Contents

1. Definition

$Let\ q_1\ be\ Quaternion\ 1\ ,\qquad q_2\ be\ Quaternion\ 2, \dots q_n,\qquad a\ be\ Real\ number$

Commutativity	$q_1 + q_2 = q_2 + q_1$ $q_1 \,.\, q_2 \ = q_2 \,.\, q_1\ (dot\ product)$ $a(q_1) = q_1(a)$ $q_1 q_2 \neq q_2 q_1$
Associativity	$q_1 + (q_2 + q_3) = (q_1 + q_2) + q_3$ $(q_1 q_2) q_3 \ = q_1(q_2 q_3)$
Distributivity, etc.	$q_1(q_2 + q_3) = q_1 q_2 + q_1 q_3$ $a(q_1 + q_2) = aq_1 + aq_2$
Dot Product	$q_1 \,.\, (q_2 + q_3) = q_1 . q_2 + q_1 . q_3$ $qq^{-1} = q^{-1}q = I, \qquad \lvert q \rvert = 1, \qquad \lvert q^{-1} \rvert = 1$ $aq^{-1} = q^{-1}a$ $(q_1 q_2)^{-1} = q_2^{-1} q_1^{-1}$
Identity Rule	$Iq = qI = q$ $if\ \lvert q_1 \rvert = \ \lvert q_2 \rvert = 1, then\ \ \lvert q_1 q_2 \rvert = 1$
Conjugate Rule	$if\ \lvert q \rvert = 1,\ \ then\ q^{-1} = \bar{q}$

2. Imaginary Rule

$$i^2 = j^2 = k^2 = ijk = -1$$

Prove:

ij = k	$ijk = -1$ $ijkk = -1k$ $-ij = -k$ $ij = k$
jk = i	$ijk = -1$ $iijk = -1i$ $-jk = -i$ $jk = i$
ki = j	$ijk = -1$ $iijki = i(-1i), \qquad ii = -1$ $-jki = 1$ $j(-j)ki = j, \qquad j(-j) = 1$ $ki = j$
ji = -k	$ij = k$ $iiji = iki$ $-ji = ij$ $ji = -ij$ $ji = -k$
kj = -i	$jk = i$ $jjkj = jij$ $-kj = jk$ $kj = -jk$ $kj = -i$
ik = -j	$ki = j$ $kkik = kjk$ $-ik = ki$ $ik = -ki$ $ik = -j$

3. Operators

$$q = [real, vector] = [x, y, z, w], \qquad [x, y, z] = vector\ part, \qquad [w] = real\ part$$

3.1. Addition and Subtraction

$$q1 + q2 = [x_1, y_1 z_1, w_1] \pm [x_2, y_2, z_2, w_2] = [x_1 \pm x_2, \qquad y_1 \pm y_2, \qquad z_1 \pm z_2, \qquad w_1 \pm w_2]$$

3.2. Identity

$$q = [0,0,0,1] = I$$

3.3. Multiplication (Note: Using IMAGINARY RULE as well)

$q_0 q_1 = [x_0, y_0, z_0, w_0] * [x_1, y_1, z_1, w_1]$

$= (\, w_0 + \ x_0 i + y_0 j + z_0 k)(w_1 + x_1 i + y_1 j + z_1 k)$

$= w_0 w_1 + w_0 x_1 i + w_0 y_1 j + w_0 z_1 k + x_0 w_1 i - x_0 x_1 + x_0 y_1 ij + x_0 z_1 ik + y_0 w_1 j + y_0 x_1 ji - y_0 y_1 + y_0 z_1 jk + z_0 w_1 k + z_0 x_1 ki + z_0 y_1 kj - z_0 z_1$

$= w_0 w_1 + w_0 x_1 i + w_0 y_1 j + w_0 z_1 k + x_0 w_1 i - x_0 x_1 + x_0 y_1 k - x_0 z_1 j + y_0 w_1 j - y_0 x_1 k - y_0 y_1 + y_0 z_1 i + z_0 w_1 k - z_0 x_1 j - z_0 y_1 i - z_0 z_1 =$

$(w_0 w_1 - x_0 x_1 - y_0 y_1 - z_0 z_1) + \ \ (real\ part)$

$(w_0 x_1 + x_0 w_1 + y_0 z_1 - z_0 y_1) i + \ \ (x\ part)$

$(w_0 y_1 - x_0 z_1 + y_0 w_1 + z_0 x_1) j + \ \ (y\ part)$

$(w_0 z_1 + x_0 y_1 - y_0 x_1 + z_0 w_1) k + \ \ (z\ part)$

Alternative form:

$$q_0 q_1 = [real, vector][real, vector] = [w_0, v_0] * [w_1, v_1] = [w_0 w_1 - v_0 . v_1, \qquad w_0 v_1 + w_1 v_0 + v_0 X v_1]$$

Matrix Form:

$$\begin{bmatrix} w_0 & -x_0 & -y_0 & -z_0 \\ x_0 & w_0 & -z_0 & y_0 \\ y_0 & z_0 & w_0 & -x_0 \\ z_0 & -y_0 & x_0 & w_0 \end{bmatrix} \begin{bmatrix} w_1 \\ x_1 \\ y_1 \\ z_1 \end{bmatrix}$$

$x \ = \ lhs.w \ * \ rhs.x \ + \ lhs.x \ * \ rhs.w \ + \ lhs.y \ * \ rhs.z \ - \ lhs.z \ * \ rhs.y;$
$y \ = \ lhs.w \ * \ rhs.y \ + \ lhs.y \ * \ rhs.w \ + \ lhs.z \ * \ rhs.x \ - \ lhs.x \ * \ rhs.z;$
$z \ = \ lhs.w \ * \ rhs.z \ + \ lhs.z \ * \ rhs.w \ + \ lhs.x \ * \ rhs.y \ - \ lhs.y \ * \ rhs.x;$
$w \ = \ lhs.w \ * \ rhs.w \ - \ lhs.x \ * \ rhs.x \ - \ lhs.y \ * \ rhs.y \ - \ lhs.z \ * \ rhs.z;$

3.4. Conjugate

$$q = [x, y, z, w], \qquad \bar{q} = [-x, -y, -z, w] \; where \; |q| = 1$$

$$q\bar{q} = \bar{q}q = I, \qquad |q\bar{q}| = 1, \qquad \bar{\bar{q}} = \overline{qq} = q$$

Prove:

$$q\bar{q} = [w_0, v_0] * [w_0, -v_0] = [w_0 w_0 - v_0. -v_0, \qquad -(w_0 v_0) + w_0 v_0 + v_0 X - v_0]$$

$$= \left[w_0 w_0 + v_0 . v_0, \qquad \vec{0}\right], \qquad since \; is \; a \; unit \; quaternion$$

$$= \left[|q|, \qquad \vec{0}\right] = \left[1, \qquad \vec{0}\right]$$

3.5. Invert

From Complex number Conjugate:

$z = a + bi, \qquad \bar{z} = a - bi$

$z\bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 + b^2$

We can solve Invert:

$$\frac{a^2 + b^2}{a^2 + b^2} = 1, \qquad \frac{z\bar{z}}{a^2 + b^2} = 1, \qquad z\left(\frac{\bar{z}}{a^2 + b^2}\right) = 1, \qquad \frac{\bar{z}}{a^2 + b^2} = z^{-1}, \qquad \frac{\bar{z}}{|z|^2} = z^{-1}$$

$Therefore \; the \; Length = \dfrac{1.0}{xx + yy + zz + ww}, \qquad q^{-1} = [-x * length, -y * length, -z * length, w]$

$As \; For \; unit \; Quaternion, \qquad q^{-1} = \bar{q}$

Prove:

$Since \; q^{-1} = [-x * length, -y * length, -z * length, w], \qquad$ for unit quaternion length $= 1$

$therefore \; q^{-1} = [-x * 1, -y * 1, -z * 1, w] = [-x, -y, -z, w] = Conjugate \; of \; q$

4. 2D Rotation

$$\text{Rectangle form: } a + bi$$
$$\text{Polar form: } re^{i\theta}$$

4.1. From Maclaurin series expansion

$$\text{Show that } e^{i\theta} = cos\theta + isin\theta$$

$$e^{i\theta} = a + bi$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos\theta = \sum_{n=0}^{\infty} \frac{\theta^{2n}(-1)^n}{2n!} = 1 - \frac{x\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$sin\theta = \sum_{n=0}^{\infty} \frac{\theta^{2n+1}(-1)^n}{2n+1!} = 1 - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{i\theta^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n\theta^n}{n!} = 1 + i\theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{\theta^{2n}(-1)^n}{2n!} + i\sum_{n=0}^{\infty} \frac{\theta^{2n+1}(-1)^n}{2n+1!}$$

$$= cos\theta + isin\theta$$

5. 3D Rotation

$$\text{Rectangle form:} \quad a + bi + cj + dk$$

$$\text{Polar form:} \quad re^{q_1 i + q_2 j + q_3 k} = e^{0,\frac{\theta}{2}\hat{n}} = \cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}$$

Quaternion Matrix:

$$From\ axis - angle\ rotation\ matrix:$$
$$R(\theta, \hat{n}) =$$
$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & n_2 n_1(1 - cos\theta) - n_3(sin\theta) & n_1 n_3(1 - cos\theta) + n_2(sin\theta) & 0 \\ n_1 n_2(1 - cos\theta) + n_3(sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2 n_3(1 - cos\theta) - n_1(sin\theta) & 0 \\ n_1 n_3(1 - cos\theta) - n_2(sin\theta) & n_2 n_3(1 - cos\theta) + n_1(sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5.1. Trigo Identity To use (mark in Red)

$$\sin\theta = sin2\frac{\theta}{2} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$\cos\theta = cos2\frac{\theta}{2} = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}$$

$$1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

5.2. Using Substitution

We know that:

$$q = (0 < \hat{n} >) = (q_0 < q_1, q_2, q_3 >) = \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}$$
$$q_0 = \cos\frac{\theta}{2}, \qquad q_1 = n_1\sin\frac{\theta}{2}, \qquad q_2 = n_2\sin\frac{\theta}{2}, \qquad q_3 = n_3\sin\frac{\theta}{2}$$

Conversion broken down into 2

Step 1:

1st row 2nd col:

$$n_2 n_1(1 - cos\theta) - n_3(sin\theta) = n_2 n_1\left(2\sin^2\frac{\theta}{2}\right) - n_3\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= 2\, n_2\left(sin\frac{\theta}{2}\right) n_1\left(sin\frac{\theta}{2}\right) - 2\, n_3\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= 2q_1 q_2 - 2q_3 q_0$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1 q_2 - 2q_3 q_0 & n_1 n_3(1 - cos\theta) + n_2(sin\theta) & 0 \\ n_1 n_2(1 - cos\theta) + n_3(sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2 n_3(1 - cos\theta) - n_1(sin\theta) & 0 \\ n_1 n_3(1 - cos\theta) - n_2(sin\theta) & n_2 n_3(1 - cos\theta) + n_1(sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, so on and so forth:

1st row 3nd col:

$$n_1 n_3(1 - cos\theta) + n_2(sin\theta) = n_1 n_3 2\sin^2\frac{\theta}{2} - n_2 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= 2q_1 q_3 - 2q_2 q_0$$

2nd row 1st col:

$$n_1 n_2 (1 - cos\theta) + n_3 (sin\theta) = n_1 n_2 2 \sin^2 \frac{\theta}{2} + n_3 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2q_1 q_2 + 2q_3 q_0$$

2nd row 3nd col:

$$n_2 n_3 (1 - cos\theta) - n_1 (sin\theta) = n_2 n_3 2 \sin^2 \frac{\theta}{2} - n_1 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2q_2 q_3 + 2q_1 q_0$$

3rd row 1st col:

$$n_1 n_3 (1 - cos\theta) - n_2 (sin\theta) = n_1 n_3 2 \sin^2 \frac{\theta}{2} - n_2 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2q_1 q_3 + 2q_2 q_0$$

3rd row 2nd col:

$$n_2 n_3 (1 - cos\theta) + n_1 (sin\theta) = n_2 n_3 2 \sin^2 \frac{\theta}{2} + n_1 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2q_2 q_3 + 2q_1 q_0$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & n_1 n_3 (1 - cos\theta) + n_2 (sin\theta) & 0 \\ n_1 n_2 (1 - cos\theta) + n_3 (sin\theta) & cos\theta + n_2^2(1 - cos\theta) & n_2 n_3 (1 - cos\theta) - n_1 (sin\theta) & 0 \\ n_1 n_3 (1 - cos\theta) - n_2 (sin\theta) & n_2 n_3 (1 - cos\theta) + n_1 (sin\theta) & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & cos\theta + n_2^2(1 - cos\theta) & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Step 2:

1st row 1st col:

$$cos\theta + n_1^2(1 - cos\theta) = \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + n_1^2 2 \sin^2 \frac{\theta}{2}$$

Step 2.5:

$$\cos^2 \frac{\theta}{2} = q_0^2 , \qquad \sin^2 \frac{\theta}{2} = n_1^2 \sin^2 \frac{\theta}{2} + n_2^2 \sin^2 \frac{\theta}{2} + n_3^2 \sin^2 \frac{\theta}{2} = q_1^2 + q_2^2 + q_3^2, \qquad n_1^2 2 \sin^2 \frac{\theta}{2} = 2q_1^2$$

Therefore Step 3:

$$\begin{aligned} cos\theta + n_1^2(1 - cos\theta) &= \cos^2 \frac{\theta}{2} - \sin^2 \frac{\theta}{2} + n_1^2 2 \sin^2 \frac{\theta}{2} \\ &= q_0^2 - (q_1^2 + q_2^2 + q_3^2) + 2q_1^2 \\ &= q_0^2 - q_1^2 - q_2^2 - q_3^2 + 2q_1^2 \\ &= q_0^2 + q_1^2 - q_2^2 - q_3^2 \end{aligned}$$

$$\begin{bmatrix} cos\theta + n_1^2(1 - cos\theta) & 2q_1q_2 - 2q_3q_0 & 2q_1q_3 - 2q_2q_0 & 0 \\ 2q_1q_2 + 2q_3q_0 & cos\theta + n_2^2(1 - cos\theta) & 2q_2q_3 + 2q_1q_0 & 0 \\ 2q_1q_3 + 2q_2q_0 & 2q_2q_3 + 2q_1q_0 & cos\theta + n_3^2(1 - cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_3q_0 & 2q_1q_3-2q_2q_0 & 0 \\ 2q_1q_2+2q_3q_0 & cos\theta+n_2^2(1-cos\theta) & 2q_2q_3+2q_1q_0 & 0 \\ 2q_1q_3+2q_2q_0 & 2q_2q_3+2q_1q_0 & cos\theta+n_3^2(1-cos\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, so on and so forth:
2nd row 2nd col:

$$\begin{aligned} cos\theta+n_2^2(1-cos\theta) &= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}+n_2^2 2\sin^2\frac{\theta}{2} \\ &= q_0^2-(q_1^2+q_2^2+q_3^2)+2q_2^2 \\ &= q_0^2-q_1^2-q_2^2-q_3^2+2q_2^2 \\ &= q_0^2-q_1^2+q_2^2-q_3^2 \end{aligned}$$

3rd row 3rd col:

$$\begin{aligned} cos\theta+n_3^2(1-cos\theta) &= \cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}+n_3^2 2\sin^2\frac{\theta}{2} \\ &= q_0^2-(q_1^2+q_2^2+q_3^2)+2q_3^2 \\ &= q_0^2-q_1^2-q_2^2-q_3^2+2q_3^2 \\ &= q_0^2-q_1^2-q_2^2+q_3^2 \end{aligned}$$

Finally:

$$\begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2q_1q_2-2q_3q_0 & 2q_1q_3-2q_2q_0 & 0 \\ 2q_1q_2+2q_3q_0 & q_0^2-q_1^2+q_2^2-q_3^2 & 2q_2q_3+2q_1q_0 & 0 \\ 2q_1q_3+2q_2q_0 & 2q_2q_3+2q_1q_0 & q_0^2-q_1^2-q_2^2+q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$