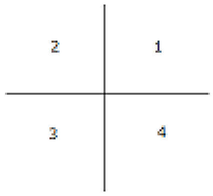
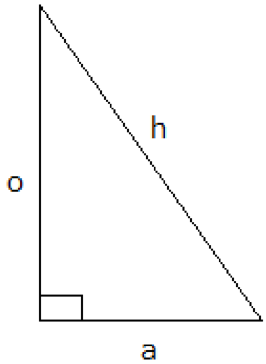
Trigonometry

Contents

1. Introduction - General Angles



2^{nd} Quadrant: Obtuse Angle $\theta \rightarrow 90° \; to \; 180°$ $\theta \rightarrow -270° \; to -180°$	1^{st} Quadrant Acute Angle $\theta \rightarrow 0° \; to \; 90°$ $\theta \rightarrow -360° \; to -270°$
3^{rd} Quadrant: Reflex Angle $\theta \rightarrow 180° \; to \; 270°$ $\theta \rightarrow -180° \; to -90°$	4^{th} Quadrant Reflex Angle $\theta \rightarrow 270° \; to \; 360°$ $\theta \rightarrow -90° \; to \; 0°$



$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{o}{h}$$

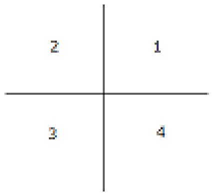
$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{a}{h}$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{o}{a}$$

2. General Angles – Special Angles

θ	$0°$	$30°$	$45°$	$60°$	$90°$
$180° = \pi$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2} \approx 0.707$	$\frac{\sqrt{3}}{2} \approx 0.866$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2} \approx 0.707$	$\frac{1}{2}$	0
$\tan\theta = \frac{\sin\theta}{\cos\theta}$	0	$\frac{\sqrt{3}}{3} = 0.577$	1	$\sqrt{3} \approx 1.732$	∞

3. General Angles – Ratios of Any angle (signs)



$A\ S\ T\ C$

2nd Quadrant $sine\ positive$ $180° - \theta$ $\sin(180° - \theta) = \sin\theta$ $\cos(180° - \theta) = -\cos\theta$ $\tan(180° - \theta) = -\tan\theta$	1st Quadrant $All\ sign\ positive$ θ $\sin\theta = \sin\theta$ $\sin(90° - y) = \cos\theta = \cos\theta$ $\tan\theta = \tan\theta$
3rd Quadrant $tan\ positive$ $180° + \theta$ $\sin(180° + \theta) = -\sin\theta$ $\cos(180° + \theta) = -\cos\theta$ $\tan(180° + \theta) = \tan\theta$	4th Quadrant $cos\ positive$ $360° - \theta\ or - \theta$ $\sin(360° - \theta) = \sin(-\theta) = -\sin\theta$ $\cos(360° - \theta) = \cos(-\theta) = \cos\theta$ $\tan(360° - \theta) = \tan(-\theta) = -\tan\theta$

$SECANT$	$\dfrac{1}{\cos\theta}, \qquad \cos\theta \neq 0$
$COSECANT$	$\dfrac{1}{\sin\theta}, \qquad \sin\theta \neq 0$
$COTANGENT$	$\dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}, \qquad \tan\theta \neq 0$
$TANGENT$	$\dfrac{\sin\theta}{\cos\theta}, \qquad \tan\theta \neq 0$

4. Exercise - General Angles

1	$\textit{Given that } \theta \text{ is acute and that } \sin\theta = \frac{1}{\sqrt{3}}, \textit{express } \frac{\sin\theta}{\cos\theta - \sin\theta} \textit{ in the for } a + \sqrt{b}$ $\sin\theta = \frac{1}{\sqrt{3}} = \frac{o}{h}$ $\textit{By pyhthagoras thm} \rightarrow a^2 = \left(\sqrt{3}\right)^2 - (1)^2 = \sqrt{2}$ $\cos\theta = \frac{\sqrt{2}}{\sqrt{3}}$ $\frac{\frac{1}{\sqrt{3}}}{\frac{\sqrt{2}}{\sqrt{3}} - \frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{\sqrt{2}+1}{(2)+\sqrt{2}-\sqrt{2}-1} = \frac{\sqrt{2}+1}{1} = 1+\sqrt{2}$
2	$\textit{Given that } 90° < y < 360° \textit{ and that } \tan y = \frac{5}{12}, \textit{ find } \sin y \textit{ and } \cos y$ $\tan y > 0, \quad \textit{in 3rd quadrant}$ $\tan y = \frac{o}{a} = \frac{5}{12}, \quad h^2 = 5^2 + 12^2 \rightarrow h = 13$ $\sin y = -\frac{5}{13}, \quad \cos y = -\frac{12}{13}$
3	$\textit{Given that } \tan y = p, \textit{and } y \textit{ is acute find } \sin y \textit{ and } \cos y \textit{ in terms of } p$ $\tan y \textit{ is acute}, \quad \textit{in 1st quadrant}$ $\tan y = \frac{o}{a} = \frac{p}{1}, \quad h^2 = p^2 + 1^2 \rightarrow h = \sqrt{p^2+1}$ $\sin y = \frac{p}{\sqrt{p^2+1}}, \quad \cos y = \frac{1}{\sqrt{p^2+1}}$
4	$\textit{Given that } \sin y = p, \textit{and } y \textit{ is acute find } \tan y, \ \sin(90° - y), \ \sin(180° + y) \textit{ in terms of } p$ $\sin y \textit{ is acute}, \quad \textit{in 1st quadrant}$ $\sin y = \frac{o}{h} = \frac{p}{1}, \quad a^2 = 1^2 - p^2 \rightarrow a = \sqrt{1-p^2}$ $\tan y = \frac{p}{\sqrt{1-p^2}}$ $\sin(90° - y) = \cos y = \sqrt{1-p^2}$ $\sin(180° + y) = -\sin\theta = -p$
5	$\textit{Given that } \tan y = p, \textit{and } y \textit{ is acute find } \tan(-y), \ \tan(\pi - y), \ \tan\left(\frac{\pi}{2} - y\right) \textit{ in terms of } p$ $\tan y \textit{ is acute}, \quad \textit{in 1st quadrant}$ $\tan y = \frac{o}{a} = \frac{p}{1}, \quad h^2 = p^2 + 1 \rightarrow h = \sqrt{p^2+1}$ $\tan(-y) = -\mathrm{p}$ $\tan(\pi - y) = -p$ $\tan\left(\frac{\pi}{2} - y\right) = -\sin\theta = -p$

5. Equations and Identities

1. $Get\ to\ the\ statement\ \sin x = k, k\ is\ a\ value$
2. $Decide\ which\ quadrant\ x\ is\ in$
3. $Find\ the\ basic\ angle\ with\ \sin x = |k|$
4. $Write\ down\ x$

a. Exercise – One Trigonometric Ratios

1	$Solve\ the\ equation\ \sin\theta = \frac{1}{2}\ for\ 0° < \theta° < 360°$ $In\ ASTC, 1st\ and\ 2nd\ quadrant\ is\ positive\ where\ \sin\theta > 0$ $therefore\ there\ is\ 2\ answers$ $\alpha = \sin^{-1}\left\|\frac{1}{2}\right\| = 30°$ $\theta = \alpha, \quad 180° - \alpha$ $\theta = 30°, \quad \theta = 150°$
2	$Solve\ the\ equation\ \tan x = 1\ for\ 0° < \theta° < 360°$ $\tan x > 0, \quad 1st\ \&\ 3rd\ quad.$ $x = \tan^{-1}\|1\| = 45°$ $\theta = x, \quad 180° + x$ $\theta = 45°, \quad \theta = 225°$
3	$Solve\ the\ equation\ \cos x = -\frac{\sqrt{3}}{2}\ for\ 0° < \theta° < 360°$ $\cos x < 0, \quad 3rd\ \&\ 4th\ quad.$ $x = \cos^{-1}\left\|-\frac{\sqrt{3}}{2}\right\| = 30°$ $\theta = 180° - x, \quad 180° + x$ $\theta = 150°, \quad \theta = 210°$
4	$Solve\ the\ equation\ \sec x = 5\ for\ 0° < \theta° < 360°$ $\frac{1}{\cos x} = 5 \rightarrow \cos x = \frac{1}{5}$ $\cos x > 0, \quad 1st\ \&\ 4th\ quad.$ $x = \cos^{-1}\left\|\frac{1}{5}\right\| = 78.5°$ $\theta = x, \quad 360° - x$ $\theta = 78.5°, \quad \theta = 281.5°$

b. Exercise – 1 Trigonometric Ratios + Rewriting to tan x

1	$Find\ all\ angles\ 0° < \theta° < 360°$ $3\cos x + 2\sin x = 0$ $2\sin x = -3\cos x$ $\frac{\sin x}{\cos x} = -\frac{3}{2} = \tan x$ $\tan x < 0, \quad 2nd\ \&\ 4th\ quad.$ $\alpha = \tan^{-1}\left\|-\frac{3}{2}\right\| = 56.3°$ $\theta = 180 - \alpha, \quad 360° - \alpha$ $\theta = 123.7°, \quad \theta = 303.7°$

c. Exercise – Compound Angle + 1 Revolution

1	$Find\ all\ angles\ 0° < \theta° < 360°$ $\cos(x + 30°) = -0.3$ $30° < x + 30° < 390°, \quad 2nd\ \&\ 3rd\ quad.$ $\alpha = \cos^{-1}\|-0.3\| = 72.5°$ $x+ = 180 - \alpha, \quad 180° + \alpha$ $x = 77.5°, \quad x = 222.5°$
2	$Find\ all\ angles\ 0° < \theta° < 360°$ $\tan(x - 50°) = -\frac{3}{4}$ $-50° < x - 50° < 310°, \quad 2nd\ \&\ 4th\ quad.$ $\alpha = \tan^{-1}\left\|-\frac{3}{4}\right\| = 36.9°$ $x - 50° = 180 - \alpha, \quad -\alpha$ $x - 50° = 143.1, \quad -36.9$ $x = 193.1°, \quad x = 13.1°$

d. Exercise – Compound Angle + Modify Number of Revolution

1	$Find\ all\ angles\ 0° < \theta° < 360°$ $\tan 2y = \sqrt{3}$ $0° < 2y < 720°, \quad 1st\ \&\ 3rd\ quad.$ $\alpha = \tan^{-1}\|\sqrt{3}\| = 60°$ $2y = \alpha, \quad 180° + \alpha, \quad \alpha + 360°, \quad 180° + \alpha + 360°$ $2y = 60°, \quad 240°, \quad 420°, \quad 600°$ $y = 30°, \quad 120°, \quad 210°, \quad 300°$
2	$Find\ all\ angles\ 0° < \theta° < 360°$ $2\sin(2x - 10°) = \sqrt{3} \rightarrow \sin(2x - 10°) = \frac{\sqrt{3}}{2}$ $-10° < 2x - 10° < 710°, \quad 1st\ \&\ 2nd\ quad.$ $\alpha = \sin^{-1}\left\|\frac{\sqrt{3}}{2}\right\| = 60°$ $2x - 10° = \alpha, \quad 180° - \alpha, \quad \alpha + 360°, \quad 180° - \alpha + 360°$ $2x - 10° = 60°, \quad 120°, \quad 420°, \quad 480°$ $x = 35°, \quad 65°, \quad 215°, \quad 245°$

e. Exercise – 2 Trigonometric Ratios

<table>
<tr><td rowspan="2">1</td><td colspan="2">$$Find\ all\ angles\ 0° < θ° < 360°$$ $$2\sin x \cos x = \sin x$$ $$2\sin x\cos x - \sin x = 0$$ $$\sin x\,(2\cos x - 1) = 0$$</td></tr>
<tr><td>$\sin x = 0, \quad 1st\ \&\ 2nd\ quad,$
$\alpha = \sin^{-1}|0| = 0°,$
$x = \alpha, \quad 180° - \alpha$
$x = 0°, \quad 180°, \quad 360°$</td><td>$\cos x = \frac{1}{2} > 0, \quad 1st\ \&\ 4th\ quad$
$\alpha = \cos^{-1}\left|\frac{1}{2}\right| = 60°$
$x = \alpha, \quad 360° - \alpha$
$x = 60°, \quad 300°$</td></tr>
<tr><td rowspan="2">2</td><td colspan="2">$$Find\ all\ angles\ 0° < θ° < 360°$$ $$\cos^2 y - \cos y = 2$$ $$\cos^2 y - \cos y - 2 = 0$$ $$(\cos y - 2)(\cos y + 1) = 0$$</td></tr>
<tr><td>$\cos y = 2, \quad (nA)$</td><td>$\cos y = -1 < 0, \quad 2nd\ \&\ 3rd\ quad$
$\alpha = \cos^{-1}|-1| = 180°$
$y = \alpha, \quad 360° - \alpha$
$y = 180°$</td></tr>
</table>

6. Identities

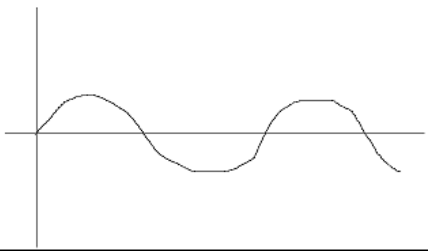
$\sin^2 θ + \cos^2 θ = 1$ $\sin^2 θ = 1 - \cos^2 θ$ $\cos^2 θ = 1 - \sin^2 θ$	Prove: $LHS \rightarrow \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 = \frac{o^2}{h^2} + \frac{a^2}{h^2} = \frac{o^2 + a^2}{h^2}$ $By\ Pythagoras\ thm \rightarrow \frac{o^2 + a^2}{h^2} = \frac{h^2}{h^2} = 1$
$1 + \tan^2 θ = \sec^2 θ$ $\tan^2 θ = \sec^2 θ - 1$	Prove: $LHS \rightarrow 1 + \frac{\sin^2 θ}{\cos^2 θ} = \frac{\cos^2 θ}{\cos^2 θ} + \frac{\sin^2 θ}{\cos^2 θ}$ $\frac{\sin^2 θ + \cos^2 θ}{\cos^2 θ} = \frac{1}{\cos^2 θ} = \sec^2 θ$
$1 + \cot^2 θ = cosec^2 θ$ $\cot^2 θ = cosec^2 θ - 1$	Prove: $LHS \rightarrow 1 + \frac{\cos^2 θ}{\sin^2 θ} = \frac{\sin^2 θ}{\sin^2 θ} + \frac{\cos^2 θ}{\sin^2 θ}$ $\frac{\sin^2 θ + \cos^2 θ}{\sin^2 θ} = \frac{1}{\sin^2 θ} = \mathrm{cosec}^2 θ$

a. Exercise – 2 Trigonometric Ratios + Identities

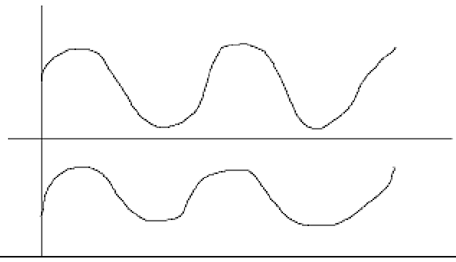
<table>
<tr><td rowspan="2">1</td><td colspan="2">$$\textit{Find all angles } 0° < \theta° < 360°$$
$$2\cos^2 y - 1 = \sin y$$
$$2(1 - \sin^2 \mathrm{y}) - 1 = \sin y$$
$$2 - 2\sin^2 \mathrm{y} - 1 - \sin y = 0$$
$$2\sin^2 \mathrm{y} + \sin y - 1 = 0$$
$$(2\sin y - 1)(\sin y + 1) = 0$$
$$\sin y = \frac{1}{2}, \qquad \sin y = -1$$</td></tr>
<tr><td>$$\sin y = \frac{1}{2}, \qquad 1st\ \&\ 2nd\ quad,$$
$$\alpha = \sin^{-1}\left|\frac{1}{2}\right| = 30°,$$
$$x = \alpha, \qquad 180° - \alpha$$
$$x = 30°, \qquad 150°$$</td><td>$$\sin y = -1 < 0, \qquad 3rd\ \&\ 4th\ quad$$
$$\alpha = \sin y^{-1}|-1| = 90°$$
$$x = 180° + \alpha, \qquad 360° - \alpha$$
$$x = 270°$$</td></tr>
</table>

7. Graphs of Trigonometric Functions

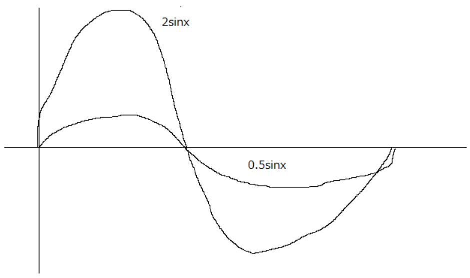
$\sin x$



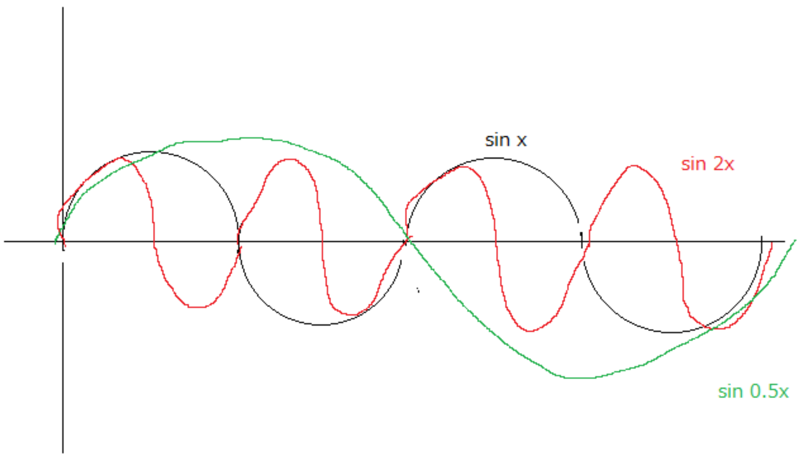
$\sin x + c$



$2\sin x$



$\sin 2x$



8. Additions Formulae

$\sin(A+B)$	$\sin A\cos B + \sin B\cos A$
$\sin(A-B)$	$\sin A\cos B - \sin B\cos A$
$\cos(A+B)$	$\cos A\cos B - \sin A\sin B$
$\cos(A-B)$	$\cos A\cos B + \sin A\sin B$
$\tan(A+B)$	$\dfrac{\tan A + \tan B}{1 - \tan A\tan B}$
$\tan(A-B)$	$\dfrac{\tan A - \tan B}{1 + \tan A\tan B}$

a. Exercise – Additions Formulae – Trigonometric Ratios

1	$Evalute\ \sin 75^\circ$ $\sin 75^\circ = \sin(30^\circ + 45^\circ)$ $\sin(30^\circ+45^\circ) = \sin 30^\circ\cos 45^\circ + \sin 45^\circ\cos 30^\circ$ $\frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{1}{4}(\sqrt{2}+\sqrt{6})$
2	$Evalute\ \cos 15^\circ$ $\cos 15^\circ = \cos(45^\circ - 30^\circ)$ $\cos(45^\circ-30^\circ) = \cos 45^\circ\, cos 30^\circ + \sin 45^\circ\sin 30^\circ$ $= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\frac{1}{2} = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4} = \frac{1}{4}(\sqrt{2}+\sqrt{6})$
3	$Given\ that\ \tan(A+B) = 5, \tan B = \frac{1}{2}, \quad Find\ A$ $\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B} \to \frac{\tan A + \frac{1}{2}}{1 - \tan A\frac{1}{2}} = 5$ $\tan A + \frac{1}{2} = 5 - \frac{5}{2}\tan A$ $\frac{7}{2}\tan A = \frac{9}{2}$ $\tan A = \frac{9}{2}\frac{2}{7} = \frac{18}{14} = \frac{9}{7}$

b. Exercise – Additions Formulae – Solving Trigonometric Equations

1	$\textit{Find all angles } 0° < \theta° < 360°$ $3\cos(x+30°) = 4\sin x$ $\cos(x+30°) = \frac{4}{3}\sin x$ $\cos x\cos 30° - \sin x \sin 30° = \frac{4}{3}\sin x$ $\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{4}{3}\sin x$ $\frac{\sqrt{3}}{2}\cos x = \frac{11}{6}\sin x$ $\tan x = \frac{6}{11}\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{11}\ ,\qquad \tan x > 0,\qquad 1st\ \&\ 3rd\ quad$ $\alpha = \tan^{-1}\left\lvert\frac{3\sqrt{3}}{11}\right\rvert = 25.3°$ $x + 30° = \alpha,\qquad 180° + \alpha$ $x = 25.3°,\qquad 205.3°$
2	$\textit{Find all angles between } 0 \textit{ and } 2\pi$ $3\cos x = 4\sin(x-2)$ $3\cos x = 4(\sin x\cos 2 - \sin 2\cos x)$ $3\cos x = 4\sin x\cos 2 - 4\sin 2\cos x$ $3\cos x + 4\sin 2\cos x = 4\sin x\cos 2$ $(3+4\sin 2)\cos x = 4\sin x\cos 2$ $\frac{(3+4\sin 2)}{4\cos 2} = \frac{\sin x}{\cos x}$ $\tan x = \frac{(3+4\sin 2)}{4\cos 2} = -3.987,\qquad 2nd\ \&\ 4th\ quad$ $\alpha = \tan^{-1}\lvert -3.987\rvert = 1.325$ $x = \pi - \alpha,\qquad 2\pi - \alpha$ $x = 1.82,\qquad 4.96$

c. Exercise – Additions Formulae – Proving Trigonometric Identities

1	$\textit{Prove that } \tan(A+45°)\ \tan(A-45°) = -1$ $LHS \to \frac{\tan A + \tan 45°}{1 - \tan A\tan 45°}\ \frac{\tan A - \tan 45°}{1 + \tan A\tan 45°}$ $= \frac{\tan A + 1}{1 - \tan A}\ \frac{\tan A - 1}{1 + \tan A} = 1$ $= \frac{\tan A^2 + \tan A - \tan A - 1}{1 + \tan A - \tan A - \tan A^2}$ $= \frac{\tan A^2 - 1}{1 - \tan A^2} = -1$

2	$$Prove\ that\ \frac{\sin(A+B)-\sin(A-B)}{\cos(A+B)-\cos(A-B)} = -\cot A$$ $$LHS \rightarrow \frac{\sin(A+B)-\sin(A-B)}{\cos(A+B)-\cos(A-B)} = \frac{\sin A\cos B+\sin B\cos A-(\sin A\cos B-\sin B\cos A)}{\cos A\cos B-\sin A\sin B-(\cos A\cos B+\sin A\sin B)}$$ $$= \frac{\sin A\cos B+\sin B\cos A-\sin A\cos B+\sin B\cos A}{\cos A\cos B-\sin A\sin B-\cos A\cos B-\sin A\sin B}$$ $$= \frac{2\sin B\cos A}{-2\sin A\sin B} = \frac{\cos A}{-\sin A} = -\cot A$$

9. Double Angle Formulae

$\sin 2A =$ $2\sin A\cos A$	Prove: $\sin(A+A) = \sin A\, cosA + \sin A\cos A$ $= 2\sin A\cos A$
$\cos 2A =$ $\cos^2 A - \sin^2 A$ $2\cos^2 A - 1$ $1 - 2\sin^2 A$	Prove: $\cos(A+A) = \cos A\, cosA - \sin A\sin A$ $= \cos^2 A - \sin^2 A$ $= \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$ $1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$
tan2A $=$ $\frac{2\tan A}{1-\tan^2 A}$	Prove: $\frac{\tan A+\tan A}{1-\tan A\tan A} = \frac{2\tan A}{1-\tan^2 A}$

a. Exercise – Double Angle Formulae – Trigonometric Ratios

<table>
<tr><td>1</td><td>

$$\sin A = \frac{3}{4}, \qquad Find\ \cos 2A\,, \cos 4A$$

$$\cos 2A = 1 - 2\sin^2 A = 1 - 2\left(\frac{9}{16}\right)$$

$$= 1 - \frac{18}{16} = \frac{1}{8}$$

$$\cos 4A = 2\cos^2 2A - 1$$

$$\cos 4A = 2\left(\frac{1}{8}\right)^2 - 1 = \frac{1}{32} - 1 = -\frac{31}{32}$$

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b. Exercise – Double Angle Formulae – Solving Trigonometric Equations

<table>
<tr><td rowspan="2">1</td><td colspan="2">

$$Find\ all\ angles\ 0° < \theta° < 360°$$

$$\sin 2x + \sin x = 0$$

$$2\sin x\cos x + \sin x = 0$$

$$\sin x\,(2\cos x + 1) = 0$$

</td></tr>
<tr><td>

$$\sin x = 0$$

$$\alpha = \sin^{-1}|0| = 0$$

$$x = 0, 180, 360$$

</td><td>

$$\cos x = -\frac{1}{2}$$

$$\alpha = \cos^{-1}\left|-\frac{1}{2}\right| = 60$$

$$x = 180 - \alpha, 180 + \alpha$$

$$x = 120, 240$$

</td></tr>
<tr><td>2</td><td colspan="2">

$$Find\ all\ angles\ 0° < \theta° < 360°$$

$$3\sin x\cos x = 1$$

$$\frac{3}{2}\sin 2x = 1$$

$$\sin 2x = \frac{2}{3}$$

$$\alpha = \sin^{-1}\left|\frac{2}{3}\right| = 41.8$$

$$2x = \alpha, \qquad 180 - \alpha, \qquad \alpha + 360, \qquad 180 - \alpha + 360$$

$$2x = 41.8, \qquad 138.2, \qquad 401.8, \qquad 408.2$$

$$x = 20.9, \qquad 69.1, \qquad 200.9, \qquad 249.1$$

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<tr><td rowspan="2">3</td><td colspan="2">

$$Find\ all\ angles\ 0° < \theta° < 360°$$

$$2\cos 2x - 3\sin x = 1$$

$$2(1 - 2\sin^2 x) - 3\sin x = 1$$

$$2 - 4\sin^2 x - 3\sin x = 1$$

$$4\sin^2 x + 3\sin x - 1 = 0$$

$$(4\sin x - 1)(\sin x + 1) = 0$$

</td></tr>
<tr><td>

$$\sin x = \frac{1}{4}$$

$$\alpha = \sin^{-1}\left|\frac{1}{4}\right| = 14.5$$

$$x = 14.5, \qquad 165.5$$

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$$\sin x = -1$$

$$\alpha = \cos^{-1}|-1| = 90$$

$$x = 180 + \alpha, 360 + \alpha$$

$$x = 270$$

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</table>

4	$\mathit{Find\ all\ angles}\ 0° < \theta° < 360°$ $\tan 2x \tan x = 5$ $\frac{2\tan x}{1-\tan^2 x}\tan x = 5$ $2\tan^2 x = 5 - 5\tan^2 x$ $7\tan^2 x = 5$ $\tan x = \pm\sqrt{\frac{5}{7}}\ \mathit{all\ quad}$ $\alpha = \tan^{-1}\left\lvert\sqrt{\frac{5}{7}}\right\rvert = 40.2$ $x = 40.2, \quad 139.8, \quad 220.2, \quad 319.8$

c. Exercise – Double Angle Formulae – Proving Trigonometric Identities

1	$\mathit{Prove}\ \sin 3A = 3\sin A - 4\sin^3 A\ \ (\mathit{Hint: write}\ 3A\ \mathit{to}\ 2A + A)$ $LHS \rightarrow\ \sin 3A = \sin(2A + A)$ $= \sin 2A\cos A + \sin A\cos 2A$ $= (2\sin A\cos A)\cos A + \sin A\,(1 - 2\sin^2 A)$ $= 2\sin A\cos^2 A + \sin A - 2\sin^3 A$ $= 2\sin A\,(1 - \sin^2 A) + \sin A - 2\sin^3 A$ $= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$ $= 3\sin A - 4\sin^3 A$
2	$\mathit{Prove}\ \frac{\tan 2x}{2} = \frac{1}{\cot x - \tan x}$ $LHS \rightarrow\ \frac{\tan 2x}{2} = \frac{1}{2}\frac{2\tan x}{1-\tan^2 x} = \frac{\tan x}{1-\tan^2 x}$ $= \frac{\tan x}{1-\tan^2 x}\frac{\frac{1}{\tan x}}{\frac{1}{\tan x}} = \frac{1}{\frac{1}{\tan x} - \tan x} = \frac{1}{\cot x - \tan x}$

10. R Formulae

- *Enables us to rewrite* $\sin x$ *and* $\cos x$ *in a same equation*

$$\boxed{\begin{gathered} \mathrm{a}\sin\theta + \mathrm{b}\cos\theta = R\sin(\theta+\alpha) \\ \mathrm{a}\sin\theta - \mathrm{b}\cos\theta = R\sin(\theta-\alpha) \\ \mathrm{a}\cos\theta + \mathrm{b}\sin\theta = R\cos(\theta-\alpha) \\ \mathrm{a}\cos\theta - \mathrm{b}\sin\theta = R\cos(\theta+\alpha) \end{gathered}}$$

$$where\ R = \sqrt{a^2+b^2}\ ,$$

$$\alpha = \tan^{-1}\left(\frac{b}{a}\right), \qquad 0° < \alpha < 90°$$

a. Exercise – R Formulae – Trigonometric Ratios

1	$4\sin x + 3\cos x = \sqrt{4^2+3^2}\sin(x+\tan^{-1}\left(\frac{3}{4}\right))$ $= 5\sin(x+36.9)$
2	$3\cos x + 4\sin x = \sqrt{3^2+4^2}\cos(x-\tan^{-1}\left(\frac{4}{3}\right))$ $= 5\sin(x+53.1)$
3	$\sqrt{3}\sin x - \cos x = \sqrt{\sqrt{3}^2+1^2}\sin\left(x - tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$ $= 2\sin(x+30)$

b. Exercise – R Formulae – Solving Trigonometric Equations

1	*Find all angles* $0° < \theta° < 360°$ $2\cos x - 5\sin x = 3.1$ $\rightarrow \sqrt{2^2+5^2}\cos\left(x + tan^{-1}\left(\frac{5}{2}\right)\right) = 3.1$ $\sqrt{29}\cos(x+68.2) = 3.1$ $\cos(x+68.2) = \frac{3.1}{\sqrt{29}}$ $\alpha = \cos^{-1}\left\lvert\frac{3.1}{\sqrt{29}}\right\rvert = 54.9, \cos x > 0, \qquad 1st, 4th\ quad$ $68.2° < x + 68.2 < 428.2°$ $x + 68.2 = \alpha + 360, \qquad 360 - \alpha$ $x + 68.2 = 54.9\ + 360, \qquad 305.1$ $x = 346.7, \qquad 236.9$

c. Exercise – R Formulae – Max and min value of R Formulae

1	$R\sin(\theta \pm \alpha)$ $\to Max\ value = R$ $\to when\ \sin(\theta \pm \alpha) = 1 \to \theta \pm \alpha = 90$ $\to Min\ value = -R$ $\to when\ \sin(\theta \pm \alpha) = -1 \to \theta \pm \alpha = 270$
2	$R\cos(\theta \pm \alpha)$ $\to Max\ value = R$ $\to when\ \sin(\theta \pm \alpha) = 1 \to \theta - \alpha = 90, \theta + \alpha = 360$ $\to Min\ value = -R$ $\to when\ \sin(\theta \pm \alpha) = -1 \to \theta \pm \alpha = 180$

11. Factor Formulae

$\sin P + \sin Q$	$2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2}$
$\sin P - \sin Q$	$2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2}$
$\cos P + \cos Q$	$2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2}$
$\cos P - \cos Q$	$-2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2}$

a. Exercise – Factor Formulae – Trigonometric Ratios

1	$Evaluate\ \cos 75 + \cos 15$ $\cos 75 + \cos 15 = 2\cos\frac{75+15}{2}\cos\frac{75-15}{2}$ $= 2\cos 45 \cos 30$ $= 2\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{6}}{2}$

b. Exercise – Factor Formulae – Solving Trigonometric Equations

1	$Solve\ \sin x + \sin 3x = -\sin 5x, \qquad Find\ all\ angles\ 0° \le θ° \le 180°$ $\sin x + \sin 3x = -\sin 5x$ $\sin x + \sin 3x + \sin 5x = 0$ $\sin 5x + \sin x + \sin 3x = 0$ $2\sin\frac{5x+x}{2}\cos\frac{5x-x}{2} + \sin 3x = 0$ $2\sin 3x\cos 2x + \sin 3x = 0$ $\sin 3x\,(2\cos 2x + 1) = 0$ $\sin 3x = 0, \qquad \cos 2x = -\frac{1}{2}$ $\alpha = \sin^{-1}\lvert 0\rvert = 0, \sin x > 0, \qquad 1st, 2nd\ quad$ $0° \le 3x \le 540°$ $3x = 0,180,360,540$ $x = 0,60,120,180$ $\alpha = \cos^{-1}\left\lvert -\frac{1}{2}\right\rvert = 60, \cos x < 0, \qquad 2nd, 3rd\ quad$ $0° \le 2x \le 360°$ $2x = 180 - \alpha, \qquad 180 + \alpha$ $2x = 120, \qquad 240$ $x = 60, \qquad 120$

c. Exercise – Factor Formulae – Proving Trigonometric Identities

1	$\frac{\sin 6x - \sin 2x}{\cos 6x - \cos 2x} = -\cot 4x$ $LHS \to \frac{\sin 6x - \sin 2x}{\cos 6x - \cos 2x} = \frac{2\cos\frac{6x+2x}{2}\sin\frac{6x-2x}{2}}{-2\sin\frac{6x+2x}{2}\sin\frac{6x-2x}{2}} = \frac{2\cos 4x\sin 2x}{-2\sin 4x\sin 2x}$ $\frac{\cos 4x}{-\cos 4x} = -\cot 4x$

12. Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

13. Cosine Rule

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

14. Area of Triangle

$$\frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C$$