



# Forecasting: principles and practice

Rob J Hyndman

3.4 Advanced methods

# Outline

**1** Time series with complex seasonality

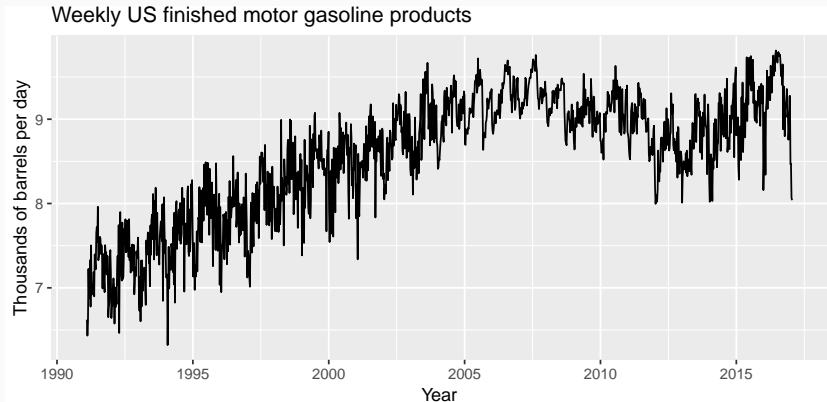
**2** Lab session 17

**3** Neural network models

**4** Lab session 18

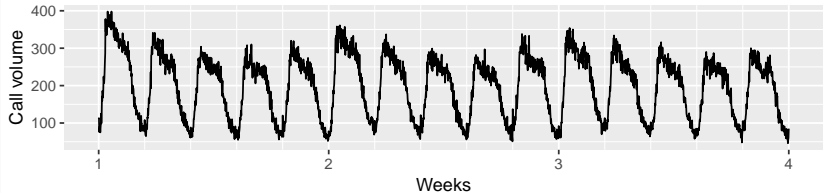
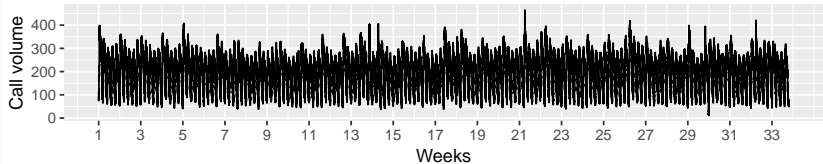
**5** Lab session 19

# Examples

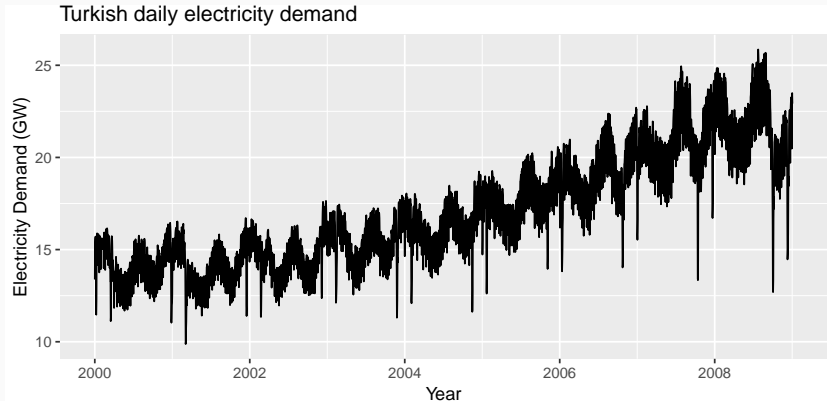


# Examples

5 minute call volume at North American bank



# Examples



## TBATS

**T**rigonometric terms for seasonality

**B**ox-Cox transformations for heterogeneity

**A**RMA errors for short-term dynamics

**T**rend (possibly damped)

**S**easonal (including multiple and  
non-integer periods)

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \left( \frac{s_{j,t}^{(i)}}{s_{j,t-1}^{(i)}} \cos \lambda_j^{(i)} + \frac{s_{j,t-1}^{*(i)}}{s_{j,t-1}^{(i)}} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t \right) + \sum_{j=1}^q \theta_j \left( -\frac{s_{j,t}^{(i)}}{s_{j,t-1}^{(i)}} \sin \lambda_j^{(i)} + \frac{s_{j,t-1}^{*(i)}}{s_{j,t-1}^{(i)}} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t \right)$$

$$s_t^{(i)} = \sum_{i=1}^{k_i} s_{i,t}^{(i)}$$

# TBATS model

$y_t$  = observation at time  $t$

Box-Cox transformation

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t$$

$$s_{j,t}^{(i)} = s_{j,t-1}^{(i)} \cos \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t$$

$$s_{j,t}^{*(i)} = -s_{j,t-1}^{(i)} \sin \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t$$

$$s_t^{(i)} = \sum_{i=1}^{k_i} s_{i,t}^{(i)}$$



# TBATS model

$y_t$  = observation at time  $t$

Box-Cox transformation

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

$M$  seasonal periods

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \sum_{j=1}^q s_{j,t-1}^{(i)} \cos \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t$$

$$s_{j,t}^{(i)} = -s_{j,t-1}^{(i)} \sin \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t$$

$$s_t^{(i)} = \sum_{i=1}^{k_i} s_{i,t}^{(i)}$$

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

Box-Cox transformation

$M$  seasonal periods

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

global and local trend

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \left( \frac{s_{j,t}^{(i)}}{s_{j,t-1}^{(i)}} \cos \lambda_j^{(i)} + \frac{s_{j,t-1}^{*(i)}}{s_{j,t-1}^{(i)}} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t \right) \\ + \sum_{j=1}^q \theta_j \left( -\frac{s_{j,t}^{(i)}}{s_{j,t-1}^{(i)}} \sin \lambda_j^{(i)} + \frac{s_{j,t-1}^{*(i)}}{s_{j,t-1}^{(i)}} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t \right)$$

$$s_t^{(i)} = \sum_{i=1}^{k_i} s_{i,t}^{(i)}$$

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

Box-Cox transformation

$M$  seasonal periods

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

global and local trend

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

ARMA error

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t$$

$$s_{j,t}^{(i)} = s_{j,t-1}^{(i)} \cos \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \sin \lambda_j^{(i)} + \gamma_1^{(i)} d_t$$

$$s_{j,t}^{(i)} = -s_{j,t-1}^{(i)} \sin \lambda_j^{(i)} + s_{j,t-1}^{*(i)} \cos \lambda_j^{(i)} + \gamma_2^{(i)} d_t$$

$$s_t^{(i)} = \sum_{k_i} s_{i,t}^{(i)}$$

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} (y_t^\omega - 1)/\omega & \text{if } \omega \neq 0; \\ \log y_t & \text{if } \omega = 0. \end{cases}$$

Box-Cox transformation

$M$  seasonal periods

$$y_t^{(\omega)} = \ell_{t-1} + \phi b_{t-1} + \sum_{i=1}^M s_{t-m_i}^{(i)} + d_t$$

global and local trend

$$\ell_t = \ell_{t-1} + \phi b_{t-1} + \alpha d_t$$

$$b_t = (1 - \phi)b + \phi b_{t-1} + \beta d_t$$

ARMA error

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t$$

Fourier-like seasonal terms

$$s_t^{(i)} = \sum_{j=1}^{k_i} s_{i,t}^{(j)}$$

# TBATS model

$y_t$  = observation at time  $t$

$$y_t^{(\omega)} = \begin{cases} \text{TBATS} & \text{if } \omega = 0; \\ \text{Trigonometric} & \text{otherwise.} \end{cases}$$

$$y_t^{(\omega)} = \ell_t + b_t + d_t + \varepsilon_t$$

$$\ell_t = \ell$$

$$b_t = (b_j)_{j=1}^M$$

$$d_t = \sum_{i=1}^p \phi_i d_{t-i} + \sum_{j=1}^q \theta_j \varepsilon_{t-j} + \varepsilon_t$$

$$s_t^{(i)} = \sum_{i=1}^{k_i} s_{i,t}^{(i)}$$

Box-Cox transformation

$M$  seasonal periods

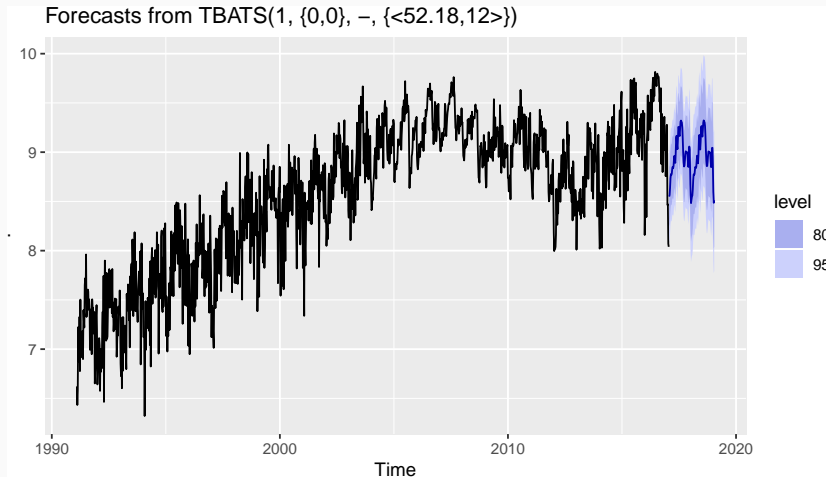
global and local trend

ARMA error

Fourier-like seasonal terms

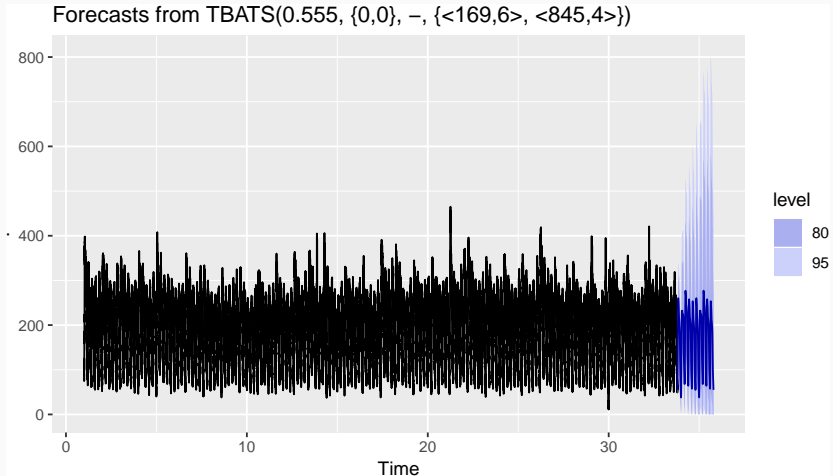
# Complex seasonality

gasoline %>% tbats %>% forecast %>% autoplot



# Complex seasonality

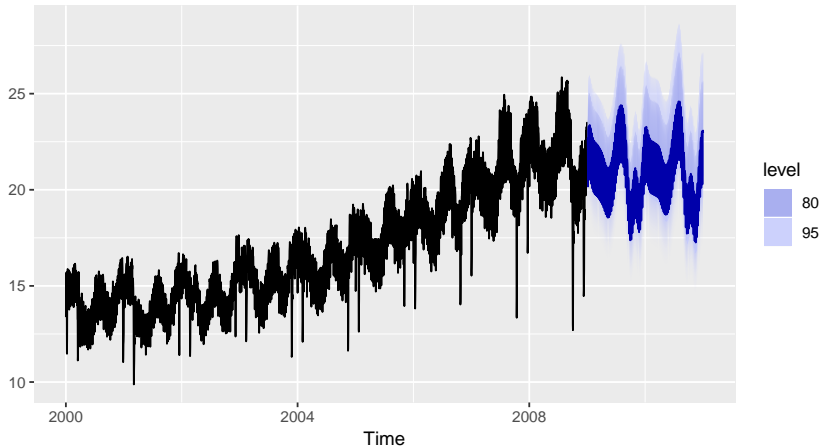
```
calls %>% tbats %>% forecast %>% autoplot
```



# Complex seasonality

```
telec %>% tbats %>% forecast %>% autoplot
```

Forecasts from TBATS(0.005, {4,2}, -, {<7,3>, <354.37,7>, <365.25,3>})





## TBATS

**T**rigonometric terms for seasonality

**B**ox-Cox transformations for heterogeneity

**A**RMA errors for short-term dynamics

**T**rend (possibly damped)

**S**easonal (including multiple and non-integer periods)

- Handles non-integer seasonality, multiple seasonal periods.
- Entirely automated
- Prediction intervals often too wide
- Very slow on long series

# Outline

**1** Time series with complex seasonality

**2** Lab session 17

**3** Neural network models

**4** Lab session 18

**5** Lab session 19

# Lab Session 17

# Outline

**1** Time series with complex seasonality

**2** Lab session 17

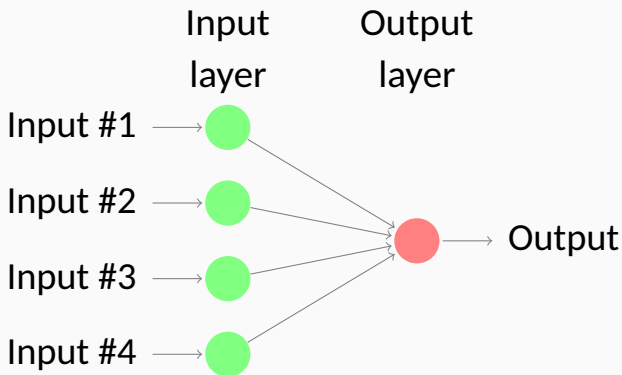
**3** Neural network models

**4** Lab session 18

**5** Lab session 19

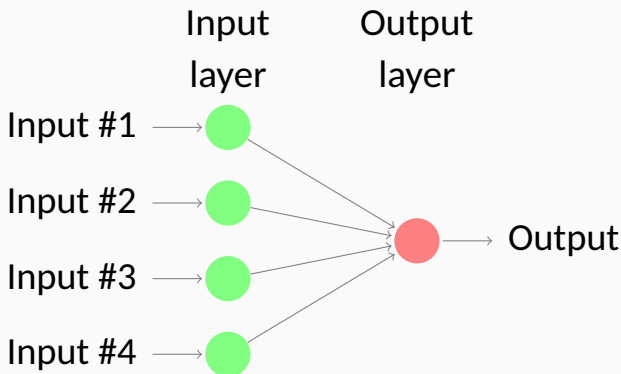
# Neural network models

## Simplest version: linear regression



# Neural network models

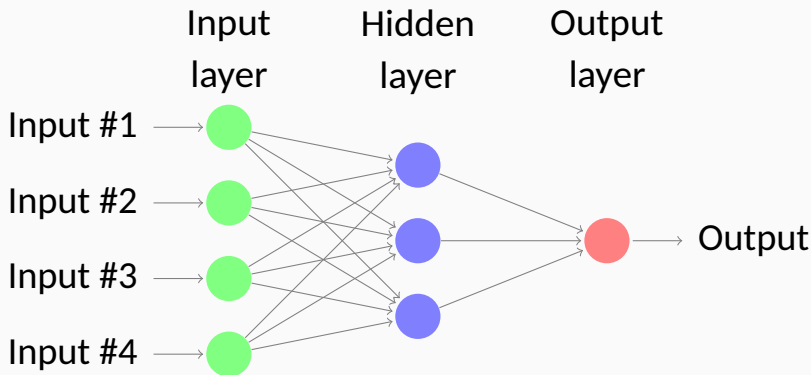
## Simplest version: linear regression



- Coefficients attached to predictors are called “weights”.
- Forecasts are obtained by a linear combination of inputs.
- Weights selected using a “learning algorithm” that

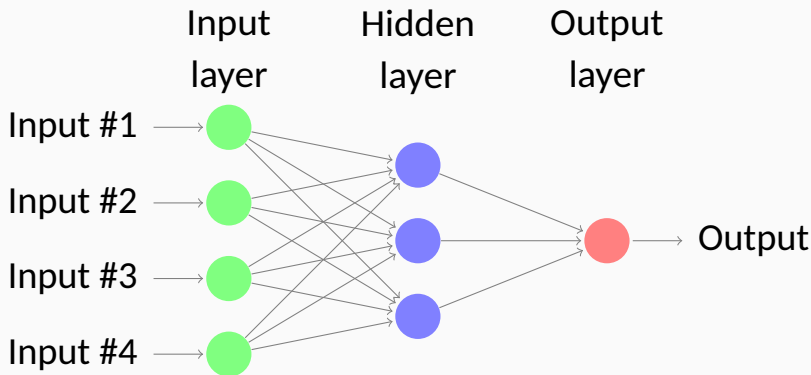
# Neural network models

## Nonlinear model with one hidden layer



# Neural network models

## Nonlinear model with one hidden layer



- A **multilayer feed-forward network** where each layer of nodes receives inputs from the previous layers.
- Inputs to each node combined using linear



# Neural network models

Inputs to hidden neuron  $j$  linearly combined:

$$z_j = b_j + \sum_{i=1}^4 w_{i,j} x_i.$$

Modified using nonlinear function such as a sigmoid:

$$s(z) = \frac{1}{1 + e^{-z}},$$

This tends to reduce the effect of extreme input values, thus making the network somewhat robust to outliers.

# Neural network models

- Weights take random values to begin with, which are then updated using the observed data.
- There is an element of randomness in the predictions. So the network is usually trained several times using different random starting points, and the results are averaged.
- Number of hidden layers, and the number of nodes in each hidden layer, must be specified in advance.

# NNAR models

- Lagged values of the time series can be used as inputs to a neural network.
- $\text{NNAR}(p, k)$ :  $p$  lagged inputs and  $k$  nodes in the single hidden layer.
- $\text{NNAR}(p, 0)$  model is equivalent to an  $\text{ARIMA}(p, 0, 0)$  model but without stationarity restrictions.
- Seasonal  $\text{NNAR}(p, P, k)$ : inputs  $(y_{t-1}, y_{t-2}, \dots, y_{t-p}, y_{t-m}, y_{t-2m}, y_{t-Pm})$  and  $k$  neurons in the hidden layer.
- $\text{NNAR}(p, P, 0)_m$  model is equivalent to an  $\text{ARIMA}(p, 0, 0)(P, 0, 0)_m$  model but without

# NNAR models in R

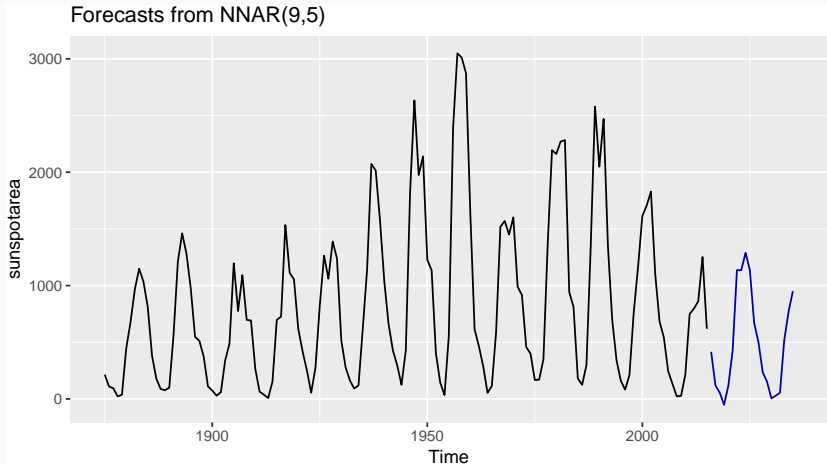
- The `nnetar()` function fits an  $NNAR(p, P, k)_m$  model.
- If  $p$  and  $P$  are not specified, they are automatically selected.
- For non-seasonal time series, default  $p$  = optimal number of lags (according to the AIC) for a linear  $AR(p)$  model.
- For seasonal time series, defaults are  $P = 1$  and  $p$  is chosen from the optimal linear model fitted to the seasonally adjusted data.
- Default  $k = (p + P + 1)/2$  (rounded to the nearest integer)

# Sunspots

- Surface of the sun contains magnetic regions that appear as dark spots.
- These affect the propagation of radio waves and so telecommunication companies like to predict sunspot activity in order to plan for any future difficulties.
- Sunspots follow a cycle of length between 9 and 14 years.

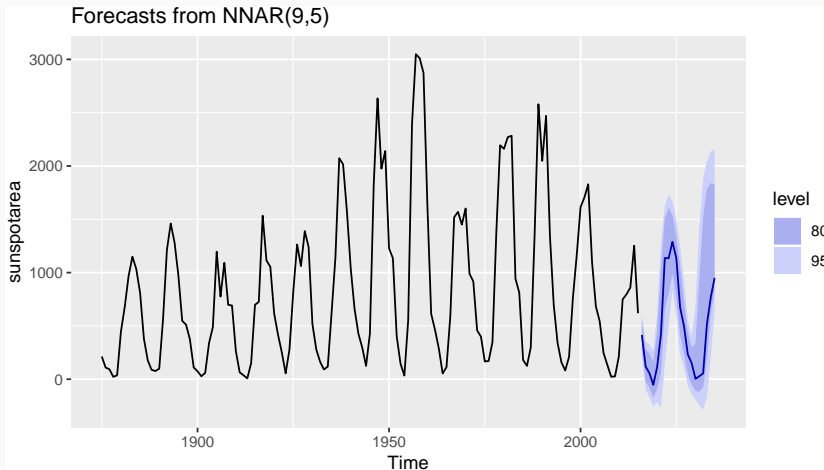
# NNAR(9,5) model for sunspots

```
fit <- nnetar(sunspotarea)  
fit %>% forecast(h=20) %>% autoplot
```



# Prediction intervals by simulation

```
fit %>% forecast(h=20, PI=TRUE) %>% autoplot
```



# Outline

**1** Time series with complex seasonality

**2** Lab session 17

**3** Neural network models

**4** Lab session 18

**5** Lab session 19



# Lab Session 18

# Outline

**1** Time series with complex seasonality

**2** Lab session 17

**3** Neural network models

**4** Lab session 18

**5** Lab session 19

# Lab Session 19