<u>Sum and Product of Roots</u>

1. What is the point of learning this?

No Idea LMAO

2. Definition

- A quadratic equation has the general form $ax^2 + bx + c = 0$
- Making the coefficient of x^2 to be 1, becomes $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
- If we are given the roots of the quadratic equation α and β
  - $(x - \alpha)(x - \beta) = x^2 - \alpha x - \beta x + \alpha\beta = x^2 - (\alpha + \beta)x + \alpha\beta$
- Comparing the coefficient
  - Sum of roots $= (\alpha + \beta) = -\frac{b}{a}$
  - Product of roots $= \alpha\beta = \frac{c}{a}$
  - Where $a = coeff^n of x^2,\ b = coeff^n of x,\ c = constant$
  - Equation 2 in words → $x^2 - (sum\ of\ roots)x + (product\ of\ roots) = 0$

3. Proof

$$Quadratic\ equation \rightarrow ax^2 + bx + c = 0$$

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

a. Sum Of Roots
   a. $\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-b-\sqrt{b^2-4ac}-b+\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$

b. Product Of Roots
   a. $\frac{-b+\sqrt{b^2-4ac}}{2a} . \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{b^2+b\sqrt{b^2-4ac}-b\sqrt{b^2-4ac}-(b^2-4ac)}{4a^2} = -\frac{4ac}{4a^2} = \frac{c}{a}$

4. Useful Identities

- $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
- $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$
  - $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = \alpha^2 + \beta^2 - 2\alpha\beta$
  - $= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta = (\alpha + \beta)^2 - 4\alpha\beta$
- $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$
  - $\alpha^4 + \beta^4 = (\alpha^2)^2 + (\beta^2)^2 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$

5. Exercise

a. Type A – Find the Value of expressions
   a. The equation $2x^2 + 6x - 3 = 0$ <u>has roots α and β</u>. Find the value of

$$a = 2, \qquad b = 6, \qquad c = -3, \qquad \alpha + \beta = -3, \qquad \alpha\beta = -\frac{3}{2}$$

i. $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-3}{-\frac{3}{2}} = 2$

ii. $(2\alpha + 1)(2\beta + 1) = 4\alpha\beta + 2(\alpha + \beta) + 1 = 4\left(-\frac{3}{2}\right) - 6 + 1 = -11$

b. Type B – Forming other equations

a. The equation $2x^2 = 1 - 4x$ has roots α and β. Form an equation whose roots are $\alpha^2 and\ \beta^2$

$$a = 2, \quad b = 4, \quad c = -1, \quad \alpha + \beta = -2, \quad \alpha\beta = -\frac{1}{2}$$

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = -2^2 - 2\left(-\frac{1}{2}\right) = 4 + 1 = 5, \quad (\alpha\beta)^2 = \frac{1}{4}$$

$$Form \to x^2 - (sum\ of\ roots)x + product\ of\ roots = 0$$

$$x^2 - 5x + \frac{1}{4} = 0 \to 4x^2 - 20x + 1 = 0$$

b. The equation $4x^2 - x + 36 = 0$ has roots α^2 and β^2.

i. Find an equation whose roots are $\frac{1}{\alpha^2}\ and\ \frac{1}{\beta^2}$

$$a = 4, \quad b = -1, \quad c = 36, \quad \alpha^2 + \beta^2 = -\left(-\frac{1}{4}\right) = \frac{1}{4}, \quad (\alpha\beta)^2 = \frac{36}{4} = 9$$

$$sum\ of\ roots\ \ \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{(\alpha\beta)^2} = \frac{\frac{1}{4}}{9} = \frac{1}{36}, \quad product\ of\ roots \frac{1}{\alpha^2}\frac{1}{\beta^2} = \frac{1}{9}$$

$$Form \to x^2 - (sum\ of\ roots)x + product\ of\ roots = 0$$

$$x - \frac{1}{36}x + \frac{1}{9} = 0 \to 36x^2 - x + 4 = 0$$

ii. Two Distinct equations whose roots are α and β

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta \to \quad (\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

Find $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = \frac{1}{4} + 2(\pm 3) = \frac{1}{4} \pm 6 = \frac{25}{4}, -5\frac{3}{4}(NA)$

$$\alpha + \beta = \sqrt{\frac{25}{4}} = \pm\frac{5}{2}, \quad (\alpha\beta)^2 = 9 \to \alpha\beta = 3$$

$$Form \to x^2 - (sum\ of\ roots)x + product\ of\ roots = 0$$

$$x^2 - \pm\frac{5}{2}x + 3 = 0$$

$$2x^2 \pm 5x + 6 = 0$$

c. Type C – Solve Unknowns

a. Given that $x^2 + (2-k)x + k = 0$ have non-zero roots which differ by 2, find the value of each root and of k.

Let roots be $\alpha, \alpha + 2$

$$a = 1, \quad b = 2 - k, \quad c = k,$$

$$Sum\ of\ roots \to \alpha + \alpha + 2 = -(2 - k)$$

$$2\alpha + 2 = -2 + k$$

$$k = 2\alpha + 4$$

$$Product\ of\ roots \to \alpha(\alpha + 2) = k$$

$$k = \alpha^2 + 2\alpha$$

$$\alpha^2 + 2\alpha = 2\alpha + 4$$

$$\alpha = 2$$