Exercise 1:

(a) $log(z_t)$ follows a stationary AR(1)-process, since $0 < \rho < 1$. Hence, $log(z_t)$ has a time independent expectation and variance.

$$\begin{aligned}\text{Erwartungswert: } E[log(z_t)] &= \rho \underbrace{E[log(z_{t-1})]}_{=E[log(z_t)]} + \underbrace{E[\varepsilon_t]}_{=0} \\ \Leftrightarrow E[log(z_t)] &= 0 \\ \Leftrightarrow log(E[z_t]) &= 0 \\ \Leftrightarrow E(z_t) &= 1 = z\end{aligned}$$

$$\begin{aligned}\text{Varianz: } var(z_t) &= \rho^2 \underbrace{var(z_{t-1})}_{=var(z_t)} + \underbrace{var(\varepsilon_t)}_{=\sigma_\varepsilon^2} \\ var(z_t) &= \frac{\sigma_\varepsilon^2}{1-\rho^2}\end{aligned}$$

(b) States k_t, z_t, control c_t. Bellmann-Equation:

$$V(k_t) = \max_{k_{t+1}} \left\{ U(\underbrace{c(k_t, k_{t+1})}_{c_t}) + \beta E_t V(k_{t+1}) \right\} \qquad Nebenbed: c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

Consider:

$$\begin{aligned}\frac{\partial V(k_t)}{\partial k_t} &= U'(c_t)\frac{\partial c_t}{\partial k_t} = U'(c_t)(f'(k_t) + (1-\delta)) \\ \Rightarrow \frac{\partial V(k_{t+1})}{\partial k_{t+1}} &= U'(c_{t+1})(f'(k_{t+1}) + (1-\delta))\end{aligned}$$

Thus, we get:

$$\begin{aligned}\frac{\partial V(k_t)}{\partial k_{t+1}} &= U'(c_t)\frac{\partial c_t}{\partial k_{t+1}} + \beta E_t \frac{\partial V(k_{t+1})}{\partial k_{t+1}} \\ &= U'(c_t)(-1) + \beta E_t \left[U'(c_{t+1})(f'(k_{t+1}) + (1-\delta))\right] \\ \Leftrightarrow c_t^{-\sigma} &= \beta E_t \left[c_{t+1}^{-\sigma}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + (1-\delta))\right]\end{aligned}$$

This is the Euler-equation.

(c) Set up the Lagrangian:

$$\begin{aligned}L &= E_t \sum_{t=0}^{\infty} \beta^{t+s} \left[\frac{c_{t+s}^{1-\sigma} - 1}{1-\sigma} - \lambda_{t+s}\left(c_{t+s} + k_{t+s+1} - z_{t+s}k_{t+s}^{\alpha} - (1-\delta)k_{t+s}\right) \right] \\ \frac{\partial L}{\partial c_t} &= E_t\left[\beta^t\left(c_t^{-\sigma} - \lambda_t\right)\right] = 0 \Leftrightarrow \lambda_t = c_t^{-\sigma} \\ \frac{\partial L}{\partial k_{t+1}} &= E_t\left[\beta^t(-\lambda_t) + \beta^{t+1}(-\lambda_{t+1})(-\alpha z_{t+1}k_{t+1}^{\alpha-1} - 1 + \delta))\right] = 0 \\ \Leftrightarrow \lambda_t &= E_t \beta \lambda_{t+1}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta) \\ \Rightarrow c_t^{-\sigma} &= \beta E_t \left[c_{t+1}^{-\sigma}(\alpha z_{t+1}k_{t+1}^{\alpha-1} + (1-\delta))\right]\end{aligned}$$

The same Euler-equation.

(d) In *steady-state* there is no uncertainty and all time-subscripts cancel: $c_t = c_{t+1} = c, k_t = k_{t+1} = k, z = E(z_t) = 1$. From the Euler equation you get:

$$\begin{aligned} c^{-\sigma} &= \beta c^{-\sigma}(\alpha k^{\alpha-1} + 1 - \delta) \\ \frac{1}{\beta} &= \alpha k^{\alpha-1} + 1 - \delta \\ \Leftrightarrow \alpha k^{\alpha-1} &= \frac{1}{\beta} - (1-\delta) \\ \Rightarrow k = \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}} & \qquad \Rightarrow c = k^\alpha + (1-\delta)k - k = k^\alpha - \delta k \end{aligned}$$

Linearization
Linearization of the Euler-equation:
From steady-state you know: $\frac{1}{\beta} = \alpha z k^{\alpha-1} + 1 - \delta \Leftrightarrow -\log(\beta) = \log(\alpha z k^{\alpha-1} + 1 - \delta)$
There are 2 methods to do this:

(1) Take logs and use Taylor-Approximation:

$$\begin{aligned} c_t^{-\sigma} &= \beta E_t\left[c_{t+1}^{-\sigma}(\alpha z_{t+1} k_{t+1}^{\alpha-1} + 1 - \delta)\right] \\ -\sigma log(c_t) &= \log(\beta) - \sigma log(E_t c_{t+1}) + log(\alpha E_t z_{t+1} E_t k_{t+1}^{\alpha-1} + 1 - \delta) \\ \sigma\left[log(E_t c_{t+1}) - log(c_t)\right] &= \log(\beta) + log(\alpha E_t z_{t+1} E_t k_{t+1}^{\alpha-1} + 1 - \delta) \end{aligned}$$

Approximate the left side:

$$\sigma\left[\log(c) - \log(c)\right] + \frac{\sigma}{c}(E_t c_{t+1} - c) - \frac{\sigma}{c}(c_t - c) = \sigma\left(E_t \widehat{c}_{t+1} - \widehat{c}_t\right)$$

Approximate the right side:

$$\begin{aligned} &\log(\beta) + \underbrace{\log(\alpha z k^{\alpha-1} + 1 - \delta)}_{-\log(\beta)} && 0 \\ &+ \frac{\alpha k^{\alpha-1}}{\underbrace{\alpha z k^{\alpha-1} + 1 - \delta}_{=1/\beta}}(E_t z_{t+1} - z)\frac{1}{\underbrace{z}_{=1}} && +\beta\alpha k^{\alpha-1} E_t \widehat{z}_{t+1} \\ &+ \beta\alpha(\alpha-1) z k^{\alpha-2}(E_t k_{t+1} - k) && +\beta\alpha(\alpha-1)k^{\alpha-1} E_t \widehat{k}_{t+1} \end{aligned}$$

Thus, the approximation yields:

$$\sigma\left(E_t \widehat{c}_{t+1} - \widehat{c}_t\right) = \beta\alpha k^{\alpha-1} E_t \widehat{z}_{t+1} + \beta\alpha(\alpha-1)k^{\alpha-1} E_t \widehat{k}_{t+1}$$

(2) Rewrite the model using $x_t = e^{\log(x_t) - \log(x) + \log(x)} = x e^{\log(x_t) - \log(x)} \approx x e^{\widehat{x}_t}$ and approximate for $\widehat{x}_t$:

$$\begin{aligned} c^{-\sigma} e^{-\sigma \widehat{c}_t} &= \beta\left[c^{-\sigma} e^{-\sigma E_t \widehat{c}_{t+1}}\left(\alpha z k^{\alpha-1} e^{E_t \widehat{z}_{t+1} + (\alpha-1)E_t \widehat{k}_{t+1}} + 1 - \delta\right)\right] \\ e^{\sigma E_t \widehat{c}_{t+1} - \sigma \widehat{c}_t} &= \beta\left(\alpha z k^{\alpha-1} e^{E_t \widehat{z}_{t+1} + (\alpha-1)E_t \widehat{k}_{t+1}} + 1 - \delta\right) \end{aligned}$$

Approximation of the LHS:

$$e^0 + e^0\sigma(E_t\widehat{c}_{t+1} - 0) - e^0\sigma(\widehat{c}_t - 0) = 1 + \sigma(E_t\widehat{c}_{t+1} - \widehat{c}_t)$$

Approximation of the RHS:

$$\begin{aligned} &\underbrace{\beta\left(\alpha z k^{\alpha-1}e^0 + 1 - \delta\right)}_{=1} \\ &+\beta\alpha z k^{\alpha-1}e^0(E_t\widehat{z}_{t+1} - 0) \\ &+\beta\alpha z k^{\alpha-1}e^0(\alpha-1)(E_t\widehat{k}_{t+1} - 0) \end{aligned}$$

This is the same linearized Euler-equation.

For the capital-accumulation-equation you get:

$$\widehat{k}_{t+1} = k^{\alpha-1}\widehat{z}_t - \frac{c}{k}\widehat{c}_t + \frac{1}{\beta}\widehat{k}_t$$

And the technological process (just take logs and subtract the steady-state):

$$\widehat{z}_t = \rho z_{t-1} + \varepsilon_t$$

(e) Put all Variables with time-subscript $t+1$ (except shocks) on the left side:

$$\begin{aligned} -1E_t\widehat{c}_{t+1} - \frac{\beta\alpha(\alpha-1)k^{\alpha-1}}{\sigma}E_t\widehat{k}_{t+1} - \frac{\beta\alpha k^{\alpha-1}}{\sigma}E_t\widehat{z}_{t+1} &= \widehat{c}_t \\ \widehat{k}_{t+1} &= -\frac{c}{k}\widehat{c}_t + \frac{1}{\beta}\widehat{k}_t + k^{\alpha-1}\widehat{z}_t \\ \widehat{z}_{t+1} &= \rho z_t + \varepsilon_{t+1} \end{aligned}$$

Thus, $\mathbf{x_{t+1}} = \begin{pmatrix} c_{t+1} \\ k_{t+1} \\ z_{t+1} \end{pmatrix}$, $\boldsymbol{\nu}_{\mathbf{t+1}} = (\varepsilon_{t+1})$, $\boldsymbol{\eta}_{\mathbf{t+1}} = \begin{pmatrix} E_t c_{t+1} - c_{t+1} \\ E_t k_{t+1} - k_{t+1} \\ E_t z_{t+1} - z_{t+1} \end{pmatrix}$. The system can then be written as:

$$\underbrace{\begin{pmatrix} -1 & -\frac{\beta\alpha(\alpha-1)k^{\alpha-1}}{\sigma} & -\frac{\beta\alpha k^{\alpha-1}}{\sigma} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} c_{t+1} \\ k_{t+1} \\ z_{t+1} \end{pmatrix}}_{\mathbf{x_{t+1}}} =$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -\frac{c}{k} & \frac{1}{\beta} & k^{\alpha-1} \\ 0 & 0 & \rho \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} c_t \\ k_t \\ z_t \end{pmatrix}}_{\mathbf{x_t}} + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{C}} \underbrace{\varepsilon_{t+1}}_{\boldsymbol{\nu}_{\mathbf{t+1}}} + \underbrace{\begin{pmatrix} 1 & \frac{\beta\alpha(\alpha-1)k^{\alpha-1}}{\sigma} & \frac{\beta\alpha k^{\alpha-1}}{\sigma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\mathbf{D}} \underbrace{\begin{pmatrix} \eta_{1,t+1} \\ \eta_{2,t+1} \\ \eta_{3,t+1} \end{pmatrix}}_{\boldsymbol{\eta}_{\mathbf{t+1}}}$$

Vectorization and Kroneckerproduct

Example for vectorization:

$$vec\begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 3 & 0 & 2 & 2 & 0 & 2 \end{pmatrix}'$$

Example for Kroneckerproduct:

$$\underbrace{\begin{pmatrix} 1 & 3 & 2 \\ 1 & 0 & 0 \\ 1 & 2 & 2 \end{pmatrix}}_{3\times 3} \otimes \underbrace{\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix}}_{3\times 2} = \underbrace{\begin{pmatrix} 1\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 3\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \\ 1\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 0\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 0\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \\ 1\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} & 2\cdot\begin{pmatrix} 0 & 5 \\ 5 & 0 \\ 1 & 1 \end{pmatrix} \end{pmatrix}}_{9\times 6}$$

Consider the matrices A: $m \times n$, B: $n \times p$ and C: $p \times k$. Show that $vec(ABC) = (C' \otimes A)\, vec(B)$.

$$\begin{aligned} ABC &= A \begin{pmatrix} b_1 & b_2 & \dots & b_p \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k} \\ c_{21} & c_{22} & \dots & c_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ c_{p1} & c_{p2} & \dots & c_{pk} \end{pmatrix} \\ &= A \underbrace{\begin{pmatrix} b_1c_{11} + b_2c_{21} + \dots + b_pc_{p1}, & b_1c_{12} + b_2c_{22} + \dots + b_pc_{p2}, & \dots, & b_1c_{1k} + b_2c_{2k} + \dots + b_pc_{pk} \end{pmatrix}}_{n\times k} \end{aligned}$$

$$vec(ABC) = \begin{pmatrix} c_{11}Ab_1 + c_{21}Ab_2 + \dots + c_{p1}Ab_p \\ c_{12}Ab_1 + c_{22}Ab_2 + \dots + c_{p2}Ab_p \\ \vdots \\ c_{1k}Ab_1 + c_{2k}Ab_2 + \dots + c_{pk}Ab_p \end{pmatrix} = \begin{pmatrix} c_{11}A & c_{21}A & \dots & c_{p1}A \\ c_{12}A & c_{22}A & \dots & c_{p2}A \\ \vdots & \vdots & \vdots & \vdots \\ c_{1k}A & c_{2k}A & \dots & c_{pk}A \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{pmatrix} \quad = (C' \otimes A)\, vec(B)$$

Exercise 5

Derivation of optimality-conditions:

- $u_c(c_t, l_t) = \theta \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} c_t^{-1}$
- $u_l(c_t, l_t) = -(1-\theta) \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} (1-l_t)^{-1}$
- $f_k = \alpha \left[\alpha k_{t-1}^\psi + (1-\alpha) l_t^\psi\right]^{\frac{1-\psi}{\psi}} k_{t-1}^{\psi-1} = \alpha \left(\frac{f_t}{k_{t-1}}\right)^{1-\psi} = \alpha \left[\alpha + (1-\alpha)\left(\frac{l_t}{k_{t-1}}\right)^\psi\right]^{\frac{1-\psi}{\psi}}$
- $f_l = (1-\alpha) \left[\alpha k_{t-1}^\psi + (1-\alpha) l_t^\psi\right]^{\frac{1-\psi}{\psi}} l_t^{\psi-1} = (1-\alpha) \left(\frac{f_t}{l_t}\right)^{1-\psi} = (1-\alpha) \left[\alpha\left(\frac{k_{t-1}}{l_t}\right)^\psi + (1-\alpha)\right]^{\frac{1-\psi}{\psi}}$
- Euler:

$$\begin{aligned} \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} c_t^{-1} &= \beta E_t \left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta}\right]^{1-\tau} c_{t+1}^{-1} \cdot \left[A_{t+1}\alpha \left(\frac{f_{t+1}}{k_t}\right)^{1-\psi} + 1 - \delta\right] \\ &= \beta E_t \left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta}\right]^{1-\tau} c_{t+1}^{-1} \cdot \left[\alpha A_{t+1}^\psi \left(\frac{y_{t+1}}{k_t}\right)^{1-\psi} + 1 - \delta\right] \end{aligned}$$

- Consumption-Leisure:

$$\frac{1-\theta}{\theta}\frac{c_t}{1-l_t} = A_t(1-\alpha)\left(\frac{f_t}{l_t}\right)^{1-\psi} = A_t^{\psi}(1-\alpha)\left(\frac{y_t}{l_t}\right)^{1-\psi}$$

- Resource-constraint:

$$y_t = c_t + k_t - (1-\delta)k_{t-1}$$

- Stochastic process: $A_t = e^{a_t}$ and $a_t = \rho a_{t-1} + \sigma \varepsilon_t$ with $\varepsilon_t \sim N(0,1)$.

Derivation of steady-state:

1. $A = A^{ss}$
2. From Euler:

$$\frac{1}{\beta} = A^{\psi}\alpha\left(\frac{y}{k}\right)^{1-\psi} + 1 - \delta$$
$$\Leftrightarrow \frac{y}{k} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha A^{\psi}}\right)^{\frac{1}{1-\psi}}$$

3. From resource-constraint:

$$\frac{c}{k} = \frac{y}{k} - \delta$$

4. Definition of Production-function:

$$\frac{y}{k} = A\left[\alpha + (1-\alpha)\left(\frac{l}{k}\right)^{\psi}\right]^{\frac{1}{\psi}} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha A^{\psi}}\right)^{\frac{1}{1-\psi}} = \frac{y}{k}$$
$$\Leftrightarrow \frac{l}{k} = \left[\left(\left(\frac{y/k}{A}\right)^{\psi} - \alpha\right)(1-\alpha)^{-1}\right]^{\frac{1}{\psi}}$$

5. Identity:

$$\frac{y}{l} = \frac{y}{k}\frac{k}{l}$$

6. Identity:

$$\frac{c}{l} = \frac{c}{k}\frac{k}{l}$$

7. From the consumption-leisure decision:

$$l\frac{c}{l} = (1-l)\frac{\theta}{1-\theta}A^{\psi}(1-\alpha)\left(\frac{y}{l}\right)^{1-\psi}$$
$$\Leftrightarrow l = \left(1 + \frac{\frac{c}{l}}{\frac{\theta(1-\alpha)}{1-\theta}A^{\psi}\left(\frac{y}{l}\right)^{1-\psi}}\right)^{-1}$$

8. Identities:

$$c = \frac{c}{l} l, \qquad k = \frac{l}{l/k}, \qquad y = \frac{y}{k} k.$$

How to calibrate?

- We have a CES-production-function. One can show that the capital-share in steady-state ist equal to

$$\begin{aligned} s(l,k) &= \frac{\alpha k^{\psi}}{\alpha k^{\psi} + (1-\alpha) l^{\psi}} = \frac{\alpha k^{\psi}}{\left[\alpha + (1-\alpha)\left(\frac{l}{k}\right)^{\psi}\right] k^{\psi}} = \frac{\alpha}{\left(\frac{y}{Ak}\right)^{\psi}} \\ &= \frac{\alpha A^{\psi}}{\left(\frac{\beta^{-1}-1+\delta}{\alpha}\right)^{\frac{\psi}{1-\psi}}} = \frac{\alpha^{\frac{1}{1-\psi}} A^{\psi}}{(\beta^{-1}-1+\delta)^{\frac{\psi}{1-\psi}}} \end{aligned}$$

- So, choose parameters such that this expression is close to 0.2-0.3.