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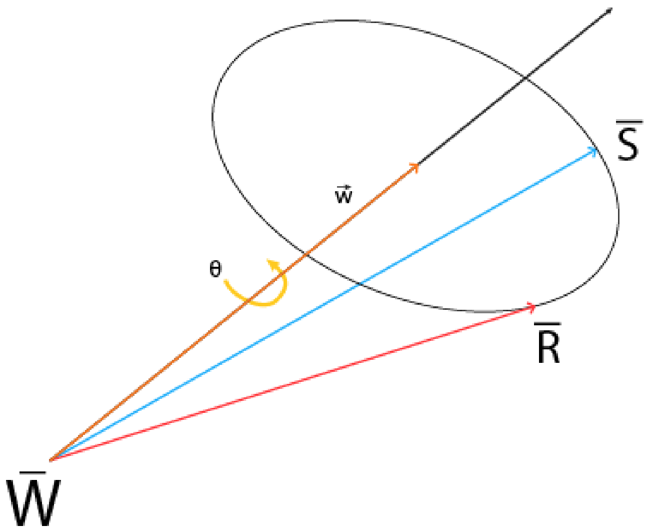
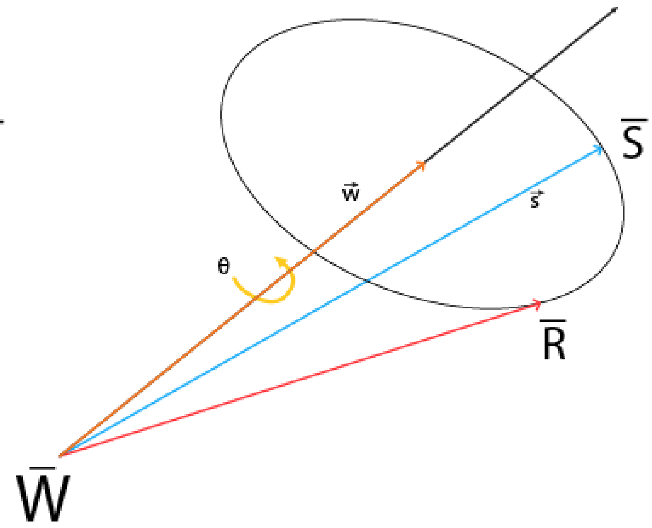
ROTATION BY AND ANGLE ABOUT AXIS



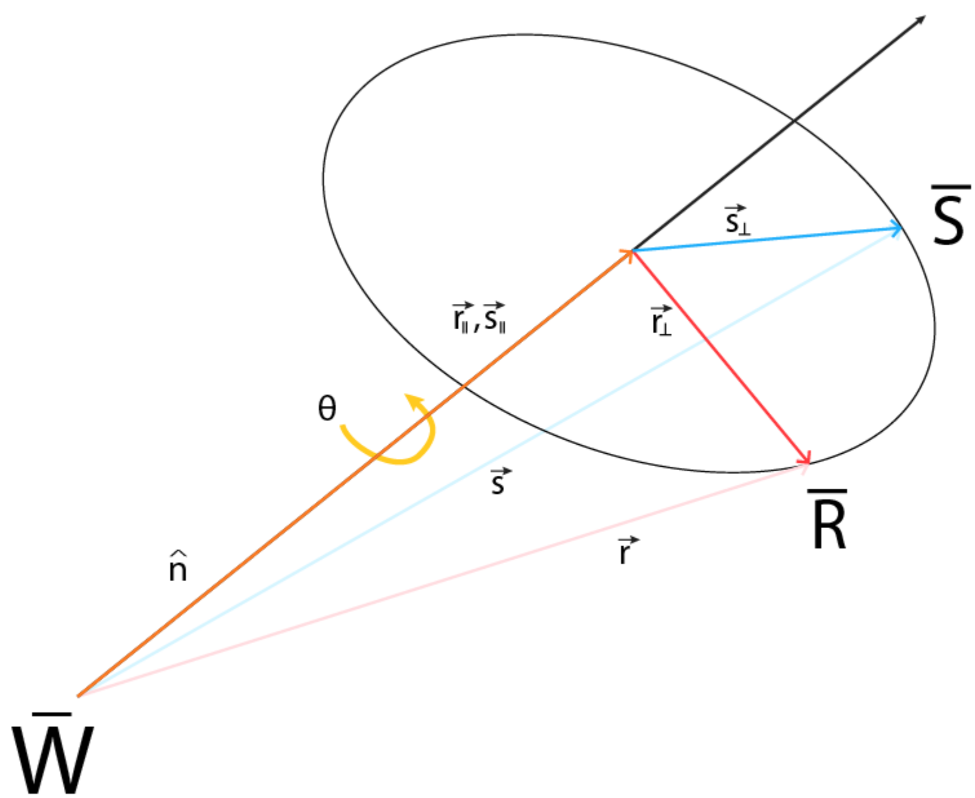


Imagine rotating about an axis $\vec{w}$, where the initial $\bar{R}$ is rotated anti-clockwise at an angle θ to $\bar{S}$

The resulting point will be $\bar{S} = \bar{W} + \vec{s}$

HOW TO FIND $\vec{s}$

Rotation does not change the area/size of the object, when rotating on an axis, it is rotating on a plane.



From the picture, we can compute these vectors

$\bar{R} - \bar{W} = \vec{r}$ (The initial vector)(Known)

$\bar{S} - \bar{W} = \vec{s}$ (The result vector)(Unknown)

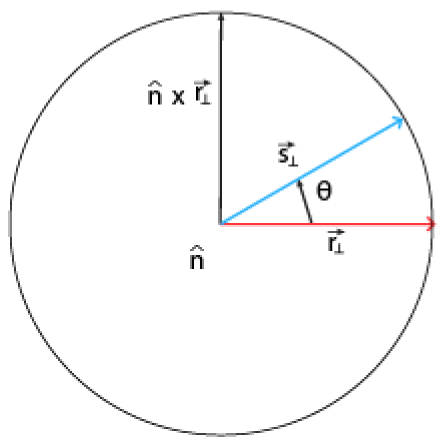
$\text{Normalize } \bar{W}, = \hat{n}$ (Normalize the rotation axis) (Known)

$\vec{s}_{ll} = \vec{r}_{ll} = (\vec{r}\,.\hat{n})\hat{n}$ (The dot product, point$\bar{R}$ and $\bar{S}$ lies on the same plane)

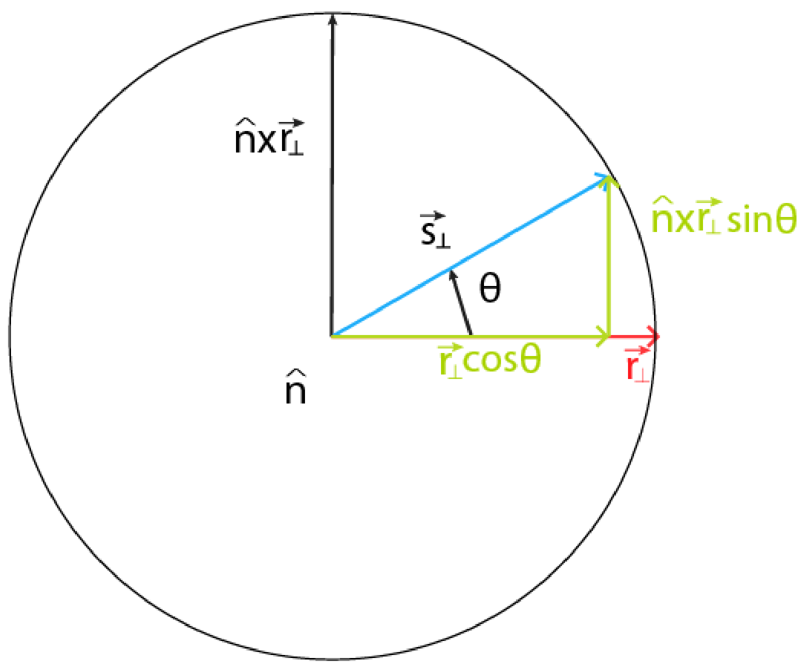
$\vec{r}_{\perp} = \vec{r} - (\vec{r}\,.\hat{n})\hat{n}$ (The dot product)

$\vec{s} = \vec{r}_{ll} + \vec{s}_{\perp}$ (The resulting vector)

HOW TO FIND $\vec{s}_{\perp}$



The bird eye view of the plane looking down from the vector $\hat{n}$. Vector $\hat{n}$ is point outwards and vector $\vec{r}_{\perp}$ is the “x” axis and the “y” axis can be computed by the cross product $\hat{n}\ X\ \vec{r}_{\perp}$.From the picture, we can deduce that $\vec{s}_{\perp}$ is a rotated vector from $\vec{r}_{\perp}$.



Therefore the "x" and "y" components of $\vec{s}_{\perp}$ is $\vec{r}_{\perp}cos\theta$ and $(\hat{n}\, X\, \vec{r}_{\perp})sin\theta$ respectively. The equation can be rewritten as:

$$\vec{s} = \vec{r}_{ll} + \vec{s}_{\perp}$$

$$\vec{s} = \vec{r}_{ll} + \vec{r}_{\perp}cos\theta + (\hat{n}\, X\, \vec{r}_{\perp})sin\theta$$

Rewrite the equation:

$$\vec{r}_{ll} + \vec{r}_{\perp}cos\theta + (\hat{n}\, X\, \vec{r}_{\perp})sin\theta$$

$$\vec{r}_{ll} + (\vec{r} - \vec{r}_{ll})cos\theta + (\hat{n}\, X\, (\vec{r} - (\vec{r}\,.\hat{n})\hat{n}))sin\theta$$

$$\vec{r}_{ll} + \vec{r}cos\theta - \vec{r}_{ll}cos\theta + (\hat{n}\, X\, \vec{r} - \hat{n}\, X\, (\vec{r}\,.\hat{n})\hat{n})sin\theta$$

$$\vec{r}cos\theta + \vec{r}_{ll} - \vec{r}_{ll}cos\theta + (\hat{n}\, X\, \vec{r} - 0)sin\theta$$

$$\vec{r}cos\theta + \vec{r}_{ll}(1 - cos\theta) + (\hat{n}\, X\, \vec{r}\,)sin\theta$$

$$\vec{r}cos\theta + (\vec{r}\,.\hat{n})\hat{n}(1 - cos\theta) + (\hat{n}\, X\, \vec{r}\,)sin\theta$$

TENSOR PRODUCT

Tensor product is used to rewrite an expression, so that it this expression can be converted to a matrix expression.

Example from the top:

$Let\ \vec{t} = (\vec{r}.\hat{n})\hat{n}$

$\vec{t} = \hat{n}(\vec{r}.\hat{n})$ (Scalar multiplication is Commutative)

$\vec{t} = \hat{n}(\hat{n}.\vec{r})$ (Dot product is Commutative)

$\vec{t} = \hat{n}(n^T\vec{r})$ (Dot product rewritten as matrix multiplication)

$\vec{t} = (\hat{n}n^T)\vec{r}$ (Matrix multiplication is Commutative)

$\vec{t} = (\hat{n} \otimes \hat{n})\vec{r}$ (It can be represent using symbols)

SKEW-SYMMETRIX MATRIX

A normal cross product can be written as:

$$\hat{n}\ X\ \vec{r} = \begin{bmatrix} nyrz - nzry \\ nzrx - nxrz \\ nxry - nyrx \end{bmatrix}$$

We can rewrite the matrix as a skew matrix:

$$\hat{n}\ X\ \vec{r} = \ \hat{n}^*\ \vec{r}$$

$$\begin{bmatrix} nyrz - nzry \\ nzrx - nxrz \\ nxry - nyrx \end{bmatrix} = \begin{bmatrix} 0 & -nz & ny \\ nz & 0 & -nx \\ -ny & nx & 0 \end{bmatrix} \begin{bmatrix} rx \\ ry \\ rz \end{bmatrix}$$

A Skew matrix $\hat{n}$ transform a vector $\vec{r}$ into a vector, orthogonal to the plane of these two vector, basically cross product of two vectors.

CONVERT TO MATRIX

With all the conversion in place we can convert to a rotation matrix. Sub vector $\vec{r}$ as an identity matrix

$$\vec{r}cos\theta \ + (\vec{r}\,.\hat{n})\hat{n}(1 - cos\theta) + \ (\hat{n}\ X\ \vec{r}\ )sin\theta$$

$$\vec{r}cos\theta \ + (\ \hat{n}n^{T})\vec{r}\ (1 - cos\theta) + \ N^{*}\vec{r}\ sin\theta$$

$$I\ cos\theta + \ (1 - cos\theta)\begin{bmatrix} nx \\ ny \\ nz \end{bmatrix}[\text{nx ny nz}]\text{I} + sin\theta\begin{bmatrix} 0 & -nz & ny \\ nz & 0 & -nx \\ -ny & nx & 0 \end{bmatrix}\text{I}$$

$$\begin{bmatrix} cos\theta & 0 & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \ (1 - cos\theta)\begin{bmatrix} nx^2 & nynx & nznx & 0 \\ nxny & ny & nzny & 0 \\ nxnz & nynz & nz^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\text{I} + sin\theta\begin{bmatrix} 0 & -nz & ny \\ nz & 0 & -nx \\ -ny & nx & 0 \end{bmatrix}\text{I}$$

Vector $\hat{n}$ is a priciple axis, where it can be any axis in the space.

$$\begin{bmatrix} cos\theta & 0 & 0 & 0 \\ 0 & cos\theta & 0 & 0 \\ 0 & 0 & cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \ 1 - cos\theta\begin{bmatrix} Rx^2 & RyRx & RzRx & 0 \\ RxRy & Ry^2 & RzRy & 0 \\ RxRz & RyRz & Rz^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \ sin\theta\begin{bmatrix} 0 & -Rz & Ry & 0 \\ Rz & 0 & -Rx & 0 \\ -Ry & Rx & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} cos\theta + Rx^2(1 - cos\theta) & RyRx(1 - cos\theta) - Rz(sin\theta) & RxRz(1 - cos\theta) + Ry(sin\theta) & 0 \\ RxRy(1 - cos\theta) + Rz(sin\theta) & cos\theta + Ry^2(1 - cos\theta) & RyRz(1 - cos\theta) - Rx(sin\theta) & 0 \\ RxRz(1 - cos\theta) - Ry(sin\theta) & RyRz(1 - cos\theta) + Rx(sin\theta) & cos\theta + Rz^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

NEGATING $\hat{u}$ AND θ

If the polygon is not rotating, $\hat{u}$ will be a zero vector, likewise, θ will be zero. The rotating matrix will result in an identity matrix.

$$\cos(0) = 1$$

$$\sin(0) = 0$$
$$\hat{u} = \hat{0}$$

$$\begin{bmatrix} 1 + Rx^2(1-1) & RyRx(1-1) - Rz(0) & RxRz(1-1) + Ry(0) & 0 \\ RxRy(1-1) + Rz(0) & 1 + Ry^2(1-1) & RyRz(1-1) - Rx(0) & 0 \\ RxRz(1-1) - Ry(0) & RyRz(1-1) + Rx(0) & 1 + Rz^2(1-1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$I = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}$$