations of play as states in a Markov chain. In order to simplify our model, we use a result from the appendix of Press and Dyson (2012) which proves that against a player with shorter-memory, any longer-memory strategy has a corresponding shorter-memory strategy which produces the same payoffs. Thus, we can restrict our view to ***memory-one*** strategies. In the Markov model, these are given by a probability of cooperation on initial round, $p_0 = p_{initial}$ and the four-vector:

$$p = (p_1, p_2, p_3, p_4) = (p_{CC}, p_{CD}, p_{DC}, p_{DD}) \tag{5.1}$$

where p_{CC}, p_{CD}, p_{DC}, p_{DD} are the probability of cooperating on the next round given the corresponding outcomes in the previous round, i.e.

$$\begin{aligned}
p_{CC} &= \mathbb{P}(\text{Cooperating on next round} \mid \text{Previous round outcome was CC}),\\
p_{CD} &= \mathbb{P}(\text{Cooperating on next round} \mid \text{Previous round outcome was CD}),\\
p_{DC} &= \mathbb{P}(\text{Cooperating on next round} \mid \text{Previous round outcome was DC}),\\
p_{DD} &= \mathbb{P}(\text{Cooperating on next round} \mid \text{Previous round outcome was DD}),
\end{aligned}$$

with $p_i \in [0, 1]$ for $i = \{0, 1, 2, 3, 4\}$. We can then compute the plays of the iterated game between two strategies p and q using the initial state, for corresponding $p_{initial}$ and $q_{initial}$ and the transition matrix, M given below:

$$M = \begin{pmatrix} p_1q_1 & p_1(1-q_1) & (1-p_1)q_1 & (1-p_1)(1-q_1) \\ p_2q_3 & p_2(1-q_3) & (1-p_2)q_3 & (1-p_2)(1-q_3) \\ p_3q_2 & p_3(1-q_2) & (1-p_3)q_2 & (1-p_3)(1-q_2) \\ p_4q_4 & p_4(1-q_4) & (1-p_4)q_4 & (1-p_4)(1-q_4) \end{pmatrix} \tag{5.2}$$

Notice here that we have swapped the second and third rows and second and third columns in q. This is to account for the switched perspective on the outcomes CD and DC, with CD for p corresponding to DC for q and vice-versa.

Via the Markov model, we can determine the distribution of outcomes of successive plays using the transition matrix given an initial distribution of plays,

$$v^0 = \begin{pmatrix} p_0q_0 & p_0(1-q_0) & (1-p_0)q_0 & (1-p_0)(1-q_0) \end{pmatrix} \tag{5.3}$$

via the following process:

$$v^1 = v^0 \cdot M \tag{5.4}$$

$$v^{n+1} = v^n \cdot M \tag{5.5}$$

5.2.1 Memory-one Strategies

Though in our formulation we have $p_i \in [0, 1]$, we will further limit our view to deterministic strategies, i.e. $p_i \in \{0, 1\}$. These can be express as 5-bit numbers as shown in the table below. We note that, both AllC and AllD can be expressed as four different 5-bit number, giving us only 26 distinct strategies.

#	Bit Form	Name	#	Bit Form	Name
0	00000	AllD	16	10000	AllD FC
1	00001	-	17	10001	-
2	00010	Swindler FD	18	10010	Swindler
3	00011	CD Alt	19	10011	DC Alt
4	00100	AllD	20	10100	-
5	00101	Tree frog FD	21	10101	Tree frog
6	00110	-	22	10110	-
7	00111	Rebel FD	23	10111	Rebel
8	01000	AllD	24	11000	Grudger
9	01001	Pavlov FD	25	11001	Pavlov
10	01010	Tit-for-Tat FD	26	11010	Tit-for-Tat
11	01011	Punisher FD	27	11011	Punisher
12	01100	AllD	28	11100	AllC
13	01101	Prodigal Son	29	11101	AllC
14	01110	Blinder	30	11110	AllC
15	01111	AllC FD	31	11111	AllC

Table 5.2: Memory-one 5-bit strategies

5.3 Iterated Prisoner's Dilemma

Naturally, we can start our explorations by running simulations of the iterated Prisoner's Dilemma. Drawing from the dependence on the competing population, demonstrated through the discrepancies between the results of Axelrod's original tournament and our reproduction of it, we may look to consider smaller sets of strategies in order to identify simpler interactions and observe the sensitivity to the competing strategies.

5.3.1 AllC, AllD, TFT

Using AllC (31) and AllD (0), we can see the mechanism behind TFT's success. It avoids exploitation from Defecting strategies with a fast punishing response but readily allows for cooperation with other cooperative strategies. Thus, TFT (26) attempts to limit the payoff of AllD and attempts to make up its own deficit through cooperative payoffs against AllC. With AllD winning in this tournament, the dependence of TFT's success on the population becomes apparent. In this environment, the advantage gained by AllD from its exploitative payoffs against AllC is greater than TFT's capacity to punish.

#	Name	Avg. Score	Wins	Rank
0	AllD	3.01	2	1
26	TFT	1.99	0	2
31	AllC	1.50	0	3

Table 5.3: Results for the Axelrod tournament with strategies: AllC, AllD and TFT

5.3.2 AllC, AllD, TFT, Grudger, CD Alt, DC Alt

Considering a larger set of strategies, using those that appear commonly across literature, we notice a different trend. Though AllD is still able to gain more points than each strategy it faces, the presence of the punishing strategies, TFT and Grudger (24), limits the total number of points AllD can obtain. This time both Grudger and TFT are able to make up for the deficit against each other, demonstrating a strategy environment that rewards mutual cooperation.

#	Name	Avg. Score	Wins	Rank
0	AllD	2.34	5	3
3	CD Alt	2.00	2	5
19	DC Alt	2.01	1	4
24	Grudger	2.66	2	1
26	TFT	2.50	0	2
31	AllC	2.00	0	6

Table 5.4: Results for the Axelrod tournament with additional strategies

5.4 Simulating other iterated games

Though the Axelrod library proves useful in simulating Axelrod's original tournaments and has the capacity to repeat similar simulations for other symmetric games, its method for conducting round-robin tournaments for asymmetric games is somewhat lacking. In order to simulate iterated play for the entire set of games, a Markov model tournament was developed during this project, using the ideas described in 5.2. As well as allowing for the simulation of two separate populations required in asymmetric games, the matrix approach forms an efficient method for computing the rounds of play for any given 2×2 bi-matrix game and pair of memory-one strategies.

5.4.1 Single round play

A simple start to our simulations, we consider the best deterministic strategy in single round play. Here, we run simulations of games between only AllC and AllD to simulate the two possible initial plays. Figure 5.1 displays the highest scoring strategy per player on a grid for the larger set off all 144 games with the lower and upper triangles of each square representing Player 1 and Player 2 respectively.

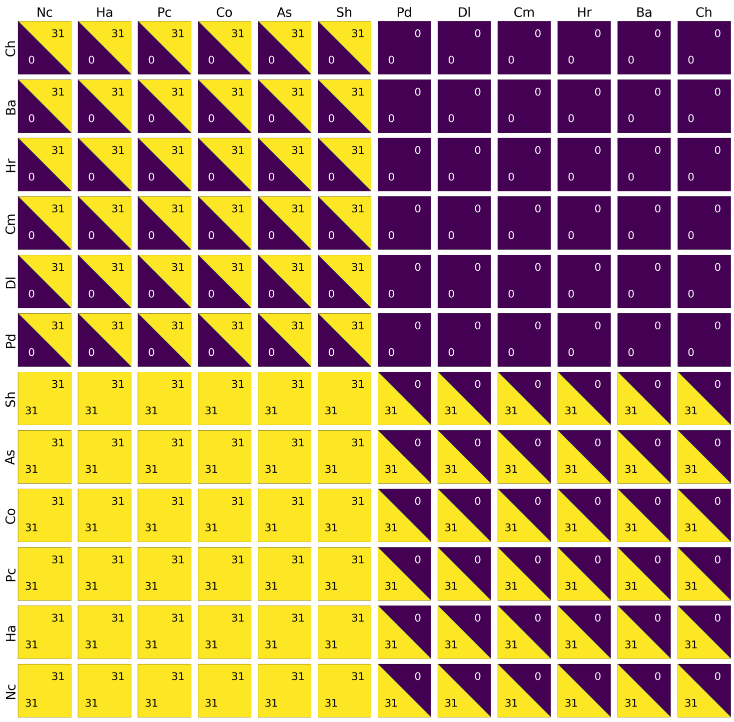


Figure 5.1: Highest-scoring pure strategies for Player 1 (bottom triangle) and 2 (top triangle) in single-round Markov tournaments (Yellow - Cooperate, Purple - Defect)

With the diagonal going from the bottom left to the top right being a line of symmetry for corresponding player swapped games, we see that the optimal strategy per player is simply swapped across this line.

Using the Axelrod payoffs, with the highest payoff being greater then the next two combined, and each strategy playing against both AllC and AllD in the tournament, the score is simply the sum of payoffs along the selected (C or D) row (column for Player 2). As expected, we find a preference for C for players with payoffs such that $R > T$ and D otherwise.

The tournament simulated in Figure 5.1 however, is different to the strategy of play between two unfixed strategies. By having a population of AllD and AllC, it can be interpreted as just an indicator of where the highest payoff is. When considering a natural form of competition, we can observe Figure 5.2.

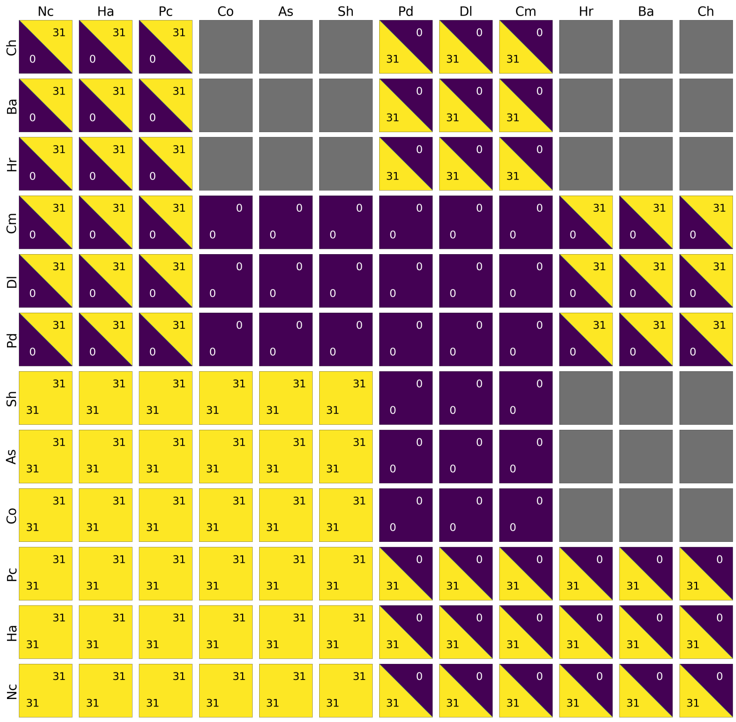


Figure 5.2: Highest-scoring pure strategies for players in a single-round using solution concepts

The results displayed here, reflect our exploration of NE in 4.3. As players aim to maximise their score whilst anticipating their opponent's move, Nash equilibria present fixed points for their respective strategies. Layer 3 as expected, when maximising scores, results in mutual cooperation. The quadrants of each layer, differing in terms of dominant strategies, have different optimal moves corresponding to the position of the NE in the payoff table. The grey regions here, represent games with either no NE or two non-Pareto-inferior NE and so both initial plays are viable. An observation here, is the restrictive nature of purely deterministic strategies. The existence of mixed equilibria in some of these games signifies the presence of optimal probabilistic strategies which we have overlooked in our simulations.

We now look to simulate iterated play across the set of all 144 games amongst the 26 unique strategies in Table 5.2. As in Axelrod's tournaments, each match-up will consist of 200 rounds of the game, with each strategy facing every other strategy as well as a copy of itself. Computing the scores for Player 1 and Player 2 separately, we display the highest scoring strategy for each Player on squares corresponding to each game of the grid of all 144 games. The results are shown in the following figure.

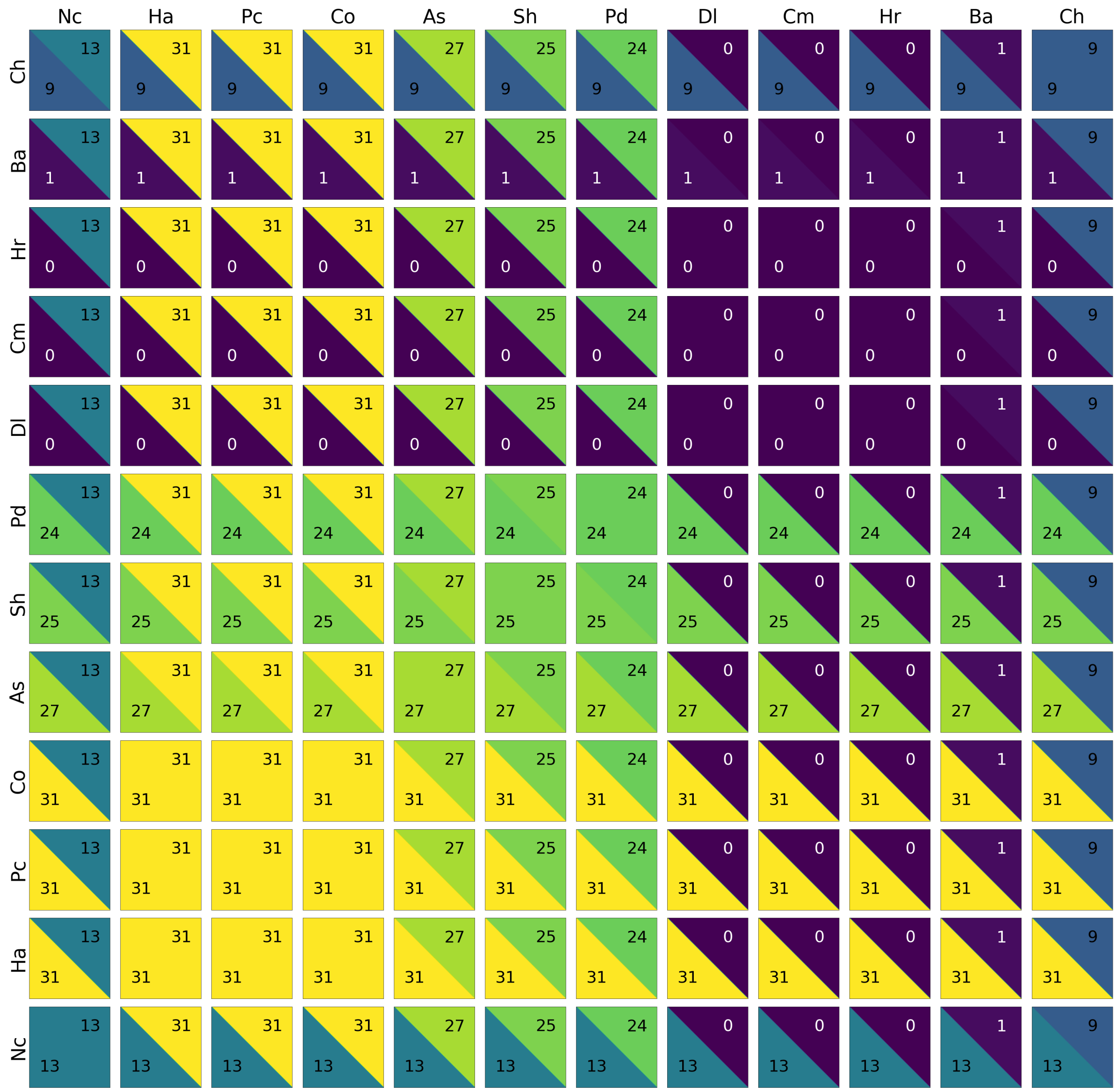


Figure 5.3: Highest-scoring 5-bit strategies for players in the Markov tournament

A clear feature of Figure 5.3 is the persistence of a single strategy for each row/column. This suggests that the choice of strategy is dependent only on a player's own payoff structure and independent of the opponent's payoffs.

Further, along the diagonal of symmetry, we see regions where AllC (31) and AllD (0) were most successful which supports our previous discussed categorizations of boring games with $R > T, P$ and $S > T$ and those where $T > R, S$ and $P > R$, showing a preference for C and D, respectively. A closely related strategy to AllD, 1, sees success in the Hr payoff structure, near the 0 region. This can be attributed to its aversion to the mutual defection payoff as oppose to 0.

Between the two regions on the diagonal, we also find the set of games constructed using only Pd, Sh and As. Here we see strategies 24, 25, 27 performing better. Observing the behaviour of these strategies through the 5-bit notation,

$$
\begin{aligned}
24 &: 11000 \\
25 &: 11001 \\
27 &: 11011,
\end{aligned}
$$

we can observe a common theme of initial cooperation, a capacity for mutual cooperation and ability to punish the opponent for Defecting in the previous round i.e. $p_{initial} = 1$, $p_{CC} = 1$ and $p_{CD} = 0$, characteristics also present in TFT (26). This provides an intuitive reason behind their success in Sh, As as well as the Prisoner's Dilemma where $R > P > S$. This particular payoff structure, provides incentive for mutual cooperation and motive to avoid the sucker's payoff, S. Here we may identify once again, a key feature of Pd with it being the only one of the three with $T > R$, giving greater incentive for defection through a dominant pure strategy.

Though the tournaments consisting of all 26 strategies provides useful insight on the success of strategies, the static nature of the competing environment may pose some limitations to the applicability of these results. For instance, we can note the existence of match-ups that prove significantly more advantageous to only one of two competing strategies. An example of this can be seen through AllD's ability to exploit AllC in section 5.3.1. With AllC scoring the least number of points, it is unable to compete with the other strategies and diminishes TFT's punishing properties.

This may be the reason behind TFT's absence in Figure 5.3, as by considering the set of all strategies, all strategic behaviour, including that which a rational player may deem ineffective (i.e. repeatedly cooperating against a defecting opponent), would be present and thus lead to wins (and points) which would otherwise not occur. In order to resolve this, one may wish to consider the competition between competitive strategies that are performing better in the tournament, excluding possible point donors.

5.4.2 "Weakest-link" tournament

Here we propose a "naive" model through the **"weakest-link"** tournament. This tournament is implemented through simulating repetitions of the Markov tournament with the lowest scoring strategy (strategies) being eliminated after each iteration. We include an additional variable dictating the minimum number of strategies competing in a tournament (set to 10 here) after which to stop and return the tournament results. This was done to preserve a wider environment of interactions between different strategies. For the asymmetric games, we consider two separate populations per player, each following it's own weakest-link process. This allows us to obtain the best strategy for each player payoff structure in the bi-matrix game.

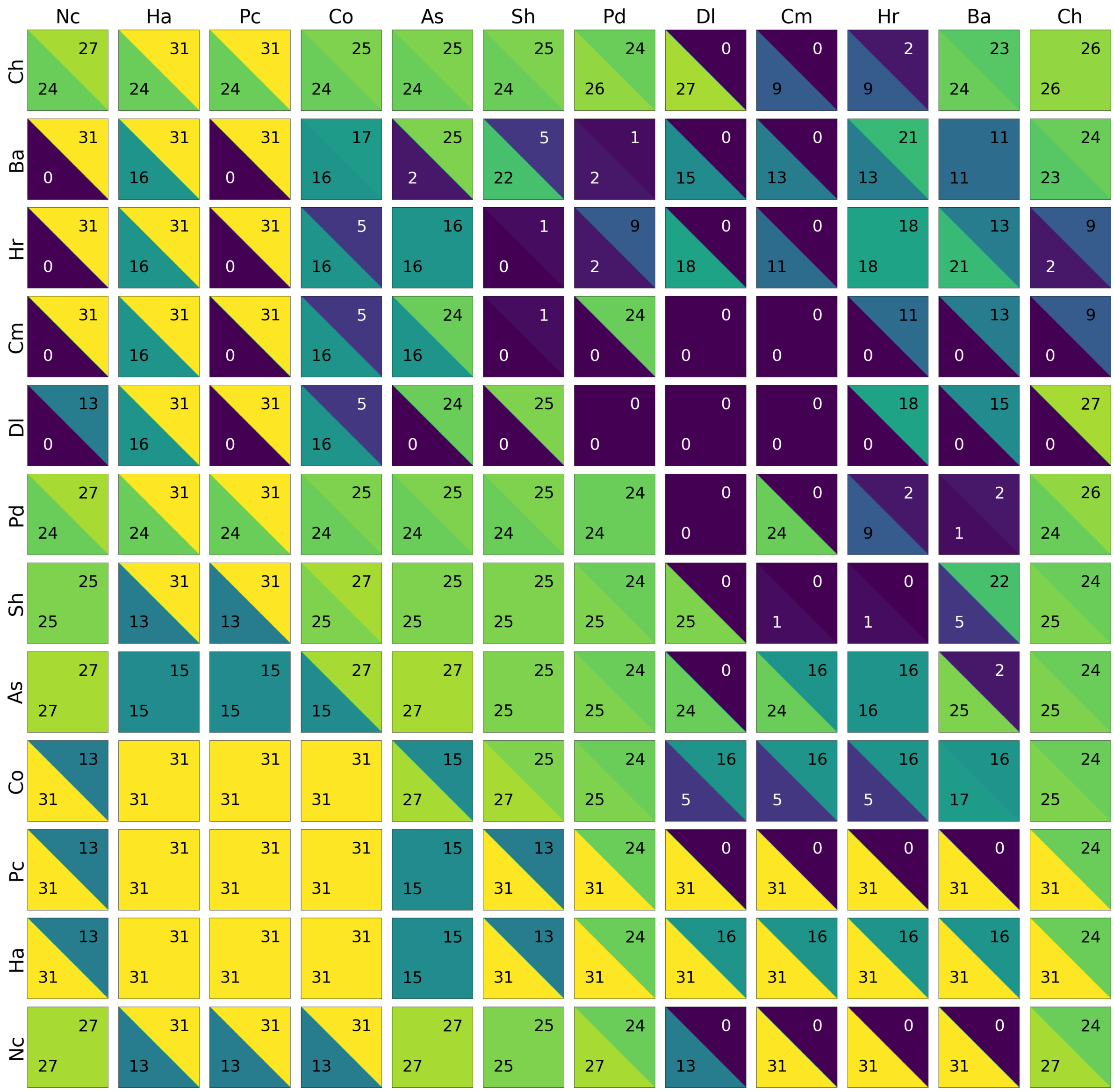


Figure 5.4: Highest-scoring 5-bit strategies for players in the “Weakest-Link” tournament

Using the weakest-link tournament, the previously identified pattern of independence of the opponent's player structure fails. With the population of competing strategies now dependent on the scores in the previous iteration for both players, the subsequent match-ups and so, the success of strategies is dependent on the opponent's payoffs.

Figure 5.4 displays a wider range of strategies, however regions on the symmetric diagonal where AllC (31) and AllD (0) win are still present as well as the strategies 24, 25 and 27. The 0 region is smaller here, with 18 winning in the symmetric Hr game.

Looking at the results in Layer 3 (boring games), we see that the square where AllC wins for both players is formed by combinations of no-conflict games. This supports our categorization of symmetric boring games into two groups: no-conflict games and strategically similar stag hunt games. With the outcome CD/DC being worse for the defecting player, AllC becomes the clear optimal strategy. With stag hunt games however, the defecting player gains a better payoff, resulting in the success of strategies that punish defecting (27, 25) in Nc, As and Sh. Thus, Bruns' categorization of stag games based on dominant strategies and Nash equilibria seems unsuitable here.

Another region of particular interest is the set of conflict games with two dominant strategies (combinations of Pd, Dl and Cm). We once again find evidence for the uniqueness of the Prisoner's Dilemma where Grudger wins, as oppose to the dominant strategy AllD in the other two symmetric games. The success of similar strategies can be seen in the strategically similar combinations of Pd with Bruns' stag games, as expected.

5.5 Conclusion

The static population tournaments provide a useful insight on the success of strategies across all 26 5-bit strategies and the weakest-link tournaments help construct a more robust, realistic alternative founded on the ideas of natural selection. The discrete nature of the weakest-link tournaments however, can be considered a crude approach to modelling this behaviour and we may prefer to develop a continuous analogue for the evolving population and the resulting dynamics.

Chapter 6

Evolutionary Game Theory: A Dynamical Approach

At the end of the first chapter of *Theory of Games and Economic Behavior*, Morgenstern and Von Neumann write:

"We repeat most emphatically that our theory is thoroughly static. A dynamic theory would unquestionably be more complete and therefore preferable."

In this chapter, we introduce a dynamical framework for analysing games. Revisiting ideas explored in Chapter 2 we begin by interpreting the resulting dynamics across the set of symmetric games. We will then look to build an intuition for applying the interpretation of symmetric games across the set of all 78 games.

6.1 An Introduction to Evolutionary Game Theory

Classical game theory provides useful insight through its analysis of characteristics of games, however, models often approach problems with the idea of an "unboundedly rational" player (Hofbauer, 1998).

A reformulation of the classical theory, evolutionary game theory considers players in a game as members of a population. As opposed to selecting/changing strategies, these members are fixed and instead, an optimal strategy emerges through the interactions between members with different strategies within the population. In biological terms, the range of strategies becomes analogous to discrete species or genotypes. With the payoffs based on the interactions between members, the success of a species, its *fitness*, influences its growth rate. As such, the distribution of strategies in a population adapts and evolves to reflect the relative success of competing members.

Given this new framework for analysis, we can begin by restating our previously established solution concepts. Consider a finite set of pure strategies, i.e. the deterministic memory-one strategies considered in the previous chapter. Let x_i denote the frequency of strategy i in a population, giving us a probability vector $(x_1, x_2, \ldots, x_n)$ of the distribution of states. We now denote the n-dimensional simplex, $\Delta_n = \{x \in \mathbb{R}^n; 0 \leq x_i \leq 1, x_1 + \ldots + x_n = 1\}$.

For a bi-matrix game (A, B) and two strategy probability vectors x, $y \in \Delta_n \subseteq \mathbb{R}^n$, we have the corresponding payoffs $x \cdot Ay$ and $x \cdot By$. Notice that in this notation, a pair of vectors $(\hat{x}, \hat{y})$ is a **Nash Equilibrium** if and only if $\forall x, y \in \Delta_n$ we have,

$$x \cdot A\hat{y} \leq \hat{x} \cdot A\hat{y} \text{ and } \hat{x} \cdot By \leq \hat{x} \cdot B\hat{y}. \tag{6.1}$$

Although the Nash equilibrium provides a useful insight on the dynamics, a property of interest to us is a populations resistance to perturbations i.e. an invasion by a minority strategy that does as well as the equilibrium and grows. Thus, we look for the population that tends back towards a previous equilibrium after the described perturbation, we call such a point *evolutionarily stable*. Formalizing this idea we have the following definition.

Definition 6.1. *A pair of probability vectors* $(\hat{x}, \hat{y}) \in \Delta_n \times \Delta_n$ *is an* ***evolutionarily stable state*** *(ESS) if within a neighbourhood of* $(\hat{x}, \hat{y})$ *i.e. for sufficiently small* $\epsilon > 0$, $\forall (x, y) \in \Delta_n \times \Delta_n$ *we have that,*

$$x \cdot A(\epsilon y + (1-\epsilon)\hat{y}) < \hat{x} \cdot A(\epsilon y + (1-\epsilon)\hat{y}) \tag{6.2}$$

$$(\epsilon x + (1-\epsilon)\hat{x}) \cdot By < (\epsilon x + (1-\epsilon)\hat{x}) \cdot B\hat{y}. \tag{6.3}$$

This notion of evolutionary stability gives rise to many proposed models for the underlying dynamics through which such states can be obtained. With the underlying dynamics being governed by a system of differential equations, models assumed different functions for quantifying the rate of growth based on a population's *fitness*. Here we look to outline and utilise the most prevalent of these models.

6.1.1 Replicator Dynamics

Introduced by Taylor and Jonker (1978), the replicator dynamics model assumes a Darwinian view of evolution with the growth rate of a population dependent on its fitness, defined here, as a measure of a population's success relative to the average fitness. The rationale behind replicator dynamics is creating a "survival of the fittest" system amongst competing populations. The corresponding model for two populations is described by the following pair of differential equations:

$$\dot{x}_i = x_i((Ay)_i - x \cdot Ay) \tag{6.4}$$

$$\dot{y}_j = y_j((x^T B)_j - x \cdot By) \tag{6.5}$$

With the average fitness of population x given by the second term in (6.4) and the first specifying the fitness of state x_i, we see that the growth rate of a state x_i is proportional to its size and the difference between its fitness and the average fitness. (6.5) defines a similar relation for states y_j in the Player 2 population y.

The intuition behind replicator dynamics is similar to that seen in the Lotka-Volterra equations in the study of ecological systems. Initially proposed by Lotka (1909), the model was extended to analyse the interactions of predator-prey populations in the book *Elements of physical biology*, Lotka (1925). With the development of evolutionary game theory and the resulting overlap with mathematical biology, Hofbauer (1981) proved the equivalence of the Lotka-Volterra equations to the above replicator dynamics equations.

6.2 Analysing Replicator Dynamics of Games

We begin our analysis by considering, once again, the Prisoner's Dilemma game. With this being symmetric, we can simplify the earlier results, requiring only (6.4) as the dynamics for x and y are identical. Pure strategies, i are given by the memory-one strategies we are interested in, however, for the sake of simplicity and convenient visualisation, we limit this to three strategies, as before, AllC (31), AllD (0) and TFT (26).

The matrix describing the average payoffs for Player 1 (rows), using the Axelrod payoffs, is given,

$$\begin{array}{c} \text{AllC} \\ \text{AllD} \\ \text{TFT} \end{array} \begin{bmatrix} 3 & 0 & 3 \\ 5 & 1 & \frac{4+n}{n} \\ 3 & \frac{n-1}{n} & 3 \end{bmatrix} \tag{6.6}$$

where n is the number of rounds. For our analysis, we specify a fixed n and use our iterated game simulations to construct these payoff matrices.

Using the ***egtplot*** python library by Mirzaev et al. (2018), we can then produce a simplex plot for the replicator dynamics for the corresponding payoffs.

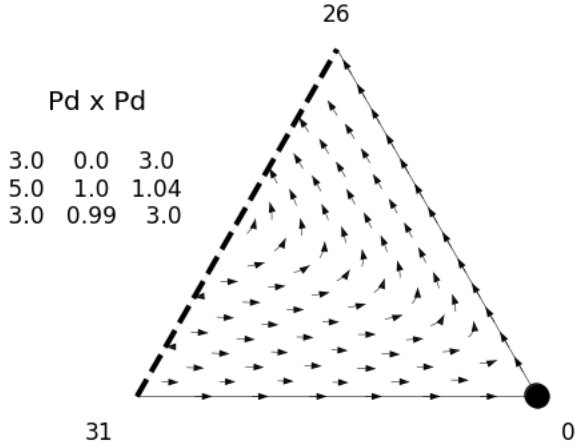


Figure 6.1: Directional field plot of the replicator dynamics of AllC-AllD-TFT in Pd (n=100)

Figure 6.1 shows the ability of a sufficiently large population of TFT to invade a population consisting only of AllD which would otherwise be a stable (Black) equilibrium. By observing the trajectories on the edge connecting TFT (26) and AllD (0), we can deduce the existence of an additional saddle point (Grey) near 0. Note that this isn't visible on Figure 6.1 due to our choice of payoffs and the precision of the egtplot library. This equilibrium signifies a bound for which a TFT population can successfully invade and lead to the extinction of the AllD population.

The dashed line on the edge connecting AllC (31) and TFT represents a line of NE, recognising that a population comprising of only AllC and TFT is strategically identical to one consisting of a population that never defects. We should note however, that there are dynamic differences in terms of stability with the NE being stable only for populations where TFT form a majority.

The dynamics displayed here are specific to the choice of strategies and we can observe a wider range of dynamic motifs between different sets of strategies. Kim et al. (2014) explores these motifs further, constructing a graph for the interactions between 5-bit strategies and computing simplex plots for a range of unique dynamic motifs present.

6.2.1 Other Symmetric Games

Our implementation of replicator dynamics will make use of the Markov tournament we developed in the previous chapter. We use the average payoff data from our simulations to form a payoff matrix, using this to generate simplex plots for the dynamics between 3 strategies.

In the following plots, we consider the replicator dynamics between the strategies AllC (31), AllD (0) and TFT (26). The trajectories of the population are plotted using arrows, with the contour map indicating their speeds.

Figure 6.2 shows the replicator dynamics for three of the boring games, Pc, Ha and Nc. With common features being the existence of an unstable equilibrium at 0 and the line of stable NE on the AllC-TFT edge, we find further evidence supporting the similarity of these games also found in our classical analysis.

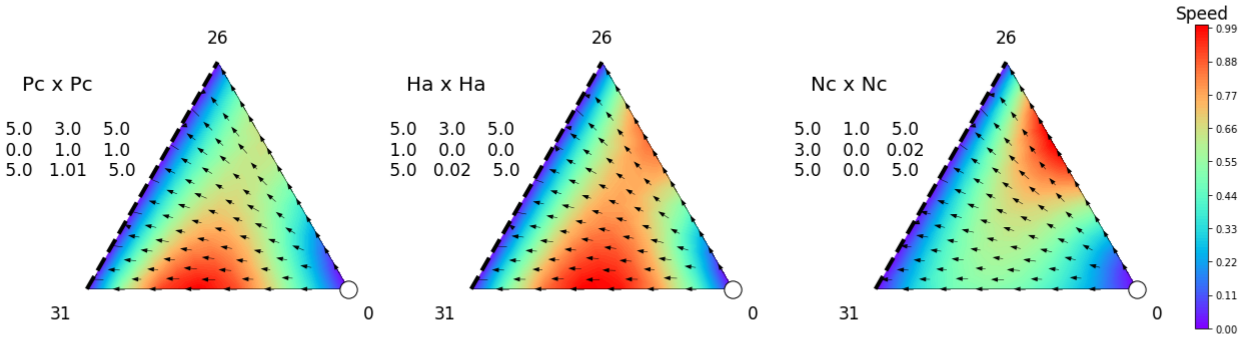


Figure 6.2: Directional field plot of the symmetric games: Pc, Ha and Nc

In Figures 6.3 and 6.4 below, we conduct similar analysis for the symmetric conflict games. Here we notice dynamic differences between games with dominant strategies (Cm, Dl and Pd) and those without (Ch, Ba and Hr).

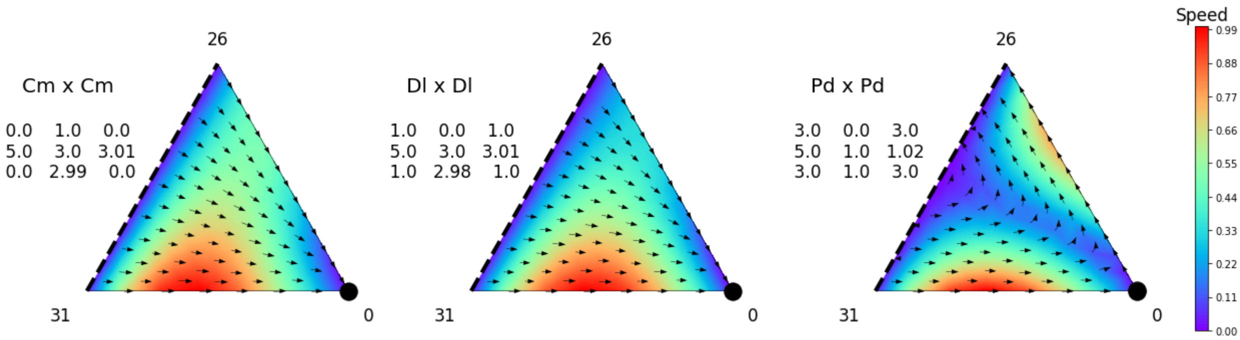


Figure 6.3: Directional field plot of the symmetric games: Cm, Dl and Pd

For games with a dominant strategy, in Figure 6.3, the dominant strategy D results in a stable equilibrium at AllD. Unlike Pd, in which we found an additional unstable equilibrium near 0, this is the only equilibrium point in Cm and Dl. Another difference is the line of NE on the AllC-TFT edge being unstable along the whole edge.

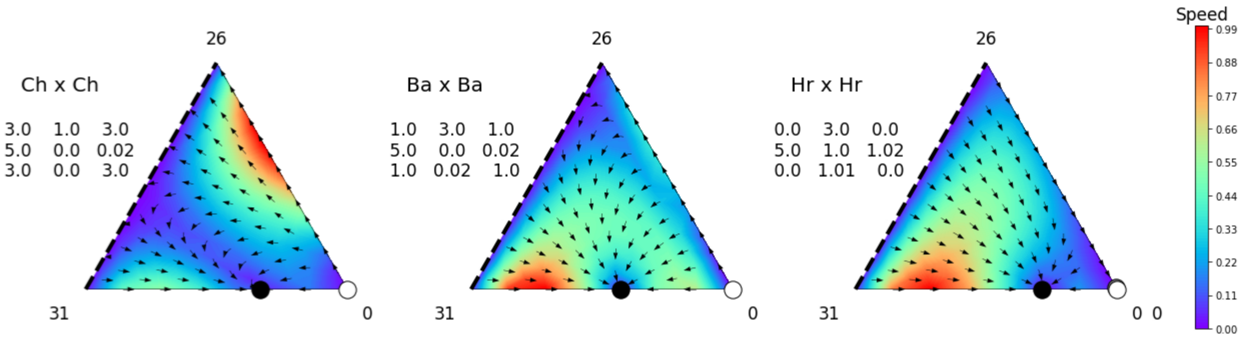


Figure 6.4: Directional field plot of the symmetric games: Ch, Ba and Hr

In conflict games without dominant strategies, Figure 6.4, the equilibrium at AllD becomes unstable and an additional stable NE emerges on the edge connecting AllC (31) and AllD (0). This stable point consisting of just AllC and AllD describes interactions where a random player and opponent may cooperate or defect following the ratio of the two populations. This is similar to the interaction of the mixed strategy NE present in Ch, Ba and Hr in our classical analysis.

Similar to the deductions in Figure 6.1, we find an additional saddle point equilibrium on the TFT-AllD edge, near 0, not accounted for here due to the precision of the code. This difference between the dynamics of Hr and those of Ch and Ba correlates with the results seen in our simulations in the previous chapter.

Limitations to the precision of the visualisations produced by egtplot can also be observed in the plots for Sh, As and Co in Figure 6.5. Here, the stable (Black) equilibrium present at 0 is contradicted by the trajectories on the edge connecting to 26 being outward. We resolve this by noting the presence of a saddle point (Grey) on the edge close to 0.

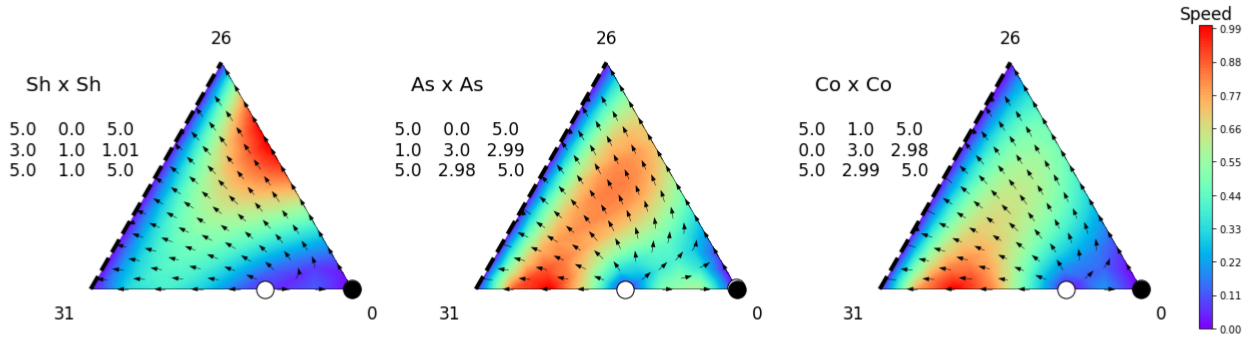


Figure 6.5: Directional field plot of the symmetric games: Sh, As and Co

The dynamic similarity of these games extends further with the TFT-AllC edge forming a stable equilibrium, representing a population that can obtain the win-win equilibrium present in their payoff tables. We note here that, the games in Figure 6.5 are Bruns' stag games, suggesting that our categorization of stag hunt games is unfitting when considering the replicator dynamics of these games. Another similarity, though at slightly different values, is the unstable equilibrium on the AllC-AllD edge. As noted in the dynamics of Ch, Ba and Hr, this is representative of the mixed NE present in these games.

6.2.2 "Weakest-Link" Evolutionary Dynamics

Though we began our observation with the strategies given 0, 26 and 31, there is little inherent importance to this population outside of observing the behaviour of TFT against pure strategies. A more useful simulation may look to observe the dynamics between equally successful strategies.

Here, we utilise the weakest-link tournament described in the previous chapter as a means to determine a set of top three strategies. We then simulate play between these strategies to compute a payoff matrix with which to plot the simplex plot of the replicator dynamics. Doing this for Pd, Ch, Ba and Sh, we obtain the following plots.

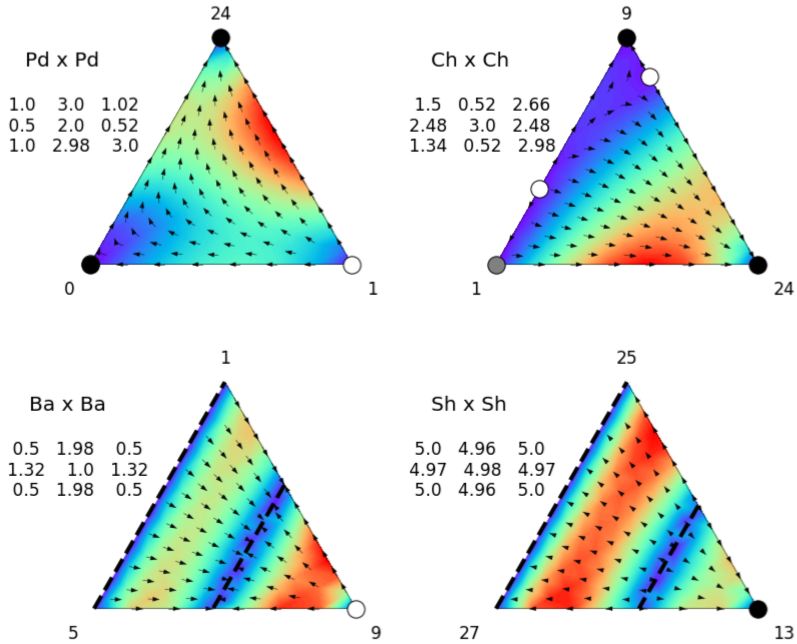


Figure 6.6: Directional field plot of symmetric games using the top three strategies in their corresponding "Weakest-Link" tournaments

Though we will not analyse the dynamics seen in Figure 6.6 in much detail, the ideas present are similar to those seen in the previous section with the simplex plots providing an intuitive method for understanding the dynamics of the population. A key point to note is the range of dynamical motifs present as we change the three competing strategies. It is also possible to expand the analysis of replicator dynamics to consider more than three strategies, however corresponding plots would require higher dimensional simplices, making the visualisation of dynamics increasingly difficult.

6.3 Replicator Dynamics of Asymmetric Games

We now look to extend the ideas used to analyse symmetric games to consider the wider set of asymmetric games. Here we look to outline key results which we use to construct a method for the analysis of asymmetric games.

Notice that an asymmetric game will yield 2 different payoff matrices, one for each of the two populations corresponding to Players 1 and 2. Unlike in our study of symmetric games, here we must consider replicator dynamics for two populations given by the pair of equations (6.4) and (6.5). Thus, visualising the dynamics becomes difficult as we must consider two simplices, where trajectories in one simplex can affect the trajectory in the other. Consequently, it is not straightforward anymore to analyse the dynamics and equilibrium landscape for both players.

Considering the 3×3 bi-matrix of payoffs, we can look to find equilibrium points. Though this task can be somewhat tedious, some NE can be found by finding equalizer strategies as seen in 2.2.1. In our new notation, the definition is as follows.

Definition 6.2. *A mixed strategy $\hat{x}$ for Player 1 is an **equalizer strategy** if Player 1's payoff, $\hat{x} \cdot Ay$ is constant for all $y \in \Delta_n$ (with an analogous definition for $\hat{y}$ for Player 2's payoff, $x \cdot B\hat{y}$)).*

Following from this definition, if we have an equalizer strategy for Player 1 in Player 2's payoff and an equalizer strategy for Player 2 in Player 1's payoff i.e. $\hat{x} \cdot By$ and $x \cdot A\hat{y}$ are both constant for all $x, y \in \Delta_n$, then we have that,

$$\hat{x} \cdot By = \hat{x} \cdot B\hat{y} \tag{6.7}$$

$$x \cdot A\hat{y} = \hat{x} \cdot A\hat{y}. \tag{6.8}$$

As the above equations satisfy the inequalities (6.1), the pair of equalizer strategies correspond to a Nash equilibrium. Note however, that not all Nash equilibria can be found using this method.

A more general result which allows us to compute the other equilibria considers the symmetric decomposition of asymmetric games, using the two resulting symmetric games to find equilibria of the original. The following result from Tuyls et al. (2018) allows us to use this method, comparing the supports of equilibria of the decomposed games.

Theorem 6.1. *Strategies x and y constitute a Nash equilibrium of an asymmetric game $G = (\Delta_1, \Delta_2, A, B)$ with the same support (i.e. $I_x = I_y$ where $I_x = \{i | x_i > 0\}$) if and only if x is a Nash equilibrium of the single population game B^T, y is a Nash equilibrium of the single population game A and $I_x = I_y$.*

Using the methods demonstrated above for finding equilibrium points, we attempt to develop an intuitive understanding of Nash equilibria in asymmetric games. Below we will follow two examples to help illustrate the application of these results, establishing a toolkit for analysis which can be applied to other games.

Example 6.1
Consider the asymmetric 3×3 game given by the bi-matrix,

$$(A, B) = \begin{bmatrix} (1,0) & (0,1) & (2,0) \\ (0,1) & (2,0) & (0,0) \\ (2,0) & (0,0) & (1,1) \end{bmatrix} \tag{6.9}$$

which we can decompose to get the subsequent two symmetric games (A, A^T) and (B^T, B) where,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{6.10}$$

We then find the Nash equilibria of games A (1-7) and B^T (8-14):

1. $x = (1, 0, 0)$, $y = (0, 0, 1)$
2. $x = (0, 1, 0)$, $y = (0, 1, 0)$
3. $x = (0, 0, 1)$, $y = (1, 0, 0)$
4. $x = (0, 1/2, 1/2)$, $y = (1/2, 1/2, 0)$
5. $x = (1/2, 0, 1/2)$, $y = (1/2, 0, 1/2)$
6. $x = (1/2, 1/2, 0)$, $y = (0, 1/2, 1/2)$
7. $x = (2/7, 3/7, 2/7)$, $y = (2/7, 3/7, 2/7)$
8. $x = (1, 0, 0)$, $y = (0, 1, 0)$
9. $x = (0, 1, 0)$, $y = (1, 0, 0)$
10. $x = (0, 0, 1)$, $y = (0, 0, 1)$
11. $x = (0, 1/2, 1/2)$, $y = (1/2, 0, 1/2)$
12. $x = (1/2, 0, 1/2)$, $y = (0, 1/2, 1/2)$
13. $x = (1/2, 1/2, 0)$, $y = (1/2, 1/2, 0)$
14. $x = (1/3, 1/3, 1/3)$, $y = (1/3, 1/3, 1/3)$

Computing these equilibria is fairly tedious hence, further detail and the complete calculations are provided in Appendix A.2.

With no common equilibrium (x, y) between A and B^T, we consider only equilibria where x and y have the same support, thus only consider 2, 5 and 7 for A and 10, 13 and 14 for B^T, with respective supports $\{2\}, \{1,3\}, \{1,2,3\}$ and $\{3\}, \{1,2\}, \{1,2,3\}$. Following from Theorem 6.1, the corresponding Nash equilibrium with matching supports are 7 and 14, which gives us the resulting Nash equilibrium of the asymmetric game as $x = (1/3, 1/3, 1/3), y = (2/7, 3/7, 2/7)$. (We can check this using the online banach solver http://banach.lse.ac.uk/).

Example 6.2
Consider the asymmetric game Chicken $\times$ Battle. For a sufficiently large number of rounds, using (6.6) with the payoff structures for Chicken and Battle, we see that the payoff matrix for strategies AllC, AllD and Tit-For-Tat is,

$$\begin{bmatrix} (3,1) & (1,5) & (3,1) \\ (5,3) & (0,0) & (0,0) \\ (3,1) & (0,0) & (3,1) \end{bmatrix}, \tag{6.11}$$

which decomposes to the two symmetric games,

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 5 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 5 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{6.12}$$

Here we will use the method of equalizer strategies to determine a NE. With full calculations provided in Appendix A.3, the equalizer strategies for Player 1 in Player 2's payoffs, B, and for Player 2 in Player 1's payoffs, A, are given,

$$\hat{x} = (1/5, 0, 4/5)$$
$$\hat{y} = (3/5, 0, 2/5),$$

resulting in a NE for the asymmetric game at $(x, y) = ((1/3, 0, 2/3), (3/5, 0, 2/5))$.

Utilising Theorem 6.1 we can build a greater understanding of the equilibrium landscape. Of particular interest to us however, are the stable equilibria, a combination of which, will provide us with an ESS for the asymmetric game. Using the simplex plots below, we see stable (Black) equilibrium points in both counterpart games on the edge connecting AllC and AllD. As they have the same support, $\{1, 2\}$, we can utilise Theorem 6.1 analysing the decomposed games or by finding equalizer strategies $(\hat{x}, \hat{y})$ where $\hat{x}_3 = \hat{y}_3 = 0$. With the full calculation provided in Appendix A.4, we find a stable NE of the asymmetric game at $(x, y) = ((3/7, 4/7, 0), (1/3, 2/3, 0))$.

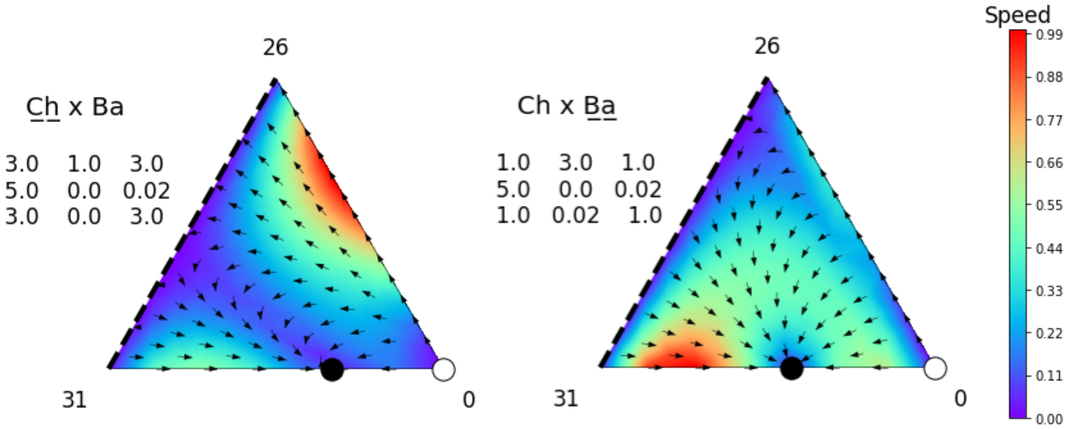


Figure 6.7: Field plots of the symmetric counterpart games of Ch$\times$Ba (Ch - left, Ba - right)

Theorem 6.1 allows for an intuitive understanding of the equilibria present in asymmetric games using the simplex plots of the symmetric decomposition. We can identify fixed points with the same support, i.e. the stable equilibria on the edge connecting 0 and 31 and the unstable equilibria at 0, swapping these across the players to find stable and unstable equilibria of the original game. By applying this theorem, one can derive similar analysis for other asymmetric games by using the symmetric plots seen in Figures 6.2 - 6.5.

Chapter 7

Conclusion

In this project, we introduced the set of two-person non-cooperative non-zero-sum games known as 2×2 *strict ordinal games*, providing examples of the use of these games in modelling conflict scenarios. We proved that there are only 78 unique games of this type and explored the relationships between games developing a topology for the set of all 2×2 ordinal games. We considered three approaches to solving games: solution concepts from classical game theory, the study of emergent communication in iterated play and evolutionary game theory and the use of replicator dynamics to find equilibrium points in an adapting and evolving population.

Chapter 2 provided background ideas on Non-cooperative game theory, stating key definitions and discussing solution concepts for N-person games. We then confined our view to two-person games in normal form demonstrating methods for finding equilibrium pairs and plotting payoff sets. In order to do this, we derived the parametric form of the parabolic side present in some payoff sets. We then explored the idea of equivalent sets of games, using this along with Burnside's lemma to compute the number of unique 2×2 games.

In Chapter 3, we used the payoff structures of the 12 unique symmetric games to construct the space of all symmetric games. By swapping the low and middle payoffs we formed connections between symmetric games, mapping the resulting geometry onto the face of a cube. We then explored the utilitarian ideas of effective play as described by Goforth and Robinson (2004), demonstrating a shift in incentive to cooperate across the space of symmetric games. Limiting our view to symmetric games with the original payoffs used by Axelrod (1980a), we then simulated tournaments across the space of symmetric games, identifying regions of success for each strategy.

Utilising a natural order for the symmetric games based on the connections made by swapping payoffs, in Chapter 4 we constructed the larger set of asymmetric games. We then provided a concise summary of the topology for the space of 2×2 games proposed by Goforth and Robinson (2004). Here we saw the payoff swap connections between games, with low and middle swap operations dividing the space into 4 toroidal layers which could be connected via high swaps. We then went on to categorise and group games by analysing the number of Nash equilibria as well as the types of pure Nash equilibria present. We also provided an outline of alternative representations of 2×2 games proposed in literature, including the use of the payoff set representation developed in earlier chapters.

In Chapter 5, we simulated iterated versions of the games found earlier. We began by recreating the initial tournament from Axelrod (1980a), discussing criteria for winning, the reproducibility of results and the dependence on the competing population. We then developed a Markov model for iterated play and described the set of deterministic memory-one strategies we would be considering in our simulations. We then simulated tournaments of iterated play for all 144 games between the 26 unique memory-one strategies, visualising, interpreting and discussing the results. Noticing a limitation of the static population, we proposed and simulated the weakest-link tournament, aiming to mimic natural selective dynamics by eliminating low-scoring strategies in subsequent repetitions of the tournaments. We concluded by discussing the advantages and disadvantages of the discrete weakest-link model for population dynamics.

Chapter 6 looked to build on the limitations discussed in Chapter 5, introducing the field of evolutionary game theory. A key idea discussed was the continuous analogue for the weakest-link tournament through replicator dynamics. Using the ideas developed in evolutionary game theory, we analysed the dynamics of populations in symmetric games, first between AllC, AllD and TFT, moving on later, to other sets of three strategies. We then looked to determine how to adapt this analysis for asymmetric games, redefining the idea of equalizer strategies and using the symmetric decomposition of the games to find Nash equilibria of the original game. Providing a few examples of this, we hoped to establish an understanding of the methods for analysing the replicator dynamics of asymmetric games.

7.1 Future Work

Throughout the project, we have noted significant points of focus. Game theory as a whole is a broad field with endless possibilities for games. Here we considered 2×2 two-person non-cooperative games with ordinal payoffs as given by the value used in Axelrod (1980a). The ideas discussed can readily be adapted to other finite N-player games, however, with increasing complexity and dimensionality, our ability to compute and visualise solutions may prove difficult.

Due to the computational limits involved, we saw Chapter 5 limit simulations to deterministic memory-one strategies. Though this provides a useful idea of possible interactions, a more stochastic model may provide further insight as found by Sigmund (2010) with strategies such as Generous TFT, a variant of TFT with $p_{DD}, p_{CD} \neq 0$. There is also scope for the investigation of tournaments with noise and player error in move selection through the use of probabilistic strategies.

A natural extension to our concepts for *strict ordinal games* is provided by Fraser and Kilgour (1986), conducting comprehensive computer-assisted analysis of the set of non-strict ordinal games. Bruns (2012) discuss this set of games with tied payoffs and their connection to the topology described in Chapter 4, however analysis of the evolutionary dynamics and iterated versions of these games remains unexplored. Exploring a greater scope of games would allow for a better understanding of the similarities and differences between various strategic situations of conflict and cooperation, facilitating the development of more rigorous models for competition.

References

Axelrod, R. (1980a). Effective choice in the prisoner's dilemma. *The Journal of Conflict Resolution*, 24(1):3–25.

Axelrod, R. (1980b). More effective choice in the prisoner's dilemma. *The Journal of Conflict Resolution*, 24(3):379–403.

Axelrod, R. and Hamilton, W. D. (1981). The evolution of cooperation. *Science*, 211(4489):1390–1396. 01.

Brams, S. J. (1993). Theory of moves. *American Scientist*, 81(6):562–570.

Brams, S. J. (2011). *Game theory and the humanities: bridging two worlds*. MIT press.

Bruns, B. (2010). Navigating the topology of 2x2 games: An introductory note on payoff families, normalization, and natural order.

Bruns, B. (2012). Escaping prisoner's dilemmas: From discord to harmony in the landscape of 2x2 games.

Bruns, B. (2015). Names for games: Locating 2×2 games. *Games*, 6(4):495–520.

Bruns, B. (2020). Transmuting social dilemmas into win-win games: Payoff families in the topology of 2x2 ordinal games.

Fraser, N. M. and Kilgour, D. M. (1986). Non-strict ordinal 2×2 games: A comprehensive computer-assisted analysis of the 726 possibilities. *Theory and Decision*, 20(2):99–121.

Fraser, N. M. and Kilgour, D. M. (1988). A taxonomy of all ordinal 2 2 games. *Theory and decision*, 24(2):99–117.

Goforth, D. and Robinson, D. (2004). *The Topology of the 2x2 Games*. Taylor and Francis, Abingdon, Oxon.

Goforth, D. and Robinson, D. (2012). Effective choice in all the symmetric 2×2 games. *Synthese*, 187(2):579–605.

Goforth, D. and Robinson, D. (2015). Conflict, no conflict, common interests, and mixed interests in 2 2 games.

Hofbauer, J. (1981). On the occurrence of limit cycles in the volterra-lotka equation. *Nonlinear Analysis: Theory, Methods & Applications*, 5(9):1003–1007.

Hofbauer, J. (1998). *Evolutionary games and population dynamics*. Cambridge University Press, Cambridge.

Kim, Y. J., Roh, M., Jeong, S.-Y., and Son, S.-W. (2014). Dynamic motifs of strategies in prisoner's dilemma games. *Journal of the Korean Physical Society*, 65(11):1709–1714.

Lotka, A. J. (1909). Contribution to the theory of periodic reactions.

Lotka, A. J. (1925). *Elements of physical biology.* Williams & Wilkins.

Mirzaev, I., Williamson, D. F., and Scott, J. G. (2018). egtplot: A python package for 3-strategy evolutionary games. *bioRxiv*, page 300004.

Morgenstern, O. and Von Neumann, J. (1953). *Theory of games and economic behavior.* Princeton university press.

Morris, P. (1994). Introduction to game theory.

Nash, J. F. et al. (1950). Equilibrium points in n-person games. *Proceedings of the national academy of sciences*, 36(1):48–49.

Osborne, M. J. and Rubinstein, A. (1994). *A course in game theory.* MIT press.

Phoplunker, A. (2020). The iterated prisoner's dilemma: Good strategies and the press-dyson representation. *Dynamics of Games*, Imperial College London.

Press, W. and Dyson, F. (2012). Iterated prisoner's dilemma contains strategies the dominate any evolutionary opponent. *Proceedings of the National Academy of Sciences*, 109(26):10409–10413.

Rapoport, A. (1974). *Game theory as a theory of conflict resolution.* D. Reidel Pub. Co, Dordrecht; Boston.

Rapoport, A. and Guyer, M. (1966). A taxonomy of 2×2 games. *Gen. Syst.*, 11:203–214.

Rousseau, J. (2014). A discourse upon the origin and the foundation of the inequality among men.

Russell, B. (2009). *Common sense and nuclear warfare.* Routledge.

Sigmund, K. (2010). *The Calculus of Selfishness.* Princeton University Press, Princeton, NJ.

Taylor, P. D. and Jonker, L. B. (1978). Evolutionary stable strategies and game dynamics. *Mathematical biosciences*, 40(1-2):145–156.

Tuyls, K., Pérolat, J., Lanctot, M., Ostrovski, G., Savani, R., Leibo, J. Z., Ord, T., Graepel, T., and Legg, S. (2018). Symmetric decomposition of asymmetric games. *Scientific reports*, 8(1):1–20.

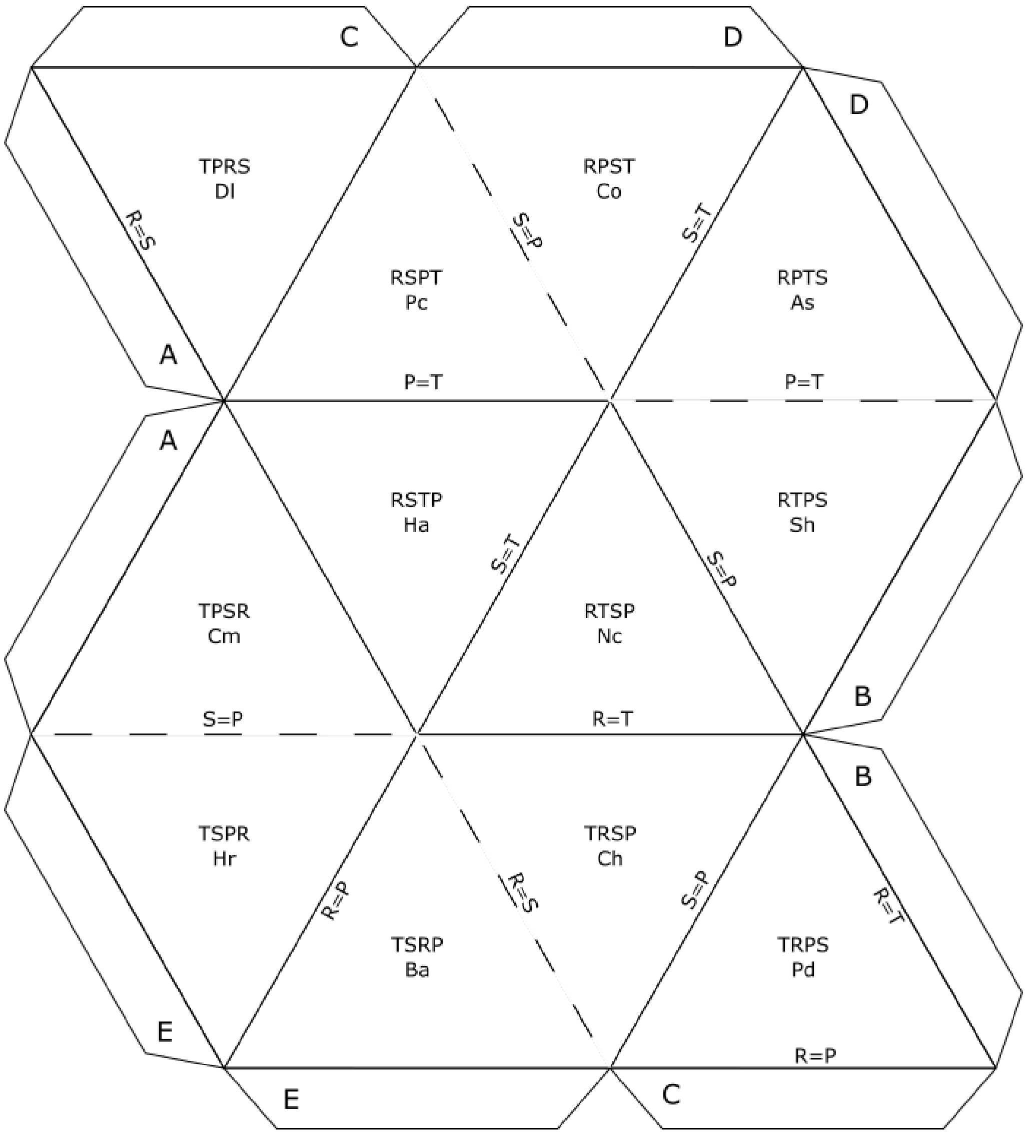
White, L. (2018). *Games, Risks and Decisions.* Imperial College London.

Wilson, R. (1971). Computing equilibria of n-person games. *SIAM Journal on Applied Mathematics*, 21(1):80–87.

Appendix A

Appendix

A.1 Geometry of Symmetric Games - "Flying Octahedron"



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A.2 Example 6.1 - Computing Nash Equilibria

Consider the asymmetric 3×3 game given by the bimatrix,

$$(A, B) = \begin{bmatrix} (1,0) & (0,1) & (2,0) \\ (0,1) & (2,0) & (0,0) \\ (2,0) & (0,0) & (1,1) \end{bmatrix}$$

which we can decompose to get the subsequent two symmetric games (A, A^T) and (B^T, B) where,

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \text{ and } B^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Beginning with A, we can trivially find the pure equilibria 1, 2 and 3, $((1,0,0),(0,0,1))$, $((0,1,0),(0,1,0))$ and $((0,0,1),(1,0,0))$.

We now use the idea of equalizer strategies to find the remaining NE. Let $\hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the equalizer strategy in A, i.e,

$$\begin{aligned} \hat{x} \cdot Ay &= \hat{x}_1(y_1 + 2y_3) + \hat{x}_2(2y_2) + \hat{x}_3(2y_1 + y_3) \\ &= y_1(\hat{x}_1 + 2\hat{x}_3) + y_2(2\hat{x}_2) + y_3(2\hat{x}_1 + \hat{x}_3) = \text{const.} \end{aligned} \tag{A.1}$$

for any distribution y. Thus, we must have $\hat{x}_1 + 2\hat{x}_3 = 2\hat{x}_2 = 2\hat{x}_1 + \hat{x}_3$. Solving this we find $\hat{x}_1 = \hat{x}_3 = 2/3\ \hat{x}_2$. Noting that $\hat{x}$ is a distribution with components summing to one and the symmetry of A, we have equilibrium 7, $((2/7, 3/7, 2/7), (2/7, 3/7, 2/7))$.

In order to find the remaining equilibria we follow a similar method to that above. Starting from the equations,

$$\begin{aligned} x \cdot Ay &= x_1(y_1 + 2y_3) + x_2(2y_2) + x_3(2y_1 + y_3) \\ &= y_1(x_1 + 2x_3) + y_2(2x_2) + y_3(2x_1 + x_3), \end{aligned} \tag{A.2}$$

we look to find equalizer strategies x, y with other supports.

Let $x_1 = 0$, considering x with support $\{2, 3\}$ so that $x_2, x_3 \neq 0$. From this, we have the equation, $y_1(2x_3) + y_2(2x_2) + y_3(x_3) = \text{const.}$ for all x, y. Since $x_3 \neq 0$, we must have either $y_1 = 0$ or $y_3 = 0$.

If $y_1 = 0$, we have $2y_2 = y_3 = 2/3$. Player 1's payoff is given $x \cdot Ay = x_1(4/3) + x_2(2/3) + x_3(2/3)$. The optimal strategy for Player 1 against strategy y is then the pure strategy, $x_1 = 1$. Thus, we must have $y_3 = 0$.

Substitution into the original equation yields equilibrium 4, $(x, y) = ((0, 1/2, 1/2), (1/2, 1/2, 0))$. Repeating this process for $x_2 = 0$ and $x_3 = 0$, we obtain the two remaining Nash equilibria of A, 5 and 6. The method described here can be repeated to calculate the Nash equilibria of B, 8 - 14.

A.3 Example 6.2 - Equalizer Strategies of Ch×Ba

Consider the asymmetric game Chicken × Battle. For a sufficiently large number of rounds, using (6.6) with the payoff structures for Chicken and Battle, we see that the payoff matrix for strategies AllC, AllD and Tit-For-Tat is,

$$\begin{bmatrix} (3,1) & (1,5) & (3,1) \\ (5,3) & (0,0) & (0,0) \\ (3,1) & (0,0) & (3,1) \end{bmatrix}, \tag{A.3}$$

which decomposes to the two symmetric games,

$$A = \begin{bmatrix} 3 & 1 & 3 \\ 5 & 0 & 0 \\ 3 & 0 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 5 & 1 \\ 3 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}. \tag{A.4}$$

As discussed in 6.3, we look to find solutions $(\hat{x}, \hat{y})$ for the pair of equations (6.7) and (6.8), that is, we look to find an equalizer strategy for Player 1 in Player 2's payoff and an equalizer strategy for Player 2 in Player 1's payoff i.e. $\hat{x} \cdot By = \text{const.}$ and $x \cdot A\hat{y} = \text{const.}$ for all $x, y \in \Delta_n$.

Beginning with Player 1's equalizer strategy $\hat{x}$, we have,

$$\begin{aligned} \hat{x} \cdot By &= \hat{x}_1(y_1 + 5y_2 + y_3) + \hat{x}_2(3y_1) + \hat{x}_3(y_1 + y_3) \\ &= y_1(\hat{x}_1 + 3\hat{x}_2 + \hat{x}_3) + y_2(5\hat{x}_1) + y_3(\hat{x}_1 + \hat{x}_3) = \text{const.} \end{aligned} \tag{A.5}$$

for all y. From the coefficients of y_1 and y_3, we have $\hat{x}_2 = 0$. Equating this to the coefficient of y_2 we have $\hat{x}_3 = 4\hat{x}_1$, giving us the equalizer strategy $\hat{x} = (1/5, 0, 4/5)$.

Similarly we now look to find $\hat{y}$,

$$x \cdot A\hat{y} = x_1(3\hat{y}_1 + \hat{y}_2 + 3\hat{y}_3) + x_2(5\hat{y}_1) + x_3(3\hat{y}_1 + 3\hat{y}_3) = \text{const.} \tag{A.6}$$

for all x. From the coefficients of x_1 and x_3, we have $\hat{y}_2 = 0$. Equating this to the coefficient of x_2 we have $2\hat{y}_1 = 3\hat{x}_3$, giving us the equalizer strategy $\hat{y} = (3/5, 0, 2/5)$, hence a resulting NE for the asymmetric game at $(\hat{x}, \hat{y}) = ((1/5, 0, 4/5), (3/5, 0, 2/5))$.

A.4 Example 6.2 - Stable Equilibria with Shared Supports

Here we look to find the NE resulting from equilibria in A and B with the shared support $\{1, 2\}$, thus for an strategy pair $(\hat{x}, \hat{y})$ we let $\hat{x}_3 = \hat{y}_3 = 0$. Following from (A.5) and (A.6) when then have,

$$\begin{aligned} \hat{x} \cdot By &= y_1(\hat{x}_1 + 3\hat{x}_2) + y_2(5\hat{x}_1) = \text{const.} \\ x \cdot A\hat{y} &= x_1(3\hat{y}_1 + \hat{y}_2) + x_2(5\hat{y}_1) = \text{const..} \end{aligned} \tag{A.7}$$

Solving this pair of equations we find the equilibrium pair $(\hat{x}, \hat{y}) = ((3/7, 4/7, 0), (1/3, 2/3, 0))$. We confirm whether this pair does form a equilibrium by assessing each player's payoff, given,

$$\begin{aligned} \hat{x} \cdot By &= y_1(\hat{x}_1 + 3\hat{x}_2 + \hat{x}_3) + y_2(5\hat{x}_1) + y_3(\hat{x}_1 + \hat{x}_3) \\ &= y_1(15/7) + y_2(15/7) + y_3(3/7) \\ x \cdot A\hat{y} &= x_1(3\hat{y}_1 + \hat{y}_2 + 3\hat{y}_3) + x_2(5\hat{y}_1) + x_3(3\hat{y}_1 + 3\hat{y}_3) \\ &= x_1(5/3) + x_2(5/3) + x_3, \end{aligned} \tag{A.8}$$

both of which are maximised by x, y with support $\{1, 2\}$.