Partial Fractions

Contents

1. Why learn Partial Fraction?

a. To be able to get the derivative or get the integral more easily in Calculus

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2. What is Partial Fraction?

a. It's a **method to simplify algebraic fraction**

b. Initially we learned how to add or subtract 2 **proper fraction**:

i. $\frac{2}{x-1} + \frac{2}{x+2} = \frac{2(x+2)+2(x-1)}{(x-1)(x+2)} = \frac{4x+4+2x-2}{(x-1)(x+2)} = \frac{6x+2}{(x-1)(x+2)}$

ii. $\frac{2}{x-1} - \frac{2}{x+2} = \frac{2(x+2)-2(x-1)}{(x-1)(x+2)} = \frac{4x+4-2x+2}{(x-1)(x+2)} = \frac{2x+6}{(x-1)(x+2)}$

c. **Partial Fraction** is to **reverse the process** by decomposing:

i. $\frac{6x+2}{(x-1)(x+2)} \rightarrow \frac{2}{x-1} + \frac{2}{x+2}$

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3. Definitions (introduction)

a. A ratio of 2 polynomials f(x) and g(x), that is $\frac{f(x)}{g(x)}$ is called an **algebraic fraction**.

i. Example: $\frac{x+1}{x^2-4}, \frac{2x}{2x^2-4}, \frac{2x^4}{2x^4-4}$

b. If the **degree of the numerator** f(x) is **less than the degree of the denominator** g(x), then the fraction $\frac{f(x)}{g(x)}$ is said to be **proper fraction**.

i. Example: $\frac{2x}{2x^2-4}$

c. If the **degree of the numerator** f(x) is **greater than or equal to the degree** of the denominator g(x), the fraction $\frac{f(x)}{g(x)}$ is said to be **improper fraction**.

i. Example: $\frac{2x^4}{2x^4-4}, \frac{2x^5}{2x^4-4}$

d. A **simple fraction** is a **proper fraction** of which the **denominator cannot be factorized**

i. Example:

✗	✓	✗	✗	✓
$\frac{x-1}{2x^2-4}$	$\frac{x-1}{(x-2)(x-3)}$	$\frac{3x^3}{1-x^3}$	$\frac{2x}{(x-1)(2x^2-4)}$	$\frac{x^2+6x+9}{(x^2+3)(x+5)}$

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4. Objective

a. To express **algebraic fraction** as **partial fraction**

  i. It must be a **proper Fraction**

  ii. The **denominator must be factorized completely**

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5. Patterns and Forms

a. The **denominator** of the single fraction determine the forms of the partial fractions

Name	Single fractions	Possible Forms of Partial Fractions
Distinct Linear	$\dfrac{\mathrm{f(x)}}{(ax+b)(cx+d)}$	$\dfrac{\mathrm{A}}{ax+b}+\dfrac{\mathrm{B}}{cx+d}$
Repeated Linear	$\dfrac{\mathrm{f(x)}}{(ax+b)(cx+d)^2}$	$\dfrac{\mathrm{A}}{ax+b}+\dfrac{\mathrm{B}}{x+d}+\dfrac{\mathrm{C}}{(cx+d)^2}$
Distinct Quadratic	$\dfrac{\mathrm{f(x)}}{(ax+b)(x^2+c)}$	$\dfrac{\mathrm{A}}{ax+b}+\dfrac{\mathrm{Bx+C}}{x^2+c}$

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6. Practice, example using different method to solve

1a. Linear	$\frac{5x+1}{(x-1)(x+2)}$

Solve:

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

Let: a = 1, b = -1, c = 1, d = 2

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x+2)}$$

$$\frac{5x+1}{(x-1)(x+2)} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$$

$$\text{Let} \rightarrow\ 5x+1 = A(x+2)+B(x-1)$$

Find A, Input x = 1:

$$5(1)+1 = A(1+2)+B(1-1)$$
$$6 = 3A$$
$$A = 2$$

Find B, Input x = -2

$$5(-2)+1 = A(-2+2)+B(-2-1)$$
$$-9 = -3B$$
$$B = 3$$

Answer:

$$\frac{5x+1}{(x-1)(x+2)} = \frac{2}{(x-1)} + \frac{3}{(x+2)}$$

Check:

$$\frac{2}{(x-1)} + \frac{3}{(x+2)} = \frac{2(x+2)+3(x-1)}{(x-1)(x+2)} = \frac{2x+4+3x-3}{(x-1)(x+2)} = \frac{5x+1}{(x-1)(x+2)}$$

1b. Linear, Factorize base	$\frac{2\mathrm{x}-13}{x^2-3x-4}$

Factorize base:

$$\frac{2\mathrm{x}-13}{x^2-3x-4}=\frac{2\mathrm{x}-13}{(x+1)(x-4)}$$

Solve:

$$\frac{2\mathrm{x}-13}{(x+1)(x-4)}=\frac{\mathrm{A}}{(x+1)}+\frac{\mathrm{B}}{(x-4)}$$

$$\frac{2\mathrm{x}-13}{(x+1)(x-4)}=\frac{\mathrm{A}(x-4)+\mathrm{B}(x+1)}{(x+1)(x-4)}$$

$$\text{Let} \rightarrow 2\mathrm{x}-13= \mathrm{A}(x-4)+\mathrm{B}(x+1)$$

Find A, Input x = -1:

$$2(-1)-13= \mathrm{A}(-1-4)+\mathrm{B}(-1+1)$$
$$-15= -5\mathrm{A}$$
$$A=3$$

Find B, Input x = 4

$$2(4)-13= \mathrm{A}(4-4)+\mathrm{B}(4+1)$$
$$-5= 5\mathrm{B}$$
$$B=-1$$

Answer:

$$\frac{2\mathrm{x}-13}{x^2-3x-4}=\frac{3}{(x+1)}-\frac{1}{(x-4)}$$

Check:

$$\frac{3}{(x+1)}-\frac{1}{(x-4)}=\frac{3(x-4)-1(x+1)}{(x+1)(x-4)}=\frac{3\mathrm{x}-12-1x-1}{(x+1)(x-4)}=\frac{2\mathrm{x}-13}{(x+1)(x-4)}$$

2a. Repeated Linear	$\frac{x+1}{(x-1)^2}$

Solve:

$$\frac{x+1}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

$$\frac{x+1}{(x-1)^2} = \frac{A(x-1)^2 + B(x-1)}{(x-1)^2} \rightarrow \frac{x+1}{(x-1)} = \frac{A(x-1)+B}{(x-1)}$$

$$\text{Let} \rightarrow x+1 = A(x-1)+B$$

Find B, Input x = 1:

$$1+1 = A(1-1)+B$$
$$B = 2$$

Find A, Input x = 0:

$$1 = -A+2$$
$$A = 1$$

Answer:

$$\frac{x+1}{(x-1)^2} = \frac{1}{(x-1)} + \frac{2}{(x-1)^2}$$

Check:

$$\frac{1}{(x-1)} + \frac{2}{(x-1)^2} = \frac{(x-1)^2 + 2(x-1)}{(x-1)(x-1)^2} = \frac{x-1+2}{(x-1)^2} = \frac{x+1}{(x-1)^2}$$

2b. Repeated Linear	$\frac{6x-9}{(x+3)^2}$

$$\frac{6x-9}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$$

$$\frac{6x-9}{(x+3)^2} = \frac{A(x+3)^2 + B(x+3)}{(x+3)^2} \rightarrow \frac{6x-9}{(x+3)} = \frac{A(x+3)+B}{(x+3)}$$

Solve:

$$\text{Let} \rightarrow 6x-9 = A(x+3)+B$$

Find B, Input x = -3:

$$-18-9 = B$$
$$B = -27$$

Find A, Input x = 0:

$$-9 = 3A-27$$
$$A = 6$$

Answer:

$$\frac{6x-9}{(x+3)^2} = \frac{6}{(x+3)} - \frac{27}{(x+3)^2}$$

Check:

$$\frac{6}{(x+3)} - \frac{27}{(x+3)^2} = \frac{6(x+3)^2 + 2(x+3)}{(x+3)(x+3)^2} = \frac{6x+18-27}{(x+3)^2} = \frac{6x-9}{(x+3)^2}$$

3a. Distinct Quadratic	$\frac{17\mathrm{x}^2+23x+12}{(3x+4)(x^2+4)}$

$$\frac{17\mathrm{x}^2+23x+12}{(3x+4)(x^2+4)} = \frac{\mathrm{A}}{(3x+4)} + \frac{\mathrm{Bx}+\mathrm{C}}{(x^2+4)}$$

$$\text{Let} \rightarrow 17\mathrm{x}^2+23x+12 = \mathrm{A}(x^2+4)+(\mathrm{Bx}+\mathrm{C})(3x+4)$$

Solve:

Find A, Input x $=-\frac{4}{3}$:

$$17\left(-\frac{4}{3}\right)^2+23\left(-\frac{4}{3}\right)+12 = \mathrm{A}\left(\left(-\frac{4}{3}\right)^2+4\right)+(\mathrm{Bx}+\mathrm{C})\left(3\left(-\frac{4}{3}\right)+4\right)$$

$$\frac{17*16}{9}-\frac{92}{3}+12 = \frac{16A}{9}+4A+0$$

$$272-276+108 = 16A+36A$$

$$104 = 52A$$

$$A = 2$$

Expand:

$$17\mathrm{x}^2+23x+12 = (\mathrm{A}+3\mathrm{B})x^2+(4\mathrm{B}+3\mathrm{C})\mathrm{x}+4A+4\mathrm{C}$$

Find B, $Coeff^n x^2$

$$17 = \mathrm{A}+3\mathrm{B}$$
$$\mathrm{B} = 5$$

Find C, $Coeff^n x^0$

$$12 = 4A+4\mathrm{C}$$
$$3 = \mathrm{A}+\mathrm{C}$$
$$\mathrm{C} = 1$$

Answer:

$$\frac{17\mathrm{x}^2+23x+12}{(3x+4)(x^2+4)} = \frac{2}{(3x+4)} + \frac{5\mathrm{x}+1}{(x^2+4)}$$

Check:

$$\frac{2}{(3x+4)}+\frac{5\mathrm{x}+1}{(x^2+4)} = \frac{2(x^2+4)+(5\mathrm{x}+1)(3x+4)}{(3x+4)(x^2+4)} = \frac{2x^2+8+15\mathrm{x}^2+20x+3x+4}{(3x+4)(x^2+4)} = \frac{17\mathrm{x}^2+23x+12}{(3x+4)(x^2+4)}$$

3b. Distinct Quadratic	$\frac{5}{(x+2)(x^2+1)}$

$$\frac{5}{(x+2)(x^2+1)} = \frac{\mathrm{A}}{(x+2)} + \frac{\mathrm{Bx}+\mathrm{C}}{(x^2+1)}$$

$$\text{Let} \rightarrow\ 5 = \mathrm{A}(x^2+1) + (\mathrm{Bx}+\mathrm{C})(x+2)$$

Solve:

Find A, Input x = -2

$$5 = \mathrm{A}(-2^2+1) + (\mathrm{Bx}+\mathrm{C})(-2+2)$$

$$5 = 5A$$

$$A = 1$$

Find C, Input x = 0

$$5 = \mathrm{A}(0^2+1) + (\mathrm{B}(0)+\mathrm{C})(0+2)$$

$$5 = 1 + 2\mathrm{C}$$

$$\mathrm{C} = 2$$

Find B, Input x = 1

$$5 = \mathrm{A}(1^2+1) + (\mathrm{B}+2)3$$
$$5 = 2 + 3B + 6$$
$$-3 = 3B$$
$$\mathrm{B} = -1$$

Answer:

$$\frac{5}{(x+2)(x^2+1)} = \frac{1}{(x+2)} + \frac{2-\mathrm{x}}{(x^2+1)}$$

Check:

$$\frac{1}{(x+2)} + \frac{2-\mathrm{x}}{(x^2+1)} = \frac{x^2+1+(2-\mathrm{x})(x+2)}{(x+2)(x^2+1)} = \frac{x^2+1+2\mathrm{x}+4-x^2-2x}{(x+2)(x^2+1)} = \frac{5}{(x+2)(x^2+1)}$$

4. Tricky Denominator (repeated)	$\frac{2\mathrm{x}}{(x-1)(x^2-1)}$

$$\frac{2\mathrm{x}}{(x-1)(x^2-1)} = \frac{2\mathrm{x}}{(x-1)(x-1)(x+1)}\frac{2\mathrm{x}}{(x-1)^2(x+1)} = \frac{\mathrm{A}}{(x+1)} + \frac{\mathrm{B}}{(x-1)} + \frac{\mathrm{C}}{(x-1)^2}$$

$$\text{Let} \rightarrow 2\mathrm{x} = \mathrm{A}(x-1)^2 + B(\mathrm{x}-1)(x+1) + C(x+1)$$

<u>Solve:</u>

Find C, Input x = 1

$$2 = \mathrm{A}(x-1)^2 + B(\mathrm{x}-1)(x+1) + C(x+1)$$

$$2 = 2C$$

$$C = 1$$

Find A, Input x = -1

$$-2 = 4\mathrm{A}$$
$$\mathrm{A} = -\frac{1}{2}$$

Find B, Input x = 0

$$0 = -\frac{1}{2} - B + 1$$
$$\mathrm{B} = \frac{1}{2}$$

<u>Answer:</u>

$$\frac{2\mathrm{x}}{(x-1)(x^2-1)} = -\frac{1}{2(x+1)} + \frac{1}{2(x-1)} + \frac{1}{(x-1)^2}$$

<u>Check:</u>

$$-\frac{1}{2(x+1)} + \frac{1}{2(x-1)} + \frac{1}{(x-1)^2} = \frac{-2(x-1)(x-1)^2 + 2(x+1)(x-1)^2 + 2(x-1)2(x+1)}{4(x+1)(x-1)(x-1)^2}$$

$$= \frac{-2(x-1)(x-1) + 2(x+1)(x-1) + 4(x+1)}{4(x+1)(x-1)(x-1)}$$

$$= \frac{-2(x^2-2x+1) + 2(x^2-1) + 4x + 4}{4(x+1)(x-1)(x-1)} = \frac{-2x^2 + 4x - 2 + 2x^2 - 2 + 4x + 4}{4(x+1)(x-1)(x-1)}$$

$$= \frac{8x}{4(x+1)(x-1)(x-1)} = \frac{2\mathrm{x}}{(x-1)(x^2-1)}$$

5. Long Division	$\frac{x^2+4x+5}{x^2+5x+6}$

Long Division:

$$x^2+5x+6 \overline{)\,x^2+4x+5}$$

$$\begin{array}{r} 1 \\ x^2+5x+6 \,\overline{)\,x^2+4x+5} \\ -\underline{\;x^2+5x+6} \\ -x-1 \end{array}$$

$$\frac{x^2+4x+5}{x^2+5x+6} = 1 + \frac{-x-1}{x^2+5x+6}$$

$$From\ right\ hand\ side \rightarrow \frac{-x-1}{x^2+5x+6} = \frac{A}{(x+3)} + \frac{B}{(x+2)}$$

$$\text{Let} \rightarrow -x-1 = A(x+2) + B(x+3)$$

Solve:

Find A, Input x = -3

$$2 = -A$$

$$A = -2$$

Find B, Input x = -2

$$1 = B$$
$$B = 1$$

Answer:

$$\frac{x^2+4x+5}{x^2+5x+6} = 1 - \frac{2}{(x+3)} + \frac{1}{(x+2)}$$

Check:

$$1 - \frac{2}{(x+3)} + \frac{1}{(x+2)} = 1 + \frac{-2x-4+x+3}{x^2+5x+6}$$

$$= \frac{x^2+5x+6-x-1}{x^2+5x+6} = \frac{x^2+4x+5}{x^2+5x+6}$$

6. Long Division (Integral)	$\int \frac{x^3 - x^2 - 6}{(x-1)(x+1)} dx$

Long Division:

$$\frac{x^3 - x^2 - 6}{(x-1)(x+1)} = \frac{x^3 - x^2 - 6}{x^2 - 1}$$

$$x^2 - 1 \overline{\smash{)}\, x^3 - x^2 - 6}$$

$$\begin{array}{r} x - 1 \\ x^2 - 1 \overline{\smash{)}\, x^3 - x^2 - 6} \\ - \quad -x^3 + x \\ \hline - \quad -x^2 + x - 6 \\ -x^2 + 1 \\ \hline x - 7 \end{array}$$

$$\frac{x^3 - x^2 - 6}{(x-1)(x+1)} = x - 1 + \frac{x - 7}{(x-1)(x+1)}$$

$$From\ right\ hand\ side \rightarrow \frac{x - 7}{(x-1)(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x+1)}$$

$$\text{Let} \rightarrow x - 7 = A(x+1) + B(x-1)$$

<u>Solve:</u>

Find A, Input x = 1

$$-6 = 2A$$

$$A = -3$$

Find B, Input x = -1

$$-8 = -2B$$
$$B = 4$$

<u>Answer:</u>

$$\int \frac{x^3 - x^2 - 6}{(x-1)(x+1)} dx = \int x - 1 - \frac{3}{(x-1)} + \frac{4}{(x+1)} dx$$

<u>Check:</u>

$$x - 1 - \frac{3}{(x-1)} + \frac{4}{(x+1)} = \frac{x^3 - x - x^2 + 1 - 3x - 3 + 4x - 4}{(x-1)(x+1)}$$

$$= \frac{x^3 - x^2 - 6}{(x-1)(x+1)}$$

7. Tips/Summary

a. **Use long division** for **improper fraction,** which is the **degree of numerator is greater than or equal than denominator. It is an extra needed step.**

   i. You may have notice the term **improper fraction, it is the exact same term** used for normal, non-polynomial fraction example like, $\frac{10}{3}$

   ii. Proper Fraction will be example like $3\frac{1}{3}$

b. See, notice and use the pattern for **linear, repeated and distinct quadratic respectively**

c. **Mainly 2 method to find A, B, C, D …**

   i. Substitute x = value

   ii. Expand the equation and use coefficient of x

8. Exercises

1	$$\frac{3x^2+23x+45}{x(x+3)}$$
2	$$\frac{x^2}{9-x^2}$$
3	$$\frac{9x^2}{(x-1)^2(x+2)}$$
4	$$\frac{8+3x-2x^2}{(x-1)(x+2)^2}$$
5	$$\frac{1+x-2x^2}{(2-x)(1+x^2)}$$
6	$$\frac{14x}{(x+2)(x-5)}$$
7	$$\frac{28}{x^2-49}$$
8	$$\frac{2x^2+1}{x(x-1)^2}$$
9	$$\frac{8x}{4x^2+4x-3}$$
10	$$\frac{2x^2+5}{(x-2)(x^2+9)}$$
11	$$\frac{x^3+4x+4}{x^2(x^2+4)}$$
12	$$\frac{x^3-5x+3}{x-2}$$
13	$$\frac{x^2+8x+9}{x^2+3x+2}$$
14	$$\frac{15x^3-11x^2-33x}{5x-2}$$
15	$$\frac{(3x-8)(x^2-6x+2)}{x^2-8x+12}$$
16	$$\frac{6x^3+4x^2-3x}{3x+2} \equiv ax^2+b+\frac{c}{3x+2}$$ Find a, b, c

9. Answers

1	$\frac{3x^2+23x+45}{x(x+3)}$

Long Division:

$$\frac{3x^2+23x+45}{x(x+3)} = \frac{3x^2+23x+45}{x^2+3x}$$

$$\begin{array}{r|l} & \ \ 3 \\ \hline x^2+3x & 3x^2+23x+45 \\ & -(3x^2+9x) \\ \hline & \qquad 14x+45 \end{array}$$

$$\frac{3x^2+23x+45}{x^2+3x} = 3 + \frac{14x+45}{x(x+3)}$$

$$From\ right\ hand\ side \rightarrow \frac{14x+45}{x(x+3)} = \frac{A}{x} + \frac{B}{(x+3)}$$

$$\text{Let} \rightarrow 14x+45 = A(x+3) + Bx$$

<u>Solve:</u>

Find A, Input x = 0

$$45 = 3A$$

$$A = 15$$

Find B, Input x = -3

$$3 = -3B$$
$$B = -1$$

<u>Answer:</u>

$$\frac{3x^2+23x+45}{x(x+3)} = 3 + \frac{15}{x} - \frac{1}{(x+3)}$$

2	$\frac{x^2}{9-x^2}$

Long Division:...

$$\frac{x^2}{9-x^2} = -1 + \frac{9}{(3-x)(3+x)}$$

$$From\ right\ hand\ side \rightarrow \frac{9}{(3-x)(3+x)} = \frac{A}{(3-x)} + \frac{B}{(3+x)}$$

$$\text{Let} \rightarrow 9 = A(3+x) + B(3-x)$$

Solve:

Find A, Input x = 3

$$9 = 6A$$

$$A = \frac{3}{2}$$

Find B, Input x = -3

$$9 = 6B$$

$$B = \frac{3}{2}$$

Answer:

$$\frac{x^2}{9-x^2} = \frac{3}{2(3-x)} + \frac{3}{2(3+x)} - 1$$

3	$\frac{9x^2}{(x-1)^2(x+2)}$

$$Let \rightarrow \frac{9x^2}{(x-1)^2(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\text{Let} \rightarrow 9x^2 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

<u>Solve:</u>

Find C, Input x = 1

$$9 = 3C$$

$$C = 3$$

Find A, Input x = -2

$$36 = 9A$$

$$A = 4$$

Find B, Input x = 0

$$0 = 4 - 2B + 6$$

$$B = -5$$

<u>Answer:</u>

$$\frac{9x^2}{(x-1)^2(x+2)} = \frac{4}{(x+2)} - \frac{5}{(x-1)} + \frac{3}{(x-1)^2}$$

4	$\frac{8+3x-2x^2}{(x-1)(x+2)^2}$

$$Let \rightarrow \frac{8+3x-2x^2}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$$

$$\text{Let} \rightarrow 8+3x-2x^2 = A(x+2)^2 + B(x+2)(x-1) + C(x-1)$$

<u>Solve:</u>

Find A, Input x = 1

$$9 = 9A$$

$$A = 1$$

Find C, Input x = -2

$$-6 = -3C$$

$$C = 2$$

Find B, Input x = 0

$$8 = 4 - 2B - 2$$

$$6 = -2B$$

$$B = -3$$

<u>Answer:</u>

$$\frac{8+3x-2x^2}{(x-1)(x+2)^2} = \frac{1}{(x-1)} - \frac{3}{(x+2)} + \frac{2}{(x+2)^2}$$

5	$\frac{1+x-2x^2}{(2-x)(1+x^2)}$

$$Let \rightarrow \frac{1+x-2x^2}{(2-x)(1+x^2)} = \frac{A}{(2-x)} + \frac{Bx+C}{(1+x^2)}$$

$$\text{Let} \rightarrow 1+x-2x^2 = A(1+x^2) + (Bx+C)(2-x)$$

<u>Solve:</u>

Find A, Input x = 2

$$-5 = 5A$$

$$A = -1$$

Find C, Input x = 0

$$1 = A + 2C$$

$$1 = -1 + 2C$$

$$2 = 2C$$

$$C = 1$$

Find B, Input x = 1

$$0 = -2 + B + 1$$

$$B = 1$$

<u>Answer:</u>

$$\frac{1+x-2x^2}{(2-x)(1+x^2)} = -\frac{1}{(2-x)} + \frac{x+1}{(1+x^2)}$$

6	$\frac{14x}{(x+2)(x-5)}$

Solve:

$$\frac{14x}{(x+2)(x-5)} = \frac{A}{(x+2)} + \frac{B}{(x-5)}$$

$$\text{Let} \rightarrow\ 14x = A(x-5) + B(x+2)$$

Find A, Input x = -2:

$$-28 = -7A$$
$$A = 4$$

Find B, Input x = 5:

$$70 = 7B$$
$$B = 10$$

Answer:

$$\frac{14x}{(x+2)(x-5)} = \frac{4}{(x+2)} + \frac{10}{(x-5)}$$

7	$\frac{28}{x^2-49}$

Solve:

$$\frac{28}{x^2-49} = \frac{A}{(x-7)} + \frac{B}{(x+7)}$$

$$\text{Let} \rightarrow\ 28 = A(x+7) + B(x-7)$$

Find A, Input x = 7:

$$28 = 14A$$
$$A = 2$$

Find B, Input x = -7:

$$28 = -14B$$
$$B = -2$$

Answer:

$$\frac{28}{x^2-49} = \frac{2}{(x-7)} - \frac{2}{(x+7)}$$

8	$\frac{2x^2+1}{x(x-1)^2}$

Solve:

$$\frac{2x^2+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$$

$$\text{Let} \rightarrow \; 2x^2 + 1 = A(x-1)^2 + Bx(x-1) + Cx$$

Find A, Input x = 0:

$$1 = A$$

Find C, Input x = 1:

$$3 = C$$

Find B, Input x =2:

$$9 = 1 + 2B + 6$$
$$2 = 2B$$
$$B = 1$$

Answer:

$$\frac{2x^2+1}{x(x-1)^2} = \frac{1}{x} + \frac{1}{(x-1)} + \frac{3}{(x-1)^2}$$

9	$\frac{8x}{4x^2+4x-3}$

Solve:

$$\frac{8x}{4x^2+4x-3} = \frac{A}{(2x+3)} + \frac{B}{(2x-1)}$$

$$\text{Let} \rightarrow \; 8x = A(2x-1) + B(2x+3)$$

Find B, Input x = 1/2:

$$4 = 4B$$
$$1 = B$$

Find A, Input x = -3/2:

$$-12 = -4A$$
$$3 = A$$

Answer:

$$\frac{8x}{4x^2+4x-3} = \frac{3}{(2x+3)} + \frac{1}{(2x-1)}$$

10	$\frac{2x^2+5}{(x-2)(x^2+9)}$

Solve:

$$\frac{2x^2+5}{(x-2)(x^2+9)} = \frac{A}{(x-2)} + \frac{Bx+C}{(x^2+9)}$$

$$\text{Let} \rightarrow\ 2x^2+5 = A(x^2+9) + (Bx+C)(x-2)$$

Find A, Input x = 2:

$$13 = 13A$$
$$1 = A$$

Find C, Input x = 0:

$$5 = 9 - 2C$$
$$-4 = -2C$$
$$2 = C$$

Find B, Input x =1:

$$7 = 10 + (B+2)(-1)$$
$$3 = B + 2$$
$$1 = B$$

Answer:

$$\frac{2x^2+5}{(x-2)(x^2+9)} = \frac{1}{(x-2)} + \frac{x+2}{(x^2+9)}$$

11	$\frac{x^3 + 4x + 4}{x^2(x^2 + 4)}$

Solve:

$$\frac{x^3 + 4x + 4}{x^2(x^2 + 4)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{(x^2 + 4)}$$

$$\text{Let} \rightarrow \; x^3 + 4x + 4 = Ax(x^2 + 4) + B(x^2 + 4) + (Cx + D)x^2$$

Find B, Input x = 0:

$$4 = 4B$$
$$1 = B$$

Expand

$$x^3 + 4x + 4 = Ax^3 + 4Ax + Bx^2 + 4 + Cx^3 + Dx^2$$

$$x^3 + 4x + 4 = (A + C)x^3 + (B + D)x^2 + 4Ax + 4$$

Find A, Coeff^n of x:

$$4 = 4A$$
$$1 = A$$

Find D, Coeff^n of x^2:

$$0 = 1 + D$$
$$-1 = D$$

Find C, Coeff^n of x^3:

$$1 = 1 + C$$
$$0 = C$$

Answer:

$$\frac{x^3 + 4x + 4}{x^2(x^2 + 4)} = \frac{1}{x} + \frac{1}{x^2} - \frac{1}{(x^2 + 4)}$$

12	$\frac{x^3 - 5x + 3}{x - 2}$

Long Division:

$$\begin{array}{rl}
 & x^2 + 2x - 1 \\
x - 2 \,\big) & x^3 - 5x + 3 \\
- & \underline{x^3 - 2x^2} \\
 & \quad 2x^2 - 5x + 3 \\
 - & \quad \underline{2x^2 - 4x} \\
 & \qquad -x + 3 \\
 - & \qquad \underline{-x + 2} \\
 & \qquad\qquad 1
\end{array}$$

Answer:

$$\frac{x^3 - 5x + 3}{x - 2} = x^2 + 2x - 1 + \frac{1}{x - 2}$$

13	$\frac{x^2+8x+9}{x^2+3x+2}$

Long Division:

$$\frac{x^2+8x+9}{x^2+3x+2} = 1 + \frac{5x+7}{(x+1)(x+2)}$$

$$Let \rightarrow \frac{5x+7}{(x+1)(x+2)} = \frac{A}{(x+1)} + \frac{B}{(x+2)}$$

$$\text{Let} \rightarrow\ 5x+7 = \ A(x+2) + B(x+1)$$

Find A, Input x = -1:

$$2 = \ A$$

Find B, Input x = -2:

$$3 = \ B$$

Answer:

$$\frac{5x+7}{(x+1)(x+2)} = \frac{2}{(x+1)} + \frac{3}{(x+2)}$$

14	$\frac{15x^3-11x^2-33x}{5x-2}$

Long Division:

$$5x-2 \,\overline{)\,15x^3-11x^2-33x}$$

$$\begin{array}{r l}
 & 3x^2 - x + 7 \\
5x-2 \,\big) & 15x^3 - 11x^2 - 33x \\
- & 15x^3 - 6x^2 \\
\hline
- & \quad -5x^2 - 33x \\
 & \quad -5x^2 + 2x \\
\hline
- & \qquad 35x \\
 & \qquad 35x - 14 \\
\hline
 & \qquad\quad -14
\end{array}$$

Answer:

$$\frac{15x^3-11x^2-33x}{5x-2} = 3x^2 - x + 7 - \frac{14}{5x-2}$$

15	$\frac{(3x-8)(x^2-6x+2)}{x^2-8x+12}$

Long Division:

$$(3x-8)(x^2-6x+2) = 3x^3 - 18x^2 + 6x - 8x^2 + 48x - 16$$

$$= 3x^3 - 26x^2 + 54x - 16$$

$$x^2 - 8x + 12 \overline{)3x^3 - 26x^2 + 54x - 16}$$

$$\begin{array}{r} 3x - 2 \qquad\qquad\quad \\ x^2 - 8x + 12 \overline{)3x^3 - 26x^2 + 54x - 16} \\ 3x^3 - 24x^2 + 36x \qquad \\ \hline -2x^2 + 18x - 16 \\ - \qquad -2x^2 + 16x - 24 \\ \hline 2x + 8 \end{array}$$

$$\frac{(3x-8)(x^2-6x+2)}{x^2-8x+12} = 3x - 2 + \frac{2x+8}{x^2-8x+12}$$

$$Let \rightarrow \frac{2x+8}{x^2-8x+12} = \frac{A}{(x-6)} + \frac{B}{(x-2)}$$

$$\text{Let} \rightarrow\ 2x + 8 = \ A(x-2) + B(x-6)$$

Find A, Input x = 6:

$$20 = \ 4A$$
$$5 = A$$

Find B, Input x = 2:

$$20 = \ -4B$$
$$-3 = B$$

<u>Answer:</u>

$$\frac{(3x-8)(x^2-6x+2)}{x^2-8x+12} = 3x - 2 + \frac{5}{(x-6)} - \frac{3}{(x-2)}$$

16	$\frac{6x^3+4x^2-3x}{3x+2} === ax^2+b+\frac{c}{3x+2}$ Find a, b, c

Long Division:

$$3x+2 \overline{)6x^3+4x^2-3x}$$

$$\begin{array}{r l} & 2x^2-1 \\ 3x+2 \,\big) & 6x^3+4x^2-3x \\ - & \underline{6x^3+4x^2} \\ & -3x \\ - & \underline{-3x-2} \\ & 2 \end{array}$$

$$\frac{6x^3+4x^2-3x}{3x+2} = 2x^2-1+\frac{2}{3x+2} = ax^2+b+\frac{c}{3x+2}$$

<u>Answer:</u>

$$a = 2, \quad b = -1, \quad c = 2$$