MONASH University

MONASH BUSINESS SCHOOL

Forecasting: principles and practice

Rob J Hyndman

3.1 Dynamic regression

Outline

Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \ldots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

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Example: ARIMA(1,1,1) errors

$$\begin{aligned} y_t &= \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t, \\ (1-\phi_1 B)(1-B)\eta_t &= (1+\theta_1 B)\varepsilon_t, \end{aligned}$$

where ε_t is white noise.

Residuals and errors

Example: η_t = ARIMA(1,1,1)

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
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Residuals and errors

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$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing η_t from ε_t.
- Only the errors η_t are assumed to be white noise.
- In ordinary regression, η_t is assumed to be white noise and so $\eta_t = \varepsilon_t$.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

1. Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
2. Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
3. *p*-values for coefficients usually too small ("spurious regression").
4. AIC of fitted models misleading.

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3. *p*-values for coefficients usually too small ("spurious regression").
4. AIC of fitted models misleading.

- Minimizing $\sum \varepsilon_t^2$ avoids these problems.
- Maximizing likelihood is similar to minimizing $\sum \varepsilon_t^2$.

Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

where η_t is an ARMA process.

- If we estimate the model while any variable is non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables to preserve interpretability.

Regression with ARIMA errors

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Original data

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$$\text{where} \quad \phi(B)(1 - B)^d \eta_t = \theta(B)\varepsilon_t$$

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$$\text{where} \quad \phi(B)(1-B)^d \eta_t = \theta(B)\varepsilon_t$$

After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t.$$

$$\text{where} \quad \phi(B)\eta_t = \theta(B)\varepsilon_t$$

$$\text{and} \quad y'_t = (1-B)^d y_t$$

Model selection

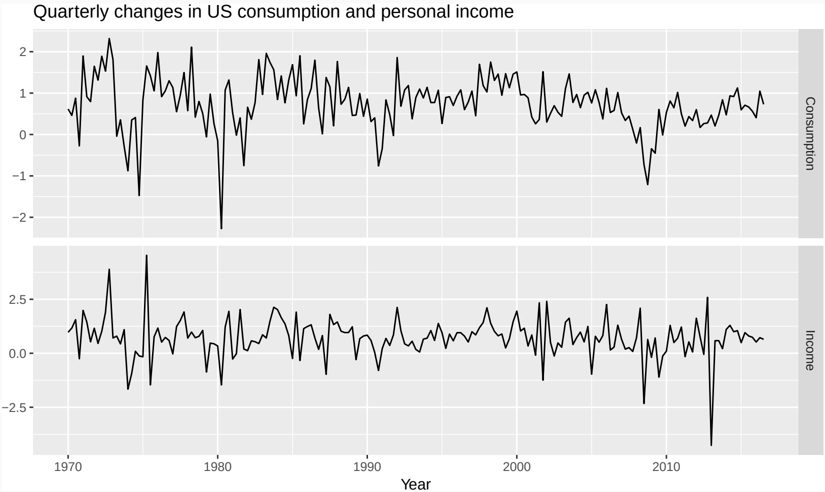
- Fit regression model with automatically selected ARIMA errors.
- Check that ε_t series looks like white noise.

Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

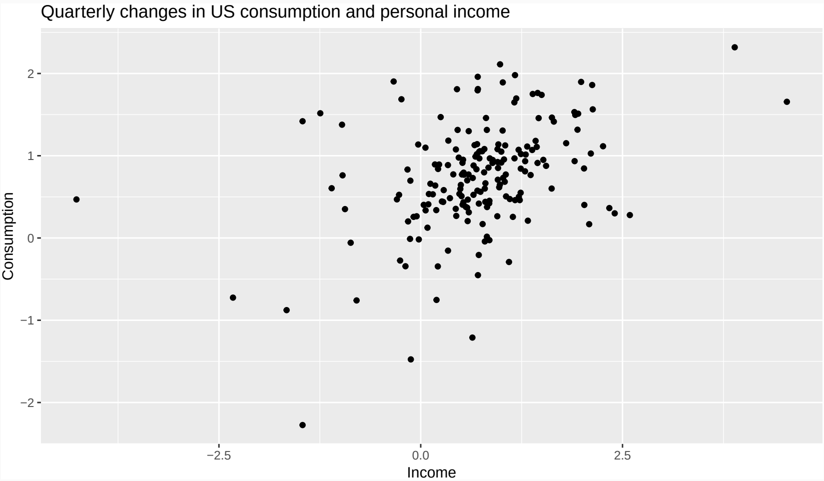
US personal consumption and income

```
autoplot(uschange[,1:2], facets=TRUE) +
  xlab("Year") + ylab("") +
  ggtitle("Quarterly changes in US consumption and personal income")
```



US personal consumption and income

```
qplot(Income,Consumption, data=as.data.frame(uschange)) +
  ggtitle("Quarterly changes in US consumption and personal income")
```



US personal consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

US personal consumption and income

```r
(fit <- auto.arima(uschange[,1], xreg=uschange[,2]))
```

```
## Series: uschange[, 1]
## Regression with ARIMA(1,0,2) errors
##
## Coefficients:
##          ar1      ma1    ma2  intercept   xreg
##        0.692  -0.576  0.198      0.599  0.203
## s.e.  0.116   0.130  0.076      0.088  0.046
##
## sigma^2 estimated as 0.322:  log likelihood=-156.9
## AIC=325.9   AICc=326.4   BIC=345.3
```

US personal consumption and income

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```

Write down the equations for the fitted model.

US personal consumption and income

```r
checkresiduals(fit, test=FALSE)
```

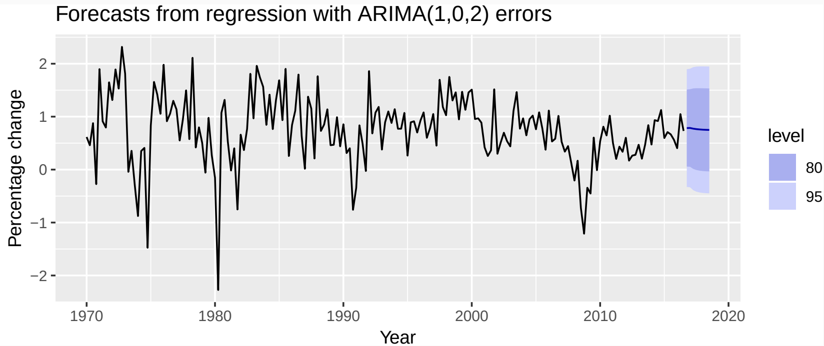
US personal consumption and income

```r
checkresiduals(fit, plot=FALSE)
```

```
##
##  Ljung-Box test
##
## data:  Residuals from Regression with ARIMA(1,0,2) errors
## Q* = 5.9, df = 3, p-value = 0.1
##
## Model df: 5.   Total lags used: 8
```

US personal consumption and income

```
fcast <- forecast(fit,
  xreg=rep(mean(uschange[,2]),8), h=8)
autoplot(fcast) + xlab("Year") +
  ylab("Percentage change") +
  ggtitle("Forecasts from regression with ARIMA(1,0,2) errors")
```



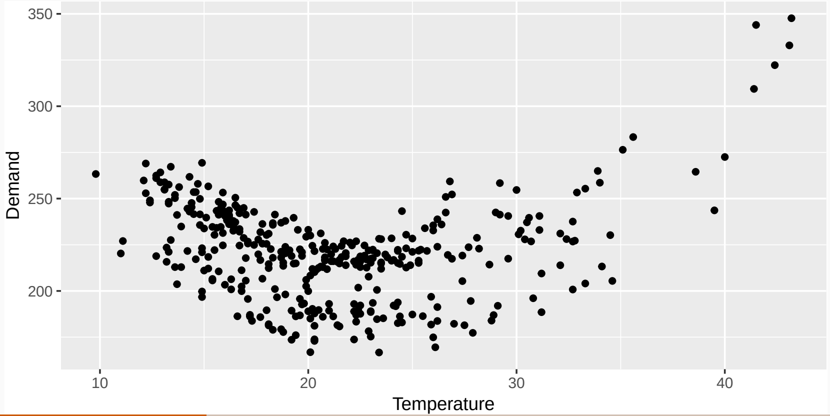
Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

Daily electricity demand

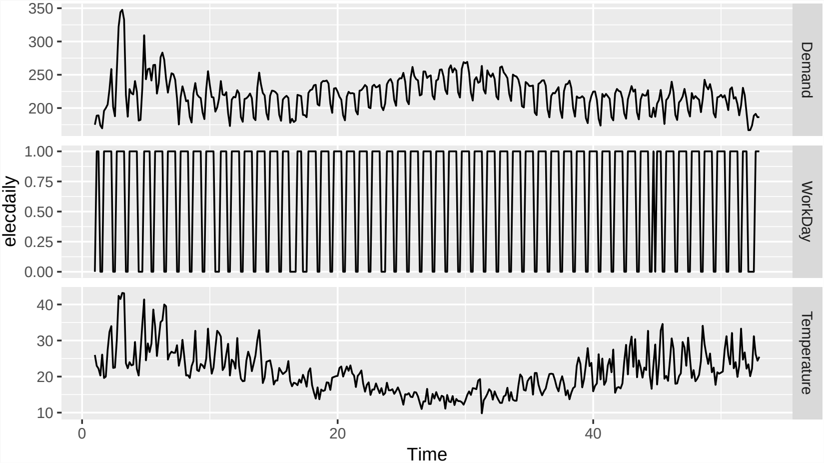
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
qplot(elecdaily[,"Temperature"], elecdaily[,"Demand"]) +
  xlab("Temperature") + ylab("Demand")
```



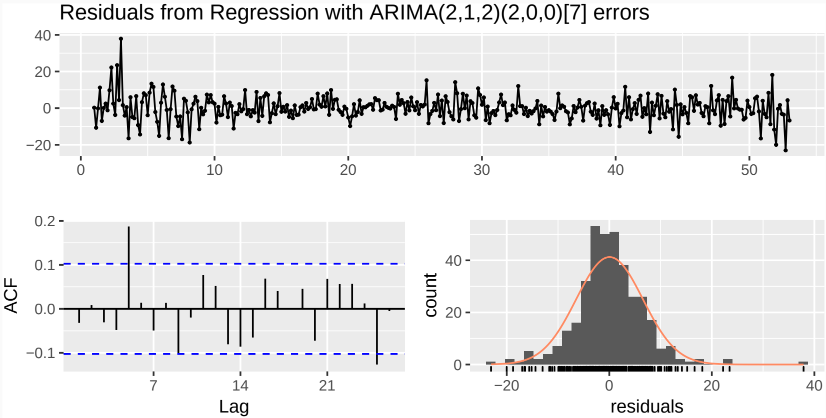
Daily electricity demand

```r
autoplot(elecdaily, facets = TRUE)
```



Daily electricity demand

```
xreg <- cbind(MaxTemp = elecdaily[, "Temperature"],
              MaxTempSq = elecdaily[, "Temperature"]^2,
              Workday = elecdaily[, "WorkDay"])
fit <- auto.arima(elecdaily[, "Demand"], xreg = xreg)
checkresiduals(fit)
```



Daily electricity demand

```
##
##  Ljung-Box test
##
## data:  Residuals from Regression with ARIMA(2,1,2)(2,0,0)[7] errors
## Q* = 28, df = 4, p-value = 1e-05
##
## Model df: 10.   Total lags used: 14
```

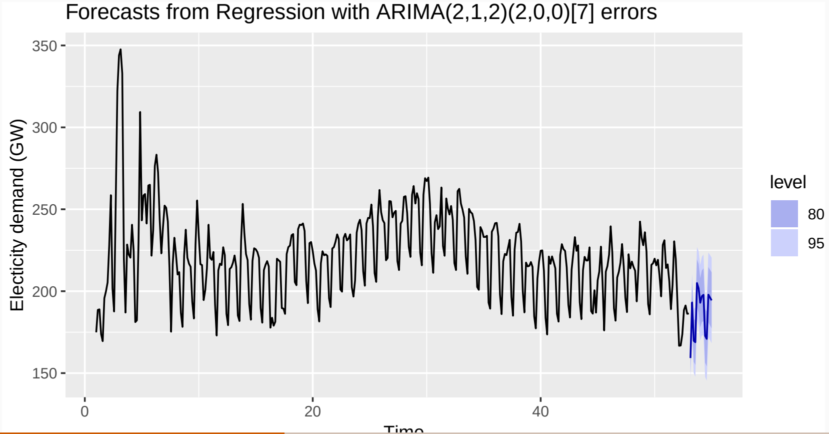
Daily electricity demand

```r
# Forecast one day ahead
forecast(fit, xreg = cbind(26, 26^2, 1))
```

```
##       Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
## 53.14          189.8 181.3 198.2 176.8 202.7
```

Daily electricity demand

```
fcast <- forecast(fit,
  xreg = cbind(rep(26,14), rep(26^2,14),
    c(0,1,0,0,1,1,1,1,1,0,0,1,1,1)))
autoplot(fcast) + ylab("Electicity demand (GW)")
```



Outline

Trend

Linear trend

$$x_t = t$$

- $t = 1, 2, \ldots, T$
- Strong assumption that trend will continue.

Dummy variables

If a categorical variable takes only two values (e.g., ‘Yes’ or ‘No’), then an equivalent numerical variable can be constructed taking value 1 if yes and 0 if no. This is called a **dummy variable**.

	A	B
1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	No	0
7	Yes	1
8	Yes	1
9	No	0
10	No	0
11	No	0
12	No	0
13	Yes	1
14	No	0

Dummy variables

If there are more than two categories, then the variable can be coded using several dummy variables (one fewer than the total number of categories).

	A	B	C	D	E
1	Monday	1	0	0	0
2	Tuesday	0	1	0	0
3	Wednesday	0	0	1	0
4	Thursday	0	0	0	1
5	Friday	0	0	0	0
6	Monday	1	0	0	0
7	Tuesday	0	1	0	0
8	Wednesday	0	0	1	0
9	Thursday	0	0	0	1
10	Friday	0	0	0	0
11	Monday	1	0	0	0
12	Tuesday	0	1	0	0
13	Wednesday	0	0	1	0
14	Thursday	0	0	0	1
15	Friday	0	0	0	0

Beware of the dummy variable trap!

- Using one dummy for each category gives too many dummy variables!
- The regression will then be singular and inestimable.
- Either omit the constant, or omit the dummy for one category.
- The coefficients of the dummies are relative to the omitted category.

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

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Outliers

- If there is an outlier, you can use a dummy variable (taking value 1 for that observation and 0 elsewhere) to remove its effect.

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Public holidays

- For daily data: if it is a public holiday, dummy=1, otherwise dummy=0.

Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K} [\alpha_k s_k(t) + \beta_k c_k(t)] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.
- Called “harmonic regression”
- `fourier()` function generates these.

Intervention variables

Spikes

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Steps

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Steps

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Change of slope

- Variables take values 0 before the intervention and values $\{1, 2, 3, \dots\}$ afterwards.

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese new year similar.

Trading days

With monthly data, if the observations vary depending on how many different types of days in the month, then trading day predictors can be useful.

$$
\begin{aligned}
z_1 &= \text{\# Mondays in month;} \\
z_2 &= \text{\# Tuesdays in month;} \\
&\vdots \\
z_7 &= \text{\# Sundays in month.}
\end{aligned}
$$

Distributed lags

Lagged values of a predictor.

Example: x is advertising which has a delayed effect

$$
\begin{aligned}
x_1 &= \text{advertising for previous month;} \\
x_2 &= \text{advertising for two months previously;} \\
&\vdots \\
x_m &= \text{advertising for } m \text{ months previously.}
\end{aligned}
$$

Outline

Dynamic harmonic regression

Combine Fourier terms with ARIMA errors

Advantages

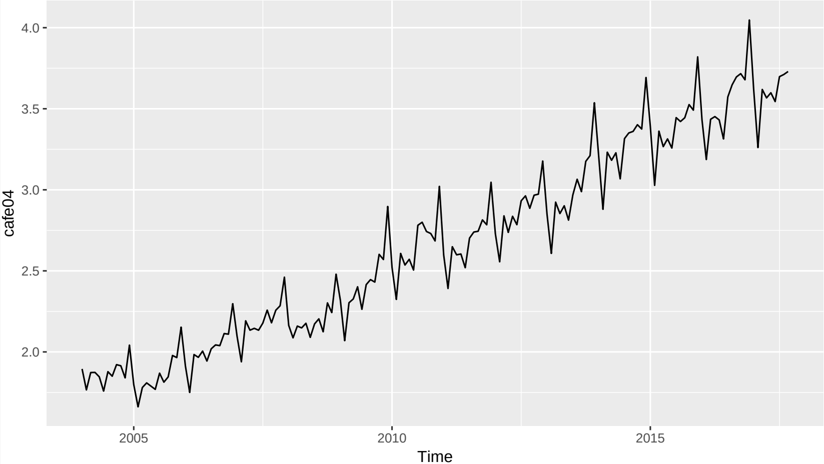
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of K (but more wiggly seasonality can be handled by increasing K);
- the short-term dynamics are easily handled with a simple ARMA error.

Disadvantages

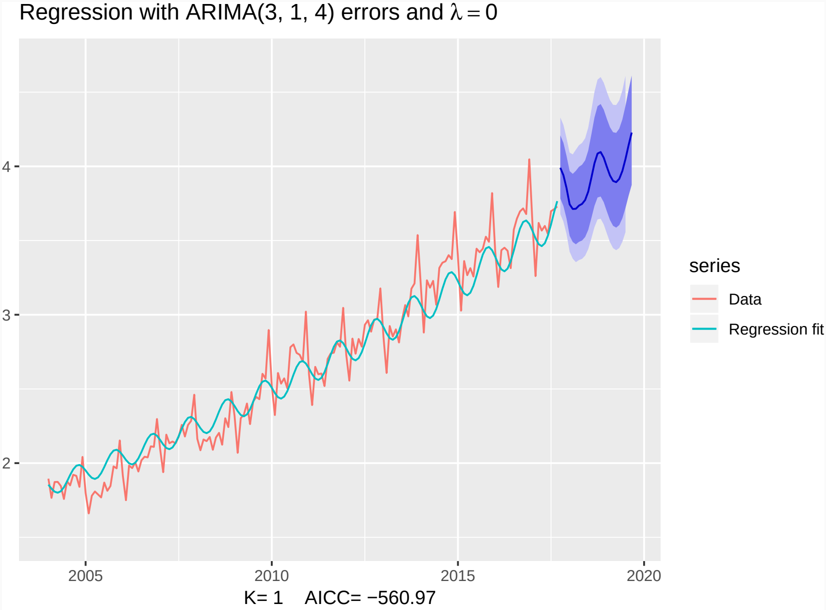
- seasonality is assumed to be fixed

Eating-out expenditure

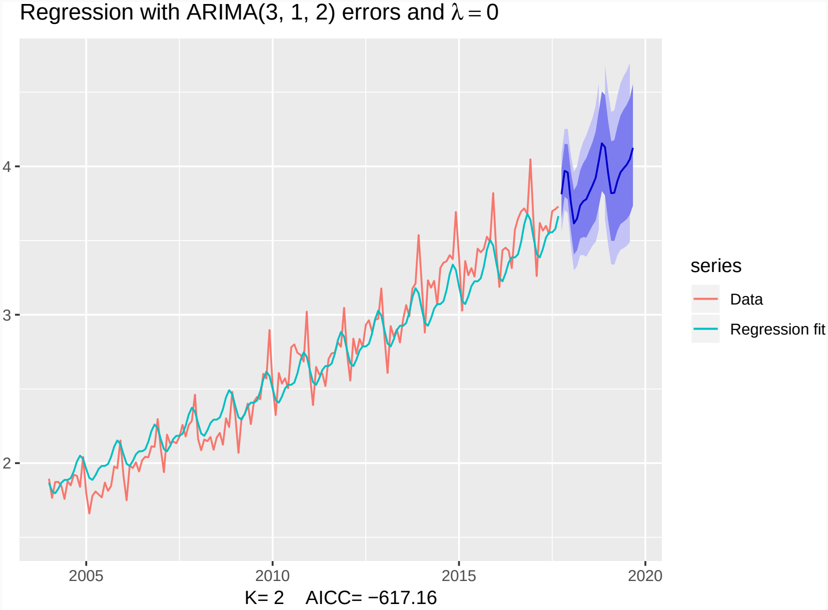
```r
cafe04 <- window(auscafe, start=2004)
autoplot(cafe04)
```



Eating-out expenditure



Eating-out expenditure



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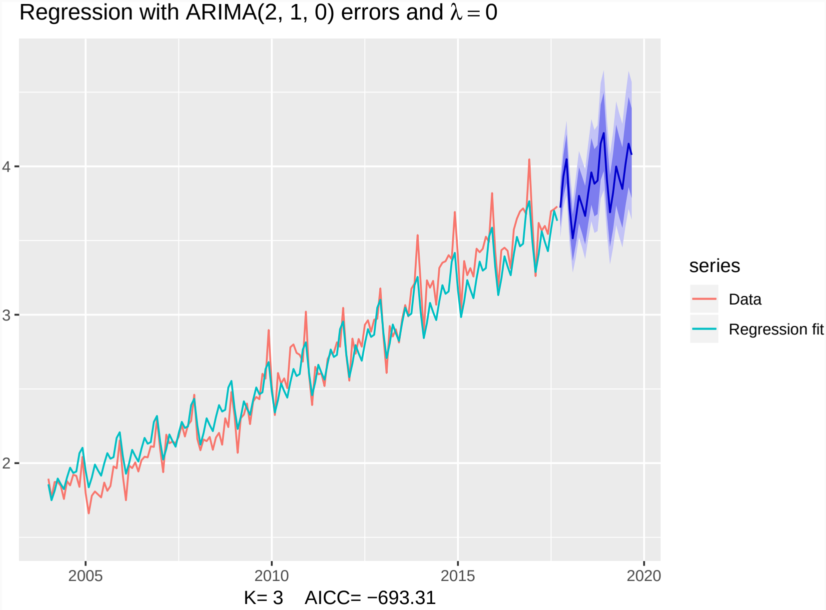

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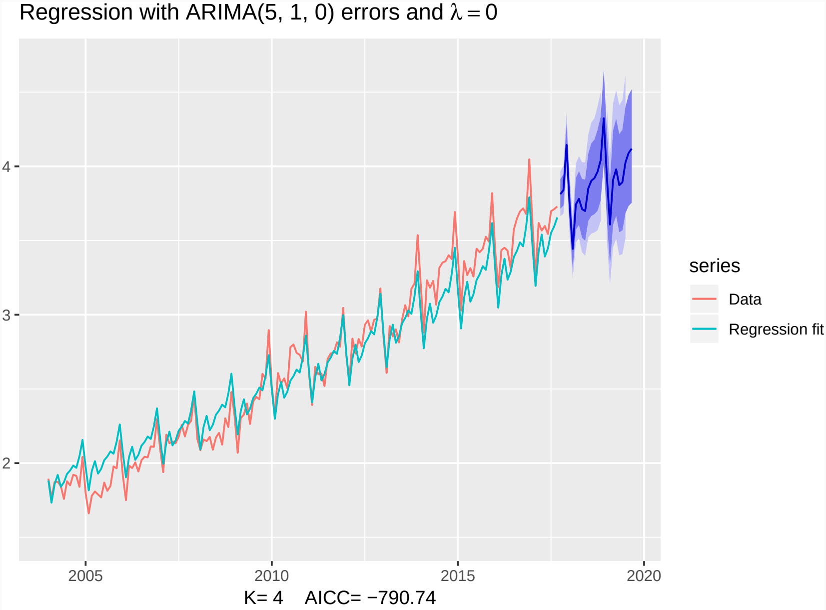

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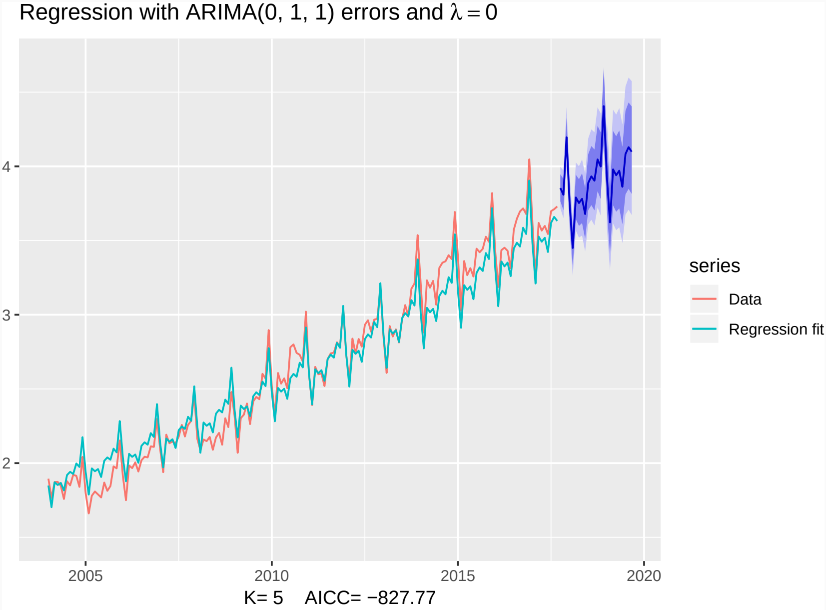
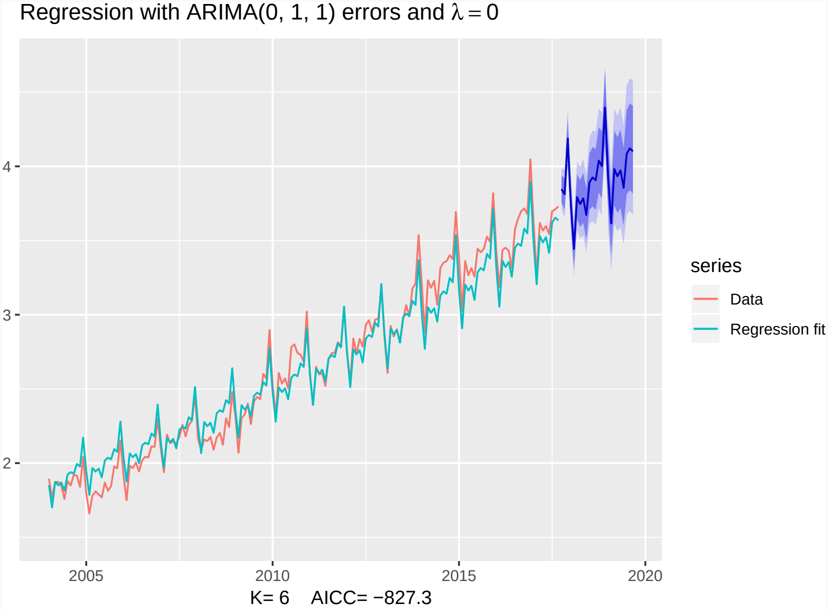
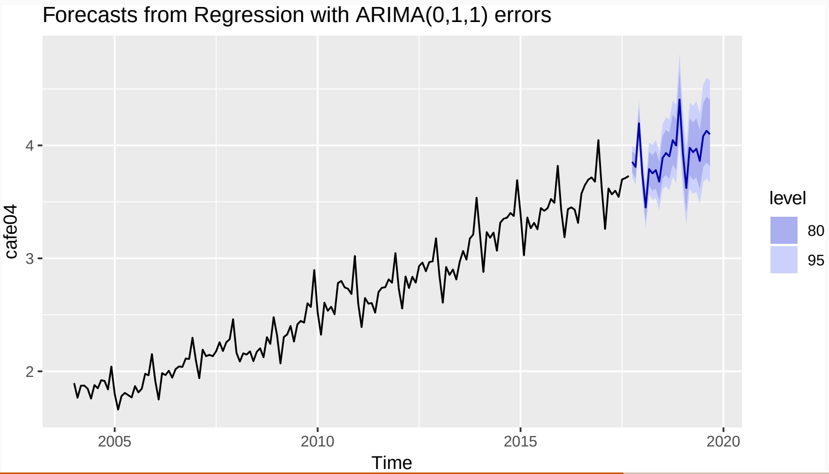


Eating-out expenditure



Eating-out expenditure

```
fit <- auto.arima(cafe04, xreg=fourier(cafe04, K=5),
                  seasonal = FALSE, lambda = 0)
fc <- forecast(fit, xreg=fourier(cafe04, K=5, h=24))
autoplot(fc)
```



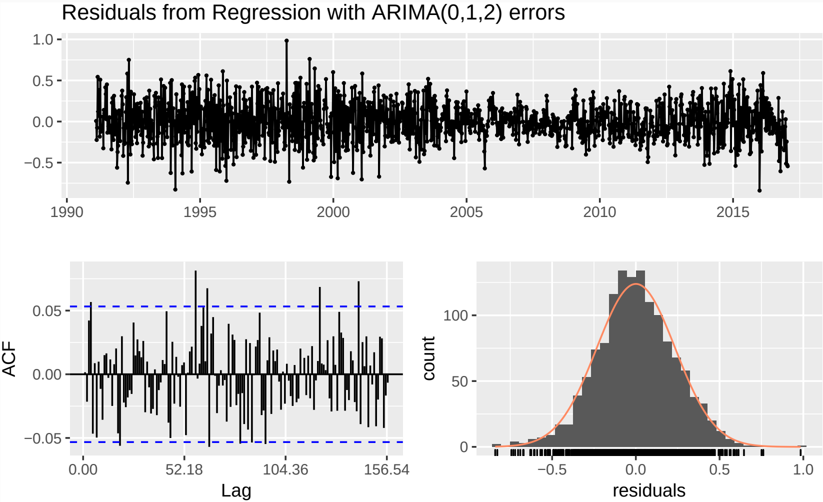
Example: weekly gasoline products

```r
harmonics <- fourier(gasoline, K = 13)
(fit <- auto.arima(gasoline, xreg = harmonics, seasonal = FALSE))
```

```
## Series: gasoline
## Regression with ARIMA(0,1,2) errors
##
## Coefficients:
##          ma1    ma2  drift  S1-52   C1-52   S2-52
##       -0.961  0.094  0.001  0.031  -0.255  -0.052
## s.e.   0.027  0.029  0.001  0.012   0.012   0.009
##        C2-52  S3-52   C3-52  S4-52   C4-52   S5-52
##       -0.017  0.024  -0.099  0.032  -0.026  -0.001
## s.e.   0.009  0.008   0.008  0.008   0.008   0.008
##        C5-52  S6-52   C6-52  S7-52  C7-52  S8-52
##       -0.047  0.058  -0.032  0.028  0.037  0.024
## s.e.   0.008  0.008   0.008  0.008  0.008  0.008
##       C8-52   S9-52  C9-52  S10-52  C10-52  S11-52
##       0.014  -0.017  0.012  -0.024   0.023   0.000
## s.e.  0.008   0.008  0.008   0.008   0.008   0.008
##       C11-52  S12-52  C12-52  S13-52  C13-52
##       -0.019  -0.029  -0.018   0.001  -0.018
## s.e.   0.008   0.008   0.008   0.008   0.008
##
## sigma^2 estimated as 0.056:  log likelihood=43.66
## AIC=-27.33   AICc=-25.92   BIC=129
```

Example: weekly gasoline products

```
checkresiduals(fit, test=FALSE)
```



Example: weekly gasoline products

```r
checkresiduals(fit, plot=FALSE)
```

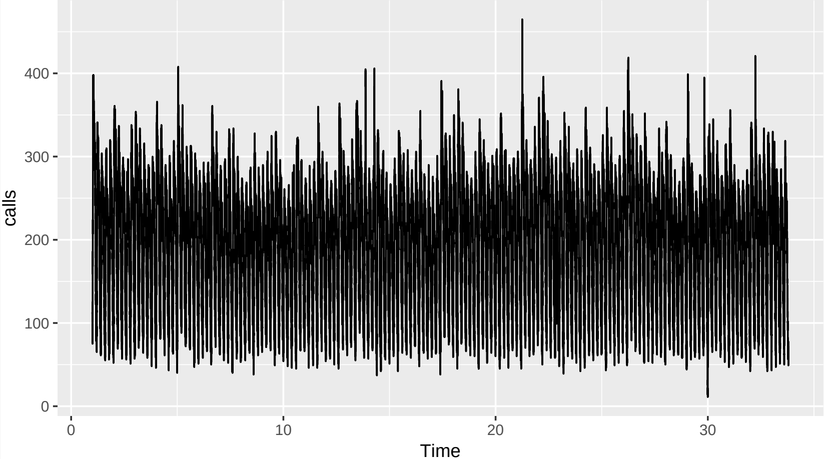
```
##
##  Ljung-Box test
##
## data:  Residuals from Regression with ARIMA(0,1,2) errors
## Q* = 130, df = 75, p-value = 6e-05
##
## Model df: 29.   Total lags used: 104.357142857143
```

Example: weekly gasoline products

```r
newharmonics <- fourier(gasoline, K = 13, h = 156)
fc <- forecast(fit, xreg = newharmonics)
autoplot(fc)
```

5-minute call centre volume

```r
autoplot(calls)
```



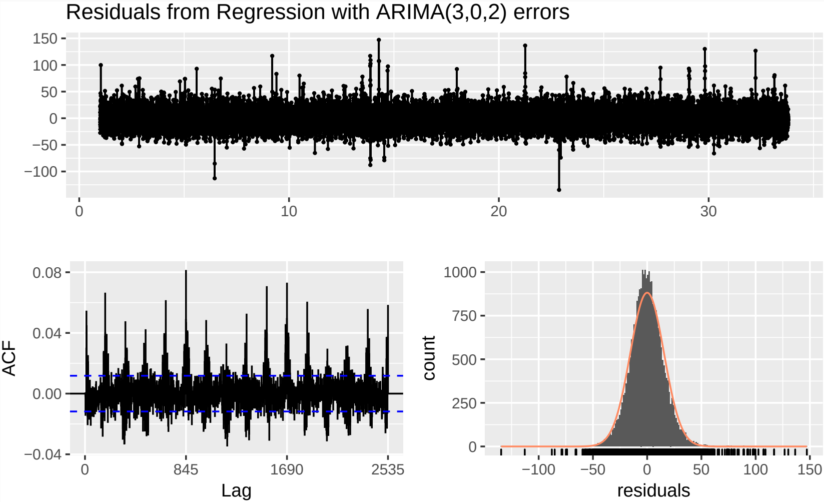
5-minute call centre volume

```r
xreg <- fourier(calls, K = c(10,0))
(fit <- auto.arima(calls, xreg=xreg, seasonal=FALSE, stationary=TRUE))
```

```
## Series: calls
## Regression with ARIMA(3,0,2) errors
##
## Coefficients:
##           ar1    ar2     ar3     ma1     ma2  intercept
##         0.841  0.192  -0.044  -0.590  -0.189    192.070
## s.e.    0.169  0.178   0.013   0.169   0.137      1.764
##         S1-169   C1-169  S2-169   C2-169   S3-169
##         55.245  -79.087  13.674  -32.375  -13.693
## s.e.     0.701    0.701   0.379    0.379    0.273
##         C3-169  S4-169  C4-169  S5-169  C5-169  S6-169
##         -9.327  -9.532  -2.797  -2.239   2.893   0.173
## s.e.     0.273   0.223   0.223   0.196   0.196   0.179
##         C6-169  S7-169  C7-169  S8-169  C8-169  S9-169
##          3.305   0.855   0.294   0.857  -1.391  -0.986
## s.e.     0.179   0.168   0.168   0.160   0.160   0.155
##         C9-169  S10-169  C10-169
##         -0.345   -1.196    0.801
## s.e.     0.155    0.150    0.150
##
## sigma^2 estimated as 243:  log likelihood=-115412
## AIC=230877   AICc=230877   BIC=231099
```

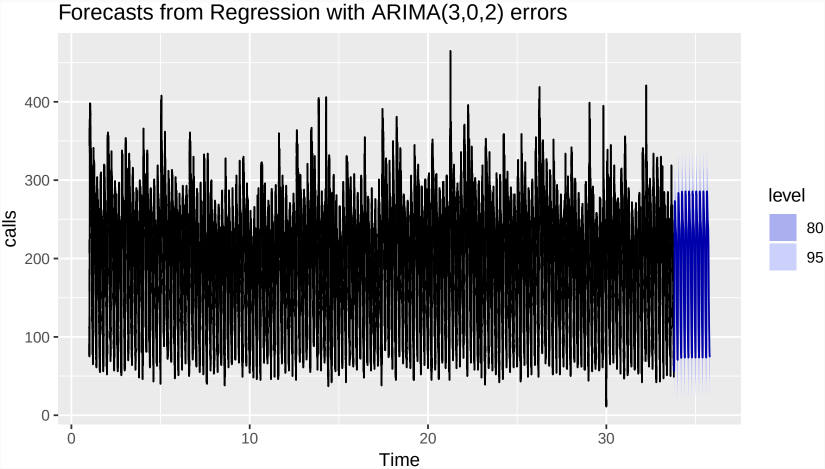
5-minute call centre volume

```
checkresiduals(fit, test=FALSE)
```



5-minute call centre volume

```r
fc <- forecast(fit, xreg = fourier(calls, c(10,0), 1690))
autoplot(fc)
```



Outline

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

- These are dynamic systems with input (x_t) and output (y_t).
- x_t is often a leading indicator.
- There can be multiple predictors.

Lagged predictors

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots$.

$$y_t = a + \nu_0 x_t + \nu_1 x_{t-1} + \cdots + \nu_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

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Rewrite model as

$$\begin{aligned} y_t &= a + (\nu_0 + \nu_1 B + \nu_2 B^2 + \cdots + \nu_k B^k) x_t + \eta_t \\ &= a + \nu(B) x_t + \eta_t. \end{aligned}$$

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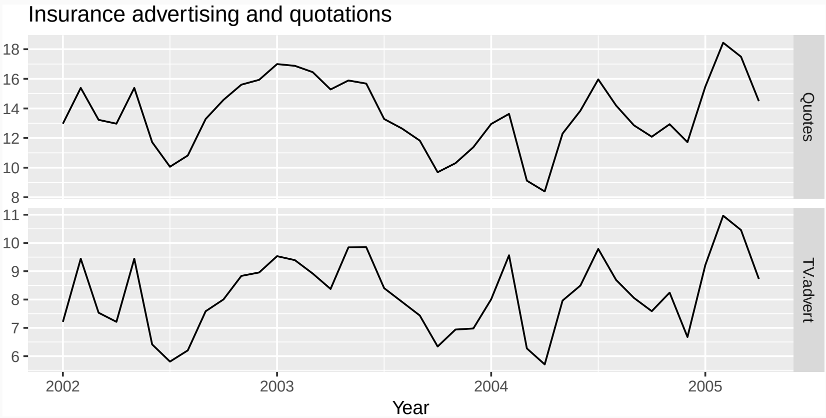
- $\nu(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t.
- x can influence y, but y is not allowed to influence x.

Example: Insurance quotes and TV adverts

```r
autoplot(insurance, facets=TRUE) +
  xlab("Year") + ylab("") +
  ggtitle("Insurance advertising and quotations")
```



Example: Insurance quotes and TV adverts

```r
Advert <- cbind(
    AdLag0 = insurance[,"TV.advert"],
    AdLag1 = lag(insurance[,"TV.advert"],-1),
    AdLag2 = lag(insurance[,"TV.advert"],-2),
    AdLag3 = lag(insurance[,"TV.advert"],-3)) %>%
  head(NROW(insurance))

# Restrict data so models use same fitting period
fit1 <- auto.arima(insurance[4:40,1], xreg=Advert[4:40,1],
  stationary=TRUE)
fit2 <- auto.arima(insurance[4:40,1], xreg=Advert[4:40,1:2],
  stationary=TRUE)
fit3 <- auto.arima(insurance[4:40,1], xreg=Advert[4:40,1:3],
  stationary=TRUE)
fit4 <- auto.arima(insurance[4:40,1], xreg=Advert[4:40,1:4],
  stationary=TRUE)
c(fit1$aicc,fit2$aicc,fit3$aicc,fit4$aicc)
```

```
## [1] 68.50 60.02 62.83 68.02
```

Example: Insurance quotes and TV adverts

```
(fit <- auto.arima(insurance[,1], xreg=Advert[,1:2],
  stationary=TRUE))
```

```
## Series: insurance[, 1]
## Regression with ARIMA(3,0,0) errors
##
## Coefficients:
##          ar1      ar2    ar3  intercept  AdLag0  AdLag1
##        1.412  -0.932  0.359      2.039   1.256   0.162
## s.e.  0.170   0.255  0.159      0.993   0.067   0.059
##
## sigma^2 estimated as 0.217:  log likelihood=-23.89
## AIC=61.78   AICc=65.28   BIC=73.6
```

Example: Insurance quotes and TV adverts

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##
## sigma^2 estimated as 0.217:  log likelihood=-23.89
## AIC=61.78   AICc=65.28   BIC=73.6
```

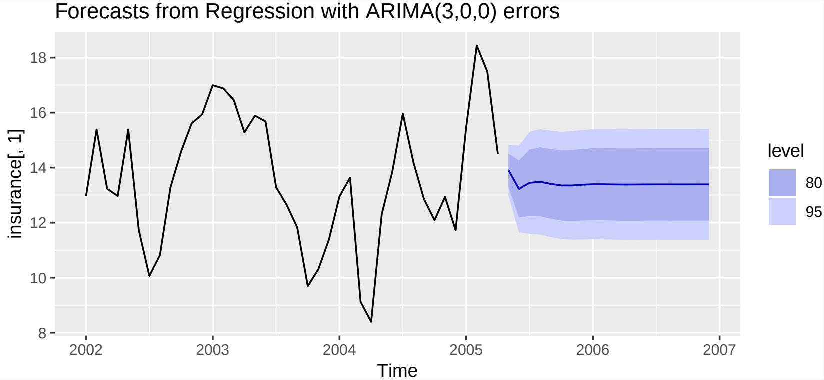
$$
\begin{aligned}
y_t &= 2.04 + 1.26x_t + 0.16x_{t-1} + \eta_t, \\
\eta_t &= 1.41\eta_{t-1} - 0.93\eta_{t-2} + 0.36\eta_{t-3} + \varepsilon_t,
\end{aligned}
$$

Example: Insurance quotes and TV adverts

```r
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(10,19)), rep(10,20)))
autoplot(fc)
```

Example: Insurance quotes and TV adverts

```
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(8,19)), rep(8,20)))
autoplot(fc)
```



Example: Insurance quotes and TV adverts

```
fc <- forecast(fit, h=20,
  xreg=cbind(c(Advert[40,1],rep(6,19)), rep(6,20)))
autoplot(fc)
```