Chapter 29

The real business cycle theory

Since the middle of the 1970s quite different approaches to the explanation of business cycle fluctuations have been pursued. We may broadly classify them as either of a New Classical or a Keynesian orientation. The New Classical approach attempts to explain output fluctuations as movements in the "natural level" of output. The Keynesian approach attempts to explain them as movements *away* from the "natural level" of output.

Within the New Classical approach, which is the topic of this chapter, the monetary mis-perception theory of Lucas (1972, 1975) was the dominating one in the 1970s. The theory came under serious empirical attack, however.[1] From the early 1980s the alternative branch within the New Classical approach, the Real Business Cycle theory, gradually took over. This theory (RBC theory for short) was initiated by Finn E. Kydland and Edward C. Prescott (1982).[2] Other major contributions include Long and Plosser (1983), Prescott (1986), and Plosser (1989).

The general philosophy of any New Classical approach to business cycle analysis is that the fluctuations can be explained by adding stochastic disturbances to the neoclassical framework with optimizing agents, rational expectations, and market clearing under perfect competition. Output and employment are seen as supply determined, the only difference compared to the standard neoclassical growth model being that there are fluctuations around the growth trend. The fluctuations are not viewed as deviations from a Walrasian equilibrium, but as a constituent part of a moving stochastic Walrasian equilibrium. Whereas in Lucas' monetary mis-perception theory from the 1970s shocks to the *money supply* were the driving force, the RBC theory is based on the idea that economic fluctuations are triggered primar-

[1] For a survey, see Blanchard (1990).

[2] In 2004 they were awarded the Nobel prize, primarily for their contributions in two areas: policy implications of time inconsistency and quantitative business cycle research.

ily by recurrent technology shocks and other supply shocks. In fact, money is completely absent from the RBC models of the 1980s. The fluctuations in employment reflect fluctuations in labor supply. Government intervention with the purpose of stabilization is seen as likely to be counterproductive. Given the uncertainty due to shocks, the market forces establish a Pareto optimal moving equilibrium. "Economic fluctuations are optimal responses to uncertainty in the rate of technological change", as Edward Prescott claims (Prescott 1986).

Below we describe the basics of the RBC model. It is this type of model we have in mind in this text when speaking of "RBC theory". It constitutes a subclass of what later has become known as Dynamic Stochastic General Equilibrium models, DSGE models. This name refers to a much broader class of quantitative models, including models with an emphasis on money and nominal price stickiness. The general philosophy behind the RBC theory in the sense the term is used here, was clearly stated in Prescott (1986).

29.1 A simple RBC model

The RBC theory is based on a non-monetary stochastic Ramsey growth model in discrete time. The presentation here is close to King and Rebelo (1999), available in *Handbook of Macroeconomics*, vol. 1B, 1999. As a rule, our notation is the same as that of King and Rebelo, but there will be a few exceptions in order not to diverge too much from the general notational principles in this text. The notation appears in Table 29.1. The most precarious differences in comparison with King and Rebelo are that we use ρ in our customary meaning as a discount rate for utility over time and θ for elasticity of marginal utility of consumption.

The firm

There are two categories of economic agents in the model: firms and households; the government sector is ignored. First we describe the firm.

Table 29.1. Notation

Variable	*King & Rebelo*	*Here*
Aggregate consumption	C_t	same
Deterministic technology level	X_t	same
Growth corrected consumption	$c_t \equiv C_t/X_t$	same
Growth corrected investment	$i_t \equiv I_t/X_t$	same
Growth corrected output	$y_t \equiv Y_t/X_t$	same
Growth corrected capital	$k_t \equiv K_t/X_t$	same
Aggregate employment (hours)	N_t	same
Aggregate leisure (hours)	$L_t \equiv 1 - N_t$	same
Effective capital intensity	$\frac{k_t}{N_t}$	$\tilde{k}_t \equiv \frac{K_t}{X_t N_t}$
Real wage	$w_t X_t$	w_t
Technology-corrected real wage	w_t	$\tilde{w}_t \equiv w_t/X_t$
Real interest rate from end of period t to end period to $t+1$	r_t	r_{t+1}
Auto-correlation coefficient in technology process	ρ	ξ
Discount factor w.r.t. utility	b	$\frac{1}{1+\rho}$
Rate of time preference w.r.t. utility	$\frac{1}{b} - 1$	ρ
Elasticity of marginal utility of cons.	σ	θ
Elasticity of marginal utility of leisure	η	same
Elasticity of output w.r.t. labor	α	same
Steady state value of c_t	c	c^*
The natural logarithm	log	same
Log deviation of c_t from steady state value	$\hat{c}_t \equiv \log \frac{c_t}{c}$	$\hat{c}_t \equiv \log \frac{c_t}{c^*}$
Log deviation of N_t from steady state value	$L_t \equiv \log \frac{L_t}{L}$	$N_t \equiv \log \frac{N_t}{N^*}$

Technology

The representative firm has the production function

$$Y_t = A_t F(K_t, X_t N_t), \tag{29.1}$$

where K_t and N_t are input of capital and labor in period t, while X_t is the exogenous deterministic "technology level", and A_t represents an exogenous random productivity factor. The production function F has constant returns to scale and is neoclassical (i.e., marginal productivity of each factor is positive, but decreasing in the same factor). In applications, often a Cobb-Douglas function is used.

It is assumed that X_t grows at a constant rate, $\gamma - 1$, i.e.,

$$X_{t+1} = \gamma X_t, \qquad \gamma > 1, \tag{29.2}$$

where γ is the deterministic technology growth factor. The productivity factor A_t is a stochastic variable which is assumed to follow a process of the form

$$A_t = A^{*1-\xi}(A_{t-1})^{\xi} e^{\varepsilon_t},$$

so that $\log A_t$ is an AR(1) process:

$$\log A_t = (1-\xi)\log A^* + \xi \log A_{t-1} + \varepsilon_t, \qquad 0 \leq \xi < 1. \tag{29.3}$$

The last term, ε_t, represents a productivity shock which is assumed to be *white noise* with variance σ_ε^2.[3] The auto-correlation coefficient ξ measures the degree of *persistence* over time of the effect on $\log A$ of a shock. If $\xi = 0$, the effect is only temporary; if $\xi > 0$, there is some persistence. The unconditional expectation of $\log A_t$ is equal to $\log A^*$ (which is thus the expected value "in the long run"). The shocks, ε_t, may represent accidental events affecting productivity, perhaps technological changes that are not sustainable, including sometimes technological mistakes (think of the introduction and later abandonment of asbestos in the construction industry). Note that negative realizations of the noise term ε_t may represent "technological regress". But it need not, since moderate negative values of ε_t are consistent with overall technological progress, though temporarily below the trend represented by the deterministic growth of X_t.

The reason that we said "not sustainable" is that sustainability would require $\xi = 1$, which conflicts with (29.3). Yet $\xi = 1$, which turns (29.3) into a random walk with drift, would correspond better to our general conception of technological change as a *cumulative* process. Technical knowledge is cumulative in the sense that a technical invention continues to be known and usable. But in the present version of the RBC model this cumulative part of technological change is represented by the deterministic trend γ in (29.2).[4] It remains somewhat vague what the stochastic A_t really embodies. Barro (2008) suggests a broad interpretation, including shifts in legal and political systems, harvest failures, wartime destruction, natural disasters, and strikes. For an open economy, shifts in terms of trade might be a possible interpretation (for example due to temporary oil price shocks).

There is an alternative version of the RBC model which is based on the specification $\xi = 1$ and $\gamma = 1$ (see Appendix). The difficulty with this specification is that it tends to generate too little fluctuation in employment

[3] We recall that a sequence of stochastic variables with zero mean, constant variance, and zero covariance across time is called *white noise*.

[4] This version of the RBC model corresponds to that of the early RBC theorists, including Prescott (1986) and King and Rebelo (1988). They adhered to the supposition $0 < \xi < 1$ and had the cumulative aspect represented by a deterministic trend.

and output. This is because, when shocks are permanent, large wealth effects offset the intertemporal substitution in labor supply.

Factor demand

The representative firm is assumed to maximize its value under perfect competition. Since there are no convex capital installation costs, the problem reduces to that of static maximization of profits each period. And since period t's technological conditions (F, X_t, and the realization of A_t) are assumed known to the firm in period t, the firm does not face any uncertainty. Profit maximization simply implies a standard factor demand (K_t, N_t), satisfying

$$A_t F_1(K_t, X_t N_t) = r_t + \delta, \qquad 0 \leq \delta \leq 1, \tag{29.4}$$

$$A_t F_2(K_t, X_t N_t) X_t = w_t, \tag{29.5}$$

where $r_t + \delta$ is the real cost per unit of the capital service and w_t is the real wage.

The household

There is a given number of households or rather dynastic families, all alike and with infinite horizon (Ramsey setup). For simplicity we ignore population growth. Thus we consider a representative household of constant size and with a constant amount of time at its disposal, say 1 time unit per period. The household's saving in period t amounts to buying investment goods that in the next period are rented out to the firms at the rental rate $r_{t+1} + \delta$. Thus the household obtains a net rate of return on financial wealth equal to the interest rate r_{t+1}.

A decision problem under uncertainty

The preferences of the household are described by the expected discounted utility hypothesis. Both consumption, C_t, and leisure, L_t, enter the period utility function. Since the total time endowment of the household is one in all periods, we have

$$L_t + N_t = 1, \qquad t = 0, 1, 2, \ldots, \tag{29.6}$$

where N_t is labor supply in period t. The fact that N has now been used in two different meanings, in (29.1) as employment and in (29.6) as labor supply, should not cause problems since in the competitive equilibrium of the model the two are the same.

The household has rational expectations. The decision problem, as seen from time 0, is to choose current consumption, C_0, and labor supply, N_0, as well as a series of *contingent plans*, $C(t, K_t)$ and $N(t, K_t)$, for $t = 1, 2, ...$, so that expected discounted utility is maximized:

$$\begin{aligned}
\max E_0(U_0) &= E_0[\sum_{t=0}^{\infty} u(C_t, 1 - N_t)(1 + \rho)^{-t}] \qquad \text{s.t.} && (29.7)\\
C_t &\geq 0, 0 \leq N_t \leq 1, \qquad \text{(control region)} && (29.8)\\
K_{t+1} &= (1 + r_t)K_t + w_t N_t - C_t, \qquad K_0 > 0 \text{ given,} && (29.9)\\
K_{t+1} &\geq 0 \qquad \text{for } t = 0, 1, 2, ... && (29.10)
\end{aligned}$$

The period utility function $u(\cdot, \cdot)$ satisfies $u_1 > 0$, $u_2 > 0$, $u_{11} < 0$, $u_{22} < 0$ and is concave.[5] The decreasing marginal utility assumption implies, first, a desire of smoothing over time both consumption and leisure; or we could say that there is aversion towards variation over time in these entities. Second, decreasing marginal utility reflects aversion towards variation in consumption and leisure over different "states of nature", i.e., risk aversion. The parameter ρ is the rate of time preference (the measure of impatience) and it is assumed positive (a further restriction on ρ will be introduced later).

When speaking of "period t", we mean the time interval $[t, t + 1)$. The symbol E_0 signifies the expected value, conditional on the information available at the end of period 0. This information includes knowledge of all variables up to period 0, including that period. There is uncertainty about future values of r_t and w_t, but the household knows the stochastic processes which these variables follow (or, what amounts to the same, the stochastic processes that lie behind, i.e., those for the productivity factor A_t).

The constraint (29.9) displays our usual way of writing, in discrete time, the dynamic accounting relation for wealth formation. An alternative way of writing the condition is:

$$K_{t+1} - K_t = r_t K_t + w_t N_t - C_t,$$

saying that private net saving is equal to income minus consumption. This way of writing it corresponds to the form used in continuous time models. We could also write

$$K_{t+1} - K_t + \delta K_t = (r_t + \delta)K_t + w_t N_t - C_t, \qquad (29.11)$$

saying that private gross investment is equal to gross income minus consumption.

[5] Concavity implies adding the assumption $u_{11}u_{22} - (u_{12})^2 \geq 0$.

Characterizing the solution to the household's problem

There are three endogenous variables, the control variables C_t and N_t, and the state variable K_t. The decision, as seen from period 0, is to choose a concrete *action* (C_0, N_0) and a series of *contingent plans* $(C(t, K_t), N(t, K_t))$ saying what to do in each of the future periods as a function of the as yet unknown circumstances, including the financial wealth, K_t, at that time. The decision is made so that expected discounted utility $E_0(U_0)$ is maximized. The pair of functions $(C(t, K_t), N(t, K_t))$ is named a *contingent plan* because it tells what consumption and labor supply will be in period t, *contingent* on the financial wealth obtained at the beginning of period t. In turn this wealth, K_t, depends on the realized path, up to period t, of the as yet unknown variable A_t. In order to choose the action (C_0, N_0) in a rational way, the household must take into account the whole future, including what the optimal contingent actions in the future will be.

To be more specific, when deciding the action (C_0, N_0), the household knows that in every new period t, it has to solve the remainder of the problem as seen from that period. Defining $\tilde{U}_t \equiv (1+\rho)^t U_t$,[6] the remainder of the problem as seen from period t $(t = 0, 1, ...)$ is:

$$\begin{aligned} \max\ E_t\tilde{U}_t &= u(C_t, 1-N_t) + (1+\rho)^{-1}E_t\left[u(C_{t+1}, 1-N_{t+1})\right. \qquad (29.12) \\ &\quad \left. +u(C_{t+2}, 1-N_{t+2})(1+\rho)^{-1} + ...\right] \\ &\quad \text{s.t. } (29.8), (29.9), \text{and } (29.10), \qquad K_t \text{ given.} \end{aligned}$$

To deal with this problem we will use the *substitution method*. First, from (29.9) we have

$$C_t = (1+r_t)K_t + w_tN_t - K_{t+1}, \qquad \text{and} \qquad (29.13)$$

$$C_{t+1} = (1+r_{t+1})K_{t+1} + w_{t+1}N_{t+1} - K_{t+2}. \qquad (29.14)$$

Substituting this into (29.12), the decision problem is reduced to an unconstrained maximization problem, namely one of maximizing the function $E_t\tilde{U}_t$ w.r.t. $(N_t, K_{t+1}), (N_{t+1}, K_{t+2}), ...$. We first take the partial derivative w.r.t. N_t in (29.12) and set it equal to 0 (thus focusing on interior solutions):

$$\frac{\partial E_t\tilde{U}_t}{\partial N_t} = u_1(C_t, 1-N_t)w_t + u_2(C_t, 1-N_t)(-1) = 0,$$

which can be written

$$u_2(C_t, 1-N_t) = u_1(C_t, 1-N_t)w_t. \qquad (29.15)$$

[6] Multiplying a utility function by a positive constant does not change the associated optimal behavior.

This first-order condition describes the trade-off between leisure in period t and consumption in the same period. The condition says that in the optimal plan, the opportunity cost (in terms of foregone current utility) associated with decreasing leisure by one unit equals the utility benefit of obtaining an increased labor income and using this increase for extra consumption (i.e., marginal cost = marginal benefit, both measured in current utility).

Similarly, w.r.t. K_{t+1} we get the first-order condition[7]

$$\frac{\partial \tilde{U}_t}{\partial K_{t+1}} = u_1(C_t, 1-N_t)(-1) + (1+\rho)^{-1}E_t[u_1(C_{t+1}, 1-N_{t+1})(1+r_{t+1})] = 0.$$

This can be written

$$u_1(C_t, 1-N_t) = (1+\rho)^{-1}E_t[u_1(C_{t+1}, 1-N_{t+1})(1+r_{t+1})], \tag{29.16}$$

where r_{t+1} is unknown in period t. This first-order condition describes the trade-off between consumption in period t and the uncertain consumption in period $t+1$, as seen from period t. The optimal plan must satisfy that the current utility loss associated with decreasing consumption by one unit equals the discounted expected utility gain next period by having $1+r_{t+1}$ extra units available for consumption, namely the gross return on saving one more unit (again, marginal cost = marginal benefit in utility terms). The condition (29.16) is an example of a *stochastic Euler equation.* If there is no uncertainty, the expectation operator E_t can be deleted. Then, ignoring the utility of leisure, (29.16) is the standard discrete-time analogue to the Keynes-Ramsey rule in continuous time.

For completeness, let us also maximize explicitly w.r.t. the future pairs (N_{t+i}, K_{t+i+1}), $i = 1, 2, \ldots$. We get

$$\frac{\partial \tilde{U}_t}{\partial N_{t+i}} = (1+\rho)^{-1}E_t\left[u_1(C_{t+i}, 1-N_{t+i})w_{t+i} + u_2(C_{t+i}, 1-N_{t+i})(-1)\right] = 0,$$

so that

$$E_t\left[u_2(C_{t+i}, 1-N_{t+i})\right] = E_t\left[u_1(C_{t+i}, 1-N_{t+i})w_{t+i}\right].$$

Similarly,

$$\begin{aligned} \frac{\partial \tilde{U}_t}{\partial K_{t+i+1}} &= E_t\Big[u_1(C_{t+i}, 1-N_{t+i})(-1) + (1+\rho)^{-1}u_1(C_{t+i+1}, 1-N_{t+i+1}) \\ \cdot(1+r_{t+i+1}) &= 0, \end{aligned}$$

[7] Generally speaking, for a given differentiable function $f(X, \alpha_1, \ldots, \alpha_n)$, where X is a stochastic variable and $\alpha_1, \ldots, \alpha_n$ are parameters, we have

$$\frac{\partial E(f(X, \alpha, \beta))}{\partial \alpha_i} = E\frac{\partial f(X, \alpha, \beta)}{\partial \alpha_i}, \quad i = 1, \ldots, n.$$

so that

$$E_t\left[u_1(C_{t+i}, 1 - N_{t+i})\right] = (1+\rho)^{-1} E_t\left[u_1(C_{t+i+1}, 1 - N_{t+i+1})(1 + r_{t+i+1})\right]$$

We see that for t replaced by $t+1$, $t+2$, ... , (29.15) and (29.16) must hold in expected values as seen from period t. The conclusion, so far, is that in general, it suffices to write down (29.15) and (29.16) and then add that for $t = 0$ these two conditions *are* part of the set of first-order conditions and for $t = 1, 2, \ldots,$ similar first-order conditions hold in expected values.

Our first-order conditions say something about *relative* levels of consumption and leisure in the same period and about the *change* in consumption over time, not about the absolute levels of consumption and leisure. The absolute levels are determined as the highest possible levels consistent with the requirement that (29.15), (29.16), and (29.10), for $t = 0, 1, 2, ...,$ hold in terms of expected values as seen from period 0. This can be shown to be equivalent to requiring the transversality condition,

$$\lim_{t\to\infty} E_0\left[(1+\rho)^{-(t-1)} u_1(C_{t-1}, 1 - N_{t-1}) K_t\right] = 0,$$

satisfied in addition to the first-order conditions.[8] Finding the resulting consumption function requires specification of the period utility function. But to solve for general equilibrium we do not need the consumption function. As in a deterministic Ramsey model, knowledge of the first-order conditions and the transversality condition is sufficient for determining the path over time of the economy.

The remaining elements in the model

It only remains to consider the market clearing conditions. Implicitly we have already assumed clearing in the factor markets, since we have used the same symbol for capital and employment, respectively, in the firm's problem (the demand side) as in the household's problem (the supply side). The equilibrium factor prices are given by (29.4) and (29.5). We will rewrite these two equations in a more convenient way. In view of constant returns to scale, we have

$$Y_t = A_t F(K_t, X_t N_t) = A_t X_t N_t F(\tilde{k}_t, 1) \equiv A_t X_t N_t f(\tilde{k}_t), \tag{29.17}$$

[8] In fact, in the budget constraint of the household's optimization problem, we could replace K_t by financial wealth and allow borrowing, so that financial wealth could be negative. Then, instead of the non-negativity constraint (29.10), a No-Ponzi-Game condition in expected value would be relevant. In a representative agent model with infinite horizon, however, this does not change anything, since the non-negativity constraint (29.10) will never be binding.

where $\tilde{k}_t \equiv K_t/(X_t N_t)$ (the effective capital intensity). In terms of the intensive production function f, (29.4) and (29.5) yield

$$r_t + \delta = A_t F_1(K_t, X_t N_t) = A_t f'(\tilde{k}_t), \tag{29.18}$$

$$w_t = A_t F_2(K_t, X_t N_t) X_t = A_t \left[f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] X_t. \tag{29.19}$$

In a closed economy, by definition, gross investment *ex post*, I_t, satisfies

$$K_{t+1} - K = I_t - \delta K_t. \tag{29.20}$$

Also, by definition, I_t equals gross saving, $Y_t - C_t$, since, by simple expenditure accounting,

$$Y_t = C_t + I_t. \tag{29.21}$$

Indeed, investment is in this model just the other side of households' saving. There is no independent investment function. To make sure that our national expenditure accounting is consistent with our national income accounting insert (29.20) into (29.21) to get

$$I_t = K_{t+1} - K_t + \delta K_t = Y_t - C_t = A_t X_t N_t f(\tilde{k}_t) - C_t \tag{29.22}$$

$$= A_t X_t N_t \left[\tilde{k}_t f'(\tilde{k}_t) + f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) \right] - C_t$$

$$= (r_t + \delta) K_t + w_t N_t - C_t, \tag{29.23}$$

where the second equality comes from (29.17) and the last equality follows from (29.18) and (29.19). The result (29.23) is identical to the dynamic budget constraint of the representative household, (29.11). Since this last equation defines aggregate saving from the national income accounting, the book-keeping is in order.

Specification of technology and preferences

To quantify the model we have to specify the production function and the utility function. We abide by the common praxis in the RBC literature and specify the production function to be Cobb-Douglas:

$$Y_t = A_t K_t^{1-\alpha} (X_t N_t)^{\alpha}, \qquad 0 < \alpha < 1. \tag{29.24}$$

Then we get

$$f(\tilde{k}_t) = \tilde{k}_t^{1-\alpha}, \tag{29.25}$$

$$r_t + \delta = (1-\alpha) A_t \tilde{k}_t^{-\alpha}, \tag{29.26}$$

$$w_t = \alpha A_t \tilde{k}_t^{1-\alpha} X_t. \tag{29.27}$$

As to the utility function we follow King and Rebelo (1999) and base the analysis on the additively separable CRRA case,

$$u(C_t, 1-N_t) = \frac{C_t^{1-\theta}-1}{1-\theta} + \omega\frac{(1-N_t)^{1-\eta}-1}{1-\eta}, \qquad \theta > 0, \eta > 0, \omega > 0. \tag{29.28}$$

Here, θ is the (absolute) elasticity of marginal utility of consumption (the desire for consumption smoothing), η is the (absolute) elasticity of marginal utility of leisure (the desire for leisure smoothing), and ω is the relative weight given to leisure. In case θ or η take the value 1, the corresponding term in (29.28) should be replaced by $\log C_t$ or $\omega \log(1 - N_t)$, respectively. In fact, most of the time King and Rebelo (1999) take both θ and η to be 1.

With (29.28) applied to (29.15) and (29.16), we get

$$\theta(1-N_t)^{-\eta} = C_t^{-\theta} w_t, \quad \text{and} \tag{29.29}$$

$$C_t^{-\theta} = \frac{1}{1+\rho} E_t\left[C_{t+1}^{-\theta}(1+r_{t+1})\right], \tag{29.30}$$

respectively.

29.2 A deterministic steady state

For a while, let us ignore shocks. That is, assume $A_t = A^*$ for all t.

The steady state solution

By a steady state we mean a path along which the growth-corrected variables like $\tilde{k}$ and $\tilde{w} \equiv w/X_t$ stay constant. With $A_t = A^*$ for all t, (29.26) and (29.27) give the steady-state relations between $\tilde{k}$, r, and $\tilde{w}$:

$$\tilde{k}^* = \left[\frac{(1-\alpha)A^*}{r^*+\delta}\right]^{1/\alpha}, \tag{29.31}$$

$$\tilde{w}^* = \alpha A^* \tilde{k}^{*1-\alpha}. \tag{29.32}$$

We may write (29.30) as

$$1+\rho = E_t\left[(\frac{C_{t+1}}{C_t})^{-\theta}(1+r_{t+1})\right]. \tag{29.33}$$

In the non-stochastic steady state the expectation operator E_t can be deleted, and r and C/X are independent of t. Hence, $C_{t+1}/C_t = \gamma$, by (29.2), and (29.33) takes the form

$$1+r^* = (1+\rho)\gamma^{\theta}. \tag{29.34}$$

In this expression we recognize the *modified golden rule* discussed in chapters 7 and 10.[9] Existence of general equilibrium in our Ramsey framework requires that the long-run real interest rate is larger than the long-run output growth rate, i.e., we need $r^* > \gamma - 1$. This condition is satisfied if and only if

$$1 + \rho > \gamma^{1-\theta}, \tag{29.35}$$

which we assume.[10] If we guess that $\theta = 1$ and $\rho = 0.01$, then with $\gamma = 1.004$ (taken from US national income accounting data 1947-96, using a quarter of a year as our time unit), we find the steady-state rate of return to be r^* $= 0.014$ or 0.056 per annum. Or, the other way round, observing the average return on the Standard & Poor 500 Index over the same period to be 6.5 per annum, given $\theta = 1$ and $\gamma = 1.004$, we estimate ρ to be 0.012.

Using that in steady state N_t is a constant, N^*, we can write (29.22) as

$$\gamma \tilde{k}_{t+1} - (1-\delta)\tilde{k}_t = A^* \tilde{k}_t^{1-\alpha} - \tilde{c}_t, \tag{29.36}$$

where $\tilde{c}_t \equiv C_t/(X_t N^*)$. Given r^*, (29.31) yields the steady-state capital intensity $\tilde{k}^*$. Then, (29.36) gives

$$\tilde{c}^* \equiv \frac{c^*}{X_t} = A^* \tilde{k}^{*1-\alpha} - (\gamma + \delta - 1)\tilde{k}^*.$$

Consumption dynamics around the steady state in case of no uncertainty

The adjustment process for consumption, absent uncertainty, is given by (29.33) as

$$(\frac{C_{t+1}}{C_t})^{-\theta}(1 + r_{t+1}) = 1 + \rho,$$

or, taking logs,

$$\log \frac{C_{t+1}}{C_t} = \frac{1}{\theta}\left[\log(1 + r_{t+1}) - \log(1+\rho)\right]. \tag{29.37}$$

This is the deterministic Keynes-Ramsey rule in discrete time with separable CRRA utility. For any "small" x we have $\log(1 + x) \approx x$ (from a first-order Taylor approximation of $\log(1 + x)$ around 0). Hence, with $x = C_{t+1}/C_t - 1$,

[9] King and Rebelo, 1999, p. 947, express this in terms of the growth-adjusted discount factor $\beta \equiv (1+\rho)^{-1}\gamma^{1-\theta}$, so that $1 + r^* = (1+\rho)\gamma^{\theta} = \gamma/\beta$.

[10] Since $\gamma > 1$, only if $\theta < 1$ (which does not seem realistic, cf. Chapter 3), is it possible that $\rho > 0$ is not sufficient for (29.35) to be satisfied.

we have $\log C_{t+1}/C_t \approx C_{t+1}/C_t - 1$, so that (29.37) implies the approximate relation

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - \rho). \tag{29.38}$$

There is a supplementary way of writing the Keynes-Ramsey rule. Note that (29.34) implies $\log(1+r^*) = \log(1+\rho) + \theta \log \gamma$. Using first-order Taylor approximations, this gives $r^* \approx \rho + \theta \log \gamma \approx \rho + \theta g$, where $g \equiv \gamma - 1$ is the trend rate of technological progress. Thus $\rho \approx r^* - \theta g$, and inserting this into (29.38) we get

$$\frac{C_{t+1} - C_t}{C_t} \approx \frac{1}{\theta}(r_{t+1} - r^*) + g.$$

Then the technology-corrected consumption level, $c_t \equiv C_t/X_t$, moves according to

$$\frac{c_{t+1} - c_t}{c_t} \approx \frac{1}{\theta}(r_{t+1} - r^*),$$

since g is the growth rate of X_t.

29.3 On the approximate solution and numerical simulation

In the special case $\theta = 1$ (the log utility case), still maintaining the Cobb-Douglas specification of the production function, the model can be solved analytically provided capital is non-durable (i.e., $\delta = 1$).[11] It turns out that in this case the solution has consumption as a constant fraction of output (i.e., there is a constant saving rate as in the Solow growth model). Further, in this special case labor supply equals a constant and is thus independent of the productivity shocks. Since in actual business cycles, employment fluctuates a lot, this might not seem to be good news for a business cycle model.

But assuming $\delta = 1$ for a period length of one quarter or one year is "far out". Given a period length of one year, δ is generally estimated to be less than 0.1. And with $\delta < 1$, labor supply *is* affected by the technology shocks. An exact analytical solution, however, can no longer be found.

One can find an *approximate* solution based on a log-linearization of the model around the steady state. Without dwelling on the more technical details we will make a few observations.

[11] Alternatively, even allowing $\delta < 1$ one could coarsely assume that it is "gross-gross output", i.e., GDP + $(1 - \delta)K$, that is described by a Cobb-Douglas production function. Then, the model could again be solved analytically.

29.3.1 Log-linearization

If x^* is the steady-state value of the variable x_t in the non-stochastic case, then one defines the new variable

$$\hat{x}_t \equiv \log(\frac{x_t}{x^*}) = \log x_t - \log x^*. \tag{29.39}$$

That is, $\hat{x}_t$ is the logarithmic deviation of x_t from its steady-state value. But this is approximately the same as x's *proportionate* deviation from its steady-state value. This is because, when x_t is in a neighborhood of its steady-state value, a first-order Taylor approximation of $\log x_t$ around x^* gives

$$\log x_t \approx \log x^* + \frac{1}{x^*}(x_t - x^*),$$

so that

$$\hat{x}_t \approx \frac{x_t - x^*}{x^*}. \tag{29.40}$$

Note that working with the transformation $\hat{x}_t$ instead of x_t implies the convenience that

$$\begin{aligned}\hat{x}_{t+1} - \hat{x}_t &= \log(\frac{x_{t+1}}{x^*}) - \log(\frac{x_t}{x^*}) = \log x_{t+1} - \log x_t \\ &\approx \frac{x_{t+1} - x_t}{x_t}.\end{aligned}$$

That is, relative changes in x have been replaced by absolute changes in $\hat{x}$.

Some of the equations of interest are exactly log-linear from start. This is true for the production conditions (29.25), (29.26), and (29.27) as well as for the first-order condition (29.29) for the household. For other equations log-linearization requires approximation. Consider for example the time constraint (29.6). With L_t denoting leisure ($\equiv 1 - N_t$), this constraint implies

$$N^* \frac{N_t - N^*}{N^*} + L^* \frac{L_t - L^*}{L^*} = 0$$

or

$$N^* \hat{N}_t + L^* \hat{L}_t \approx 0, \tag{29.41}$$

by the principle in (29.40). From (29.29), taking into account that $1 - N_t = L_t$, we have

$$\begin{aligned}\theta L_t{}^{-\eta} &= C_t^{-\theta} w_t \equiv (c_t X_t)^{-\theta} \tilde{w}_t X_t \\ &= c_t{}^{-\theta} \tilde{w}_t X_t^{1-\theta}.\end{aligned} \tag{29.42}$$

In steady state this takes the form

$$\theta L^{*-\eta} = c^{*-\theta} \tilde{w}^* X_t^{1-\theta}. \tag{29.43}$$

We see that when $\gamma > 1$ (sustained technological progress), we *need* $\theta = 1$ for a steady state to exist (which explains why in their calibration King and Rebelo assume $\theta = 1$). This quite "narrow" theoretical requirement is an unwelcome feature and is due to the additively separable utility function assumed by King and Rebelo.

Combining (29.43) with (29.42) gives

$$\left(\frac{L_t}{L^*}\right)^{-\eta} = \left(\frac{c_t}{c^*}\right)^{-\theta} \frac{\tilde{w}_t}{\tilde{w}^*}.$$

Taking logs on both sides we get

$$-\eta \log \frac{L_t}{L^*} = \log \frac{\tilde{w}_t}{\tilde{w}^*} - \theta \log \frac{c_t}{c^*}$$

or

$$-\eta \hat{L}_t = \hat{w}_t - \theta \hat{c}_t.$$

In view of (29.41), this implies

$$\hat{N}_t = -\frac{L^*}{N^*}\hat{L}_t = \frac{1-N^*}{N^*\eta}\hat{w}_t - \frac{1-N^*}{N^*\eta}\theta\hat{c}_t. \tag{29.44}$$

This result tells us that the elasticity of labor supply w.r.t. a temporary change in the real wage depends negatively on η.[12] Indeed, calling this elasticity ε, we have

$$\varepsilon = \frac{1-N^*}{N^*\eta}. \tag{29.45}$$

Departing from the steady state, a one per cent increase in the wage ($\hat{w}_t = 0.01$) leads to an ε per cent increase in the labor supply, by (29.44) and (29.45). The number ε measures a kind of compensated wage elasticity of labor supply (in an intertemporal setting), relevant for evaluating the pure substitution effect of a temporary rise in the wage. King and Rebelo (1999) reckon N^* in the US to be 0.2 (that is, out of available time one fifth is working time). With $\eta = 1$ (as in most of the simulations run by King and Rebelo), we then get $\varepsilon = 4$. This elasticity is much higher than what the micro-econometric evidence suggests (at least for men), namely typically an elasticity below 1 (Pencavel 1986). But with labor supply elasticity as low as

[12] This is not surprising, since η reflects the desire for leisure smoothing across time.

1, the RBC model is far from capable of generating a volatility in employment comparable to what the data show.

For some purposes it is convenient to have the endogenous time-dependent variables appearing separately in the stationary dynamic system. Then, to describe the supply of output in log-linear form, let $y_t \equiv Y_t/X_t \equiv A_t f(\tilde{k}_t)N_t$ and $k_t \equiv K_t/X_t \equiv \tilde{k}_t N_t$. From (29.24),

$$y_t = A_t k_t^{1-\alpha} N_t^{\alpha},$$

and dividing through by the corresponding expression in steady state, we get

$$\frac{y_t}{y^*} = \frac{A_t}{A^*}(\frac{k_t}{k^*})^{1-\alpha}(\frac{N_t}{N^*})^{\alpha}.$$

Taking logs on both sides we end up with

$$\hat{y}_t = \hat{A}_t + (1-\alpha)\hat{k}_t + \alpha\hat{L}_t. \tag{29.46}$$

For the demand side we can obtain at least an approximate log-linear relation. Indeed, dividing trough by X_t in (29.21) we get

$$c_t + i_t = y_t,$$

where $i_t \equiv I_t/X_t$. This can also be written

$$\frac{c^*}{y^*}\frac{c_t - c^*}{c^*} + \frac{i^*}{y^*}\frac{i_t - i^*}{i^*} = \frac{y_t - y^*}{y^*},$$

which, using the hat notation from (29.40), can be written

$$\frac{c^*}{y^*}\hat{c}_t + \frac{i^*}{y^*}\hat{\imath}_t \approx \hat{y}_t. \tag{29.47}$$

to be equated with the right hand side of (29.46).

29.3.2 Numerical simulation

After log-linearization the model can be reduced to two coupled *linear* stochastic first-order difference equations in k_t and c_t, where k_t is predetermined, and c_t is a jump variable. There are different methods available for solving such an approximate dynamic system analytically.[13] Alternatively, based on

[13] For details one may consult Campbell (1994, p. 468 ff.), Obstfeld and Rogoff (1996, p. 503 ff.), or Uhlig (1999).

a specified set of parameter values one can solve the system by numerical simulation on a computer.

In any case, when it comes to checking the quantitative performance of the model, RBC theorists generally stick to *calibration,* that is, the method based on a choice of parameter values such that the model matches a list of data characteristics. In the present context this means that:

(a) the structural parameters $(\alpha, \delta, \rho, \theta, \eta, \omega, \gamma, N^*)$ are given values that are taken or constructed partly from national income accounting and similar data, partly from micro-econometric studies of households' and firms' behavior;

(b) the values of the parameters, ξ and σ_ε, in the stochastic process for the productivity variable A are chosen either on the basis of data for the Solow residual[14] over a long time period, or one or both values are chosen to yield, as closely as possible, a correspondence between the statistical moments (standard deviation, auto-correlation etc.) predicted by the model and those in the data.

The first approach to $(\xi, \sigma_\varepsilon)$ is followed by, e.g., Prescott (1986). It has been severely criticized by, among others, Mankiw (1989). In the short and medium term, the Solow residual is very sensitive to labor hoarding and variations in the degree of utilization of capital equipment. It can therefore be argued that it is the business cycle fluctuations that explain the fluctuations in the Solow residual, rather than the other way round.[15] The second approach, used by, e.g., Hansen (1985) and Plosser (1989), has the disadvantage that it provides no independent information on the stochastic process for productivity shocks. Yet such information is necessary to assess whether the shocks can be the driving force behind business cycles.[16]

[14] Given (29.24), take logs on both sides to get

$$\log Y_t = \log A_t + (1-\alpha)\log K_t + \alpha \log X_t + \alpha \log L_t.$$

Then the Solow residual; SR, may be defined by

$$\log SR_t \equiv \log A_t + \alpha \log X_t = \log Y_t - (1-\alpha)\log K_t - \alpha \log L_t.$$

[15] King and Rebelo (1999, p. 982-993) believe that the problem can be overcome by refinement of the RBC model.

[16] At any rate, *calibration* is different from econometric estimation and testing in the formal sense. Criteria for what constitutes a good fit are not offered. The calibration method should rather be seen as a first check whether the model is logically capable of

As hitherto we abide to the approach of King and Rebelo (1999) which like Prescott's is based on the Solow residual. The parameters chosen are shown in Table 19.2. Remember that the time unit is a quarter of a year.

Table 29.2. Parameter values

α	δ	ρ	θ	η	ω	γ	N^*	ξ	σ_ε
0.667	0.025	0.0163	1	1	3.48	1.004	0.2	0.979	0.0072

Given these parameter values and initial values of k and A in conformity with the steady state, the simulation is ready to be started. The shock process is activated and the resulting evolution of the endogenous variables generated through the "propagation mechanism" of the model calculated by the computer. From this evolution the analyst next calculates the different relevant statistics: standard deviation (as an indicator of volatility), autocorrelation (as an indicator of degree of persistence), and cross correlations with different leads and lags (reflecting the co-movements and dynamic interaction of the different variables). These model-generated statistics can then be compared to those calculated on the basis of the empirical observations.

In order to visualize the economic mechanisms involved, so-called *impulse-response functions* are calculated. Shocks before period 0 are ignored and the economy is assumed to be in steady state until this period. Then, a positive once-for-all shock to A occurs so that productivity is increased by, say, 1 % (i.e., given $A_{-1} = A^* = 1$, we put $\varepsilon_0 = 0,01$ in (29.3) with $t = 0$). The resulting path for the endogenous variables is calculated under the assumption that no further shocks occur (i.e., $\varepsilon_t = 0$ for $t = 1, 2, \ldots$). An "inpulse-response diagram" shows the implied time profiles for the different variables.

Remark. The text should here show some graphs of impulse-response functions. These graphs are not yet available. Instead the reader is referred to the graphs in King and Rebelo (1999), p. 966-970. As expected, the time profiles for output, consumption, employment, real wages, and other variables differ, depending on the size of ξ in (29.3). Comparing the case $\xi = 0$ (a purely temporary productivity shock) and the case $\xi = 0.979$ (a highly persistent productivity shock), we see that the responses are more drawn out over time in the latter case. This persistence in the endogenous

matching main features of the data (say the first and second moments of key variables). Calibration delivers a quantitative example of the working of the model. It does not deliver an econometric test of the validity of the model or of a hypothesis based on the model. Neither does it provide any formal guide as to what aspects of the model should be revised (see Hoover, 1995, pp. 24-44).

variables is, however, just inherited from the assumed persistence in the shock. And amplification is limited. When ξ is high, in particular when $\xi = 1$ (a permanent productivity shock), wealth effects on labor supply are strong and dampen the substitution effect.

29.4 The two basic propagation mechanisms

We have added technology shocks to a standard neoclassical growth model (utility-maximizing households, profit-maximizing firms, rational expectations, market clearing under perfect competition). The conclusion is that correlated fluctuations in output, consumption, investment, work hours, output per man-hour, real wages, and the real interest rate are generated. So far so good. There are two basic "propagation" mechanisms (transmission mechanisms) that drive the fluctuations:

1. *The capital accumulation mechanism.* To understand this mechanism in its pure form, let us abstract from the endogenous labor supply and assume an inelastic labor supply. A positive productivity shock increases marginal productivity of capital and labor. If the shock is not purely temporary, the household feels more wealthy. Both output, consumption and saving (due to intertemporal substitution in consumption) go up. The increased capital stock implies higher output also in the next periods. Hence output shows positive auto-correlation (persistence). And output, consumption, and investment move together (co-movement).

2. *Intertemporal substitution in labor supply.* An immediate implication of increased marginal productivity of labor is a higher real wage. To the extent that this increased real wage is only temporary, the household is motivated to supply more labor in the current period and less later. This is the phenomenon of intertemporal substitution in leisure. By the adherents of the RBC theory the observed fluctuations in work hours are seen as reflecting this.

29.5 Limitations

During the last 15-20 years there has been an increasing scepticism towards the RBC theory. The main limitation of the theory derives from its insistence upon interpreting fluctuations in employment as reflecting fluctuations in labor supply. The critics maintain that, starting from market clearing based on flexible prices, it is not surprising that difficulties matching the business cycle facts arise.

We may summarize the objections to the theory in the following four points:

a. *Where are the productivity shocks?* As some critics ask: "If productivity shocks are so important, why don't we read about them in the Wall Street Journal?" For example, it definitely seems hard to interpret the absolute economic contractions (decreases in GDP) that sometimes occur in the real world as due to productivity shocks. If the elasticity of output w.r.t. productivity shocks does not exceed one (as it does not seem to, empirically, according to Campbell 1994), then a *backward* step in technology at the aggregate level is needed. Although sound technological knowledge as such is always increasing, mistakes could be made in choosing technologies. At the *disaggregate* level, one can sometimes identify technological mistakes, like the use of DDT and its subsequent ban in the 1960's due to its damaging effects on health. But it is very hard to think of technological drawbacks at the *aggregate* level, capable of explaining the observed economic recessions. Think of the large and long-lasting contraction of GDP in the US during the Great Depression (27 % reduction between 1929 and 1933 according to Romer, 2001, p. 171). Sometimes the adherents of the RBC theory refer also to other kinds of supply shocks: changes in taxation, changes in environmental legislation etc. (Hansen and Prescott, 1993). But the problem is that significant changes in taxation and regulation occur rather infrequently and are therefore not a convincing candidate for the driving force in the stochastic process (29.3).

b. *Lack of internal propagation.* Given the micro-econometric evidence that we have, the two mechanisms above seem far from capable at generating the *large* fluctuations that we observe. Both mechanisms imply too little "amplification" of the shocks. Intertemporal substitution in labor supply does not seem able of generating much amplification. This is related to the fact that changes in real wages tend to be permanent rather than purely transitory. Permanent wage increases tend to have little or no effect on labor supply (the wealth effect tends to offset the substitution and income effect). Given the very minor temporary movements in the real wage that occur at the empirical level, a high intertemporal elasticity of substitution in labor supply[17] is required to

[17] Recall that this is defined as the percentage increase (calculated along a given indifference curve) in the ratio of labor supplies in two succeeding periods prompted by a one

generate the large fluctuations in employment observed in the data. But the empirical evidence suggests that this requirement is not met. Indeed, micro-econometric studies of labor supply indicate that this elasticity, at least for men, is quite small (in the range 0 to 1.5, typically below 1).[18] Yet, Prescott (1986) and Plosser (1989) assume it is around 4.

c. *Correlation puzzles.* Sometimes the sign, sometimes the size of correlation coefficients seem persevering wrong (see King and Rebelo, p. 957, 961). As Akerlof (2003, p. 414) points out, there is a conflict between the empirically observed pro-cyclical behavior of workers' quits[19] and the theory's prediction that quits should increase in cyclical *downturns* (since variation in employment is voluntary according to the theory). Considering a dozen of OECD countries Danthine and Donaldson (1993) find that the required positive correlation between labor productivity and output is visible only in data for the U.S. (and not strong), whereas the correlation is markedly negative for the majority of the other countries.

d. *Disregard of non-neutrality of money.* According to many critics, the RBC theory conflicts with the empirical evidence on the real effects of monetary policy.

Numerous, and more and more imaginative, attempts at overcoming the criticisms have been made; King and Rebelo (1999, p. 974-993) present some of these. In particular, adherents of the RBC theory have looked for mechanisms that may raise the size of labor supply elasticities at the aggregate level over and above that at the individual level found in microeconometric studies.

29.6 Conclusion

It seems advisory to make a distinction between on the one hand *RBC theory* (based on perfect competition and market clearing in an environment where productivity shocks are the driving force behind the fluctuations) and on the other hand the quantitative methods introduced by Lucas, Prescott, and others. A significant amount of recent research on business cycle fluctuations has

percentage increase in the corresponding wage ratio, cf. Chapter 5.

[18] *Handbook of Labor Economics*, vol. 1, 1986, Table 1.22, last column. See also Hall (1999, p. 1148 ff.).

[19] See Chapter 28.

left the RBC theory, but apply similar *quantitative methods*. These methods are now often summed up under the heading *DSGE models*. This approach consists in an attempt at building small quantitative *Dynamic Stochastic General Equilibrium* models. The economic content of such a model *can* be New Classical (as with Lucas and Prescott). Alternatively it can be more or less Keynesian or New Keynesian, based on a combination of imperfect competition and other market imperfections (also in the financial markets), and nominal and real price rigidities (see, e.g., Jeanne, 1998, Smets and Wouters, 2003, and Danthine and Kurmann, 2004).

Medium-term theory attempts to throw light on business cycle fluctuations and to clarify what kinds of counter-cyclical economic policy, if any, may be functional. This is probably the area within macroeconomics where there is most disagreement − and has been so for a long time. Some illustrating quotations:

> Indeed, if the economy did not display the business cycle phenomena, there would be a puzzle. ... costly efforts at stabilization are likely to be counterproductive. Economic fluctuations are optimal responses to uncertainty in the rate of technological change (Prescott 1986).
>
> My view is that real business cycle models of the type urged on us by Prescott have nothing to do with the business cycle phenomena observed in the United States or other capitalist economies. ... The image of a big loose tent flapping in the wind comes to mind (Summers 1986).

29.7 Bibliographic notes

In dealing with the intertemporal decision problem of the household we applied the substitution method. More advanced approaches include the discrete time Maximum Principle (see Chapter 8), the Lagrange method (see, e.g., King and Rebelo, 1999), or Dynamic Programming (see, e.g., Ljungqvist and Sargent, 2004).

29.8 Appendix: Technological change as a random walk with drift

In contrast to Prescott (1986) and King and Rebelo (1999), Plosser (1989) assumes that technological change is a random walk with drift. The representative firm has the production function

$$Y_t = Z_t F(K_t, N_t),$$

where Z_t is a measure of the level of technology, and the production function F has constant returns to scale. In the numerical simulation Plosser used a Cobb-Douglas function.

The technology variable Z_t (total factor productivity) is an exogenous stochastic variable. In contrast to the process for the logarithm of A_t above, where we had $\xi < 1$, we now assume a "unit root", i.e., $\xi = 1$. So the process assumed for $z_t \equiv \log Z_t$ is

$$z_t = \beta + z_{t-1} + \varepsilon_t, \tag{29.48}$$

a *random walk*. This corresponds to our general conception of technical knowledge as *cumulative.* If the deterministic term $\beta \neq 0$, the process is called a random walk *with drift*. In the present setting we can interpret β as some underlying given trend in productivity, suggesting $\beta > 0$.[20] Negative occurrences of the noise term ε_t need not in this case represent "technological regress", but just a technology development below trend (which will be the case if $-\beta \leq \varepsilon_t < 0$).

This version of the RBC model also faces difficulties. Indeed, embedded in a Walrasian equilibrium framework the specification (29.48) tends to generate too little fluctuation in employment and output. This is because, when shocks are permanent, large wealth effects offset the intertemporal substitution in labor supply.

29.9 Exercises

[20]The growth rate in total factor productivity is $(Z_t - Z_{t-1})/Z_{t-1}$. From (29.48) we have $E_{t-1}(z_t - z_{t-1}) = \beta$, and $z_t - z_{t-1} = \log Z_t - \log Z_{t-1} \approx (Z_t - Z_{t-1})/Z_{t-1}$ by a 1. order Taylor approximation of $\log Z_t$ about Z_{t-1}. Hence, $E_{t-1}(Z_t - Z_{t-1})/Z_{t-1} \approx \beta$. In Plosser's model all technological change is represented by change in Z_t, i.e., in (29.2) Plosser has $\gamma = 1$.