



# Forecasting: principles and practice

Rob J Hyndman

3.2 Forecasting with multiple seasonality

# Outline

**1** Time series with complex seasonality

2 STL with multiple seasonal periods

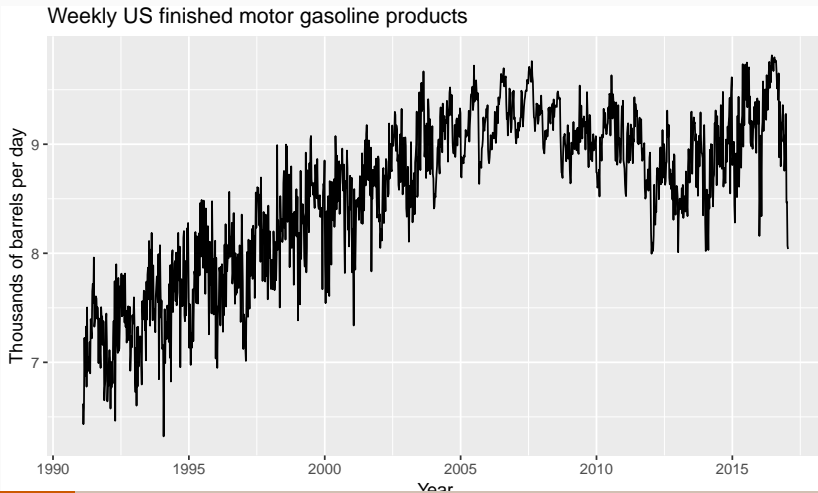
3 Dynamic harmonic regression

4 TBATS model

5 Lab session 21

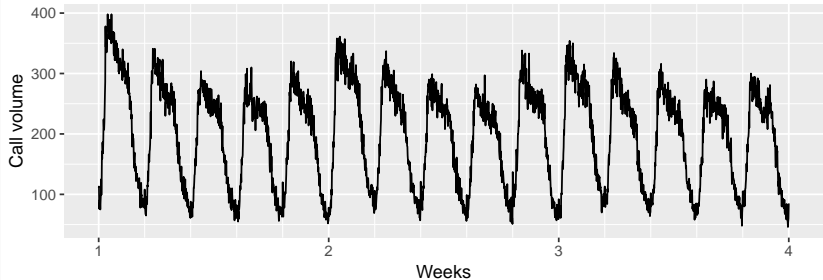
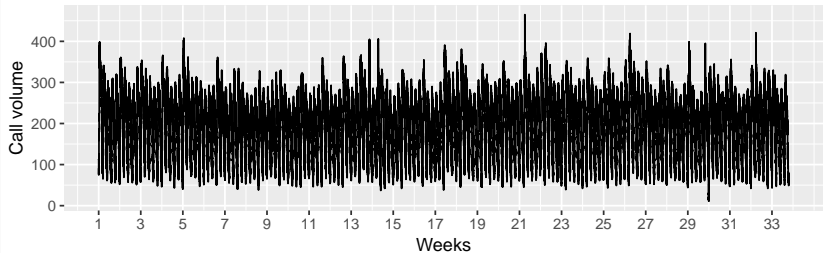
# Examples

```
autoplot(gasoline) +  
  xlab("Year") + ylab("Thousands of barrels per day") +  
  ggtitle("Weekly US finished motor gasoline products")
```

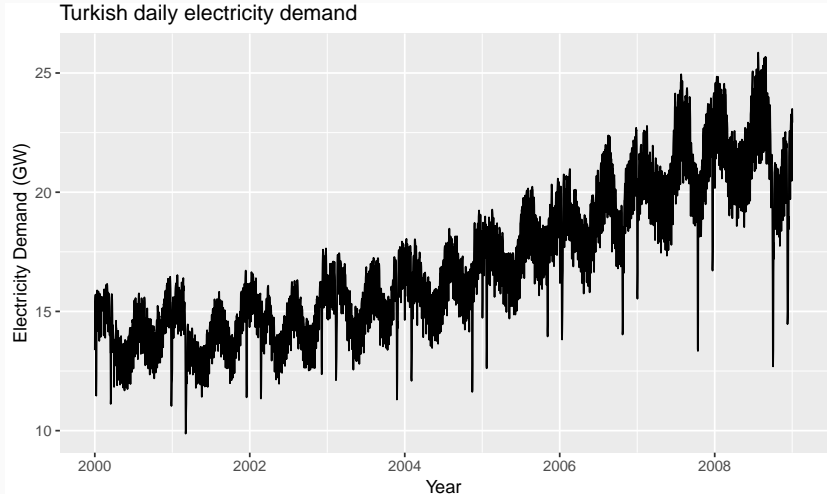


# Examples

5 minute call volume at North American bank



# Examples

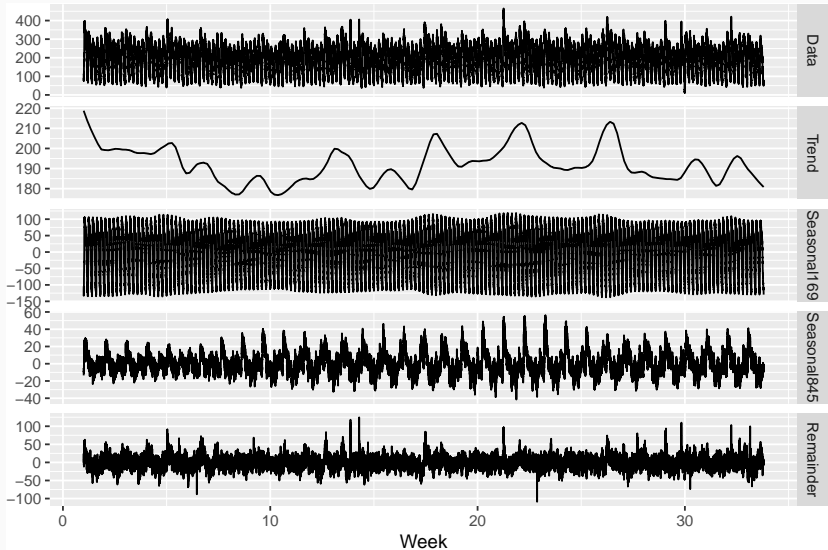


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- 2 STL with multiple seasonal periods
- 3 Dynamic harmonic regression
- 4 TBATS model
- 5 Lab session 21

# STL with multiple seasonal periods

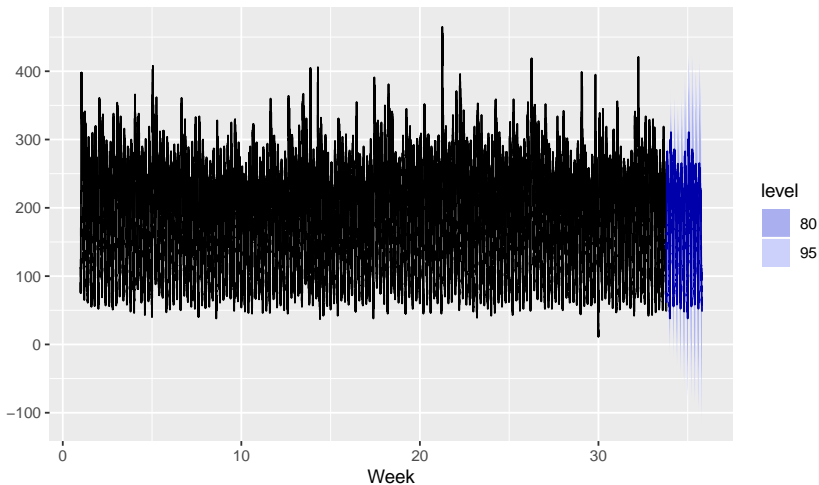
```
calls %>% mstl() %>% autoplot() + xlab("Week")
```



# STL with multiple seasonal periods

```
calls %>% stlf() %>% autoplot() + xlab("Week")
```

Forecasts from STL + ETS(M,N,N)

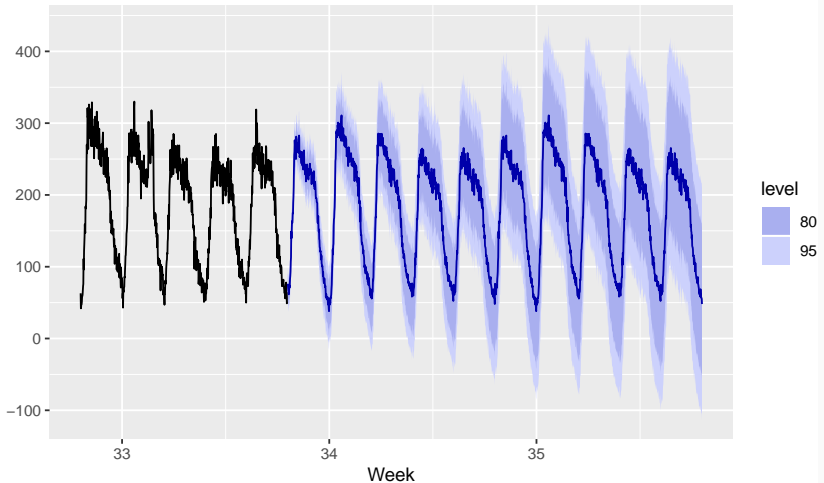




# STL with multiple seasonal periods

```
calls %>% stlf() %>% autoplot(include=5*169) + xlab("Week")
```

Forecasts from STL + ETS(M,N,N)

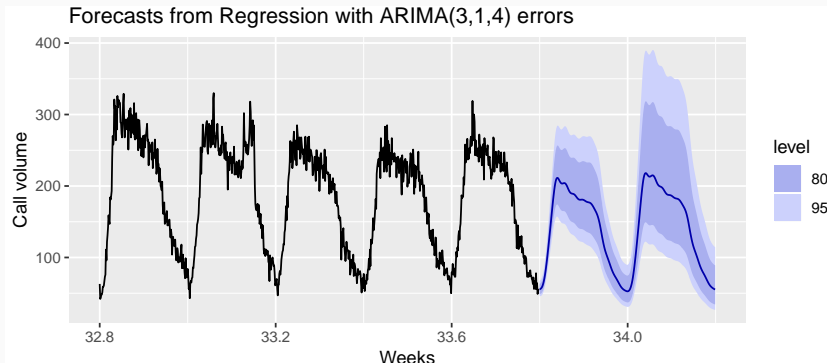


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# Dynamic harmonic regression

```
fit <- auto.arima(calls, seasonal=FALSE, lambda=0,  
  xreg=fourier(calls, K=c(10,10)))  
fit %>%  
  forecast(xreg=fourier(calls, K=c(10,10), h=2*169)) %>%  
  autoplot(include=5*169) +  
    ylab("Call volume") + xlab("Weeks")
```



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- 1 Time series with complex seasonality
- 2 STL with multiple seasonal periods
- 3 Dynamic harmonic regression
- 4 **TBATS model**
- 5 Lab session 21

## TBATS

**T**rigonometric terms for seasonality

**B**ox-Cox transformations for heterogeneity

**A**RMA errors for short-term dynamics

**T**rend (possibly damped)

**S**easonal (including multiple and non-integer periods)

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

$$y_t^{(\omega)} = \ell_{t-1} + \phi \mathbf{b}_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

$$\ell_t = \ell_{t-1} + \phi \mathbf{b}_{t-1} + \alpha d_t$$

$$\mathbf{b}_t = (1 - \phi)\mathbf{b} + \phi \mathbf{b}_{t-1} + \beta d_t$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t$$

$$s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)} \quad \begin{aligned} s_{j,t}^{(i)} &= s_{j,t-1}^{(i)} \cos \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t \\ s_{j,t}^{(i)} &= -s_{j,t-1}^{(i)} \sin \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t \end{aligned}$$

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Box-Cox transformation

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$M$  seasonal periods

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$M$  seasonal periods

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

global and local trend

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

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ARMA error

$$s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$$
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$M$  seasonal periods

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ARMA error

$$s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$$

Fourier-like seasonal terms

$$s_{j,t}^{(i)} = s_{j,t-1}^{(i)} \cos \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t$$

$$s_{j,t}^{(i)} = -s_{j,t-1}^{(i)} \sin \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t$$

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \end{cases}$$

Box-Cox transformation

$$y_t^{(\omega)} = \ell_t + b_t + d_t$$

$M$  seasonal periods

$$\ell_t = \ell_t$$

$$b_t = (1$$

$$d_t = \sum_{i=1}^p$$

$$s_t^{(i)} = \sum_{j=1}^{k_i} s_{j,t}^{(i)}$$

**TBATS**  
**Trigonometric**  
**Box-Cox**  
**ARMA**  
**Trend**  
**Seasonal**

global and local trend

ARMA error

Fourier-like seasonal terms

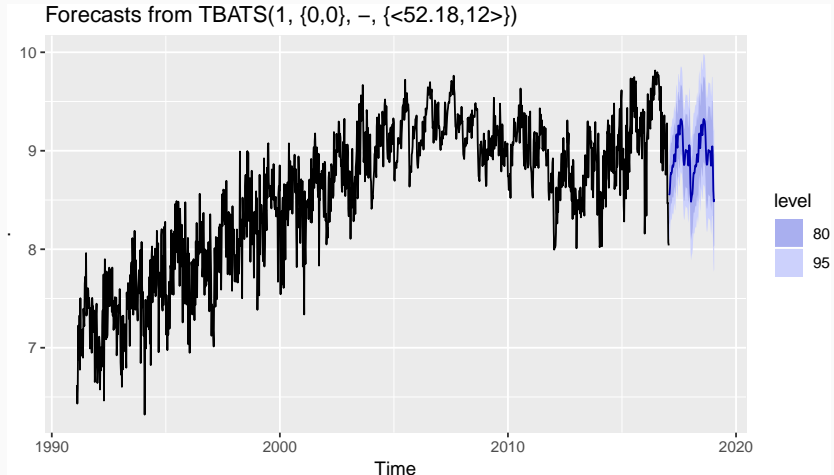
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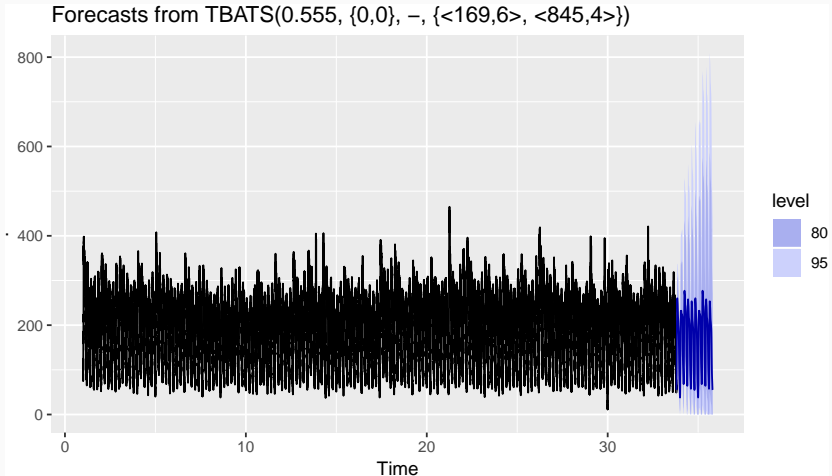
# Complex seasonality

```
gasoline %>% tbats() %>% forecast() %>% autoplot()
```



# Complex seasonality

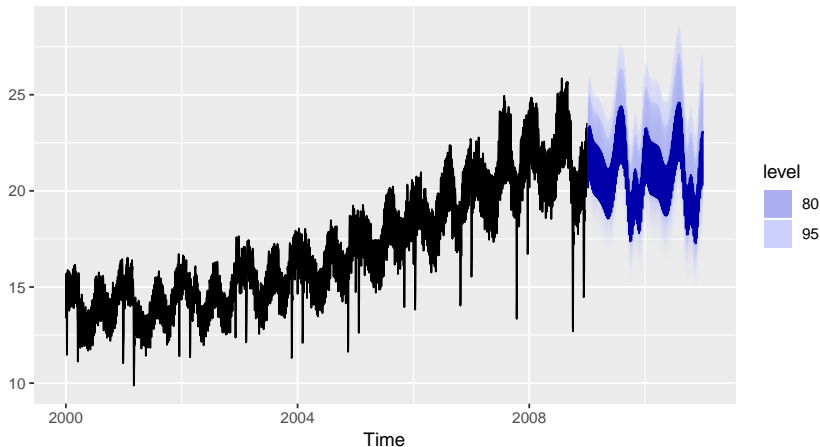
```
calls %>% tbats() %>% forecast() %>% autoplot()
```



# Complex seasonality

```
telec %>% tbats() %>% forecast() %>% autoplot()
```

Forecasts from TBATS(0.005, {4,2}, -, {<7,3>, <354.37,7>, <365.25,3>})



## TBATS

**T**rigonometric terms for seasonality

**B**ox-Cox transformations for heterogeneity

**A**RMA errors for short-term dynamics

**T**rend (possibly damped)

**S**easonal (including multiple and non-integer periods)

- Handles non-integer seasonality, multiple seasonal periods.
- Entirely automated
- Prediction intervals often too wide
- Very slow on long series



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# Lab Session 21