MONASH University

MONASH BUSINESS SCHOOL

Forecasting: principles and practice

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2.5 Seasonal ARIMA models

Outline

Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	$\uparrow$	$\uparrow$
	Non-seasonal part of the model	Seasonal part of of the model

where m = number of observations per year.

Seasonal ARIMA models

E.g., ARIMA$(1, 1, 1)(1, 1, 1)_4$ model (without constant)

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Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$\underbrace{(1-\phi_1 B)}_{\text{Non-seasonal AR(1)}}\underbrace{(1-\Phi_1 B^4)}_{\text{Seasonal AR(1)}}\underbrace{(1-B)}_{\text{Non-seasonal difference}}\underbrace{(1-B^4)}_{\text{Seasonal difference}}y_t = \underbrace{(1+\theta_1 B)}_{\text{Non-seasonal MA(1)}}\underbrace{(1+\Theta_1 B^4)}_{\text{Seasonal MA(1)}}\varepsilon_t.$$

Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)$_4$ model (without constant)

$$(1-\phi_1 B)(1-\Phi_1 B^4)(1-B)(1-B^4)y_t \;=\; (1+\theta_1 B)(1+\Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = {} & (1+\phi_1)y_{t-1} - \phi_1 y_{t-2} + (1+\Phi_1)y_{t-4} \\ & - (1+\phi_1+\Phi_1+\phi_1\Phi_1)y_{t-5} + (\phi_1+\phi_1\Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1+\phi_1\Phi_1)y_{t-9} - \phi_1\Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1\varepsilon_{t-1} + \Theta_1\varepsilon_{t-4} + \theta_1\Theta_1\varepsilon_{t-5}. \end{aligned}$$

Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1)$_m$	with log transformation
ARIMA(0,1,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,0)(0,1,1)$_m$	with log transformation
ARIMA(0,2,2)(0,1,1)$_m$	with log transformation
ARIMA(2,1,2)(0,1,1)$_m$	with no transformation

Understanding ARIMA models

Long-term forecasts

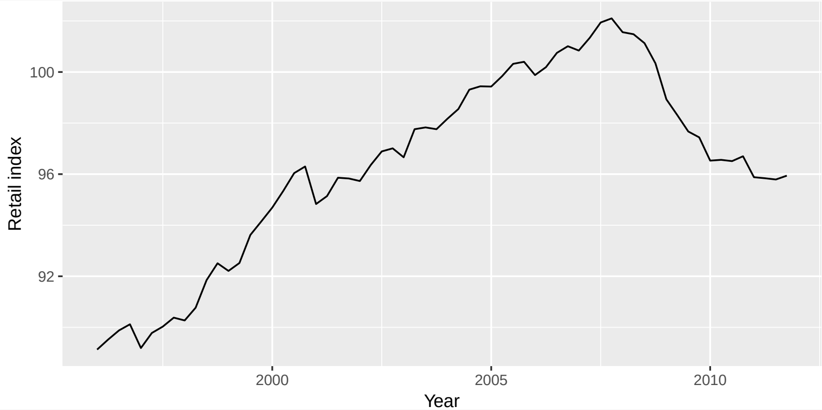
zero	$c = 0, d + D = 0$	
non-zero constant	$c = 0, d + D = 1$	$c \neq 0, d + D = 0$
linear	$c = 0, d + D = 2$	$c \neq 0, d + D = 1$
quadratic	$c = 0, d + D = 3$	$c \neq 0, d + D = 2$

Forecast variance and $d + D$

- The higher the value of $d + D$, the more rapidly the prediction intervals increase in size.
- For $d + D = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

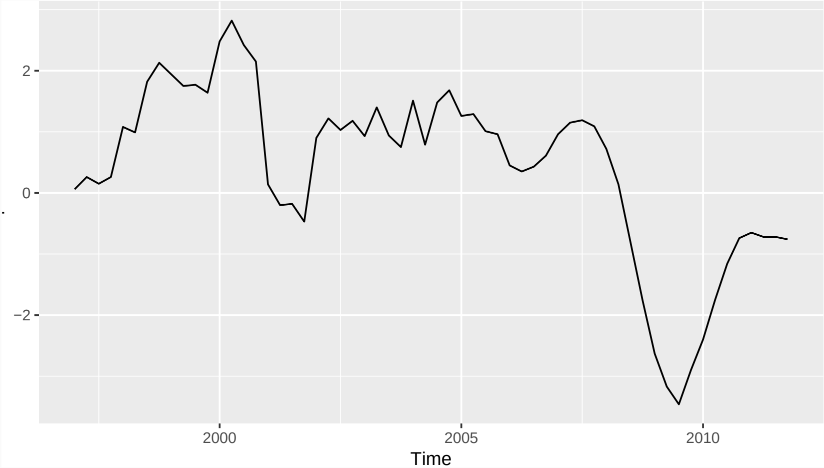
European quarterly retail trade

```r
autoplot(euretail) +
  xlab("Year") + ylab("Retail index")
```



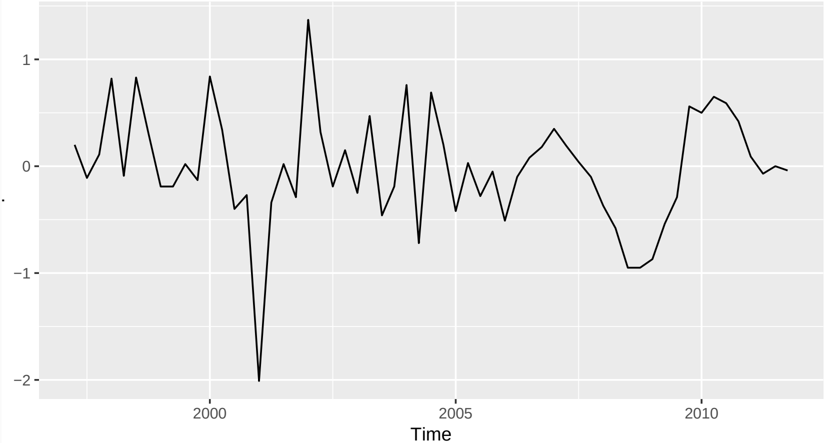
European quarterly retail trade

```r
euretail %>% diff(lag=4) %>% autoplot()
```



European quarterly retail trade

```
euretail %>% diff(lag=4) %>% diff() %>%
  autoplot()
```



European quarterly retail trade

```r
(fit <- auto.arima(euretail))
```

```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1     ma1    ma2    sma1
##        0.736  -0.466  0.216  -0.843
## s.e.   0.224   0.199  0.210   0.188
##
## sigma^2 estimated as 0.159:  log likelihood=-29.6
## AIC=69.2   AICc=70.4   BIC=79.6
```

European quarterly retail trade

```r
(fit <- auto.arima(euretail, stepwise=TRUE,
  approximation=FALSE))
```

```
## Series: euretail
## ARIMA(1,1,2)(0,1,1)[4]
##
## Coefficients:
##          ar1      ma1     ma2     sma1
##        0.736  -0.466  0.216  -0.843
## s.e.   0.224   0.199  0.210   0.188
##
## sigma^2 estimated as 0.159:  log likelihood=-29.6
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```
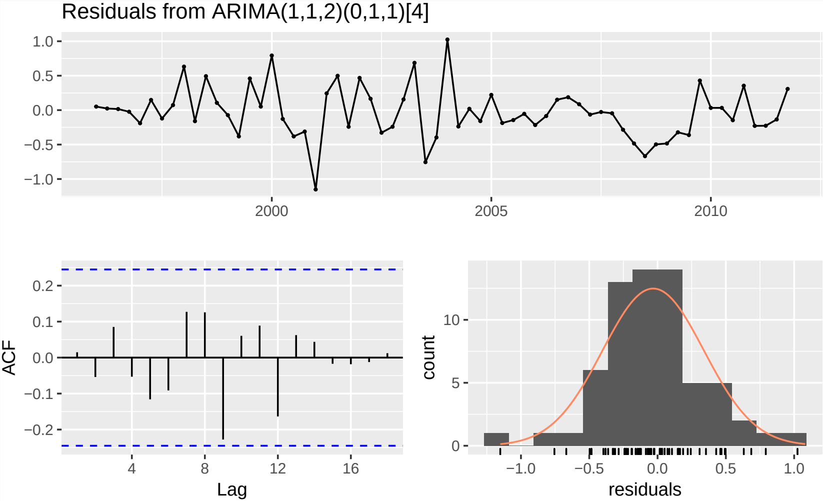

European quarterly retail trade

```r
checkresiduals(fit, test=FALSE)
```



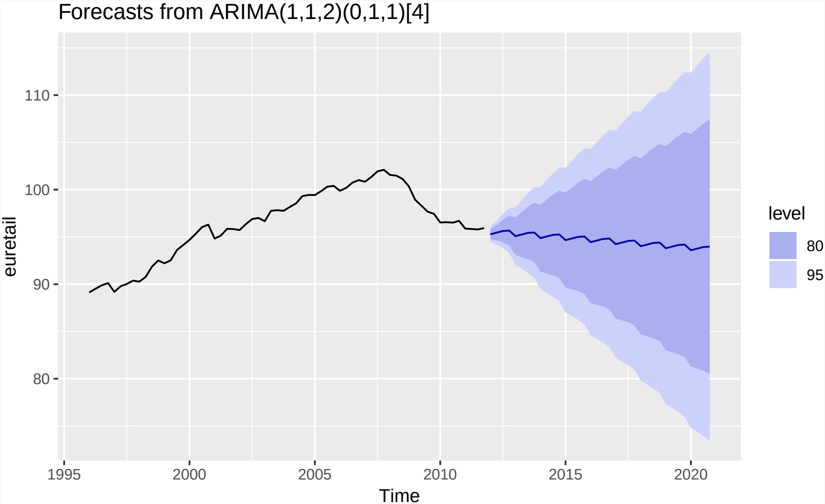
European quarterly retail trade

```r
checkresiduals(fit, plot=FALSE)
```

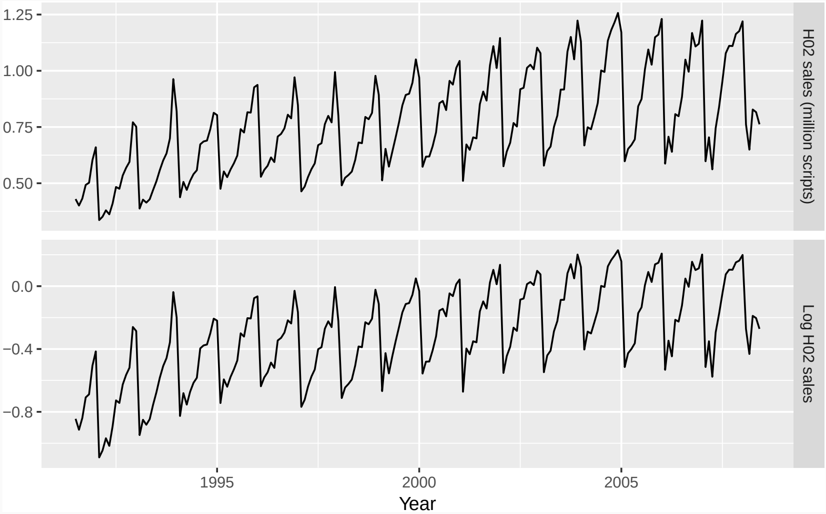
```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(1,1,2)(0,1,1)[4]
## Q* = 5, df = 4, p-value = 0.3
##
## Model df: 4.   Total lags used: 8
```

European quarterly retail trade

```r
forecast(fit, h=36) %>% autoplot()
```

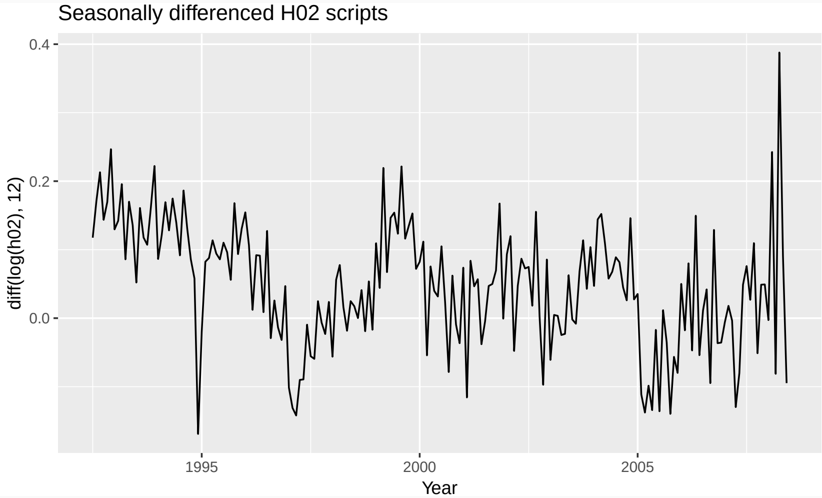


Cortecosteroid drug sales



Cortecosteroid drug sales

```
autoplot(diff(log(h02),12), xlab="Year",
  main="Seasonally differenced H02 scripts")
```



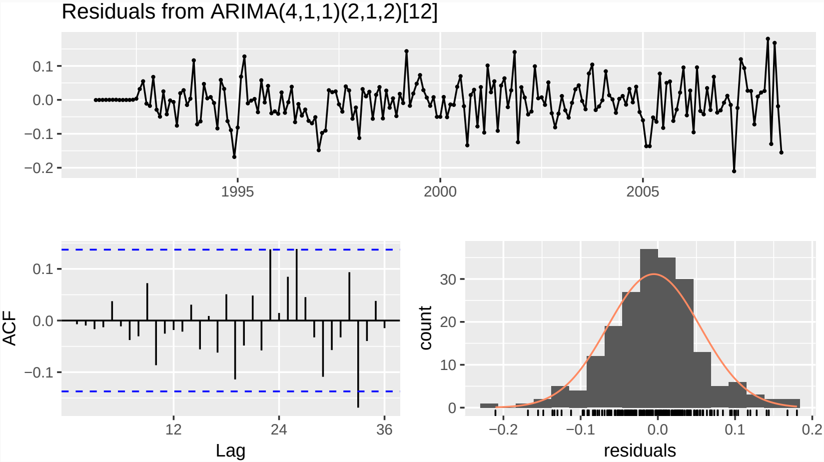
Cortecosteroid drug sales

```
(fit <- auto.arima(h02, lambda=0, max.order=9,
  stepwise=FALSE, approximation=FALSE))
```

```
## Series: h02
## ARIMA(4,1,1)(2,1,2)[12]
## Box Cox transformation: lambda= 0
##
## Coefficients:
##          ar1    ar2    ar3     ar4     ma1   sar1
##       -0.042  0.210  0.202  -0.227  -0.742  0.621
## s.e.   0.217  0.181  0.114   0.081   0.207  0.242
##         sar2    sma1   sma2
##       -0.383  -1.202  0.496
## s.e.   0.118   0.249  0.214
##
```

Cortecosteroid drug sales

```r
checkresiduals(fit)
```



##

Cortecosteroid drug sales

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(4,1,1)(2,1,2)[12]
## Q* = 20, df = 20, p-value = 0.4
##
## Model df: 9.   Total lags used: 24
```

Cortecosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
getrmse <- function(x,h,...)
{
  train.end <- time(x)[length(x)-h]
  test.start <- time(x)[length(x)-h+1]
  train <- window(x,end=train.end)
  test <- window(x,start=test.start)
  fit <- Arima(train,...)
  fc <- forecast(fit,h=h)
  return(accuracy(fc,test)[2,"RMSE"])
}
getrmse(h02,h=24,order=c(3,0,0),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,2),seasonal=c(2,1,0),lambda=0)
getrmse(h02,h=24,order=c(3,0,1),seasonal=c(1,1,0),lambda=0)
```

Cortecosteroid drug sales

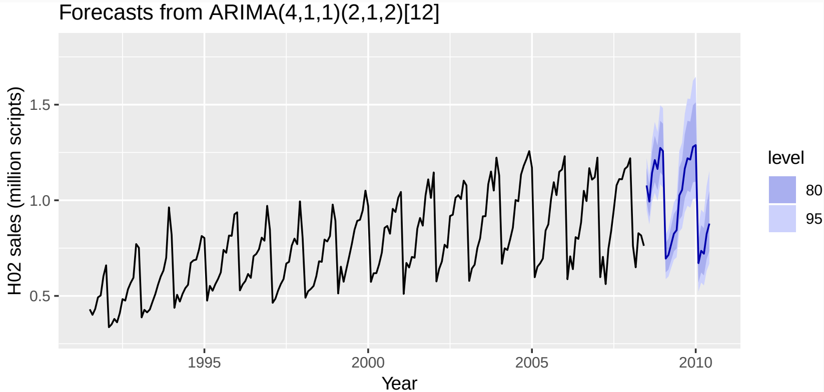
Model	RMSE
ARIMA(4,1,2)(2,1,2)[12]	0.0614
ARIMA(4,1,1)(2,1,2)[12]	0.0615
ARIMA(3,0,1)(0,1,2)[12]	0.0622
ARIMA(3,0,1)(1,1,1)[12]	0.0630
ARIMA(2,1,4)(0,1,1)[12]	0.0632
ARIMA(2,1,3)(0,1,1)[12]	0.0634
ARIMA(4,1,2)(1,1,2)[12]	0.0634
ARIMA(3,1,2)(2,1,2)[12]	0.0636
ARIMA(3,0,3)(0,1,1)[12]	0.0639
ARIMA(2,1,5)(0,1,1)[12]	0.0640
ARIMA(3,0,1)(0,1,1)[12]	0.0644
ARIMA(3,0,2)(0,1,1)[12]	0.0644
ARIMA(3,0,2)(2,1,0)[12]	0.0645

Cortecosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

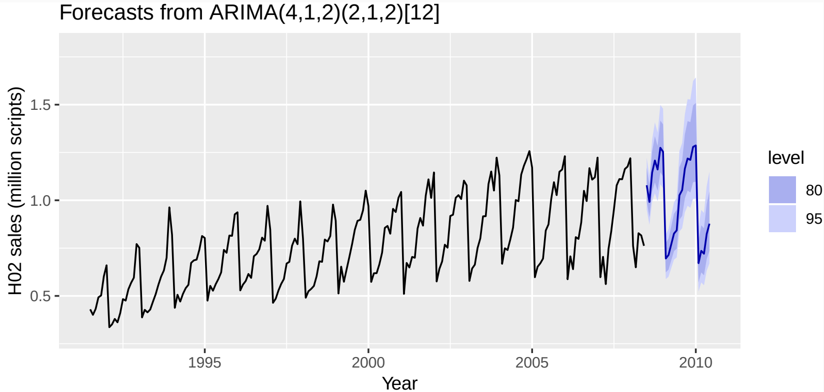
Cortecosteroid drug sales

```
fit <- Arima(h02, order=c(4,1,1), seasonal=c(2,1,2),
  lambda=0)
autoplot(forecast(fit)) + xlab("Year") +
  ylab("H02 sales (million scripts)") + ylim(0.3,1.8)
```



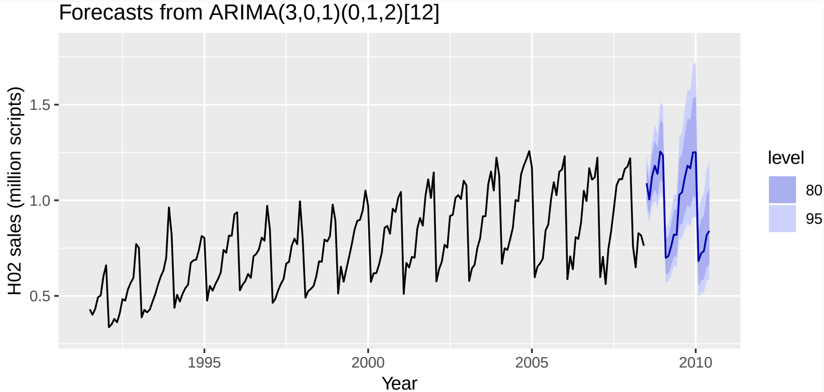
Cortecosteroid drug sales

```
fit <- Arima(h02, order=c(4,1,2), seasonal=c(2,1,2),
  lambda=0)
autoplot(forecast(fit)) + xlab("Year") +
  ylab("H02 sales (million scripts)") + ylim(0.3,1.8)
```



Cortecosteroid drug sales

```
fit <- Arima(h02, order=c(3,0,1), seasonal=c(0,1,2),
  lambda=0)
autoplot(forecast(fit)) + xlab("Year") +
  ylab("H02 sales (million scripts)") + ylim(0.3,1.8)
```



Outline

ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$
		$\theta_2 = 1 - \alpha$
ETS(A,A,N)	ARIMA(1,1,2)	$\phi_1 = \phi$
		$\theta_1 = \alpha + \phi\beta - 1 - \phi$
		$\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0)$_m$	
ETS(A,A,A)	ARIMA(0,1,$m+1$)(0,1,0)$_m$	
ETS(A,A,A)	ARIMA(1,0,$m+1$)(0,1,0)$_m$	

Outline

Lab Session 16