

Forecasting: principles and practice

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1.4 Exponential smoothing

Outline

- 1 Simple exponential smoothing
- 2 Trend methods
- 3 Lab session 7
- 4 Seasonal methods
- 5 Lab session 8
- 6 Taxonomy of exponential smoothing methods

Simple methods

Time series y_1, y_2, \dots, y_T .

Random walk forecasts

$$\hat{y}_{T+h|T} = y_T$$

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Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T} \sum_{t=1}^T y_t$$

Simple methods

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Average forecasts

$$\hat{y}_{T+h|T} = \frac{1}{T} \sum_{t=1}^T y_t$$

- Want something in between that weights most recent data more highly.
- Simple exponential smoothing uses a weighted moving average with weights that decrease exponentially.

Simple Exponential Smoothing

Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1 - \alpha)y_{T-1} + \alpha(1 - \alpha)^2 y_{T-2} + \dots$$

where $0 \leq \alpha \leq 1$.

Simple Exponential Smoothing

Forecast equation

$$\hat{y}_{T+1|T} = \alpha y_T + \alpha(1 - \alpha)y_{T-1} + \alpha(1 - \alpha)^2 y_{T-2} + \dots$$

where $0 \leq \alpha \leq 1$.

Weights assigned to observations for:				
Observation	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
y_T	0.2	0.4	0.6	0.8
y_{T-1}	0.16	0.24	0.24	0.16
y_{T-2}	0.128	0.144	0.096	0.032
y_{T-3}	0.1024	0.0864	0.0384	0.0064
y_{T-4}	$(0.2)(0.8)^4$	$(0.4)(0.6)^4$	$(0.6)(0.4)^4$	$(0.8)(0.2)^4$
y_{T-5}	$(0.2)(0.8)^5$	$(0.4)(0.6)^5$	$(0.6)(0.4)^5$	$(0.8)(0.2)^5$

Simple Exponential Smoothing

Component form

Forecast equation $\hat{y}_{t+h|t} = \ell_t$

Smoothing equation $\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$

- ℓ_t is the level (or the smoothed value) of the series at time t .
- $\hat{y}_{t+1|t} = \alpha y_t + (1 - \alpha)\hat{y}_{t|t-1}$
Iterate to get exponentially weighted moving average form.

Weighted average form

$$\hat{y}_{T+1|T} = \sum_{j=0}^{T-1} \alpha(1 - \alpha)^j y_{T-j} + (1 - \alpha)^T \ell_0$$

Optimisation

- Need to choose value for α and ℓ_0
- Similarly to regression — we choose α and ℓ_0 by minimising SSE:

$$\text{SSE} = \sum_{t=1}^T (y_t - \hat{y}_{t|t-1})^2.$$

- Unlike regression there is no closed form solution — use numerical optimization.

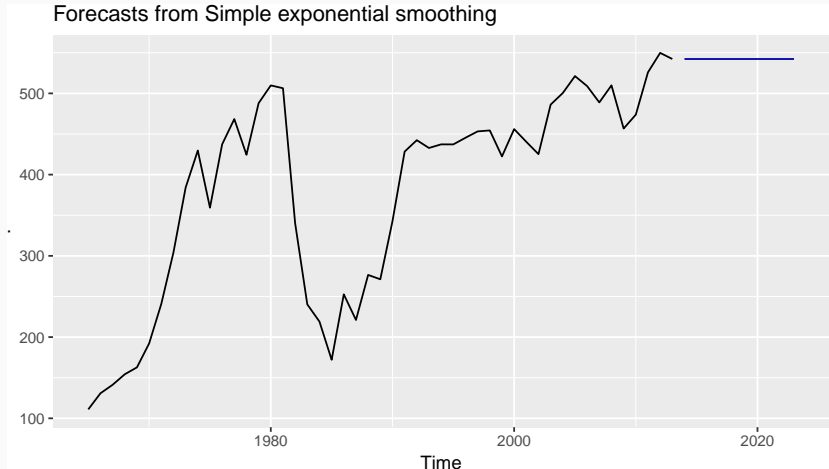
Example: Oil production

```
fc <- ses(oil, h=5)
summary(fc[["model"]])
```

```
## Simple exponential smoothing
##
## Call:
##   ses(y = oil, h = 5)
##
##   Smoothing parameters:
##     alpha = 0.9999
##
##   Initial states:
##     l = 110.8832
##
##   sigma:  49.05
##
##   AIC  AICc  BIC
## 576.2 576.7 581.8
```

Example: Oil production

```
oil %>% ses(PI=FALSE) %>% autoplot
```



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Holt's linear trend

Component form

Forecast $\hat{y}_{t+h|t} = \ell_t + hb_t$

Level $\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$

Trend $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},$

Holt's linear trend

Component form

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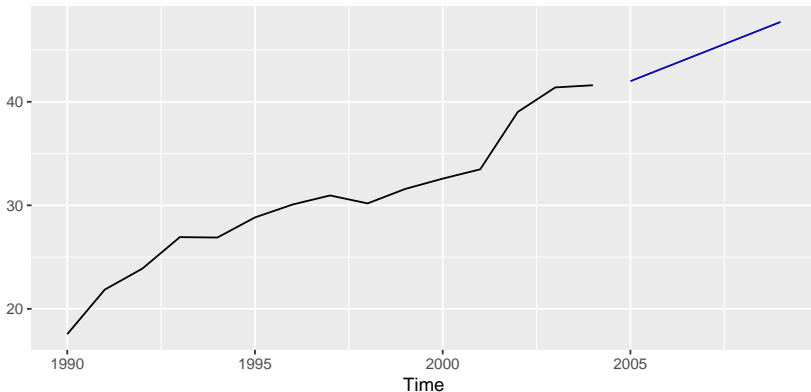
Trend $b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1},$

- Two smoothing parameters α and β^*
($0 \leq \alpha, \beta^* \leq 1$).
- ℓ_t level: weighted average between y_t and one-step ahead forecast for time t ,
($\ell_{t-1} + b_{t-1} = \hat{y}_{t|t-1}$)
- b_t slope: weighted average of $(\ell_t - \ell_{t-1})$ and b_{t-1} , current and previous estimate of slope.

Holt's method in R

```
window(ausair, start=1990, end=2004) %>%  
  holt(h=5, PI=FALSE) %>%  
  autoplot()
```

Forecasts from Holt's method



Damped trend method

Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

Damped trend method

Component form

$$\hat{y}_{t+h|t} = \ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t$$

$$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}.$$

- Damping parameter $0 < \phi < 1$.
- If $\phi = 1$, identical to Holt's linear trend.
- As $h \rightarrow \infty$, $\hat{y}_{T+h|T} \rightarrow \ell_T + \phi b_T / (1 - \phi)$.
- Short-run forecasts trended, long-run forecasts constant.

Example: Sheep in Asia

```
livestock2 <- window(livestock, start=1970, end=2000)
fc1 <- ses(livestock2)
fc2 <- holt(livestock2)
fc3 <- holt(livestock2, damped = TRUE)
```

```
accuracy(fc1, livestock)
```

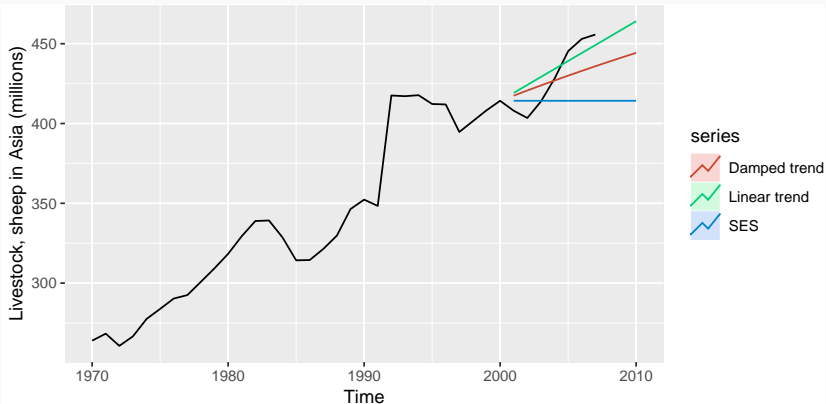
```
accuracy(fc2, livestock)
```

```
accuracy(fc3, livestock)
```

	SES	Linear trend	Damped trend
Test RMSE	25.46	11.88	14.73
Test MAE	20.38	10.71	13.30
Test MAPE	4.60	2.54	3.07
Test MASE	2.26	1.19	1.48

Example: Sheep in Asia

```
autoplot(window(livestock, start=1970)) +  
  autolayer(fc1, series="SES", PI=FALSE) +  
  autolayer(fc2, series="Linear trend", PI=FALSE) +  
  autolayer(fc3, series="Damped trend", PI=FALSE) +  
  ylab("Livestock, sheep in Asia (millions)")
```



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Lab Session 7

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Holt-Winters additive method

Holt and Winters extended Holt's method to capture seasonality.

Component form

$$\hat{y}_{t+h|t} = \ell_t + hb_t + s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$$

$$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m},$$

- $k = \text{integer part of } (h - 1)/m$. Ensures estimates from the final year are used for forecasting.
- Parameters: $0 \leq \alpha \leq 1$, $0 \leq \beta^* \leq 1$, $0 \leq \gamma \leq 1 - \alpha$ and $m = \text{period of seasonality (e.g. } m = 4 \text{ for quarterly data)}$.

Holt-Winters multiplicative method

For when seasonal variations are changing proportional to the level of the series.

Component form

$$\hat{y}_{t+h|t} = (\ell_t + hb_t)s_{t+h-m(k+1)}.$$

$$\ell_t = \alpha \frac{y_t}{s_{t-m}} + (1 - \alpha)(\ell_{t-1} + b_{t-1})$$

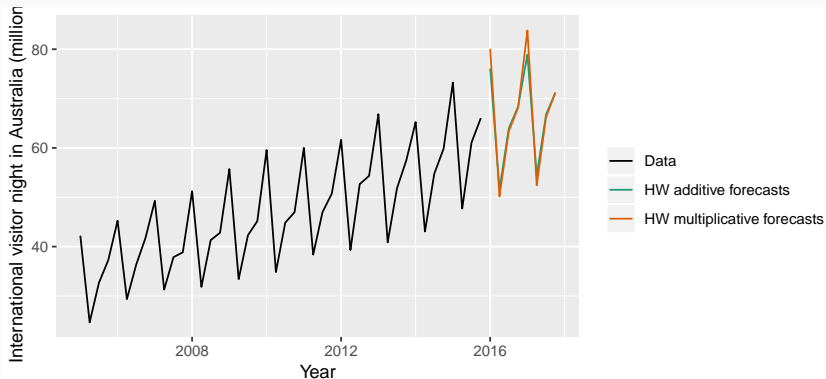
$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + b_{t-1})} + (1 - \gamma)s_{t-m}$$

- k is integer part of $(h - 1)/m$.
- **Additive:** s_t in absolute terms: within each year $\sum_i s_i \approx 0$.
- **Multiplicative:** s_t in relative terms: within each year $\sum_i s_i \approx m$.

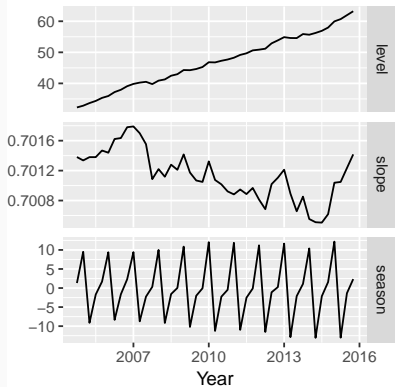
Example: Visitor Nights

```
aust <- window(austourists,start=2005)
fc1 <- hw(aust,seasonal="additive")
fc2 <- hw(aust,seasonal="multiplicative")
```

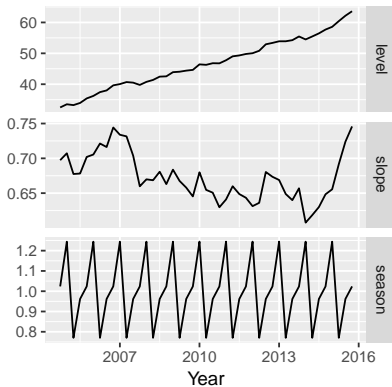


Estimated components

Additive states



Multiplicative states



Holt-Winters damped method

Often the single most accurate forecasting method for seasonal data:

$$\hat{y}_{t+h|t} = [\ell_t + (\phi + \phi^2 + \dots + \phi^h)b_t]s_{t+h-m(k+1)}$$

$$\ell_t = \alpha(y_t/s_{t-m}) + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$$

$$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$$

$$s_t = \gamma \frac{y_t}{(\ell_{t-1} + \phi b_{t-1})} + (1 - \gamma)s_{t-m}$$

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Lab Session 8

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Exponential smoothing methods

Trend Component		Seasonal Component		
		N (None)	A (Additive)	M (Multiplicative)
N	(None)	(N,N)	(N,A)	(N,M)
A	(Additive)	(A,N)	(A,A)	(A,M)
A _d	(Additive damped)	(A _d ,N)	(A _d ,A)	(A _d ,M)

(N,N): Simple exponential smoothing

(A,N): Holt's linear method

(A_d,N): Additive damped trend method

(A,A): Additive Holt-Winters' method

(A,M): Multiplicative Holt-Winters' method

(A_d,M): Damped multiplicative Holt-Winters' method

Recursive formulae

Trend	Seasonal		
	N	A	M
N	$\hat{y}_{t+h t} = \ell_t$	$\hat{y}_{t+h t} = \ell_t + s_{t-m+h_m^+}$	$\hat{y}_{t+h t} = \ell_t s_{t-m+h_m^+}$
	$\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$	$\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)\ell_{t-1}$	$\ell_t = \alpha(y_t/s_{t-m}) + (1 - \alpha)\ell_{t-1}$
		$s_t = \gamma(y_t - \ell_{t-1}) + (1 - \gamma)s_{t-m}$	$s_t = \gamma(y_t/\ell_{t-1}) + (1 - \gamma)s_{t-m}$
A	$\hat{y}_{t+h t} = \ell_t + hb_t$	$\hat{y}_{t+h t} = \ell_t + hb_t + s_{t-m+h_m^+}$	$\hat{y}_{t+h t} = (\ell_t + hb_t)s_{t-m+h_m^+}$
	$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + b_{t-1})$	$\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$	$\ell_t = \alpha(y_t/s_{t-m}) + (1 - \alpha)(\ell_{t-1} + b_{t-1})$
	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)b_{t-1}$
		$s_t = \gamma(y_t - \ell_{t-1} - b_{t-1}) + (1 - \gamma)s_{t-m}$	$s_t = \gamma(y_t/(\ell_{t-1} - b_{t-1})) + (1 - \gamma)s_{t-m}$
Ad	$\hat{y}_{t+h t} = \ell_t + \phi_h b_t$	$\hat{y}_{t+h t} = \ell_t + \phi_h b_t + s_{t-m+h_m^+}$	$\hat{y}_{t+h t} = (\ell_t + \phi_h b_t)s_{t-m+h_m^+}$
	$\ell_t = \alpha y_t + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$	$\ell_t = \alpha(y_t - s_{t-m}) + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$	$\ell_t = \alpha(y_t/s_{t-m}) + (1 - \alpha)(\ell_{t-1} + \phi b_{t-1})$
	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$	$b_t = \beta^*(\ell_t - \ell_{t-1}) + (1 - \beta^*)\phi b_{t-1}$
		$s_t = \gamma(y_t - \ell_{t-1} - \phi b_{t-1}) + (1 - \gamma)s_{t-m}$	$s_t = \gamma(y_t/(\ell_{t-1} - \phi b_{t-1})) + (1 - \gamma)s_{t-m}$

R functions

- Simple exponential smoothing: no trend.

```
ses(y)
```

- Holt's method: linear trend.

```
holt(y)
```

- Damped trend method.

```
holt(y, damped=TRUE)
```

- Holt-Winters methods

```
hw(y, damped=TRUE, seasonal="additive")
```

```
hw(y, damped=FALSE, seasonal="additive")
```

```
hw(y, damped=TRUE, seasonal="multiplicative")
```

```
hw(y, damped=FALSE, seasonal="multiplicative")
```

- Combination of no trend with seasonality not possible using these functions.