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Solution of Conformable Fractional Partial Differential Equations by Reduced Differential Transform Method

Omer Acan, Omer Firat, Yildiray Keskin, Galip Oturanc

Abstract

In this paper, conformable fractional reduced differential transform method is introduced by using Conformable Calculus and reduced differential transform method. And given its application to fractional partial differential equations. Fractional partial differential equations have special importance in engineering and sciences. Moreover, this technique doesn't require any discretization, linearization or small perturbations and therefore it reduces significantly the numerical computations.

Keywords

Conformable Fractional Derivative, Conformable Calculus, Reduced Differential Transform Method, Conformable Fractional Partial Differential Equations

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Reduced Differential Transform Method for Nonlinear Partial Differential Equations within Conformable Fractional Derivative Operators

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Abstract

In this paper, we propose a new reduced differential transformation method based on conformable fractional derivative to solve fractional order differential equations. The properties of conformable fractional reduced differential transformation method are first presented. These properties of the method are used to solve some fractional order partial differential equations numerically and used distinct fractional order for more understanding of the proposed conformable fractional reduced differential transformation method. Finally, the graphics of the numerical and exact solutions are plotted.

Keywords: Numerical Solution, Reduced Differential Transform Method, Conformable Fractional Derivative, Partial Differential Equations

1. Introduction

Despite as old as calculus, Fractional derivative is nowadays one of the most intensively developing areas of mathematics and applied braches of science. Fractional calculus implies the calculus of the differentiation and integration given by fractional order. There are many good textbooks for he fractional calculus [1–3].

In recent years, different variants of fractional derivatives has attracted the attention of many researches [4–6]. In the fundamental idea of the fractional calculus is collected into two distinct approximation. The first one is known as Riemann-Liouville approach constructed on the taking n-times integral operator and altered $n!$ to the gamma function. Hence non-integer order of the fractional integral is described and then integral was used to identify the Riemann and Caputo sense. The other is Grunwald-Letnikov method which constructed on the n-times taking derivative. The provided derivatives by this way seemed so sophisticated and missing several fundamental properties such as derivative of product and chain rule. For a positive integer n, the fractional derivative definitions of Riemann-Liouville, Caputo and Grunwald-Letnikov are as in the following

(i) Riemann-Liouville Fractional Derivative Definition:

$$D_a^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x \frac{f(t)}{(x-t)^{n-\alpha-1}} dt, \;\; \alpha \in (n-1, n].$$

(ii) Caputo-Fractional Derivative Definition:

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$$D_a^{\alpha} f\left(x\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\int_a^x \frac{f^{(n)}\left(t\right)}{\left(x-t\right)^{n-\alpha-1}}dt,\ \alpha\in\left(n-1,n\right].$$

(iii) Grunwald-Letnikov Fractional Derivative Definition:

$$D_a^{\alpha} f\left(x\right)=\lim_{n\to 0}\frac{1}{h^{\alpha}}\sum_{j=0}^{\frac{x-\alpha}{h}}\left(-1\right)^j\binom{\alpha}{j}f\left(x-jh\right),\ \alpha\in\left(n-1,n\right].$$

Recently, there is to much interest in new fractional derivative called conformable fractional derivative. For example, Khalil at al. [7] define a new well-behaved simple fractional derivative called the conformable fractional derivative depending just on the limit definition of the derivative. Iyiola and Olayinka, [8]obtained analytical solution of space-time fractional Fornberg–Whitham equation in series form using the relatively new method called q-homotopy analysis method. Abdeljawad [9] provide fractional versions of the chain rule, exponential functions, Gronwall's inequality, integration by parts, Taylor power series expansions and Laplace transforms. Cenesiz and Kurt [10] give the solutions of time and space fractional heat differential equations by conformable fractional derivative. Kurt and Cenesiz [11] get approximate analytical solution of the time conformable fractional Burger's equation determined by Homotopy Analysis Method (HAM). this new subject gives academicians an opportunity to study further in many engineering, physical and applied mathematics problems.

The Reduced Differential Transform Method (RDTM) was first proposed by Keskin and Oturanc [12–14] in 2009. This method is widely used by many researchers to study fractional and non-fractional, linear and non-linear, partial differential equations (PDEs). The method introduces a reliable and efficient process for a wide variety of engineering, scientific and physics applications, such as fractional and non-fractional, linear, non-linear, homogeneous and non-homogeneous PDEs [12–21].

By using conformable derivative and RDTM, conformable Fractional Reduced Differential Transform Method (CFRDTM) will be introduced for the first time into the literature. This method can be applied to fractional PDEs and the results illustrated. For this method, in section 2, we give some definitions and basic theorems of conformable calculus and then in section 3, we introduced CFRDTM, In sections 4 and 5, we give applications, numerical results and discussions of CFRDTM. and in the final section we give the conclusion.

2. Conformable Fractional Calculus

In this section we present some basic definitions and important properties of conformable fractional calculus [6,7,9,22,23].

Definition 2.1. [7,23]

Given a function $f:[0,\infty)\to\mathbb{R}$. Then the conformable fractional derivative of f order α is defined by Fractional Calculus

$$(T_{\alpha} f)(t)=\lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}$$

for all $t>0$, $\alpha\in(0,1]$.

Theorem 2.1. [7]

Let $\alpha\in(0,1]$ and f,g be α-differentiable at a point $t>0$. Then

(i) $T_{\alpha}(af+bg)=a(T_{\alpha} f)+b(T_{\alpha} g)$ for $a,b\in\mathbb{R}$,

(ii) $T_{\alpha}(t^p)=pt^{p-\alpha}$, for all $p\in\mathbb{R}$,

(iii) $T_\alpha(f(t)) = 0$, for all constant functions $f(t) = \lambda$,

(iv) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$,

(v) $T_\alpha(f/g) = \dfrac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$,

(vi) If, in addition, f is differentiable, then $T_\alpha(f) = t^{1-\alpha}\dfrac{df}{dt}(t)$.

Now it can be given CFRDTM .

3. Conformable Fractional Reduced Differential Transform Method

In this section CFRDTM, explained in literature firstly, is given as follows:

In this study, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k^\alpha(x)$ stand for the conformable fractional reduced differential transformed (CFRDT) function.

Definition 3.1.

Assume $u(x,t)$ is analytic and differentiated continuously with respect to time t and space x in the its domain. CFRDT of $u(x,t)$ is defined as

$$U_k^\alpha(x) = \frac{1}{\alpha^k k!}\left[\left(T_\alpha^{(k)} u\right)\right]_{t=t_0}$$

where some $0 < \alpha \le 1$, α is a parameter describing the order of conformable fractional derivative, $T_\alpha^{(k)} u = \underbrace{\left(T_\alpha T_\alpha \cdots T_\alpha\right)}_{k \text{ times}} u(x,t)$ and the t dimensional spectrum function $U_k^\alpha(x)$ is the CFRDT function.

Definition 3.2.

Let $U_k^\alpha(x)$ be the CFRDT of $u(x,t)$. Inverse CFRDT of $U_k^\alpha(x)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty} U_k^\alpha(x)(t-t_0)^{\alpha k} = \sum_{k=0}^{\infty}\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)} u\right]_{t=t_0}(t-t_0)^{\alpha k}$$

CFRDT of initial conditions for integer order derivatives are defined as

$$U_k^\alpha(x) = \begin{cases} \dfrac{1}{(\alpha k)!}\left[\dfrac{\partial^{\alpha k}}{\partial t^{\alpha k}} u(x,t)\right]_{t=t_0} & \text{if } \alpha k \in \mathbb{Z}^+ \\ 0 & \text{if } \alpha k \notin \mathbb{Z}^+ \end{cases} \quad \text{for } k = 0,1,2,\ldots,\left(\frac{n}{\alpha} - 1\right)$$

where n is the order of conformable fractional PDE.

Theorem 3.1.

Let a,b be a constants. If $u(x,t) = av(x,t) \pm bw(x,t)$, then $U_k^\alpha(x) = aV_k^\alpha(x) \pm bW_k^\alpha(x)$.

Proof.

CFRDT of $v(x,t)$ and $w(x,t)$ can be written as the following:

$$V_k^\alpha(x) = \frac{1}{\alpha^k k!}\left[T_\alpha^{(k)} v\right]_{t=t_0}$$

$$W_k^\alpha(x) = \frac{1}{\alpha^k k!}\left[T_\alpha^{(k)} w\right]_{t=t_0}$$

Because of Theorem 2.1 *(i)*, it is that

$$
\begin{aligned}
U_k^\alpha(x) &= \frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}(av \pm bw)\right]_{t=t_0} \\
&= \frac{a}{\alpha^k k!}\left[T_\alpha^{(k)} v\right]_{t=t_0} \pm \frac{b}{\alpha^k k!}\left[T_\alpha^{(k)} w\right]_{t=t_0} \\
&= aV_k^\alpha(x) \pm bW_k^\alpha(x)
\end{aligned}
$$

Theorem 3.2.

If $u(x,t) = v(x,t)w(x,t)$, then $U_k^\alpha(x) = \sum_{s=0}^{k} V_s^\alpha(x) W_{k-s}^\alpha(x)$.

Proof.

By the help of definition 3.2, $v(x,t)$ and $w(x,t)$ can be written that

$$
v(x,t) = \sum_{k=0}^{\infty} V_k^\alpha(x)(t-t_0)^{\alpha k}
$$

$$
w(x,t) = \sum_{k=0}^{\infty} W_k^\alpha(x)(t-t_0)^{\alpha k}.
$$

Then, $u(x,t)$ is obtained as

$$
\begin{aligned}
U_k^\alpha(x) &= \sum_{k=0}^{\infty} V_k^\alpha(x)(t-t_0)^{\alpha k} \sum_{k=0}^{\infty} W_k^\alpha(x)(t-t_0)^{\alpha k} \\
&= \left[V_0^\alpha(x) + V_1^\alpha(x)(t-t_0)^\alpha + V_2^\alpha(x)(t-t_0)^{2\alpha} + \cdots + V_n^\alpha(x)(t-t_0)^{n\alpha} + \cdots\right] \\
&\quad \times\left[W_0^\alpha(x) + W_1^\alpha(x)(t-t_0)^\alpha + W_2^\alpha(x)(t-t_0)^{2\alpha} + \cdots + W_n^\alpha(x)(t-t_0)^{n\alpha} + \cdots\right] \\
&= V_0^\alpha(x)W_0^\alpha(x) + \left(V_0^\alpha(x)W_1^\alpha(x) + V_1^\alpha(x)W_0^\alpha(x)\right)(t-t_0)^\alpha \\
&\quad + \left(V_0^\alpha(x)W_2^\alpha(x) + V_1^\alpha(x)W_1^\alpha(x) + V_2^\alpha(x)W_0^\alpha(x)\right)(t-t_0)^{2\alpha} + \cdots \\
&= \sum_{k=0}^{\infty}\sum_{s=0}^{k} V_s^\alpha(x) W_{k-s}^\alpha(x)(t-t_0)^{k\alpha}.
\end{aligned}
$$

Hence, $U_k^\alpha(x)$ is found as

$$
U_k^\alpha(x) = \sum_{s=0}^{k} V_s^\alpha(x) W_{k-s}^\alpha(x).
$$

Theorem 3.3.

If $u(x,t) = T_\alpha v(x,t)$, then $U_k^\alpha(x) = \alpha(k+1)V_{k+1}^\alpha(x)$.

Proof.

CFRDT of $v(x,t)$ can be written that

$$
V_k^\alpha(x) = \frac{1}{\alpha^k k!}\left[T_\alpha^{(k)} v\right]_{t=t_0}.
$$

For $u(x,t) = T_\alpha v(x,t)$,

$$
\begin{aligned}
U_k^\alpha(x) &= \frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}(T_\alpha v)\right]_{t=t_0} \\
&= \frac{1}{\alpha^k k!}\left[T_\alpha^{(k+1)} v\right]_{t=t_0} \\
&= \alpha(k+1)\frac{1}{\alpha^{k+1}(k+1)!}\left[T_\alpha^{(k+1)} v\right]_{t=t_0} \\
&= \alpha(k+1)V_k^\alpha(x).
\end{aligned}
$$

Theorem 3.4.

If $u(x,t)=x^m(t-t_0)^n$, then $U_k^\alpha(x)=x^m\delta\left(k-\frac{n}{\alpha}\right)$. Where $\delta(k)=\begin{cases}1 & if \quad k=0\\ 0 & if \quad k\neq 0\end{cases}$.

Proof.

CFRDT of $u(x,t)=x^m(t-t_0)^n$ is

$$U_k^\alpha(x)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}\left(x^m(t-t_0)^n\right)\right]_{t=t_0}.$$

If the conformable fractional derivative of $u(x,t)=x^m(t-t_0)^n$, with respect to t, is calculated for k times, where $\alpha\in(0,1]$, then

$$U_k^\alpha(x)=\frac{1}{\alpha^k k!}\left[x^m\left(n(n-\alpha)\ldots(n-(k-1)\alpha)(t-t_0)^{n-k\alpha}\right)\right]_{t=t_0}.$$

is obtained. If $k=\frac{n}{\alpha}$, then

$$
\begin{aligned}
U_k^\alpha(x) &= \frac{1}{\alpha^{\frac{n}{\alpha}}\left(\frac{n}{\alpha}\right)!}x^m\left(n(n-\alpha)\ldots\left(n-\left(\frac{n}{\alpha}-1\right)\alpha\right)(t-t_0)^{n-k\alpha}\right) \\
&= \frac{1}{\alpha^{\frac{n}{\alpha}}\left(\frac{n}{\alpha}\right)!}x^m\left(n(n-\alpha)\ldots(\alpha)\right) \\
&= \frac{\alpha^{\frac{n}{\alpha}}}{\alpha^{\frac{n}{\alpha}}\left(\frac{n}{\alpha}\right)!}x^m\left(\frac{n}{\alpha}\left(\frac{n}{\alpha}-1\right)\ldots(1)\right) \\
&= x^m
\end{aligned}
$$

.

Otherwise, for $t=t_0$ it is that

$$U_k^\alpha(x)=0.$$

Hence

$$U_k^\alpha(x)=x^m\delta\left(k-\frac{n}{\alpha}\right).$$

is obtained. Where

$$\delta(k)=\begin{cases}1 & if \quad k=0\\ 0 & if \quad k\neq 0\end{cases}.$$

To handle fractional PDE, we first write a system in operator form with initial condition

$$L\left(u(x,t)\right)+R\left(u(x,t)\right)+N\left(u(x,t)\right)=g(x,t) \tag{3.1}$$

$$u(x,0)=f(x) \tag{3.2}$$

where $L=\partial^{\alpha}/\partial t^{\alpha}$, $0<\alpha\le 1$, α is a parameter describing the order of the time-fractional derivative in the conformable sense, R is a linear operator which has partial derivatives, N is a non-linear term and $g(x,t)$ is an inhomogeneous term.

At the moment CFRDTM can be applied to conformable fractional PDEs. Consider the equations of (3.1). According to above definitions and theorems of CFRDTM, we can construct the following iteration formula:

$$\alpha(k+1)U_{k}^{\alpha}(x)=G_{k}^{\alpha}(x)-R\left(U_{k}^{\alpha}(x)\right)-N\left(U_{k}^{\alpha}(x)\right) \tag{3.3}$$

where $R\left(U_{k}^{\alpha}(x)\right), N\left(U_{k}^{\alpha}(x)\right)$ and $G_{k}^{\alpha}(x)$ are the CFRDTs of the functions $R\left(u(x,t)\right)$, $N\left(u(x,t)\right)$ and $g(x,t)$ respectively.

From initial condition (3.2), we write

$$U_{0}^{\alpha}(x)=f(x). \tag{3.4}$$

Substituting (3.4) into (3.3) and by straight forward iterative calculations, we get the following $U_{k}^{\alpha}(x)$ values. Then the inverse transformation of the set of values $\left\{U_{k}^{\alpha}(x)\right\}_{k=0}^{n}$ gives approximation solution as,

$$\tilde{u}_n\left(x,t\right)=\sum_{k=0}^{n}U_k^{\alpha}\left(x\right)t^{k\alpha},\quad n=1,2,3,... \tag{3.5}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t)=\lim_{n\to\infty}\tilde{u}_n(x,t). \tag{3.6}$$

4. Applications for CFRDTM

To illustrate the effectiveness of the given CFRDTM two examples are considered in this section. The accuracy of the method is assessed by comparison with the exact solutions.

Example 4.1:

Firstly, Consider the linear time-fractional diffusion equation[21,24]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^{2}u(x,t)}{\partial x^{2}}\ t>0, x\in R, 0<\alpha\le 1 \tag{4.1}$$

subject to the initial condition

$$u(x,0)=\sin(x). \tag{4.2}$$

where $u = u(x,t)$ is a function of the variables x and t.
Exact solution of the problem is given as follows.[21,24]:

$$u(x,t) = \sum_{k=0}^{\infty} \frac{(-1)^k \sin(x) t^{k\alpha}}{\Gamma(k\alpha+1)}. \tag{4.3}$$

Now solve this problem by using CFRDTM. By taking the conformable fractional reduced differential transform of (4.1), it can be obtained that

$$\alpha(k+1) U_{k+1}^{\alpha}(x) = \frac{\partial^2}{\partial x^2} U_k^{\alpha}(x) \tag{4.4}$$

where the t-dimensional spectrum function $U_k^{\alpha}(x)$ is the conformable fractional reduced differential transform function.

From the initial condition (4.2) we write

$$U_0^{\alpha}(x) = \sin(x) \tag{4.5}$$

Substituting (4.5) into (4.4), we obtain the following $U_k^{\alpha}(x)$ values successively

$$U_1^{\alpha}(x) = -\frac{\sin(x)}{\alpha},\ \ U_2^{\alpha}(x) = \frac{\sin(x)}{2!\alpha^2}, U_3^{\alpha}(x) = -\frac{\sin(x)}{3!\alpha^3}, \ldots$$

Then, the inverse transformation of the set of values $\left\{U_k^{\alpha}(x)\right\}_{k=0}^{n}$ gives the following approximation solution

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k^{\alpha}(x) t^{k\alpha} = \sum_{k=0}^{n} \frac{(-1)^k \sin(x)}{k!\alpha^k} t^{k\alpha}. \tag{4.6}$$

For $\alpha = 1$ CFRDTMT solution is $u(x,y) = \lim_{n\to\infty} \tilde{u}_n(x,y) = \sin(x) e^{-t}$. The CFRDTMT solution coincide with exact solution for $\alpha = 1$. This exact solution is $u(x,t) = \sin(x) e^{-t}$. and our solutions are in good agreement with the exact values.

Example 4.2:

We consider the one-dimensional linear inhomogeneous time-fractional equation[21,25]

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + x\frac{\partial u(x,t)}{\partial x} + \frac{\partial^2 u(x,t)}{\partial x^2} = 2t^{\alpha} + 2x^2 + 2, \quad t > 0, x \in R, 0 < \alpha \leq 1 \tag{4.6}$$

with the initial conditions

$$u(x,0) = x^2. \tag{4.7}$$

Exact solution of the problem is given as follows [21,25]:

$$u(x,t) = x^2 + \frac{2\Gamma\left(\alpha+1\right)}{\Gamma\left(2\alpha+1\right)} t^{2\alpha}. \tag{4.3}$$

Solve the problem by CFRDTM. By taking the conformable fractional reduced differential transform of (4.6), it can be obtained that

$$\alpha\left(k+1\right)U_{k+1}^{\alpha}\left(x\right) + x\frac{\partial}{\partial x}U_k^{\alpha}\left(x\right) + \frac{\partial^2}{\partial x^2}U_k^{\alpha}\left(x\right) = 2\delta\left(k-1\right) + \delta\left(k\right)\left(x^2+2\right) \tag{4.8}$$

where the t-dimensional spectrum function $U_k^{\alpha}\left(x\right)$ is the conformable fractional reduced differential transform function.

From the initial condition (4.7) we write

$$U_0^{\alpha}\left(x\right) = x^2 \tag{4.9}$$

Substituting (4.9) into (4.8), we obtain the following $U_k^{\alpha}\left(x\right)$ values successively

$$U_1^{\alpha}\left(x\right) = 0, \;\; U_2^{\alpha}\left(x\right) = \frac{1}{\alpha}, U_k^{\alpha}\left(x\right) =, \; k = 3,4,5...$$

Taking the inverse differential transform of $U_k^{\alpha}\left(x\right)$ then produces

$$u(x,t) = \sum_{k=0}^{\infty} U_k^{\alpha}\left(x\right) t^{k\alpha} = x^2 + \frac{t^{2\alpha}}{\alpha}.$$

CFRDTMT solution coincide with exact solution for $\alpha = 1$. This exact solution is $u(x,t) = x^2 + t^2$ and our solutions are in good agreement with the exact values.

5. Numerical results and discussion

In order to demonstrate the efficiency of the CFRDTM for time-fractional PDEs comparison of fourth-order approximate solution and exact solution for examples are given Figs 5.1 and 5.2 graphically describes the three dimensional variation of third order approximate solution with respect to x and t and different values of α. The numerical results show that the CFRDTM study very well for this problem, even if it is used third-order or fourth-order approximate solution. The accuracy of the method can be improved by using higher-order approximate solutions.

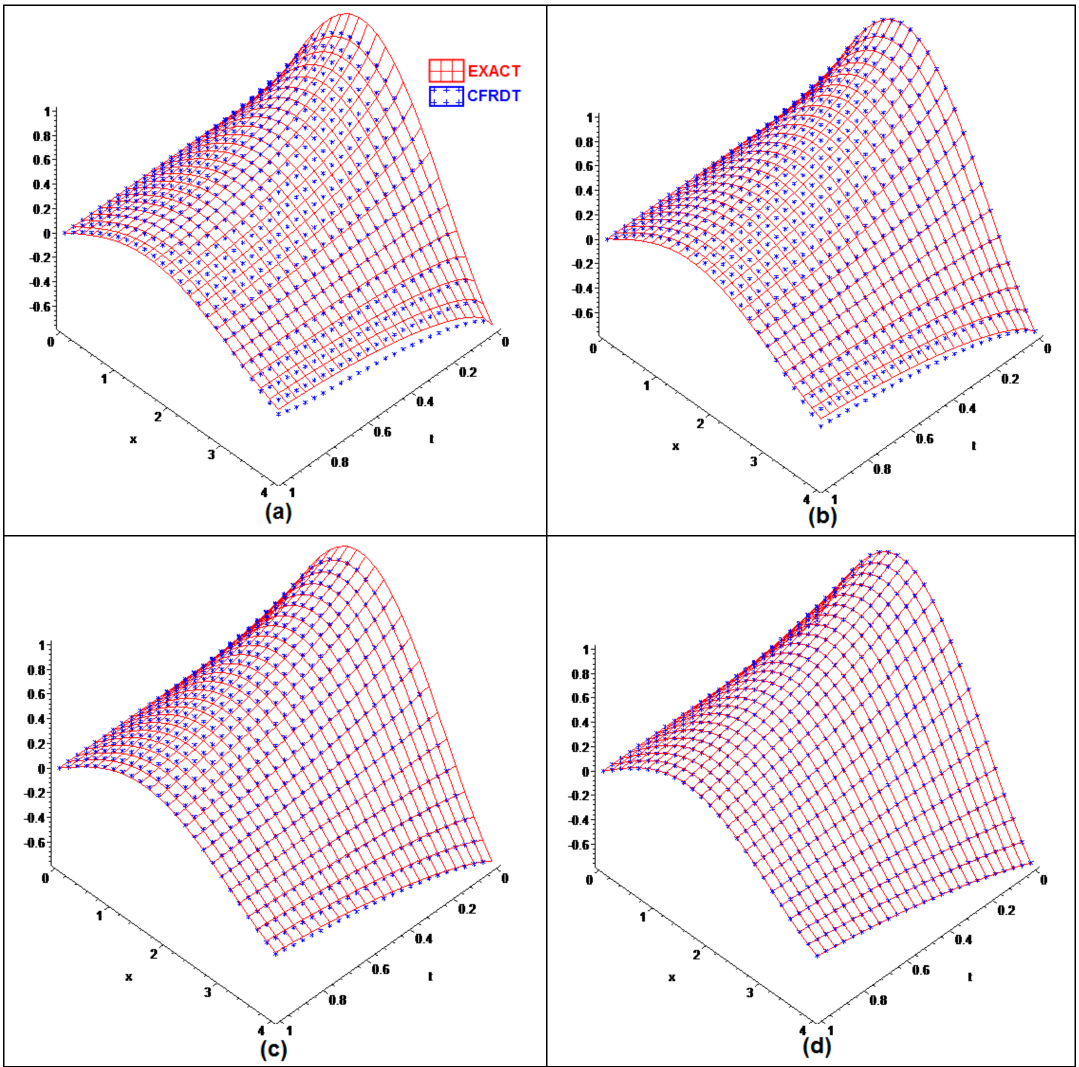

Fig. 5.1. The surfaces show compression Exact and the fourth-order CFRDTM solutions of Eq. (4.1) for (a) $\alpha = 0.7$,(b) $\alpha = 0.8$, (c) $\alpha = 0.9$, (d) $\alpha = 1$.

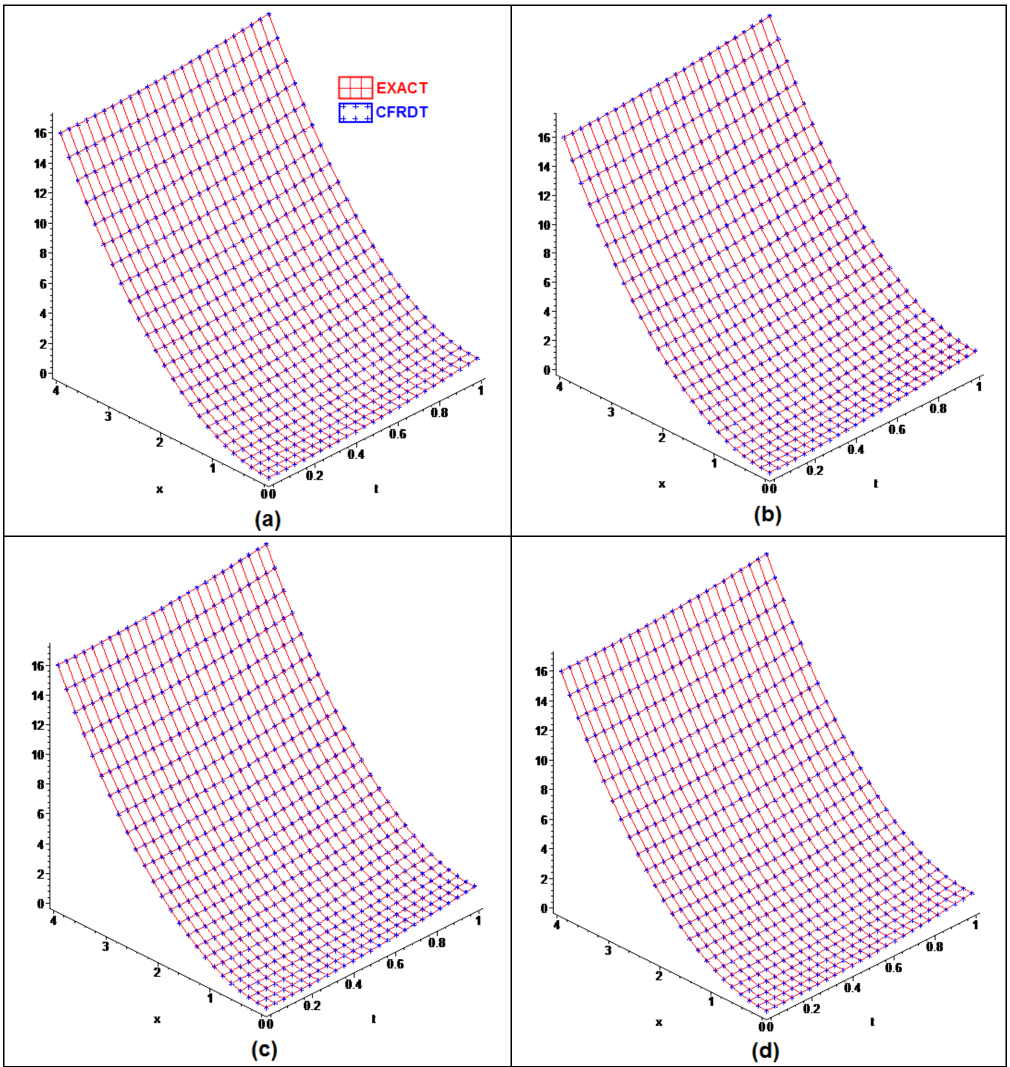

Fig. 5.2. The surfaces show compression Exact and the fourth-order CFRDTM solutions of Eq. (4.6) for (a) $\alpha = 0.7$,(b) $\alpha = 0.8$, (c) $\alpha = 0.9$, (d) $\alpha = 1$.

6. Conclusion

In this paper, it is presented conformable fractional reduced differential transform method (CFRDTM) to find the numerical solution of fractional PDEs. Then, it applied this new method to conformable fractional PDEs. In the examples, numerical solution obtained by the help of CFRDTM can be written so as to exact solution. Otherwise, the number of terms in solution is increased to improve the accuracy of the obtained solution. The applications show that CFRDTM is reliable and introduces a significant improvement in solving fractional PDEs.

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