CONTENTS

DEFINITION

$$Let\ q1\ be\ Quaternion\ 1\ , \quad q2\ be\ Quaternion\ 2, \ldots qn, \quad a\ be\ real\ number$$

COMMUTATIVITY

$q1 + q2 = q2 + q1$

$q1 \,.\, q2 \ = q2 \,.\, q1\ (dot\ product)$

$a(q1) = q1(a)$

$q1q2 \neq q2q1$

ASSOOCIATIVITY

$q1 + (q2 + q3) = (q1 + q2) + q3$

$(q1q2)q3 \ = q1(q2q3)$

DISTRIBUTIBITY + ETC

$q1(q2 + q3) = q1q2 + q1q3$

$a(q1 + q2) = aq1 + aq2$

$q1 \,.\, (q2 + q3) = q1.\, q2 + q1.\, q3\ (dot\ product)$

$qq^{-1} = q^{-1}q = I, \qquad |q| = 1, \qquad |q^{-1}| = 1$

$aq^{-1} = q^{-1}a$

$(q1q2)^{-1} = q2^{-1}q1^{-1}$

$Iq = qI = q\ (identity\ rule)$

$if\ |q_1| = \ |q_2| = 1, then\ \ |q_1 q_2| = 1$

$if\ |q| = 1, \ \ then\ q^{-1} = \bar{q}\ (conjugate\ rule\ )$

IMAGINARY RULE

$$i^2 = j^2 = k^2 = ijk = -1$$

Prove:

IJ = K

$$ijk = -1$$

$$ijkk = -1k$$

$$-ij = -k$$

$$ij = k$$

JK = I

$$ijk = -1$$

$$iijk = -1i$$

$$-jk = -i$$

$$jk = i$$

KI = J

$$ijk = -1$$

$$iijki = i - 1i, \qquad ii = \ -1$$

$$-jki = 1$$

$$j(-j)ki = j, \qquad j(-j) = 1$$

$$ki = j$$

JI = -K

$$ij = k$$

$$iiji = iki$$

$$-ji = ij$$

$$ji = -ij$$

$$ji = -k$$

KJ = -I

$$jk = i$$

$$jjkj = jij$$

$$-kj = jk$$

$$kj = -jk$$

$$kj = -i$$

$$ki = j$$

$$kkik = kjk$$

$$-ik = ki$$

$$ik = -ki$$

$$ik = -j$$

OPERATORS

$$q = [real, vector] = [x, y, z, w], \qquad [x, y, z] = vector\ part, \qquad [w] = real\ part$$

ADDITION AND SUBTRATION

$q1 + q2 = [x_1, y_1 z_1, w_1] \pm [x_2, y_2, z_2, w_2] = [x_1 \pm x_2, \qquad y_1 \pm y_2, \qquad z_1 \pm z_2, \qquad w_1 \pm w_2]$

IDENTITY

$q = [0,0,0,1] = I$

MULTIPLICATION (NOTE: USING IMAGINARY RULE AS WELL)

$q_0 q_1 = [x_0, y_0, z_0, w_0] * [x_1, y_1, z_1, w_1]$

$= (w_0 + x_0 i + y_0 j + z_0 k)(w_1 + x_1 i + y_1 j + z_1 k)$

$= w_0 w_1 + w_0 x_1 i + w_0 y_1 j + w_0 z_1 k + x_0 w_1 i - x_0 x_1 + x_0 y_1 ij + x_0 z_1 ik + y_0 w_1 j + y_0 x_1 ji - y_0 y_1 + y_0 z_1 jk + z_0 w_1 k + z_0 x_1 ki + z_0 y_1 kj - z_0 z_1$

$= w_0 w_1 + w_0 x_1 i + w_0 y_1 j + w_0 z_1 k + x_0 w_1 i - x_0 x_1 + x_0 y_1 k - x_0 z_1 j + y_0 w_1 j - y_0 x_1 k - y_0 y_1 + y_0 z_1 i + z_0 w_1 k - z_0 x_1 j - z_0 y_1 i - z_0 z_1 =$

$(w_0 w_1 - x_0 x_1 - y_0 y_1 - z_0 z_1) + \quad (real\ part)$

$(w_0 x_1 + x_0 w_1 + y_0 z_1 - z_0 y_1) i + \quad (x\ part)$

$(w_0 y_1 - x_0 z_1 + y_0 w_1 + z_0 x_1) j + \quad (y\ part)$

$(w_0 z_1 + x_0 y_1 - y_0 x_1 + z_0 w_1) k + \quad (z\ part)$

Alternative form:

$$q_0 q_1 = [real, vector][real, vector] = [w_0, v_0] * [w_1, v_1] = [w_0 w_1 - v_0 . v_1, \qquad w_0 v_1 + w_1 v_0 + v_0 X v_1]$$

Matrix Form:

$$\begin{bmatrix} w_0 & -x_0 & -y_0 & -z_0 \\ x_0 & w_0 & -z_0 & y_0 \\ y_0 & z_0 & w_0 & -x_0 \\ z_0 & -y_0 & x_0 & w_0 \end{bmatrix} \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

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x = lhs.w * rhs.x + lhs.x * rhs.w + lhs.y * rhs.z - lhs.z * rhs.y;
y = lhs.w * rhs.y + lhs.y * rhs.w + lhs.z * rhs.x - lhs.x * rhs.z;
z = lhs.w * rhs.z + lhs.z * rhs.w + lhs.x * rhs.y - lhs.y * rhs.x;
w = lhs.w * rhs.w - lhs.x * rhs.x - lhs.y * rhs.y - lhs.z * rhs.z;
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CONJUGATE

$q = [x, y, z, w], \qquad \bar{q} = [-x, -y, -z, w]\ where\ |q| = 1$

$q\bar{q} = \bar{q}q = I, \qquad |q\bar{q}| = 1, \qquad \bar{\bar{q}} = \overline{\overline{q}\,} = q$

Prove:

$$q\bar{q} = [w_0, v_0] * [w_0, -v_0] = [w_0 w_0 - v_0 . -v_0, \quad -(w_0 v_0) + w_0 v_0 + v_0 X - v_0]$$

$$= \left[w_0 w_0 + v_0 . v_0, \quad \vec{0}\right], \quad since\ is\ a\ unit\ quaternion$$

$$= \left[|q|, \quad \vec{0}\right] = \left[1, \quad \vec{0}\right]$$

INVERT

<u>From Complex number Conjugate:</u>

$z = a + bi, \qquad \bar{z} = a - bi$

$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$

<u>We can solve Invert:</u>

$$\frac{a^2 + b^2}{a^2 + b^2} = 1, \qquad \frac{z\bar{z}}{a^2 + b^2} = 1, \qquad z\left(\frac{\bar{z}}{a^2 + b^2}\right) = 1, \qquad \frac{\bar{z}}{a^2 + b^2} = z^{-1}, \qquad \frac{\bar{z}}{|z|^2} = z^{-1}$$

$Therefore\ the\ Length = \dfrac{1.0}{xx + yy + zz + ww}, \qquad q^{-1} = [-x * length, -y * length, -z * length, w]$

$As\ For\ unit\ Quaternion, \qquad q^{-1} = \bar{q}$

<u>Prove:</u>

$Since\ q^{-1} = [-x * length, -y * length, -z * length, w], \qquad \text{for unit quaternion length} = 1$

$therefore\ q^{-1} = [-x * 1, -y * 1, -z * 1, w] = [-x, -y, -z, w] = Conjugate\ of\ q$

Rectangle form: $a + bi$
Polar form: $re^{i\theta}$

MACLAURIN SERIES EXPANSION

Show that $e^{i\theta} = cos\theta + isin\theta$

$$e^{i\theta} = a + bi$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{\theta^{2n}(-1)^n}{2n!} = 1 - \frac{x\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$sin\theta = \sum_{n=0}^{\infty} \frac{\theta^{2n+1}(-1)^n}{2n+1!} = 1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i\theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}(-1)^n}{2n!} + i\sum_{n=0}^{\infty} \frac{\theta^{2n+1}(-1)^n}{2n+1!}$$

$$= cos\theta + isin\theta$$

3D ROTATION

$$\text{Rectangle form:} \quad a + bi + cj + dk$$

$$\text{Polar form:} \quad re^{q_1 i + q_2 j + q_3 k} = e^{0, \frac{\theta}{2}\hat{n}} = \cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}$$

QUATERNION MATRIX

$$From\ axis - angle\ rotation\ matrix:$$

$$R(\theta, \hat{n}) =$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & n_2 n_1(1 - cos\theta) - n_3(sin\theta) & n_1 n_3(1 - cos\theta) + n_2(sin\theta) & 0 \\ n_1 n_2(1 - cos\theta) + n_3(sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2 n_3(1 - cos\theta) - n_1(sin\theta) & 0 \\ n_1 n_3(1 - cos\theta) - n_2(sin\theta) & n_2 n_3(1 - cos\theta) + n_1(sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

TRIGO IDENTITY

$$\sin\theta = sin2\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = cos2\frac{\theta}{2} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

SUBSTITUTION

We know that:

$$q = (0 < \hat{n} >) = (q_0 < q_1, q_2, q_3 >) = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$$

$$q_0 = \cos\frac{\theta}{2}, \qquad q_1 = n_1\sin\frac{\theta}{2}, \qquad q_2 = n_2\sin\frac{\theta}{2}, \qquad q_3 = n_3\sin\frac{\theta}{2}$$

Step 1:
1st row 2nd col:

$$n_2 n_1(1 - cos\theta) - n_3(sin\theta) = n_2 n_1 2\sin^2\frac{\theta}{2} - n_3 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2n_2\, sin\frac{\theta}{2} n_1\, sin\frac{\theta}{2} - 2n_3\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_1 q_2 - 2q_3 q_0$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1 q_2 - 2q_3 q_0 & n_1 n_3(1 - cos\theta) + n_2(sin\theta) & 0 \\ n_1 n_2(1 - cos\theta) + n_3(sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2 n_3(1 - cos\theta) - n_1(sin\theta) & 0 \\ n_1 n_3(1 - cos\theta) - n_2(sin\theta) & n_2 n_3(1 - cos\theta) + n_1(sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore:
1st row 3nd col:

$$n_1 n_3(1 - cos\theta) + n_2(sin\theta) = n_1 n_3 2\sin^2\frac{\theta}{2} - n_2 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_1q_3 - 2q_2q_0$$

2nd row 1st col:

$$n_1n_2(1 - cos\theta) + n_3(sin\theta) = n_1n_2 2\sin^2\frac{\theta}{2} + n_3 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_1q_2 + 2q_3q_0$$

2nd row 3nd col:

$$n_2n_3(1 - cos\theta) - n_1(sin\theta) = n_2n_3 2\sin^2\frac{\theta}{2} - n_1 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_2q_3 + 2q_1q_0$$

3rd row 1st col:

$$n_1n_3(1 - cos\theta) - n_2(sin\theta) = n_1n_3 2\sin^2\frac{\theta}{2} - n_2 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_1q_3 + 2q_2q_0$$

3rd row 2nd col:

$$n_2n_3(1 - cos\theta) + n_1(sin\theta) = n_2n_3 2\sin^2\frac{\theta}{2} + n_1 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_2q_3 + 2q_1q_0$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & n_1n_3(1 - cos\theta) + n_2(sin\theta) & 0 \\ n_1n_2(1 - cos\theta) + n_3(sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2n_3(1 - cos\theta) - n_1(sin\theta) & 0 \\ n_1n_3(1 - cos\theta) - n_2(sin\theta) & n_2n_3(1 - cos\theta) + n_1(sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & cos\theta + n_2^2(1 - cos\theta) & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2:

1st row 1st col:

$$cos\theta + n_1^2(1 - cos\theta) = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + n_1^2 2\sin^2\frac{\theta}{2}$$

Step 2.5:

$$\sin^2\frac{\theta}{2} = n_1^2\sin^2\frac{\theta}{2} + n_2^2\sin^2\frac{\theta}{2} + n_3^2\sin^2\frac{\theta}{2} = q_1^2 + q_2^2 + q_3^2, \qquad n_1^2 2\sin^2\frac{\theta}{2} = 2q_1^2$$

Step 3:

$$\begin{aligned} cos\theta + n_1^2(1 - cos\theta) &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + n_1^2 2\sin^2\frac{\theta}{2} \\ &= q_0^2 - (q_1^2 + q_2^2 + q_3^2) + 2q_1^2 \\ &= q_0^2 - q_1^2 - q_2^2 - q_3^2 + 2q_1^2 \\ &= q_0^2 + q_1^2 - q_2^2 - q_3^2 \end{aligned}$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & cos\theta + n_2^2(1 - cos\theta) & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & cos\theta + n_2^2(1 - cos\theta) & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore:

2nd row 2nd col:

$$\begin{aligned} cos\theta + n_2^2(1 - cos\theta) &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + n_2^2 2\sin^2\frac{\theta}{2} \\ &= q_0^2 - (q_1^2 + q_2^2 + q_3^2) + 2q_2^2 \\ &= q_0^2 - q_1^2 - q_2^2 - q_3^2 + 2q_2^2 \\ &= q_0^2 - q_1^2 + q_2^2 - q_3^2 \end{aligned}$$

3rd row 3rd col:

$$\begin{aligned} cos\theta + n_3^2(1 - cos\theta) &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} + n_3^2 2\sin^2\frac{\theta}{2} \\ &= q_0^2 - (q_1^2 + q_2^2 + q_3^2) + 2q_3^2 \\ &= q_0^2 - q_1^2 - q_2^2 - q_3^2 + 2q_3^2 \\ &= q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{aligned}$$

Finally:

$$\begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & q_0^2 - q_1^2 - q_2^2 + q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$