MONASH University

MONASH BUSINESS SCHOOL

Forecasting: principles and practice

Rob J Hyndman

2.4 Non-seasonal ARIMA models

Outline

1 Autoregressive models

2 Moving average models

3 Non-seasonal ARIMA models

4 Partial autocorrelations

5 Estimation and order selection

6 ARIMA modelling in R

7 Forecasting

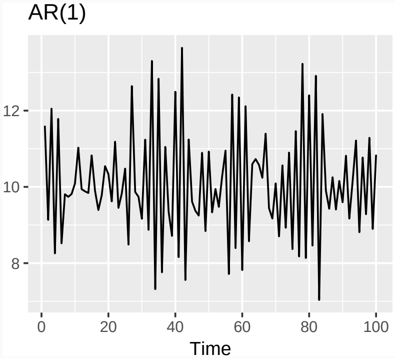
8 Lab session 11

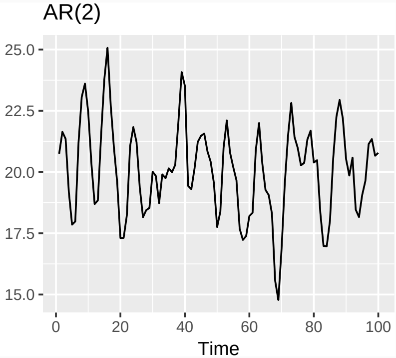
Autoregressive models

Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + e_t,$$

where e_t is white noise. This is a multiple regression with **lagged values** of y_t as predictors.

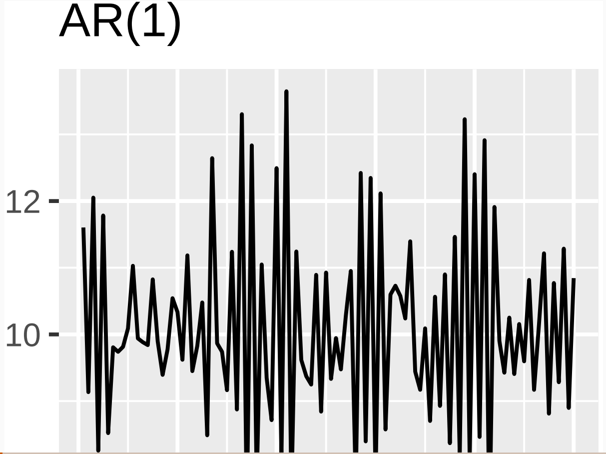




AR(1) model

$$y_t = 2 - 0.8y_{t-1} + e_t$$

$e_t \sim N(0, 1), \quad T = 100.$



AR(1) model

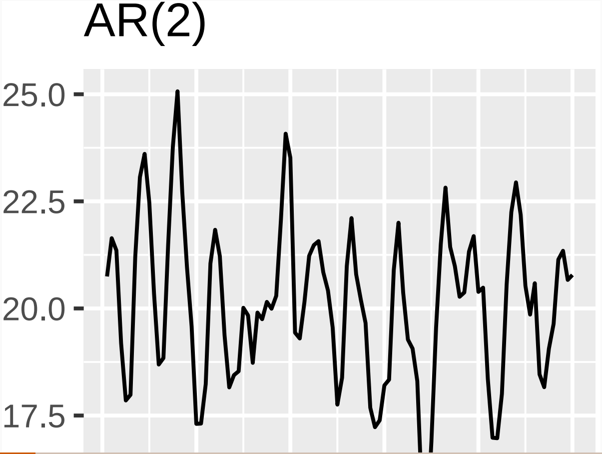
$$y_t = c + \phi_1 y_{t-1} + e_t$$

- When $\phi_1 = 0$, y_t is **equivalent to WN**
- When $\phi_1 = 1$ and $c = 0$, y_t is **equivalent to a RW**
- When $\phi_1 = 1$ and $c \neq 0$, y_t is **equivalent to a RW with drift**
- When $\phi_1 < 0$, y_t tends to **oscillate between positive and negative values**.

AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + e_t$$

$e_t \sim N(0, 1), \qquad T = 100.$



Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

General condition for stationarity

Complex roots of $1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p$ lie outside the unit circle on the complex plane.

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- For $p = 1$: $-1 < \phi_1 < 1$.
- For $p = 2$:
  $-1 < \phi_2 < 1 \qquad \phi_2 + \phi_1 < 1 \qquad \phi_2 - \phi_1 < 1.$
- More complicated conditions hold for $p \geq 3$.
- Estimation software takes care of this.

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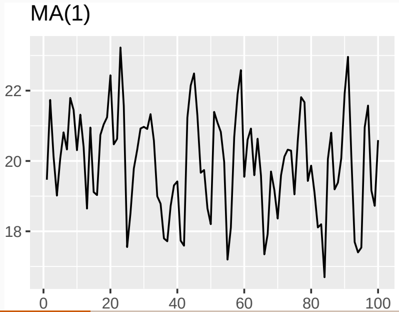
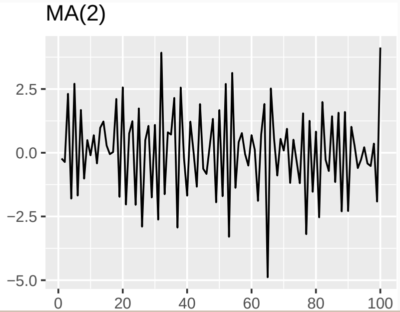
Moving Average (MA) models

Moving Average (MA) models:

$$y_t = c + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q},$$

where e_t is white noise. This is a multiple regression with ***past errors*** as predictors. *Don't confuse this with moving average smoothing!*

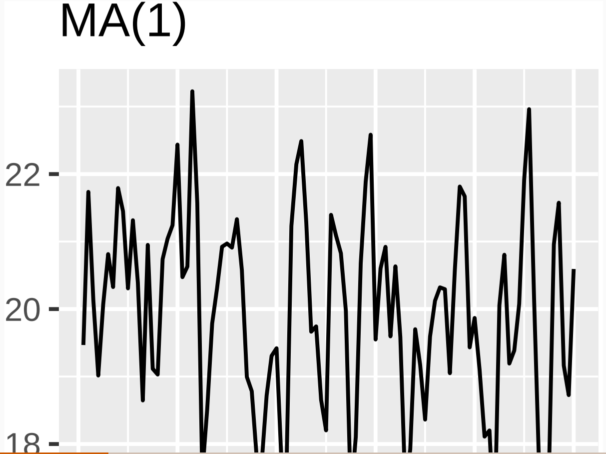




MA(1) model

$$y_t = 20 + e_t + 0.8e_{t-1}$$

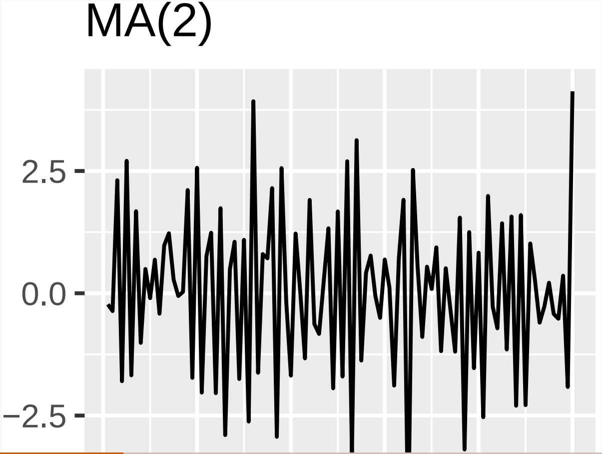
$e_t \sim N(0, 1), \quad T = 100.$



MA(2) model

$$y_t = e_t - e_{t-1} + 0.8e_{t-2}$$

$e_t \sim N(0, 1), \quad T = 100.$



Invertibility

- Any MA(q) process can be written as an AR(∞) process if we impose some constraints on the MA parameters.
- Then the MA model is called "invertible".
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

Invertibility

General condition for invertibility

Complex roots of $1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ lie outside the unit circle on the complex plane.

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- For $q = 1$: $-1 < \theta_1 < 1$.
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ARIMA models

Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 e_{t-1} + \cdots + \theta_q e_{t-q} + e_t.$$

ARIMA models

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- Predictors include both **lagged values of y_t and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

ARIMA models

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- Predictors include both **lagged values of y_t and lagged errors.**
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- Conditions on coefficients ensure invertibility.

Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing**.
- $(1 - B)^d y_t$ follows an ARMA model.

ARIMA models

Autoregressive Integrated Moving Average models

ARIMA(p, d, q) model

AR:	p = order of the autoregressive part
I:	d = degree of first differencing involved
MA:	q = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR(p): ARIMA(p,0,0)

Backshift notation for ARIMA

- ARMA model:

$$
\begin{aligned}
& y_t = c + \phi_1 B y_t + \cdots + \phi_p B^p y_t + e_t + \theta_1 B e_t + \cdots + \theta_q B^q e \\
\text{or} \quad & (1 - \phi_1 B - \cdots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \cdots + \theta_q B
\end{aligned}
$$

- ARIMA(1,1,1) model:

$$
\underset{\substack{\uparrow \\ \text{AR(1)}}}{(1 - \phi_1 B)} \quad \underset{\substack{\uparrow \\ \text{First} \\ \text{difference}}}{(1 - B) y_t} \quad = \quad c + \underset{\substack{\uparrow \\ \text{MA(1)}}}{(1 + \theta_1 B) e_t}
$$

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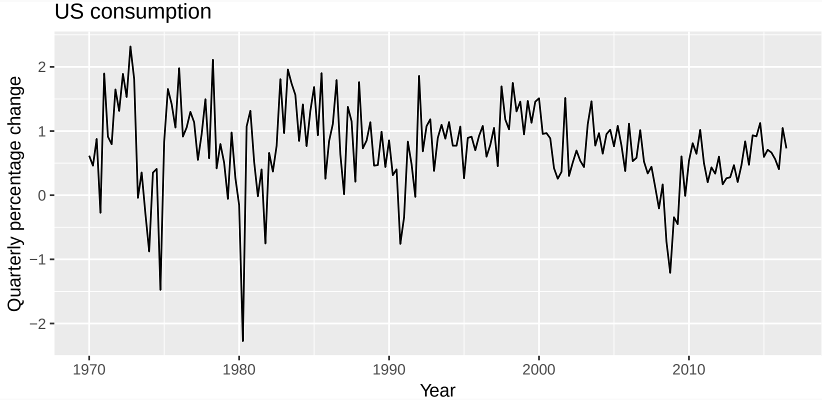
$$

Written out:

$$
y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 e_{t-1} + e_t
$$

US personal consumption



US personal consumption

```r
(fit <- auto.arima(uschange[,"Consumption"],
    seasonal=FALSE))
```

```
## Series: uschange[, "Consumption"]
## ARIMA(2,0,2) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ma1     ma2    mean
##       1.3908  -0.5813  -1.1800  0.5584  0.7463
## s.e.  0.2553   0.2078   0.2381  0.1403  0.0845
##
## sigma^2 estimated as 0.3511:  log likelihood=-165.14
## AIC=342.28   AICc=342.75   BIC=361.67
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```
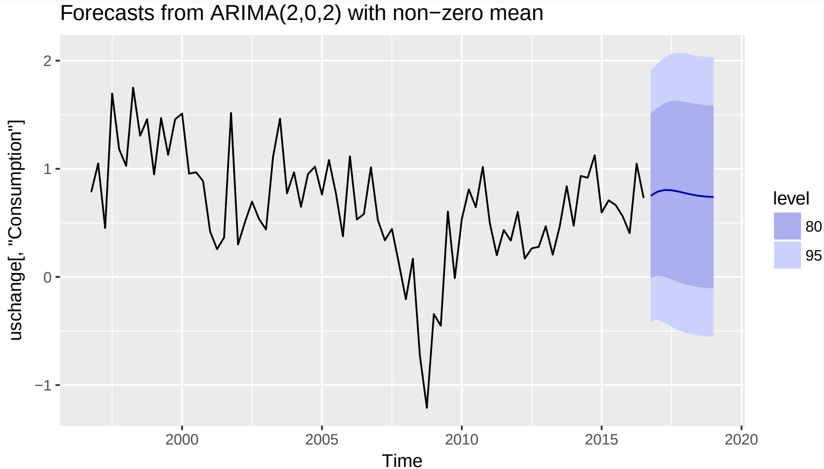

ARIMA(0,0,3) or MA(3) model:

$$y_t = 0.756 + e_t + 0.254e_{t-1} + 0.226e_{t-2} + 0.269e_{t-3},$$

US personal consumption

```
fit %>% forecast(h=10) %>% autoplot(include=80
```



Understanding ARIMA models

- If $c = 0$ and $d = 0$, the long-term forecasts will go to zero.
- If $c = 0$ and $d = 1$, the long-term forecasts will go to a non-zero constant.
- If $c = 0$ and $d = 2$, the long-term forecasts will follow a straight line.
- If $c \neq 0$ and $d = 0$, the long-term forecasts will go to the mean of the data.
- If $c \neq 0$ and $d = 1$, the long-term forecasts will follow a straight line.

Understanding ARIMA models

Forecast variance and d

- The higher the value of d, the more rapidly the prediction intervals increase in size.
- For $d = 0$, the long-term forecast standard deviation will go to the standard deviation of the historical data.

Cyclic behaviour

- For cyclic forecasts, $p > 2$ and some restrictions on coefficients are required.
- If $p = 2$, we need $\phi_1^2 + 4\phi_2 < 0$. Then average cycle of length

$$(2\pi)/\left[\text{arc cos}(-\phi_1(1-\phi_2)/(4\phi_2))\right].$$

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Partial autocorrelations

Partial autocorrelations measure relationship between y_t and y_{t-k}, when the effects of other time lags — $1, 2, 3, \dots, k-1$ — are removed.

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$$\begin{aligned} \alpha_k &= k\text{th partial autocorrelation coefficient} \\ &= \text{equal to the estimate of } b_k \text{ in regression:} \\ & \qquad y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}. \end{aligned}$$

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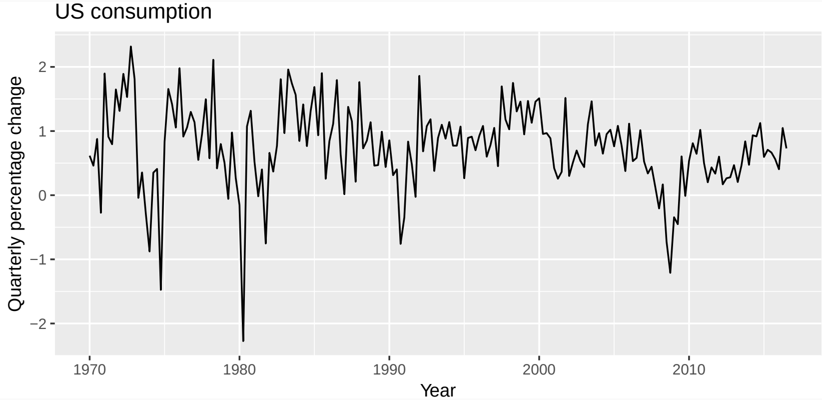
- Varying number of terms on RHS gives α_k for different values of k.
- There are more efficient ways of calculating α_k.
- $\alpha_1 = \rho_1$

Example: US consumption



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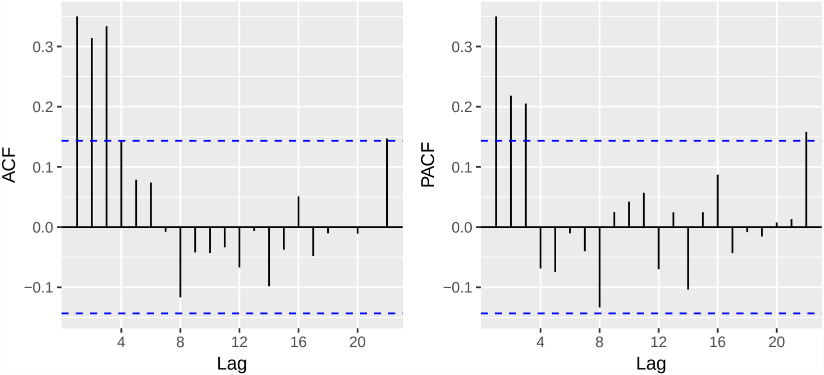

ACF and PACF interpretation

ARIMA(p,d,0) model if ACF and PACF plots of differenced data show:

- the ACF is exponentially decaying or sinusoidal;
- there is a significant spike at lag p in PACF, but none beyond lag p.

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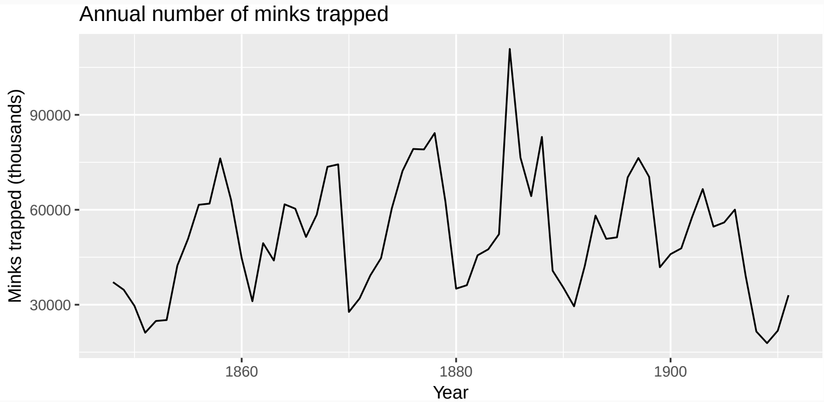
ARIMA(0,d,q) model if ACF and PACF plots of differenced data show:

- the PACF is exponentially decaying or sinusoidal;
- there is a significant spike at lag q in ACF, but none beyond lag q.

Example: Mink trapping



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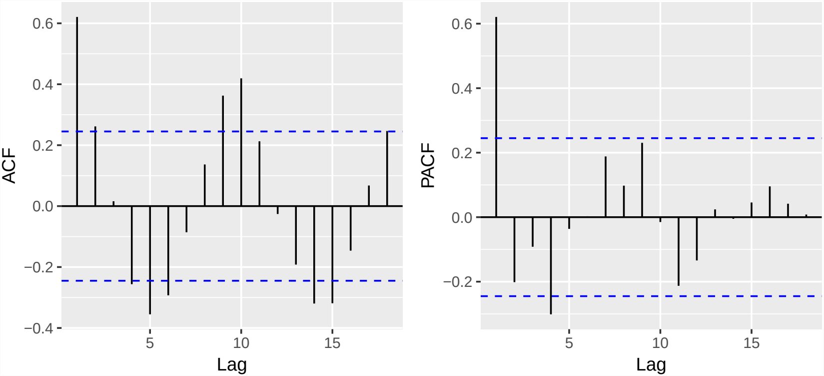

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Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters c, $\phi_1, \dots, \phi_p$, $\theta_1, \dots, \theta_q$.

Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$.

- MLE is very similar to least squares estimation obtained by minimizing
$$\sum_{t-1}^{T} e_t^2.$$
- The `Arima()` command allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

Information criteria

Akaike's Information Criterion (AIC):

$$\text{AIC} = -2\log(L) + 2(p + q + k + 1),$$

where L is the likelihood of the data,

$k = 1$ if $c \neq 0$ and $k = 0$ if $c = 0$.

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Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p+q+k+1)(p+q+k+2)}{T-p-q-k-2}.$$

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Bayesian Information Criterion:

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How does auto.arima() work?

A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders: p, q, d

Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences d and D via unit root tests.
- Select p, q by minimising AICc.
- Use stepwise search to traverse model space.

How does auto.arima() work?

$$\text{AICc} = -2\log(L) + 2(p+q+k+1)\left[1 + \frac{(p+q+k+2)}{T-p-q-k-2}\right].$$

where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

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where L is the maximised likelihood fitted to the *differenced* data, $k = 1$ if $c \neq 0$ and $k = 0$ otherwise.

Step1: Select current model (with smallest AICc) from:
ARIMA$(2, d, 2)$
ARIMA$(0, d, 0)$
ARIMA$(1, d, 0)$
ARIMA$(0, d, 1)$

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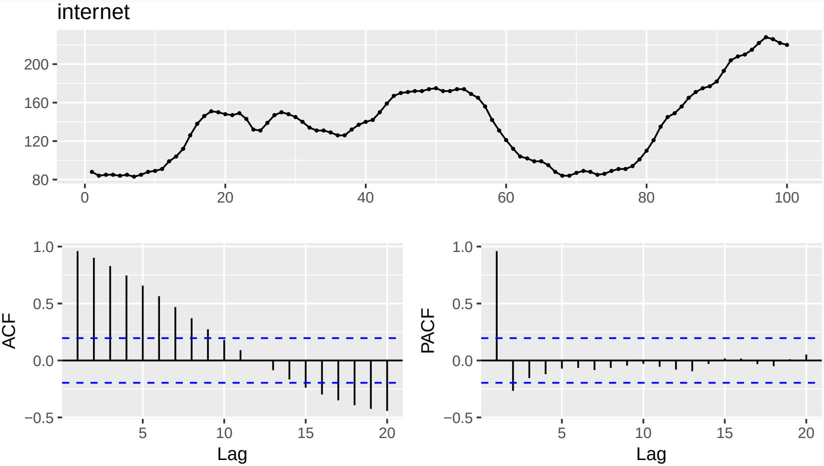
Step1: Select current model (with smallest AICc) from:
ARIMA$(2,d,2)$
ARIMA$(0,d,0)$
ARIMA$(1,d,0)$
ARIMA$(0,d,1)$

Step 2: Consider variations of current model:

- vary one of $p, q,$ from current model by ± 1;
- p, q both vary from current model by ± 1;
- Include/exclude c from current model.

Choosing your own model

```r
ggtsdisplay(internet)
```



Choosing your own model

```r
tseries::adf.test(internet)
```

```
##
##  Augmented Dickey-Fuller Test
##
## data:  internet
## Dickey-Fuller = -2.6421, Lag order = 4, p-value = 0.3107
## alternative hypothesis: stationary
```

```r
tseries::kpss.test(internet)
```

```
##
##  KPSS Test for Level Stationarity
##
## data:  internet
## KPSS Level = 0.72197, Truncation lag parameter = 2, p-
value =
## 0.01155
```

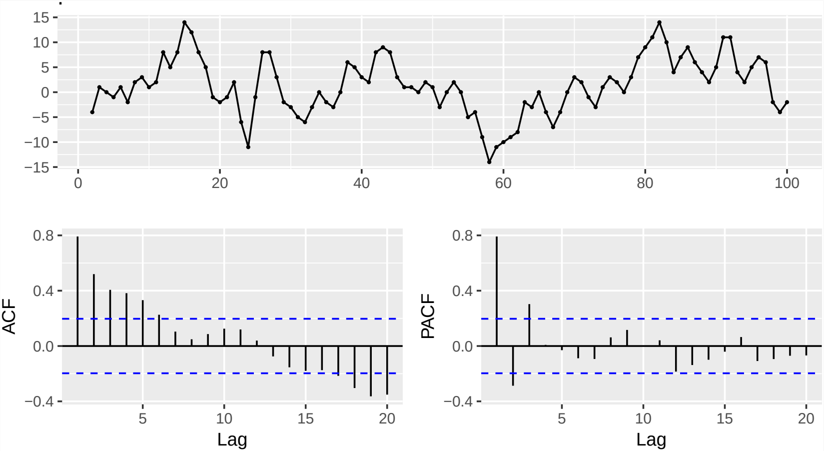
Choosing your own model

```
tseries::kpss.test(diff(internet))
```

```
##
##  KPSS Test for Level Stationarity
##
## data:  diff(internet)
## KPSS Level = 0.26352, Truncation lag parameter = 2, p-
value = 0.1
```

Choosing your own model

```
internet %>% diff %>% ggtsdisplay
```



Choosing your own model

```r
(fit <- Arima(internet,order=c(3,1,0)))
```

```
## Series: internet
## ARIMA(3,1,0)
##
## Coefficients:
##          ar1      ar2     ar3
##       1.1513  -0.6612  0.3407
## s.e.  0.0950   0.1353  0.0941
##
## sigma^2 estimated as 9.656:  log likelihood=-
252
## AIC=511.99   AICc=512.42   BIC=522.37
```

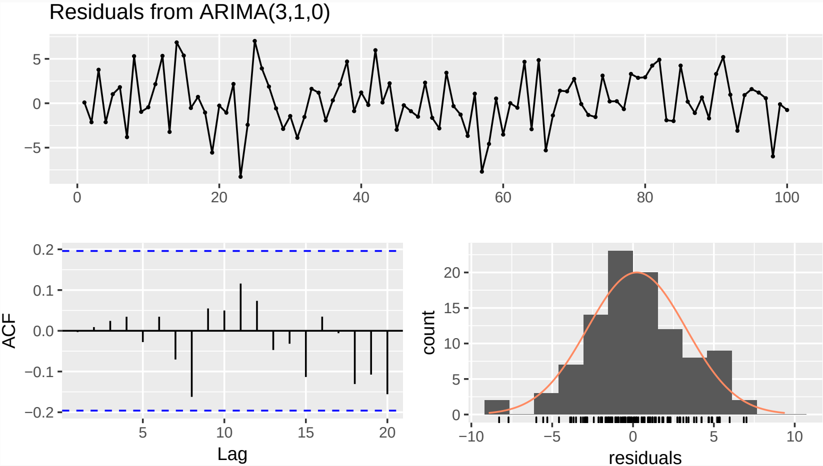
Choosing your own model

```r
(fit2 <- auto.arima(internet))
```

```
## Series: internet
## ARIMA(1,1,1)
##
## Coefficients:
##          ar1     ma1
##       0.6504  0.5256
## s.e.  0.0842  0.0896
##
## sigma^2 estimated as 9.995:  log likelihood=-
254.15
## AIC=514.3   AICc=514.55   BIC=522.08
```

Choosing your own model

```
checkresiduals(fit, plot=TRUE)
```



#

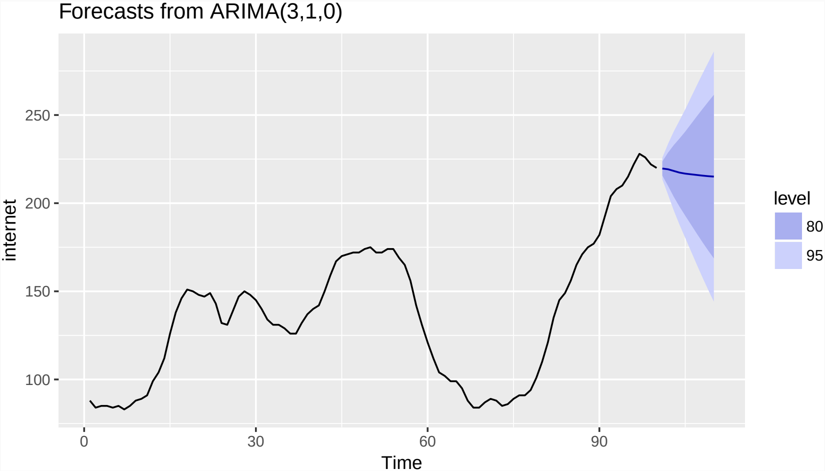
Choosing your own model

```r
checkresiduals(fit, plot=FALSE)
```

```
##
##  Ljung-Box test
##
## data:  Residuals from ARIMA(3,1,0)
## Q* = 4.4913, df = 7, p-value = 0.7218
##
## Model df: 3.   Total lags used: 10
```

Choosing your own model

```
fit %>% forecast %>% autoplot
```



Modelling procedure with `Arima`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
3. If the data are non-stationary: take first differences of the data until the data are stationary.
4. Examine the ACF/PACF: Is an AR(p) or MA(q) model appropriate?
5. Try your chosen model(s), and use the AICc to search for a better model.
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.

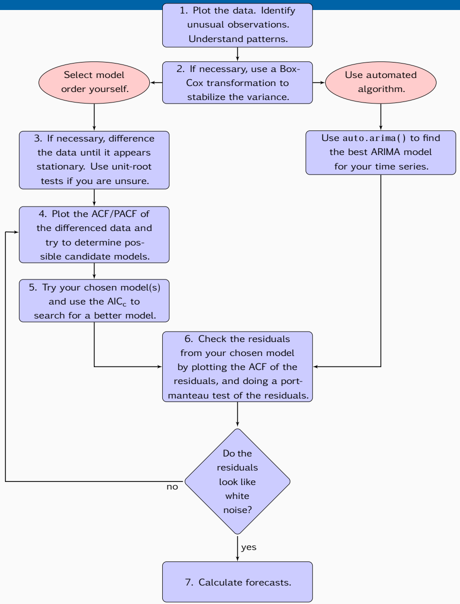
Modelling procedure with `auto.arima`

1. Plot the data. Identify any unusual observations.
2. If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.

3. Use `auto.arima` to select a model.

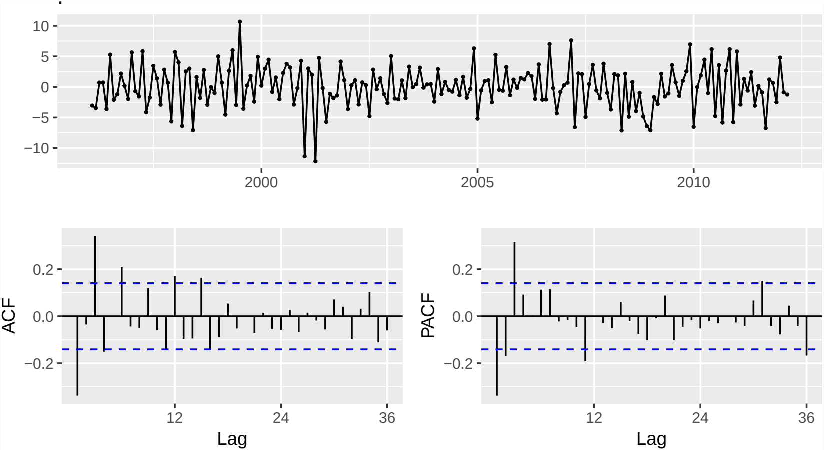
6. Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
7. Once the residuals look like white noise, calculate forecasts.

Modelling procedure



Seasonally adjusted electrical equipment

```
eeadj %>% diff %>% ggtsdisplay
```



##Seasonally adjusted electrical equipment

Outline

1 Autoregressive models
2 Moving average models
3 Non-seasonal ARIMA models
4 Partial autocorrelations
5 Estimation and order selection
6 ARIMA modelling in R
7 Forecasting
8 Lab session 11

Point forecasts

1. Rearrange ARIMA equation so y_t is on LHS.
2. Rewrite equation by replacing t by $T + h$.
3. On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with $h = 1$. Repeat for $h = 2, 3, \ldots$.

Prediction intervals

95% Prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where $v_{T+h|T}$ is estimated forecast variance.

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- $v_{T+1|T} = \hat{\sigma}^2$ for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = e_t + \sum_{i=1}^{q} \theta_i e_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[1 + \sum_{i=1}^{h-1} \theta_i^2\right], \qquad \text{for } h = 2, 3, \dots.$$

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- AR(1): Rewrite as MA(∞) and use above result.
- Other models beyond scope of this workshop.

Prediction intervals

- Prediction intervals **increase in size with forecast horizon**.
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed**.
- Prediction intervals tend to be too narrow.
    - the uncertainty in the parameter estimates has not been accounted for.
    - the ARIMA model assumes historical patterns will not change during the forecast period.
    - the ARIMA model assumes uncorrelated future errors

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Lab Session 11