Logarithm

Contents

1. Why learn Logarithm?

a. ?

Real-life example:

- pH scale (acidic to alkaline)

- Richter scale (measure earthquake magnitude)

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2. Introduction

a. Expressing a number in index form with base 10
   i. Example 100 = 10^2
   ii. Example 1000 = 10^3
b. How about 40 with base 10? What is x?
   i. 40 = 10^x

3. Definitions

Index Form	$a^x = y$
Logarithmic Form	$a^x = y \rightarrow x = \log_a y$ $base\ a \rightarrow \log_a$ $power \rightarrow x$ $value \rightarrow y$ $base\ 10 = lg$ $where, \quad a > 0, \quad a \neq 1, \quad y > 0$

4. Change of Base (exercise)

$10^x = 40$	$x = lg40 \ \approx 1.6\ (using\ calculator)$
$3^2 = 9$	$2 = \log_3 9$ $2 = \frac{lg9}{lg3}\ \ (using\ calculator)$

a. Index form to Logarithmic Form

$4^2 = 16$	$2 = \log_4 16$
$2^5 = 32$	$5 = \log_2 32$
$10^3 = 1000$	$3 = lg1000$
$2^{-1} = \frac{1}{2}$	$-1 = \log_2 \frac{1}{2}$ $1 = -\log_2 2$

$9^{\frac{1}{2}} = 3$	$\frac{1}{2} = \log_9 3$

b. Logarithmic Form to Index form

$\log_2 8 = 3$	$8 = 3^2$
$\log_5 25 = 2$	$25 = 5^2$
$\log_2 \sqrt{2} = \frac{1}{2}$	$\sqrt{2} = 2^{\frac{1}{2}}$
$\log_2 \frac{1}{4} = -2$	$\frac{1}{4} = 2^{-2}$
$\log_{16} \frac{1}{2} = -\frac{1}{4}$	$\frac{1}{2} = 16^{-\frac{1}{4}}$
$\log_3 x = 2$	$x = 3^2$

5. Rule

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<u>1 Logarithm Base</u>

$\log_a a = 1$

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Example:

$$log_2 2 = 1$$
$$log_{\frac{1}{3}} \frac{1}{3} = 1$$
$$log_{0.5} 0.5 = 1$$

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<u>2 Logarithm 1</u>

$\log_a 1 = 0$

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Example:

$$log_2 2 = 1$$
$$log_{\frac{1}{3}} \frac{1}{3} = 1$$
$$log_{0.5} 0.5 = 1$$

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<u>3 Logarithm Product Rule</u>

$\log_b xy = \log_b x + \log_b y$

Note:
$\log_b(x + y) \neq \log_b x + \log_b y$

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Example 1:
$$\log_2(3)(5) = \log_2 3 + \log_2 5$$
Example 1 Solve:
$$\log_2 3 + \log_2 5 = \log_2 15$$
$$\log_2 3 + \log_2 5$$
$$= \log_2 15$$
$$= 3.906890595608518529324058373 4372$$
$$Check \rightarrow 2^{3.9068905956085185293240583734372} = 15$$

Example 2:
$$\log_2 7x = \log_2 7 + \log_2 x$$
Example 3:
$$\log_2 x(x + 3) = \log_2 x + \log_2(x + 3)$$

Example 4:
$$\log_6 3 + \log_6 2 = \log_6 6 = 1$$
Example 5:
$$\log_2 40 + \log_2 0.1 + \log_2 25$$
$$\log_2(40 * 0.1 * 25) = \log_2 1 = 0$$

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<u>4 Logarithm Quotient Rule</u>

$\log_b x - \log_b y = \log_b \frac{x}{y}$

Note:
$\frac{\log_b x}{\log_b y} \neq \log_b \frac{x}{y}$

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Example 1:
$$\log_2 \frac{7}{2} = \log_2 7 - \log_2 2$$
Example 2:
$$\log_2 \frac{x}{5} = \log_2 x - \log_2 5$$
Example 3:
$$\log_2 \frac{x}{(x + 3)} = \log_2 x - \log_2(x + 3)$$
Example 4:
$$\log_4 8 - \log_4 2 = \log_4 4 = 1$$

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	Example 5: $$\log_4 x^4 - \log_4 x^3 = \log_4 x$$
5 Logarithm Power Rule $\log_b x^r = \mathrm{r}\log_b x$	Prove: $$\log_b x * x = \log_b x + \log_b x = 2\log_b x$$ Example 1: $$\log_2 x^{-3} = -3\log_2 x$$ Example 2: $$\log_2 \sqrt{x} = \frac{1}{2}\log_2 x$$
Logarithm Base Switch Rule	$$\log_b a = \frac{1}{\log_a b}$$
Logarithm Base Change Rule	$$\log_b a = \frac{\log_c a}{\log_c b}$$
Logarithm 0	$$\log_b 0 = undefined$$
Logarithm Negative	$$\log_b x = undefined\,, \qquad x \leq 0$$
6 common Logarithm	$$\lg y = x$$ $$y = 10^x$$ $$lg10 = 1$$
7 Natural Logarithm	$$\log \textit{to the base } e, \qquad e = 2.718 \ldots$$ $$\log_e \quad = \ln$$ $$\rightarrow \ln y = e^x, \qquad \ln e = 1$$

6. Exercise – Evaluate the following

1	$$\log_2 2\sqrt{2} = \log_2 2^{\frac{3}{2}} = \frac{3}{2}$$
2	$$log_a\left(\frac{1}{a^2}\right) = -2\log_a a = -2$$
3	$$\log_6 54 - 2\log_6 3 = \log_6 \frac{54}{9} = \log_6 6 = 1$$
4	$$\log_5 4 + 2\log_5 3 - 2\log_5 6 = \log_5 \frac{4 * 9}{36} = \log_5 1 = 0$$

7. Exercise – Solve

1	$5^x = 10$ $x = \log_5 10 = \frac{lg10}{lg5} \approx 1.43$
2	$2^x = 3^{x-2}$ $lg2^x = lg3^{x-2}$ $xlg2 = (x-2)lg3$ $xlg2 = xlg3 - 2lg3$ $x(lg2 - lg3) = -2lg3$ $x = \frac{-2lg3}{(lg2 - lg3)} = \frac{-2lg3}{\lg\frac{2}{3}} \approx 5.42$
3	$e^{2x-1} = 6$ $2x - 1 = ln6$ $x = \frac{ln6 + 1}{2} \approx 1.40$
4	$3^x \times 2^{2x} = 7(5^x)$ $\frac{3^x 4^x}{5^x} = 7 \rightarrow \frac{12}{5}^x = 7$ $x = \log_{\frac{12}{5}} 7 = \frac{lg7}{\lg\frac{12}{5}} \approx 2.22$
5	$3^{y+1} = 0.45$ $y = \frac{lg0.45}{\lg 3} - 1 \approx -1.73$
6	$e^{3x-4} = 5.47$ $x = \frac{ln5.47 + 4}{3} \approx 1.90$
7	$3^{y+1} = 4^y$ $3 = (\frac{4}{3})^y \rightarrow y = \frac{\lg 3}{\lg\frac{4}{3}} \approx 3.82$
8	$5^{x-1}3^{x+2} = 10$ $15^x \frac{9}{5} = 10 \rightarrow 15^x = \frac{50}{9}$ $x = \frac{\lg\frac{50}{9}}{\lg 15} \approx 1.18$
9	$2^{2x}5^{x+1} = 7$ $20^x = \frac{7}{5} \rightarrow x = \frac{\lg\frac{7}{5}}{\lg 20} \approx 0.112$

8. Exercise – Exponential Equations

1	$81^x + 2(9^{x+1}) = 40 \to 9^{2x} + 18(9^x) - 40 = 0$ $let\ y = 9^x \to y^2 + 18y - 40 = 0$ $(y+20)(y-2) = 0$ $y = -20\ (N.A.\ y < 0), \quad y = 2$ $9^x = 2$ $x = \frac{lg2}{lg9} \approx 0.315$
2	$e^{2x} + 4e^{-2x} = 4 \to e^{2x} + \frac{4}{e^{2x}} - 4 = 0$ $let\ y = e^{2x} \to y + \frac{4}{y} - 4 = 0 \to y^2 - 4y + 4 = 0$ $(y-2)(y-2) = 0$ $y = 2$ $e^{2x} = 2$ $x = \frac{ln2}{2} \approx 0.347$
3	$3^{x+1} - 2 = 8 * 3^{x-1} \to 3 * 3^x - \frac{8}{3} * 3^x - 2 = 0$ $let\ y = 3^x \to 3y - \frac{8}{3}y - 2 = 0 \to \frac{1}{3}y = 2 \to y = 6$ $3^x = 6$ $x = \frac{lg6}{lg3} \approx 1.63$
4	$9^y + 5(3^y - 10) = 0$ $let\ x = 3^y \to x^2 + 5x - 50 = 0$ $(x+10)(y-5) = 0$ $x = -10, \quad x = 5$ $3^y = 5$ $y = \frac{lg5}{lg3} \approx 1.46$
5	$10^{2x+1} - 7(10^x) = 26$ $let\ y = 10^x \to 10y^2 + 10y - 26 = 0$ $(y-2)(10y+13)$ $y = 2, \quad x = -\frac{13}{10}$ $10^x = 2$ $y = lg2 \approx 0.301$
6	$e^{3x} + 2e^x = 3e^{2x} \to e^{2x} - 3e^x + 2 = 0$ $let\ y = e^x \to y^2 - 3y + 2 = 0$ $(y-1)(y-2)$ $y = 1, \quad y = 2$ $e^x = 1, \quad e^x = 2$ $x = ln1, \quad x = ln2$ $x = 0, \quad x = 0.693$

7	$$2e^x = 7\sqrt{e^x} - 3 \rightarrow 2e^x - 7\sqrt{e^x} + 3 = 0$$ $$let\ y = e^{\frac{1}{2}x} \rightarrow 2y^2 - 7y + 3 = 0 \rightarrow 2y^2 - 7y + 3 = 0$$ $$(2y-1)(y-3)$$ $$y = \frac{1}{2}, \qquad y = 3$$ $$e^{\frac{1}{2}x} = \frac{1}{2}, \qquad e^{\frac{1}{2}x} = 3$$ $$x = 2\ln\frac{1}{2}, \qquad x = 2ln3$$ $$x = -1.39, \qquad x = 2.20$$

9. Exercise – Solving Unknown

1	$$The\ equation\ of\ a\ curve\ is\ y\ = ax^n,$$ $$Given\ that\ the\ points\ (2{,}9)\ and\ (3{,}4) lie\ on\ the\ curve,$$ $$calculate\ the\ value\ of\ a\ and\ of\ n$$ $$9 = a2^n, \qquad 4 = a3^n$$ $$\frac{a3^n}{a2^n} = \frac{4}{9} \rightarrow \frac{3^n}{2} = \frac{4}{9}$$ $$n = \frac{\lg\left(\frac{4}{9}\right)}{\lg\frac{3}{2}} = -2, \qquad 9 = \frac{a}{4} \rightarrow a = 36$$

10. Exercise – Logarithm Equations

1	$$\log_3(2x+1) = 2 + \log_3(3x-11)$$ $$\log_3(2x+1) - \log_3(3x-11) = 2$$ $$\log_3\frac{(2x+1)}{(3x-11)} = 2$$ $$\frac{(2x+1)}{(3x-11)} = 9$$ $$2x + 1 = 27x - 99$$ $$-25x = -100$$ $$x = 4$$
2	$$lg2x - 3lg2 = \frac{1}{2}\lg(x-3)$$ $$lg\frac{2x}{2^3} = \lg(x-3)^{\frac{1}{2}}$$ $$\frac{4x^2}{64} = \mathrm{x} - 3 \rightarrow \frac{x^2}{16} = \mathrm{x} - 3$$ $$x^2 - 16x + 48 = 0 \rightarrow x^2 - 16x + 48 = 0$$ $$(x-12)(x-4)$$ $$x = 12, \qquad x = 4$$

3	$\log_7(17y+15) = 2+\log_7(2y-3)$ $\frac{(17y+15)}{(2y-3)} = 49$ $17y+15 = 98y-147$ $162 = 81y$ $y = 2$
4	$\log_5(8y-6)-\log_5(y-5) = \log_4 16$ $\log_5\frac{(8y-6)}{(y-5)} = 2$ $\frac{(8y-6)}{(y-5)} = 25$ $8y-6 = 25y-125$ $119 = 17y$ $y = 7$
5	$lgx+\lg\left(5(x+1)\right) = 2$ $5x^2+5x = 100$ x^2+x-20 $(x-4)(x+5)$ $x = 4, \qquad x = -5$
6	$2\lg(x+2)+\lg 4 = lgx+4lg3$ $\lg 4(x+2)^2 = lgx3^4$ $4\mathrm{x}^2+16x+16 = 81x$ $4\mathrm{x}^2+65x+16 = 0$ $(x-16)(4x-1)$ $x = 16, \qquad x = \frac{1}{4}$
7	$\log_x 72 = 3-\log_x 3$ $216 = x^3$ $x = 6$
8	$(\log_3 y)^2+\log_3 y^2 = 8$ $(\log_3 y)^2+2\log_3 y-8 = 0$ $(\log_3 y+4)(\log_3 y-2)$ $y = 3^{-4}, \qquad y = 3^2$ $y = \frac{1}{81}, \qquad y = 9$
9	$3^p = 9(27)^q$ $3^p = 3^2 3^{3q}$ $p = 2+3q$ $\log_2 7-\log_2(11q-2p) = 1$ $\frac{7}{11q-2p} = 2 \rightarrow 7 = 22q-4p$ $\rightarrow 7 = 22q-4(2+3q) \rightarrow 7 = 22q-8-12q \rightarrow 15 = -2q$ $q = -\frac{15}{2}, \qquad p = 2-\frac{45}{2}$

9	$\ln(3x-y)=2ln6-ln9$ $3x-y=\frac{36}{9}\to 3x-4=y$ $\frac{(e^x)^2}{e^y}=e\to 2x-y=1\to 2x-1=y$ $\to 3x-4=2x-1\to x=3$ $y=5$

11. Exercise – Logarithm Change of base

1	$3\log_5 y-\log_{25} y=10$ $3\log_5 y-\frac{\log_5 y}{\log_5 25}=10$ $3\log_5 y-\frac{1}{2}\log_5 y=10$ $\frac{5}{2}\log_5 y=10$ $\log_5 y=4$ $y=625$
2	$u=\log_4 x\,,in\ terms\ of\ u\ find:$ $x=4^u$ $\log_4\frac{16}{x}=\log_4 4^2-\frac{1}{2}\log_4 4^u=2-u$ $\log_x 8=\frac{\log_4 8}{\log_4 x}=\frac{\log_4 8}{u}=\frac{\frac{\log_2 8}{\log_2 4}}{u}=\frac{\frac{3}{2}}{u}=\frac{3}{2u}$
3	$given\ that\ \log_p(x^2y)=8,\quad \log_p\left(\frac{y^2}{x}\right)=6\,,\to evaluate\ \log_p xy\,,\quad and\ \log_p\frac{y}{x}$ $\log_p(x^2y)=8\to 2\log_p x+\log_p y=8$ $\log_p\left(\frac{y^2}{x}\right)=6\to 2\log_p y-\log_p x=6$ $2x+y=8,\quad 2y-x=6$ $y=8-2x,\quad 2y-6=x$ $2(8-2x)-6=x\to 16-4x-6=x$ $10=5x\to x=2,\quad y=4$ $\log_p xy=\log_p x+\log_p y=2+4=6$ $\log_p\frac{y}{x}=\log_p y-\log_p x=4-2=2$

4	$\log_{16}(3x-1)=\log_4(3x)+\log_4 0.5$ $\frac{\log_4(3x-1)}{\log_4 16}=\log_4(3x)+\log_4 1/2$ $(3x-1)^{\frac{1}{2}}=\frac{3}{2}x\rightarrow 3x-1=\frac{9}{4}x^2\rightarrow 12x-4=9x^2$ $9x^2-12x+4=0$ $(3x-2)(3x-2)$ $x=\frac{2}{3}$
5	$\log_2(x)-\log_4(x-4)=2$ $\log_2\left(\frac{x}{(x-4)^{\frac{1}{2}}}\right)=2\rightarrow\frac{x}{(x-4)^{\frac{1}{2}}}=4$ $x=4(x-4)^{\frac{1}{2}}\rightarrow x^2=16(x-4)$ $x^2-16x+64=0$ $(x-8)(x-8)$ $x=8$
6	$\log_3(y)+4\log_y 3=4$ $\log_3(y)+\frac{\log_3 81}{\log_3 y}=4$ $\log_3(y)^2+\log_3 81=4\log_3 y$ $\log_3(y)^2-4\log_3 y+4=0$ $(\log_3 y-2)(\log_3 y-2)$ $\log_3 y=2$ $y=9$
7	$\lg(p-q)=\lg p-\lg q\ , express\ p\ in\ terms\ of\ q$ $\lg(p-q)=\lg p-\lg q$ $\lg(p-q)=\lg\frac{p}{q}$ $p-q=\frac{p}{q}\rightarrow pq-q^2=p$ $pq-p=q^2$ $p(q-1)=q^2$ $p=\frac{q^2}{q-1}$

8	*Given that* $\lg x = p$ *and* $\lg y\ \ q$ *find* $\lg xy^2\,,\qquad \lg\left(\frac{10x}{y}\right),\qquad \lg\sqrt{10x^3y}\,,\qquad \lg(\frac{100\sqrt{x}}{y^2})$ $\lg xy^2 = \lg x + \lg y^2 = p + 2q$ $\lg\left(\frac{10x}{y}\right) = \lg 10 + \lg x - \lg y = 1 + p - q$ $\lg\sqrt{10x^3y} = \frac{1}{2}(\lg 10 + 3\lg x + \lg y) = \frac{1}{2}(1 + 3p + q)$ $\lg\left(\frac{100\sqrt{x}}{y^2}\right) = \lg 100 + \frac{1}{2}\lg x - 2\lg y = 2 + \frac{1}{2}p - 2q$
9	*Given that* $\log_b xy^2 = m$ *and* $\log_b x^3\, y = n$ *express* $\log_b \frac{y}{x},\qquad \log_b\sqrt{xy}\,,\qquad$ *in terms of* m *and* n $\log_b xy^2 = \log_b x + 2\log_b y = m \rightarrow \log_b x = m - 2\log_{\mathrm{b}} y$ $\log_b x^3\, y = 3\log_b x + \log_b y = n$ $\cancel{3(m - 2\log_{\mathrm{b}} y) + \log_b y = n}$ $\cancel{3m - 6\log_{\mathrm{b}} y + \log_b y = n}$ $\cancel{3m - 5\log_{\mathrm{b}} y = n}$ $\cancel{\log_{\mathrm{b}} y = \frac{3m-n}{5}}$ $\cancel{\log_b x = m - 2\frac{3m-n}{5} = m - \frac{6m-2n}{5} = \frac{-m-2n}{5}}$ $\cancel{\log_b \frac{y}{x} = \frac{3m-n}{5} - \frac{-m-2n}{5} = \frac{3m-n+m+2n}{5} = \frac{4m+n}{5}}$ $\cancel{\log_b\sqrt{xy} = \frac{1}{2}\frac{-m-2n}{5}\,\frac{3m-n}{5} = \frac{-m-2n}{50}}$