Catalogue no. 12-001-X
ISSN 1492-0921

Survey Methodology

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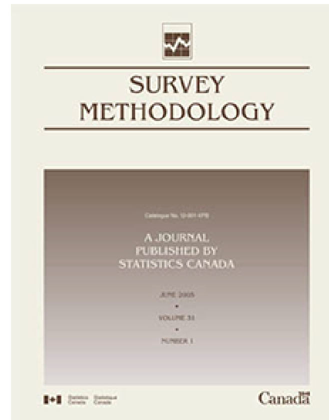

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Release date: January 6, 2022

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Survey Methodology, December 2021
Vol. 47, No. 2, pp. 361-370
Statistics Canada, Catalogue No. 12-001-X

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Yong You[1]

Abstract

In this paper, we consider the Fay-Herriot model for small area estimation. In particular, we are interested in the impact of sampling variance smoothing and modeling on the model-based estimates. We present methods of smoothing and modeling for the sampling variances and apply the proposed models to a real data analysis. Our results indicate that sampling variance smoothing can improve the efficiency and accuracy of the model-based estimator. For sampling variance modeling, the HB models of You (2016) and Sugasawa, Tamae and Kubokawa (2017) perform equally well to improve the direct survey estimates.

Key Words: EBLUP; Hierarchical Bayes; Gibbs sampling; Log-linear model; Relative error; Sampling variance; Small area.

1. Introduction

Small area estimation is popular and important in survey data analysis. Model-based estimates have been widely used in practice to provide reliable estimates for small areas. In practice, area level models are usually used whenever direct survey estimates and area level auxiliary variables are available. Various area level models have been proposed to improve the precision of the direct survey estimates, see Rao and Molina (2015). Among the area level models, the Fay-Herriot model (Fay and Herriot, 1979) is a basic area level model widely used in small area estimation. The Fay-Herriot model has two components, namely, a sampling model for the direct survey estimates and a linking model for the small area parameter of interest. The sampling model assumes that a direct survey estimator y_i is design unbiased for the small area parameter θ_i such that

$$y_i = \theta_i + e_i, \quad i = 1, \ldots, m, \tag{1.1}$$

where e_i is the sampling error associated with the direct estimator y_i and m is the number of small areas. It is customary to assume that e_i's are independently normal random variables with mean $\mathrm{E}(e_i) = 0$ and sampling variance $\mathrm{Var}(e_i) = \sigma_i^2$. The linking model assumes that the small area parameter θ_i is related to auxiliary variables $x_i = (x_{i1}, \ldots, x_{ip})'$ through a linear regression model given as

$$\theta_i = x_i'\beta + v_i, \quad i = 1, \ldots, m, \tag{1.2}$$

where $\beta = (\beta_1, \ldots, \beta_p)'$ is a $p \times 1$ vector of regression coefficients, and the v_i's are area-specific random effects assumed to be independent and identically distributed with $\mathrm{E}(v_i) = 0$ and $\mathrm{Var}(v_i) = \sigma_v^2$. The assumption of normality is generally included. Random effects v_i and sampling errors e_i are mutually

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independent. The model variance σ_v^2 is unknown and needs to be estimated. Combining models (1.1) and (1.2) leads to a linear mixed model given as

$$y_i = x_i' \beta + v_i + e_i, \quad i = 1, \ldots, m. \tag{1.3}$$

Model (1.3) involves both design-based random errors e_i and model-based random effects v_i. For the Fay-Herriot model, the sampling variance σ_i^2 is usually assumed to be known. This is a very strong assumption. In practice, unbiased direct estimates of the sampling variances are generally available. To make use of the direct sampling variance estimates, two approaches are available in practice, namely, smoothing and modeling. For the smoothing approach, smoothed estimates of the sampling variances are used in the Fay-Herriot model and then treated as known. The smoothing approach requires external variables and external models such as use of the generalized variance function (GVF) and design effects. You and Hidiroglou (2012) particularly studied the GVF and design effects methods for sampling variance smoothing for proportions. In this paper, we will use a GVF model proposed in You and Hidiroglou (2012) for the sampling variance smoothing.

As an alternative to smoothing, sampling variance modeling is also commonly used in practice. Let s_i^2 denote the direct estimator for the sampling variance σ_i^2. We consider a custom model for s_i^2 as $d_i s_i^2 \sim \sigma_i^2 \chi_{d_i}^2$, where $d_i = n_i - 1$ and n_i is the sample size for the i^{th} area. Rivest and Vandal (2002) and Wang and Fuller (2003) used empirical best linear unbiased prediction (EBLUP) method to obtain the model-based estimates. You and Chapman (2006) considered a hierarchical Bayes (HB) approach and combined the sampling variance model $d_i s_i^2 \sim \sigma_i^2 \chi_{d_i}^2$ with the small area model (1.3) to construct an integrated model. The integrated model borrows strength for small area estimates and sampling variance estimates simultaneously. The integrated HB modeling approach with $d_i s_i^2 \sim \sigma_i^2 \chi_{d_i}^2$ has thus been widely used in practice, for example, You (2008, 2016), Dass, Maiti, Ren and Sinha (2012), Sugasawa, Tamae and Kubokawa (2017), Ghosh, Myung and Moura (2018), and Hidiroglou, Beaumont and Yung (2019).

In this paper, we consider both the smoothing and modeling approaches for the sampling variances. In Section 2, we present the EBLUP method based on both the smoothed and direct estimates of the sampling variances. In Section 3, we present the Fay-Herriot HB model and three other HB models based on sampling variance modeling. We compare the effects of sampling variance smoothing and modeling in Section 4 through a real data analysis, and we offer some suggestions in Section 5.

2. Fay-Herriot model using EBLUP approach

Under the Fay-Herriot model (1.3), assuming σ_i^2 and σ_v^2 known in the model, we obtain the best linear unbiased prediction (BLUP) estimator of θ_i as $\tilde{\theta}_i = \gamma_i\, y_i + (1 - \gamma_i)\, x_i' \tilde{\beta}$, where $\gamma_i = \sigma_v^2 / (\sigma_v^2 + \sigma_i^2)$ and $\tilde{\beta} = \left(\sum_{i=1}^m (\sigma_i^2 + \sigma_v^2)^{-1} x_i\, x_i' \right)^{-1} \left(\sum_{i=1}^m (\sigma_i^2 + \sigma_v^2)^{-1} x_i\, y_i \right)$. To estimate the variance component σ_v^2, we have to first assume σ_i^2 known. There are several methods available to estimate σ_v^2, and we use REML method to estimate σ_v^2. Then the EBLUP of the small area parameter θ_i is obtained as

$$\hat{\theta}_i = \hat{\gamma}_i y_i + (1 - \hat{\gamma}_i) x_i' \hat{\beta}, \tag{2.1}$$

where $\hat{\gamma}_i = \hat{\sigma}_v^2 / (\hat{\sigma}_v^2 + \sigma_i^2)$ and $\hat{\sigma}_v^2$ is the REML estimator. The estimator for the mean squared error (MSE) of $\hat{\theta}_i$ is given by $\operatorname{mse}(\hat{\theta}_i) = g_{1i} + g_{2i} + 2g_{3i}$, where $g_{1i} = \hat{\gamma}_i \sigma_i^2$ is the leading term, g_{2i} accounts for the variability due to estimation of the regression parameter β, and g_{3i} is due to the estimation of the model variance σ_v^2; see Rao and Molina (2015) for details.

We may use the smoothed or direct estimate of σ_i^2 in (2.1). For sampling variance smoothing, we use a log-linear regression model on the direct sampling variance s_i^2 as suggested in You and Hidiroglou (2012), and the smoothing model is defined as:

$$\log(s_i^2) = \eta_0 + \eta_1 \log(n_i) + \varepsilon_i, \quad i = 1, \ldots, m, \tag{2.2}$$

where the model error term is $\varepsilon_i \sim N(0, \psi^2)$, and ψ^2 is unknown. Let $\hat{\eta}_0$ and $\hat{\eta}_1$ denote the ordinary least square estimates of the regression coefficients η_0 and η_1, and $\hat{\psi}^2$ be the estimated residual variance of the log-linear regression model (2.2). A smoothed estimator of the sampling variance σ_i^2 can be obtained as

$$\tilde{\sigma}_i^2 = \exp(\hat{\eta}_0 + \hat{\eta}_1 \log(n_i)) \exp(\hat{\psi}^2 / 2).$$

The smoothed sampling variances $\tilde{\sigma}_i^2$ can then be used in the EBLUP estimator (2.1) and its MSE computation. This procedure is a common practice, see Rao and Molina (2015).

If direct sampling variance estimate s_i^2 is used in the place of the true sampling variance σ_i^2 in (2.1), then an extra term accounting for the uncertainty of using s_i^2 is needed in the MSE estimator. This term, denoted as g_{4i}, is given as $g_{4i} = 4(n_i - 1)^{-1} \hat{\sigma}_v^4 s_i^4 (\hat{\sigma}_v^2 + s_i^2)^{-3}$; see Rivest and Vandal (2002) and Rao and Molina (2015), page 150. However, using s_i^2 directly in the EBLUP could lead to an over estimation of the model variance σ_v^2 (You, 2010; Rubin-Bleuer and You, 2016), as well as less accurate estimates. We will compare the EBLUP estimates with the HB estimates based on the smoothed and direct sampling variances in Section 4.

3. Fay-Herriot model using HB approach with sampling variance modeling

In this section we first present the Fay-Herriot model in a HB framework. Then we consider three models for the sampling variance modeling. The first model is the one considered in You and Chapman (2006) in which an inverse gamma model is used for the sampling variance σ_i^2 with known vague parameter values. The second model is introduced in You (2016) whereby a log-linear model with random error is used for σ_i^2. The third model is one proposed by Sugasawa et al. (2017) where an inverse gamma model is used for σ_i^2 but with different parameter settings.

HB Model 1: Fay-Herriot model in HB, denoted as FH-HB:

- $y_i \mid \theta_i, \sigma_i^2 \sim \text{ind } N(\theta_i, \sigma_i^2), \quad i = 1, \ldots, m;$
- $\theta_i \mid \beta, \sigma_v^2 \sim \text{ind } N(x_i' \beta, \sigma_v^2), \quad i = 1, \ldots, m;$
- Flat priors for unknown parameters: $\pi(\beta) \propto 1, \;\; \pi(\sigma_v^2) \propto 1.$

Note that in the FH-HB model, the sampling variance σ_i^2 is assumed to be known. Either a smoothed sampling variance $\tilde{\sigma}_i^2$ or a direct sampling variance estimate s_i^2 will be used in place of σ_i^2.

HB Model 2: You-Chapman Model (You and Chapman, 2006), denoted as YCM:

- $y_i \mid \theta_i, \sigma_i^2 \sim \text{ind } N(\theta_i, \sigma_i^2), \quad i = 1, \ldots, m;$
- $d_i s_i^2 \mid \sigma_i^2 \sim \text{ind } \sigma_i^2 \chi_{d_i}^2, \;\; d_i = n_i - 1, \quad i = 1, \ldots, m;$
- $\theta_i \mid \beta, \sigma_v^2 \sim \text{ind } N(x_i' \beta, \sigma_v^2), \quad i = 1, \ldots, m;$
- $\pi(\sigma_i^2) \sim \text{IG}(a_i, b_i)$, where $a_i = 0.0001$, $b_i = 0.0001$, $i = 1, \ldots, m;$
- Flat priors for unknown parameters: $\pi(\beta) \propto 1, \;\; \pi(\sigma_v^2) \propto 1.$

The full conditional distributions for the Gibbs sampling procedure under both FH-HB and YCM can be found in You and Chapman (2006).

HB Model 3: You (2016) Log-linear model on sampling variances, denoted as YLLM:

- $y_i \mid \theta_i, \sigma_i^2 \sim \text{ind } N(\theta_i, \sigma_i^2), \quad i = 1, \ldots, m;$
- $d_i s_i^2 \mid \sigma_i^2 \sim \text{ind } \sigma_i^2 \chi_{d_i}^2, \; d_i = n_i - 1, \quad i = 1, \ldots, m;$
- $\theta_i \mid \beta, \sigma_v^2 \sim \text{ind } N(x_i' \beta, \sigma_v^2), \quad i = 1, \ldots, m;$
- $\log(\sigma_i^2) \sim N\left(\delta_1 + \delta_2 \log(n_i), \tau^2\right), \quad i = 1, \ldots, m;$
- Flat priors for unknown parameters: $\pi(\beta) \propto 1, \;\; \pi(\delta_1, \delta_2) \propto 1, \;\; \pi(\sigma_v^2) \propto 1, \;\; \pi(\tau^2) \propto 1.$

Note that model YLLM uses a log-linear model for the sampling variance σ_i^2, and extends the model proposed by Souza, Moura and Migon (2009) for sampling variances by using $\log(n_i)$ and adding a random effect to the regression part in the model. The full conditional distributions for the Gibbs sampling procedure are given in the Appendix.

HB Model 4: Sugasawa, Tamae and Kubokawa (2017) model shrinking both means and variances, denoted as STKM:

- $y_i \mid \theta_i, \sigma_i^2 \sim \text{ind } N(\theta_i, \sigma_i^2), \quad i = 1, \ldots, m;$
- $d_i s_i^2 \mid \sigma_i^2 \sim \text{ind } \sigma_i^2 \chi_{d_i}^2, \; d_i = n_i - 1, \quad i = 1, \ldots, m;$
- $\theta_i \mid \beta, \sigma_v^2 \sim \text{ind } N(x_i' \beta, \sigma_v^2), \quad i = 1, \ldots, m;$

- $\pi(\sigma_i^2) \sim \mathrm{IG}(a_i, b_i\gamma)$, where a_i and b_i are known constants, $a_i = O(1)$, $b_i = O(n_i^{-1})$;
- Flat priors for unknown parameters: $\pi(\beta) \propto 1$, $\pi(\sigma_v^2) \propto 1$, $\pi(\gamma) \propto 1$.

Note that in STKM, for the inverse gamma model of σ_i^2, we choose $a_i = 2$ and $b_i = n_i^{-1}$ as suggested by Sugasawa et al. (2017). Ghosh et al. (2018) also used the same setting in their study of comparing HB estimators. The full conditional distributions for STKM can be found in Sugasawa et al. (2017).

Note that the Chi-squared sampling variance modeling $d_i s_i^2 \sim \sigma_i^2 \chi_{d_i}^2$ in the above HB Models 2-4 is based on normality and simple random sampling (Rivest and Vandal, 2002). For complex survey designs, the degrees of freedom d_i may need to be determined more carefully. There is no sound theoretical result for determining the degrees of freedom (Dass et al., 2012). The approximation formula based on non-normal unit level errors provided by Wang and Fuller (2003) and the simulation based guideline of Maples, Bell and Huang (2009) could be useful but require unit level data and an extensive simulation study. A careful determination of the degrees of freedom may provide a reasonably useful approximation. Moreover, Bayesian model fit analysis can also be helpful for model determination.

4. Application

In this section, we apply the models in Sections 2 and 3 to the Canadian Labour Force Survey (LFS) data and compare the EBLUP and HB estimates. The LFS releases monthly unemployment rate estimates for large areas such as the nation and provinces as well as local areas such as Census Metropolitan Areas (CMAs) and Census Agglomerations (CAs) across Canada. The direct LFS estimates for some local areas are not reliable exhibiting very large coefficient of variations (CVs) due to small sample sizes. Model-based estimators are considered to improve the direct LFS estimates. As an illustration, we apply the Fay-Herriot model to the May 2016 unemployment rate estimates at the CMA/CA level, and compare the model-based estimates and the direct estimates with the census estimates to compare the effects of sampling variance smoothing and modeling. Hidiroglou et al. (2019) also compared the model-based LFS estimates with the census estimates. For the unemployment rate estimation, the local area employment insurance monthly beneficiary rate is used as an auxiliary variable in the model. For comparison of point estimates, we compute the absolute relative error (ARE) of the direct and model estimates with respect to the census estimates for each CMA/CA as follows:

$$\mathrm{ARE}_i = \left| \frac{\theta_i^{\mathrm{Census}} - \theta_i^{\mathrm{Est}}}{\theta_i^{\mathrm{Census}}} \right|,$$

where θ_i^{Est} is the direct or the EBLUP/HB estimate and $\theta_i^{\mathrm{Census}}$ is the corresponding census value of the unemployment rate. Then we take the average of AREs over CMA/CAs. For CV, we compute the average CVs of the direct and model-based estimates. We prefer a model with smaller ARE and smaller CV.

We first apply the models to all the 117 CMA/CAs with sample size $\geq 2,$ and then apply them to 92 CMA/CAs with sample size $\geq 5,$ and finally 79 CMA/CAs with sample size $\geq 7.$ Table 4.1 presents the average ARE and the corresponding average CV (in brackets). In Table 4.1, the model with Smoothed sv indicates that a smoothed sampling variance is used, Direct sv indicates that a direct sampling variance estimate is used.

With Smoothed sv, both FH-EBLUP and FH-HB substantially improve the direct survey estimates with much smaller ARE and CV. In particular, FH-HB has the smallest ARE, and FH-EBLUP has the smallest CV. For example, over the 117 areas, the direct LFS estimator has ARE 0.263 with average CV 0.329, FH-EBLUP Smoothed sv has ARE 0.124 with average CV 0.087, FH-HB Smoothed sv has ARE 0.118 with average CV 0.116. The good performance of FH-EBLUP and FH-HB with Smoothed sv indicates that the smoothing GVF (2.2) is very useful and effective in improving the model-based estimates.

With Direct sv, both FH-EBLUP and FH-HB perform the worst among all the models, with almost identical results under this scenario. The other three HB models perform better than the FH-EBLUP and FH-HB using direct sv. YLLM and STKM perform better than YCM with smaller ARE and smaller CV. YLLM and STKM perform very similarly for all the CMA/CA groups, and YLLM consistently has slightly smaller ARE than STKM, but YLLM has slightly larger CV than STKM. For example, over the 117 areas, YLLM has ARE 0.135, STKM has ARE 0.137, and YLLM has average CV 0.123, and STKM has average CV 0.122. YCM has ARE 0.148 with CV 0.136, FH-HB has ARE 0.171 with CV 0.221.

Table 4.1
Comparison of average absolute relative error (ARE) and average CV in parenthesis

CMA/CAs	Direct LFS	FH-EBLUP Smoothed sv	FH-HB Smoothed sv	FH-EBLUP Direct sv	FH-HB Direct sv	YCM Direct sv	YLLM Direct sv	STKM Direct sv
Average over 117 CMA/CAs	0.263	0.124	0.118	0.170	0.171	0.148	0.135	0.137
(sample size ≥ 2)	(0.329)	(0.087)	(0.116)	(0.238)	(0.221)	(0.136)	(0.123)	(0.122)
Average over 92 CMA/CAs	0.216	0.124	0.116	0.133	0.132	0.132	0.125	0.127
(sample size ≥ 5)	(0.262)	(0.076)	(0.103)	(0.123)	(0.123)	(0.121)	(0.117)	(0.116)
Average over 79 CMA/CAs	0.181	0.122	0.113	0.126	0.122	0.122	0.118	0.120
(sample size ≥ 7)	(0.232)	(0.057)	(0.094)	(0.115)	(0.115)	(0.115)	(0.114)	(0.113)

Now we present a Bayesian model comparison using conditional predictive ordinate (CPO) for the four HB models with Direct sv. CPOs are the observed likelihoods based on the cross-validation predictive distribution $f\left(y_i \mid y_{\text{obs}(i)}\right).$ We compute the CPO values for each observed data point $y_{i,\text{obs}}$ and larger CPO indicates that $y_{i,\text{obs}}$ supports the model and a better model fit. For model choice, we can compute the CPO ratio of model A against model B. If this ratio is greater than 1, then $y_{i,\text{obs}}$ supports model A. We compute the CPO ratio for YCM/FH-HB, YLLM/FH-HB and STKM/FH-HB, and count the number of the CPO ratios are larger than 1. We can also plot the CPO values or summarize the CPO values by taking the average of the estimated CPOs. For more detail on CPO, see for example, Gilks, Richardson and Spiegelhalter (1996), page 153, You and Rao (2000), and Molina, Nandram and Rao (2014). Table 4.2 presents the CPO mean and median values over the 117 CMA/CAs and the number of CPO ratios larger than 1.

Table 4.2
Summary of CPO values and CPO ratios over 117 CMA/CAs

	FH-HB Direct sv	**YCM Direct sv**	**YLLM Direct sv**	**STKM Direct sv**
CPO Mean	0.1053	0.1222	0.1242	0.1238
CPO Median	0.0976	0.1004	0.1045	0.1051
# of CPO ratio >1	-	72	78	76

It is clear from Table 4.2 that YCM, YLLM and STKM have larger CPO values than FH-HB, which indicate that the HB model with sampling variance modeling is preferred when the direct sampling variance estimates are used, and YLLM and STKM are better than YCM. For CPO ratios, among the 117 areas, 72 areas/observations support YCM, 78 areas support YLLM and 76 areas support STKM. Therefore more observations support YCM, YLLM and STKM over FH-HB, and YLLM has the most number of CPO ratios that are larger than 1. The CPO comparison is consistent with the results reported in Table 4.1. For other model checking and evaluation methods, see Hidiroglou et al. (2019).

5. Conclusion

In this paper, we compare the model-based estimates under the Fay-Herriot model when sampling variances are smoothed and modeled. As in Hidiroglou et al. (2019), our results indicate that the Fay-Herriot model can provide great improvement for the direct survey estimates for LFS rate estimation, even though more complex models such as unmatched models or time series models could be used (e.g., You, 2008). Among all the estimators, FH-EBLUP and FH-HB using smoothed sampling variances perform the best in terms of ARE and CV reduction. Both FH-EBLUP and FH-HB using direct sampling variance estimates perform the worst. For HB modeling approach, both YLLM and STKM perform very well and are better than YCM, and YLLM is slightly better than STKM in our study. Thus if direct sampling variance estimates are used, YLLM or STKM model is suggested. Alternatively, smoothed sampling variances should be used in the Fay-Herriot model to overcome the sampling variance modeling difficulty as discussed in Section 3. The smoothed sampling variances based on the GVF model given by (2.2) in Section 2 can perform very well as shown in our study.

Appendix

Full conditional distributions and sampling procedure for YLLM

- $\left[\theta_i \,|\, y, \beta, \sigma_i^2, \sigma_v^2\right] \sim N\left(\gamma_i y_i + (1-\gamma_i)\, x_i'\beta, \gamma_i \sigma_i^2\right)$, where $\gamma_i = \sigma_v^2 / \sigma_v^2 + \sigma_i^2$, $i = 1, \ldots, m$;
- $\left[\, \beta \,|\, y, \theta, \sigma_i^2, \sigma_v^2 \,\right] \sim N_p\left(\left(\sum_{i=1}^m x_i x_i'\right)^{-1} \left(\sum_{i=1}^m x_i \theta_i\right), \sigma_v^2 \left(\sum_{i=1}^m x_i x_i'\right)^{-1}\right)$;
- $\left[\, \sigma_v^2 \,|\, y, \theta, \beta, \sigma_i^2 \,\right] \sim \mathrm{IG}\left(\frac{m}{2} - 1, \frac{1}{2} \sum_{i=1}^m (\theta_i - x_i'\beta)^2\right)$;
- $\left[\sigma_i^2 \,|\, y, \theta, \beta, \sigma_v^2, \delta, \tau^2\right] \propto f(\sigma_i^2) \cdot h(\sigma_i^2)$, where $f(\sigma_i^2)$ and $h(\sigma_i^2)$ are $f(\sigma_i^2) \sim \mathrm{IG}\left(\frac{d_i + 1}{2}, \frac{(y_i - \theta_i)^2 + d_i s_i^2}{2}\right)$, and $h(\sigma_i^2) = \exp\left(-\frac{(\log(\sigma_i^2) - z_i'\delta)^2}{2\tau^2}\right)$;
- $\left[\, \delta \,|\, y, \theta, \beta, \sigma_i^2, \sigma_v^2, \tau^2 \,\right] \sim N_2\left(\left(\sum_{i=1}^m z_i z_i'\right)^{-1} \left(\sum_{i=1}^m z_i \log(\sigma_i^2)\right), \tau^2 \left(\sum_{i=1}^m z_i z_i'\right)^{-1}\right)$;
- $\left[\, \tau^2 \,|\, y, \theta, \beta, \sigma_i^2, \sigma_v^2, \delta \,\right] \sim \mathrm{IG}\left(\frac{m}{2} - 1, \frac{1}{2} \sum_{i=1}^m \left(\log(\sigma_i^2) - z_i'\delta\right)^2\right)$.

We use Metropolis-Hastings rejection step to update σ_i^2:

(1) Draw σ_i^{2*} from $\mathrm{IG}\left(\frac{d_i + 1}{2}, \frac{(y_i - \theta_i)^2 + d_i s_i^2}{2}\right)$;

(2) Compute the acceptance probability $\alpha(\sigma_i^{2*}, \sigma_i^{2(k)}) = \min\left\{ h(\sigma_i^{2*}) / h(\sigma_i^{2(k)}), 1 \right\}$;

(3) Generate u from Uniform (0, 1), if $u < \alpha(\sigma_i^{2*}, \sigma_i^{2(k)})$, the candidate σ_i^{2*} is accepted, $\sigma_i^{2(k+1)} = \sigma_i^{2*}$; otherwise σ_i^{2*} is rejected, and set $\sigma_i^{2(k+1)} = \sigma_i^{2(k)}$.

Acknowledgements

I would like to thank the Editor, the Associate Editor and one referee for their constructive comments and suggestions to improve the paper.

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