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A Comparative Study of Numerical Methods for Solving $(n+1)$ Dimensional and Third-Order Partial Differential Equations

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In this study, we compare the reduced differential transform method and the variational iteration method. This has been achieved by handling $(2+1)$ dimensional and third-order type of the Zakharov–Kuznetsov ZK(m, m) partial differential equations. Two numerical examples have also been carried out to validate and demonstrate efficiency of the two methods. Also, it is shown that the reduced differential transform method has advantage over variational iteration method. This method is very fast, efficient, applicable and has powerful effects in linear and nonlinear problems.

Keywords: Partial Differential Equations (PDEs), Zakharov Kuznetsov ZK(m, m) Equations, Numerical Solution, Reduced Differential Transform Method (RDTM), Variational Iteration Method (VIM).

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1. INTRODUCTION

Linear and non-linear partial differential equations have been widely investigated and are still being investigated due to their numerous essential applications in many fields such as science, physics, mechanics, engineering technique fields and so on.[1,2] Many researchers have paid attention to the solutions of linear and non-linear partial differential equations by various methods, such as, the reduced differential transform method (RDTM),[3–10] the variational iteration method (VIM),[11–13] the differential transform method (DTM),[14–18] the homotopy analysis method (HAM),[19–21] the Adomian decomposition method (ADM)[22–24] and the sinc-Galerkin method.[25–27] The RDTM was first proposed by Keskin[3,4] in 2009. This method is now widely used by many researchers to study linear and non-linear partial differential equations. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications, such as linear, non-linear, homogeneous and inhomogeneous partial differential equations.[3–10]

Zakharov–Kuznetsov (ZK) equations[28–30]

$$u_t + auu_x + (\nabla^2 u)_x = 0 \tag{1}$$

where $\nabla^2 = \partial_x^2 + \partial_y^2$ or $\nabla^2 = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the isotropic Laplacian. This means that the ZK equation is given by

$$u_t + auu_x + (u_{xx} + u_{yy})_x = 0 \tag{2}$$

and

$$u_t + auu_x + (u_{xx} + u_{yy} + u_{zz})_x = 0 \tag{3}$$

in $(2+1)$ and $(3+1)$-dimensional spaces. The ZK equation was first derived for describing weakly nonlinear ion-acoustic wave in a strongly magnetized lossless plasma in two dimensions.[28] After a further discussion of the analytical properties of the ZK equation some constructive results were given.

Recently, in 2005, Wazwaz[31] studied the type of ZK equation, that is, the $(2+1)$ and $(3+1)$ dimensional ZK(m, m) equations in the form

$$u_t + a(u^m)_x + b(u^m)_{xxx} + k(u^m)_{yyx} = 0, \quad b, k > 0 \tag{4}$$

and

$$u_t + a(u^m)_x + b(u^m)_{xxx} + k(u^m)_{yyx} + r(u^m)_{zzx} = 0, \quad b, k, r > 0 \tag{5}$$

and in 2007, Lin and Zhang[32] studied the $(3+1)$ dimensional modified ZK equation in the form

$$u_t + au^p u_x + (u_{xx} + u_{yy} + u_{zz})_x = 0 \tag{6}$$

In this study, the ZK(m, m) equation was solved by RDTM and VIM up to the second Iteration using the Maple 15 software. RDTM and VIM were compared on ZK(m, m) equation. In this comparison, the existing exact solution

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in the literature, the numerical solutions obtained, the absolute error values and the solution speed of methods were dealt with. Graphical and table results of solutions were analyzed. While solving the ZK(m, m) equation and obtaining the tables and graphics; Maple 15 software was used in a PC featured Intel® Core(TM) i5-2430 M CPU @ 2.40 GHz and 4.00 GB of RAM.

The main purpose of this study paper has been organized as follows: Section 2 deals with the analysis of the methods. In Section 3, we apply the RDTM and VIM to solve types of ZK(m, m) equations in the form (5) equations. Conclusions are given in Section 4.

2. ANALYSIS OF THE METHODS

We will briefly introduce RDTM and VIM for ZK(m, m) in this section. For the purpose of illustration of the methodology to the proposed methods, we write the ZK(m, m) equation in the standard operator form

$$L(u(x,y,t)) + N(u(x,y,t)) = 0 \tag{7}$$

with initial condition

$$u(x,y,0) = f(x,y) \tag{8}$$

where $L(u(x,y,t)) = u_t$ is a linear operator which has partial derivatives, $N(u(x,y,t)) = a(u^m)_x + b(u^m)_{xxx} + k(u^m)_{yyx}$ is a nonlinear term.

2.1. Reduced Differential Transform Method (RDTM) for ZK(m, m)

The basic definitions of the RDTM[3] are as follows:

Definition 2.1 (Keskin and Oturanç (2009)). If function $u(x,y,t)$ is analytic and differentiated continuously with respect to time t and spaces x and y in the domain of interest, then let

$$U_k(x,y) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0} \tag{9}$$

where the t-dimensional spectrum function $U_k(x,y)$ is the transformed function. In this paper, the lowercase $u(x,y,t)$ represent the original function while the uppercase $U_k(x,y)$ stand for the transformed function.

The differential inverse transform of $U_k(x,y)$ is defined as follows:

$$u(x,y,t) = \sum_{k=0}^{\infty} U_k(x,y)t^k \tag{10}$$

Then combining Eqs. (9) and (10) we write

$$u(x,y,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k}u(x,y,t)\right]_{t=0} t^k \tag{11}$$

From the above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion. The following theorem of the fundamental operators of RDTM is given below (for details see Refs. [3–5];

Theorem 2.1 (Keskin and Oturanç (2009)). *Assume that the reduced differential transform functions of $u(x,y,t)$, $v(x,y,t)$ and $w(x,y,t)$ are $U_k(x,y)$, $V_k(x,y)$ and $W_k(x,y)$ respectively. Then*;

(i) *If $w(x,y,t) = u(x,y,t) \pm \alpha v(x,y,t)$, then $W_k(x,y) = U_k(x,y) \pm \alpha V_k(x,y)$ (α is a constant),*

(ii) *If $w(x,y,t) = u(x,y,t)v(x,y,t)$, then $W_k(x,y) = \sum_{r=0}^{k} V_r(x,y)U_{k-r}(x,y) = \sum_{r=0}^{k} U_r(x,y)V_{k-r}(x,y)$,*

(iii) *If $w(x,y,t) = \partial^r/\partial t^r u(x,y,t)$, then $W_k(x,y) = (k+1)\dots(k+r)U_{k+1}(x,y) = ((k+r)!/k!)U_{k+r}(x,y)$,*

(iv) *If $w(x,y,t) = (\partial/(\partial x^m \partial y^n))u(x,y,t)$, then $W_k(x,y) = (\partial/(\partial x^m \partial y^n))U_k(x,y)$.*

According to the RDTM and Theorem 2.1, we can construct the following iteration formula:

$$(k+1)U_{k+1}(x,y) = -N(U_k(x,y)) \tag{12}$$

where $N_k = N(U_k(x,y))$ is the transformations of the function $N(u(x,y,t))$ respectively. For the easy to follow of the reader, we can give the first few nonlinear term are

$$N_0 = a\frac{\partial}{\partial x}U_0^m(x,y) + b\frac{\partial^3}{\partial x^3}U_0^m(x,y) + k\frac{\partial^3}{\partial y^2 \partial x}U_0^m(x,y),$$

$$\begin{aligned} N_1 = {} & a\frac{\partial}{\partial x}(mU_0^{(m-1)}(x,y)U_1(x,y)) \\ & + b\frac{\partial^3}{\partial x^3}(mU_0^{(m-1)}(x,y)U_1(x,y)) \\ & + k\frac{\partial^3}{\partial y^2 \partial x}(mU_0^{(m-1)}(x,y)U_1(x,y)), \end{aligned}$$

$$\begin{aligned} N_2 = {} & a\frac{\partial}{\partial x}\bigg(\frac{m}{2}(mU_0^{(m-2)}(x,y)U_1^2(x,y) \\ & + mU_0^{(m-1)}(x,y)U_2(x,y) - U_0^{(m-2)}(x,y)U_1^2(x,y))\bigg) \\ & + b\frac{\partial^3}{\partial x^3}\bigg(\frac{m}{2}(mU_0^{(m-2)}(x,y)U_1^2(x,y) \\ & + mU_0^{(m-1)}(x,y)U_2(x,y) - U_0^{(m-2)}(x,y)U_1^2(x,y))\bigg) \\ & + k\frac{\partial^3}{\partial y^2 \partial x}\bigg(\frac{m}{2}(mU_0^{(m-2)}(x,y)U_1^2(x,y) \\ & + mU_0^{(m-1)}(x,y)U_2(x,y) - U_0^{(m-2)}(x,y)U_1^2(x,y))\bigg) \end{aligned}$$

Maple Code for Nonlinear Function as given[3]

```
restart:
NF := Nu(x,y,t)∧ p: ♯ Nonlinear
  Func.
k := 2: ♯ Order
u[t] := sum(u[a]*t∧ a, a=0..k):
NF[t] := subs(Nu(x,y,t)=u[t],NF):
s := expand(NF[t],t):
dt := unapply(s,t):
for i from 0 to k do
```

```
    n[i]:=((D@@i)(dt)(0)/i!):
    print(N[i],n[i]); ♯ Transform
      Func.
  od:
```

From initial condition (8), we write

$$U_0(x,y) = f(x,y) \tag{13}$$

Substituting (13) into (12) and by a straight forward iterative calculations, we get the following $U_k(x,y)$ values. Then the inverse transformation of the set of values $\{U_k(x,y)\}_{k=0}^{n}$ gives approximation solution as,

$$\tilde{u}_n(x,y,t) = \sum_{k=0}^{n} U_k(x,y)t^k \tag{14}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,y,t) = \lim_{n\to\infty} \tilde{u}_n(x,y,t) \tag{15}$$

2.2. Variational Iteration Method (VIM) for ZK(m, m)

Consider the differential Eq. (7) with initial condition (8). According to VIM,[33–35] we write

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^t \lambda(\tau)[L(u_n(x,y,\tau)) + N(\tilde{u}_n(x,y,\tau))]\,d\tau \tag{16}$$

where λ is a general Lagrangian multiplier and it can be optimally determined by the aid of variational theory. Here $\tilde{u}_n$ is a restricted variation where $\delta\tilde{u}_n = 0$. It is first required to determine λ which is the Lagrangian multiplier will be optimally identified. Using the determined Lagrangian multiplier and any selective function u_0, u_{n+1}, which is the successive approximations of $u(x,y,t)$ for $n \geq 0$, will be obtained readily. Hence, we get the solution as

$$u(x,y,t) = \lim_{n\to\infty} u_n(x,y,t) \tag{17}$$

3. NUMERICAL APPLICATIONS

In this section, RDTM and VIM have been applied in the two distinguished $(2+1)$ dimensional and third-order ZK(3, 3) and ZK(2, 2) equations. The examples of approximate solutions and the absolute error of ZK(3, 3) and ZK(2, 2) equations have been analysed and illustrated in the tables and graphics. The absolute error in the following examples is obtained as shown in the following formula

$$\text{Absolute error} = |\text{numerical} - \text{Exact}|$$

Example 3.1. Considering the following ZK(3, 3) equation:[36]

$$u_t + (u^3)_x + 2(u^3)_{xxx} + 2(u^3)_{yyx} = 0 \tag{18}$$

subject to initial condition

$$u(x,y,0) = \frac{3}{2}\lambda \sinh\frac{1}{6}(x+y) \tag{19}$$

the exact solution of (18) is given as;[36]

$$u(x,y,t) = \frac{3}{2}\lambda \sinh\frac{1}{6}(x+y-\lambda t) \tag{20}$$

3.1. Solution by RDTM

If we apply RDTM and Theorem 2.1 in (18) it gives us

$$U_{k+1}(x,y) = -\frac{1}{(k+1)}\left(\frac{\partial}{\partial x}U_k^3(x,y) + 2\frac{\partial^3}{\partial x^3}U_k^3(x,y) + 2\frac{\partial^3}{\partial y^2 \partial x}U_k^3(x,y)\right) \tag{21}$$

where $U_k(x,y)$'s are the transformed functions. By the initial condition (19) we write

$$U_0(x,y) = \frac{3}{2}\lambda \sinh\frac{1}{6}(x+y) \tag{22}$$

Now, substituting (22) into (23) respectively, we obtain

$$U_1(x,y) = -\frac{3}{8}\lambda^3 \cosh\left(\frac{x+y}{6}\right)\left(9\cosh^3\left(\frac{x+y}{6}\right) - 8\right) \tag{23}$$

$$U_2(x,y) = \frac{3}{64}\lambda^2 \sinh\left(\frac{x+y}{6}\right)\left(765\cosh^4\left(\frac{x+y}{6}\right) - 729\cosh^2\left(\frac{x+y}{6}\right) + 91\right) \tag{24}$$

$$U_3(x,y) = -\frac{1}{256}\lambda^7 \cosh\left(\frac{x+y}{6}\right) \times \begin{pmatrix} -382293\cosh^4\left(\frac{x+y}{6}\right) \\ +188181\cosh^6\left(\frac{x+y}{6}\right) \\ +234468\cosh^2\left(\frac{x+y}{6}\right) \\ -39851 \end{pmatrix} \tag{25}$$

By (14)

$$\tilde{u}_3(x,y,t) = \sum_{k=0}^{3} U_k(x,y)t^k \tag{26}$$

Substituting (22), (23), (24) and (25) in (26), we get

$$\tilde{u}_3(x,y,t) = \frac{\lambda^7}{256}\left(-234468\cosh^3\left(\frac{x+y}{6}\right) + 382293\cosh^5 \times \left(\frac{x+y}{6}\right) + 39851\cosh\left(\frac{x+y}{6}\right) - 188181\cosh^5\right.$$

$$\times\left(\frac{x+y}{6}\right)\Bigg)t^3+\frac{\lambda^5}{256}\left(9180\sinh\left(\frac{x+y}{6}\right)\cosh^4\right.$$
$$\times\left(\frac{x+y}{6}\right)-8748\sinh\left(\frac{x+y}{6}\right)\cosh^2\left(\frac{x+y}{6}\right)$$
$$\times 1092\sinh\left(\frac{x+y}{6}\right)\Bigg)t^2+\frac{\lambda^3}{256}\left(-864\cosh^3\right.$$
$$\times\left(\frac{x+y}{6}\right)+768\cosh\left(\frac{x+y}{6}\right)1092\sinh\left(\frac{x+y}{6}\right)\Bigg)$$
$$\times t+\frac{3}{2}\lambda\sinh\left(\frac{x+y}{6}\right) \tag{27}$$

3.2. Solution by VIM

For solving by VIM we obtain the recurrence relation

$$u_{n+1}(x,y,t)=u_n(x,y,t)-\int_0^t\left\{\frac{\partial u_n(x,y,\tau)}{\partial\tau}+\frac{\partial u_n^3(x,y,\tau)}{\partial x}+2\frac{\partial^3 u_n^3(x,y,\tau)}{\partial x^3}+2\frac{\partial^3 u_n^3(x,y,\tau)}{\partial y^2\partial x}\right\}d\tau \tag{28}$$

By the initial condition (19) we write

$$u_0(x,y,t)=u(x,y,0)=\frac{3}{2}\lambda\sinh\frac{1}{6}(x+y) \tag{29}$$

Now, substituting (30) with (29) respectively, we obtain

$$u_1(x,y,t)=-\frac{3}{8}\lambda\left(9\lambda^2\cosh^3\left(\frac{x}{6}+\frac{y}{6}\right)-8\lambda^2\cosh\left(\frac{x}{6}+\frac{y}{6}\right)\right)t+\frac{3}{2}\lambda\sinh\left(\frac{x}{6}+\frac{y}{6}\right) \tag{30}$$

$$u_2(x,y,t)=\frac{1}{2048}\lambda\left(\begin{array}{l}351000\lambda^8\cosh^4\left(\frac{x}{6}+\frac{y}{6}\right)\sinh\left(\frac{x}{6}+\frac{y}{6}\right)\\ -65664\lambda^8\cosh^2\left(\frac{x}{6}+\frac{y}{6}\right)\sinh\left(\frac{x}{6}+\frac{y}{6}\right)\\ +295245\lambda^8\cosh^8\left(\frac{x}{6}+\frac{y}{6}\right)\sinh\left(\frac{x}{6}+\frac{y}{6}\right)\\ +1536\lambda^8\sinh\left(\frac{x}{6}+\frac{y}{6}\right)-578340\lambda^8\cosh^6\\ \times\left(\frac{x}{6}+\frac{y}{6}\right)\sinh\left(\frac{x}{6}+\frac{y}{6}\right)\end{array}\right)\times t^4+\cdots \tag{31}$$

$$u_3(x,y,t)=\frac{1}{1160800811089920}\lambda\times\left(7155490965527070346950000\lambda^{26}\times\cosh^{25}\left(\frac{x}{6}+\frac{y}{6}\right)\sinh\left(\frac{x}{6}+\frac{y}{6}\right)-\cdots\right)t^{13}$$
$$+\cdots+\frac{1}{1160800811089920}\lambda\times\left(-3917702737428480\lambda^2\cosh^3\left(\frac{x}{6}+\frac{y}{6}\right)+3482402433269760\lambda^2\times\cosh\left(\frac{x}{6}+\frac{y}{6}\right)\right)t+\frac{3}{2}\lambda\sinh\left(\frac{x}{6}+\frac{y}{6}\right) \tag{32}$$

ZK(3, 3) equation was solved by RDTM and VIM using Maple 15 software in a PC featured Intel® Core(TM) i5-2430 M CPU @ 2.40 GHz and 4.00 GB of RAM. Both methods were calculated until the third iteration. It was obtained that while the time spent by PC is 0,171 second for RDTM, it is 3,463 seconds for VIM. Now to have better data than we have, let's analyze these solutions with the exact solution in the literature and absolute error values in some interval by the aid of Table I and Figure 1.

EXAMPLE 3.2. Now we consider the following ZK(2, 2) equation:[36]

$$u_t+(u^2)_x+\frac{1}{8}(u^2)_{xxx}+\frac{1}{8}(u^2)_{yyx}=0 \tag{33}$$

subject to initial condition

$$u(x,y,0)=\frac{4}{3}\lambda\sinh^2(x+y) \tag{34}$$

the exact solution of (33) is given as[36]

$$u(x,y,t)=\frac{4}{3}\lambda\sinh^2(x+y-\lambda t) \tag{35}$$

3.3. Solution by RDTM

Similarly If we apply RDTM and Theorem 2.1 in (33) it gives us

$$U_{k+1}(x,y)=-\frac{1}{(k+1)}\left(\frac{\partial}{\partial x}U_k^2(x,y)+\frac{1}{8}\frac{\partial^3}{\partial x^3}U_k^2(x,y)+\frac{1}{8}\frac{\partial^3}{\partial y^2\partial x}U_k^2(x,y)\right) \tag{36}$$

where $U_k(x,y)$'s are the transformed functions. By the initial condition (34) we write

$$U_0(x,y)=\frac{4}{3}\lambda\sinh^2(x+y) \tag{37}$$

Now, substituting (37) into (36), we obtain the following $U_k(x,y)$ values successively

$$U_1(x,y)=-\frac{32\lambda^2}{9}\sinh(x+y)\cosh(x+y)\times(10\cosh^2(x+y)-7) \tag{38}$$

$$U_2(x,y)=\frac{64\lambda^3}{27}(1200\cosh^6(x+y)-2080\cosh^4(x+y)+968\cosh^2(x+y)-79) \tag{39}$$

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Table I. Comparison of three iteration approximate solutions with existing exact solution for ZK(3, 3) equation. ($\lambda = 0{,}001$).

x	y	t	EXACT solution	RDTM solution	VIM solution	Abs-error RDTM	Abs-error VIM
0.0	0.0		−0.0000000500000	−0.0000000000750	−0.0000000000750	$0.4992500001 \times 10^{-7}$	$0.4992500001 \times 10^{-7}$
0.0	0.5		0.0001250945526	0.0001251446462	0.0001251446462	0.500936×10^{-7}	0.500936×10^{-7}
0.0	1.0		0.0002511083202	0.0002511589208	0.0002511589208	0.506006×10^{-7}	0.506006×10^{-7}
0.5	0.0		0.0001250945526	0.0001251446462	0.0001251446462	0.500936×10^{-7}	0.500936×10^{-7}
0.5	0.5	0.2	0.0002511083202	0.0002511589208	0.0002511589208	0.506006×10^{-7}	0.506006×10^{-7}
0.5	1.0		0.0003788669048	0.0003789183534	0.0003789183534	0.514486×10^{-7}	0.514486×10^{-7}
1.0	0.0		0.0002511083202	0.0002511589208	0.0002511589208	0.506006×10^{-7}	0.506006×10^{-7}
1.0	0.5		0.0003788669048	0.0003789183534	0.0003789183534	0.514486×10^{-7}	0.514486×10^{-7}
1.0	1.0		0.0005092580328	0.0005093106746	0.0005093106746	0.526418×10^{-7}	0.526418×10^{-7}
0.0	0.0		−0.00000007500000	−0.00000000001125	−0.00000000001125	$0.7488750003 \times 10^{-7}$	$0.7488750003 \times 10^{-7}$
0.0	0.5		0.0001250694658	0.0001251446062	0.0001251446062	0.751404×10^{-7}	$0.7514040001 \times 10^{-7}$
0.0	1.0		0.0002510829723	0.0002511588732	0.0002511588732	0.759008×10^{-7}	$0.7590080004 \times 10^{-7}$
0.5	0.0		0.0001250694658	0.0001251446062	0.0001251446062	0.751404×10^{-7}	$0.7514040001 \times 10^{-7}$
0.5	0.5	0.3	0.0002510829723	0.0002511588732	0.0002511588732	0.759008×10^{-7}	$0.7590080004 \times 10^{-7}$
0.5	1.0		0.0003788411198	0.0003789182925	0.0003789182925	0.771728×10^{-7}	$0.7717280009 \times 10^{-7}$
1.0	0.0		0.0002510829723	0.0002511588732	0.0002511588732	0.759008×10^{-7}	$0.7590080004 \times 10^{-7}$
1.0	0.5		0.0003788411198	0.0003789182925	0.0003789182925	0.771728×10^{-7}	$0.7717280009 \times 10^{-7}$
1.0	1.0		0.0005092316313	0.0005093105939	0.0005093105939	0.789627×10^{-7}	$0.7896270020 \times 10^{-7}$
		In the solution of ZK(3, 3) equation, the computation time of methods				**RDTM : 0,171 second** **VIM : 3,463 seconds**	

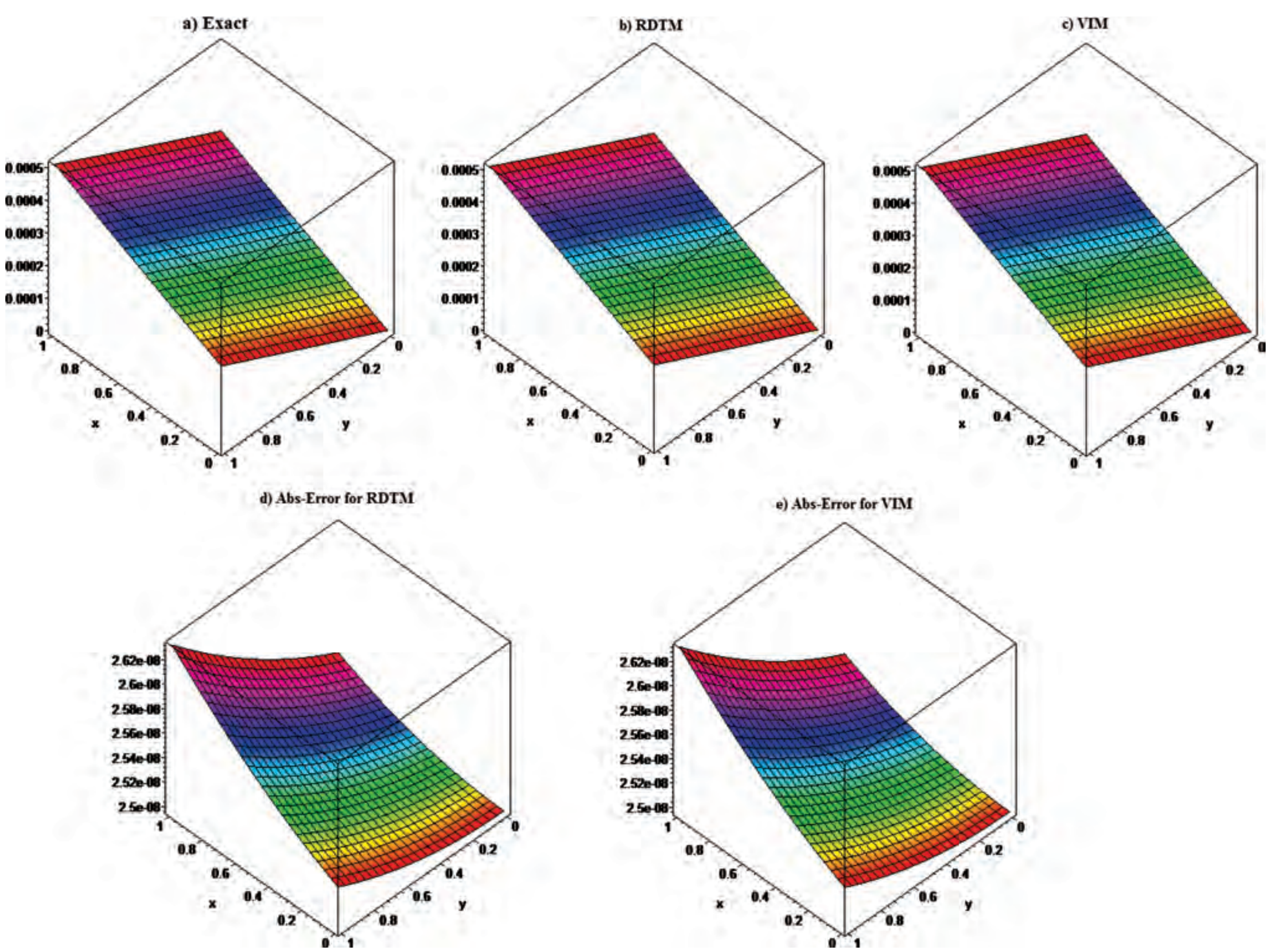

Fig. 1. (a) Exact solution of ZK(3, 3) equation (b) RDTM solution of ZK(3, 3) equation (c) VIM solution of ZK(3, 3) equation (d) the absolute error between RDTM solution and exact solution (e) the absolute error between VIM solution and exact solution. $t = 0{,}1$, $\lambda = 0{,}001$, $0 \le x \le 1$, $0 \le y \le 1$.

$$\begin{aligned} U_3(x,y) = & -\frac{4096\lambda^4}{243}\sinh(x+y)\cosh(x+y) \\ & \times (23800\cosh^6(x+y) - 42900\cosh^4(x+y) \\ & + 22665\cosh^2(x+y) - 3142) \end{aligned} \tag{40}$$

By (14)

$$\tilde{u}_3(x,y,t) = \sum_{k=0}^{3} U_k(x,y)t^k \tag{41}$$

Substituting (37), (38), (39) and (40) in (41), we have

$$\begin{aligned} \tilde{u}_4(x,y,t) = & -\frac{4096\lambda^4}{243}\sinh(x+y)\cosh(x+y) \\ & \times (23800\cosh^6(x+y) - 42900\cosh^4(x+y) \\ & + 22665\cosh^2(x+y) - 3142)t^3 \\ & + \frac{64\lambda^3}{27}(1200\cosh^6(x+y) - 2080\cosh^4(x+y) \\ & + 968\cosh^2(x+y) - 79)t^2 \\ & - \frac{32\lambda^2}{9}\sinh(x+y)\cosh(x+y) \\ & \times (10\cosh^2(x+y) - 7)t \\ & + \frac{4}{3}\lambda\sinh^2(x+y) \end{aligned} \tag{42}$$

3.4. Solution by VIM

For solving by VIM we obtain the recurrence relation

$$\begin{aligned} & u_{n+1}(x,y,t) \\ & \quad = u_n(x,y,t) - \int_0^t \left\{ \frac{\partial u_n(x,y,\tau)}{\partial \tau} + \frac{\partial u_n^2(x,y,\tau)}{\partial x} \right. \\ & \qquad \left. + \frac{1}{8}\frac{\partial^3 u_n^2(x,y,\tau)}{\partial x^3} + \frac{1}{8}\frac{\partial^3 u_n^2(x,y,\tau)}{\partial y^2 \partial x} \right\} d\tau \end{aligned} \tag{43}$$

By the initial condition (35) we write

$$u_0(x,y,t) = u(x,y,0) = \frac{4}{3}\lambda\sinh^2(x+y) \tag{44}$$

Now, substituting (46) with (45) respectively, we obtain

$$\begin{aligned} u_1(x,t) = & \frac{4}{9}\lambda^2\sinh(x+y)(-80\cosh^3(x+y) \\ & + 56\cosh(x+y))t + \frac{4}{3}\lambda\sinh^2(x+y) \end{aligned} \tag{45}$$

$$\begin{aligned} u_2(x,t) = & -\frac{4\lambda^4}{243}(-340480\sinh(x+y)\cosh(x+y) \\ & + 3481600\sinh(x+y)\cosh^7(x+y) \\ & - 5836800\sinh(x+y)\cosh^5(x+y) \\ & + 2810880\sinh(x+y)\cosh^3(x+y))t^3 + \cdots \end{aligned} \tag{46}$$

$$\begin{aligned} & u_3(x,t) \\ & = -\frac{4\lambda^8}{413343}(-216832430571520000 \\ & \times \cosh^9(x+y)\sinh(x+y) \\ & - 188468533657600000\cosh^{13}(x+y)\sinh(x+y) \\ & + 282685285072896000\cosh^{11}(x+y)\sinh(x+y) \\ & + 1817174343680000\cosh^3(x+y)\sinh(x+y) \\ & + 50425600409600000\cosh^{15}(x+y)\sinh(x+y) \\ & - 50575389491200\sinh(x+y)\cosh(x+y) \\ & + 89554327017881600\cosh^7(x+y)\sinh(x+y) \\ & - 19121568114278400\cosh^5(x+y)\sinh(x+y)) \\ & \times t^7 + \cdots \end{aligned} \tag{47}$$

Table II. Comparison of three iteration approximate solutions with existing exact solution for ZK(2, 2) equation. ($\lambda = 0{,}0001$).

x	y	t	EXACT solution	RDTM solution	VIM solution	Abs-error RDTM	Abs-error VIM
0.0	0.0		0.00000000000005	0.00000000000090	0.00000000000090	0.900000×10^{-12}	0.900000×10^{-12}
0.0	0.5		0.00003620224186	0.0000361815180	0.0000361815179	0.207241×10^{-7}	0.207241×10^{-7}
0.0	1.0		0.0001841367078	0.0001839302192	0.0001839302191	0.2064889×10^{-6}	0.2064889×10^{-6}
0.5	0.0		0.00003620224186	0.0000361815180	0.0000361815179	0.207241×10^{-7}	0.207241×10^{-7}
0.5	0.5	0.2	0.0001841367078	0.0001839302192	0.0001839302191	0.2064889×10^{-6}	0.2064889×10^{-6}
0.5	1.0		0.0006044840856	0.0006028025823	0.0006028025833	$0.16815035 \times 10^{-5}$	$0.16815024 \times 10^{-5}$
1.0	0.0		0.0001841367078	0.0001839302192	0.0001839302191	0.2064889×10^{-6}	0.2064889×10^{-6}
1.0	0.5		0.0006044840856	0.0006028025823	0.0006028025833	$0.16815035 \times 10^{-5}$	$0.16815024 \times 10^{-5}$
1.0	1.0		0.001753809417	0.001741101068	0.001741101237	$0.12708350 \times 10^{-4}$	$0.127081795 \times 10^{-4}$
0.0	0.0		0.00000000000012	0.0000000000019	0.0000000000019	0.190000×10^{-11}	0.190000×10^{-11}
0.0	0.5		0.00003620067502	0.0000361696074	0.00003616960749	0.310676×10^{-7}	0.310676×10^{-7}
0.0	1.0		0.0001841318725	0.0001838226047	0.0001838226047	0.3092675×10^{-6}	0.3092675×10^{-6}
0.5	0.0		0.00003620067502	0.0000361696074	0.00003616960749	0.310676×10^{-7}	0.310676×10^{-7}
0.5	0.5	0.3	0.0001841318725	0.0001838226047	0.0001838226047	0.3092675×10^{-6}	0.3092675×10^{-6}
0.5	1.0		0.0006044707294	0.0006019584203	0.0006019584259	0.2512309×10^{-5}	$0.251230934 \times 10^{-5}$
1.0	0.0		0.0001841318725	0.0001838226047	0.0001838226047	0.3092675×10^{-6}	0.3092675×10^{-6}
1.0	0.5		0.0006044707294	0.0006019584203	0.0006019584259	0.2512309×10^{-5}	$0.251230934 \times 10^{-5}$
1.0	1.0		0.001753773033	0.001734903389	0.001734904240	$0.18869644 \times 10^{-4}$	$0.18868793 \times 10^{-4}$
		In the solution of ZK(2, 2) equation, the computation time of methods				**RDTM: 0,047 second** **VIM: 0,328 second**	

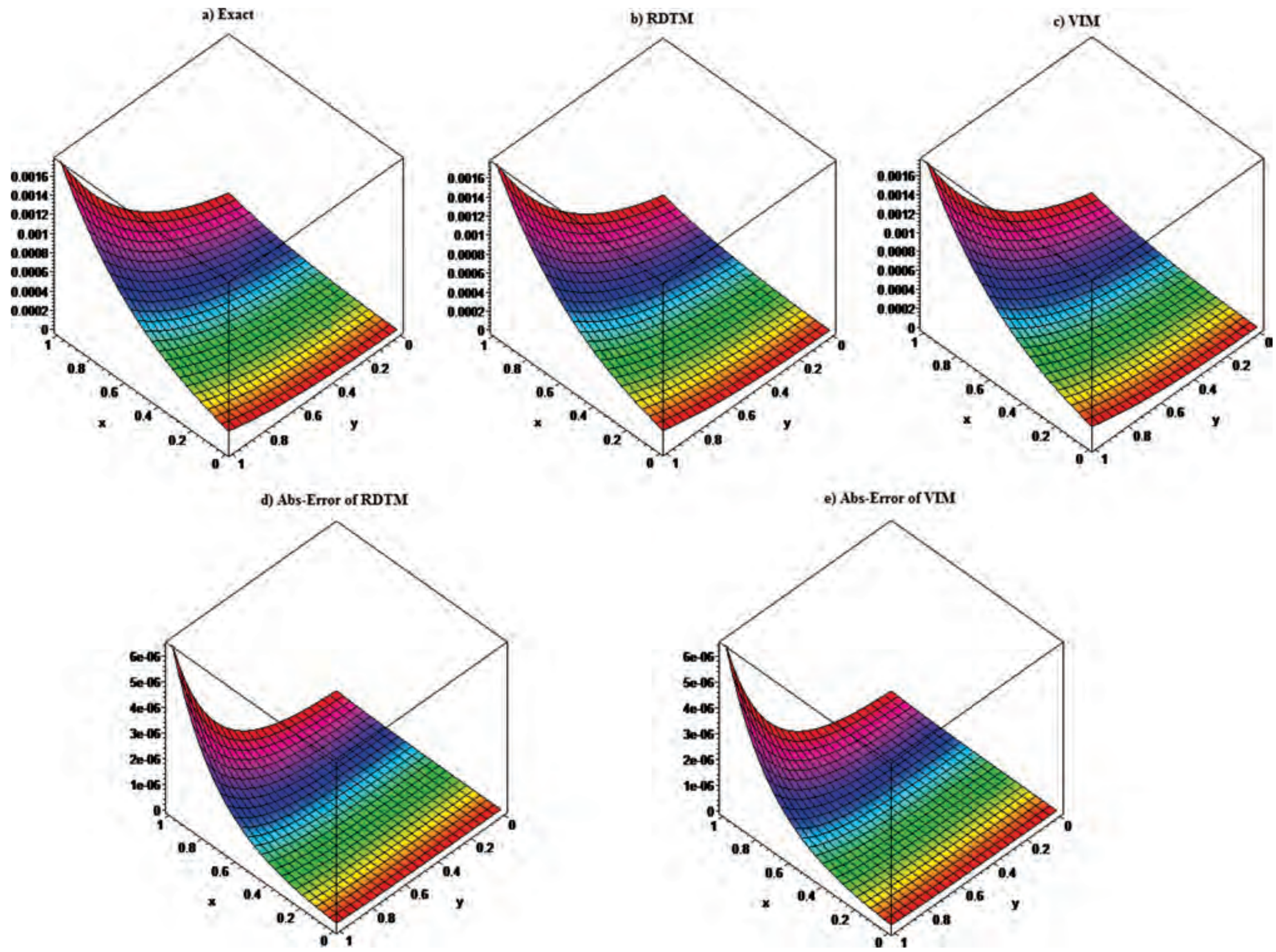

Fig. 2. (a) Exact solution of ZK(2, 2) equation (b) RDTM solution of ZK(2, 2) equation (c) VIM solution of ZK(2, 2) equation (d) the absolute error between RDTM solutions and exact solution (e) the absolute error between VIM solutions and exact solution. $t = 0{,}1$, $\lambda = 0{,}0001$, $0 \le x \le 1$, $0 \le y \le 1$.

Similarly, ZK(3, 3) equation was solved by RDTM and VIM using Maple 15 software in a PC featured Intel® Core(TM) i5-2430M CPU @ 2.40 GHz and 4.00 GB of RAM. Both methods were calculated until the third iteration. It was obtained that while the time spent by PC is 0,047 second for RDTM, it is 0,328 second for VIM. Now to have better data than we have, let's analyze these solutions with the exact solution in the literature and absolute error values in some interval by the aid of Table II and Figure 2.

4. CONCLUSION

In this study, the RDTM and VIM have been successfully applied to two different $(2+1)$ dimensional and third-order ZK(3, 3) and ZK(2, 2) PDEs. In these examples it was shown that, the absolute errors of RDTM and VIM are almost the same. But in the solution of the equation, it takes more time than RDTM since VIM works with integral and it also tires the PC. When the computation time is compared, in the firs example RDTM nearly 20 times faster than VIM and in the second example RDTM nearly 7 times faster than VIM with the same certainty. As a result, RDTM is more advantageous, fast, useful, and effective than VIM.

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