Remainder and Factor Theorem

Contents

1. Why learn?

- Learn about Polynomials, meaning many terms in Greek
- A polynomial in the variable x is a collection of terms, each of the form ax^n where **a is a constant** and the power **n is a non-negative integer.**

2. Polynomials

Example of polynomials:

$$8x^3 - x^2 + 3x + 4 \ (4\ terms, powers\ in\ descending\ order)$$

$$2 - 5x + 3x^2 (3\ terms, powers\ in\ ascending\ order)$$

Example of non-polynomials:

$$x^2 - 2x^{\frac{1}{2}} + 6 \ (fractional\ powers\ of\ x)$$

$$x^2 + x^{-1} - 2 \ (negative\ powers\ of\ x)$$

a. Coefficient:

$$In\ the\ term\ ax^n, a\ is\ called\ the\ coefficient\ of\ x^n$$

Example

$$8x^3 - x^2 + 3x + 4$$

$$\rightarrow coeff^n of\ x^3 = 8$$

$$\rightarrow coeff^n of\ x^2 = -1$$

$$\rightarrow coeff^n of\ x^1 = 3$$

$$\rightarrow coeff^n of\ x^0 = 4$$

b. Degree:

$$The\ degree\ of\ a\ polynomial\ in\ x\ is\ the\ highest\ power\ of\ x$$

$$\rightarrow the\ degree\ of\ 8x^3 - x^2 + 3x + 4\ is\ 3$$

$$\rightarrow the\ degree\ of\ 8x^3 - x^5 + 3x + 4\ is\ 5$$

$$\rightarrow the\ degree\ of\ 4\ is\ 0$$

c. Value:

$$The\ value\ of\ a\ polynomial\ is\ its\ magnitude\ when\ x\ takes\ on\ a\ number.$$

$$For\ easy\ reference, polynomials\ are\ often\ denoted\ by\ function\ notations\ like\ f(x)$$

$$let\ f(x) = 4x^3 - 6x^2 + x$$

$$f(2) = 4(2)^3 - 6(2)^2 + 2 = 10 \rightarrow the\ value\ of\ f(x)\ is\ 10$$

3. Identities

$$What\ is\ the\ difference\ between\ an\ Equation\ and\ an\ Identity$$

a. Equation

$$Consider\ the\ polnomials\ x^2+2x\ and\ 4x+3$$

$$x^2+2x=4x+3\ is\ only\ true\ when\ x=-1\ and\ x=3$$

$$x^2+2x=4x+3\ os\ and\ equation\ with\ finite\ numbers\ of\ solution$$

b. Identical

$$Consider\ the\ polnomials\ x^2-4\ and\ (x-2)(x+2)$$

$$x^2-4=(x-2)(x+2)\ is\ true\ for\ all\ value\ of\ x$$

$$They\ are\ identical,$$

$$x^2-4=(x-2)(x+2)\ is\ and\ identity\ with\ infinite\ numbers\ of\ solutions$$

c. Solving – By comparing coefficients

$$Given\ that\ 3x^3+4x^2-17x-11=(Ax+1)(x+B)(x-2)+C$$

$$For\ all\ value\ of\ x, Find\ A,B,C$$

1	$3x^3+4x^2-17x-11=(Ax^2+ABx+x+B)(x-2)+C$ $3x^3+4x^2-17x-11=Ax^3+ABx^2+x^2+Bx-2Ax^2-2ABx-2x-2B+C$ $3x^3+4x^2-17x-11=Ax^3+(AB+1-2A)x^2+(B-2AB-2)x-2B+C$ $\rightarrow coeff^n of x^3 \rightarrow A=3$ $\rightarrow coeff^n of x^2 \rightarrow AB+1-2A=4$ $3B+1-6=4$ $3B=9$ $B=3$ $\rightarrow coeff^n of constant \rightarrow -2B+C=-11$ $-6+C=-11$ $C=-5$

d. Solving – By Substitution

1	$Choose\ x=2, find\ C$ $C=3(2)^3+4(2)^2-17(2)-11$ $C=-5$ $Choose\ x=0, find\ B$ $-2B+C=-11$ $-2B-5=-11$ $B=3$ $Choose\ x=1, find\ A$ $(Ax+1)(x+B)(x-2)+C=3+4-17-11$ $(A+1)(1+B)(1-2)+C=-21$ $(A+1)(1+3)(-1)-5=-21$ $-4(A+1)=-16$ $A=3$

4. Division of polynomials

a. Long Division

$$x^2+5x+6\,\overline{)\,x^2+4x+5}$$

$$\begin{array}{rr}
 & 1 \\
x^2+5x+6 & \overline{)\,x^2+4x+5} \\
- & x^2+5x+6 \\ \cline{2-2}
 & -x-1
\end{array}$$

$$x^2-1\,\overline{)\,x^3-x^2-6}$$

$$\begin{array}{rr}
 & x-1 \\
x^2-1 & \overline{)\,x^3-x^2-6} \\
- & -x^3+x \\ \cline{2-2}
- & -x^2+x-6 \\
 & -x^2+1 \\ \cline{2-2}
 & x-7
\end{array}$$

b. Synthetic Divisor (works for Linear Divisor)

$$Divide\ 2x^4-3x^3+x^2+1\ by\ x-2$$

$$\begin{array}{c|c}
\begin{array}{ccccc} x^4 & x^3 & x^2 & x^1 & x^0 \\ 2 & -3 & 1 & 0 & 1 \\ & & & & \end{array} & \begin{array}{l} x-2=0 \\ x=2 \end{array} \\ \hline
\begin{array}{ccccc} 2 & & & & \end{array} &
\end{array}$$

$$\rightarrow 2*2=4$$

$$\begin{array}{c|c}
\begin{array}{ccccc} x^4 & x^3 & x^2 & x^1 & x^0 \\ 2 & -3 & 1 & 0 & 1 \\ & 4 & & & \end{array} & \begin{array}{l} x-2=0 \\ x=2 \end{array} \\ \hline
\begin{array}{ccccc} 2 & 1 & & & \end{array} &
\end{array}$$

$$\rightarrow 1*2=2$$

$$\begin{array}{c|c}
\begin{array}{ccccc} x^4 & x^3 & x^2 & x^1 & x^0 \\ 2 & -3 & 1 & 0 & 1 \\ & 4 & 2 & & \end{array} & \begin{array}{l} x-2=0 \\ x=2 \end{array} \\ \hline
\begin{array}{ccccc} 2 & 1 & 3 & & \end{array} &
\end{array}$$

$$\rightarrow 3*2=6$$

$$\begin{array}{cccccc|c} & x^4 & x^3 & x^2 & x^1 & x^0 & x-2=0 \\ & 2 & -3 & 1 & 0 & 1 & x=2 \\ + & & 4 & 2 & 6 & & \\ \hline & 2 & 1 & 3 & 6 & & \end{array}$$

$\rightarrow 6 * 2 = 12$

$$\begin{array}{cccccc|c} & x^4 & x^3 & x^2 & x^1 & x^0 & x-2=0 \\ & 2 & -3 & 1 & 0 & 1 & x=2 \\ + & & 4 & 2 & 6 & 12 & \\ \hline & 2 & 1 & 3 & 6 & 13 & \end{array}$$

$2x^4 - 3x^3 + x^2 + 1 \; by \; x - 2 = (x-2)(2x^3 + x^2 + 3x + 8) + 13$

5. Remainder theorem

$$What\ if\ the\ question\ only\ wants\ the\ remainder?$$

$$Divide\ 2x^4 - 3x^3 + x^2 + 1\ by\ x - 2$$

$$let\ f(x) = 2x^4 - 3x^3 + x^2 + 1$$

$$f(2) = 13$$

a. Confirming the REMAINDER THOREM

$$f(x) = (x - a)Q(x) + R$$

$$if\ a\ polynomial\ f(x) is\ divided\ by\ a\ linear\ divisor\ (x - a) \rightarrow remainder = f(a)$$

$$if\ a\ polynomial\ f(x) is\ divided\ by\ a\ linear\ divisor\ (ax - b) \rightarrow remainder = f\left(\frac{b}{a}\right)$$

$$Let\ f(x) = 4x^3 - 5x + 1, find\ the\ remainder\ when\ divided\ by:$$

1	$x - 2 \rightarrow x = 2$ $f(2) = 4(2)^3 - 5(2) + 1 = 23$
2	$2x - 1 \rightarrow x = \frac{1}{2}$ $f\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 5\left(\frac{1}{2}\right) + 1 = -1$
3	$3x + 2 \rightarrow x = -\frac{2}{3}$ $f\left(-\frac{2}{3}\right) = 4\left(-\frac{2}{3}\right)^3 - 5\left(-\frac{2}{3}\right) + 1 = -3\frac{4}{27}$
4	$x \rightarrow x = 0$ $f(0) = 4(0)^3 - 5(0) + 1 = 1$

b. Exercise on REMAINDER THOREM

1	$The\ expression\ 4x^2 - px + 7\ leaves\ a\ remainder\ of - 2\ when\ divided\ by (x - 3), Find\ p$ $f(x) = 4x^2 - px + 7$ $f(3) = 4(3)^2 - 3p + 7 = -2$ $36 - 3p = -9$ $-3p = -45$ $p = 15$
2	$The\ expression\ ax^3 + bx^2 + 6,$ $leaves\ a\ remainder\ of\ 10\ and\ 1\ when\ divided\ by (x - 2)\ and\ (x + 1)\ respectively,$ $Find\ a\ and\ b$ $f(x) = ax^3 + bx^2 + 6$ $f(2) = 8a + 4b + 6 = 10$ $f(2) = 2a + b = 1$ $f(-1) = -a + b + 6 = 1$ $f(-1) = -a + b = -5$ $b = -5 + a$ $b = -3, \quad a = 2$

6. Factor Theorem

$$if\ the\ divisor\ (x-a)\ is\ a\ facetor\ of\ f(x) \rightarrow f(x) has\ no\ remainder$$

$$\rightarrow f(x) = (x-a)Q(x) + 0$$
$$\rightarrow f(a) = 0$$

1	$Determine\ whether\ (x+1) is\ a\ factor\ of$ $f(x) = 3x^4 + x^3 - x^2 + 3x + 2$ $f(-1) = 0$ $f(x) = x^6 + 2x(x-1) - 4$ $f(-1) = 1$
2	$Find\ value\ of\ k\ if\ (x-2) is\ a\ facetor\ of\ x^3 - 2kx^2 + 3x + k$ $f(2) = x^3 - 2kx^2 + 3x + k = 0$ $8 - 8k + 6 + k = 0$ $k = 2$

7. Solutions of Cubic Equation

Note: Only Step 1 is a new step, Step 2 and 3 is what we have learnt above or learnt before

Step 1

$$let\ f(x) = px^3 + qx^2 + rx + s$$

$$Obtain\ the\ 1st\ factor (x-k) by\ trial\ and\ error$$

$$i.e. find\ the\ value\ of\ k\ such\ that\ f(k) = 0\ using\ calculator$$

Step 2

$$Factorize\ f(x) into\ (x-k)(ax^2 + bx + c) = 0$$

$$by\ long\ division\ or\ synthetic\ division$$

Step 3

$$if\ f(x) = 0 \rightarrow x - k = 0\ or\ ax^2 + bx + c = 0$$

$$solve\ ax^2 + bx + c = 0\ by\ factorizing\ or\ formulae$$

1	$f(x) = 2x^3 + 3x^2 - 11x - 6 = 0$ Tips: $try\ to\ use\ \pm 1 \pm 2 \pm 3 \pm 6,\ (factor\ of\ 6) for\ trial\ and\ error$ Step 1, trial and error $f(1) = 2x^3 + 3x^2 - 11x - 6 = -12 \neq 0\ (skip)$ $f(2) = 2x^3 + 3x^2 - 11x - 6 = 0$ $(x-2) is\ a\ factor$ Step 2, $\frac{2x^3 + 3x^2 - 11x - 6}{(x-2)} = (x-2)(2x^2 + 7x + 3)$ Step 3, $f(x) = (x-2)(2x^2 + 7x + 3)$ $f(x) = (x-2)(2x+1)(x+3)$ $x = 2, \quad x = -\frac{1}{2}, \quad x = -3$