Example for Kroneckerproduct:

$$\underbrace{\begin{pmatrix}1 & 3 & 2\\ 1 & 0 & 0\\ 1 & 2 & 2\end{pmatrix}}_{3\times 3} \otimes \underbrace{\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix}}_{3\times 2} = \underbrace{\begin{pmatrix} 1\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 3\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 2\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix}\\ 1\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 0\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 0\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix}\\ 1\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 2\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix} & 2\cdot\begin{pmatrix}0 & 5\\ 5 & 0\\ 1 & 1\end{pmatrix}\end{pmatrix}}_{9\times 6}$$

Consider the matrices A: $m \times n$, B: $n \times p$ and C: $p \times k$. Show that $vec(ABC) = (C' \otimes A)\, vec(B)$.

$$\begin{aligned} ABC &= A \begin{pmatrix} b_1 & b_2 & \dots & b_p \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1k}\\ c_{21} & c_{22} & \dots & c_{2k}\\ \vdots & \vdots & \vdots & \vdots\\ c_{p1} & c_{p2} & \dots & c_{pk} \end{pmatrix}\\ &= A \underbrace{\begin{pmatrix} b_1c_{11} + b_2c_{21} + \dots + b_pc_{p1}, & b_1c_{12} + b_2c_{22} + \dots + b_pc_{p2}, & \dots, & b_1c_{1k} + b_2c_{2k} + \dots + b_pc_{pk}\end{pmatrix}}_{n\times k} \end{aligned}$$

$$vec(ABC) = \begin{pmatrix} c_{11}Ab_1 + c_{21}Ab_2 + \dots + c_{p1}Ab_p\\ c_{12}Ab_1 + c_{22}Ab_2 + \dots + c_{p2}Ab_p\\ \vdots\\ c_{1k}Ab_1 + c_{2k}Ab_2 + \dots + c_{pk}Ab_p \end{pmatrix} = \begin{pmatrix} c_{11}A & c_{21}A & \dots & c_{p1}A\\ c_{12}A & c_{22}A & \dots & c_{p2}A\\ \vdots & \vdots & \vdots & \vdots\\ c_{1k}A & c_{2k}A & \dots & c_{pk}A \end{pmatrix} \begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_p \end{pmatrix} = (C' \otimes A)\, vec(B)$$

Exercise 5

Derivation of optimality-conditions:

- $u_c(c_t, l_t) = \theta \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} c_t^{-1}$
- $u_l(c_t, l_t) = -(1-\theta) \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} (1-l_t)^{-1}$
- $f_k = \alpha \left[\alpha k_{t-1}^\psi + (1-\alpha) l_t^\psi\right]^{\frac{1-\psi}{\psi}} k_{t-1}^{\psi-1} = \alpha \left(\frac{f_t}{k_{t-1}}\right)^{1-\psi} = \alpha \left[\alpha + (1-\alpha)\left(\frac{l_t}{k_{t-1}}\right)^\psi\right]^{\frac{1-\psi}{\psi}}$
- $f_l = (1-\alpha) \left[\alpha k_{t-1}^\psi + (1-\alpha) l_t^\psi\right]^{\frac{1-\psi}{\psi}} l_t^{\psi-1} = (1-\alpha) \left(\frac{f_t}{l_t}\right)^{1-\psi} = (1-\alpha) \left[\alpha \left(\frac{k_{t-1}}{l_t}\right)^\psi + (1-\alpha)\right]^{\frac{1-\psi}{\psi}}$
- Euler:

$$\begin{aligned} \left[c_t^\theta (1-l_t)^{1-\theta}\right]^{1-\tau} c_t^{-1} &= \beta E_t \left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta}\right]^{1-\tau} c_{t+1}^{-1} \cdot \left[A_{t+1}\alpha \left(\frac{f_{t+1}}{k_t}\right)^{1-\psi} + 1 - \delta\right]\\ &= \beta E_t \left[c_{t+1}^\theta (1-l_{t+1})^{1-\theta}\right]^{1-\tau} c_{t+1}^{-1} \cdot \left[\alpha A_{t+1}^\psi \left(\frac{y_{t+1}}{k_t}\right)^{1-\psi} + 1 - \delta\right] \end{aligned}$$

- Consumption-Leisure:

$$\frac{1-\theta}{\theta}\frac{c_t}{1-l_t} = A_t(1-\alpha)\left(\frac{f_t}{l_t}\right)^{1-\psi} = A_t^{\psi}(1-\alpha)\left(\frac{y_t}{l_t}\right)^{1-\psi}$$

- Resource-constraint:

$$y_t = c_t + k_t - (1-\delta)k_{t-1}$$

- Stochastic process: $A_t = e^{a_t}$ and $a_t = \rho a_{t-1} + \sigma\varepsilon_t$ with $\varepsilon_t \sim N(0,1)$.

Derivation of steady-state:

1. $A = A^{ss}$
2. From Euler:

$$\frac{1}{\beta} = A^{\psi}\alpha\left(\frac{y}{k}\right)^{1-\psi} + 1 - \delta$$
$$\Leftrightarrow \frac{y}{k} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha A^{\psi}}\right)^{\frac{1}{1-\psi}}$$

3. From resource-constraint:

$$\frac{c}{k} = \frac{y}{k} - \delta$$

4. Definition of Production-function:

$$\frac{y}{k} = A\left[\alpha + (1-\alpha)\left(\frac{l}{k}\right)^{\psi}\right]^{\frac{1}{\psi}} = \left(\frac{\beta^{-1} - 1 + \delta}{\alpha A^{\psi}}\right)^{\frac{1}{1-\psi}} = \frac{y}{k}$$
$$\Leftrightarrow \frac{l}{k} = \left[\left(\left(\frac{y/k}{A}\right)^{\psi} - \alpha\right)(1-\alpha)^{-1}\right]^{\frac{1}{\psi}}$$

5. Identity:

$$\frac{y}{l} = \frac{y}{k}\frac{k}{l}$$

6. Identity:

$$\frac{c}{l} = \frac{c}{k}\frac{k}{l}$$

7. From the consumption-leisure decision:

$$l\frac{c}{l} = (1-l)\frac{\theta}{1-\theta}A^{\psi}(1-\alpha)\left(\frac{y}{l}\right)^{1-\psi}$$
$$\Leftrightarrow l = \left(1 + \frac{\frac{c}{l}}{\frac{\theta(1-\alpha)}{1-\theta}A^{\psi}\left(\frac{y}{l}\right)^{1-\psi}}\right)^{-1}$$

8. Identities:

$$c = \frac{c}{l}l, \qquad k = \frac{l}{l/k}, \qquad y = \frac{y}{k}k.$$

How to calibrate?

- We have a CES-production-function. One can show that the capital-share in steady-state ist equal to

$$\begin{aligned} s(l,k) &= \frac{\alpha k^{\psi}}{\alpha k^{\psi} + (1-\alpha)l^{\psi}} = \frac{\alpha k^{\psi}}{\left[\alpha + (1-\alpha)\left(\frac{l}{k}\right)^{\psi}\right] k^{\psi}} = \frac{\alpha}{\left(\frac{y}{Ak}\right)^{\psi}} \\ &= \frac{\alpha A^{\psi}}{\left(\frac{\beta^{-1}-1+\delta}{\alpha}\right)^{\frac{\psi}{1-\psi}}} = \frac{\alpha^{\frac{1}{1-\psi}} A^{\psi}}{(\beta^{-1}-1+\delta)^{\frac{\psi}{1-\psi}}} \end{aligned}$$

- So, choose parameters such that this expression is close to 0.2-0.3.