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Numerical Study of Gas Dynamics Equation arising in Shock Fronts

Mohammad Tamsir[a]*, Omer Acan[b], Jai Kumar[c], Akhilesh Kumar Singh[d]

[a]*Department of Mathematics & Statistics, DDU Gorakhpur University, Gorakhpur-273009, India*
[b]*Department of Mathematics, Science Faculty, Selcuk University, Konya-42003, Turkey*
[c]*Mission Development Group, ISRO Satellite Center, Bangalore-560017, India*
[d]*Department of Mathematics, Jaypee Institute of Information Technology, Noida-201301, India*

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ABSTRACT

In this paper, numerical solution of gas dynamics equation arising in shock fronts is obtained by using a modified cubic B-spline differential quadrature method (MCB-DQM). Present method is used in space to obtain a system of first-order ordinary differential equations which is solved by an optimal five stage fourth-order strong stability preserving Runge-Kutta scheme (SSP-RK54) in time. Two numerical examples of gas dynamics equations with known exact solutions are carried out. Obtained results are compared with the exact solutions and a good agreement is found.

Keywords: Gas dynamics equation; MCB-DQM; SSP-RK54; Thomas algorithm

1. Introduction

Consider one-dimensional gas dynamics equation arising in shock fronts [5]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-u(1-u)=g(x,t),\ x\in[a,b],0<t \tag{1}$$

with the initial condition

$$u(x,0)=\varphi(x), \tag{2}$$

and the boundary conditions

$$u(a,t)=h_1(t),\ u(b,t)=h_2(t), \tag{3}$$

where g, φ, h_1 and h_2 are known functions.

In last decades, various analytical and numerical methods have been used to solve one dimensional gas dynamics equation such as decomposition method [1], Homotopy perturbation

*E-mail address: tamsiriitm@gmail.com

method [2], Variational Iteration Method [3], Elzaki transform homotopy perturbation method [4], fractional homotopy analysis transform method [5]. In 1972, Bellman *et al.* [6] proposed an efficient technique "differential quadrature method". It was further improved by Quan and Chang [7]. In this method, the weighting coefficients are determined using several test functions such as Cubic B-spline function, sinc function, Lagrange interpolation polynomials, Legendre polynomials, modified cubic B-spline function [7-11, 14-16] etc.

In this paper, computational solution of the homogenous and non-homogeneous gas dynamic equations using MCB-DQM [14,15,16] is obtained. Modified cubic B-spline functions are used as a basis functions in differential quadrature method. Two test examples are considered to check the accuracy and efficiency of the method. Rest of the paper is prepared as; In Section 2, the present method is described. In Section 3, implementation procedure is illustrated. In Section 4, two numerical examples are provided, while Section 5 concludes our study.

2. Description of the method

This section describes the modified cubic B-spline differential quadrature method. We assume that N knots are uniformly distributed as $a = x_1 < x_2, \ldots.. < x_N = b$ with $x_{i+1} - x_i = h$. The first and second order spatial derivatives of the $u(x,t)$ at any time on the knot x_i for $i = 1, 2, \ldots.., N$ are expressed as

$$\frac{\partial u(x_i,t)}{\partial x} = \sum_{j=1}^{N} a_{ij} u(x_j,t), \text{ for } i = 1, 2, \ldots, N \tag{4}$$

$$\frac{\partial^2 u(x_i,t)}{\partial x^2} = \sum_{j=1}^{N} b_{ij} u(x_j,t), \text{ for } i = 1, 2, \ldots, N \tag{5}$$

where a_{ij} and b_{ij} weighting coefficients of the first and second order derivatives with respect to x, respectively. The cubic B-spline basis functions at the knots are defined as in [14]

$$\varphi_j(x) = \frac{1}{h^3}\begin{cases} (x - x_{j-2})^3, & x \in (x_{j-2}, x_{j-1}] \\ (x - x_{j-2})^3 - 4(x - x_{j-1})^3, & x \in (x_{j-1}, x_j] \\ (x_{j+2} - x)^3, & x \in (x_{j+1}, x_{j+2}] \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

where $\{\psi_0, \psi_1, \ldots, \psi_N, \psi_{N+1}\}$ is taken so that it forms a basis over $[a,b]$. The values of cubic B-splines and its derivatives at the nodal points are given in Table 1.

The cubic B-spline basis functions are modified in order to get diagonally a dominant matrix of equations. The modified cubic B-spline basis functions at the knots are defined as in [14]

$$\left.\begin{aligned}
&\psi_1(x)=\varphi_1(x)+2\varphi_0(x)\\
&\psi_2(x)=\varphi_2(x)-\varphi_0(x)\\
&\psi_j(x)=\varphi_j(x) \quad \text{for } j=3,\ldots,N-2\\
&\psi_{N-1}(x)=\varphi_{N-1}(x)-\varphi_{N+1}(x)\\
&\psi_N(x)=\varphi_N(x)+2\varphi_{N+1}(x)
\end{aligned}\right\}, \tag{7}$$

where $\{\varphi_0,\varphi_1,\ldots,\varphi_N\}$ forms a basis over $[a,b]$.

Table 1. Coefficients of cubic B-spline and its derivative at knots x_j (refer [14])

	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
$\varphi_j(x)$	0	1	4	1	0
$\varphi_j'(x)$	0	$3/h$	0	$-3/h$	0
$\varphi_j''(x)$	0	$6/h^2$	$-12/h^2$	$-6/h^2$	0

The approximate value of first order derivative at i^{th} node point is given by

$$\psi_k'(x_i)=\sum_{j=1}^{N} a_{ij}\psi_k(x_j),\ k=1,2,\ldots,N. \tag{8}$$

From Eq. (7) and Table 1, Eq. (8) reduces into a tridiagonal system of equations:

$$A\,\vec{a}[i]=\vec{R}[i],\ for\ i=1,2,\ldots,N, \tag{9}$$

where A is the coefficient matrix, $\vec{a}[i]$ denotes the weighting coefficient vector corresponding to grid point x_i, and $\vec{R}[i]$ denotes the coefficient vector corresponding to x_i, $i=1,2,\ldots,N$.

Now, the tridiagonal system (9) is solved by using Thomas algorithm to compute the weighting coefficients a_{ij} while the weighting coefficients b_{ij} are obtained by using the expression [8]

$$\left.\begin{aligned}
&b_{ij}=2a_{ij}\left(a_{ij}-\frac{1}{x_i-x_j}\right),\ for\ i=j\\
&b_{ii}=-\sum_{i=1,i\neq j}^{N} b_{ij}
\end{aligned}\right\}. \tag{10}$$

3. Implementation of method

Using Eq. (4) and Eq. (5) into Eq. (1), gas dynamic equation (1) can be expressed as:

$$\frac{\partial u(x_i,t)}{\partial t}=-u(x_i)\sum_{j=1}^{N}a_{ij}u(x_j)+u(x_i)\left[1-u(x_i)\right]+f(x_i), i=1,2,...,N. \tag{11}$$

Eq. (11) is reduced into a set of first-order ODE

$$\frac{\partial u_i}{\partial t}=F(u_i), i=1,2,...,N, \tag{12}$$

where F is a nonlinear differential operator. The resulting system of ODE is solved by using SSP-RK54 method [12].

4. Results and discussions

In section, we discussed two test examples with their MCB-DQM solution. The existence of the exact solutions helps to measure the accuracy of the numerical method. The accuracy and the efficiency of the scheme are measured by evaluating the L_2 and L_∞ error norms, defined as:

$$\left.\begin{aligned} L_2 &= \left(\frac{\sum_{j=1}^{N}\left[U_j^{exact}-U_j^{numerical}\right]^2}{\sum_{j=1}^{N}\left(U_j^{exact}\right)^2}\right), \\ L_\infty &= \max_{1\le j\le N}|U_j^{exact}-U_j^{numerical}|, \end{aligned}\right\}. \tag{13}$$

The rate of convergence of the scheme is defined as [13]

$$rate=\frac{\log\left(E^h/E^{h/2}\right)}{\log(2)}, \tag{14}$$

where E^h and $E^{h/2}$ are the error norms with the grid size h and $\frac{h}{2}$, respectively.

Example 1. Consider the nonlinear gas dynamics equation as in [5]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-u(1-u)=0. \tag{15}$$

The initial and boundary conditions are taken from the exact solution:

$$u(x,t)=e^{-x+t},\ t\ge 0. \tag{16}$$

Here, $[0,1]$ is considered as the computational domain. A comparison is carried out between the solution obtained by the MCB-DQM and the exact solution and is shown in the Table 2. L_2 and L_∞ errors are given in Table 3 at different time levels for different h with $\Delta t=0.0001$. It is observed that present method results are accurate. From the Table 4, it can be seen that the method gives quadratic rate of convergence. Comparison of MCB-DQM and exact solution is depicted in Fig. 1 for $h=0.01$, $\Delta t=0.0001$ at $t=1$.

Table 2. Comparison of numerical and exact solution with $h = 0.04$, $\Delta t = 0.0001$ for Ex 1.

t	x	MCB-DQM	Exact
0.01	0.2	0.827042	0.827042
	0.4	0.677125	0.677125
	0.6	0.554383	0.554383
	0.8	0.453890	0.453890
0.1	0.2	0.904835	0.904837
	0.4	0.740818	0.740818
	0.6	0.606531	0.606531
	0.8	0.496581	0.496585
1.0	0.2	2.226570	2.225760
	0.4	1.822850	1.822300
	0.6	1.492570	1.491970
	0.8	1.221800	1.221520

Table 3. Comparison of L_2 and L_∞ errors for different h with $\Delta t = 0.0001$ at different t for Ex 1.

t	$h = 0.04$		$h = 0.06$		$h = 0.1$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	1.4264E-04	3.3414E-04	3.3324E-04	7.8605E-04	7.7529E-04	1.9977E-03
0.50	2.5267E-04	5.1439E-04	5.4877E-04	9.2150E-04	1.3348E-03	2.1948E-03
0.75	4.1236E-04	7.2251E-03	6.7617E-04	1.0684E-03	1.6715E-03	2.5317E-03
1.00	5.7850E-04	9.2084E-03	8.6607E-04	1.3579E-03	2.1069E-03	2.8299E-03

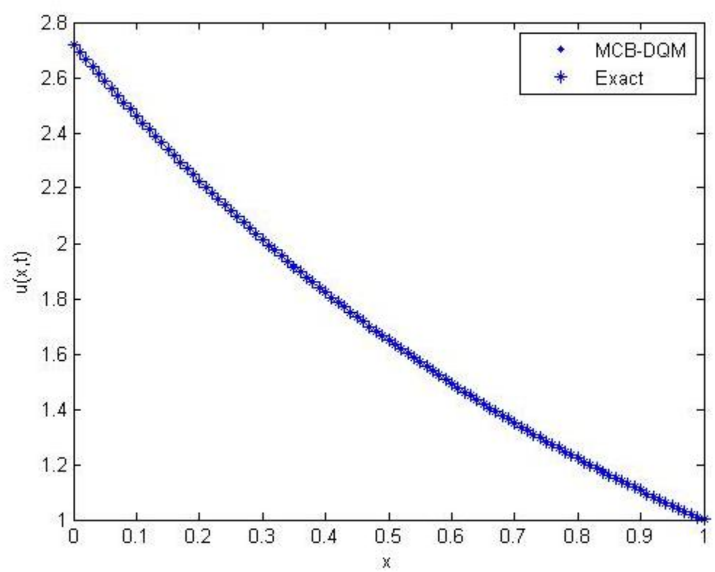

Fig. 1. Comparison of exact and numerical solution at $t = 1.0$ with $h = 0.01$, $\Delta t = 0.0001$ for Ex. 1.

Table 4. Rate of convergence for Ex. 1 at $\Delta t = 0.0001$ and $t = 1$.

N	L_2	ROC	L_∞	ROC
4	1.104405E-02	-	1.610034E-02	-
8	3.292043E-03	1.7462	4.584291E-03	1.8123
16	8.660694E-04	1.9264	1.357896E-03	1.7553
32	2.199950E-04	1.9770	3.552125E-04	1.9346
64	5.494492E-05	2.0014	9.023130E-05	1.9770
128	1.348860E-05	2.0262	2.134105E-05	2.0800
256	3.226155E-06	2.0639	5.104248E-06	2.0639

Example 2. Consider the nonlinear gas dynamics equation as in [5]

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - u(1-u)\log a = 0,\ a > 0, \tag{17}$$

initial and boundary conditions are taken from the exact solution

$$u(x,t) = a^{-x+t},\ t \geq 0. \tag{18}$$

The computational domain is considered over the region $[0,1]$. The numerical and exact solutions are shown for various values of x and t with $h = 0.04$, $\Delta t = 0.0001$ in Table 5. L_2 and L_∞ errors are shown in Table 6 at different times for various mesh sizes h with $\Delta t = 0.0001$. It is found that the proposed scheme results are accurate. From the Table 7, it is observed that the rate of convergence of the scheme is quadratic. Physical behavior of analytical and present solution is depicted at $t = 1.0$ with $h = 0.04$, $\Delta t = 0.0001$ in Fig. 2 and Fig. 3 corresponding to different values of a. From the Fig. 2 and Fig. 3, we notice that as the value of a increases the numerical and exact solutions also grow.

Table 5. Comparison of numerical and exact solutions with $h = 0.04$, $\Delta t = 0.0001$ at $a = 2$ for Ex. 2.

t	x	MCB-DQM	Exact
0.01	0.2	0.876666	0.876666
	0.4	0.763183	0.763183
	0.6	0.664389	0.664389
	0.8	0.578384	0.578384
0.1	0.2	0.933032	0.933033
	0.4	0.812252	0.812252
	0.6	0.707107	0.707107
	0.8	0.615567	0.615572
1.0	0.2	1.741560	1.741220
	0.4	1.516050	1.515820
	0.6	1.319880	1.319600
	0.8	1.148900	1.148780

Table 6. Comparison of L_2 and L_∞ errors at different t for different h with $\Delta t = 0.0001$, at $a = 2$ for Ex. 2.

t	$h = 0.04$		$h = 0.06$		$h = 0.1$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.25	6.6813E-05	1.5029E-04	1.5618E-04	3.6347E-04	3.6147E-04	9.0045E-04
0.50	1.1837E-04	2.5951E-04	2.3804E-04	4.4844E-04	5.8123E-04	1.0092E-03
0.75	1.8710E-04	3.2139E-04	2.6619E-04	4.6183E-04	6.6040E-04	1.1735E-03
1.00	2.5016E-04	3.8615E-04	3.0511E-04	4.7697E-04	7.5598E-04	1.1726E-03

Table 7. Rate of convergence for Ex. 2 at $a = 2$, $\Delta t = 0.0001$ and $t = 1$.

N	L_2	ROC	L_∞	ROC
4	3.980525E-03	-	6.341321E-03	-
8	1.148940E-03	1.7927	1.750636E-03	1.8569
16	3.051104E-04	1.9129	4.769755E-04	1.8759
32	7.845660E-05	1.9594	1.172578E-04	2.0242
64	1.968959E-05	1.9945	2.904893E-05	2.0131
128	4.873841E-06	2.0143	7.280006E-06	1.9965
256	1.181332E-06	2.0446	1.769941E-06	2.0402

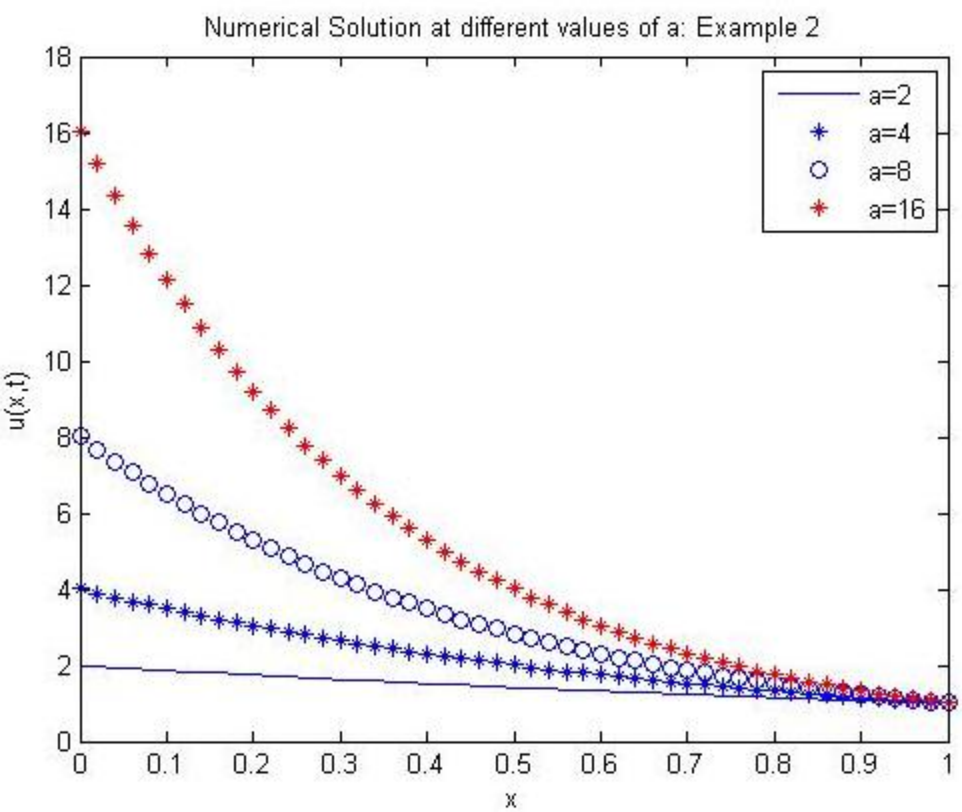

Fig. 2. MCB-DQM solutions at $t = 1.0$ with $h = 0.02$, $\Delta t = 0.0001$ for Ex. 2.

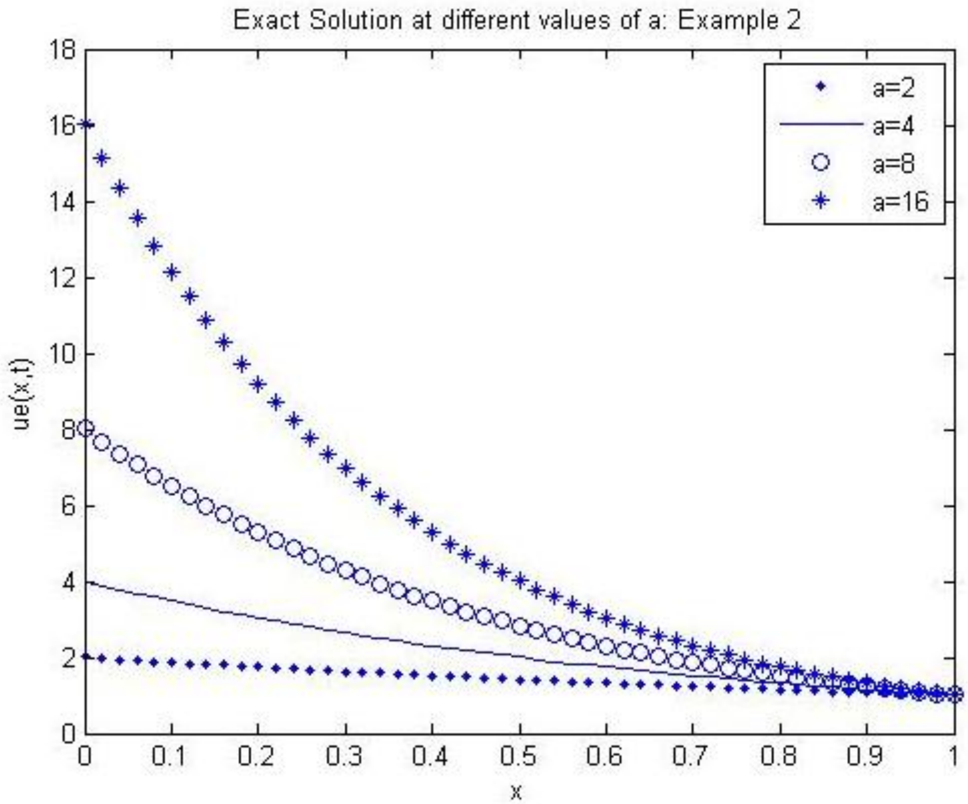

Fig. 3. Exact solutions at $t = 1.0$ with $h = 0.02, \Delta t = 0.0001$ for Ex. 2.

5. Conclusions

In this paper, a numerical approach is described for one dimensional gas dynamics equation using modified cubic B-spline differential quadrature method. The accuracy and efficiency of the method is studied through two test problems. A good agreement is found between the MCB-DQM solution and the exact solution. It is also found that the present method is easy to implement for the partial differential equations and it reduces the data complexity. Therefore, easiness of the implementation and low memory storage are main advantages of the present method.

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