

## SUMUDU TRANSFORM SERIES EXPANSION METHOD FOR SOLVING THE LOCAL FRACTIONAL LAPLACE EQUATION IN FRACTAL THERMAL PROBLEMS

by

**Zheng-Hong GUO<sup>a</sup>, Omer ACAN<sup>b</sup>, and Sunil KUMAR<sup>c\*</sup>**

<sup>a</sup> School of Information Science and Engineering, Heibei North University,  
Zhangjiakou, China

<sup>b</sup> Department of Mathematics, Science Faculty, Selcuk University, Konya, Turkey

<sup>c</sup> Department of Mathematics, National Institute of Technology,  
Jamshedpur, Jharkhand, India

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*In this article, the Sumudu transform series expansion method is used to handle the local fractional Laplace equation arising in the steady fractal heat-transfer problem via local fractional calculus.*

**Key words:** *heat transfer, Laplace equation, local fractional calculus, analytical solution, Sumudu transform series expansion method*

### Introduction

The local fractional partial differential equations [1-3] had important applications in mathematical physics, e. g., heat transfer [4-7], fluid flow [8], damped vibration [9], and others [10-12]. Recently, the local fractional Laplace equation arising in the steady fractal heat-transfer problem via local fractional calculus was written in the form [13]:

$$H_x^{(2\beta)}(x, y) + H_y^{(2\beta)}(x, y) = 0, \quad 0 < \beta \leq 1 \quad (1a)$$

subject to the initial conditions:

$$H_x^{(\beta)}(x, 0) = g(x) \quad (1b)$$

and

$$H(x, 0) = \varpi(x) \quad (1c)$$

In eqs. (1a) and (1b), the local fractional partial derivative is defined [1]:

$$H_y^{(\beta)}(x, y) = \frac{\partial^\beta H(x, y)}{\partial y^\beta} \Big|_{y=y_0} = \lim_{y \rightarrow y_0} \frac{\Delta^\beta [H(x, y) - H(x, y_0)]}{(y - y_0)^\beta} \quad (2a)$$

where

$$\Delta^\beta [H(x, y) - H(x, y_0)] \cong \Gamma(1 + \beta) \Delta [H(x, y) - H(x, y_0)]$$

\* Corresponding author; e-mail: skitbhu28@gmail.com

The Sumudu transform of  $K(x)$  of order  $\beta (0 < \beta \leq 1)$  via local fractional integral is defined [14]:

$$LFS_{\beta}\{K(x)\} = \frac{1}{\Gamma(1+\beta)} \int_0^{\infty} E_{\beta}(-h^{-\beta} x^{\alpha}) \frac{K(x)}{h^{\beta}} (dx)^{\beta}, \quad 0 < \beta \leq 1 \quad (2b)$$

where

$$I_x^{(\beta)} K(x) = \frac{1}{\Gamma(1+\beta)} \int_{x_0}^x K(x) (dx)^{\beta} = \frac{1}{\Gamma(1+\beta)} \lim_{\Delta x \rightarrow 0} \sum_{j=0}^{j=N-1} K(x_j) (\Delta x)^{\beta} \quad (2c)$$

with the partitions of the interval  $[x, x_0]$  are  $(x_j, x_{j+1})$ ,  $j = 0, \dots, N-1$ ,  $\Delta x = x_{j+1} - x_j$ .

The analytic solution of the local fractional Laplace equation was investigated in [15, 16]. The goal of the article is to use the Sumudu transform series expansion method [14] to find the non-differentiable solution of the local fractional Laplace equation in fractal heat transfer.

### Solving the local fractional Laplace equation in fractal heat transfer

In this section, we used the idea of the Sumudu transform series expansion method [14] to deal with the local fractional Laplace equation in fractal heat transfer.

By considering the solution of eq. (1a) in the form [14]:

$$H(x, y) = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_i(x) \quad (3a)$$

we have:

$$H(h, y) = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_i(h) \quad (3b)$$

$$LFS_{\beta}\{H_x^{(2\beta)}(x, y)\} = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_{i+2}(h) \quad (3c)$$

$$LFS_{\beta}\{H_y^{(2\beta)}(x, y)\} = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \left[ \varpi_i^{(2\beta)}(x) \right] (h) \quad (3d)$$

With the help of eqs. (3c) and (3d), eq. (1a) can be rewritten in the form:

$$\sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_{i+2}(h) = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \left[ \varpi_i^{(2\beta)}(x) \right] (h) \quad (4a)$$

which reduces to the following:

$$\varpi_{i+2}(h) = \left[ \varpi_i^{(2\beta)}(x) \right] (h) \quad (4b)$$

Due to eqs. (1b), (1c), and (4b), we have:

$$\begin{cases} \varpi_{i+2}(h) = \left[ \varpi_i^{(2\beta)}(x) \right] (h) \\ \varpi_0(h) = \varpi(h) \end{cases} \quad (5a)$$

and

$$\begin{cases} \varpi_{i+2}(h) = \left[ \varpi_i^{(2\beta)}(x) \right](h) \\ \varpi_1(h) = \mathcal{G}(h) \end{cases} \quad (5b)$$

where  $LFS_\beta\{\varpi(x)\} = \varpi(h)$  and  $LFS_\beta\{\mathcal{G}(x)\} = \mathcal{G}(h)$ .

Taking  $\varpi(x) = \sin_\beta(x^\beta)$  and  $\mathcal{G}(x) = 0$ , we have:

$$\begin{cases} \varpi_{i+2}(h) = - \left[ \varpi_i^{(2\beta)}(x) \right](h) = \varpi_i(h) \\ \varpi_0(h) = LFS_\beta\{\sin_\beta(x^\beta)\} = \frac{h^\beta}{1+h^{2\beta}} \end{cases} \quad (6a)$$

which yields to:

$$\varpi_0(h) = \varpi_2(h) = \varpi_4(h) = \dots = \frac{h^\beta}{1+h^{2\beta}} \quad (6b)$$

Therefore, from eq. (3b) we obtain the Sumudu transform series solution:

$$H(h, y) = \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_i(h) = \frac{h^\beta}{1+h^{2\beta}} \left[ 1 + \frac{y^{2\beta}}{\Gamma(1+2\beta)} + \frac{y^{4\beta}}{\Gamma(1+4\beta)} + \dots \right] \quad (7)$$

Taking the inverse LFST operator of eq. (7) gives the solution of eq. (1a) with the non-differentiable series terms:

$$\begin{aligned} LFS_\beta^{-1}[H(h, y)] &= LFS_\beta^{-1} \left[ \sum_{i=0}^{\infty} \frac{y^{i\beta}}{\Gamma(1+i\beta)} \varpi_i(h) \right] \\ &= \sin_\beta(x^\beta) \left[ 1 + \frac{y^{2\beta}}{\Gamma(1+2\beta)} + \frac{y^{4\beta}}{\Gamma(1+4\beta)} + \dots \right] \end{aligned} \quad (7)$$

where the operator  $LFS_\beta^{-1}$  is the inverse LFST operator [14].

## Conclusions

In our work, we considered the steady fractal heat-transfer problem via local fractional derivative. The Sumudu transform series expansion method was considered to handle the local fractional Laplace equation in fractal heat transfer. The obtained result shows the present technology is accuracy and efficient to handle the fractal heat-transfer problems.

## Nomenclature

$x, y$  – space co-ordinates, [m]  
 $LFS_\beta$  – LFST operator, [–]  
 $LFS_\beta^{-1}$  – inverse LFST operator, [–]

Greek symbol

$\beta$  – fractal dimensional order, [–]

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