Chapter 3

Analytic Functions and Power Series

In Section 3.1 we discuss the fundamental difference between the derivative of a function of a real and that of a complex variable, by providing necessary and sufficient conditions for differentiability. As a consequence, a continuously differentiable function (briefly C^1-function) f defined in an open set D is analytic if $f_{\bar{z}}(z) = 0$ on D. There is an interesting relationship between functions that are analytic in a domain and real-valued functions that are harmonic in that domain. In Section 3.2 we discuss this relationship at an introductory level, especially in finding harmonic conjugates. In addition, we discuss the polar form of the Cauchy-Riemann equations and the Laplace equation which have many implications in various branches of applied mathematics. In Section 3.3, we investigate 'infinite polynomials'-a generalization of 'polynomials' and the infinite series of complex numbers. The discussion allow us to characterize the behavior of the power series in a very natural fashion. Therefore, the aim of this section is to study the convergence of the power series. In Section 3.4, we examine various known properties of the so-called standard or elementary functions from a real variable to a complex variable. Section 3.5 illustrates the relationship between $w = \log z$ $(z \neq 0)$ and $z = e^w$ and considers its local properties. Several standard functions are associated with exponential functions or logarithmic functions. As a consequence, in Section 3.5, we define the inverse trigonometric and the inverse hyperbolic functions.

3.1 Differentiability and Cauchy-Riemann Equations

Differentiation in $\mathbb{C}$ is set against a background of limits, continuity and so on. To some extent the rules for differentiation of a function of a complex variable are similar to those of differentiation of a function of a real variable. Since $\mathbb{C}$ is merely $\mathbb{R}^2$ with additional structures of addition and multiplication of complex numbers, we can immediately transfer *most* of the concepts of $\mathbb{R}^2$ into those for the complex field $\mathbb{C}$. In fact, we have already done so when we defined the concept of distance (modulus),

$$|z - z'| = \sqrt{(x - x')^2 + (y - y')^2} \quad (z = x + iy \text{ and } z' = x' + iy')$$

which is the same as the *Euclidean distance* between the points $z = (x, y)$ and $z' = (x', y')$ in $\mathbb{R}^2$. We now start with

3.1. Definition. We say that a complex function f defined in a non-empty open set D

is *differentiable* (or *complex differentiable*) at $z_0 \in D$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \tag{3.2}$$

exists. When this is the case, the value of the limit, denoted by $f'(z_0)$, is called the derivative of f at z_0. The number $f'(z_0)$ is generally a complex number. The function f is said to be *differentiable in (on) D* if it is differentiable at every point of D. A function which is differentiable in the entire complex plane is called an *entire function*.

We observe that "formally" the limit definition of the derivative given by (3.2) is identical in form to that of the derivative of a real (or complex)-valued function of a real variable. In terms of '$\epsilon - \delta$' notation, the limit in (3.2) exists iff given any $\epsilon > 0$, there exists a $\delta = \delta(\epsilon, z_0) > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \quad \text{whenever} \quad 0 < |z - z_0| < \delta.$$

Letting

$$\eta(z) = \begin{cases} \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0) & \text{for } 0 < |z - z_0| < \delta \\ 0 & \text{for } z = z_0, \end{cases}$$

we have $\lim_{z \to z_0} \eta(z) = 0 = \eta(z_0)$. Therefore η is continuous at z_0 and we get an explicit expression for $f(z)$ in the form

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\eta(z) \tag{3.3}$$

for $|z - z_0| < \delta$. In conclusion, we have

3.4. Proposition. *Let $D \subseteq \mathbb{C}$ be open, $f : D \to \mathbb{C}$ and $z_0 \in D$. Then $f'(z_0)$ exists iff there exists a function $\eta : D \to \mathbb{C}$ which is continuous at z_0 and satisfies (3.3) for all $z \in D$. Equivalently, f is differentiable at z_0 iff*

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + E(z) \tag{3.5}$$

where E is a function defined in a neighborhood of a such that

$$\lim_{z \to z_0} |E(z)/(z - z_0)| = 0.$$

By (3.5), we obtain a linear function $L(z) = f(z_0) + f'(z_0)(z - z_0)$ which approximates $f(z)$ up to an "error" term $E(z)$, which, for z close to z_0, is small, in absolute value in comparison with $|z - z_0|$. This may be treated as an alternate and geometric description of differentiability at a point. At this place it is necessary to make an important point to the reader. Since it is not possible to graph a complex functions in the usual way as we do with real-valued functions, it is not meaningful to visualize $f'(z_0)$ as a 'slope' of some curve as we do in the real case.

It is immediate that the derivative of a constant function is zero; for if $f(z) \equiv c$, then for each $z_0 \in \mathbb{C}$ we have

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

3.6. Theorem. *A real-valued function of a complex variable either has derivative zero or the derivative does not exist.*

Proof. Suppose that $f(z)$ is a real-valued function of complex variable whose derivative exists at a point a. Then

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.$$

If we take the limit $h \to 0$ along the real axis, then $f'(a)$ is real. If we take the limit $h \to 0$ along the imaginary axis, then $f'(a)$ becomes a purely imaginary number. We must have $f'(a) = 0$. ∎

There are several other immediate consequences of Definition 3.1 which are worth mentioning. First along this line is to allow $z \to z_0$ in (3.3) and obtain

3.7. Theorem. *If f has derivative at z_0, then it is continuous at z_0.*

As in the real case, continuity of f does not necessarily imply differentiability of f in complex case also. The function $f(z) = |z|$ demonstrates this. For if $h = h_1 + ih_2$ (h_1, h_2 real), then

$$\frac{f(h) - f(0)}{h} = \frac{|h|}{h} \longrightarrow \begin{cases} 1 & \text{for } h = h_1 + i \cdot 0,\ 0 < h_1 \to 0 \\ -1 & \text{for } h = h_1 + i \cdot 0,\ 0 > h_1 \to 0 \\ -i & \text{for } h = 0 + ih_2,\ 0 < h_2 \to 0 \\ i & \text{for } h = 0 + ih_2,\ 0 > h_2 \to 0. \end{cases} \tag{3.8}$$

Thus, $|z|$ is not differentiable at $z = 0$ but is continuous at 0. Is f differentiable at other points in $\mathbb{C}$?

Note that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is (real) differentiable on $\mathbb{R} \setminus \{0\}$ and continuous on $\mathbb{R}$ whereas its complex analog map $f : \mathbb{C} \to \mathbb{C}$ defined by $f(z) = |z|$ is nowhere differentiable although it is continuous in $\mathbb{C}$ (see Example 3.10).

Next we give an example of a function which is differentiable at a single point in $\mathbb{C}$ and nowhere else. For example, if $f(z) = |z|^2$ then f is clearly continuous in $\mathbb{C}$. On the other hand (with $h \neq 0$), we have

$$\frac{f(h) - f(0)}{h} = \frac{h\overline{h}}{h} = \overline{h} \to 0 \quad \text{as } h \to 0$$

so that f is differentiable at the origin with derivative $f'(0) = 0$. Is f differentiable at any other point? Now let $z_0 \neq 0$ be an arbitrary point in $\mathbb{C} \setminus \{0\}$. With $0 \neq h$ as neighboring variable point of 0, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{|z_0 + h|^2 - |z_0|^2}{h} = \frac{\overline{z}_0 h + z_0 \overline{h} + h\overline{h}}{h} = \overline{z}_0 + z_0 \left(\frac{\overline{h}}{h}\right) + \overline{h}.$$

As $z_0 \neq 0$, (3.8) shows that the expression on the right tends to different values as $h \to 0$. Thus, the function $f(z) = |z|^2$ cannot be differentiable at $z_0 \neq 0$. How about the translated function g defined by $g(z) = f(z - a)$? Clearly, $g(z) = |z - a|^2$ (a is a fixed point in $\mathbb{C}$) is differentiable only at a and nowhere else.

Similarly, it is easy to verify that the function $f(z) = z \operatorname{Re} z$ is differentiable at $z = 0$ only. Problems like this can be done also by using the Cauchy-Riemann equations. Thus, there exist complex functions which are differentiable at a given point but not in any neighborhood of that point.

Many formulas for the derivatives of complex functions are the same as those for the real counterparts. For example, it follows that the monomials $f(z) = z^n$ ($n \in \mathbb{N}$) are entire, i.e. differentiable in the entire complex plane. In fact, for each fixed $z_0 \in \mathbb{C}$, we have

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^n - z_0^n}{z - z_0} = \lim_{z \to z_0} [z^{n-1} + z^{n-2} z_0 + \cdots + z_0^{n-1}] = n z_0^{n-1}.$$

In consonance with Real Analysis we can write this in the form

$$f'(z) = \frac{d}{dz}(z^n) = nz^{n-1}$$

for every $z \in \mathbb{C}$ and $n = 0, 1, 2, \ldots$. By Definition 3.1 we have the following standard theorems on differentiation in complex counterpart.

3.9 Theorem. *If f and g are differentiable at z_0, then their sum $f + g$, difference $f - g$, product fg, quotient f/g (where $g(z_0) \neq 0$) and the scalar multiplication cf are also differentiable at z_0 and*

$$(f \pm g)' = f' \pm g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}, \quad (cf)' = cf',$$

where c is a complex constant.

More generally, a finite linear combinations (of the form $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n$, $c_j \in \mathbb{C}$, $j = 1, 2, \ldots, n$) and finite products of functions differentiable at z_0 are also differentiable at z_0.

Proof. For $z \neq z_0$, consider the following

$$\frac{(f \pm g)(z) - (f \pm g)(z_0)}{z - z_0} = \left\{\frac{f(z) - f(z_0)}{z - z_0}\right\} \pm \left\{\frac{g(z) - g(z_0)}{z - z_0}\right\}$$

$$\frac{(fg)(z) - (fg)(z_0)}{z - z_0} = f(z)\left\{\frac{g(z) - g(z_0)}{z - z_0}\right\} + g(z_0)\left\{\frac{f(z) - f(z_0)}{z - z_0}\right\}$$

and, for $g(z_0) \neq 0$,

$$\frac{\left(\frac{f}{g}\right)(z) - \left(\frac{f}{g}\right)(z_0)}{z - z_0} = \frac{g(z_0)\left(\frac{f(z) - f(z_0)}{z - z_0}\right) - f(z_0)\left(\frac{g(z) - g(z_0)}{z - z_0}\right)}{g(z)g(z_0)}.$$

The assertions then follow from the above equalities and the properties of the limit by making $z \to z_0$ and noting that if f, g are differentiable at z_0 they are continuous at z_0 so that $f(z) \to f(z_0)$, $g(z) \to g(z_0)$ as $z \to z_0$. ∎

Let $f(z) = 1$ and $g(z) = z^n$ ($z \neq 0$, $n \in \mathbb{N}$). Then $g'(z) = nz^{n-1}$ and Theorem 3.9 help us to get $h'(z)$, where $h(z) = f(z)/g(z) = 1/z^n$, $z \neq 0$.

$$h'(z) = \frac{d}{dz}\left(\frac{1}{z^n}\right) = -nz^{-n-1}$$

and so $h(z) = z^{-n}$ is differentiable in $\mathbb{C} \setminus \{0\}$. Further, Theorem 3.9 shows that each polynomial of the form $p(z) = a_0 + a_1 z + \cdots + a_n z^n$, where n is a non-negative integer, is differentiable in $\mathbb{C}$ and has the derivative

$$p'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}, \quad z \in \mathbb{C},$$

obtained by term-by-term differentiation of the polynomial $p(z)$. In particular, every polynomial is an entire function. This fact and Theorem 3.9 immediately imply that a rational function of the form $R(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are polynomials in z, is differentiable at all

points except where the denominator vanishes and the formula for $R'(z)$ is obtained by using Theorem 3.9.

3.10 Example. Let $f(z) = \operatorname{Re} z$ and z_0 be an arbitrary fixed point in $\mathbb{C}$. Then, for $h = h_1 + ih_2$ ($\neq 0$),

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\operatorname{Re}(h)}{h} = \begin{cases} 1 & \text{for } h = h_1 + i0 \in \mathbb{R} \setminus \{0\} \\ 0 & \text{for } h = 0 + ih_2 \in i\mathbb{R} \setminus \{0\} \end{cases}$$

so that

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

does not exist. Therefore, $f(z) = \operatorname{Re} z$ is nowhere differentiable even though it is continuous in $\mathbb{C}$.

Similarly, we see that $\operatorname{Im} z$, $|z|$, $\bar{z}$ and $\operatorname{Arg} z$ are all nowhere differentiable in $\mathbb{C}$. Is each of the functions listed here continuous in $\mathbb{C}$? Is each of these functions infinitely often real differentiable in $\mathbb{R}^2$? Is a product of two nowhere differentiable functions always nowhere differentiable? How about $g(z) = (z)^2$? ∎

3.11 Definition. A function f is said to be *analytic, or holomorphic* at a point $a \in \mathbb{C}$ if it is differentiable at every point of some neighborhood of a. Similarly, f is analytic in (on) an arbitrary set S if it is differentiable at every point of some open set containing S. An *entire function* is then the one which is analytic in (on) the whole complex plane.

Formally, we say that a point at which a function ceases to be analytic is called a *singularity* or a *singular point*. For a precise definition, we refer to Section 7.1. Can we say $f(z) = \bar{z}$ has every point in $\mathbb{C}$ as its singularity? The answer is no, see Example 7.1.

Observe that even though $f(z) = |z - a|^2$ is differentiable at a, it is not analytic at this point because there does not exist a neighborhood of a in which $|z - a|^2$ is differentiable at each point of the neighborhood. Moreover, this function is nowhere analytic in $\mathbb{C}$.

The meaning of sentences such as "f is analytic for $|z| \leq R$, $R > 0$" should now be clear (meaning that f is analytic in some domain containing the closed region $\bar{S}_R$). Further, it is also clear from the definition that if D is an open set then the phrase "analytic in (on) D" means "differentiable at all points of D". The set of all analytic functions (mappings) in the open set D is denoted by $\mathcal{H}(D)$, $\mathcal{H}$ standing for 'holomorphic', an alternative for 'analytic'. As usual, we write

$$f^{(0)} = f, \quad f^{(1)} = f', \quad f^{(2)} = f'', \quad \text{and so on.}$$

Because of the inherent two-dimensional ($\mathbb{R}^2$) character of a complex variable ($\mathbb{C}$), the usual rules of the derivative from calculus may be used when differentiating analytic functions. The proofs extend without much change beyond the real case by replacing the real variables x, y by complex variables z, w. As an example, a direct consequence of Definition 3.1 gives the following *chain rule* for differentiation of composite functions.

3.12 Theorem. *Let $f : D_1 \to \mathbb{C}$, $g : D_2 \to \mathbb{C}$ be such that $f(D_1) \subseteq D_2$. If f is differentiable at z_0 and g is differentiable at $w_0 = f(z_0)$, then the composition $(g \circ f)(z) = g(f(z))$ is differentiable at z_0 and*

$$(g \circ f)'(z_0) = g'(w_0)f'(z_0) = (g' \circ f)(z_0)f'(z_0).$$

In consonance with Real Analysis we can write this in the form

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$$(g \circ f)'(z_0) = g'(w_0)f'(z_0) = (g' \circ f)(z_0)f'(z_0).$$

Further, if $f \in \mathcal{H}(D_1)$ and $g \in \mathcal{H}(D_2)$ then $g \circ f \in \mathcal{H}(D_1)$.

Proof. Let $w = f(z)$, $z \in D_1$ and assume the hypotheses. By (3.3), since f is differentiable at z_0, there exists a $\delta_1 > 0$ such that

$$f(z) - f(z_0) = (z - z_0)\eta_f(z) \quad \text{for } z \in \Delta(z_0; \delta_1) \subset D_1,$$

where η_f ($\equiv f'(z_0) + \eta(z)$ in (3.3)) is continuous in $\Delta(z_0; \delta_1)$ with

$$\lim_{z \to z_0} \eta_f(z) = f'(z_0).$$

Further, since g is continuous at w_0, there exists a $\delta_2 > 0$ such that

$$g(w) - g(w_0) = (w - w_0)\eta_g(w) \quad \text{for } w \in \Delta(w_0; \delta_2) \subset D_2,$$

where η_g is continuous in $\Delta(w_0; \delta_2)$ with $\lim_{w \to w_0} \eta_g(w) = g'(w_0)$. Now choose $\delta > 0$ such that $\delta < \delta_1$ and

$$|z - z_0| < \delta_1 \Rightarrow |f(z) - f(z_0)| < \delta_2.$$

Then for $z \in \Delta(z_0; \delta)$, we have by substitution

$$\begin{aligned} g(f(z)) - g(f(z_0)) &= (f(z) - f(z_0))\eta_g(f(z)) \\ &= (z - z_0)\eta_f(z) \cdot \eta_g(f(z)) \\ &= (z - z_0)\eta_{g \circ f}(z). \end{aligned}$$

By Corollary 2.21, $\eta_g \circ f$ is continuous at z_0 and its value at z_0 is $\eta_g(w_0) = g'(w_0)$. By Theorem 2.18, the product $\eta_f(z) \cdot \eta_g(f(z))$, i.e. $\eta_{g \circ f}(z)$, is continuous at z_0 and

$$\lim_{z \to z_0} \eta_{g \circ f}(z) = \lim_{z \to z_0} [\eta_f(z) \cdot \eta_g(f(z))] = f'(z_0) \cdot g'(w_0) = f'(z_0)g'(f(z_0)).$$

The assertion now follows from (3.3). ∎

Theorem 3.9 leads quickly to the following useful properties of analytic functions.

3.13. Theorem. *Linear combinations and finite products of analytic functions in an open set D are all analytic in D. If f and g are analytic in D, then the quotient f/g is analytic in D except for those z in D at which g vanishes.*

3.14. Corollary. *If f and g are entire then so are $f \pm g$, fg; and $f \circ g$ is entire when it is defined.*

We know that if $F(z) = U(x, y) + iV(x, y)$ where U and V are real-valued functions defined in a neighborhood of z_0, then the partial derivative $U_x(x, y)$ of U with respect to x at (x, y) (where $z = x + iy$), if it exists, is defined to be

$$U_x(x, y) = \lim_{h \to 0} \frac{U(x + h, y) - U(x, y)}{h}. \tag{3.15}$$

Similarly, the partial derivative $U_y(x, y)$ of U with respect to y at (x, y), if it exists, is defined to be

$$U_y(x, y) = \lim_{h \to 0} \frac{U(x, y + h) - U(x, y)}{h}. \tag{3.16}$$

Sometimes (3.15) and (3.16) are denoted by the Leibnitz notation,

$$\frac{\partial U}{\partial x}(z) = \frac{\partial U}{\partial x}(x, y) \quad \text{and} \quad \frac{\partial U}{\partial y}(z) = \frac{\partial U}{\partial y}(x, y),$$

respectively. Note that h in (3.15) and (3.16) is a non-zero real number near 0. The partial derivatives $F_x(x, y)$ and $F_y(x, y)$ of the complex-valued function of the complex variable $F(z) = U(x, y) + iV(x, y)$ at $z = x + iy$ are defined by

$$F_x(z) = U_x(x, y) + iV_x(x, y) \quad \text{and} \quad F_y(z) = U_y(x, y) + iV_y(x, y),$$

respectively, provided the partial derivatives on the right side of the corresponding equations exist. We simply write these expressions as

$$F_x = U_x + iV_x, \quad \text{and} \quad F_y = U_y + iV_y.$$

If $F_y = iF_x$, then, by equating imaginary and real parts, we have a pair of famous partial differential equations

$$U_x = V_y, \quad U_y = -V_x.$$

Conversely, the later two equations imply $F_y = iF_x$. These two extremely important partial differential equations are called the *Cauchy-Riemann*[6] (briefly we write the C-R) equations. The former is actually referred to as the C-R equations in Cartesian form. We note that the *existence of the derivative* for real-valued functions of single real variable is a mild smooth condition while the same for complex-valued functions of a complex variable leads to the above pair of partial differential equations. We usually write

$$f(z) = f(x + iy) = u(x, y) + iv(x, y) = \operatorname{Re} f(z) + i \operatorname{Im} f(z)$$

or $f(z) = u(z) + iv(z)$ instead. However, we may abuse the notation slightly by writing $f(z) = f(x, y)$, but when we write it like this, we actually identify $u(x, y) + iv(x, y) \in \mathbb{C}$ with $(u(x, y), v(x, y)) \in \mathbb{R}^2$ and use results from two variable calculus for our investigation.

If $\Omega \subseteq \mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ is a continuous function, then u is called a C^1- (or *continuously differentiable*) function in Ω if u_x and u_y exist and are continuous in Ω. More generally, if $k \in \mathbb{N}$ then u is said to be in $C^k(\Omega)$ (or simply a C^k function or k-times continuously differentiable function) if all the partial derivatives of u up to and including order k exist and are continuous in Ω. We indicate this by writing $u \in C^k$. Note that $C^0(\Omega)$ denotes the set of all continuous functions in Ω. A function $f : \Omega \to \mathbb{C}$ is said to belong to $C^k(\Omega)$, or simply call it a C^k-function, if both u and v belong to $C^k(\Omega)$.

From the inspection of functions such as $|z|$, $|z|^2$ and $z \operatorname{Re} z$, we conclude that it is not necessarily an easy task to determine whether a given function does or does not have a derivative. Let us start deriving a simple criterion which helps us to handle this problem.

3.17. Theorem. *If $f(z) = u(x, y) + iv(x, y)$ is differentiable at z_0, then the C-R equations hold at $z_0 = x_0 + iy_0$:*

$$if_x(z_0) = f_y(z_0);$$

[6] These two equations are named in honor of French a mathematician, Augustin-Louis Cauchy (1789-1857), who discovered them, and in honor of a German mathematician, Georg Friedrich Bernhard Riemann (1826-1866), who made them fundamental in the development of the theory of complex analysis. Riemann is considered one of the three founders of complex function theory; the others being Cauchy and Weierstrass.

or equivalently,

$$u_x(x_0, y_0) = v_y(x_0, y_0) \quad \text{and} \quad u_y(x_0, y_0) = -v_x(x_0, y_0).$$

Proof. Let $f : \Omega \to \mathbb{C}$ where $\Omega \subseteq \mathbb{C}$ is a neighborhood of z_0. If $f'(z_0)$ exists for some point $z_0 = x_0 + iy_0$, then the limit

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and is independent of the path along which $h = h_1 + ih_2 \to 0$. In particular, we have

$$f'(z_0) = \lim_{h_1 \to 0} \frac{f(x_0 + h_1, y_0) - f(x_0, y_0)}{h_1 + i0} = \frac{\partial f}{\partial x}(z_0),$$

and

$$f'(z_0) = \lim_{h_2 \to 0} \frac{f(x_0, y_0 + h_2) - f(x_0, y_0)}{0 + ih_2} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

A comparison of the two expressions for $f'(z_0)$ shows that the complex differentiability of f at z_0 implies that not only the partial derivatives of f (with respect to x and y) exist at z_0, but also that they satisfy the C-R equations–in complex form

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0), \quad \text{i.e.} \quad if_x(z_0) = f_y(z_0). \tag{3.18}$$

Equating the real and imaginary parts yields the C-R equations–in Cartesian form $u_x(z_0) = v_y(z_0)$ and $u_y(z_0) = -v_x(z_0)$. ∎

Another convenient notation is to treat the pair of conjugate complex variables z and $\overline{z}$ as two independent variables by writing

$$x = \frac{z + \overline{z}}{2}, \quad y = -i\left(\frac{z - \overline{z}}{2}\right).$$

Now we introduce the following differential operators:

$$\frac{\partial}{\partial z} := \frac{\partial}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \overline{z}} := \frac{\partial}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

It follows that if $f = u + iv$, then

$$f_z = \frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left([u_x + v_y] + i[v_x - u_y]\right) \tag{3.19}$$

and similarly,

$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\left([u_x - v_y] + i[u_y + v_x]\right). \tag{3.20}$$

Therefore, the C-R equations (3.18) are exactly equivalent to $f_{\overline{z}}(z_0) = 0$ which is also referred to as the complex form of the C-R equations. With this notation, we have the following alternate form of Theorem 3.17.

3.21. Theorem. *A necessary condition for a complex-valued function $f = u + iv$ to be differentiable at z_0 is that $f_{\overline{z}}(z_0) = 0$. In particular, if $f \in \mathcal{H}(D)$, then C-R equations hold at every point $z \in D$.*

The C-R equations are more helpful in proving non-differentiability. For example, consider $f(z) = \overline{z}$ and $g(z) = \operatorname{Re} z$. Then $f_{\overline{z}}(z) = 1 \neq 0$ and writing g as

$$g(z) = \frac{z + \overline{z}}{2},$$

it follows that $g_{\overline{z}}(z) = 1/2 \neq 0$. Thus, both f and g are nowhere differentiable.

On the other hand, the converse of Theorem 3.17 (equivalently, Theorem 3.21) is not true. We can demonstrate this by a number of examples (for instance, see Examples 3.23 and 3.24). That is, the necessary condition stated in Theorem 3.17 for differentiability at a point is not generally sufficient for differentiability at that point.

3.22. Remark. Each of the functions

$$f_1(z) = |z|, \quad f_2(z) = \operatorname{Re} z \quad \text{and} \quad f_3(z) = \operatorname{Im} z,$$

is a non-constant real-valued function defined in $\mathbb{C}$ (see Theorem 3.6). Each of them is nowhere analytic. If we rewrite these functions as

$$f_1(x, y) = \sqrt{x^2 + y^2}, \quad f_2(x, y) = x, \quad f_3(x, y) = y,$$

then, except f_1, each of these functions are real differentiable in $\mathbb{R}^2$.

3.23. Example. It is easy to see that the function f defined by

$$f(z) = |\operatorname{Re} z \operatorname{Im} z|^{1/2}$$

satisfies the C-R equations at the origin, but is not differentiable at this point. To see this, we may rewrite the given function as

$$f(z) = \frac{|z^2 - \overline{z}^2|^{1/2}}{2} \quad \text{or } f = u + iv \text{ with } u(x, y) = |xy|^{1/2} \text{ and } v(x, y) = 0.$$

Note that f is identically zero on the real and imaginary axes. Therefore, it is trivial to see that $u_x(0, 0) = u_y(0, 0) = v_x(0, 0) = v_y(0, 0) = 0$. For example,

$$u_x(0, 0) = \lim_{s \to 0} \frac{u(s, 0) - u(0, 0)}{s} = 0.$$

Thus, the C-R equations hold at $z = 0$. However, taking $h = re^{i\theta} \neq 0$ with $r \to 0$, we find that

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{r \to 0} \frac{|r^2 \cos\theta \sin\theta|^{1/2}}{r(\cos\theta + i\sin\theta)} = \frac{e^{-i\theta}|\sin 2\theta|^{1/2}}{\sqrt{2}}$$

which is clearly depending upon θ (e.g. take $\theta = 0$ and $\theta = \pi/4$). We conclude that f is not differentiable at $z = 0$ even though f satisfies the C-R equations at the origin. Here, since $v(x, y) = 0$, v is a C^∞-function in $\mathbb{R}^2$. *Are the partial derivatives u_x and u_y continuous at the origin?* How about the functions

so that $if_x(0,0) = f_y(0,0)$ and the C-R equations hold at the origin. Then, taking $h = re^{i\theta} \neq 0$ with $r \neq 0$, it follows that for $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \frac{h^4}{|h|^4} = e^{i4\theta}$$

and therefore, as $h \to 0$ along different paths, the difference quotient does not yield a unique value. Thus, f is not differentiable at the origin. Again, can f be differentiable at other points in $\mathbb{C}$? For $z_0 \neq 0$, we observe that

$$f_{\bar{z}}(z_0) = -2\left(\frac{z_0}{\bar{z}_0}\right)^3$$

which gives that $|f_{\bar{z}}(z_0)| = 2$ for $z_0 \neq 0$. Thus, f cannot be differentiable at $z_0 \neq 0$ and hence, it is nowhere differentiable. ●

Next we give sufficient conditions for a complex-valued function to be differentiable. To present this we need the following lemma which is a well-known result from two variable calculus.

3.25. Lemma. *Let $u : \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$, where Ω is an open set containing the point (x_0, y_0). Suppose that*

(i) *u_x and u_y exist at every point in a neighborhood of (x_0, y_0), and*

(ii) *u_x and u_y are continuous at (x_0, y_0).*

Then, for sufficiently small s and t in $\mathbb{R}$,

$$u(x_0 + s, y_0 + t) - u(x_0, y_0) = su_x(x_0 + s^*, y_0 + t) + tu_y(x_0, y_0 + t^*)$$

with $|s^*| < |s|$ and $|t^*| < |t|$.

Proof. We consider

$$u(x_0 + s, y_0 + t) - u(x_0, y_0) = [u(x_0 + s, y_0 + t) - u(x_0, y_0 + t)] + [u(x_0, y_0 + t) - u(x_0, y_0)]$$

By the mean value theorem of calculus applied to the function

$$\phi(x) = u(x, y_0 + t)$$

for x between x_0 and $x_0 + s$, we have

$$u(x_0 + s, y_0 + t) - u(x_0, y_0 + t) = su_x(x_0 + s^*, y_0 + t)$$

with $|s^*| < |s|$. (Indeed when we restrict u to a horizontal line segment in the disk $\Delta(z_0; \delta)$, then it reduces to a differentiable function of the real variable x with derivative u_x, whilst in the vertical segment in this disk u is a function of y alone and has the derivative u_y. Note that corresponding to (x_0, y_0), there exist points x^* in

$$[x_0, x_0 + s] = \{(1 - t)x_0 + t(x_0 + s) : t \in [0, 1]\} \subseteq \Delta(z_0; \delta),$$

that is

$$\begin{cases} x_0 < x^* < x_0 + s & \text{if } s > 0 \\ x_0 + s < x^* < x_0 & \text{if } s < 0, \end{cases}$$

so that $|x^* - x_0| < |s|$. Thus, corresponding to z_0, there exist points $x^* \in [x_0, x_0 + s]$ with $|s^*| < |s|$, satisfying the desired equation). Similarly, we have

$$u(x_0, y_0 + t) - u(x_0, y_0) = t u_y(x_0, y_0 + t^*)$$

with $|t^*| < |t|$. Adding this with the previous expression gives the desired result. ■

3.26. Theorem. *Let $f : \Omega \to \mathbb{C}$, where Ω is an open set in $\mathbb{C}$ containing the point z_0, and $f(z) = u(x, y) + iv(x, y)$ for $z = x + iy \in \Omega$. Suppose that*

(i) *u_x, u_y, v_x, and v_y exist at each point in a neighborhood of z_0 and are continuous at z_0*

(ii) *the C-R equations are valid at z_0.*

Then f is differentiable at z_0 and $f'(z_0) = u_x(z_0) + iv_x(z_0)$.

Proof. Let $z_0 = x_0 + iy_0$. We wish to show that $f'(z_0)$ exists. Since Ω is open and $z_0 \in \Omega$, there exists a $\delta > 0$ such that $\Delta(z_0; \delta) \subset \Omega$. Choose $h = s + it \neq 0$ with $0 < |h| < \delta$, so that $z_0 + h \in \Delta(z_0; \delta)$. Now we consider

$$\begin{aligned} f(z_0 + h) - f(z_0) &= f(x_0 + s, y_0 + t) - f(x_0, y_0) \\ &= f(x_0 + s, y_0 + t) - f(x_0, y_0 + t) + f(x_0, y_0 + t) - f(x_0, y_0). \end{aligned}$$

The mean value theorem applied to both real and imaginary parts of $f(z)$ (see Lemma 3.25) shows that

$$f(z_0 + h) - f(z_0) = s f_x(x_0 + s^*, y_0 + t) + t f_y(x_0, y_0 + t^*) \tag{3.27}$$

with $|s^*| < |s|$ and $|t^*| < |t|$. Remember that

$$h \to 0 \iff s \to 0 \text{ and } t \to 0 \implies s^* \to 0 \text{ and } t^* \to 0.$$

By assumption u_x, u_y, v_x and v_y exist in some neighborhood of z_0 and are continuous at z_0 and therefore, f_x and f_y exist in that neighborhood and are continuous at z_0. In particular, as f_x is continuous, we note that

$$f_x(x_0 + s^*, y_0 + t) \to f_x(x_0, y_0) \text{ as } h \to 0;$$

or equivalently

$$f_x(x_0 + s^*, y_0 + t) = f_x(x_0, y_0) + \alpha(h), \tag{3.28}$$

where $\alpha(h) \to 0$ as $h \to 0$. Similarly, as f_y is continuous at z_0, we find that

$$f_y(x_0, y_0 + t^*) = f_y(x_0, y_0) + \beta(h),$$

where $\beta(h) \to 0$ as $h \to 0$. Further, since the C-R equations are satisfied at z_0, we have $f_y(x_0, y_0) = i f_x(x_0, y_0)$ and so the last equation becomes

$$f_y(x_0, y_0 + t^*) = i f_x(x_0, y_0) + \beta(h). \tag{3.29}$$

By (3.28) and (3.29), (3.27) gives that for $h = s + it$ with $0 < |h| < \delta$

$$\begin{aligned} \frac{f(z_0 + h) - f(z_0)}{h} &= \frac{s[f_x(x_0, y_0) + \alpha(h)]}{s + it} + \frac{t[i f_x(x_0, y_0) + \beta(h)]}{s + it} \\ &= f_x(x_0, y_0) + \alpha(h)\left(\frac{s}{s + it}\right) + \beta(h)\left(\frac{t}{s + it}\right). \end{aligned}$$

Note that $s/(s + it)$ and $t/(s + it)$ are bounded by 1, and each of $\alpha(h)$ and $\beta(h)$ approaches zero as $h \to 0$. Consequently, taking the limit as $h \to 0$ in the above equation, we see that

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists and is equal to $f_x(x_0, y_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$. ■

If the hypotheses of Theorem 3.26 hold at each point of a neighborhood $\Delta(z_0; \delta)$ of z_0, then f is analytic in $\Delta(z_0; \delta)$. Further, to minimize the notational difficulties, we could assume $z_0 = 0$ in Theorem 3.26, because the result for $z_0 \neq 0$ can be retrieved from the case $z_0 = 0$ by merely applying the latter to the function $g(z) = f(z + z_0)$ which exhibits the same behavior at the origin that the function $f(z)$ does for $z_0 \neq 0$.

3.30. Example. Consider $f = u + iv$, where $u = x^2y^2$ and $v = 2x^2y^2$. Then all the partial derivatives exist and are continuous in $\mathbb{C}$. However, the C-R equations are satisfied only when $2xy^2 = 4x^2y$ and $2x^2y = -4xy^2$; that is when

$$xy(y - 2x) = 0 \text{ and } xy(x + 2y) = 0.$$

These two equations show that the C-R equations hold only along the lines $x = 0, y = 0$ and nowhere else. Thus, $f = u + iv$ is differentiable only along points on the real and imaginary axes and nowhere else. Note that, every neighborhood of every point on the line $x = 0$ (and $y = 0$, respectively) will contain points off the line at which the C-R equations does not hold. It follows that $f = u + iv$ is nowhere analytic. ●

If f is constant in some domain D, then $f'(z) = 0$ in D. As a consequence of the C-R equations, we shall now establish the converse part of it (see also Corollaries 4.21 and 12.3 for a proof). For more about the basic properties concerning the range of non-constant analytic functions, we refer to Section 12.6.

3.31. Theorem. *Suppose that f is analytic in a domain D. We have*

(i) *if $f'(z) = 0$ in D, then f is constant*

(ii) *if one of $|f|$, $\operatorname{Re} f$, $\operatorname{Im} f$, $\operatorname{Arg} f$ is constant in D, then f is constant. Equivalently, we say that if the range $f(D)$ lies in either a circle or a vertical line or horizontal line or any line with constant slope, then f is constant.*

Proof. (i) For $z = x + iy \in D$ and $f = u + iv$, we have

$$f' = u_x + iv_x = v_y - iu_y.$$

Suppose $f'(z) = 0$ in D. Then all the partial derivatives of u and v on D are zero, i.e. $u_x(z) = u_y(z) = v_x(z) = v_y(z) = 0$ for $z \in D$. We know from Real Analysis that if ϕ is a real-valued differentiable function of a real variable and such that $\phi'(x) = 0$ on (a, b), then ϕ is a constant on (a, b). Let $z_0 = x_0 + iy_0$ be an arbitrary point in D and for $a < b$, let

$$L = \{x + iy_0 : x \in (a, b)\}.$$

If the horizontal line segment L lies in D, then ϕ defined by

$$\phi(x) = u(x, y_0)$$

satisfies $\phi'(x) = u_x(x, y_0) = 0$ on L and so u is constant on L. It follows that u is constant on each horizontal line segment in D. Likewise u is constant on each vertical line segment in D. Since D is open and connected, each pair of points in D can be connected by a polygonal step path consisting entirely of such line segments. Therefore, u is constant in D. Similarly, v is constant in D. Consequently, f is constant throughout D (see also Corollary 4.21).

(ii) Now suppose that $|f(z)| = k$, so that $u^2 + v^2 = k^2$. If $k = 0$ it is obvious that $f = 0$ in D. So, let $k \neq 0$. Differentiating partially with respect to x and y, we see that

$$uu_x + vv_x = 0, \quad uu_y + vv_y = 0.$$

By the C-R equations, these become

$$uu_x - vu_y = 0, \quad uu_y + vu_x = 0.$$

Squaring and adding, we obtain

$$0 = (u^2 + v^2)(u_x^2 + u_y^2) = k^2|f'(z)|^2$$

so that $f'(z) = 0$ in D. Therefore from (i) we deduce that f is constant.

Suppose u is constant. Then $u_x = u_y = 0$; therefore, by the C-R equations, $v_x = v_y = 0$ so that $f'(z) = 0$ in D. Hence f is constant.

If $\operatorname{Arg} f = \alpha$, a constant, then $g(z) = e^{-i\alpha} f(z)$ is a real-valued analytic function and so, by part (i), $g(z)$ is constant. ∎

Suppose f and g are analytic in a domain D such that $f'(z) = g'(z)$ in D then applying part (i) of Theorem 3.31 to $f - g$ we conclude that f and g differ from each other by a constant.

The hypothesis that D is connected in Theorem 3.31 is not superfluous. For example, if $D = \mathbb{C}\backslash\{z : 1 \leq |z| \leq 3\}$ and if $f : D \to \mathbb{C}$ is defined by

$$f(z) = \begin{cases} 0 & \text{for } |z| < 1 \\ i & \text{for } |z| > 3, \end{cases}$$

then $f \in \mathcal{H}(D)$ and $f'(z) = 0$ in D, yet f is non-constant in D.

Here is another example. Define

$$f(z) = \begin{cases} 0 & \text{for } \operatorname{Re} z > a \\ i & \text{for } \operatorname{Re} z < a, \end{cases}$$

where $a \in \mathbb{R}$ is fixed. Then $f \in \mathcal{H}(D)$, where $D = \mathbb{C}\backslash\{z : \operatorname{Re} z = a\}$, yet f is not constant in D. Note that, in each of these two examples, $\operatorname{Re} f(z) = u(x, y) = 0$ in D but $\operatorname{Im} f(z) = v(x, y)$ is not constant.

3.2 Harmonic Functions

We begin with the formal definition of a harmonic function. A real-valued function $\phi = \phi(x, y)$ of real variables x and y is said to be *harmonic* in an open subset Ω of $\mathbb{C}$ if it has continuous partial derivatives of second order and satisfies the Laplace's equation[7] in two variables $\nabla^2\phi = 0$ throughout Ω, where ∇^2 is the second order differential operator given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \phi_{xx} + \phi_{yy}.$$

[7]This equation that bears the name of the French mathematician, Pierre Simon de Laplace (1749-1827), had been found by Leonhard Euler in 1752 in connection with Euler's studies on hydrodynamics. Laplace contributed significantly to the fields of celestial mechanics and probability theory and his parents were farmers.

The operator $\phi \mapsto \nabla^2\phi$ is called *Laplace operator* or simply a *Laplacian*. A function v is called a conjugate harmonic function (or, more briefly, a *harmonic conjugate*) for a harmonic function u in Ω whenever $f = u + iv$ is analytic in Ω. Note that the word conjugate here is not the same as conjugate of a complex number z. Further, the harmonic conjugate v is unique, up to an addition of a real constant. Indeed, if v_1 is another harmonic conjugate of u, so that $F = u + iv_1$ is also analytic in Ω, then the difference $F - f = i(v_1 - v)$ becomes analytic in Ω. But then by Theorem 3.31(ii), $v_1 - v$ is a constant.

Since $-if = v + i(-u)$, we also observe that $-u$ is a harmonic conjugate of v whenever v is a harmonic conjugate of u.

3.32. Remark. As can be observed from the C-R equations, we cannot choose two arbitrary harmonic functions u and v and claim that the resulting function $f = u + iv$ is analytic. For example, $u(x, y) \equiv x$ and $v(x, y) \equiv -y$ are harmonic functions in $\mathbb{C}$, but $f = u + iv = x + i(-y) = \overline{z}$ is nowhere analytic. On the other hand,

$$v + iu = -y + ix = i(x + iy) = iz$$

which is analytic in $\mathbb{C}$. Further, it would be appropriate to have the Laplace equation to be satisfied not just for any set of points but for an open set or a domain or more importantly a simply connected domain. For example, if $u = x^3 - y^3$ then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x - y) = 0 \quad \text{only when } y = x$$

and the set $\{z = x + iy : y = x\}$ is not open in $\mathbb{C}$ and so, u cannot be treated as a harmonic function in any open subset of $\mathbb{C}$. ●

Let us now discuss some facts about this topic and construct simple examples of harmonic functions. We first introduce a complex Laplacian. Suppose that we are given a harmonic function

$$u(x, y) = u\left(\frac{z + \overline{z}}{2}, \frac{z - \overline{z}}{2i}\right).$$

Then, formally treating z and $\overline{z}$ as independent variables, as before

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}[u_x - iu_y]$$

so that

$$\begin{aligned} \frac{\partial^2 u}{\partial z \partial \overline{z}} &= \frac{1}{2}\left[\frac{\partial}{\partial x}(u_x - iu_y) \cdot \frac{\partial x}{\partial \overline{z}} + \frac{\partial}{\partial y}(u_x - iu_y) \cdot \frac{\partial y}{\partial \overline{z}}\right] \\ &= \frac{1}{4}\left[(u_{xx} + u_{yy}) + i(u_{xy} - u_{yx})\right], \end{aligned}$$

where we have used the notation

$$u_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \quad u_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \quad u_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \quad u_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right).$$

Therefore any harmonic function u satisfies the differential equation

$$\nabla_c^2 u = \frac{\partial^2 u}{\partial z \partial \overline{z}} = 0.$$

The operator ∇_c^2 is sometimes called the *complex Laplacian*. Thus if u and its first and second partial derivatives are continuous in an open subset $\Omega \subseteq \mathbb{C}$, then

$$\nabla_c^2 u = (1/4)\,\nabla^2 u.$$

Hence, the Laplacian (operator) in complex form is given by

$$\nabla^2 = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}\right). \tag{3.33}$$

For example, if $f = u + iv$ is a complex-valued function in a domain Ω then

$$\nabla^2(zf) = 4\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}(zf)\right) = 4\frac{\partial}{\partial \bar{z}}\left(f + \frac{\partial f}{\partial z}\right) = 4f_{\bar{z}} + 4\,\nabla^2 f$$

which shows that if f and zf are harmonic in Ω, then $f_{\bar{z}} = 0$ in Ω and hence f is analytic in Ω, by Theorem 3.26. *We say that a complex-valued function is harmonic in an open set if both its real and imaginary parts are harmonic thereat.*

Harmonic functions play an important role in both mathematics and physics. There are (at least) two important reasons why harmonic functions coupled with the C-R equations are discussed as an important part of complex analysis as demonstrated in the next two theorems. First, we recall that if $f = u + iv$ is analytic in an open set Ω, then the C-R equations hold throughout Ω:

$$u_x = v_y, \quad u_y = -v_x. \tag{3.34}$$

The C-R equations have some interesting consequences. For instance, if the second partial derivatives exist and are continuous (in fact, we shall later see that the derivative of an analytic function in a domain is itself analytic there–more generally infinitely differentiable, and so u and v both have continuous partial derivatives of all orders) then by differentiating the first equation with respect to x and second with respect to y, we get

$$u_{xx} = v_{xy}, \quad u_{yy} = -v_{yx}.$$

The continuity of these partial derivatives implies that the mixed derivatives are equal (which is an important fact from two variable calculus) and, in particular, $v_{xy} = v_{yx}$ and therefore,

$$u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0.$$

In a similar way, differentiating the first equation in (3.34) with respect to y and the second in (3.34) with respect to x, we find that v satisfies the Laplace equation $\nabla^2 v = 0$. In conclusion we have

3.35. Theorem. *Let Ω be an open subset of $\mathbb{C}$. Then the real and imaginary parts of an analytic function in Ω are harmonic in Ω.*

At this point it should be noted that the mixed second partial derivatives do not coincide in general. For instance, for the real-valued function $u(x,y)$ defined by

$$u(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } x = y = 0, \end{cases}$$

it is easily seen that $u_{xy}(0,0) \neq u_{yx}(0,0)$.

3.36. Example. If $\phi(x,y)$ is harmonic in a domain D, then, by Theorem 3.26 with $u\,(=\phi_x)$ and $v\,(=-\phi_y)$, we find that $f = \phi_x - i\phi_y \in \mathcal{H}(D)$. Similarly, if $\psi(x,y)$ is harmonic in D then $g = \psi_x - i\psi_y \in \mathcal{H}(D)$. In particular, we have

$$\begin{aligned} f \pm g &= (\phi_x \pm \psi_x) - i(\phi_y \pm \psi_y) \in \mathcal{H}(D) \\ f \pm ig &= (\phi_x \pm \psi_y) - i(\phi_y \mp \psi_x) \in \mathcal{H}(D) \\ if \mp g &= (\phi_y \mp \psi_x) + i(\phi_x \pm \psi_y) \in \mathcal{H}(D). \end{aligned}$$

This example gives a way of obtaining analytic functions from harmonic functions. •

Theorem 3.35 provides numerous examples of harmonic functions as we shall soon see. However, the function u defined by $u(x,y) = x^2 + y^2$ cannot be a real part of an analytic function since $\nabla^2 u = 4$. We raise the question: *Given a real-valued harmonic function u in an open set Ω, can it be written as a real or imaginary part of an analytic function $f : \Omega \to \mathbb{C}$? If so, under what condition this is possible.* We need some preparation to answer this question.

3.37. Definition. A domain D in $\mathbb{C}$ is *simply connected* if its complement with respect to $\mathbb{C}_\infty$ (i.e. $\mathbb{C}_\infty \setminus D$) is a connected subset of $\mathbb{C}_\infty$.

Topologically, a simply connected domain D in $\mathbb{C}$ can be continuously shrunk to a point in D. Note that the punctured unit disk $A = \{z : 0 < |z| < 1\}$ can be shrunk to an arbitrarily small domain, but not to a point in A. Intuitively, a simply connected domain in $\mathbb{C}$ means that it does not contain any "holes". Consequently, according to Definition 3.37, exterior of circles (i.e. complement of closed disks in $\mathbb{C}$) are not simply connected in $\mathbb{C}$. Note that, in the extended plane, the exterior of a circle is simply connected because it can be "shrunk" to the point at ∞. Some simple examples of simply connected domains in $\mathbb{C}$ are

(i) disks $\Delta(a;r)$

(ii) half-planes $\{z : \operatorname{Re}(e^{i\theta}z) > \beta\}$

(iii) convex domains such as in (i), (ii) and infinite parallel strips, such as $D = \{z : \alpha < \operatorname{Re}(e^{i\theta}z) < \beta\}$ for some α and β with $\alpha < \beta$

(iv) domains that are starlike with respect to the origin such as $\mathbb{C}\setminus[0,\infty)$ and $\mathbb{C}\setminus\{iy : |y| > 1\}$

(v) the whole complex plane.

The annular domain $E = \{z : 1 < |z| < 2\}$ is not simply connected, whereas the domain E with $\{z : 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z = 0\}$ removed from it (i.e. $E\setminus\{(1,2)\}$), is simply connected.

We note that there are many equivalent definitions of simply connectedness. One such equivalent form of it is the following (see also Definition 4.50):

3.38. Definition. Let D be a domain in $\mathbb{C}$ and D_∞ be the set corresponding to D on the Riemann sphere S. Then D is called simply connected if $D_\infty^c = S\setminus D_\infty$ is connected and contains the north pole.

It would have been nice if every harmonic function is a real part of some analytic function. But this is not true in general. However, it is indeed true locally; but globally provided Ω is a simply connected domain. More precisely, we have

3.39. Theorem. *Let Ω be a simply connected domain and let ϕ be harmonic in Ω. Then ϕ has a harmonic conjugate in Ω.*

It is convenient to supply a proof of Theorem 3.39 in Chapter 4 since a motivation for an explicit construction for the harmonic conjugate comes from (complex) integration. On the other hand, one can give a simple and a direct proof of Theorem 3.39 especially when

(i) $\phi(x,y)$ is a real-valued harmonic polynomial in $\mathbb{C}$

(ii) $\phi(x,y)$ is a real valued harmonic function in Ω where Ω is either an open disk or open rectangle, see Corollary 3.57.

3.40. Example. Consider $u(x,y) = 4xy - x^3 + 3xy^2$. Then

$$u_x = 4y - 3x^2 + 3y^2 \quad \text{and} \quad u_y = 4x + 6xy$$

from which it is easy to see that u is harmonic in $\mathbb{R}^2$. To find the harmonic conjugate $v(x,y)$ in $\mathbb{C}$ for $u(x,y)$, we may proceed as follows. If $f = u + iv$ is the corresponding analytic function, then

$$\begin{aligned} f'(z) &= u_x - iu_y \\ &= 4y - 3x^2 + 3y^2 - i(4x + 6xy) \\ &= -3[x^2 - y^2 + 2ixy] - 4i(x + iy) \\ &= -3z^2 - 4iz \end{aligned}$$

which gives $f(z) = -z^3 - 2iz^2 + ik$, where k is a real constant. The harmonic conjugate $v(x,y)$ may be obtained by taking the imaginary part of the last expression.

Alternately, use the C-R equations to obtain

$$v_y = u_x = 4y - 3x^2 + 3y^2. \tag{3.41}$$

Integrate (3.41) with respect to y to obtain

$$v = \int v_y dy + \phi(x) = 2y^2 - 3x^2y + y^3 + \phi(x), \tag{3.42}$$

where ϕ is a function of x. If ϕ in (3.42) is differentiable with respect to x the equation $v_x = -6xy + \phi'(x)$ $(= -u_y)$ is obtained which, together with the C-R equation $u_y = -v_x$, gives

$$\phi'(x) = 6xy - u_y = 6xy - (4x + 6xy) = -4x$$

so that $\phi(x) = -2x^2 + k$, where k is some real constant. Hence, by (3.42), the harmonic conjugate function $v(x,y)$ is given by

$$v = (2y^2 - 3x^2y + y^3) + (-2x^2 + k).$$

Now we can use $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$ to write f as a function of z. ●

3.43. Theorem. *If u and v are harmonic conjugates to each other in some domain then u and v must be constant there.*

Proof. By the definition and the hypotheses, $f = u + iv$ and $g = v + iu$ are analytic in D. But

$$f - ig = 2u, \quad \text{and} \quad f + ig = 2iv.$$

By Theorem 3.9, these imply that the real-valued functions of a complex variable, namely u and v are analytic in D. Therefore, by Theorem 3.6, $f'(z) = 0$ in D. Since D is a domain, it follows that f and g are constants and hence, u and v are constants. ■

3.44. Polar form of the C-R equations and Laplacian. There are ways to obtain the C-R equations in polar form. Here is a direct method. Let $f(z) = u(r,\theta) + iv(r,\theta)$ be differentiable at a point z. Then $f'(z)$ exists and equals

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$

Set $z = re^{i\theta}$. The nearby points of z along the radial direction may be given by $z + h = (r + \Delta r)e^{i\theta}$. Therefore, for $h = \Delta r e^{i\theta} \neq 0$,

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta)}{\Delta r e^{i\theta}} - \frac{u(r,\theta) + iv(r,\theta)}{\Delta r e^{i\theta}} \\ &= \frac{1}{e^{i\theta}}\left\{\frac{u(r+\Delta r,\theta) - u(r,\theta)}{\Delta r}\right\} + \frac{i}{e^{i\theta}}\left\{\frac{v(r+\Delta r,\theta) - v(r,\theta)}{\Delta r}\right\}. \end{aligned}$$

Allowing $\Delta r \to 0$ shows that

$$f'(z) = \frac{1}{e^{i\theta}}[u_r + iv_r]. \tag{3.45}$$

Next choose nearby points of z along the circular path through the point z so that

$$z + h = re^{i(\theta+\Delta\theta)} = re^{i\theta}e^{i\Delta\theta}, \quad \text{i.e.} \quad h = re^{i\theta}(e^{i\Delta\theta} - 1)$$

and for $\Delta\theta \neq 0$,

$$\begin{aligned} \frac{f(z+h) - f(z)}{h} &= \frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{re^{i\theta}(e^{i\Delta\theta} - 1)} + i\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{re^{i\theta}(e^{i\Delta\theta} - 1)} \\ &= \frac{1}{ire^{i\theta}}\left\{\left(\frac{u(r,\theta+\Delta\theta) - u(r,\theta)}{\Delta\theta}\right) + i\left(\frac{v(r,\theta+\Delta\theta) - v(r,\theta)}{\Delta\theta}\right)\right\}\frac{i\Delta\theta}{e^{i\Delta\theta} - 1}. \end{aligned}$$

Now allowing $\Delta\theta \to 0$, it follows that

$$f'(z) = \frac{1}{ire^{i\theta}}[u_\theta + iv_\theta] = \frac{1}{re^{i\theta}}[v_\theta - iu_\theta]. \tag{3.46}$$

Comparing (3.45) and (3.46) produces the C-R equations in polar form:

$$u_r = \frac{v_\theta}{r} \quad \text{and} \quad v_r = -\frac{u_\theta}{r}. \tag{3.47}$$

Alternatively, let $f(z) = f(re^{i\theta})$, $r \neq 0$, be differentiable at a point z_0. Then both

$$f'(z_0) = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z}\bigg|_{z=z_0} = (u_r + iv_r)e^{-i\theta}\big|_{z_0}$$

and
$$f'(z_0) = \frac{\partial f}{\partial \theta}\frac{\partial \theta}{\partial z}\bigg|_{z=z_0} = (u_\theta + iv_\theta)\frac{e^{-i\theta}}{ir}\bigg|_{z_0}$$
must be the same. Therefore, equating the right hand side of the last two equations yields (3.47).

If we differentiate the first equation in (3.47) with respect to r and the second in (3.47) with respect to θ we get
$$\frac{\partial}{\partial r}(v_\theta) = u_r + ru_{rr}, \quad \frac{\partial}{\partial \theta}(v_r) = -\frac{1}{r}u_{\theta\theta}$$
and so, using the continuity of second partial derivatives, these two equations give
$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0.$$
which is the polar form of the Laplace equation. If $u(r,\theta)$ depends on r alone, then the above Laplace equation becomes
$$u_{rr} + \frac{1}{r}u_r = 0.$$
For example, we have

- $r^n \cos n\theta$ and $r^n \sin n\theta$ are harmonic for any positive integer n
- $\ln r = \ln|z|$ is harmonic in the punctured disk $\mathbb{C}\backslash\{0\}$. We note that $\ln|z|$ has no harmonic conjugate in $\mathbb{C}\backslash\{0\}$, though it does have in $\mathbb{C}\backslash[0,\infty)$, see Section 3.5.

Next we attempt to develop methods for finding an analytic function f whose real part is a given harmonic function $u(x,y)$ which is a rational function in x and y. For this we start with
$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$
so that
$$\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y),$$
where $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/(2i)$. Adding the above two equations we find that
$$f(x+iy) + \overline{f(x+iy)} = 2u(x,y).$$
Recall that analytic functions are completely characterized by the condition:

(3.48)
$$f_{\overline{z}} = \frac{\partial f}{\partial \overline{z}} = 0.$$
For example, if we substitute z for f then we see that $f_{\overline{z}} = 0$ and $f_z = 1$. Thus we remark that $\overline{f(z)}$, the conjugate of an analytic function $f(z)$, has the derivative zero with respect to z and so $\overline{f(z)}$ may be considered as a function of $\overline{z}$ alone. Therefore, we can denote $\overline{f(z)}$ simply by $\overline{f}(\overline{z})$. That is
$$\overline{f(x+iy)} = \overline{f}(x-iy), \quad \text{and} \quad 2u(x,y) = f(x+iy) + \overline{f}(x-iy).$$
In particular, the last equation yields
$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \overline{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \overline{f}(0)]$$

which gives
$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \overline{f}(0).$$
Also we observe that $u(0,0) = (1/2)[f(0) + \overline{f}(0)] = \operatorname{Re} f(0)$ and so
$$\overline{f(0)} = \operatorname{Re} f(0) - i \operatorname{Im} f(0) = u(0,0) - ik,$$
where $k = \operatorname{Im} f(0)$ is a real number. Consequently, the function f may be computed by the formula
$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ik.$$
Similarly, it follows that in a neighborhood of z_0, the analytic function $f = u + iv$ associated with the harmonic function $u(x,y)$ is given by
$$f(z) = 2u\left(\frac{z+\overline{z}_0}{2}, \frac{z-\overline{z}_0}{2i}\right) - u(x_0,y_0) + ik,$$
where k is a real constant. We shall illustrate the application of this method with two examples.

3.49. Example. Consider $u(x,y) = x^3 - 3xy^2$. Then u is defined in $\mathbb{R}^2$ and u satisfies Laplace's equation for all points in $\mathbb{R}^2$. Using the above method, the corresponding analytic function f is given by
$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ik = 2\left[\left(\frac{z}{2}\right)^3 - 3\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2\right] + ik = z^3 + ik,$$
where k is a real number. •

3.50. Example. We wish to find the most general cubic form
$$u(x,y) = ax^3 + bx^2y + cxy^2 + dy^3 \quad (a,b,c,d\text{-real}),$$
which satisfies Laplace's equation, and to determine an analytic function which has u as its real part. To do this, for $(x,y) \in \mathbb{R}^2$, we compute
$$u_{xx} + u_{yy} = 2[(3a+c)x + (b+3d)y]$$
and so, $\nabla^2 u = 0$ in $\mathbb{R}^2$ provided $3a + c = 0 = b + 3d = 0$. Thus, the most general harmonic polynomial of degree three takes the form
$$u = ax^3 - 3dx^2y - 3axy^2 + dy^3.$$
Further, using the above method of construction of analytic function, we have
$$\begin{aligned} f(z) &= 2u\left(\frac{z}{2}, \frac{z}{2i}\right) + ik \\ &= 2\left[a\left(\frac{z}{2}\right)^3 - 3d\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) - 3a\left(\frac{z}{2}\right)\left(\frac{z}{2i}\right)^2 + d\left(\frac{z}{2i}\right)^3\right] + ik \\ &= (a+id)z^3 + ik \end{aligned}$$
which is the required analytic function having u as its real part, where k is a real constant. •

3.51. Discussion on finding harmonic conjugates. Let us carefully look at the problem of whether a real-valued harmonic function u, a C^2-solution of Laplace's equation in some domain Ω, is a real or imaginary part of some analytic function. For the time being we suppose that a harmonic function u is given and we have found an analytic function $F(z)$ in Ω such that $F(z) = u + iv$, $z \in \Omega$. Then, the C-R equations give $v_x = -u_y$ and $v_y = u_x$ which show that, v_x and v_y are completely determined from the given function u. Therefore, v may be found up to an additive constant. Again, in view of the C-R equations, we see that

$$u_{xx} + u_{yy} = 0 \iff \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial y}(-u_y) \iff \frac{\partial}{\partial x}(v_y) = \frac{\partial}{\partial y}(v_x).$$

Set $f = -u_y$ and $g = u_x$. This amounts to rephrasing our problem as follow.

3.52. Problem. Given $f, g \in C^1(\Omega)$ with $f_y = g_x$, can we find a function $v \in C^2(\Omega)$ such that $v_x = f$ and $v_y = g$ in Ω? If so under what conditions on Ω, is this possible?

First, we aim at giving a partial solution to this problem by discussing a special case when Ω is an open rectangle (and the proof is similar when Ω is an open disk).

3.53. Theorem. *Let $\Omega = \{(x,y) \in \mathbb{R}^2 : |x-a| < \delta, |y-b| < \delta'\}$ be an open rectangle in $\mathbb{R}^2$. Suppose that $f, g \in C^1(\Omega)$ such that*

$$f_y = g_x \quad \text{in } \Omega. \tag{3.54}$$

Then there exists a function $v \in C^2(\Omega)$ satisfying the conditions

$$v_x = f \quad \text{and} \quad v_y = g \tag{3.55}$$

in Ω. (We may take v as real-valued whenever f and g are also).

Proof. Choose an arbitrary point $(x,y) \in \Omega$ and define

$$v(x,y) = \int_a^x f(s,b)\,ds + \int_b^y g(x,t)\,dt.$$

Observe that any other v may differ from the above merely by a constant. What motivates us to define $v(x,y)$ in this way? (see Remark 3.56). We need to show that v has the desired properties. By the fundamental theorem of calculus for C^1-functions,

$$v_y(x,y) = g(x,y).$$

Next, once again using the fundamental theorem of calculus for the first integral and the theorem on differentiation under the integral sign, it follows that

$$\begin{aligned} v_x(x,y) &= f(x,b) + \frac{\partial}{\partial x}\left(\int_b^y g(x,t)\,dt\right) \\ &= f(x,b) + \int_b^y \frac{\partial}{\partial x} g(x,t)\,dt \\ &= f(x,b) + \int_b^y \frac{\partial}{\partial y} f(x,t)\,dt \quad (\text{since } g_x = f_y) \\ &= f(x,b) + f(x,y) - f(x,b) \end{aligned}$$

so that $v_x(x,y) = f(x,y)$. Thus, we have found a function v with $v_y = g \in C^1(\Omega)$ and $v_x = f \in C^1(\Omega)$ which means that $v \in C^2(\Omega)$. ■

3.56. Remark. The function $v(x,y)$ that we need to define must satisfy (3.55). Indeed, integrating $v_y(x,y) = g(x,y)$ with respect to y, we get

$$v(x,y) = \int_b^y g(x,t)\,dt + \phi(x)$$

where ϕ is some C^1-function of x. Since we also need $v_x = f$, by the first equation in (3.55), we compute that

$$v_x(x,y) = \int_b^y \frac{\partial}{\partial x} g(x,t)\,dt + \phi'(x) = \int_b^y \frac{\partial}{\partial y} f(x,t)\,dt + \phi'(x)$$

which gives

$$f(x,y) = f(x,y) - f(x,b) + \phi'(x); \quad \text{i.e.} \quad \phi'(x) = f(x,b).$$

This has the solution

$$\phi(x) = \int_a^x f(s,b)\,ds + c,$$

where c is some constant. Hence, $v(x,y)$ must be of the form

$$v(x,y) = \int_a^x f(s,b)\,ds + \int_b^y g(x,t)\,dt + c.$$

●

3.57. Corollary. *If Ω is either an open rectangle (with sides parallel to the axes) or open disk and if u is a real-valued harmonic function in Ω, then there exists an analytic function F in Ω such that $u = \operatorname{Re} F$.*

Proof. Set $f = -u_y$ and $g = u_x$. Then $f, g \in C^1(\Omega)$ and, since $\nabla^2 u = 0$ in Ω, we have $f_y = g_x$ in Ω. By Theorem 3.53, there exists a real-valued function $v \in C^2(\Omega)$ such that

$$v_x = f = -u_y \quad \text{and} \quad v_y = g = u_x \quad \text{in } \Omega.$$

By the theorem on sufficient conditions for analytic functions (see Theorem 3.26), we conclude that $F = u + iv$ is analytic in Ω. ■

Theorem 3.53 is called an antiderivative theorem (for real-valued functions). Now it is natural to raise the following

3.58. Problem. Given an analytic function F in Ω, can we find an analytic function G such that $G'(z) = F(z)$?

If the answer to this problem is yes, then we call the function G a *primitive* or *anti-derivative* for F. Any other anti-derivative of F would differ from G by a constant. Indeed, if both G_1 and G_2 are primitives of a function F, then

$$(G_1(z) - G_2(z))' = G_1'(z) - G_2'(z) = F(z) - F(z) = 0$$

and so by Theorem 3.31, we see that $G_1(z) - G_2(z)$ is a constant.

Here is a simple illustration. The primitive of the polynomial function p defined by $p(z) = \sum_{k=0}^{n} a_k z^k$ is $P(z) = \sum_{k=0}^{n}(a_k/(k+1))z^{k+1} + K$, where K is an arbitrary constant.

Is there any restriction on Ω for the existence of primitives? First we give an affirmative answer when Ω is an open rectangle or convex or open disk in $\mathbb{C}$. However, we shall later discuss a more general theorem which provides an affirmative answer to this problem whenever Ω is a simply connected domain.

3.59. Theorem. (Antiderivative Theorem) *Let Ω be either an open rectangle (with sides parallel to the axes) or an open disk. Then every analytic function $F(z)$ in Ω possesses a primitive in Ω.*

Proof. Set $F(z) = u(z) + iv(z)$. Since F is analytic in Ω,

$$u_y = -v_x \iff f_y = g_x,$$

where $u = f \in C^1(\Omega)$ and $-v = g \in C^1(\Omega)$. By Theorem 3.53, there exists $U \in C^2(\Omega)$ such that

$$U_x = f\ (= u) \quad \text{and} \quad U_y = g\ (= -v) \quad \text{in } \Omega.$$

As F is analytic, we have $v_y = u_x$ where $v, u \in C^1(\Omega)$. Again, by Theorem 3.53, it follows that there exists $V \in C^2(\Omega)$ such that

$$V_x = v \quad \text{and} \quad V_y = u \quad \text{in } \Omega.$$

Finally, define $G(z) = U + iV$. Then, $U, V \in C^2(\Omega)$ and the C-R equations are satisfied in Ω. Hence, by Theorem 3.26, G is analytic in Ω. Note that $G'(z) = U_x + iV_x = u + iv = F(z)$. ∎

3.3 Power Series as an Analytic Function

In our earlier sections we have seen that the polynomial of degree $n \geq 1$ given by

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n \quad (a_n \neq 0) \tag{3.60}$$

can be differentiated term-by-term to get

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}. \tag{3.61}$$

Consider a series of functions of the form $\sum_{n\geq 0} a_n(\zeta - \zeta_0)^n$, where ζ is a complex variable and ζ_0, a_n, $n = 0, 1, 2, \ldots$, are fixed constants. Such a series will be called a (formal) *power series* with center ζ_0 and coefficients a_n. The substitution $z = \zeta - \zeta_0$ transforms the above series into the power series

$$\sum_{n\geq 0} a_n z^n. \tag{3.62}$$

So, for our discussion it is enough to consider power series with center 0 as the results about general power series can be obtained by translation. Now the polynomial (3.60) may be thought of as a power series at 0 with coefficients $a_k = 0$ for all $k > n$ and so we say that the family of polynomials is contained in the family of power series. Further, we also note that a power series defined by (3.62) is a special case of the limit of infinite sequences of functions, namely, $\{f_n(z)\}$, where $f_n(z) = \sum_{k=0}^{n-1} a_k z^k$. In any case, (3.61) suggests that for the power series (3.62) with sum $f(z)$, we should have

$$f'(z) = \sum_{n\geq 1} n a_n z^{n-1}$$

and in that case we call the R.H.S the *derived series* of $\sum_{n\geq 0} a_n z^n$. This fact of course requires justification and before giving a proof, we discuss some important facts about the convergence of (3.62) to appreciate fully the ideas involved.

Now, we consider the polynomials

$$\begin{aligned} p_n(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!}, \\ q_n(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \\ r_n(z) &= 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}. \end{aligned}$$

We know that these are entire. As $n \to \infty$, we obtain that

$$p_n(z) \to \cos z, \quad q_n(z) \to \sin z, \quad r_n(z) \to e^z$$

and in all three cases, the corresponding limit functions are also entire. How about the polynomial $1 + z + z^2 + \cdots + z^n$ when $n \to \infty$? How about $1 + rz + r^2 z^2 + \cdots + r^n z^n$ $(r > 1)$ as $n \to \infty$?

We start with some interesting facts due to Abel: Suppose $0 \neq z_1 \in \mathbb{C}$ is such that the power series $\sum_{n\geq 0} a_n z_1^n$ converges. The terms of the series are then bounded (see Section 1.6). Indeed, as $|a_n z_1^n| \to 0$ as $n \to \infty$, there exists an $M > 0$ such that $|a_n z_1^n| < M$ for all $n \geq 0$. We have then, for all z with $|z| < |z_1|$,

$$|a_n z^n| = |a_n z_1^n| \left|\frac{z}{z_1}\right|^n < M \left|\frac{z}{z_1}\right|^n \quad \text{for all } n.$$

Since $|z/z_1| < 1$, the geometric series $\sum_{n\geq 0} |z/z_1|^n$ converges, so that, by the comparison test, $\sum_{n\geq 0} |a_n z^n|$ converges for all z with $|z| < |z_1|$. For example, if $f(z) = \sum_{n\geq 1} z^n/n$ then the series converges for $|z| < 1$, since $f(-1) = \sum_{n\geq 1}(-1)^n/n$ converges.

If the series $\sum_{n\geq 0} a_n z_2^n$ diverges, then, for all z with $|z| > |z_2|$, we have

$$|a_n z^n| \geq |a_n z_2^n| \quad \text{for each } n \geq 0.$$

So, by the comparison test, the series $\sum_{n\geq 0} a_n z^n$ does not converge. For example, $f(z) = \sum_{n\geq 1} z^n/n$ diverges for $|z| > 1$, since $f(1) = \sum_{n\geq 1} 1/n$ diverges.

The above facts allow us to characterize the behavior of the power series (3.62) in a very natural fashion. Indeed, translating the series about '0' into a series about $a \in \mathbb{C}$, the above discussion gives (see Figure 3.1)

3.63. Theorem. *If the series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges at some point $z_1 (\neq a)$, then the series converges (absolutely) at all points in the disk $\Delta(a; |z_1 - a|)$. If the series diverges at $z_2 (\neq a)$, then it diverges for all z with $|z - a| > |z_2 - a|$.*

Because of the interesting information about the convergence from Theorem 3.63, it is natural to ask: *what is the largest disk about a on which the series $\sum_{n\geq 0} a_n(z-a)^n$ converges?* We now make this issue more precise in the form of a definition for a series about the origin. The *radius of convergence* R of the given power series (3.62) is defined by

$$R = \sup\{\rho : \sum_{n\geq 0} a_n z^n \text{ converges for all } z \text{ satisfying } |z| \leq \rho\}.$$

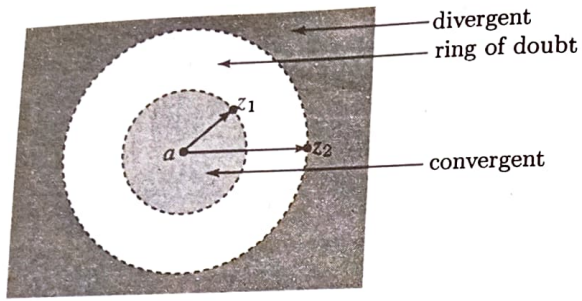

Figure 3.1: Description for Abel's test.

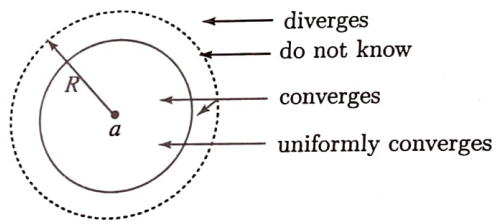

Figure 3.2: Illustration for the disk of convergence.

Note that $R = 0$ if $\sum_{n\geq 0} a_n z^n$ converges only for $z = 0$. If $R = \infty$, then the series $\sum_{n\geq 0} a_n z^n$ converges for all $z \in \mathbb{C}$. Thus, by definition, if $0 < R < \infty$, $\sum |a_n z^n|$ converges for all z such that $|z| < R$ and diverges for all z such that $|z| > R$. The series may converge for some or all points on the circle $|z| = R$. The circle $|z| = R$ is then called the *circle of convergence* because this is the greatest circle about $a = 0$ inside which $\sum_{n\geq 0} |a_n z^n|$ converges at each point (see Figure 3.2).

The conventions $0^{-1} = \infty$ and $\infty^{-1} = 0$ are observed so that R is the unique number in $[0, \infty]$.

3.64. Theorem. (Root Test) *Let $L^{-1} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then*

(i) *if $L = \infty$, the series $\sum_{n\geq 0} a_n z^n$ converges absolutely for all finite z and uniformly in any bounded set*

(ii) *if $L = 0$, the series converges only at $z = 0$ and diverges at all other points other than 0*

(iii) *if $0 < L < \infty$, the series converges absolutely for $|z| < L$, uniformly for $|z| \leq r$, $r < L$ and diverges for $|z| > L$*

(iv) *$L = R$.*

Proof. As the series converges for $z = 0$, we need to consider only the case $z \neq 0$. So, for $z \neq 0$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n z^n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{L}.$$

By Theorem 1.51, the series $\sum_{n\geq 0} a_n z^n$ converges absolutely for $|z| < L$ and diverges for $|z| > L$. The uniform convergence of the series for $|z| \leq r$ $(r < L)$ follows from the Weierstrass M-test (see Theorem 2.59). Thus, (i) to (iii) follows. We leave (iv) as a simple exercise. ■

3.65. Remark. Note that if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists, then we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

In this case, the radius of convergence R is simply determined from

$$R^{-1} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$ ●

3.66. Example. For each fixed $z \neq 0$, we have $\lim_{n\to\infty} |z|^{1/n} = 1$. This can be proved as follows: Let $\delta_n = n^{1/n} - 1$. Then for each $n > 1$, δ_n is a positive real number. Now, for $n \geq 2$, we have

$$n = (1 + \delta_n)^n = \sum_{k=0}^{n} \binom{n}{k} \delta_n^k > 1 + \frac{n(n-1)}{2}\delta_n^2 > \frac{n(n-1)}{2}\delta_n^2 \geq \frac{n^2}{4}\delta_n^2.$$

This implies that $0 \leq \delta_n^2 < \frac{2}{n}$ and so we have $\delta_n \to 0$ as $n \to \infty$; i.e. $\lim_{n\to\infty} n^{1/n} = 1$. Applying this result for $|z| \geq 1$, we get $|z|^{1/n} \to 1$, since

$$1 \leq |z|^{1/n} \leq n^{1/n} \quad \text{for sufficiently large } n.$$

If $|z| < 1$, apply the preceding argument for $1/z$. ●

3.67. Remark. The above example shows that $\{z_n\}$, where $z_n = n^{1/n} - 1$, is a null sequence. Since $n^{1/n} \to 1$, we also have

$$\frac{1}{n}\left[1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}\right] \to 1 \quad \text{as} \quad n \to \infty.$$ ●

The determination of the radius of convergence for the series (3.62) will not always require an application of Theorem 3.64, i.e. the *Root test*. In most of the cases, the following theorem called the *Ratio test* will serve the purpose instead.

3.68. Theorem. (Ratio Test) *If $a_n \neq 0$ for all but finitely many values of n, then the radius of convergence R of (3.62) is related by*

$$l := \liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \leq \frac{1}{R} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| =: L.$$

In particular, if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists either as finite or $+\infty$ then

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

Proof. Proof of this theorem is an immediate consequence of Theorem 1.48. Again it suffices to consider the case $z \neq 0$. In this event,

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = |z|L \quad \text{and} \quad \liminf_{n\to\infty} \left|\frac{a_{n+1}z^{n+1}}{a_n z^n}\right| = |z|l.$$

By Theorem 1.48, the series $\sum_{n\geq 0} a_n z^n$ converges absolutely for $|z| < 1/L$ and diverges for $|z| > 1/l$. Thus, R must be at least $1/L$ and at most $1/l$; that is R is related by $1/L \leq R \leq 1/l$. Consequently, if $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists then $L = l$ and hence

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \limsup_{n\to\infty}|a_n|^{1/n} = \liminf_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}|a_n|^{1/n} = \frac{1}{R}.$$

This completes the proof. ∎

Note that if the sequence $\{|a_{n+1}|/|a_n|\}$ oscillates, the limit does not exist and therefore the Ratio test becomes of no use. For instance, for series such as

$$\sum_{n\geq 0} a_n z^n = 3 + z + 3z^2 + z^3 + 3z^4 + \cdots + 3z^{2k} + z^{2k+1} + \cdots$$

the ratio $|a_{n+1}|/|a_n|$ is alternately 1/3 and 3, so the limit does not exist. In this particular series, even though the Ratio test indicates that the value of R is lying between 1/3 and 3, it does not yield the exact value of R. However, as $|a_{2k}| = 3$ and $|a_{2k+1}| = 1$, Example 3.66 gives $R^{-1} = \limsup_{n\to\infty}|a_n|^{1/n} = 1$.

Thus in the above two examples *the Ratio test* gives no information whereas the *Root test* gives the radius of convergence. We look now at a string of examples.

(i) Since (see Example 3.66) $n^{1/n} \to 1$ as $n \to \infty$, the series $\sum_{n\geq 1} nz^n$ converges for $|z| < 1$ and diverges for $|z| > 1$. For $|z| = 1$, $|nz^n| = n \to \infty$ and so the series diverges for $|z| = 1$. In this case we note that the series diverges at all points on the circle of convergence. Also, we observe that the Root test is clearly applicable.

(ii) Consider the series $\sum_{n\geq 0}(-1)^n z^{pn} = 1 - z^p + z^{2p} - \cdots$, where $p \geq$ is a fixed positive integer. Then

$$a_{np} = \begin{cases} -1 & \text{if } n = 1, 3, \ldots \\ 1 & \text{if } n = 0, 2, \ldots \end{cases}, \quad \text{i.e. } a_m = \begin{cases} (-1)^{m/k} & \text{if } m = 0, k, 2k, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

In this case $\limsup_{n\to\infty}|a_n|^{1/n} = 1$. Hence, $R = 1$ and the series converges absolutely for $|z| < 1$, uniformly for $|z| \leq \rho < 1$, and diverges for $|z| > 1$. Since the sequence of terms of the series does not approach zero when $|z| = 1$, it follows that the series diverges for $|z| = 1$. Note that the Ratio test is not applicable directly, but we could obtain the radius of convergence by translating the given series into a new series with a new variable ζ, $\zeta = z^p$.

(iii) By means of the Ratio test we at once see that each of the series

$$\sum_{n\geq 1}(-1)^n n^k z^n, \quad \sum_{n\geq 1}(-1)^{n-1}\frac{z^n}{n^k}, \quad \sum_{n\geq 1} n^k z^n \quad (k = 1, 2, \ldots)$$

has 1 as the radius of convergence. For $k \geq 2$, the second of the above converges absolutely on $|z| = 1$ since, on $|z| = 1$,

$$\sum_{n\geq 1}\left|\frac{(-1)^n z^n}{n^k}\right| = \sum_{n\geq 1}\frac{1}{n^k}$$

converges for $k \geq 2$. For $k = 1$, the series becomes

$$\sum_{n\geq 1}(-1)^{n-1}\frac{z^n}{n} = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots, \quad |z| < 1.$$

This series is called the *Logarithmic series* (see Section 3.5). On the other hand, the series

$$\sum_{n\geq 1}\frac{z^n}{n} \tag{3.69}$$

at $z = -1$ becomes $-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots$ and so converges for $z = -1$, but non-absolutely. At $z = 1$, this series is $\sum_{n\geq 1}\frac{1}{n}$ which is the harmonic series and we know that this is divergent. What happens on the rest of $\partial\Delta$? To see this, we let $z = e^{i\theta}$ $(\theta \neq 0, 2\pi)$ and $a_k = z^k = e^{ik\theta}$. Then

$$\left|\sum_{k=1}^{n} z^k\right| = \left|\frac{1 - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - e^{i\theta}|}$$

showing that the partial sums are bounded provided $\theta \neq 0, 2\pi$. Hence, by Theorem 1.52(b) with $b_n = n^{-1}$, the series (3.69) converges for $|z| = 1$ except at $z = 1$. Consequently, the logarithmic series $\sum_{n\geq 1}(-1)^{n-1}z^n/n$ converges for $|z| = 1$ except at $z = -1$.

(iv) Consider a series $\sum_{n\geq 0} q^n z^{kn}$, where $k \geq$ is a fixed positive integer and $q \neq 0$, independent of n. Note that

$$a_n = \begin{cases} q^{n/k} & \text{if } n = 0, k, 2k, 3k, \ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then, according to the Root test, one has $\limsup_{n\to\infty}|a_n|^{1/n} = |q|^{1/k}$ and so, the power series has the radius of convergence $R = |q|^{-1/k}$. The series converges absolutely for $|z| < |q|^{-1/k}$, uniformly for $|z| \leq \rho < |q|^{-1/k}$, and diverges for $|z| > |q|^{-1/k}$.

For instance, if $q = -1$ then $a_n = (-1)^n$ for $n \geq 0$ and the corresponding series $\sum_{n\geq 0}(-1)^n z^{kn}$ converges absolutely for $|z| < 1$, uniformly for $|z| \leq \rho < 1$, and diverges for $|z| > 1$. In this special case, the terms of the series does not approach zero when $|z| = 1$, and therefore, the series diverges for $|z| = 1$.

If we set $q = 1/2$ then we get the series

$$\sum_{n\geq 0} 2^{-n} z^{kn}$$

which converges absolutely for $|z| < 2^{1/k}$, uniformly for $|z| \leq \rho < 2^{1/k}$, and diverges for $|z| > 2^{1/k}$. For $|z| = 2^{1/k}$, $|2^{-n}z^{nk}| = 1$, showing that the series diverges. In particular, for $k = 2$ the power series is

$$\sum_{n\geq 0} 2^{-n} z^{2n}, \quad \text{with } a_n = \begin{cases} 0 & \text{if } n = 1, 3, 5, \ldots \\ 2^{-n/2} & \text{if } n = 0, 2, 4, \ldots \end{cases},$$

and so $R = \sqrt{2}$ is the radius of convergence for this series.

If we set $q = 5$, then the corresponding series

$$\sum_{n\geq 0} 5^n z^{kn} = \sum_{n\geq 0}(5z^k)^n$$

converges absolutely for $|z| < 5^{-1/k}$, uniformly for $|z| \leq \rho < 5^{-1/k}$, and diverges for $|z| > 5^{-1/k}$. For $|z| = 5^{-1/k}$, $|5^n z^{nk}| = 1$, showing that the series diverges. In particular, for $k = 3$ the above discussion gives the power series

$$\sum_{n\geq 0} 5^n z^{3n}, \quad \text{with } a_n = \begin{cases} 5^{n/3} & \text{if } n = 0, 3, 6, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

and so $R = 5^{-1/3}$ is the radius of convergence for this series.

3.70. Theorem. *A power series $\sum_{n\geq 0} a_n z^n$ and the k-times derived series defined by $\sum_{n\geq k} n(n-1)\cdots(n-k+1)a_n z^{n-k}$ have the same radius of convergence.*

Proof. Let $A_n = n(n-1)\cdots(n-k+1)a_n = n!a_n/(n-k)!$, $k \geq 1$. Then

$$|A_n|^{1/n} = \left|\frac{n!}{(n-k)!}a_n\right|^{1/n} = \left|\frac{n!}{(n-k)!}\right|^{1/n} \cdot |a_n|^{1/n}.$$

Using the particular case of Theorem 3.68 (see also Example 3.66) we have

$$\limsup_{n\to\infty}\left[\frac{n!}{(n-k)!}\right]^{1/n} = \lim_{n\to\infty}\left[\frac{(n+1)!/(n+1-k)!}{n!/(n-k)!}\right] = \lim_{n\to\infty}\frac{n+1}{n+1-k} = 1.$$

Therefore, $\limsup_{n\to\infty}|A_n|^{1/n} = \limsup_{n\to\infty}|a_n|^{1/n}$ which proves our theorem. ■

Since a power series of the form (3.62) with non-zero radius of convergence R converges (absolutely) for $|z| < R$, we can study its behavior as a function f defined by the sum $f(z) = \sum_{n\geq 0} a_n z^n$. The power series obtained by differentiating this series term-by-term gives

$$f'(z) = \sum_{n\geq 1} n a_n z^{n-1}, \quad |z| < R.$$

Next we have

3.71. Theorem. *If $\sum_{n\geq 0} a_n z^n$ has radius of convergence $R > 0$, then $f(z) = \sum_{n\geq 0} a_n z^n$ is analytic in $|z| < R$, $f^{(k)}(z)$ exists for every $k \in \mathbb{N}$ and*

$$f^{(k)}(z) = k!a_k + \sum_{n\geq k+1}\frac{n!}{(n-k)!}a_n z^{n-k} \quad (|z| < R). \tag{3.72}$$

where $a_k = f^{(k)}(0)/k$.

For example, the geometric series $(1-z)^{-1} = \sum_{n\geq 0} z^n$ which converges for $|z| < 1$, after k-times differentiation yields

$$\frac{1}{(1-z)^{k+1}} = \sum_{n\geq k}\binom{n}{k}z^{n-k} = \sum_{m\geq 0}\frac{(m+k)!}{k!m!}z^m \quad \text{for } |z| < 1.$$

In particular, $z(1-z)^{-2} = \sum_{n\geq 1} nz^n$ for $|z| < 1$.

Proof. Let $f(z) = \sum_{n\geq 0} a_n z^n$ with the radius of convergence R. We have to prove the existence of $f'(z)$ in Δ_R. By Theorem 3.70 with $k = 1$, the series

$$\sum_{n\geq 1} n a_n z^{n-1}$$

converges for $|z| < R$ and defines a function, say $g(z)$, in $|z| < R$. We show that

$$f'(z) = \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = g(z) \quad \text{for all } z \in \Delta_R.$$

Let $z \in \Delta_R$ be fixed and let $h \in \mathbb{C}$, $0 < |h| < (R - |z|)/2$. Then

$$|z+h| \leq |z| + |h| < |z| + \frac{R-|z|}{2} = \frac{|z|+R}{2} < \frac{R+R}{2} = R.$$

Now we consider

$$\frac{f(z+h)-f(z)}{h} = \sum_{n\geq 1} a_n\left(\frac{(z+h)^n - z^n}{h}\right),$$

where z and $z+h$ are such that $\max\{|z|, |z+h|\} \leq r < R$ and $|h|$ is positive. So, we must show that as $h \to 0$,

$$\left|\frac{f(z+h)-f(z)}{h} - g(z)\right| = \left|\sum_{n\geq 2} a_n\left(\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right)\right| \to 0.$$

First we note that, for $\alpha \neq 1$ and $n \geq 2$, we have the identity

$$\frac{1-\alpha^n}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1}$$

and differentiating with respect to α shows that

$$\frac{1-\alpha^n}{1-\alpha} - n\alpha^{n-1} = (1-\alpha)[1 + 2\alpha + 3\alpha^2 + \cdots + (n-1)\alpha^{n-2}].$$

Replacing α by z/w $(w \neq z)$ gives

$$\frac{w^n - z^n}{w-z} - nz^{n-1} = (w-z)[w^{n-2} + 2w^{n-3}z + \cdots + (n-1)z^{n-2}]$$

so that for $\max\{|w|, |z|\} \leq r$,

$$\begin{aligned}\left|\frac{w^n - z^n}{w-z} - nz^{n-1}\right| &\leq |w-z|\,[|w|^{n-2} + 2|w|^{n-3}|z| + \cdots + (n-1)|z|^{n-2}] \\ &\leq |w-z|r^{n-2}[1 + 2 + 3 + \cdots + n - 1] \\ &= |w-z|\frac{n(n-1)}{2}r^{n-2}.\end{aligned}$$

By Theorem 3.70, the derived series $\sum_{n\geq 2} n(n-1)|a_n|r^{n-2}$ is convergent for each r such that $|z| \leq r < R$. Using this we see that as $h \to 0$,

$$\left|\sum_{n\geq 2} a_n\left(\frac{(z+h)^n - z^n}{h} - nz^{n-1}\right)\right| \leq |h|\sum_{n\geq 2}|a_n|\frac{n(n-1)}{2}r^{n-2} \to 0.$$

Consequently, $f'(z)$ exists and equals $g(z)$. Since z is arbitrary, this holds at any interior point in the disk of convergence.

A repeated application of this argument shows that all the derivatives $f', f'', \ldots, f^{(k)}, \ldots$ exist in $|z| < R$ and (3.72) holds. Putting $z = 0$ in (3.72) we have the formula for the coefficient a_k: $f^{(k)}(0)/k = a_k$ and we are done. ■

3.74. Remark. Now it is clear that entire functions are just those functions which are defined by the sums of the series $\sum_{n\geq 0} a_n z^n$ where $\limsup_{n\to\infty} |a_n|^{1/n} = 0$. ●

In the next theorem (called the **Uniqueness Theorem** for power series), we show that a power series representing a given function, when obtained by whatever method, is unique.

3.75. Theorem. *Suppose $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$ converge for $|z| < R_1$ and $|z| < R_2$, respectively. If $f(z_k) = g(z_k)$ for a sequence $\{z_k\}$ of nonzero complex numbers in $0 < |z| < \delta$ such that $z_k \to 0$ as $k \to \infty$, or if $f(z) = g(z)$ for all z with $|z| < \delta$, where $0 < \delta < \min\{R_1, R_2\}$, then $a_n = b_n$ for every $n \in \mathbb{N}_0$.*

Proof. Since f, g are continuous in $|z| \leq \delta$, being analytic thereon,

$$z_k \to 0 \text{ and } f(z_k) = g(z_k) \implies \lim_{k\to\infty} f(z_k) = \lim_{k\to\infty} g(z_k) \implies f(0) = g(0).$$

Thus, $a_0 = b_0$. To complete the proof, we use the method of induction. Now suppose that $a_j = b_j$ for $j = 0, 1, \ldots, m-1$. Then

$$\begin{aligned} f(z_k) = g(z_k) &\implies \sum_{n\geq m} a_n z_k^n = \sum_{n\geq m} b_n z_k^n \\ &\implies \sum_{n\geq m} a_n z_k^{n-m} = \sum_{n\geq m} b_n z_k^{n-m}, \quad \text{since } z_k \neq 0, \\ &\implies F(z_k) = G(z_k), \end{aligned}$$

where

$$F(z) = \sum_{n\geq m} a_n z^{n-m} \text{ and } G(z) = \sum_{n\geq m} b_n z^{n-m}.$$

Since the radius of convergence of $\sum_{n\geq 0} a_n z^n$ and $\sum_{n\geq m} a_n z^{n-m}$ are the same, F is continuous for $|z| \leq \delta$. Similarly, G is continuous for $|z| \leq \delta$. Thus (see Theorem 2.22),

$$F(z_k) = G(z_k) \implies \lim_{k\to\infty} F(z_k) = \lim_{k\to\infty} G(z_k) \implies a_m = b_m$$

and we conclude that $a_n = b_n$ for every $n \geq 0$. Note that if $f(z) = g(z)$ for $|z| < \delta$, we can find a sequence $\{z_k\}_{k\geq 1} \in \{z : |z| < \delta\}$ such that $z_k \to 0$ as $k \to \infty$ (see Theorem 2.22). The proof is complete. ■

3.76. Corollary. *Suppose $f(z) = \sum_{n\geq 0} a_n z^n$, with the power series having radius of convergence $R > 0$. If 0 is a limit point of the set of zeros of f, then $f \equiv 0$ for $|z| < R$.*

It will be shown, more generally later (Theorem 4.103) that if f, g are two analytic functions which agree at a sequence of points $\{z_k\}$ having a limit point in their common domain of analyticity, then $f \equiv g$ therein.

3.4 Exponential and Trigonometric Functions

In this section we describe the complex analogous of exponential and trigonometric functions of elementary calculus. There are many approaches which lead to definitions of these functions although our approach is intuitive and direct. Let us first recall the following facts from calculus:

(a) Trigonometric functions are defined by means of the ratios of the sides of a right triangle; for instance, $\sin^2 x + \cos^2 x = 1$, $x \in \mathbb{R}$.

(b) $\frac{d}{dx}(\sin x) = \cos x$, $\frac{d}{dx}(\cos x) = -\sin x$, $\frac{d}{dx}(e^x) = e^x$;

(c) Further,

$$\sin x = x - \frac{x^2}{3!} + \frac{x^5}{5!} - \cdots, \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{and} \quad e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

The properties (b) and (c) show that $\sin x$ and $\cos x$ are the unique solution of the second order ordinary differential equation

$$f''(x) + f(x) = 0$$

subject to the conditions $f(0) = f'(0) - 1 = 0$ and $f(0) = f'(0) - 1 = 1$, respectively. On the other hand e^x is defined to be the solution of

$$f'(x) = f(x), \quad f(0) = 1.$$

In the same way we begin with a function $f(z) = \sum_{n\geq 0} a_n z^n$ satisfying

(i) $f'(z) = f(z)$ for $z \in \mathbb{C}$, i.e. f is analytic in $\mathbb{C}$

(ii) $f(x) = e^x$, $x \in \mathbb{R}$.

Using (i) we find that $a_{n-1} = na_n$. Since $f(0) = e^0 = 1 = a_0$, induction on n yields $a_n = 1/n!$. Thus we define the complex exponential function as

$$f(z) = \exp(z) = e^z = \sum_{n\geq 0} \frac{z^n}{n!}. \tag{3.77}$$

Conversely, if the complex exponential function is given by (3.77) then (i)-(ii) follow easily. By the Ratio test, the series (3.77) converges absolutely for all $z \in \mathbb{C}$ and hence, e^z is an entire function. In the same way one can obtain the series representations of $\sin z$ and $\cos z$.

As $(\exp z)' = \exp z$, we have the following: if $g \in \mathcal{H}(D)$, then so does $F(z) = \exp(g(z))$ and $F'(z) = g'(z)\exp(g(z))$ for $z \in D$.

Next, for a fixed $\zeta \in \mathbb{C}$, we define

$$g_\zeta(z) = e^z e^{\zeta - z} (= f(z)f(\zeta - z)).$$

Then for every $z \in \mathbb{C}$, we have $g'_\zeta(z) = 0$. Therefore, by Theorem 3.31, g_ζ is constant. Since $g_\zeta(0) = e^0 e^\zeta = e^\zeta$, we must have

$$e^z e^{\zeta - z} = e^\zeta.$$

Taking $z = z_1$ and $\zeta = z_1 + z_2$, we see that for every $z_1, z_2 \in \mathbb{C}$,

$$e^{z_1 + z_2} = e^{z_1} e^{z_2}. \tag{3.78}$$

The result (3.78) is called the *"Addition theorem"* for the exponential function which is also a consequence of the Uniqueness theorem (Section 4.11). This property can also be obtained

directly using the Cauchy product of series:

$$\begin{aligned} e^{z_1}e^{z_2} &= \sum_{n\geq 0}\frac{z_1^n}{n!}\sum_{m\geq 0}\frac{z_2^m}{m!} = \sum_{k\geq 0}\left[\sum_{p=0}^{k}\frac{z_1^p}{p!}\frac{z_2^{k-p}}{(k-p)!}\right] \\ &= \sum_{k\geq 0}\left[\frac{1}{k!}\sum_{p=0}^{k}\binom{k}{p} z_1^p z_2^{k-p}\right] \\ &= \sum_{k\geq 0}\frac{(z_1+z_2)^k}{k!} \\ &= e^{z_1+z_2}. \end{aligned}$$

For any positive integer m, by induction, (3.78) with $z_1 = z_2 = z$ shows that

$$e^{mz} = (e^z)^m. \tag{3.79}$$

For $z = iy$, $y \in \mathbb{R}$, (3.77) gives

$$e^{iy} = \sum_{n\geq 0}\frac{i^n y^n}{n!} = \sum_{k\geq 0}(-1)^k\frac{y^{2k}}{(2k)!} + i\sum_{k\geq 0}\frac{(-1)^k y^{2k+1}}{(2k+1)!}.$$

Thus, we have the Euler formula

$$e^{iy} = \cos y + i\sin y, \quad y \in \mathbb{R}. \tag{3.80}$$

Therefore, for $z_1 = x$ and $z_2 = iy$, $x, y \in \mathbb{R}$, (3.78) shows that

$$e^{x+iy} = e^x(\cos y + i\sin y); \quad e^z = e^x e^{iy}. \tag{3.81}$$

Since $e^z e^{-z} = e^{z-z} = e^0 = 1$, $e^z \neq 0$ for each $z \in \mathbb{C}$. Further,

$$|e^z|^2 = e^z\overline{(e^z)} = e^z e^{\bar{z}} = e^{z+\bar{z}} = e^{2\operatorname{Re} z}$$

so that $|e^z|^2 = (e^{\operatorname{Re} z})^2$, by (3.79). Thus, from (3.81) and the above, we write

$$|e^z| = e^{\operatorname{Re} z}; \quad \arg(e^z) = \operatorname{Im} z \pmod{2\pi}. \tag{3.82}$$

In particular, we have $|e^z| = 1 \iff z \in i\mathbb{R}$. For m, n integers and $k = 0, 1, \ldots, n-1$, we have

$$(e^z)^{1/n} = (e^x e^{iy})^{1/n} = e^{x/n}[e^{i(y+2k\pi)/n}] = e^{(z+2k\pi i)/n}$$

and so we write

$$(e^z)^{m/n} = e^{m(z+2k\pi i)/n}. \tag{3.83}$$

From (3.80) we see in particular that, if z is real then

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \quad \text{and} \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \tag{3.84}$$

Thus the extension of the exponential to the complex plane suggests to make (3.84) as definitions of $\cos z$ and $\sin z$ for any $z \in \mathbb{C}$ although these definitions can be realized as a consequence of the Uniqueness theorem. Hence we make the definition–the famous Euler formula:

$$e^{iz} = \cos z + i\sin z, \quad z \in \mathbb{C}.$$

Now consider $f(z) = \sin z$ for $z \in \mathbb{C}$. Then, by the definition of $\sin z$ and the fact that $(\exp z)' = \exp z$, (3.84) gives

$$f'(z) = \frac{ie^{iz} + ie^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

and so using the definition of $\cos z$,

$$f''(z) = \frac{ie^{iz} - ie^{-iz}}{2} = -\frac{e^{iz} - e^{-iz}}{2i} = -\sin z.$$

Consequently, we have

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k \\ (-1)^k & \text{if } n = 2k+1 \end{cases}, \quad k \in \mathbb{N}_0.$$

Similarly, if $f(z) = \cos z$, we obtain $f'(z) = -\sin z$ and $f''(z) = -\cos z$ so that

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n = 2k+1 \\ (-1)^k & \text{if } n = 2k \end{cases}, \quad k \in \mathbb{N}_0.$$

Now we use the Maclaurin series expansion (see just above Corollary 3.73), namely

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n,$$

to get the series expansions for $\cos z$ and $\sin z$. Therefore, for any $z \in \mathbb{C}$, the cosine and sine series are now

(3.85) $$\cos z = \sum_{n \geq 0} (-1)^n \frac{z^{2n}}{(2n)!}$$

and

(3.86) $$\sin z = \sum_{n \geq 0} (-1)^n \frac{z^{2n+1}}{(2n+1)!},$$

respectively. Also, note that (3.85) and (3.86) follow from the definition of e^{iz} and (3.84). Each of the series converges absolutely for every $z \in \mathbb{C}$, because

$$\sum_{n \geq 0} \frac{|z|^{2n}}{(2n)!} \quad \text{and} \quad \sum_{n \geq 0} \frac{|z|^{2n+1}}{(2n+1)!}$$

are the subseries (i.e. terms chosen from those of) of the convergent series $\sum_{n \geq 0} [|z|^n/n!]$ for $z \in \mathbb{C}$. The remaining circular functions are then defined by

$$\tan z = \frac{\sin z}{\cos z}, \quad \sec z = \frac{1}{\cos z} \quad \left(z \neq \frac{(2k+1)\pi}{2}\right);$$
$$\cot z = \frac{\cos z}{\sin z}, \quad \csc z = \frac{1}{\sin z} \quad (z \neq k\pi),$$

where $k \in \mathbb{Z}$. These functions are analytic except at points where the denominators vanish. The hyperbolic functions are defined by

$$\cos(iz) = \cosh z \quad \text{and} \quad \sin(iz) = i \sinh z$$

so that

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2},$$
$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \operatorname{sech} z = \frac{1}{\cosh z} \quad \left(z \neq \frac{(2k+1)\pi i}{2}\right),$$
$$\coth z = \frac{\cosh z}{\sinh z}, \quad \operatorname{csch} z = \frac{1}{\sinh z} \quad (z \neq k\pi i),$$

where $k \in \mathbb{Z}$. Again these functions are analytic except at points where the denominators vanish. For $z_1, z_2 \in \mathbb{C}$, the identity

(3.87) $$e^{i(z_1 \pm z_2)} = e^{iz_1} e^{\pm iz_2}$$

yields the addition formulae (which will be proved later also by using the Uniqueness theorem)

(3.88) $$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

and

(3.89) $$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2.$$

Putting $z_1 = z_2 = z$ in (3.88) and (3.89), we get

$$\cos^2 z + \sin^2 z = 1; \quad \cos 2z = \cos^2 z - \sin^2 z; \quad \sin 2z = 2 \sin z \cos z.$$

Recall the known facts about the sine and cosine functions of a real variable indicated by the following chart:

Interval	cos	sin
$[0, \pi/2]$	$1 \searrow 0$	$0 \nearrow 1$
$[\pi/2, \pi]$	$0 \searrow -1$	$1 \searrow 0$
$[\pi, 3\pi/2]$	$-1 \nearrow 0$	$0 \searrow -1$
$[3\pi/2, 2\pi]$	$0 \nearrow 1$	$-1 \nearrow 0$

as also the properties $\cos 2k\pi = 1$, $\sin 2k\pi = 0$, $k \in \mathbb{N}_0$. Therefore, we have from (3.88) and (3.89)

(3.90) $$\cos(z + 2k\pi) = \cos z \quad \text{and} \quad \sin(z + 2k\pi) = \sin z$$

where $k \in \mathbb{Z}$. The above facts together with the equation in (3.84) immediately yield the following necessary and sufficient conditions for certain identities to become true:

(a) $\cos z = 1 \iff z = 2\pi k$,
(b) $\sin z = 1 \iff z = (4k+1)\pi/2$,
(c) $\tan z = 1 \iff z = (4k+1)\pi/4$,
(d) $\cos z_1 = \cos z_2 \iff$ either $z_1 + z_2 = 2\pi k$ or $z_1 - z_2 = 2\pi k$,
(e) $\sin z_1 = \sin z_2 \iff$ either $z_1 + z_2 = (2k+1)\pi$ or $z_1 - z_2 = 2\pi k$,

where $k \in \mathbb{Z}$. Note that (d) $\iff$ (e). This fact can be easily seen by taking $z_1 = \zeta_1 - \pi/2$ and $z_2 = \zeta_2 - \pi/2$.

3.91. Definition. A function $f : D \to \mathbb{C}$ is said to be *periodic* if there exists an $\omega \neq 0$ such that $f(z+\omega) = f(z)$ for all $z \in D$. The complex number ω is then called a *period* of f. The function f is called *doubly periodic* in D if there are complex numbers ω_1, ω_2 which are linearly independent over $\mathbb{R}$ such that $f(z+\omega_1) = f(z+\omega_2) = f(z)$ for all $z \in D$.

Since $e^{2\pi k i} = 1$, we find that $e^{z+2k\pi i} = e^z e^{2k\pi i} = e^z$. Suppose $\omega = u + iv$ is a period for e^z. Then by definition $e^{z+\omega} = e^z$; i.e. $e^\omega = 1$, and $|e^\omega| = e^u = 1$ and so $u = 0$. Hence

$$e^{iv} = 1; \quad \text{i.e. } \cos v = 1 \text{ and } \sin v = 0.$$

By (3.90), this occurs only for $v = 2k\pi$, where k is some integer. This shows that ω is a period of e^z iff $\omega = 2k\pi i$. In particular, e^z is a periodic function with period $2\pi i$. Similarly, we have

$$\sin(z+\omega) = \sin z, \quad \text{or} \quad \cos(z+\omega) = \cos z \iff \omega = 2k\pi, \quad k \in \mathbb{Z}.$$

3.92. Example. Consider the function $f(z) = e^z$. Then we have

$$f(z_1) = f(z_2) \Longrightarrow e^{z_1 - z_2} = 1 \Longrightarrow z_1 - z_2 = 2k\pi i \quad \text{for some} \quad k \in \mathbb{Z}.$$

This shows that f is one-to-one in a domain D iff D does not have even a single pair of distinct points z_1 and z_2 satisfying

$$z_1 = z_2 + 2k\pi i, \quad k = \pm 1, \pm 2, \ldots.$$

For instance, f is univalent in $|z| < \pi$; but not in $\mathbb{C}$. Similarly, we see that f is univalent within any horizontal strip of the form:

$$b < \operatorname{Im} z < a, \quad 0 < a - b \leq 2\pi.$$

In fact we see that f is univalent in each strip

$$D_k = \{z : 2(k-1)\pi < \operatorname{Im} z < 2k\pi\}, \quad k \in \mathbb{Z},$$

and maps each strip onto the w-plane with a cut along (i.e. omitting) the positive real axis: $0 \leq x < \infty$. ●

3.93. Example. Note that for $z = x + iy$, we have

$$e^z = \begin{cases} e^x e^{ib} & \text{if } y = b, \text{ a constant,} \\ e^a e^{iy} & \text{if } x = a, \text{ a constant.} \end{cases}$$

As a further illustration of the exponential function, we have

(i) The exponential function e^z maps each horizontal line $y = b$ onto a ray from the origin to infinity, namely, $\{re^{ib} : r \in (0, \infty)\}$.

(ii) The exponential function e^z maps each vertical line $x = a$ onto a circle centered at the origin and radius e^a. Observe that each point on this circle is the image of infinitely many points on the vertical lines as e^{iy} is periodic. Note that these two families of curves (and also their images) are orthogonal to each other.

(iii) The line $y = b$ and the line $x = a$ intersect at (a, b). How about their images in the w-plane?

(iv) The rectangle $D = \{z : a_1 < \operatorname{Re} z < a_2,\ b_1 < \operatorname{Im} z < b_2\}$ is mapped into

$$D' = \{w : e^{a_1} < |w| < e^{a_2},\ b_1 < \arg w < b_2\}.$$

In particular, $\exp z$ maps the infinite strip $\{z : 0 < \operatorname{Im} z < \pi/2\}$ onto the open first quadrant $\{w : \operatorname{Re} w > 0,\ \operatorname{Im} w > 0\}$. ●

3.5 Logarithmic Functions

It is natural to think of a logarithmic function as the inverse of the exponential function. Recall that, when x is real,

(a) $e^x > 0$ for all $x \in \mathbb{R}$

(b) $e^x \to \infty$ as $x \to \infty$

(c) $e^{-x} = 1/e^x$ which by (a) gives $e^x \to 0$ as $x \to -\infty$

(d) $\dfrac{d}{dx}(e^x) = e^x$ and so e^x is monotonically increasing for all x.

This shows that e^x defines a strictly increasing differentiable function from $\mathbb{R}$ onto $\mathbb{R}^+$, the set of positive reals. Hence it has a continuous strictly increasing inverse function called the *natural logarithm* (with base e)

$$\ln : \mathbb{R}^+ \to \mathbb{R}$$

with the property that $\ln x = y$ is the solution of $e^y = x$. In particular, for each $x > 0$ there is exactly one y such that $e^y = x$.

We would like to mimic the real variable case and say that a complex number w in $\mathbb{C}$ is called *a logarithm of a complex number* z in $\mathbb{C}$, denoted by $w = \log z$, if $e^w = z$ holds. Since $e^w \neq 0$ for any $w \in \mathbb{C}$, the number 0 has no logarithm; hence, the equation $e^w = z$ has no solution in $\mathbb{C}$ when $z = 0$ and $z = \infty$. Consider an arbitrary fixed $z \neq 0$ in polar form

$$z = |z| e^{i \operatorname{Arg} z} = re^{i\theta} \quad (r = |z| > 0, \ -\pi < \theta \leq \pi).$$

Let us now solve the equation $w = \log z$. If we let $w = u + iv$, u, v real, then $e^w = z$ becomes $e^{u+iv} = re^{i\theta}$ which yields

$$e^u = r, \quad e^{(v-\theta)i} = 1; \quad \text{i.e. } u = \ln r, \quad v = \theta + 2k\pi, \ k \in \mathbb{Z}.$$

Accordingly, we therefore have for $z \neq 0$,

$$w = \log z = \ln |z| + i(\operatorname{Arg} z + 2k\pi), \quad k \in \mathbb{Z}. \tag{3.94}$$

The *principal value* of a logarithm, sometimes denoted by Log, corresponds to the value of the principal value of the argument; that is for $z \neq 0$,

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z, \quad -\pi < \operatorname{Arg} z \leq \pi. \tag{3.95}$$

This agrees with the meaning of '$\operatorname{Log} z$' for positive real numbers $z = x$, for which $\operatorname{Arg} z = 0$, with which the reader is already familiar. We use the notation $*\log z$ to denote the infinite set of values

$$\{\ln |z| + i(\operatorname{Arg} z + 2k\pi) : k \in \mathbb{Z}\}.$$

It is often convenient to single out a particular member in this set. Thus, we formulate the following

3.96. Definition. If $w \in \mathbb{C}$ satisfies $e^w = z$ for an arbitrary fixed $z \neq 0$, then w defined by

$$w = *\log z = \begin{cases} \ln |z| + i(\operatorname{Arg} z + 2k\pi) \\ \quad \text{(or)} \\ \operatorname{Log} z + 2k\pi i \\ \quad \text{(or)} \\ \ln |z| + i \arg z, \end{cases} \tag{3.97}$$

is called the *logarithm* of z, where $k \in \mathbb{Z}$.

Note that $z \mapsto \operatorname{Log} z$ is a function defined in $\mathbb{C}\backslash\{0\}$ and every non-zero complex number has infinitely many logarithms which differ from each other by integral multiples of $2\pi i$. This shows that $w = \log z$ is not a function in general and is in fact an infinitely-valued association or relation, with infinitely many values for each $z \neq 0$. For each k, the strip of the type

$$D_k = \{u + iv : (2k-1)\pi < v \leq (2k+1)\pi, \ \ k \in \mathbb{Z}\}$$

is called a fundamental region for $\log z$ (Note that a region in the z-plane whose image just covers the w-plane once is called a *fundamental region* for the function $w = \log z$). Since the union is the w-plane, the set $\{D_k\}_{k=-\infty}^{\infty}$ is a complete set of fundamental regions for the logarithm. In particular, this observation proves that for each $k \in \mathbb{Z}$, $\exp(D_k) = \mathbb{C}\backslash\{0\}$. It is also easy to see that for each $k \in \mathbb{Z}$, $\exp : D_k \to \mathbb{C}\backslash\{0\}$ is one-to-one and onto.

We have

(a) $\operatorname{Log}(\pm i) = \pm i\pi/2$, $\operatorname{Log}(1+i) = \ln\sqrt{2} + i\pi/4$

(b) $\operatorname{Log}((1 \pm i)/\sqrt{2}) = \pm i\pi/4$, $\ln(-1) = i\pi$

(c) $\operatorname{Log}(2-3i) = \ln\sqrt{13} - i\operatorname{Arctan}(3/2)$

(d) $\operatorname{Log}(-2+3i) = \ln\sqrt{13} + i(\pi - \operatorname{Arctan}(3/2))$

(e) $\operatorname{Log} i^{1/4} = i\pi/8$, $\operatorname{Log}(\alpha z) = \ln\alpha + \operatorname{Log} z$ $(\alpha > 0)$.

From these examples, we can see that it is not even true in general that

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

when we limit our attention to principal values. However, for $z_1 \neq 0$ and $z_2 \neq 0$, the statements

$$\log(z_1 z_2) = \log z_1 + \log z_2 \pmod{2\pi} \quad \text{and} \quad \log(z_1/z_2) = \log z_1 - \log z_2 \pmod{2\pi}$$

hold. Suppose that $z_1 = |z_1|e^{i\operatorname{Arg} z_1}$ and $z_2 = |z_2|e^{i\operatorname{Arg} z_2}$. Then, for the complex numbers z_1, z_2, we write

$$z_1 z_2 = |z_1 z_2| e^{i\operatorname{Arg}(z_1 z_2)}.$$

There is a $k_1 \in \{-1, 0, 1\}$ (see Exercise 1.54(a)) such that

$$\begin{aligned}\operatorname{Log}(z_1 z_2) &= \ln|z_1 z_2| + i\operatorname{Arg}(z_1 z_2)\\ &= \ln|z_1| + \ln|z_2| + i(\operatorname{Arg} z_1 + \operatorname{Arg} z_2 + 2\pi k_1)\\ &= \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi i k_1.\end{aligned}$$

Thus,

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 \iff \operatorname{Arg} z_1 + \operatorname{Arg} z_2 \in (-\pi, \pi].$$

In particular, this statement leads to

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

for all $z_1, z_2 \in \mathbb{C}$ with $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$. A serious trouble ensues if the conditions $\operatorname{Re} z_1 > 0$ and $\operatorname{Re} z_2 > 0$ are dropped in the last equation. This point can also be checked on

taking $z_1 = i$ and $z_2 = i$. Similarly, we see that the condition $\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \in (-\pi, \pi]$ is met whenever $\operatorname{Im} z_1 < 0$ and $\operatorname{Im} z_2 > 0$. This means that

$$\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2$$

for all z_1, z_2 with $\operatorname{Im} z_1 < 0 < \operatorname{Im} z_2$.

In fact, by Exercise 1.54(b), we similarly have

$$\operatorname{Log}(z_1/z_2) = \operatorname{Log} z_1 - \operatorname{Log} z_2 + 2\pi i k_2,$$

where $k_2 = k_2(z_1, z_2)$ is defined in Exercise 1.54(b). For instance, let $z_1 = i$ and $z_2 = -1$. Then

$$\operatorname{Log} z_1 = i\pi/2, \ \ \operatorname{Log} z_2 = i\pi \ \ \text{and} \ \ \operatorname{Log}(z_1 z_2) = -i\pi/2.$$

Therefore, $\operatorname{Log}(z_1 z_2) = \operatorname{Log} z_1 + \operatorname{Log} z_2 - 2\pi i$ since $k_1(i, -1) = -1$. Similarly, we have

(a) $\operatorname{Log} 1 = \operatorname{Log}(-1) + \operatorname{Log}(-1) + 2\pi i$, since $k_1(-1, -1) = 1$.

(b) $\operatorname{Log}(-1-i) = \operatorname{Log} i + \operatorname{Log}(-1+i) - 2\pi i$, since $k_1(i, -1+i) = -1$.

(c) $\operatorname{Log}(r_1 r_2 e^{i3\pi/2}) = \operatorname{Log}(r_1 e^{i3\pi/4}) + \operatorname{Log}(r_2 e^{i3\pi/4}) - 2\pi i$.

It is often convenient to single out a particular set in the set $* \log z$. One immediate possibility is to consider the principal value of the logarithm $\operatorname{Log} z$ and to study analyticity of the function $\operatorname{Log} z$ whose domain of definition is $\mathbb{C}\backslash\{0\}$. For $z \neq 0$, let

$$z = re^{i\theta} \ \ \text{and} \ \ u + iv = \operatorname{Log} z,$$

where $|z| = r$ and $-\pi < \theta = \operatorname{Arg} z \leq \pi$. Recall that

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z.$$

If $\operatorname{Log} z = u(x, y) + iv(x, y)$, u, v real, then

$$u(r, \theta) = \ln r \ \ \text{and} \ \ v(r, \theta) = \theta. \tag{3.98}$$

The function $\operatorname{Log} z$ is not continuous at points on the negative real axis, since $\operatorname{Arg} z$ fails to possess a limit at the points along this axis; for, let $z_0 = x_0 < 0$. Then, for $z = x_0 + iy$ with $y > 0$, we have

$$\lim_{z \to z_0} \operatorname{Arg} z = \lim_{\substack{y \to 0 \\ y > 0}} \operatorname{Arg}(x_0 + iy) = \pi$$

and for $z = x_0 + iy$ with $y < 0$, we have

$$\lim_{z \to z_0} \operatorname{Arg} z = \lim_{\substack{y \to 0 \\ y < 0}} \operatorname{Arg}(x_0 + iy) = -\pi.$$

Since every neighborhood of z_0 intersects both second and third quadrants making it possible to approach z_0 in two different ways (see Figure 3.3), $\operatorname{Arg} z$ cannot converge. However, $\operatorname{Log} z$ is single-valued and continuous in the domain D, where $D = \mathbb{C}\backslash\{x + iy \in \mathbb{C} : y = 0, x \leq 0\}$. Now we can compute the derivative of the logarithm. From (3.98), it follows that

$$u_r = \frac{v_\theta}{r} = \frac{1}{r} \ \ \text{and} \ \ v_r = -\frac{u_\theta}{r} = \theta.$$

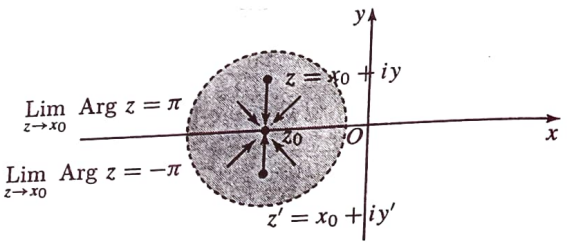

Figure 3.3: Description for continuity of $\operatorname{Arg} z$ at $z_0 = x_0 < 0$.

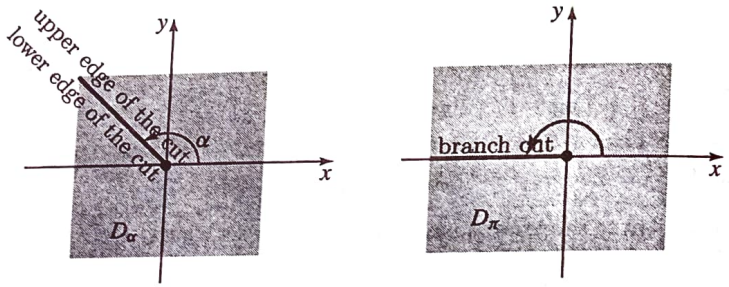

Figure 3.4: Slit plane $D_\alpha = \mathbb{C}\backslash\{Re^{i\alpha} : R > 0, \alpha - 2\pi < \arg_\alpha < \alpha\}$.

Thus u and v satisfy the C-R equations (3.47). Moreover, the partial derivatives u_r, v_r etc., are all continuous in D. This implies the existence of the derivative of $\operatorname{Log} z$ in D and so $\operatorname{Log} z$ is analytic therein. Since

$$\frac{dw}{dz} = e^{-i\theta}\frac{\partial w}{\partial r},$$

we readily have

$$\frac{d}{dz}(\operatorname{Log} z) = e^{-i\theta}\frac{\partial}{\partial r}(\ln r + i\theta) = \frac{e^{-i\theta}}{r} = \frac{1}{re^{i\theta}} = \frac{1}{z}, \quad z \in D.$$

In the same way, we see that $\arg_\alpha z$ with $z = re^{i\theta}$ $(r > 0,\ \alpha - 2\pi < \theta < \alpha)$, is continuous in the 'slit plane' $D_\alpha = \mathbb{C}\backslash\{Re^{i\alpha} : R \geq 0\}$ where $\alpha \in \mathbb{R}$ is fixed (see Figure 3.4). Thus, in D_α, the function $f_\alpha := \log_\alpha$ defined by

(3.99)
$$f_\alpha(z) := \log_\alpha z = \ln|z| + i\arg_\alpha z,$$

where $\arg_\alpha z$ denotes the unique choice of $\arg z$ in the interval $(\alpha - 2\pi, \alpha)$, is continuous with derivative

$$\frac{d}{dz}(f_\alpha(z)) = \frac{1}{z}, \quad z \in D_\alpha.$$

Clearly, f_α is not continuous at points on the ray $\{re^{i\alpha} : 0 \leq r < \infty\}$. Evidently, $\exp(f_\alpha(z)) = z$. For the analytic property of multiple-valued functions, we regard a multiple-valued function as a collection of single-valued functions. More explicitly we have

3.100. Definition. Suppose F is a multiple-valued function defined in S. A *branch* of F is a single-valued analytic function f in some domain $D \subset S$ obtained from F in such a way that at each point of D, f assumes exactly one of the possible values of F.

For example, from our discussion above, for each fixed $\alpha \in \mathbb{R}$, the function f_α defined by (3.99) represents a branch of $\log z$ in the slit plane D_α. We have created a cut along $\theta = \alpha$, so that the restriction on $\theta = \arg_\alpha z$ makes the function f_α single-valued and analytic. This cut corresponding to $\theta = \alpha$, i.e. the semi infinite line $\{Re^{i\alpha} : R \geq 0\}$, $\alpha \in \mathbb{R}$ fixed, is regarded as a *branch cut* for the branch f_α of $\log z$. For $\alpha = \pi$, f_α becomes $\operatorname{Log} z$ in D_π and is often called the *principal branch* of $\log z$. The *branch cut* for $\operatorname{Log} z$ is then the negative-real axis from the origin, i.e. $\{-R : R \geq 0\}$.

3.101. Definition. A multiple-valued function F defined in S is said to have a *branch point* at $z_0 \in \mathbb{C}$ if, when z describes an arbitrarily small circle about z_0, then for every branch f of F, $f(z)$ does not return to its original value.

For example every $z \neq 0$ is not a branch point of $\log z$, since on a sufficiently small circle enclosing the point z, every branch of $\log z$ returns to its original value.

3.102. Definition. A function $f \in \mathcal{H}(D)$, where D is a domain D, is called an *analytic/holomorphic branch* of $\log z$ in D if f is analytic/holomorphic in D and $\exp(f(z)) = z$ for each $z \in D$.

Let $\phi \in \mathcal{H}(D)$ such that $0 \notin \phi(D)$. We say $f \in \mathcal{H}(D)$ is an *analytic-logarithm* of ϕ if $\exp(f(z)) = \phi(z)$ for each $z \in D$. Clearly, if $\phi(z) = z$ in D then the concept of analytic-logarithm coincides with that of the (analytic) branch of $\log z$. Analytic/holomorphic branches of $\log \phi(z)$ are called *logarithmic functions.*

For instance, in the slit plane D_α, the function f_α defined by (3.99) is an example of a logarithmic function.

3.103. Theorem. *Let D be a domain in $\mathbb{C}$. Then, any two logarithmic functions f, $\tilde{f} : D \to \mathbb{C}$ are related by*

(3.104)
$$\tilde{f}(z) = f(z) + 2\pi k i$$

for some $k \in \mathbb{Z}$. Conversely, if a logarithmic function $f : D \to \mathbb{C}$ is related by (3.104) with $\tilde{f} : D \to \mathbb{C}$, then $\tilde{f}$ is also a logarithmic function.

Proof. Let f and $\tilde{f}$ be two logarithmic functions in D. Then by definition

$$\exp\{\tilde{f}(z)\} = \exp\{f(z)\}, \quad \text{i.e.} \quad \exp\{\tilde{f}(z) - f(z)\} = 1 \quad \text{for all } z \in D,$$

which implies that

$$F(z) = \frac{\tilde{f}(z) - f(z)}{2\pi i} \in \mathbb{Z}.$$

Note that F is an analytic function in D and, in particular, F is continuous in the open and connected set D. But F is an integer-valued and so F maps a connected set onto a connected subset of $\mathbb{Z}$, viz. a single point, k, say. Thus, f and $\tilde{f}$ are related by (3.104).

Conversely, if f is a logarithmic function in D and $\tilde{f}: D \to \mathbb{C}$ is related by (3.104) for some $k \in \mathbb{Z}$ then for all $z \in D$, we have

$$\exp\{\tilde{f}(z)\} = \exp\{f(z)\}\exp\{2\pi ik\} = \exp\{f(z)\} = z$$

and so $\tilde{f}$ becomes a logarithmic function in D, by definition.

Theorem 3.103 shows that, given a branch of logarithm in a domain, one can obtain all the branches of logarithm on that domain. For example, we let $\alpha = 3\pi/2$ and consider (3.99). Then $f_{3\pi/2}$ is an analytic branch of $\log z$ for $z \in D_{3\pi/2}$. Thus to find a numerical value of $f_{3\pi/2}(z_0)$, where $z_0 = -1 - i$, we write

$$f_{3\pi/2}(-1-i) = \ln\sqrt{2} + i\arg_{3\pi/2}(-1-i) = \ln\sqrt{2} + \frac{5\pi i}{4}.$$

From Theorem 3.103, we also note that the value of $f_{3\pi/2}(-1-i)$ for any other branch of $\log z$ which is analytic in the slit plane $D_{3\pi/2}$ is given by

$$\ln\sqrt{2} + \frac{5\pi i}{4} + 2k\pi i, \quad \text{for some } k \in \mathbb{Z}.$$

As another characterization of logarithmic functions, we have

3.105. Theorem. *Let $f: D \to \mathbb{C}$ be analytic in a domain D not containing 0. Then f is a branch of $\log z$ in D iff $f'(z) = 1/z$ for all $z \in D$ and $\exp\{f(a)\} = a$ for at least one $a \in D$.*

Proof. Suppose that f is a branch of $\log z$ in D. Then, $\exp\{f(z)\} = z$ for all $z \in D$ so that

$$f'(z)\exp\{f(z)\} = 1, \quad \text{i.e.} \quad f'(z) = 1/z \quad \text{for all } z \in D.$$

To prove the converse, we need to show that

$$\exp\{f(z)\} = z \quad \text{for all } z \in D. \tag{3.106}$$

To do this, we consider $g(z) = z\exp\{-f(z)\}$. Then g is analytic in D and satisfies

$$g'(z) = [1 - zf'(z)]\exp\{-f(z)\} = 0 \quad \text{for all } z \in D.$$

By Theorem 3.31, g is a constant, say k. Thus, $z\exp\{-f(z)\} = k$ for all $z \in D$. Since $\exp\{f(a)\} = a$, we must have $k = 1$. ∎

3.107. Example. We now demonstrate that there can be no branch of logarithm in the domain $\Omega = \mathbb{C} \setminus \{0\}$. This fact can be proved by a number of ways.

Suppose on the contrary that $f(z)$ is a branch of $\log z$ in Ω. The restriction of f to the slit plane $D_\pi = \mathbb{C} \setminus (-\infty, 0]$ is then a branch of logarithm in D_π. As $\operatorname{Log} z$ is the principal branch in D_π, Theorem 3.103 shows that for $z \in D_\pi$, f must be of the form

$$f(z) = \operatorname{Log} z + 2k\pi i$$

for some $k \in \mathbb{Z}$. But then, f being analytic in Ω, f is continuous at $x_0 < 0$ so that

$$f(x_0) = \lim_{\substack{y\to 0\\ y>0}} f(x_0 + iy) = \lim_{\substack{y\to 0\\ y>0}} \operatorname{Log}(x_0 + iy) + 2k\pi i = (2k+1)\pi i$$

and

$$f(x_0) = \lim_{\substack{y\to 0\\ y<0}} f(x_0 + iy) = \lim_{\substack{y\to 0\\ y<0}} \operatorname{Log}(x_0 + iy) + 2k\pi i = (2k-1)\pi i$$

which is a contradiction. Thus, no such f can exist. ●

It follows from Theorem 3.105 that every branch f of $\log z$ in D is infinitely differentiable in D. A natural question is whether f can be represented by a power series valid in appropriate disks in D. For example, the following result conveniently stated in a different form shows that

$$\operatorname{Log} z = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n}(z-1)^n \quad \text{for } |z-1| < 1.$$

3.108. Theorem. *In the unit disk Δ, the power series $\sum_{n\geq 1}(z^n/n)$ represents the logarithmic function $-\operatorname{Log}(1-z)$.*

Proof. Let $f(z) = -\operatorname{Log}(1-z)$. Note that the radius of convergence of the given series is 1 and so by Theorem 3.71, it represents an analytic function, say $g(z)$, in the unit disk $\Delta = \{z : |z| < 1\}$. Then, as we have seen earlier, $f'(z) = (1-z)^{-1}$ and since the power series can be differentiated term-by-term for $z \in \Delta$, by Theorem 3.71, we must have

$$g'(z) = \sum_{n\geq 1} z^{n-1} = \frac{1}{1-z}$$

so that $f'(z) - g'(z) = 0$. Therefore, by Theorem 3.31, $f - g$ is constant in Δ. As $f(0) = 0 = g(0)$, it follows that $f(z) - g(z) = 0$ in Δ. ∎

3.109. Example. Suppose we wish determine the sum of the series $1 + \sum_{n=2}^{\infty}(z-4)^n/n$. We know that the series converges for $|z-4| < 1$ and so, we may let its sum by $f(z)$:

$$f(z) = 1 + \sum_{n=2}^{\infty} \frac{1}{n}(z-4)^n, \quad |z-4| < 1.$$

The term-by-term differentiation gives

$$f'(z) = \sum_{n=2}^{\infty}(z-4)^{n-1} = \frac{z-4}{1-(z-4)} = -1 + \frac{1}{5-z}, \quad |z-4| < 1.$$

Antiderivative gives

$$f(z) = -z - \log_\alpha(5-z) + c$$

where the branch cut must lie to the right of the disk $|z-4| < 1$, i.e. in the half-plane $\operatorname{Re} z > 5$ (see Figure 3.5). As $f(4) = 1$, we see that $c = 5 + \log_\alpha 1$ which gives

$$f(z) = -(z-5) - \log_\alpha(5-z) + \log_\alpha 1.$$

For instance if $\alpha = 0$, then $\log_0 1 = \ln 1 + i\operatorname{Arg} 1 = 0$. In this choice, we get

$$f(z) = -(z-5) - \operatorname{Log}(5-z).$$ ●

We conclude this section with a discussion on the generalized 'power functions'. Given $0 \neq z \in \mathbb{C}$, the principal value of z^a, i.e. the a-th power of z, is defined by $z^a = \exp(a\operatorname{Log} z)$

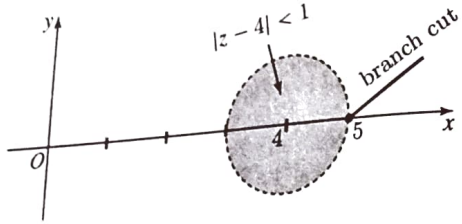

Figure 3.5: Choosing suitable branch.

and this is what most mathematicians mean by z^a on most occasions, especially if z is real and positive. The other values of z^a could be obtained from $\exp(a \log z)$ in which case z^a is multi-valued because $\arg z$ (and hence $\log z$) is multiple-valued.

When a is an integer, $\exp(a \log z) = z^a$ is single-valued. For let $a = n$, n an integer. Then, because $\log z = \ln|z| + i(\operatorname{Arg} z + 2\pi k)$.

$$\begin{aligned} z^a &= e^{n[\ln|z| + i(\theta + 2\pi k)]} \quad (\theta = \operatorname{Arg} z,\ k \in \mathbb{Z}) \\ &= e^{n \operatorname{Log} z} e^{i2\pi(kn)} = e^{n \operatorname{Log} z}, \end{aligned}$$

and hence z^a produces the single point. Note that $\operatorname{Log} z$ is single-valued. Conversely, z^a is single-valued only if a is an integer.

When a is a real rational number with the reduced form m/n, then $\exp(a \log z)$ produces exactly n distinct values and may be represented as $* \sqrt[n]{z^m}$ as in Remark 1.25. For,

$$z^a = e^{(m/n)[\operatorname{Log} z + i2k\pi]} = e^{(m/n)\operatorname{Log} z} e^{i(m/n)2k\pi}, \quad k \in \mathbb{Z}$$

and since $e^{(m/n)\operatorname{Log} z}$ is single-valued and $e^{i(m/n)2k\pi}$ are the n-th roots of unity, we infer that z^a has exactly n values corresponding to $k = 0, 1, 2, \ldots, n-1$.

When a is an irrational (real) number or an imaginary number, then z^a is infinite-valued. That is the set of all values of z^a is an infinite set. For example,

$$1^{\sqrt{2}} = \exp\{\sqrt{2}(\operatorname{Log} 1 + i2k\pi)\} = e^{2\sqrt{2}k\pi i}, \quad k \in \mathbb{Z}.$$

The values corresponding to different k's are distinct.

As another example all the values of i^i are obtained from the expression

$$\exp\{i(\operatorname{Log} i + 2k\pi i)\} = \exp\{i((\pi i/2) + 2k\pi i)\} = e^{-(4k+1)\pi/2},$$

where $k \in \mathbb{Z}$, and so $e^{-\pi/2}$ is the principal value of i^i. Similarly, all the values of $(-i)^i$ and $i^{1/\pi}$ are obtained from

$$(-i)^i = e^{-(4k-1)\pi/2} \quad \text{and} \quad i^{1/\pi} = e^{i(4k+1)/2},$$

where $k \in \mathbb{Z}$, respectively.

We emphasize that the principal branch and other branches of z^a are obtained from $\operatorname{Log} z$ and $\log z$, respectively. Different branches of $\log z$ yields different branches of z^a. For instance, $z^a = e^{a \operatorname{Log} z}$ defines a principal branch of z^a which is analytic in $D_\pi = \mathbb{C} \setminus \{z : z = x, x \le 0\}$. More generally, we have

3.110. Theorem. *If we choose f_α defined by (3.99) to be a branch of $\log z$, then, in the cut plane D_α, $(z^a)' = az^{a-1}$ (same branches).*

Proof. By hypothesis, $z^a = \exp(af_\alpha(z))$ is analytic in the cut plane D_α, where $\alpha \in \mathbb{R}$ is fixed. By the Chain rule

$$\frac{d}{dz}(z^a) = e^{af_\alpha(z)}\left(\frac{a}{z}\right) = e^{af_\alpha(z)}\left(\frac{a}{e^{f_\alpha(z)}}\right) = ae^{(a-1)f_\alpha(z)} = az^{a-1}$$

which gives the result. ∎

3.111. Corollary. *For $n \in \mathbb{N}$, $z^{1/n}$ is analytic in the domain of a branch of $\log z$ and $(z^{1/n})' = (1/n)z^{1/n-1}$.*

Observe that if $a = 1/n$ $(n \ge 2)$, then

$$\left[\exp\left(\frac{1}{n}f_\alpha(z)\right)\right]^n = \exp(f_\alpha(z)) = z$$

making $\exp((1/n)f_\alpha(z))$ a branch of the n-th root function in D_α, the branch of the n-th root function in D_α associated with $f_\alpha(z)$. A proof similar to Theorem 3.108 gives

3.112. Theorem. (Binomial Series) *For $z \in \Delta = \{z : |z| < 1\}$,*

$$(1+z)^a = 1 + \sum_{n \ge 1} \frac{a(a-1)\cdots(a-n+1)}{n!} z^n,$$

where a is an arbitrary complex number.

Proof. Since the radius of convergence of the series is 1, the sum converges for all z in the unit disk Δ (and defines an analytic function in Δ). Call it $f(z)$. Now the power series can be differentiated term-by-term. Thus,

$$(1+z)f'(z) = af(z).$$

Further the function $(1+z)^a = \exp\{a \operatorname{Log}(1+z)\}$ is well-defined and analytic in Δ. Let

$$g(z) = \frac{f(z)}{(1+z)^a} = f(z)\exp\{-a \operatorname{Log}(1+z)\}.$$

Thus, g is analytic in Δ and

$$\begin{aligned} g'(z) &= f'(z)\exp\{-a \operatorname{Log}(1+z)\} - f(z)\exp\{-a \operatorname{Log}(1+z)\}\frac{a}{1+z} \\ &= \exp\{-a \operatorname{Log}(1+z)\}\left[f'(z) - \frac{a}{1+z}f(z)\right] = 0. \end{aligned}$$

Theorem 3.31(i) gives that $g(z)$ is constant in Δ. So, $f(z) = k(1+z)^a$, where k is some constant. Since $f(0) = 1$ and since $1^a = e^{a \operatorname{Log} 1} = e^0 = 1$, it follows that $k = 1$. ∎

Note. Unless otherwise stated explicitly, from now onwards, z^a will denote the value of $\exp\{a \operatorname{Log} z\}$. Further, in the particular case $z = e$ the expression $z^a \equiv e^a$ is single-valued for all values of a, since $e^a = e^{\operatorname{Re} a + i \operatorname{Im} a}$ is defined to be

$$e^{a \operatorname{Log} e} = e^{(\operatorname{Re} a + i \operatorname{Im} a)\operatorname{Log} e} = e^{\operatorname{Re} a \operatorname{Log} e}(\cos(\operatorname{Im} a \operatorname{Log} e) + i \sin(\operatorname{Im} a \operatorname{Log} e)),$$

which is clearly a unique complex number. This note is to caution the reader when comparing with Euler's notation (see Equation (3.81)).

3.6 Inverse Functions

The *inverse trigonometric* function $w = \text{trig}^{-1} z$ or $\text{arctrig}\, z$ is defined by the equation

$$z = \text{trig}\, w. \tag{3.113}$$

Similarly we define the *inverse hyperbolic* function $w = \text{hyp}^{-1} z$ or $\text{archyp}\, z$ by the equation

$$z = \text{hyp}\, w. \tag{3.114}$$

Here 'trig w' ("hyp w") denotes any of the trigonometric (hyperbolic) functions $\sin w, \cos w$ etc. ($\sinh w, \cosh w$, etc.). Recall that, as e^z is $2\pi i$-periodic, defining the logarithm function as its inverse led to a multivalued function. As trigonometric and hyperbolic functions are periodic, it is natural to expect a similar situation when we define their inverses. Indeed, we will see that the functions w defined by the equations (3.113) and (3.114) are multiple-valued, since trig and hyp functions are periodic. For instance, the fact that

$$\cos w = \cos(-w)$$

shows that the first and third quadrants, and the second and fourth quadrants map into the same regions by the cosine. First, we wish to show that given a complex number z, there exist infinitely many solutions of $z = \sin w$. To find w, in terms of z, we note that

$$z = \frac{e^{iw} - e^{-iw}}{2i}, \quad \text{i.e.} \quad (e^{iw})^2 - 2ize^{iw} - 1 = 0,$$

which is a quadratic equation in e^{iw}. Note that $e^{iw} \neq 0$. Solving for e^{iw}, we have

$$e^{iw} = iz + Z_j, \quad j = 1, 2,$$

where Z_1 and Z_2 denote the two numbers such that $Z_1^2 = Z_2^2 = 1 - z^2$. Therefore,

$$w = \frac{1}{i}\log(iz + Z_j), \quad j = 1, 2$$

(This also gives that $\sin(\mathbb{C}) = \mathbb{C}$). This function is called the *inverse sine function* and is denoted by $\sin^{-1}(z) := \arcsin z$:

$$w = \sin^{-1}(z) = -i\log(iz \pm \sqrt{1 - z^2}). \tag{3.115}$$

For instance, if $z = 0$, (3.115) becomes $w = -i\log(\pm 1)$. If we select $+1$ and the principal value for $\log 1$, we have $w = 0$ and if we select -1 and principal value for $\log(-1)$, we have $w = -i\pi$. Therefore if Z denotes any of the two values Z_1, Z_2,

$$\begin{aligned} \sin w &= \sin(-i\log(iz + Z)) \\ &= \frac{\exp(\log(iz + Z)) - \exp(-\log(iz + Z))}{2i} \\ &= \frac{1}{2i}\left[(iz + Z) - \frac{1}{iz + Z}\right] = z. \end{aligned}$$

Note that $\log 1 = 2k\pi i$ and $\log(-1) = \pi i + 2k\pi i$ ($k \in \mathbb{Z}$) also leads us to $\sin w = z$. This shows that any of the values

$$-i\log(iz + Z_1) + 2k\pi, \quad -i\log(iz + Z_2) + 2k\pi \quad (k \in \mathbb{Z})$$

are the values and the only values of $\sin^{-1} z$. Because of this, for a given z, there are an infinite number of solutions for the equation $z = \sin w$ (and so for (3.113) and (3.114) in general). For instance,

$$\sin^{-1}(i) = -i\log(-1 \pm \sqrt{2}) = -i[\text{Log}(-1 \pm \sqrt{2}) + 2k\pi i], \quad k \in \mathbb{Z},$$

so that

$$\sin^{-1}(i) = \begin{cases} -i[\ln(1 + \sqrt{2}) + (2k+1)\pi i] & \text{corresponding to } - \text{ sign}, \\ -i[\ln(1/(\sqrt{2} + 1)) + 2k\pi i] & \text{corresponding to } + \text{ sign}. \end{cases}$$

Therefore (since $\ln(1/|x|) = -\ln|x|$),

$$\sin^{-1}(i) = i(-1)^k \ln(\sqrt{2} + 1) + k\pi, \quad k \in \mathbb{Z}.$$

Similarly, all possible values of $w = \sin^{-1}(2)$ are given by

$$w = \frac{1}{i}\left[\ln(2 \pm \sqrt{3}) + i(\pi/2 + 2k\pi)\right] = (2k + 1/2)\pi \pm i\ln(2 + \sqrt{3}), \quad k \in \mathbb{Z}$$

(Note that $(2 + \sqrt{3})(2 - \sqrt{3}) = 1$ and so $\ln(2 - \sqrt{3}) = -\ln(2 + \sqrt{3})$).

The idea used to derive the values of $\sin^{-1} z$ would help us to derive the following:

$$\begin{aligned} \cos^{-1} z &= -i\log(z \pm \sqrt{z^2 - 1}) = -i\log(z + i\sqrt{1 - z^2}) \\ \tan^{-1} z &= \frac{1}{2i}\log\left(\frac{1 + iz}{1 - iz}\right) \\ \sinh^{-1} z &= \log(z + \sqrt{z^2 + 1}) \\ \cosh^{-1} z &= \log(z + \sqrt{z^2 - 1}) \\ \tanh^{-1} z &= \frac{1}{2}\log\left(\frac{1 + z}{1 - z}\right), \end{aligned}$$

where $\sqrt{z^2 - 1}$, $\sqrt{1 - z^2}$ stands for one of the complex numbers Z such that $Z^2 = z^2 - 1$ or $Z^2 = 1 - z^2$ as the case may be. Besides '=' signifies one of the values.

The inverse trig and hyp functions can be made single-valued by choosing a particular branch of the log function (and the particular branch of the square root function if necessary). When the principal branch of log is chosen in the formulas trig^{-1} and hyp^{-1}, the resulting single-valued function is called the principal branch of trig^{-1} and hyp^{-1}, respectively. For example, with $\sqrt{1 - z^2} = \exp((1/2)\,\text{Log}(1 - z^2))$ the principal square root function, we have

$$\text{Sin}^{-1} z := \text{Arcsin}\, z = -i\,\text{Log}(iz + \sqrt{1 - z^2}),$$

is called *the principal branch* of $\sin^{-1} z := \arcsin z$. The derivatives of the (single-valued) inverse trig and hyp functions can be obtained from the above formulas or from differentiating $z = \text{trig}\, w$ and $z = \text{hyp}\, w$ implicitly (see Exercise 3.141).

3.7 Exercises

3.116. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

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(a) The function $f : D \longrightarrow \mathbb{C}$ is analytic at $z_0 \in D$ if and only if there is a $\delta > 0$ such that if $\{z_n\}$ is a sequence in $\Delta(z_0;\delta)\setminus\{z_0\}$ with $z_n \to z_0$, then

$$\lim_{n\to\infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0).$$

(b) The function $f(x+iy) = x^3 + ax^2y + bxy^2 + cy^3$ is analytic in $\mathbb{C}$ only if $a = 3i$, $b = -3$ and $c = -i$.

(c) If $f = u + iv$ is analytic in a domain D such that $au + bv$ is constant, where a, b are real constants (not both zero), then f is constant.

(d) Let D be a domain in $\mathbb{C}$ and $\overline{D} = \{z \in \mathbb{C} : \overline{z} \in D\}$. Then, $f \in \mathcal{H}(D)$ iff $g \in \mathcal{H}(\overline{D})$, where $g(z) = \overline{f(z)} = \overline{f}(\overline{z})$.

(e) Let f_j $(j = 1, 2, \ldots, n)$ be analytic in a domain D such that $\sum_{j=1}^{n} |f_j(z)|^2$ is constant in D. Then each f_j is a constant function.

(f) If u is a real-valued function in a disk Δ_R such that $u^{-1} + iu$ is analytic in Δ_R, then u is constant throughout the disk.

(g) There exists a homeomorphism (bijective and bicontinuous map) between the unit disk Δ and the upper half-plane.

(h) The mean value theorem of a real-valued function of a real variable does not hold in general for complex-valued functions.

(i) A real-valued function u in domain D cannot be analytic in D unless it is a constant function.

(j) There exists a non-constant real-valued function u in an open set D which is analytic in D.

(k) The function $f(x+iy) = x^2 - y^2 + x + i(2xy + y)$ represents an analytic function in $\mathbb{C}$ whereas $g(x+iy) = x^2 + y^2 + x + i(2xy + y)$ is not.

(l) If f and g are functions that satisfy the C-R equations at a point $a \in \mathbb{C}$, then $f + g$ and fg also satisfy the C-R equations at a.

(m) The function f defined by

$$f(z) = u + iv = \begin{cases} \dfrac{\operatorname{Im}(z^2)}{\overline{z}} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

satisfies the C-R equations at the origin, yet it is not differentiable there.

(n) The function f defined by

$$f(z) = \begin{cases} 0 & \text{if } z = 0 \\ \exp(-1/z^4) & \text{if } z \neq 0 \end{cases}$$

is not continuous at the origin but satisfies the C-R equations at the origin.

(o) The function $f(z)$ is analytic in a domain D iff both real and imaginary parts of $f(z)$ and $zf(z)$ are harmonic in D.

(p) If Ω is a domain which is symmetric about the real axis and if f is differentiable at $a \in \Omega$ as well as at $\overline{a} \in \Omega$, then $f(\overline{z})$ is not differentiable at a.

(q) The function $f(x+iy) = x^3 + i(y-1)^3$ is nowhere analytic but is continuous in $\mathbb{C}$ and differentiable on $\{x + iy : x + y = 1\} \cup \{x + iy : x - y = -1\}$.

(r) An entire function f such that $f(x+iy) = u(x) + iv(y)$ must be of the form $f(z) = \alpha z + \beta$ with $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{C}$.

(s) If $f = u + iv$ and $g = u + iV$ are two analytic functions defined in a domain Ω, then $\operatorname{Im}(f - g)$ is necessarily a constant.

(t) If $f(z)$ and $\overline{f(z)}$ are analytic functions in a domain, then f is necessarily a constant.

(u) There exist no analytic functions f such that $\operatorname{Re} f(z) = y^2 - 2x$.

(v) There exist no analytic functions f such that $\operatorname{Im} f(z) = x^3 - y^3$.

(w) An analytic function $f = u + iv$ in a domain Ω such that $v = u^2$ is necessarily a constant.

(x) A real-valued function $u(x, y)$ is harmonic in D iff $u(x, -y)$ is harmonic in D.

(y) If $f : D \to D'$ is analytic and $u : D' \to \mathbb{R}$ is harmonic, then the composition $u \circ f$ is harmonic in D.
Note: If $u(x, y)$ is harmonic in $\mathbb{R}^2$, then $u(x^2 - y^2, 2xy)$ is harmonic in $\mathbb{R}^2$. Similarly, if $f : D \to \mathbb{C}\setminus\{0\}$ is analytic then $\log|f(z)|$ is harmonic in D. In particular, if $f(z) = z$ then $\log|z|$ is harmonic in $\mathbb{C}\setminus\{0\}$.

(z) Let $\{f_n(z)\}$ be a sequence of analytic functions in a domain D such that $f_n \to f$ uniformly in D. Then, $f_n' \to f'$ uniformly in D.

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3.117. Determine whether each of the following statements is true or false. Justify your answer with a proof or a counterexample.

(a) $\lim_{n\to\infty}(n!)^{1/n} = \infty$.

(b) If $a_n \geq 0$, $b_n \geq 0$ and $b_n \to b$, then $\limsup_{n\to\infty} a_n b_n = b \limsup_{n\to\infty} a_n$.

(c) The set of all power series with positive radius of convergence forms a vector space over the field $\mathbb{C}$.

(d) A power series has an anti-derivative everywhere inside its disk of convergence.

(e) If R is the radius of convergence of $\sum_{n\geq 0} a_n z^n$, then the radii of convergence of $\sum_{n\geq 0} a_n z^{2n}$ and $\sum_{n\geq 0} a_n^2 z^n$ are $\sqrt{R}$ and R^2, respectively.

(f) The radius of convergence of $\sum_{n\geq 0}(\operatorname{Re} a_n) z^n$ is always greater than or equal to the radius of convergence of $\sum_{n\geq 0} a_n z^n$.

(g) If $k \geq 1$ is a fixed positive integer, then the series $\sum_{n\geq 1} \frac{z^{kn}}{n}$ converges for $|z| = 1$ except when $z = \omega_j$, $j = 0, 1, \ldots, k-1$, the k-th roots of unity.

(h) The radius of convergence of the series $\sum_{n\geq 1} \frac{(-1)^n}{n(n+1)} z^{n(n+1)}$ is 1.

(i) The series $\sum_{n\geq 0}((z+1)/(z-1))^n$ converges for $\operatorname{Re} z \leq 0$ and diverges for $\operatorname{Re} z > 0$.

(j) The series $\sum_{n=0}^{\infty} \left(\frac{z}{1-z}\right)^n$ converges for $\operatorname{Re} z < 1/2$.

(k) The series $\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^4}{z^n}\right)$ converges for $|z| > 1$ and diverges everywhere else.

(l) Each of the series $\sum_{n=1}^{\infty} \frac{nz^n}{1-z^n}$ and $\sum_{n=1}^{\infty} \frac{z^n}{(1-z^n)^2}$ converges uniformly in the closed disk $|z| \leq r$, $r \in (0, 1)$. The two series are actually the same.

(m) If f is analytic in a domain D and if a, b are constants, then $f(z) = ae^{bz}$ and $f'(z) = bf(z)$ are equivalent statements in D.

(n) e^z assumes all values except zero; i.e., the equation $e^z = \omega$ is solvable for any $0 \neq \omega \in \mathbb{C}$.

(o) For $z = x + iy$,

(i) $e^z > 0$ when y is an even multiple of π

(ii) $e^z < 0$ when y is an odd multiple of π

(iii) e^z is purely imaginary when y is an odd multiple of $\pi/2$

(iv) $|e^{iz}| < 1$ when $y > 0$

(v) $|e^{iz}| > 1$ when $y < 0$

(vi) $|e^z - 1| \leq e^{|z|} - 1 \leq |z|e^{|z|}$ for any $z \in \mathbb{C}$

(vii) $|\exp(z^2)| \leq \exp(|z|^2)$

(viii) $|\sin z|^2 + |\cos z|^2 = 1 \iff z = x$, a real.

(p) For $z = x + iy$,

$$\sinh y \leq \left|\begin{matrix}\cos\\ \sin\end{matrix}(x \pm iy)\right| \leq \cosh y.$$

(q) If f is any one of the six hyperbolic (circular) functions, then, for all $z \in \mathbb{C}$, $\overline{f(z)} = f(\bar{z})$.

(r) Each of the trigonometric, hyperbolic functions $\cos z$, $\sin z$, $\cosh z$ and $\sinh z$ takes every value $c \in \mathbb{C}$ countably many times.

(s) For x real, $\cos^{-1}(x) + \sin^{-1}(x) = \pi/2$.

(t) The roots of the equation $\tan z + \cot z = 2$ are at $z = (4k+1)\pi/4$, $k \in \mathbb{Z}$.

(u) If $f(z) = e^{1/z}$, then $|f(z)|$ is constant on $|z - r| = r$ $(r > 0)$.

(v) If $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that $\operatorname{Re}(z_k) > 0$ and $\operatorname{Re}(z_1 \cdots z_k) > 0$ for each $k = 1, 2, \ldots, n$, then $\operatorname{Log}\left(\prod_{k=1}^n z_k\right) = \sum_{k=1}^n \operatorname{Log} z_k$.

(w) $\operatorname{Re} z^{1/2} > 0$ for all $z \in \mathbb{C}\setminus\{0\}$.

(x) If $f(z) = z^i = e^{i \operatorname{Log} z}$ (principal branch is chosen), then there exists a constant $C > 0$ such $|f(z)| \leq C$ for all z in the domain of definition of f.
Note: What happens if i in this problem is replaced by ib, where b is finite real number.

(y) There does not exist an analytic function f in a neighborhood of 0 whose square is z.

(z) For $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$, we have $|z^a| \leq |z|^{\operatorname{Re} a} e^{\pi|\operatorname{Im} a|}$.

3.118. Let $f(z) = x^2 + iy^2$. Does it satisfy the C-R equations at the origin? Is f differentiable at the origin? Is f analytic at the origin?

3.119. If $f(z) = |x^2 - y^2| + 2i|xy|$, then show that

$$f(z) = \begin{cases} z^2 & \text{if } 0 < \operatorname{Arg} z < \pi/4 \text{ and } -\pi < \operatorname{Arg} z < -3\pi/4 \\ -z^2 & \text{if } \pi/2 < \operatorname{Arg} z < 3\pi/4 \text{ and } \quad -\pi/2 < \operatorname{Arg} z < -\pi/4. \end{cases}$$

3.120. An entire function $f = u + iv$ has the feature that $u_x v_y - u_y v_x = 1$ in $\mathbb{C}$. Demonstrate that f has the form $f(z) = az + b$, where a and b are some constants with $|a| = 1$.

3.121. Suppose that $f = u + iv$ is entire such that $u_x + v_y = 0$ in $\mathbb{C}$. Demonstrate that f has the form $f(z) = az + b$ where a, b are some constants with $\operatorname{Re} a = 0$.

3.122. Suppose that $f(z)$ is defined in the unit disk Δ such that both $f^2(z)$ and $f^3(z)$ are analytic in Δ. Prove or disprove that $f(z)$ is analytic in Δ.

3.123. If f is analytic in a neighborhood N of the origin such that $f(z_1 + z_2) = f(z_1) + f(z_2)$ for all $z_1, z_2 \in N$, then show that $f(z) = az$ for some complex constant a.

3.124. If $a \in \mathbb{R}$, then show that $u(x, y) = e^{-2axy}\cos a(x^2 - y^2)$ is harmonic in $\mathbb{R}^2$. Find all its harmonic conjugates $v(x, y)$ in $\mathbb{R}^2$. Write $f = u + iv$ as a function of z with $f(0) = 1$.

3.125. If $f \in \mathcal{H}(\Omega)$ and $|f|$ is harmonic in Ω, then show that f is a constant function.

3.126. Is it always possible to find a function v which is a harmonic conjugate to u in the same domain where u is harmonic? Justify your answer by giving an example.

3.127. Let R_1 and R_2 be the radius of convergence of the power series $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n\geq 0} b_n z^n$, respectively. Find the radius of convergence of $f(z) \pm g(z)$ and $f(z)g(z)$. Also find the radius of convergence of $\sum_{n\geq 0} n^\alpha a_n z^n$, and $\sum_{n\geq 0} \alpha^n a_n z^n$, where $\alpha > 0$.

3.128. Find the region of convergence and uniform convergence of the series:

(i) $\displaystyle\sum_{n\geq 0}\left(\frac{z}{1+z}\right)^n$ (ii) $\displaystyle\sum_{n\geq 0}\frac{z^n}{1+z^{2n}}$.

3.129. If $f(z) = z(1-z)^{-2}$, then use the relation $(1-z)^2 f(z) = z$ to compute the Maclaurin series of $f(z)$.

3.130. We know that $f(z) = e^z$ is a solution of the differential equation $f'(z) - f(z) = 0$. Suppose that g is a solution of the first order differential equation $g'(2z) + g(z) = 0$ such that g is analytic at 0. Then show that g is entire.

3.131. Let $a \in \mathbb{R}$ be fixed. Find the set of $z \in \mathbb{C}$ for which $\sum_{n=1}^\infty n^{i(z^2+a)}$ represents an analytic function.

3.132. Using the 'ϵ-δ' definition verify the continuity of $f(z) = z^2$ at $z_0 \in \mathbb{C}$. For $z \neq 0$, let $z = re^{i\theta}$ $(r = |z|,\ \theta = \operatorname{Arg} z)$. Find the region of continuity for $g_1(z) = \sqrt{r}e^{i\theta/2}$ and $g_2(z) = -i\sqrt{r}e^{i\theta/2}$.

3.133. If z_1 and z_2 satisfy one of the following conditions

(a) $\operatorname{Re} z_1 > 0$, $\operatorname{Re} z_2 > 0$

(b) $\operatorname{Im} z_1 < 0 < \operatorname{Im} z_2$

(c) $\operatorname{Im}(z_1 z_2) > 0$, $\operatorname{Im}(z_1 z_2) \geq 0$

(d) $\operatorname{Im}(z_1 z_2) < 0$, $\operatorname{Im} z_2 < 0$,

then show that $z_1^a z_2^a = (z_1 z_2)^a$ (Assume that all the powers involved have their principal values). **Note:** If $z_1 = 1 = z_2 = -1$, then $\sqrt{z_1 z_2} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$ whereas $\sqrt{z_1}\sqrt{z_2} = \sqrt{-1}\sqrt{-1} = i \cdot i = -1$. Thus, one needs to be careful when performing algebraic manipulations with real/complex powers.

3.134. Let $z \notin (-\infty, 0]$. Using the principal logarithms on each side, show that $\overline{z^a} = (\bar{z})^{\bar{a}}$.

3.135. Suppose $f(z) = z^z$. Using the principal logarithm find $f'(z)$, $f'(i)$, $\operatorname{Re} f(z)$ and $\operatorname{Im} f(z)$.

3.136. Let a be a constant. Consider the principal value of z^a and a non-principal value of z^{-a}.

(a) In general, is $z^a z^{-a} = 1$ always true?

(b) Suppose the principal values of both z^a and z^{-a} are used. Is $z^a z^{-a} = 1$ true?

3.137. Let $F(z)$ be the branch of the function $(z+i)^{1/4}$ such that $F(1)$ equals the value $-2^{1/8}e^{i\pi/16}$. Find $F(0)$, $F(i)$ and $F(-1)$.

3.138. Find all the values of $\log(z^{1/2})$ and $\log(z^{1/3})$ in Cartesian form.

3.139. At what points is each of the functions $\operatorname{Log}(1-z^2)$, $\operatorname{Log}(1+z^2)$, $\operatorname{Log}(1-iz^2)$, $\operatorname{Log}(1+iz^2)$, analytic? Explain.

3.140. If $f(z)$ is the principal branch of $z^{1/3}$, find $f'(-1-i)$.

3.141. Prove the following formulas for the inverse trigonometric functions:

$$\frac{d}{dz}(\operatorname{Arcsin}^{-1} z) = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}(\operatorname{Arcsinh}^{-1} z) = \frac{1}{\sqrt{z^2-1}}$$

$$\frac{d}{dz}(\operatorname{Arccos}^{-1} z) = \frac{1}{\sqrt{1-z^2}}, \qquad \frac{d}{dz}(\operatorname{Arccosh}^{-1} z) = \frac{1}{\sqrt{z^2-1}}$$

$$\frac{d}{dz}(\operatorname{Arctan}^{-1} z) = \frac{1}{1+z^2}, \qquad \frac{d}{dz}(\operatorname{Arctanh}^{-1} z) = \frac{1}{1-z^2}.$$

3.142. Find all solutions of

(a) $e^z = -2i$ (b) $e^z = 1+i$ (c) $\sin z = 2$.