Wireless Communication Technologies

Rutgers University – Dept. of Electrical and Computer Engineering
ECE559 (Advanced Topics in Communication Engineering)
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Lecture 11

Minimum Shift Keying

Minimum shift keying (MSK) is a special type of continuous phase-frequency shift keying (CPFSK) with h=0.5. A modulation index of 0.5 corresponds to the minimum frequency spacing that allows two FSK signals to be coherently orthogonal, and the name minimum shift keying implies the minimum frequency separation (i.e. bandwidth) that allows orthogonal detection.
MSK has one of two possible frequencies over any symbol interval:

$$S(t) = A\cos[(2\pi f_c + \frac{\pi x_k}{2T})t + \underbrace{\frac{\pi}{2}\sum_{n=-\infty}^{k-1} x_n - \frac{\pi}{2}(kx_n)}_{excess\_phase}] \tag{11.1}$$

In traditional FSK we use signals of two different frequencies of f_0 and f_1 to transmit a message $m=0$ or $m=1$ over a time of T_b seconds,

$$S_0(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_0 t) \qquad 0 \le t \le T_b \tag{11.2}$$

$$S_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t) \qquad 0 \le t \le T_b \tag{11.3}$$

We assume that $f_0 > f_1 > 0$. If we choose the frequencies so that in each time interval T_b there is an integer number of periods, $f_0 = \frac{k_0}{T_b}; f_1 = \frac{k_1}{T_b}$,

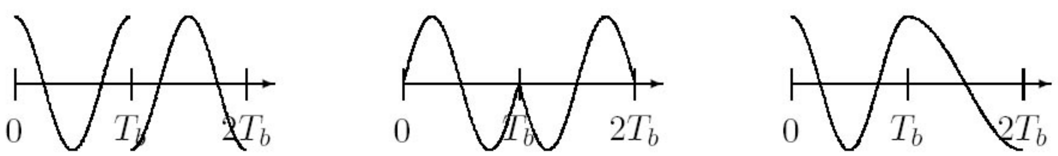

Figure 1. Signals with different degrees of discontinuity

With k_0 and k_1 integers, the signal is guaranteed to have continuous phase. Figure 1 shows an example of a signal that is discontinuous, a signal with discontinuous phase and a signal with continuous phase. As phase-continuous signals in general have better spectral properties than signals that are not phase-continuous, we prefer to transmit signals that have this property.

If either f_0 or f_1 are chosen such that there is a no-integer number of periods the traditional FSK modulator will output a signal with discontinuities in the phase. In order to maintain phase continuity, we can let the transmitter have memory. We choose the signals for a general CPSFSK transmitter to be

$$S_0(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_0 t + \theta(0)) \qquad 0 \le t \le T_b \qquad (11.4)$$

$$S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_1 t + \theta(0)) \qquad 0 \le t \le T_b \qquad (11.5)$$

We keep the phase continuous by letting $\theta(0)$ be equal to the argument of the cosine pulse for the previous bit interval. For the signals over an arbitrary bit interval, $kT_b \le t < (k+1)T_b$, the general phase memory term is $\theta(kT_b)$.

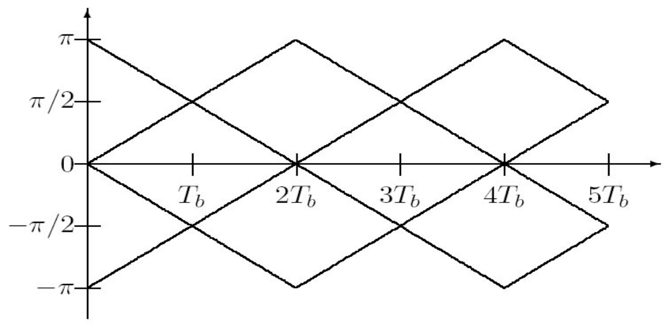

Figure 2. Phase trellis for h=1/2

In Figure 2 we depict the phase variation over time in a phase trellis, here, we have assumed h=1/2 and $\theta(0) = 0$ or $\theta(0) = \pi$. We see that for every multiple of the bit time the phase can only take on one of two values, the values being 0 and π for $t = 2kT_b$, and $\pm\frac{\pi}{2}$ for $t = (2k+1)T_b$.

CPFSK with deviation ratio h=1/2 is called MSK. The frequency difference $f_0 - f_1 = \frac{1}{2T_b}$ that results from choosing h=1/2 is the smallest possible difference if the signals of the two frequencies are to be orthogonal over one bit interval. An example of an MSK signal with $k_0 = 1$ and

$k_1 = 1/2$ is given in Figure 3.

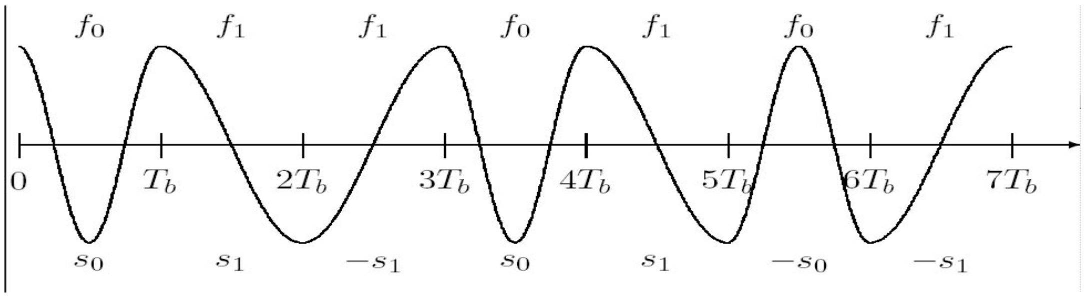

Figure 3. Example of an MSK signal

BFSK detection of MSK is perhaps the most natural first choice for a detector principle. It has the same bit error probability as ordinary BPSK.

$$P_e = Q(\sqrt{\frac{2E_b}{N_0}}) \tag{11.6}$$

That means MSK is approximate the same as BPSK in power efficiency.

Gaussian Minimum Shift keying (GMSK)

Gaussian Minimum Shift Keying (GMSK) is a modification of MSK (i.e. CPFSK with h = 1/2). A filter used to reduce the bandwidth of a baseband pulse train prior to modulation is called a pre-modulation filter. The Gaussian pre-modulation filter smooths the phase trajectory of the MSK signal thus limiting the instantaneous frequency variations. The result is an FM modulated signal with a much narrower bandwidth. This bandwidth reduction does not come for free since the pre-modulation filter smears the individual pulses in pulse train. As a consequence of this smearing in time, adjacent pulses interfere with each other generating what is commonly called inter-symbol interference or ISI. In the applications where GMSK is used, the trade-off between power efficiency and bandwidth efficiency is well worth the cost.

BER for GMSK is

$$P_e = Q(\sqrt{\frac{2\alpha E_b}{N_0}}) \tag{11.7}$$

where α is a constant related to BT_b.

The value of BT_b	The values of α
0.25	0.68
∞	0.85

Table1. GMSK parameter α related to BT_b

Note that the case where $BT \to \infty$ corresponds to MSK (i.e. the filter is allpass for a fixed symbol interval T_s).

Recall the probability of error for plain MSK is given by

$$P_e \approx Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{11.8}$$

By comparing it with (11.7), we can conclude that $P_e^{GMSK} > P_e^{MSK}$. This arises from the trade off between power and bandwidth efficient: GMSK achieves a better bandwidth efficiency than MSK at the expense of power efficiency.

Error Probabilities on Flat and Slow Fading Channel

We transmit a signal as:

$$T_x : S_i(t) = \sqrt{\frac{2E_s}{T_s}} \cos(2\pi f_c t + \frac{2\pi}{M}(i-1)), 0 \le t < T_s \tag{11.9}$$

In flat fading channel, the received signal is modeled as:

$$x(t) = g(t)s_i(t) + w(t) \tag{11.10}$$

Where, $g(t)$ is the attenuation parameter in amplitude of signal while $w(t)$ is AWGN with zero-mean and power spectral density of $\frac{N_0}{2}$.

For slow flat fading channel, channel changes very slowly during a symbol interval, (i.e. $T_s << T_c$), $g(t)$ is effectively constant over a symbol duration.

Let $g(t) = \alpha$

$$x(t) = \alpha s_i(t) + w(t) \tag{11.11}$$

For a constant α, ML decoding rule still remain same. Optimum detector should minimize $\left\| \underline{x} - \alpha \underline{s_k} \right\|$ over $k = 1,2,...M$. Receive structure is the same to project $x(t)$ onto $\{\phi_i(t)\}_{i=1}^N$ followed by correlation detection.

$$P_e \le \sum_{\substack{k=1 \\ k \ne i}}^{M} Q\left(\frac{\alpha d_{ik}}{\sqrt{2N_0}}\right) \tag{11.12}$$

Typically, α is Rayleigh or Ricean according to NLOS or LOS. So, the average probability of error:

$$\overline{P}_e \le \sum_{\substack{k=1 \\ k \ne i}}^{M} \int_0^\infty Q\left(\frac{\alpha d_{ik}}{\sqrt{2N_0}}\right) f_\alpha(\bullet) d\alpha \tag{11.13}$$

1) Analysis of BPSK

For a special case, BPSK, $M = 2$, the SNR is given as

$$\gamma_b = \frac{\alpha^2 E_b}{N_0} \tag{11.14}$$

Let $\beta = \alpha^2$, we know that β is an exponential random variable if α is a Rayleigh distribution. So:

$$\overline{P}_e = \int_0^\infty Q\left(\sqrt{\frac{2\beta E_b}{N_0}}\right) f_\beta(\bullet) d\beta \tag{11.15}$$

In order to state the distribution of γ_b, we need its mean:

$$\gamma_b = \beta \frac{E_b}{N_0} \tag{11.16}$$

$$E[\gamma_b] = \overline{\gamma_b} = E[\beta]\frac{E_b}{N_0} \tag{11.17}$$

After known its mean, we can write the distribution as follows:

$$f(\gamma_b) = \frac{1}{\overline{\gamma_b}} \exp\left(-\frac{\gamma_b}{\overline{\gamma_b}}\right), \qquad \gamma_b \geq 0 \tag{11.18}$$

Then, re-write (11.15) as:

$$\overline{P}_e = \int_0^\infty \underbrace{Q(\sqrt{2\gamma_b})}_{u} \underbrace{\frac{1}{\overline{\gamma_b}} e^{-\gamma_b/\overline{\gamma_b}} d\gamma_b}_{dv} \tag{11.19}$$

Integrating by parts using:

$$du = \frac{2}{2\gamma_b} Q(\sqrt{2\gamma_b}) = -\frac{1}{\sqrt{2\pi}} e^{-\gamma_b} \gamma_b^{-1/2} \tag{11.20}$$

$$v = -\exp\left(-\frac{\gamma_b}{\overline{\gamma_b}}\right) \tag{11.21}$$

Substitute: $$u = \gamma_b \frac{1+\overline{\gamma_b}}{\overline{\gamma_b}} \tag{11.22}$$

$$\overline{P}_e = \left[-e^{-\frac{\gamma_b}{\overline{\gamma_b}}} Q(\sqrt{2\gamma_b})\right]_0^\infty - \int_0^\infty e^{-\frac{\gamma_b}{\overline{\gamma_b}}} \frac{1}{2\sqrt{2\pi}} e^{-\gamma_b} \gamma_b^{-1/2} d\gamma_b \tag{11.23}$$

$$=\frac{1}{2}-\frac{1}{2\sqrt{\pi}\sqrt{\frac{1+\overline{\gamma_b}}{\overline{\gamma_b}}}}\int_0^{\infty} e^{-u}u^{-1/2}\,du$$

$$=\frac{1}{2}(1-\sqrt{\frac{\overline{\gamma_b}}{1+\overline{\gamma_b}}})$$

2) Analysis of BFSK

For Binary Frequency Shift Keying (BFSK), since

$$P_e = Q(\sqrt{\frac{E_b}{N_0}}) \tag{11.24}$$

In slow flat fading channel, the probability of bit error is given by:

$$P_e = \frac{1}{2}(1-\sqrt{\frac{\overline{\gamma_b}}{2+\overline{\gamma_b}}}) \tag{11.25}$$

Comparing (11.25) to (11.23), we can get the conclusion that coherent BPSK is about 3dB better than BFSK.

Non-Coherent Detection

In the above discussion we have assumed accurate phase information, however we must realize that in practical conditions fading actually destroys all phase information. Thus, in practice non-coherent modulation may be preferable.

In this case, we assume the Transmit signal is:

$$S_i(t)=\sqrt{\frac{2E}{T}}\cos(2\pi f_i t) \qquad 0\le t\le T \tag{11.26}$$

And the received signal is given as:

$$x(t)=\sqrt{\frac{2E}{T}}\cos(2\pi f_i t+\theta)+w(t) \tag{11.27}$$

That is:

$$x(t)=\sqrt{\frac{2E}{T}}(\cos(2\pi f_i t)\cos\theta-\sin(2\pi f_i t)\sin\theta)+w(t) \tag{11.28}$$

Where, θ is an unknown phase and $w(t)$ is AWGN with zero-mean and $\frac{N_0}{2}$. We usually assume that θ is uniformly distribution over $[0,2\pi]$. How to detect $S_i(t)$? This can be accomplished by using a Quadrature receiver.

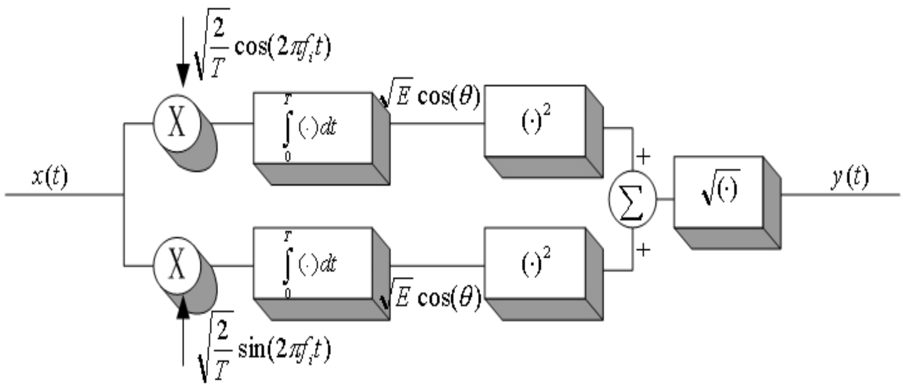

Figure 4. Quadrature Receiver

Also, we can use an envelope detector to achieve this aim:

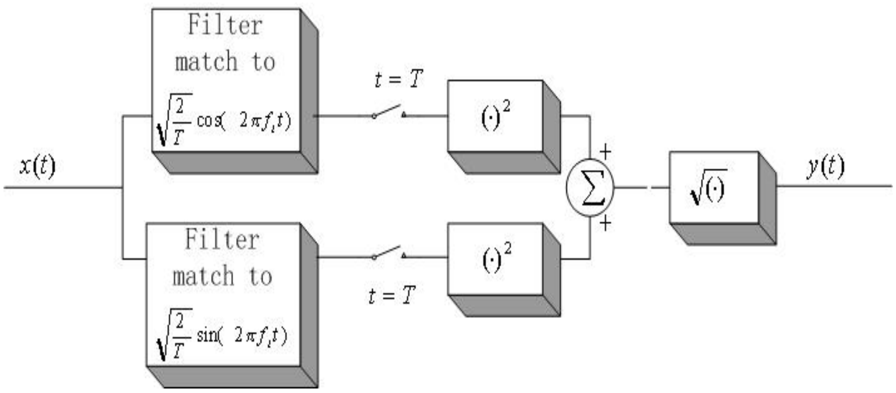

Figure 5. Envelope Detector

It is easy to prove that a Quadrature receiver and an envelope detector can be implemented interchangeabley.

Non-Coherent Orthogonal Modulation

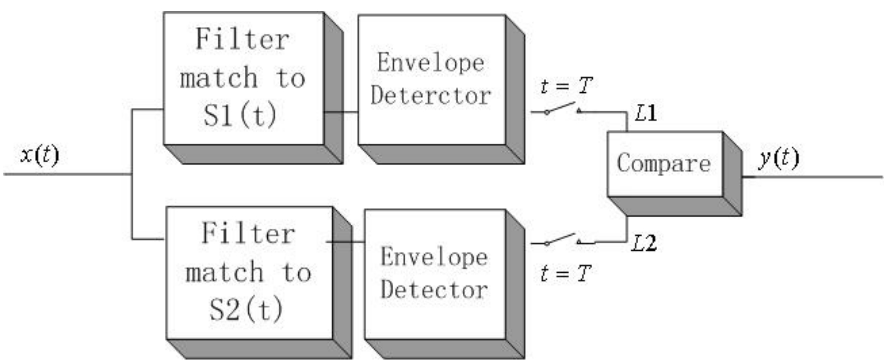

Figure 6. Non-coherent Receiver for BFSK

The optimum decision is given by comparing the output of the two branches of the non-coherent receiver. We can get the P_e as:

$$P_e = \frac{1}{2}\exp(-\frac{E_b}{2N_0}) \qquad (11.29)$$

As for M-ary FSK System with non-coherent detecting,

$$P_e = \frac{1}{2(M-1)}\sum_{i=2}^{M}(-1)^{i}\exp(-\frac{(i-1)kE_b}{iN_0}) \qquad k = \log_2 M \qquad (11.30)$$

The BER curves for noncoherent M-ary FSK as a function of M and SNR are depicted in Figure 7.

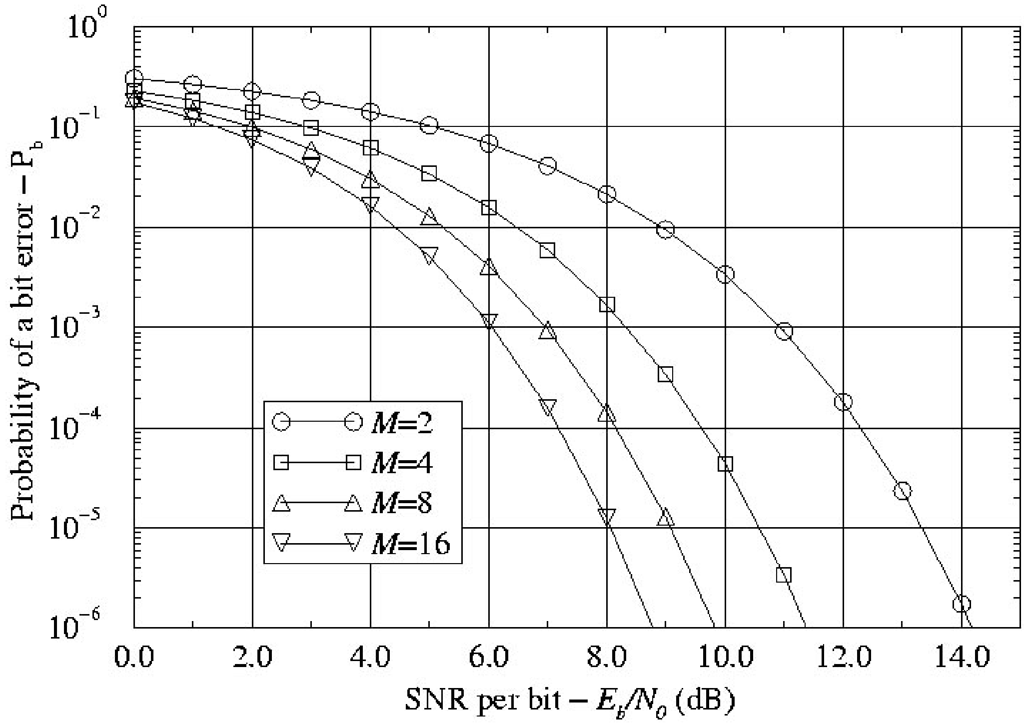

Figure 7. Non-coherent FSK BER

Lecture 12

Differential phase shift keying (DPSK)

DPSK is a non-coherent form of PSK avoiding the need for a coherent reference signal at the receiver. Instead, the received signal of the k^{th} symbol interval is compared to the phase of the received signal of the $(k-1)^{th}$ symbol interval. This method of modulation is appropriate in the presence of slow fading where the difference between two symbol intervals is small. Generate differentially encoded sequence $\{d_k\}$ from $\{m_k\}$ as follows

1. Sum d_{k-1} and m_k modulo 2.

2. Set k_k to be the compliment of result of step 1.

3. Use d_k to shift carrier phase (i.e. d_k =1, $\theta = 0$; d_k =0, $\theta = \pi$).

m_k	1	0	0	1	0	0	1	1
d_{k-1}	1	1	0	1	1	0	1	1
d_k	1	0	1	1	0	1	1	1
θ	0	π	0	0	π	0	0	0

Table 2. DPSK Carrier Phase Change Process

Since we use formula (12.1) to generate d_k

$$d_k = \overline{m_k \oplus d_{k-1}} \tag{12.1}$$

So, symbol d_k is unchanged from previous symbol, if the incoming symbol is '1'. Otherwise, it will be changed. DPSK signal over an interval $2T_b$ are

$$S_1(t) = \begin{cases} \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t), 0 \le t \le T_b \\ \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t), T_b \le t \le 2T_b \end{cases} \tag{12.2}$$

$$S_2(t) = \begin{cases} \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t), 0 \le t \le T_b \\ \sqrt{\dfrac{E_b}{2T_b}} \cos(2\pi f_c t + \pi), T_b \le t \le 2T_b \end{cases} \tag{12.3}$$

Over interval $T = 2T_b$, $S_1(t) \perp S_2(t)$, so we can view DPSK as a non-coherent orthogonal modulation. From (12.4), (12.5), (12.6), we compared the performance of three binary signaling schemes, the results are depicted in Figure 8.

$$P^{DPSK}{}_e = \frac{1}{2}\exp(-\frac{E}{2N_0}) = \frac{1}{2}\exp(-\frac{E_b}{N_0}) \tag{12.4}$$

$$P_e^{BPSK} = Q(\sqrt{\frac{2E_b}{N_0}}) \tag{12.5}$$

$$P_e^{NC-FSK} = \frac{1}{2}\exp(-\frac{E_b}{N_0}) \tag{12.6}$$

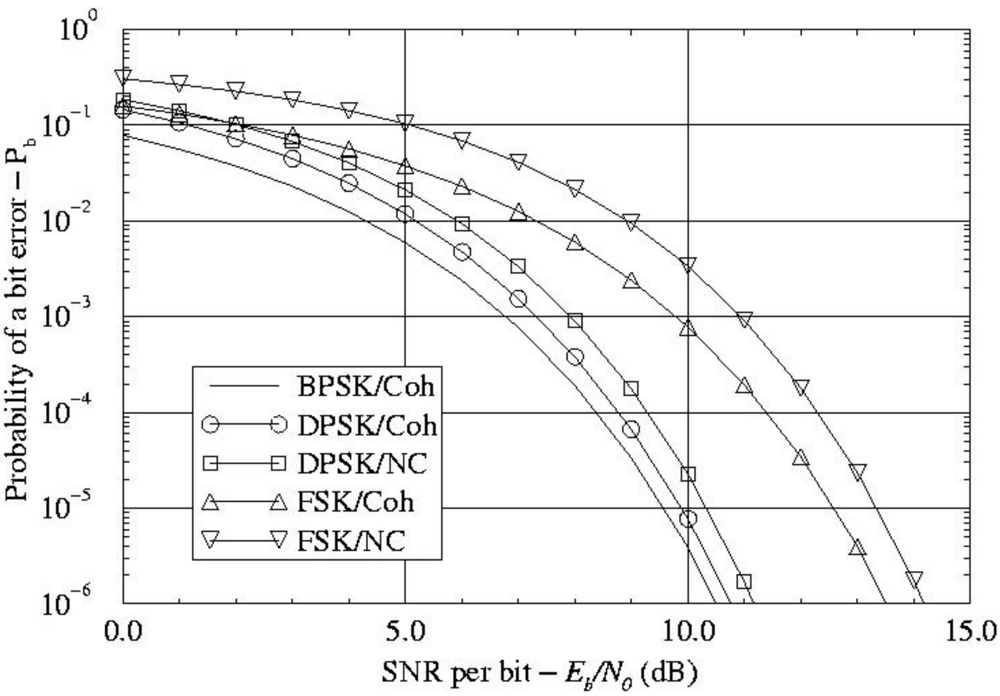

Figure 8. Comparison of Binary Signaling Schemes

Observe that for a bit error rate of $P_e \leq 10^{-3}$ the difference in SNR between BPSK and B=DPSK is less than 3 dB and that his difference becomes less than 1dB at a $P_e \leq 10^{-5}$. We can conclude that, at a high SNR, $P_e^{DPSK} \rightarrow P_e^{BPSK}$.

Digital Signaling Over Frequency Selective Fading Channels

The information signal over a communication channel is modeled as

$$v(t) = A\sum_k b(t-kT, \underline{x_k}) \tag{12.7}$$

For our analysis, we will restrict ourselves to linear modulation schemes (i.e. information sequence is manipulated through linear operations only):

$$b(t, \underline{x_k}) = x_k h_a(t) \tag{12.8}$$

where $\{x_k\}$ is the complex symbol sequence and $h_a(t)$ denotes the linear modulation operation. The information signal transmitted through a communication channel $c(t)$ results in the received complex signal

$$\omega(t) = \sum_{k=0}^{\infty} x_k h(t-kT) + z(t) \tag{12.9}$$

where $z(t)$ is a sample function of an Additive White Gaussian Noise (AWGN) process with zero mean and power spectral density N_0 and $h(t)$ denotes the time convolution of the channel impulse response and the linear modulation:

$$h(t) = \int_{-\infty}^{\infty} h_a(\tau)c(t-\tau)d\tau \tag{12.10}$$

For causal channels, this integral is nonzero only for time t greater than zero.

We further assume that the length of the filter is finite, meaning that $h(t)$ is nonzero only for a bounded time interval LT:

$$h(t) = 0 \;\text{ for }\; t \le 0 \;\text{ and }\; h(t) = 0 \;\text{ for }\; t \ge LT \tag{12.11}$$

The foregoing process is described in Figure 9.

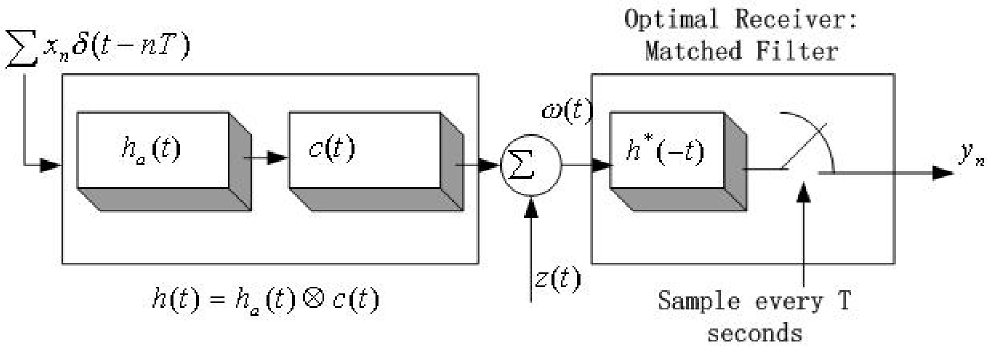

Figure 9. Matched Filter in an Additive Noise Channel

In order to build a matched filter to correctly detect the information we need to have the channel impulse response $c(t)$.

If we know $h(t)$, the matched filter can be implemented as follows:

$$y(t) = \sum_{k=-\infty}^{\infty} x_k f\left(t-kT\right) + v(t) \tag{12.12}$$

where $f(t)$ is the composite pulse response and $v(t)$ is the filtered noise. These two components are given by:

$$f(t) = \int_{-\infty}^{\infty} h^*(\tau)h(\tau+t)d\tau \tag{12.13}$$

$$v(t) = \int_{-\infty}^{\infty} h^*(\tau)z(\tau+t)d\tau \tag{12.14}$$

The receiver then samples the output of the matched filter $y(t)$ to get y_n given as:

$$y_n = y(nT) = \underbrace{x_n f_0}_{Desired\_Singal} + \underbrace{\sum_{\substack{k=-\infty \\ k\neq n}}^{\infty} x_k f_{n-k}}_{ISI(must\_let\_it\_be\_0)} + \underbrace{v_n}_{Noise\_term} \tag{12.15}$$

To achieve the same performance as in AWGN, the ISI term must be zero:

$$\sum_{\substack{k=-\infty \\ k\neq n}}^{\infty} x_k f_{n-k} = 0 \Rightarrow f_k = \delta_{k_0} f_0 \qquad \delta_{ij} = \begin{cases} 0, i \neq j \\ 1, i = j \end{cases} \tag{12.16}$$

If we meet Nyquist's criterion, the ISI portion would be zero. In order to meet this condition we must know the channel impulse response $c(t)$.

Optimum Receiver

$\omega(t)$ can be represented by a set of basis functions $\phi_n(t)$ as follows:

$$\omega(t) = \lim_{N\to\infty} \sum_{n-1}^{N} \omega_n \phi_n(t) \tag{12.17}$$

Note that if $\omega(t)$ is a random process. Then we should use Karhunen-Loeve's expansion and the limit would be in the mean sense. The main point is that once we have done the mapping from a continuous time function to a countable set of samples we can them continue with our developments in discrete space. We thus have:

$$\omega_n = \sum_{k=-\infty}^{\infty} x_k h_{nk} + z_n \tag{12.18}$$

$$h_{nk} = \int_0^T h(t-kT)\phi_n^*(t)dt \tag{12.19}$$

$$z_n = \int_0^T z(t)\phi_n^{\ *}(t)dt \tag{12.20}$$

Since we are working in N-dimensional space, we can continue our developments using N-dimensional vectors. Note that $\underline{\omega} = (\omega_1, \omega_2 ... \omega_n)$ is a multivariate Gaussian with PDF

$$p(\underline{\omega} \mid \underline{x}, H) = \prod_{n=1}^{N} \frac{1}{\pi N_0} \exp(-\frac{1}{N_0} \mid \omega_n - \sum_{k=-\infty}^{\infty} x_k h_{nk} \mid^2) \tag{12.21}$$

where

$$H = [h_1, h_2 ... h_N]^T \tag{12.22}$$

$$\underline{h_n} = (...h_{n,-3}, h_{n,-2}, h_{n,-1} h_{n,0}, h_{n,1}, h_{n,2}, h_{n,3} ...) \tag{12.23}$$

The optimum receiver is given by the condition

Choose $\underline{x}$ if $\log[p(\underline{\omega}) \mid \underline{x}, H] > \log[p(\underline{\omega}) \mid \hat{\underline{x}}, H] \qquad \forall\, \hat{\underline{x}} \neq \underline{x}$

$$\equiv \arg\{\{\max_{\underline{x}}\}\mu(\underline{x})\} = -\sum_{n=1}^{N}\left|\omega_n - \sum_{k=-\infty}^{\infty} x_k h_{nk}\right|^2 \tag{12.24}$$

From above, we can make the following conclusions

1. In order to implement the optimum receiver we must have knowledge of the f_n which will allow us to equalize the channel. Thus, we need to estimate the channel.
2. An additional problem results by inspecting

$$y(t) = \sum_{k=-\infty}^{\infty} x_k f(t - kT) + v(t) \tag{12.25}$$

where the noise function

$$v(t) = \int_{-\infty}^{\infty} h^*(\tau) z(\tau + t) d\tau \tag{12.26}$$

is Gaussian but not white. Thus, the noise samples at the output of the filter are correlated. To combat the crippling effects of correlated noise, we apply a whitening filter to the sampled sequence y_n. The output of the white filter v_k is given by

$$v_k = \sum_{n=0}^{L} g_n x_{k-n} + \eta_k \tag{12.27}$$

where $g(\cdot)$ embodies the filter for the channel and the whitening filter.

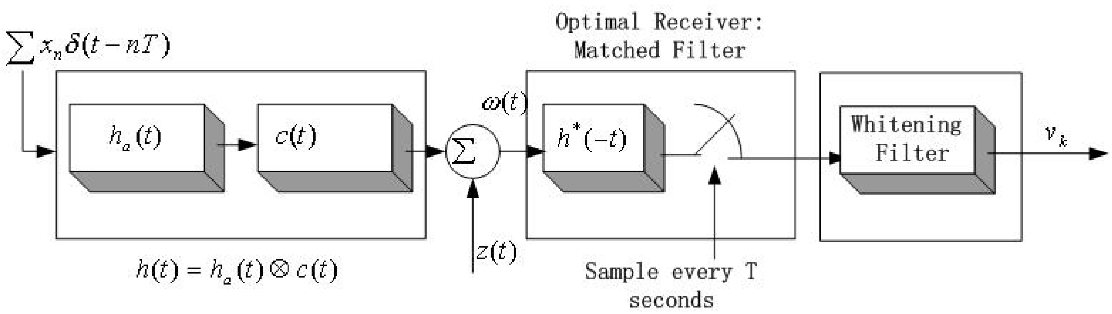

Figure 10. Whitening Filter Using after a Matched Filter

3. A third point is that design of ISI filters is extremely sensitive to timing information. To overcome this sensitivity, we introduce two schemes:
   a. Pulse Shaping: In the particular case of raised cosine pulses. We can derive the length of pulse by sampling at even points.
   b. Fractional Sampling: Sample output at a higher than 2/T rate and you achieve less sensitivity to timing errors.

Equalization Schemes

We use a discrete model for the channel described in the previous section. Namely, we will describe the memory-limited channel as a linear combination the delayed channel-inputs $\{a_n\}$ weighted by appropriate channel coefficients $\{h_j\}$.

Discrete Channel Model:

The discrete channel model that affects information input signal $\{a_n\}$ is given as

$$r_n = \sum_{k=0}^{L} a_{n-k} h_k + \eta_n \tag{12.28}$$

The objective of an equalizer is to determine an estimate $\{\hat{a}_n\}$ of the symbol $\{a_n\}$ that meets a defined set of criteria. This process is depicted in below.

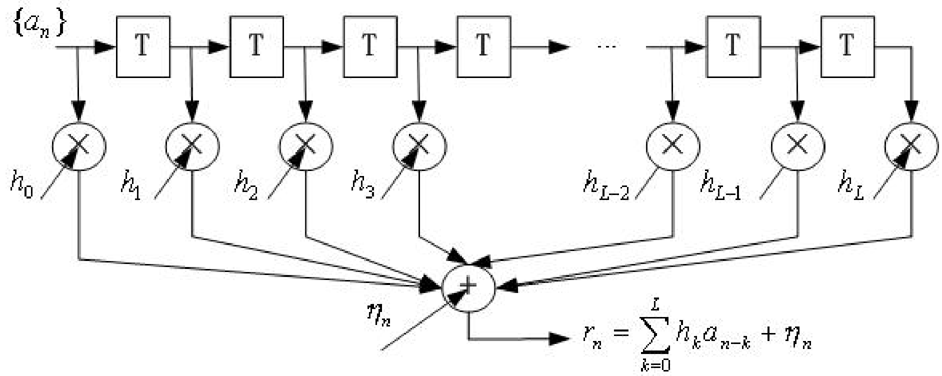

Figure 11. Discrete Time Model of the AWGN linear channel

There are two main types of equalization schemes

1. Symbol by symbol equalization
2. Sequence estimation

Symbol by Symbol Equalizers

Symbol by symbol equalizers can be either linear or nonlinear.

1) Zero Forcing Equalizer

Equalizer concept in which a frequency response is corrected by processing a signal through the inverse channel response, thus forcing inter symbol interference to zero and, theoretically, removing dispersion impairment.

Let the output of the channel be given as before

$$r_n = \sum_{k=0}^{L} a_{n-k} h_k + \eta_n \tag{12.29}$$

The output of the equalizer is given by

$$\hat{a}_n = \sum_{-M}^{M} c_j r_{n-j} \tag{12.30}$$

In a zero forcing equalizer, the equalizer coefficients c_n are chosen to force the samples of the combined channel and equalizer impulse response to zero at all but one of the NT spaced sample points in the tapped delay line filter. By letting the number of coefficients increase without bound, an infinite length equalizer with zero ISI at the output can be obtained. When each of the delay elements provide a time delay equal to the symbol duration T, the frequency response $H_{eq}(f)$ of the equalizer is periodic with a period equal to the symbol rate $1/T$. The combined response of the channel with the equalizer must satisfy Nyquist's first criterion:

$$H_{ch}(f)H_{eq}(f) = 1, |f| < 1/2T \tag{12.31}$$

where $H_{ch}(f)$ is the folded frequency response of the channel. Thus, an infinite length, zero, ISI equalizer is simply an inverse filter which inverts the folded frequency response of the channel. This infinite length equalizer is usually implemented by a truncated length version.
The zero forcing equalizer has the disadvantage that the inverse filter may excessively amplify noise at frequencies where the folded channel spectrum has high attenuation. The ZF equalizer thus neglects the effect of noise altogether and is not often used for wireless links.

2) MMSE Equalizer

A more robust equalizer is the LMS equalizer where the criterion used is the minimization of the mean square error (MSE) between the desired equalizer output and the actual equalizer output. We define the estimation error:

$$\varepsilon_n = \underbrace{a_n}_{sent\_symbol} - \underbrace{\hat{a}_n}_{estimated\_symbol} \tag{12.32}$$

The function to be minimized is given as

$$J = \min_{\underline{c}}\{E[\varepsilon_n^2]\} = \min_{\underline{c}}\{E[(a_n - \sum_{j=-M}^{M} c_j r_{n-j})^2]\} \tag{12.33}$$

The error is minimized by choosing $\{c_j\}$, so as to make the error vector orthogonal to the input sequence: (i.e. $E[e_n r_{n-l}] = 0$, $|l| \le M$).

In order to implement the MMSE equalizer, typically we use steepest descent algorithms:

$$C_j(n+1) = C_j(n) - \frac{1}{2}\mu \frac{\partial E[\varepsilon^2(n)]}{\partial C_j} = C_j(n) + \mu R_{\varepsilon x}(j), j = 0, \pm 1, \pm 2 \ldots \pm M \tag{12.34}$$

$$\therefore C_j(n+1) = C_j(n) + \varepsilon(n) r_{n-j} \tag{12.35}$$

In this algorithm, we need use training sequences to estimate $\varepsilon(n)$.

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