Cost Register Automata

2019.12.20

Overview

Contents

Introduction

A Cost Register Automaton (CRA)

- process a data word (a sequence of tagged values) and outputs a value in a data set that is computed using a given set of operations over the data set;
- provides a machine-based characterization of the class of *regular transductions* from strings to costs.

Contents

Data Transductions

Notations

- Σ is a finite alphabet whose elements are called tags;
- D is a data set of data values.

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Definition

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Example

Suppose $\Sigma = \{a, \#\}, D = \mathbb{N}$. Then

$$(a, 0)(a, 2)(\#, 0)(a, 1)(\#, 2)$$

is a data word.

Data Transductions

Definition

A data transduction is a partial function of type

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A multi-valued transduction is a partial function of type

$$(\Sigma \times D)^* \to \mathcal{P}(D),$$

where $\mathcal{P}$ is the powerset operator.

Data Transductions

Definition

For a data word $w \in (\Sigma \times D)^*$, $w|_\Sigma$ denotes the elementwise projection of w to the tag component. Formally,

$$(a_1, d_1)(a_2, d_2) \cdots (a_n, d_n)|_\Sigma = a_1 a_2 \cdots a_n$$

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Definition

The rate of f is the language $\mathsf{R}(f) \subseteq \Sigma^*$ defined as follows:

$$\mathsf{R}(f) = \{\sigma \in \Sigma^* | \forall w \text{ with } w|_\Sigma = \sigma, f(w) \text{ is defined}\}.$$

Contents

Notations

- X is a set of variables;
- $\mathcal{O}$ is a family of constants and operations that are allowed on the data set D;
  - $\mathcal{O}_n$ denotes the set of n-ary operations that are contained in $\mathcal{O}$;
  - $\mathcal{O}_0$ is the set of constants in $\mathcal{O}$, in particular.

Syntax and Semantics of CRAs

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Definition

$\mathbb{E}_{\mathcal{O}}[X]$ denotes the set of expressions over X, defined by the following rules:

- $c \in \mathbb{E}_{\mathcal{O}}[X]$, if constanc $c \in \mathcal{O}$;
- $x \in \mathbb{E}_{\mathcal{O}}[X]$, if variable $x \in X$;
- $op(t_1, \ldots, t_n) \in \mathbb{E}_{\mathcal{O}}[x]$, if n-ary operation $op \in \mathcal{O}$, $t_i \in \mathbb{E}_{\mathcal{O}}[X]$ for all $i = 1, \ldots, n$.

Syntax and Semantics of CRAs

Definition (CRA)

A (nondeterministic, copyful) Cost Register Automaton (NCRA) over the tag alphabet Σ, data values D, and data operations $\mathcal{O}$ is a tuple

$$\mathcal{A} = (Q, X, \Delta, I, F),$$

where

- Q is a finite set of states,
- X is a finite set of registers,
- $\Delta \subseteq Q \times \Sigma \times U_{\mathcal{O}} \times Q$ is the set of transitions with $U_{\mathcal{O}}$ the set of register updates $X \to \mathbb{E}_{\mathcal{O}}[X \cup \{\mathtt{val}\}]$,
- $I : Q \rightharpoonup (X \to \mathbb{E}_{\mathcal{O}}[\emptyset])$ is the initialization function, and
- $F : Q \rightharpoonup \mathbb{E}_{\mathcal{O}}[X]$ is the finalization function.

Syntax and Semantics of CRAs

Branches

abbreviations	NCRA	UCRA	DCRA
the automaton is	nondeterministic	unambiguous	deterministic

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Definition

A CRA is said to be copyless if

1. $\forall (p, a, \theta, q) \in \Delta, \forall x \in X$, there is at most one occurrence of x in the list of expressions $\theta(x_1), \ldots, \theta(x_n)$, where $x_1, \ldots, x_n$ is an enumeration of X, and
2. $\forall q \in \mathrm{dom}(F), \forall x \in X$, there is at most one occurrence of x in the expression $F(q)$.

Definition

A function $\alpha : X \to D$ is called a variable assignment. It extends uniquely to a homomorphism $\hat{\alpha} : \mathbb{E}_{\mathcal{O}}[X] \to D$.

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Notations

- `val` is a special symbol, referring to the value of the current data item;
- $\alpha[\texttt{val} \mapsto d]$ means the extension of α that maps `val` to d.

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Definition

An expression $t \in \mathbb{E}_{\mathcal{O}}[X]$ denotes a function $[\![t]\!] : D^X \to D$, defined as:

for a variable assignment $\alpha : X \to D$, put $[\![t]\!](\alpha) = \hat{\alpha}(t)$.

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Definition

An expression $t \in \mathbb{E}_{\mathcal{O}}[X]$ denotes a function $[\![t]\!] : D^X \to D$, defined as:

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An expression $t \in \mathbb{E}_{\mathcal{O}}[X \cup \{\mathtt{val}\}]$ denotes a function $[\![t]\!] : D^X \times D \to D$,

$$[\![t]\!](\alpha, d) = \hat{\beta}(t) \text{ where } \beta = \alpha[\mathtt{val} \mapsto d] : X \cup \{\mathtt{val}\} \to D$$

Syntax and Semantics of CRAs

Definition

For an input sequence $w = (a_1, d_1)(a_2, d_2) \cdots (a_n, d_n) \in (\Sigma \times D)^*$, we define a w-run in $\mathcal{A}$ to be a sequence

$$(q_0, \alpha_0) \xrightarrow{(a_1,d_1)} (q_1, \alpha_1) \xrightarrow{(a_2,d_2)} (q_2, \alpha_2) \xrightarrow{(a_3,d_3)} \cdots \xrightarrow{(a_n,d_n)} (q_n, \alpha_n)$$

with $q_i \in Q$ and $\alpha_i : X \to D$ for all i so that:

1. *Initialization:* $q_0 \in \mathrm{dom}(I)$ and $\alpha_0(x) = [\![I(q_0)(x)]\!]$ for every register $x \in X$;
2. *Transition:* $\forall (p, \alpha) \xrightarrow{(a,d)} (q, \beta)$, $\exists \theta \in U_{\mathcal{O}}$ with $(p, a, \theta, q) \in \Delta$ s.t.

$$\forall x \in X,\ \beta(x) = [\![\theta(x)]\!](\alpha, d);$$

3. *Finalization:* $q_n \in \mathrm{dom}(F)$.

The value of the run is $[\![F(q_n)]\!](\alpha_n)$.

Contents

Example 1

Suppose

$\Sigma = \{a, b\}, D = \mathbb{N}, \mathcal{O} = \{0, +\}$

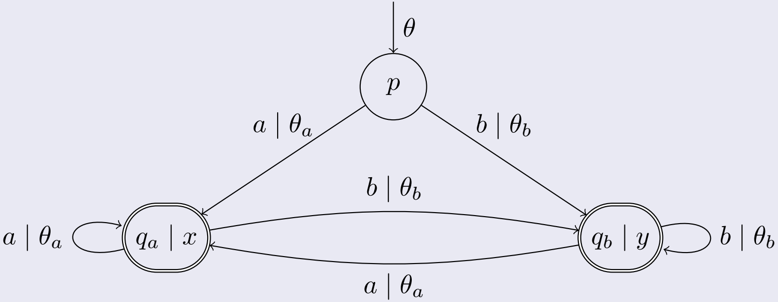
Transduction

$$f : (\Sigma \times D)^* \rightharpoonup D, \text{ defined on all nonempty sequences.}$$

If a sequence ends with an a-labeled (b-labeled) value, then f outputs the sum of all a-labeled (b-labeled) values in the sequence.

Example 1

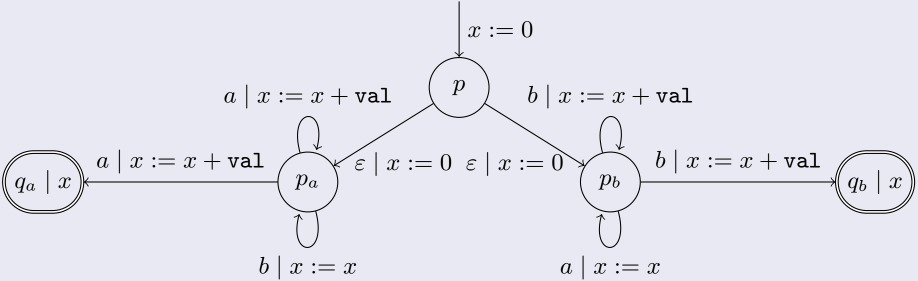
Copyless DCRA



$$\theta = \begin{cases} x := 0 \\ y := 0 \end{cases} \qquad \theta_a = \begin{cases} x := x + \texttt{val} \\ y := y \end{cases} \qquad \theta_b = \begin{cases} x := x \\ y := y + \texttt{val} \end{cases}$$

Example 2

Copyless UCRA



Example 3

Suppose

$\Sigma = \{a, \#\}$, $D = \mathbb{N}$, $\mathcal{O} = \{0, +, \max\}$.

Transduction

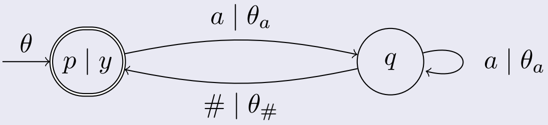
$$f : (\Sigma \times D)^* \rightharpoonup D, \quad \text{rate: } (a^+\#)^*$$

f outputs the maximum cost over all input blocks, where

- a block is a maximal subsequence of the input that is of the form $aa \ldots a\#$,
- the cost of a block is the sum of the a-labeled values.

Example 3

Copyless DCRA



$$\theta = \begin{cases} x := 0 \\ y := 0 \end{cases} \qquad \theta_a = \begin{cases} x := x + \texttt{val} \\ y := y \end{cases} \qquad \theta_\# = \begin{cases} x := 0 \\ y := \max(y, x) \end{cases}$$

Example 4

Suppose

$\Sigma = \{a, b\}, D = \mathbb{N}, \mathcal{O} = \{0, \ominus\}$. $x \ominus y = \max(x - y, 0)$.

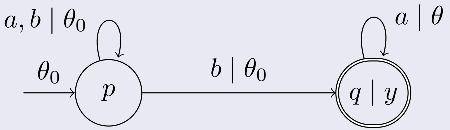
Transduction

$$f : (\Sigma \times D)^* \rightharpoonup D,$$

defined on sequences that contain at least one a-labeled value.
f outputs the maximum drawdown in the input signal after the last occurrence of a b-labeled value.

Example 4

Copyful UCRA



$$\theta_0 = \begin{cases} x := 0 \\ y := 0 \end{cases} \qquad \theta = \begin{cases} x := \max(x, \mathtt{val}) \\ y := \max(y, \max(x, \mathtt{val}) \ominus \mathtt{val}) \end{cases}$$

Example 5

Suppose

$\Sigma = \{a\}, D = \mathbb{Q}, \mathcal{O} = \{0, op\}$.

$$op(x, y) = \lambda \cdot x + y, \quad \lambda \in (0, 1).$$

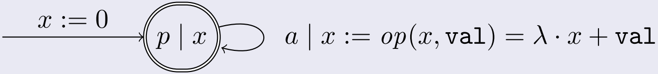
Transduction

For w with $w|_D = d_1 d_2 \dots d_n \in D^*$, f outputs

$$\lambda^{n-1} \cdot d_1 + \cdots + \lambda \cdot d_{n-1} + d_n.$$

Example 5

CRA



Example 5

Transduction

$$g(w_1 w_2 \dots w_n) = f(w_n \dots w_2 w_1).$$

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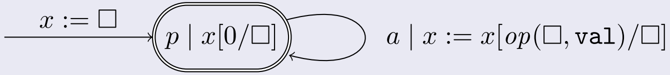
CRA with a hole



$x[t/\square]$ denotes the result of substituting t for $\square$ in the term that x holds.