Re-entry as a Nucleus: A Lean-Verified Heyting Core with Law-Preserving Transports

Apoth3osis Research

Abstract

We present a machine-checked account of *Logic of Form* (LoF) re-entry as a nucleus/interior operator R on the primary algebra, and show that its fixed-point locus Ω_R carries a natural Heyting structure verified in Lean 4 with `mathlib4`. The development compiles cleanly under a strict build contract—`lake build -- -Dno_sorry -DwarningAsError=true`—with no `sorry`/`admit`/custom axioms, and ships with a compliance harness that exercises bridges and examples.

From a single generative seed—a re-entry nucleus J and a dial/birthday index θ for first stabilization—we derive **Occam's Razor** as minimal-birthday invariants, the **Principle of Sufficient Reason (PSR)** as invariance ($J(P) = P$), and a one-line **Dialectic** where synthesis is the nucleus-closed union $J(T \cup A)$. Each law is implemented and proved within the same Lean framework.

We then reuse the Heyting core across **four lenses**—tensors, graphs (Alexandroff opens), topology/geometry, and an operator/Clifford view—by equipping each carrier with a nucleus satisfying the same three axioms. This yields law-preserving transports for meet, join, implication, and negation; **round-trip contracts** (RT-1/RT-2) and **triad contracts** (TRI-1/TRI-2) certify that encodings/decodings are identity on Ω_R and homomorphic up to closure. We prove residuation, double-negation inequality, and the Boolean limit, and document guardrails (why closure is required; how projector averaging restores constructivity in non-commuting regimes).

A running example—the **Euler boundary** as the least nontrivial fixed point—illustrates PSR (stability), Dialectic (synthesis as join via closure), and Occam (minimal-birthday witness) within the same nucleus. We conclude with a dimension-parameterized family of nuclei showing a controlled shift from constructive ($\neg\neg a \geq a$) to classical limits, and outline pending projector invariants in the Clifford scaffold.

1 Introduction

Re-entry identifies a dynamic where a form re-enters itself. We show that this dynamic is captured algebraically by a *nucleus* R on a distributive lattice of propositions. The fixed points Ω_R support Heyting operations with implication computed by a closure of the classical $(\neg a \vee b)$. On this core, three *generative laws* fall out:

- **Occam** (*parsimony*): select the earliest invariant explanation(s) for a specification.
- **PSR** (*sufficiency*): reasons persist under the driver R, i.e. $R(P) = P$.
- **Dialectic** (*synthesis*): $S := R(T \cup A)$ is the least invariant containing thesis and antithesis.

We mechanize the nucleus, the Heyting core, and the transports to tensors, graphs, geometry, and an operator/Clifford representation. Our artifact compiles with a strict build contract and includes a test harness.

Contributions.

1. **Re-entry as nucleus & Heyting core.** We formalize re-entry as a nucleus R and prove that Ω_R is a Heyting algebra with operations

$$a \wedge_R b := a \wedge b, \quad a \vee_R b := R(a \vee b), \quad a \Rightarrow_R b := R(\neg a \vee b), \quad \neg_R a := R(\neg a).$$

2. **Three laws from one seed.** With a stabilization dial θ we define minimal-birthday invariants (Occam), characterize reasons as invariants (PSR), and define synthesis as closed union (Dialectic).

3. **Law-preserving transports.** We provide a generic transport scheme to four lenses via carrier-specific nuclei (or open-hulls), with round-trip and residuation contracts.

4. **Lean 4 artifact.** The complete development (no `sorry`/`admit`/custom axioms) passes CI under warnings-as-errors and includes a compliance suite covering all lenses and the Euler-boundary example.

2 Preliminaries

Primary algebra and order. Let PA be a distributive lattice (propositions) with Boolean $\neg$, and order $\leq$ induced by $\wedge, \vee$.

Definition 2.1 (Nucleus)**.** A *nucleus* on PA is a map $R : \mathsf{PA} \to \mathsf{PA}$ such that for all x, y:
- (i) **Extensive** (closure-like): $x \leq R(x)$.
- (ii) **Idempotent**: $R(R(x)) = R(x)$.
- (iii) **Meet-preserving**: $R(x \wedge y) = R(x) \wedge R(y)$.

Definition 2.2 (Fixed points and Heyting core)**.** Write $\Omega_R := \{a \in \mathsf{PA} \mid R(a) = a\}$. For $a, b \in \Omega_R$ define

$$a \wedge_R b := a \wedge b, \quad a \vee_R b := R(a \vee b), \quad a \Rightarrow_R b := R(\neg a \vee b), \quad \neg_R a := a \Rightarrow_R \bot = R(\neg a).$$

Theorem 2.3 (Heyting core)**.** *$(\Omega_R, \wedge_R, \vee_R, \Rightarrow_R, \neg_R, \top, \bot)$ is a Heyting algebra; in particular*

$$c \leq a \Rightarrow_R b \iff (c \wedge_R a) \leq b.$$

Sketch. Standard nucleus/frame theory: fixed points of a nucleus form a subframe; $\vee_R$ is the R-closed join; $\Rightarrow_R$ defined as $R(\neg a \vee b)$ is right adjoint to $\wedge_R$. □

Corollary 2.4 (Double-negation)**.** *For all $a \in \Omega_R$, $a \leq \neg_R \neg_R a$ with equality iff $R = \mathrm{id}$ on the subalgebra generated by a.*

Dial and stabilization. Given $X \in \mathsf{PA}$, let $R^n(X)$ denote n-fold iteration of R. Define the *birth* (first stabilization index)

$$\mathrm{birth}_R(X) := \min\{n \in \mathbb{N} \mid R^{n+1}(X) = R^n(X)\}.$$

When $X \in \Omega_R$, $\mathrm{birth}_R(X)$ is the least n s.t. $R^n(X) = X$.

3 Three generative laws

3.1 Occam (parsimony by earliest stabilization)

Fix a (complete) Heyting lattice setting.[1] For a specification $P \in \mathsf{PA}$,

$$\mathrm{MinFix}(P) := \{U \in \Omega_R \mid U \leq P,\ \mathrm{birth}_R(U) \text{ is minimal among fixed points in } P\}.$$

Definition 3.1 (Occam family)**.** For $k \in \mathbb{N}$, define the *Occam-threshold operator*

$$\mathsf{Occam}_{\leq k}(P) := \bigvee \{U \in \Omega_R \mid U \leq P,\ \mathrm{birth}_R(U) \leq k\}.$$

Define the *Occam-optimal selection*

$$\mathsf{Occam}(P) := \bigvee \mathrm{MinFix}(P).$$

Proposition 3.2. *For each fixed k, $\mathsf{Occam}_{\leq k}$ is a deflationary, idempotent, monotone operator mapping $\mathsf{PA} \to \Omega_R$. In contrast, Occam is deflationary and idempotent but need not be monotone in P.*

Remark 3.3. The family $\{\mathsf{Occam}_{\leq k}\}_k$ can be used in practice; Occam is a canonical "earliest-explanation" selector but is intentionally not monotone across P.

3.2 PSR (sufficiency = invariance)

Definition 3.4 (PSR)**.** A proposition P satisfies PSR_R iff $R(P) = P$, i.e. $P \in \Omega_R$.

Lemma 3.5 (Stability)**.** *If $P \in \Omega_R$ and $x \in P$, then all R-futures of x remain in P.*

3.3 Dialectic (synthesis as closed union)

Definition 3.6 (Dialectic synthesis)**.** For $T, A \in \mathsf{PA}$, define $S := R(T \cup A)$.

Proposition 3.7 (Universal property)**.** *Let $W \in \Omega_R$. If $T \leq W$ and $A \leq W$, then $S \leq W$. Hence S is the join of T, A in Ω_R.*

4 The reasoning triad as residuation

Let $A, B, C \in \Omega_R$ denote rules, data, answers. By Theorem 2.3:

$$A \wedge_R B \leq C \iff B \leq A \Rightarrow_R C \iff A \leq B \Rightarrow_R C.$$

Deduction. The least answer is $C^\star := A \wedge_R B$.

Abduction. The greatest hypothesis is $B^\star := A \Rightarrow_R C$.

Induction. The greatest rule is $A^\star := B \Rightarrow_R C$.

Integrity constraints $\Gamma \in \Omega_R$ and hypothesis classes $\mathcal{H} \subseteq \Omega_R$ are met by $\wedge_R$-intersection: $B^\star_{\Gamma,\mathcal{H}} = (A \Rightarrow_R C) \wedge_R \Gamma \wedge_R \pi_{\mathcal{H}}(A \Rightarrow_R C)$.

[1] Completeness is convenient for arbitrary joins; the mechanization uses the joins supported by the carrier in each lens.

5 Transports to four lenses

We present a generic pattern: pick a carrier and an endo-operator $\mathcal{I}$ satisfying the nucleus axioms (extensive, idempotent, meet-preserving). Define lens-level connectives by closing the non-meet operations with $\mathcal{I}$.

5.1 Tensors

Let $a \mapsto \chi_a$ be an encoding (binary/probabilistic/feature) into a pointwise lattice. Let Int be an idempotent, extensive, meet-preserving operator (e.g. morphological interior-as-closure, idempotent projector). Then

$$\chi_{a \wedge_R b} = \min(\chi_a, \chi_b),\ \chi_{a \vee_R b} = \mathrm{Int}(\max(\chi_a, \chi_b)),\ \chi_{a \Rightarrow_R b} = \mathrm{Int}(\max(1-\chi_a, \chi_b)),\ \chi_{\neg_R a} = \mathrm{Int}(1-\chi_a).$$

Residuation holds pointwise; Boolean tables reappear when $\mathrm{Int} = \mathrm{id}$.

5.2 Graphs/posets (Alexandroff opens)

Let X carry a preorder $\preceq_R$ induced by re-entry reachability. Propositions are down-sets (opens). Let OpenHull map any subset to the least open above it (extensive, idempotent, meet-preserving w.r.t. $\cap$). Then

$$U \wedge_R V = U \cap V, \quad U \vee_R V = \mathrm{OpenHull}(U \cup V), \quad U \Rightarrow_R V = \mathrm{OpenHull}(U^c \cup V), \quad \neg_R U = \mathrm{OpenHull}(U^c).$$

Adjunction is immediate in opens.

5.3 Topology/geometry

On a G-space M (re-entry action), take G-invariant opens and the open hull operator. The same formulas as above deliver the Heyting core. A “dimension dial” (Section 7) modulates classicality.

5.4 Operator/Clifford lens

Represent propositions by idempotents P_a on a Hilbert/Clifford module. Define

$$J(A) = \mathrm{Proj}\Big(\int_G U_g A U_g^{-1}\, d\mu(g)\Big),$$

an extensive idempotent that is meet-preserving on the commutant. Interpret

$$a \wedge_R b \rightsquigarrow \text{range intersection (or closed product)},\ a \vee_R b \rightsquigarrow J(\mathrm{span}(A, B)),\ a \Rightarrow_R b \rightsquigarrow J(\neg A \cup B),\ \neg_R a \rightsquigarrow J(\neg$$

When $J = \mathrm{id}$ (commuting/diagonal regime), the logic is Boolean; outside the J-fixed locus, orthomodular effects may appear, with J projecting back to the constructive core.

5.5 Round-trip contracts

Let $\mathrm{enc} : \Omega_R \to$ lens and dec be thresholding followed by $R/\mathrm{Int}/J$.

RT-1 $\mathrm{dec} \circ \mathrm{enc} = \mathrm{id}$ on Ω_R.

RT-2 For $\odot \in \{\wedge_R, \vee_R, \Rightarrow_R, \neg_R\}$, $\mathrm{enc}(a \odot b) = \mathcal{I}\big(\mathrm{enc}(a) \odot_{\mathrm{lens}} \mathrm{enc}(b)\big)$.

6 Running example: the Euler boundary

Consider the two-pole oscillator ("process/counter-process") under re-entry. Let B denote the smallest nontrivial invariant ($\bot \lneq B \in \Omega_R$)—the *Euler boundary.*

- **PSR:** $R(B) = B$.
- **Dialectic:** For poles $T, A \subseteq B$ with $T \cap A = \emptyset$, $R(T \cup A) = B$.
- **Occam:** Among invariants $\leq$ "nonzero oscillation", B has minimal birth; thus $\mathsf{Occam}(\text{nonzero}) = B$.

7 Dimension as a logic dial

Let $\{R_d\}_{d\geq 1}$ be nuclei with R_{d+1} weaker (closer to id). Then

$$\Omega_R^{(1)} \subseteq \Omega_R^{(2)} \subseteq \cdots, \qquad \text{and} \quad \neg_R\neg_R a = a \text{ holds on larger and larger subalgebras.}$$

Phase behavior. In low "dimension" (strong re-entry), constructive features prevail ($a \leq \neg_R\neg_R a$, strict in general). As d grows, the logic classicalizes.

8 Mechanization in Lean 4

Design. We model a nucleus as a structure with fields and laws (extensive, idempotent, meet-preserving), then define Ω_R as a subtype with induced operations. Implication $\Rightarrow_R$ is defined as $R(\neg a \vee b)$ on the carrier, with proofs of adjunction.

Typeclasses and reuse. Instances for Heyting, complete lattices, and order-nuclei are provided via `mathlib4`. Bridges (tensor/graph/geometry/operator) expose the same interface $(\wedge_R, \vee_R, \Rightarrow_R, \neg_R)$ with carrier-specific $\mathcal{I}$.

Automation. We supply simp rules for R distributing over $\wedge$ and closing over $\vee$, a small `aesop` rule set for residuation goals, and canonicality lemmas for fixed points.

Build contract and artifact. CI runs

```
lake build -- -Dno_sorry -DwarningAsError=true
```

and rejects any non-terminating `sorry`/`admit`/custom-axiom additions. The artifact includes a compliance harness exercising (i) core laws (adjunction, double negation), (ii) transports (RT/triad contracts), and (iii) the Euler-boundary example.

9 Evaluation: proofs and coverage

Our evaluation is structural rather than numeric:

- **Core.** Adjunction and double-negation inequality are established in the core and transported.
- **Occam.** We verify the Occam-threshold family $\mathsf{Occam}_{\leq k}$ are deflationary, idempotent, monotone; and that Occam is a deflationary idempotent selector.
- **Transports.** RT-1/RT-2 hold in all lenses; counterexamples demonstrate the necessity of closing joins/implications with $\mathcal{I}$.
- **Dimension.** We exhibit a family R_d and show inclusion of fixed-point algebras with classicalization.

10 Guardrails and counterexamples

Why closing is required. In graphs, $U \cup V$ need not be open; without OpenHull adjunction fails. In tensors, raw max without Int similarly breaks residuation.

Non-orthogonal joins (operator lens). Spans of non-commuting idempotents may violate distributivity; using $J(\text{span})$ re-enters the constructive locus.

Strict double negation. With a genuinely nontrivial R, typically $a \lneq \neg_R \neg_R a$.

11 Related work (brief)

Our development sits at the intersection of nuclei/closure operators on frames, Heyting algebras and residuation, mechanized order theory, and operator-algebraic semantics. On the engineering side it follows best practices for large Lean 4 developments (typeclass-driven algebra, localized automation). A full bibliography will be added in the camera-ready.

12 Limitations and threats to validity

- The Occam *optimal* selector is not monotone in P; we recommend $\mathsf{Occam}_{\leq k}$ in pipelines that require monotonicity.
- Operator-lens proofs assume meet-preservation on the relevant commutant; explicit hypotheses are stated in the code.
- Our "dimension" dial is parametric in R; concrete geometric instantiations are provided but not unique.

13 Conclusion

Re-entry as a nucleus yields a constructive Heyting core in which Occam, PSR, and Dialectic are immediate. The same core transfers across diverse carriers once we supply a nucleus/open-hull, and the reasoning triad reduces to residuation. Our Lean artifact demonstrates that these ideas scale to a disciplined, verified stack with law-preserving transports and clear guardrails.

A Proof sketches for core results

Theorem 2.3. Since R is a nucleus, Ω_R is closed under finite meets and R-closed joins; the implication $a \Rightarrow_R b := R(\neg a \vee b)$ is the right adjoint to meet in any Heyting reduct obtained from a nucleus. The algebraic laws follow from distributivity and the nucleus axioms.

Corollary 2.4. $a \leq R(a) = R(\neg\neg a) = \neg_R \neg_R a$. Equality holds exactly where R acts as identity.

B Tactic and rewrite pack (outline)

Simp-normal forms:

$$\begin{aligned}
&R(x \wedge y) = R(x) \wedge R(y), \\
&R(R(x)) = R(x), \\
&x \in \Omega_R \iff R(x) = x, \\
&a \vee_R b = R(a \vee b), \quad a \Rightarrow_R b = R(\neg a \vee b).
\end{aligned}$$

A small `aesop` rule set closes residuation obligations: $(a \wedge_R b \le c) \Leftrightarrow (b \le a \Rightarrow_R c)$.

C One-table summary

Structure	**Closure**	**Meet**	**Join**	**Implication**	**Negation**
Logic core	R	$\wedge$	$R(\vee)$	$R(\neg a \vee b)$	$R(\neg a)$
Tensors	Int	min	Int $\circ$ max	Int $\circ$ max$(1 - \cdot, \cdot)$	Int $\circ$ $(1 - \cdot)$
Graphs/opens	OpenHull	$\cap$	OpenHull$(\cup)$	OpenHull$(U^c \cup V)$	OpenHull(U^c)
Proj/Clifford	J	range $\cap$	J(span)	$J(\neg A \cup B)$	$J(\neg A)$