



Forecasting: principles and practice

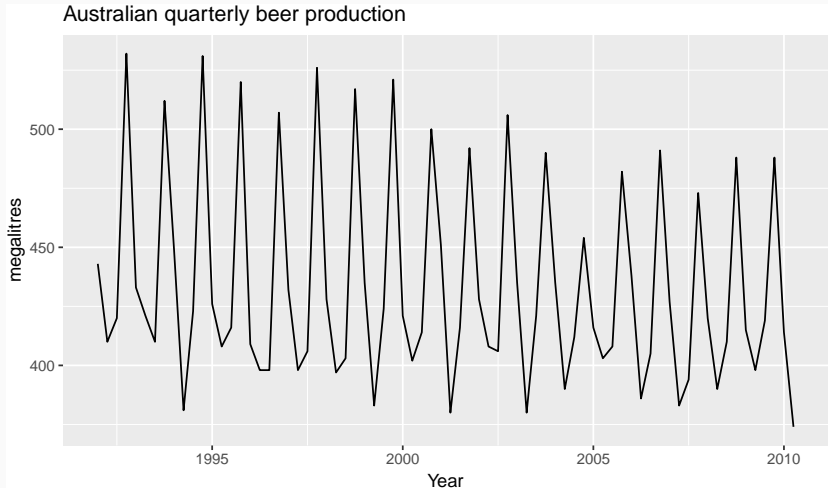
Rob J Hyndman

1.3 Forecast evaluation

Outline

- 1 Benchmark methods**
- 2 Forecasting residuals
- 3 Lab session 3
- 4 Evaluating forecast accuracy
- 5 Lab session 4
- 6 Time series cross-validation
- 7 Lab session 5
- 8 Prediction intervals

Some simple forecasting methods



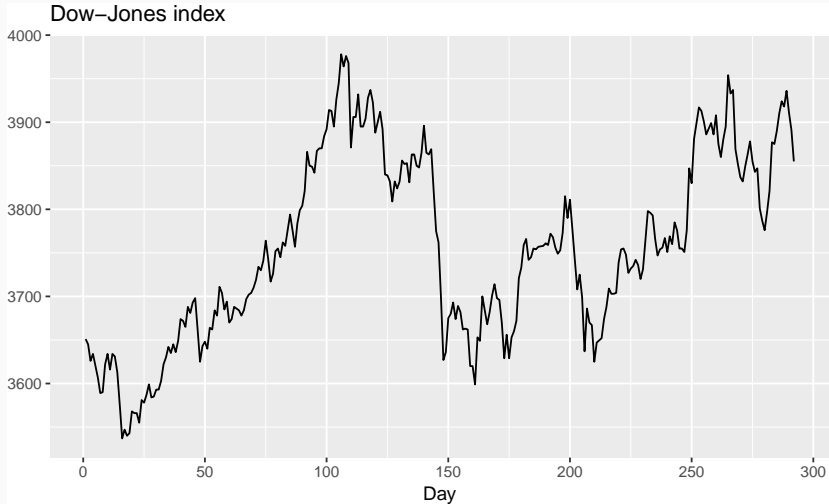
How would you forecast these data?

Some simple forecasting methods



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Some simple forecasting methods

Average method

- Forecast of all future values is equal to mean of historical data $\{y_1, \dots, y_T\}$.
- Forecasts: $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \dots + y_T)/T$

Some simple forecasting methods

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Naïve method

- Forecasts equal to last observed value.
- Forecasts: $\hat{y}_{T+h|T} = y_T$.
- Consequence of efficient market hypothesis.

Some simple forecasting methods

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Naïve method

- Forecasts equal to last observed value.
- Forecasts: $\hat{y}_{T+h|T} = y_T$.
- Consequence of efficient market hypothesis.

Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts: $\hat{y}_{T+h|T} = y_{T+h-km}$ where m = seasonal period and k is integer part of $(h - 1)/m$.

Some simple forecasting methods

Drift method

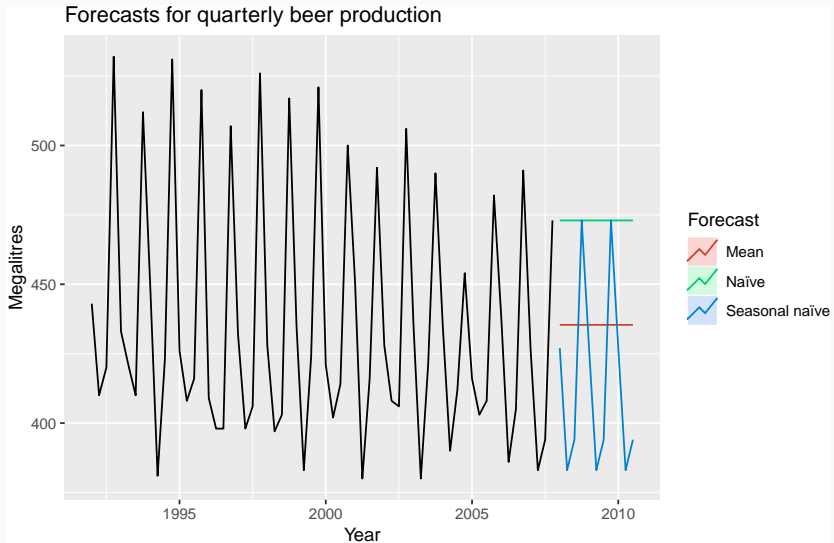
- Forecasts equal to last value plus average change.

- Forecasts:

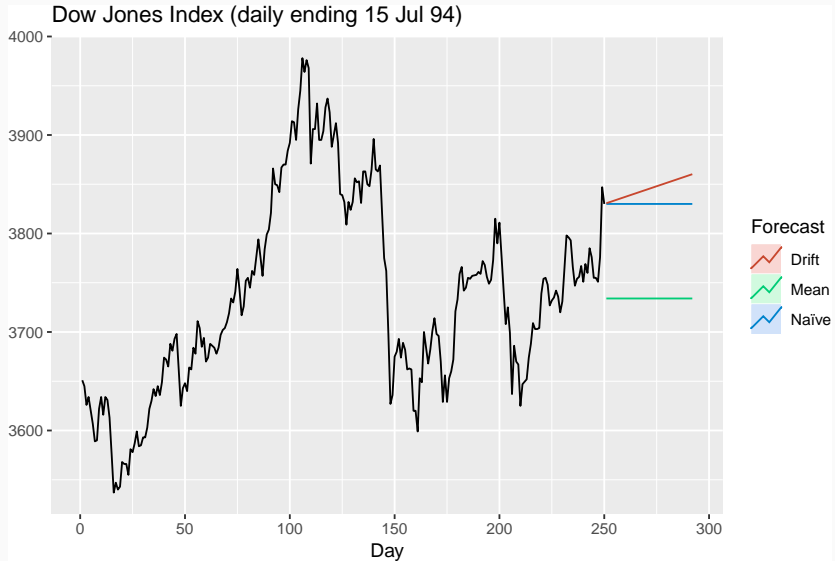
$$\begin{aligned}\hat{y}_{T+h|T} &= y_T + \frac{h}{T-1} \sum_{t=2}^T (y_t - y_{t-1}) \\ &= y_T + \frac{h}{T-1} (y_T - y_1).\end{aligned}$$

- Equivalent to extrapolating a line drawn between first and last observations.

Some simple forecasting methods



Some simple forecasting methods



Some simple forecasting methods

- Mean: `meanf(y, h=20)`
- Naïve: `naive(y, h=20)`
- Seasonal naïve: `snaive(y, h=20)`
- Drift: `rwf(y, drift=TRUE, h=20)`

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Fitted values

- $\hat{y}_{t|t-1}$ is the forecast of y_t based on observations y_1, \dots, y_t .
- We call these “fitted values”.
- Sometimes drop the subscript: $\hat{y}_t \equiv \hat{y}_{t|t-1}$.
- Often not true forecasts since parameters are estimated on all data.

For example:

- $\hat{y}_t = \bar{y}$ for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T - 1)$ for drift method.

Forecasting residuals

Residuals in forecasting: difference between observed value and its fitted value: $e_t = y_t - \hat{y}_{t|t-1}$.

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Assumptions

- 1 $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2 $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

Forecasting residuals

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Assumptions

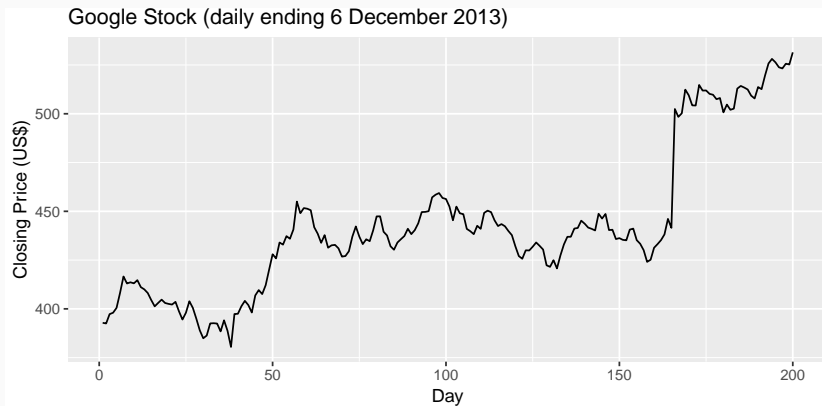
- 1 $\{e_t\}$ uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2 $\{e_t\}$ have mean zero. If they don't, then forecasts are biased.

Useful properties (for prediction intervals)

- 3 $\{e_t\}$ have constant variance.
- 4 $\{e_t\}$ are normally distributed.

Example: Google stock price

```
autoplot(goog200) +  
  xlab("Day") + ylab("Closing Price (US$)") +  
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



Example: Google stock price

Naïve forecast:

$$\hat{y}_{t|t-1} = y_{t-1}$$

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$$e_t = y_t - y_{t-1}$$

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Naïve forecast:

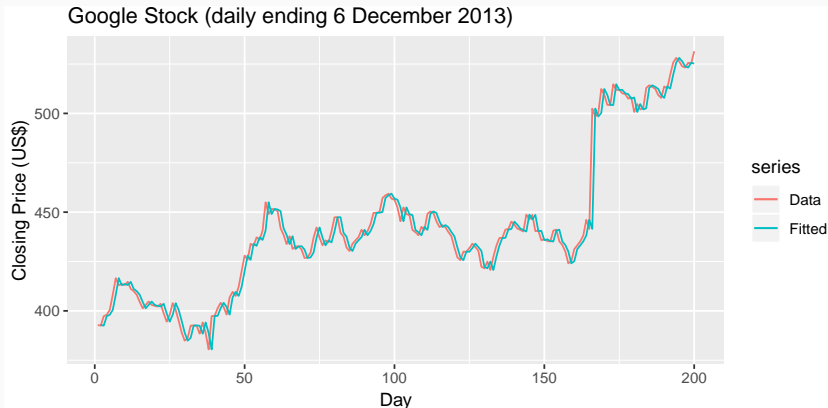
$$\hat{y}_{t|t-1} = y_{t-1}$$

$$e_t = y_t - y_{t-1}$$

Note: e_t are one-step-forecast residuals

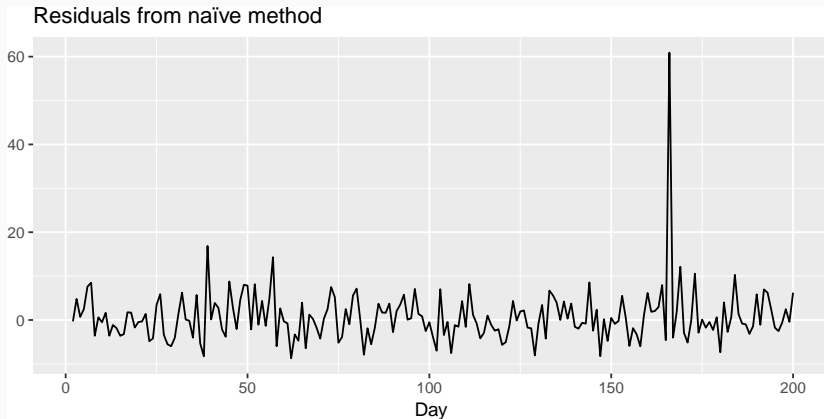
Example: Google stock price

```
fits <- fitted(naive(goog200))  
autoplot(goog200, series="Data") +  
  autolayer(fits, series="Fitted") +  
  xlab("Day") + ylab("Closing Price (US$)") +  
  ggtitle("Google Stock (daily ending 6 December 2013)")
```



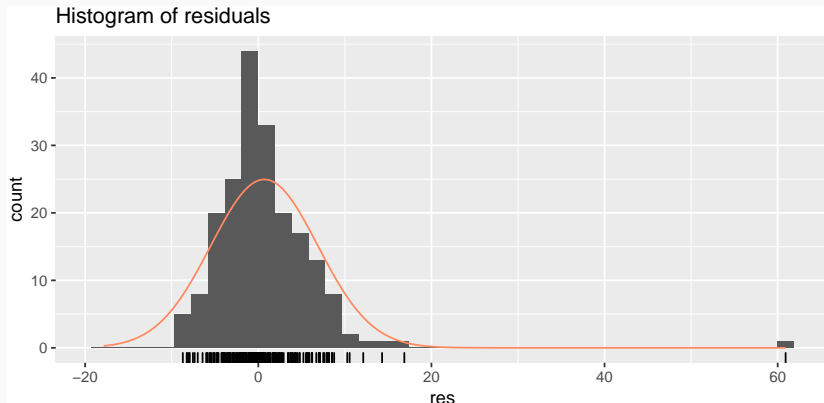
Example: Google stock price

```
res <- residuals(naive(goog200))  
autoplot(res) + xlab("Day") + ylab("") +  
  ggtitle("Residuals from naïve method")
```



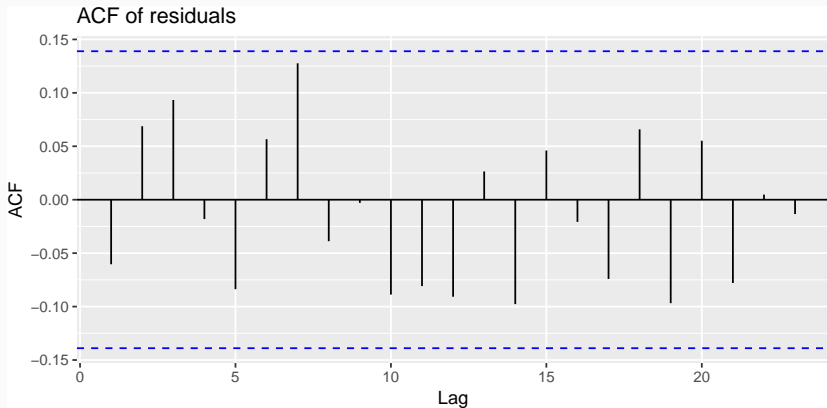
Example: Google stock price

```
gghistogram(res, add.normal=TRUE) +  
  ggtitle("Histogram of residuals")
```



Example: Google stock price

```
ggAcf(res) + ggtitle("ACF of residuals")
```



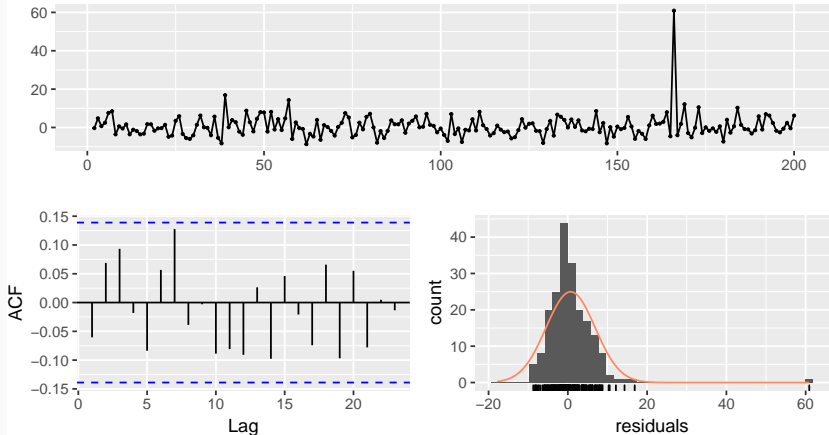
ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.
- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.
- We *expect* these to look like white noise.

checkresiduals function

```
checkresiduals(naive(goog200))
```

Residuals from Naive method



Portmanteau tests

Test whether set of r_k values are significantly different from a zero set.

Ljung-Box test

$$Q^* = T(T+2) \sum_{k=1}^h (T-k)^{-1} r_k^2$$

where h is max lag being considered and T is number of observations.

- My preferences: $h = 10$ for non-seasonal data, $h = 2m$ for seasonal data.
- If each r_k close to zero, Q will be **small**.
- p-value measures probability of results if residuals are WN.

checkresiduals function

```
checkresiduals(naive(goog200))
```

```
##
```

```
##  Ljung-Box  test
```

```
##
```

```
## data:  Residuals from Naive method
```

```
## Q* = 11.088, df = 10, p-value = 0.3507
```

```
## Model df: 0.    Total lags used: 10
```

Outline

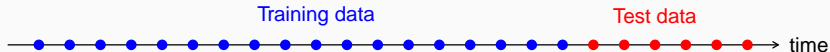
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Lab Session 3

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Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

Forecast errors

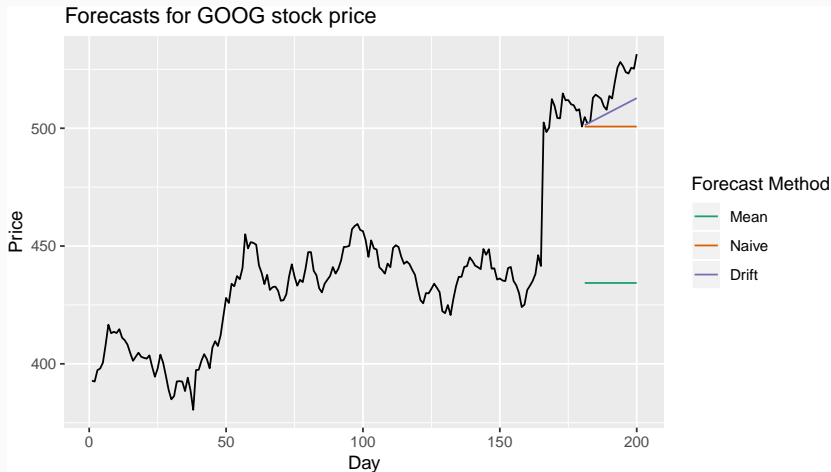
Forecast “error”: the difference between an observed value and its forecast.

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.
- These are *true* forecast errors as the test data is not used in computing $\hat{y}_{T+h|T}$.

Measures of forecast accuracy



Measures of forecast accuracy

y_{T+h} = $(T + h)$ th observation, $h = 1, \dots, H$

$\hat{y}_{T+h|T}$ = its forecast based on data up to time T .

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T}$$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e_{T+h}^2)$$

$$\text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)}$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

Measures of forecast accuracy

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MAE = $\text{mean}(|e_{T+h}|)$

MSE = $\text{mean}(e_{T+h}^2)$

RMSE = $\sqrt{\text{mean}(e_{T+h}^2)}$

MAPE = $100\text{mean}(|e_{T+h}|/|y_{T+h}|)$

- MAE, MSE, RMSE are all scale dependent.
- MAPE is scale independent but is only sensible if $y_t \gg 0$ for all t , and y has a natural zero.

Measures of forecast accuracy

Mean Absolute Scaled Error

$$\text{MASE} = T^{-1} \sum_{t=1}^T |y_t - \hat{y}_{t|t-1}| / Q$$

where Q is a stable measure of the scale of the time series $\{y_t\}$.

Proposed by Hyndman and Koehler (IJF, 2006).

For non-seasonal time series,

$$Q = (T - 1)^{-1} \sum_{t=2}^T |y_t - y_{t-1}|$$

works well. Then MASE is equivalent to MAE relative to a naïve method.

Measures of forecast accuracy

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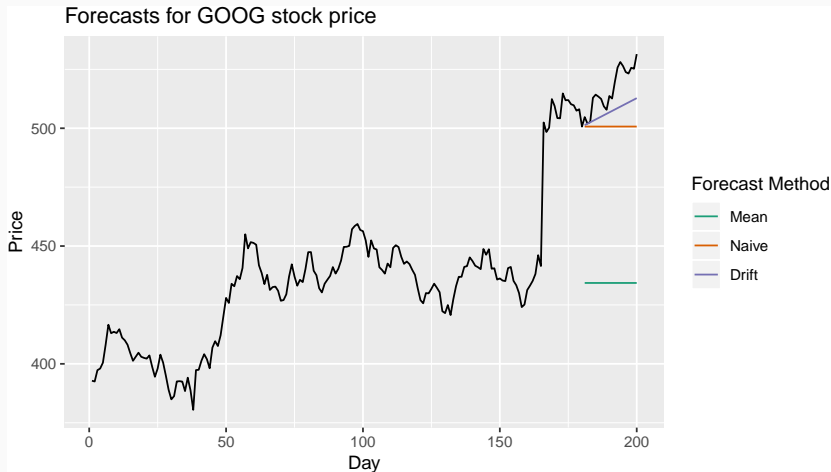
Proposed by Hyndman and Koehler (IJF, 2006).

For seasonal time series,

$$Q = (T - m)^{-1} \sum_{t=m+1}^T |y_t - y_{t-m}|$$

works well. Then MASE is equivalent to MAE relative to a seasonal naïve method.

Measures of forecast accuracy



Measures of forecast accuracy

```
googtrain <- window(goog200, start=1992, end=c(2007,4))  
googfc1 <- meanf(googtrain, h=10)  
googfc2 <- rwf(googtrain, h=10)  
googfc3 <- snaive(googtrain, h=10)  
accuracy(googfc1, goog200)  
accuracy(googfc2, goog200)  
accuracy(googfc3, goog200)
```

	RMSE	MAE	MAPE	MASE
Mean method	82.89	82.43	15.93	21.61
Naïve method	18.29	16.04	3.08	4.21
Drift method	11.34	9.71	1.86	2.55

Poll: true or false?

- 1 Good forecast methods should have normally distributed residuals.
- 2 A model with small residuals will give good forecasts.
- 3 The best measure of forecast accuracy is MAPE.
- 4 If your model doesn't forecast well, you should make it more complicated.
- 5 Always choose the model with the best forecast accuracy as measured on the test set.

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Lab Session 4

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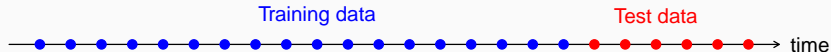
Time series cross-validation

Traditional evaluation

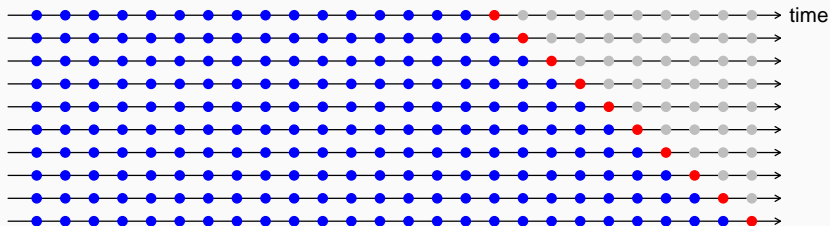


Time series cross-validation

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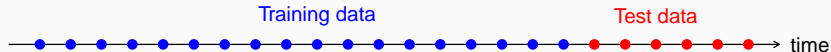


Time series cross-validation

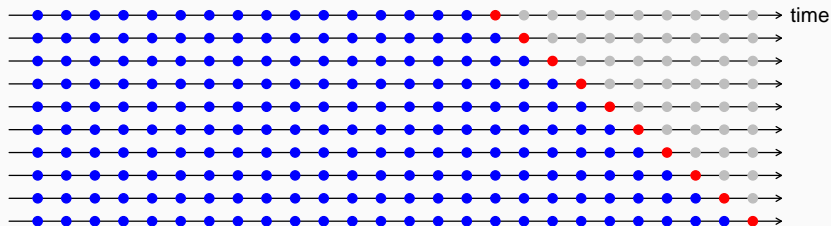


Time series cross-validation

Traditional evaluation



Time series cross-validation



- Forecast accuracy averaged over test sets.
- Also known as “evaluation on a rolling forecasting origin”

tsCV function:

```
e <- tsCV(goog200, rwf, drift=TRUE, h=1)
sqrt(mean(e^2, na.rm=TRUE))
```

```
## [1] 6.233245
```

```
sqrt(mean(residuals(rwf(goog200, drift=TRUE))^2,
          na.rm=TRUE))
```

```
## [1] 6.168928
```

A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.

Pipe function

Ugly code:

```
e <- tsCV(goog200, rwf, drift=TRUE, h=1)
sqrt(mean(e^2, na.rm=TRUE))
sqrt(mean(residuals(rwf(goog200, drift=TRUE))^2,
            na.rm=TRUE))
```

Better with a pipe:

```
goog200 %>%
  tsCV(forecastfunction=rwf, drift=TRUE, h=1) -> e
e^2 %>% mean(na.rm=TRUE) %>% sqrt
goog200 %>% rwf(drift=TRUE) %>% residuals -> res
res^2 %>% mean(na.rm=TRUE) %>% sqrt
```

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Prediction intervals

- A forecast $\hat{y}_{T+h|T}$ is (usually) the mean of the conditional distribution $y_{T+h} \mid y_1, \dots, y_T$.
- A prediction interval gives a region within which we expect y_{T+h} to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

where $\hat{\sigma}_h$ is the st dev of the h -step distribution.

- When $h = 1$, $\hat{\sigma}_h$ can be estimated from the residuals.

Prediction intervals

Drift forecasts with prediction interval:

```
rwf(goog200, level=95, drift=TRUE)
```

##	Point Forecast	Lo 95	Hi 95
## 201	532.1750	520.0232	544.3268
## 202	532.8717	515.6438	550.0996
## 203	533.5684	512.4165	554.7204
## 204	534.2652	509.7810	558.7494
## 205	534.9619	507.5207	562.4031
## 206	535.6586	505.5251	565.7922
## 207	536.3553	503.7285	568.9822
## 208	537.0521	502.0884	572.0158

Prediction intervals

- Point forecasts are often useless without prediction intervals.
- Prediction intervals require a stochastic model (with random errors, etc).
- Multi-step forecasts for time series require a more sophisticated approach (with PI getting wider as the forecast horizon increases).
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.

Prediction intervals

Assume residuals are normal, uncorrelated, $\text{sd} = \hat{\sigma}$:

Mean forecasts: $\hat{\sigma}_h = \hat{\sigma} \sqrt{1 + 1/T}$

Naïve forecasts: $\hat{\sigma}_h = \hat{\sigma} \sqrt{h}$

Seasonal naïve forecasts $\hat{\sigma}_h = \hat{\sigma} \sqrt{k + 1}$

Drift forecasts: $\hat{\sigma}_h = \hat{\sigma} \sqrt{h(1 + h/T)}$.

where k is the integer part of $(h - 1)/m$.

Note that when $h = 1$ and T is large, these all give the same approximate value $\hat{\sigma}$.