7. Real business cycle

We have argued that a change in fundamentals, modifies the long-run solution of the system, while a temporary change –not permanently affecting fundamentals– leads to a 'short-run' dynamics that eventually end with the system returning to the steady state (when the latest is asymptotically stable). In RBC models business cycles are represented as the dynamics originated by temporary departures from the steady state, mostly occurring due to random shocks to factors productivity (see our discussion in section 1). The idea that random shocks are responsible of macroeconomic fluctuations goes back to Ragnar Frisch (1933), but it has been effectively exploited for the first time by Brock and Mirman (1972) to study optimal growth in economies with productivity shocks.

Since shocks are represented by realizations of a random process $(A_t)_{t\in\mathcal{T}}$, the Ramsey model is proposed in its stochastic version. Moreover, for pedagogical reasons, we start presenting a version of the model that has a closed-form solution.[28] Finally, exactly because it is customary to represent business cycles as fluctuations about the the steady state, comparison with empirical evidence requires working with detrended data, so that the unique balanced growth solution is indeed the steady state. However, one should be aware that model evaluation is sensitive to the detrending method (see figure 7 below for an illustration).

The stochastic version of the RCK programming problem, presented in section 4.2; at the initial date,

$$\max_{(C_t,N_t,K_{t+1},\lambda_t)_{t\in\mathcal{T}}} \mathcal{L}_0 = \mathbb{E}_0 \sum_{t\in\mathcal{T}} \left\{\beta^t u(C_t,N_t) + \lambda_t \left[A_t F(K_t,N_t) + K_t(1-\delta) - K_{t+1} - C_t\right]\right\}$$

where $(C_t, K_{t+1})_{t\in\mathcal{T}}$ are restricted to the space of nonnegative sequences, $0 \leq N_t \leq 1$ at all t in $\mathcal{T}$, $K_0 > 0$ is given, A is a stochastic process with known distribution and $A_0 = 1$, $(\lambda_t)_{t\in\mathcal{T}}$ is a sequence of Lagrange multipliers. At an interior solution, first order conditions imply, at all t in $\mathcal{T}$,

$$\text{(E)} \qquad \mathbb{E}_t \frac{\beta u_{c,t+1}}{u_{c,t}}[1 + \underbrace{A_{t+1}F_{k,t+1} - \delta}_{r_{t+1}}] = 0$$

$$\text{(L)} \qquad \frac{u_{n,t}}{u_{c,t}} = \underbrace{A_t F_{n,t}}_{w_t}$$

$$\text{(F)} \qquad C_t = A_t F(K_t, N_t) + K_t(1-\delta) - K_{t+1}$$

These dynamic equations, $(E), (L), (R)$, describe the optimal behavior of $(C_t, N_t, K_{t+1})_{t\in\mathcal{T}}$. Other equations can be added to keep track of other variables of interest. For example $(Y_t, I_t)_{t\in\mathcal{T}}$,

$$\text{(Y)} \qquad Y_t = A_t F(K_t, N_t)$$

$$\text{(I)} \qquad I_t = K_{t+1} - K_t(1-\delta)$$

To see whether the RCK model delivers reasonable business cycle predictions, one usually proceed by computing the balance growth (or steady-state) and analyze the model dynamics as

[28] This section borrows from chapter 4 in Romer (1996) and from King and Rebelo (1999).

a technological shock, changing A, moves the system away from its long-run equilibrium. The dynamic behavior of the variables is described by the above difference equations, which normally do not admit a 'closed form' solution; loosely speaking, even by transforming variables in logs, it is in general impossible to write the system as one of independent equations, each of which described by a AR model, with stochastic term depending on A. The only exception is that of a Cobb-Douglas economy with full capital depreciation, presented in the next subsection. Thus, in general, one has to appeal to local analysis, studying the model behavior around a solution. This essentially boils down to analyze how the log-linear approximation of the dynamical system behaves around its steady-state.

7.1. **A simplified model, admitting a closed form solution.** To attain a closed form solution, we introduce the following assumptions,

- the per-period utility u is log-linear, $u(C_t, N_t) = \ln C_t + b \ln(1 - N_t), \quad b > 0$,
- the technology is Cobb-Douglas, $Y_t = A_t N_t^{\alpha} K_t^{1-\alpha}, \quad 0 < \alpha < 1$,
- capital fully depreciate at all dates, $\delta = 1$.

The 'log-log' parametrization of preferences has been suggested by Prescott and Plosser in the '80s, based on the results obtained on asset-price postwar data. It has the special feature that the intra-temporal substitution and income effects on labor supply of a contemporaneous change of the real wage (*i.e.* the ratio of wage to commodity price) cancel out. Indeed, when the real wage increases, the relative price of leisure to consumption also increases, hence the household demands less leisure (the intra-temporal substitution effect). Moreover, a higher real wage makes household income increase (for any given level of labor supply), making feasible to keep the same level of consumption expenditure with less labor (and more leisure), hence the household demands more leisure (the intra-temporal income effect). These two effects on labor supply cancel out when the utility function is log-linear. To see this, given the effects are all intra-temporal, consider the static household's problem: maximize $u(w, N) = \ln C + b \ln(1 - N)$ such that $C = wN$. First order conditions imply that the labor supply is,

$$N = \frac{1}{1 + b}$$

Thus, in the static model, the household supplies a fixed level of labor, independently of the real wage rate, which is decreasing in the elasticity of substitution between consumption and leisure b.[29] More generally, we know that in a dynamic model this might not be so because of the intertemporal substitution of labor supply, which we can summarize by the equation,

$$\mathbb{E}_t \frac{1 - N_t}{1 - N_{t+1}} = \mathbb{E}_t \frac{W_{t+1}/(\beta R_{t+1})}{W_t}$$

where, as usual, $W_t := F_{n,t}$ and $R_t := 1 + A_t F_{k,t} - \delta$ are the shadow real wage and gross interest rate.

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$$\left(\frac{\partial C}{\partial(1 - N)} \frac{1 - N}{C} \right)_{\bar{u}} = b$$

To solve the model, we use the intuition coming from the static labor supply and guess that there is a solution in which labor is constant over time. The solution is computed using the following optimality conditions (see example 4.1), at all t in $\mathcal{T}$,

$$\text{(E)} \qquad \frac{1}{C_t} = \beta(1-\alpha)\mathbb{E}_t\left[\frac{1}{C_{t+1}}\frac{Y_{t+1}}{K_{t+1}}\right]$$

$$\text{(L)} \qquad \frac{bC_t}{1-N_t} = \alpha\frac{Y_t}{N_t}$$

$$\text{(F)} \qquad C_t + K_{t+1} = Y_t$$

The first is the Euler equation, the second is the optimal labor supply and the third is the feasibility condition (recall that $\delta = 1$ implies $I_t = K_{t+1}$).

Notice from (L) that if labor supply is constant, also the propensity to consume must be so. But then, also the propensity to save and invest must be constant,

$$\frac{I_t}{Y_t} = \frac{K_{t+1}}{Y_t} = 1 - \frac{C_t}{Y_t}$$

Thus, we guess the system has a solution with constant labor supply and constant propensity to consume and to invest,

$$i)\ C_t = \mu Y_t, \quad ii)\ K_{t+1} = (1-\mu)Y_t.$$

We now verify our guess, checking if the fractions gamma and the fixed level of labor supply can be consistently determined. Start by substituting i) into (E),

$$\frac{1}{C_t} = \beta(1-\alpha)\mathbb{E}_t\left[\frac{1}{C_{t+1}}\frac{Y_{t+1}}{K_{t+1}}\right]$$

$$\frac{1}{\mu Y_t} = \beta(1-\alpha)\mathbb{E}_t\left[\frac{1}{\mu Y_{t+1}}\frac{Y_{t+1}}{K_{t+1}}\right]$$

$$\frac{1}{\mu Y_t} = \frac{\beta(1-\alpha)}{\mu}\mathbb{E}_t\left[\frac{1}{K_{t+1}}\right]$$

Use ii) into K_{t+1},

$$\frac{1}{Y_t} = \beta(1-\alpha)\mathbb{E}_t\left[\frac{1}{(1-\mu)Y_t}\right]$$

We are left to find the constant, optimal labor supply. From (L) and i),

$$b\mu Y_t = \alpha\frac{Y_t(1-N_t)}{N_t}$$

$$b\mu = \alpha\frac{1-N_t}{N_t}$$

implying that a solution with (N_t, C_t, K_t) having a constant labor supply and constant propensities to consume and invest (or save) exists:

$$\text{(a)} \qquad N_t = \frac{\alpha}{\alpha + b\mu}, \quad \mu = 1 - \beta(1-\alpha)$$

$$\text{(b)} \qquad C_t = \mu A_t N_t^{\alpha} K_t^{1-\alpha}$$

$$\text{(c)} \qquad K_{t+1} = (1-\mu)A_t N_t^{\alpha} K_t^{1-\alpha}$$

Next, we use this solution, in explicit form, to study the effects of a productivity shock. Assume,

$$\text{(A)} \qquad A_{t+s} = A_{t-1+s}^{\rho} e^{\varepsilon_{t+s}}, \quad \varepsilon_t \overset{iid}{\simeq} N(0, \sigma^2), \quad 0 < \rho < 1, \quad A_0 = 1.$$

Shocks are governed by the independent and identically, normally distributed variables $(e_t)_{t \in \mathcal{T}}$. This *white noise* variable well captures the idea that shocks are expected to be zero in every period; if they realize, the planner is able to compute their effects on A, just exploiting her knowledge of the model (A) governing A.

Suppose we start at $t = 0$ at the steady state and that a (totally unexpected) shock realizes in $t = 1$ and lasts only one period, $(\varepsilon_0, \varepsilon_1, \varepsilon_2, ...) = (0, 1, 0, ...)$. Consumption, investment and capital jump up, following the positive change in the productivity, while employment stays constant. Interestingly, the following period there is no shock, but this variables adjust back to the steady-state slowly, with hump-shaped dynamics. More precisely, every variable x (the log-consumption, the log-capital and the log-income) follows an $AR(2)$ process,

$$x_t = \kappa + \gamma_1 x_{t-1} + \gamma_1 x_{t-2} + \varepsilon_t$$

with identical coefficients $0 < \gamma_1 < 1$, $\gamma_2 < 0$, and different constant term κ. We detail this next.

First, notice that (b) and (c) imply,

$$\text{(d)} \qquad K_{t+1} = \mu(1 - \mu) C_t$$

Let the natural logs of each variable be denoted in small cases (*i.e.* $x_t = \ln X_t$).

$$a_{t+1} = \rho a_t + \varepsilon_t$$

and, from (d),

$$k_{t+1} = \underbrace{\ln \mu + \ln(1 - \mu)}_{z_0} + c_t = z_0 + c_t$$

Let n_* denote the constant, steady state level of labor supply. From (c), the law of motion of capital becomes,

$$k_{t+1} = \ln(1 - \mu) + a_t + \alpha n_* + (1 - \alpha) k_t$$

Since this holds at all t,

$$k_t = \ln(1 - \mu) + a_{t-1} + \alpha n_* + (1 - \alpha) k_{t-1}$$

from which we can derive a_{t-1},

$$a_{t-1} = k_t - \ln(1 - \mu) - \alpha n_* - (1 - \alpha) k_{t-1}$$

Hence,

$$a_t = \rho \left[\overbrace{k_t - \ln(1 - \mu) - \alpha n_* - (1 - \alpha) k_{t-1}}^{a_{t-1}} \right] + \varepsilon_t$$

Use this to substitute into the law of motion of capital,

$$\begin{aligned}
k_{t+1} &= \ln(1-\mu) + \rho\left[k_t - \ln(1-\mu) - \alpha n_* - (1-\alpha)k_{t-1}\right] + \varepsilon_t + \alpha n_* + (1-\alpha)k_t \\
&= (1-\rho)\underbrace{\left[\ln(1-\mu) + \alpha n_*\right]}_{z_1} + \rho k_t - \rho(1-\alpha)k_{t-1} + \varepsilon_t + (1-\alpha)k_t \\
&= (1-\rho)z_1 + (1-\alpha+\rho)k_t - \rho(1-\alpha)k_{t-1} + \varepsilon_t
\end{aligned}$$

Next, we derive the consumption process; from (d), taking logs,

$$\begin{aligned}
c_t &= k_{t+1} - z_0 \\
&= (1-\rho)z_1 + (1-\alpha+\rho)k_t - \rho(1-\alpha)k_{t-1} + \varepsilon_t - z_0 \\
&= (1-\rho)z_1 + (1-\alpha+\rho)(z_0 + c_{t-1}) - \rho(1-\alpha)(z_0 + c_{t-2}) + \varepsilon_t - z_0 \\
&= (1-\alpha+\rho)c_{t-1} - \rho(1-\alpha)c_{t-2} + (1-\rho)(z_1 - \alpha z_0) + \varepsilon_t
\end{aligned}$$

Thus, by redefining constants κ_1, κ_2 in the obvious way,

$$\begin{aligned}
k_{t+1} &= \kappa_1 + (1-\alpha+\rho)k_t - \rho(1-\alpha)k_{t-1} + \varepsilon_t \\
c_t &= \kappa_2 + (1-\alpha+\rho)c_{t-1} - \rho(1-\alpha)c_{t-2} + \varepsilon_t
\end{aligned}$$

Moreover, it is easy to check that output has a process,

$$y_t = \kappa_3 + (1-\alpha+\rho)y_{t-1} - \rho(1-\alpha)y_{t-2} + \varepsilon_t$$

We conclude that all the three variables considered follow the same AR(2) model, with different constant terms κ. Thus, after a positive temporary shock, consumption and investments respond to an output increase. Similarly to actual data, their response is hump-shaped; which is essentially due to the combination of a positive coefficient in the first-order lagged variable and a negative coefficient in the second-order one. However, contrary to actual data, model data perfectly comove with output: the γ coefficients are the same. This implies that consumption is too volatile and investment is too little. The reason for this is that saving rate is constant, similarly to Solow's growth model: propensity to consume is,

$$\frac{C_t}{Y_t} = \mu = 1 - \beta(1-\alpha)$$

In addition, labor supply is constant, instead of being strongly procyclical as in actual data, and real wage is strongly procyclical, instead of being essentially constant.

Finally, much of the persistence of a shock in affecting economic activity depends on ρ, that is on the autoregressive coefficient of the factor productivity. If $\rho = 0$ the model governing consumption, capital and output reduces to an $AR(1)$; hence the technological shock still has some, but not much, persistence. For reasonable values of the other parameters, most of the adjustment, following a one-time shock, occurs within the first two or three periods. This, more limited, persistency is due to the increase of total savings (at constant propensity to save) in response of a positive income shock.

Now, suppose we modify the economy assuming there is partial depreciation of capital, $0 \le \delta < 1$. Indeed, quarterly data show δ is closed to zero (approximately 1-2.5 per cent in the US). Then, labor supply is not constant anymore. To see what this brings up, suppose there is no depreciation, $\delta = 0$, and no growth; so that investment is zero when there is no shock. A shock raises productivity of capital and induces some investment. No matter if saving rate is constant, this is possible because labor supply is not constant; as $W_2/(R_2W_1)$ decreases, driven by the the change of productivity of labor and capital, labor supply is anticipated. Thus, even if contemporaneous income- and substitution- effects cancel out, labor supply and demand increase busting up production and investment. Moreover, when $0 \leq \delta < 1$, it turns out that the saving rate is not constant (when labor supply is not constant, our previous reasoning –based on (L)– shows that saving rate C/Y varies over time with N): it is optimal for the saving rate to increase in response to a positive income shock, providing an essential source of 'consumption smoothing'.

Summing up, capital depreciation suffices to modify the prediction of the simple Cobb-Douglas economy, so as to generate more plausible consumption, investment and employment fluctuations. Yet, it opens up the problem of how to solve for dynamics. We explore this issue next.

7.2. **The log-linear model with inelastic labor supply.** For dynamic models with no closed form solution, we proceed by considering its log-linear transformation. Variables are taken in natural logs and functions are approximated to the first order, by Taylor's formula, around the steady state.

To illustrate this procedure, we first consider a stochastic version of the Ramsey-Cass-Koopmans model (P), with inelastic labor supply:

(Ps)

$$\max_{(\ldots.C_t,K_{t+1}\ldots.)} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t)$$

$$\begin{aligned}
&C_t + K_{t+1} = f(K_t; A_t) + (1-\delta)K_t, \quad t \in \mathcal{T} \\
&A_{t+1} = A_t^{\rho} e^{\varepsilon_{t+1}}, \quad \varepsilon_{t+1} \overset{iid}{\simeq} N(0, \sigma^2), \quad 0 \leq \rho < 1, \quad A_0 = 1, \quad t \in \mathcal{T} \\
&K_0 > 0 \;\; \text{given}
\end{aligned}$$

where the variables are in capital letters for convention as well as for notational simplicity (later we will indicate in small characters their logarithmic transformations).

Finally, to capture economic interactions, we adopt the following parametrization,

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

$$Y_t = f(K_t; A_t) = A_t K_t^{\alpha}, \quad 0 < \alpha < 1$$

Notice that the utility is of the iso-elastic (or constant relative risk aversion) type (CRRA) ($\gamma = -c(u''(c)/u'(c))$), independent of c), encompassing the logarithmic utility for $\gamma \to 1$.

At all t in $\mathcal{T}$, first order conditions are,

$$\text{(Focs)} \qquad \begin{aligned} &1 = \mathbb{E}_t\left[\beta\left(\frac{C_t}{C_{t+1}}\right)^{\gamma} R_{t+1}\right] \\ &C_t + K_{t+1} = A_t K_t^{\alpha} + (1-\delta)K_t \\ &A_t = A_{t-1}^{\rho} e^{\varepsilon_t} \\ &R_{t+1} := \alpha A_{t+1} K_{t+1}^{\alpha-1} + 1 - \delta = \alpha\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta \end{aligned}$$

Computing the steady state (C_*, K_*, R_*) we find,

$$\text{(SS)} \qquad \begin{aligned} &1 = \beta R_* \\ &C_* = A_* K_*^{\alpha} - \delta K_* \\ &A_* = 1, \quad \text{this is a restriction we impose} \\ &R_* := \alpha A_* K_*^{\alpha-1} + 1 - \delta = \alpha\frac{Y_*}{K_*} + 1 - \delta \end{aligned}$$

Log-linearizations typically hinge on first-order (linear) approximations around the steady state. Some common tricks will be used. First, consider any variable X_t and a constant (steady-state) value X.

$$x_t := ln\left(\frac{X_t}{X}\right) = lnX_t - lnX$$

The following is correct to the first-order:

(i) $X_t \approx (1 + x_t)X$

(ii) $x_t \approx \frac{dX_t}{X}$, $dX_t := X_t - X$.

In words, the rate of change of a variable around its steady-state value is approximately equal to the difference of the logarithmic transformation of the variable and the logarithmic value of its steady-state. We now derive approximations (i) and (ii). For (i), first, notice that $e^{x_t} = X_t/X$, just taking the exponential of the above definition. The steady state value of x_t is zero; hence a first order approximation of e^{x_t} around $x_t = 0$ yields,

$$\frac{X_t}{X} = e^{x_t} \approx e^0 + e^0(x_t - 0) = 1 + x_t$$

For (ii), simply observe that, by (i), $x_t \approx (X_t - X)/X =: dX_t/X$.

Using these approximations we can express (Focs), respectively, as follows,

$$\text{(c)} \qquad 0 = \mathbb{E}_t\left[\gamma(c_t - c_{t+1}) + r_{t+1}\right]$$

$$\text{(k)} \qquad c_t = -\frac{K_*}{C_*}k_{t+1} + \frac{K_*}{C_*}R_* k_t + \frac{Y_*}{C_*}a_t$$

$$\text{(a)} \qquad a_{t+1} = \rho a_t + \varepsilon_{t+1}$$

$$\text{(r)} \qquad r_t = \pi\left[a_t - (1-\alpha)k_t\right], \quad \pi := [1 - \beta(1-\delta)]$$

These are derived using Taylor first-order approximations around the steady-state. So, each function $F(X)$ is substituted by $F(X_*) + DF(X_*) \cdot (X - X_*)$. We detail the derivation, in remark 7.1 below.

What about investments? Recall that the national account of this economy is $Y_t = C_t + I_t$; using the law of motion of capital, the latter can be rewritten as $I_t = K_{t+1} - (1-\delta)K_t$. This equation, log-linearized, yields,

$$i_t = \frac{K_*}{I_*}\left[k_{t+1} - (1-\delta)k_t\right]$$

Moreover, since $I_* = \delta K_*$,

$$i_t = \frac{1}{\delta}\left[k_{t+1} - (1-\delta)k_t\right]$$

Thus, one can explicitly incorporate investments in the model by adding this equation, along with the investment variable i_t.

To sum up, the log-linearized system $(c) - (r)$ forms a system of first order, linear, difference equations. We can also reduce the equilibrium conditions to a 2×2 system of linear, difference equations: substituting in for the interest rate r_{t+1}, and for the innovation variable a_{t+1}, respectively, using equations (r) and (a),

$$\text{(c)} \qquad 0 = \mathbb{E}_t\left[\gamma(c_t - c_{t+1}) + \pi\left(\rho a_t + \varepsilon_{t+1} - (1-\alpha)k_{t+1}\right)\right], \ \pi := [1 - \beta(1-\delta)]$$

$$\text{(k)} \qquad c_t = -\frac{K_*}{C_*}k_{t+1} + \frac{K_*}{C_*}R_* k_t + \frac{Y_*}{C_*}a_t$$

Remark 7.1 (Deriving the log-linearized system)**.** *Start from* (c)*. Note that the right hand side is the function* F *we want to approximate, whose value is* 1 *both in and out of the steady-state; hence,*

$$\begin{aligned}
1 &\approx \left(\frac{C_*}{C_*}\right)^{\gamma}\beta R_* + \mathbb{E}_t\left[\gamma\beta C_t^{\gamma-1}C_{t+1}^{-\gamma}R_{t+1}dC_t - \gamma\beta C_t^{\gamma}C_{t+1}^{-\gamma-1}R_{t+1}dC_{t+1} + \beta C_t^{\gamma}C_{t+1}^{-\gamma}dR_{t+1}\right]_{SS} \\
0 &= \mathbb{E}_t\left[\gamma\beta C_*^{\gamma-1}C_*^{-\gamma}R_*(dC_t/C_*)C_* - \gamma\beta C_*^{\gamma}C_*^{-\gamma-1}R_*(dC_{t+1}/C_*)C_* + \beta C_*^{\gamma}C_*^{-\gamma}(dR_{t+1}/R_*)R_*\right] \\
&\approx \mathbb{E}_t\left[\gamma\beta R_*(c_t - c_{t+1}) + \beta R_* r_{t+1}\right], \quad \textit{where at SS } 1 = \beta R_*
\end{aligned}$$

Next, consider (k)

$$\begin{aligned}
0 &\approx C_*c_t + K_*k_{t+1} - (1-\delta)K_*k_t - \left(\alpha A_t K_t^{\alpha-1}dK_t + K_t^{\alpha}dA_t\right)_{SS} \\
&= C_*c_t + K_*k_{t+1} - (1-\delta)K_*k_t - \alpha A_* K_*^{\alpha}\left(\alpha k_t + a_t\right) \\
&= C_*c_t + K_*k_{t+1} - \left[(1-\delta)K_* + A_*K_*^{\alpha}\right]k_t - A_*K_*^{\alpha}a_t \\
&= C_*c_t + K_*k_{t+1} - R_*K_*k_t - Y_*a_t
\end{aligned}$$

where in the forth equation we used $R_* = \alpha A_* K_*^{\alpha-1} + 1 - \delta$ *and the definition of output at the steady state,* $Y_* = A_*K_*^{\alpha}$*.*

Next, for (a)*, take the logarithmic transformation of* $A_t = A_{t-1}^{\rho}e^{\varepsilon_t}$*:* $\ln A_t = \rho \ln A_{t-1} + \varepsilon_t$*. Subtract from the left hand side* $\ln A_*$ *and from the right hand side* $\rho \ln A_*$ *(whose values are both zero). This, as claimed, yields:*

$$ln\left(\frac{A_t}{A_*}\right) = \rho \ln\left(\frac{A_{t-1}}{A_*}\right) + \varepsilon_t.$$

Finally, we recover (r).

$$
\begin{aligned}
0 &= -R_t + \alpha A_t K_t^{\alpha-1} + 1 - \delta \\
&\approx -dR_t + \left[\alpha(\alpha-1)A_t K_t^{\alpha-2} dK_t + \alpha K_t^{\alpha-1} dA_t\right]_{SS} \\
&= -\frac{dR_t}{R_*}R_* + \alpha(\alpha-1)A_* K_*^{\alpha-2}\frac{dK_t}{K_*}K_* + \alpha K_*^{\alpha-1}\frac{dA_t}{A_*}A_* \\
&= -R_* r_t + \alpha A_* K_*^{\alpha-1}\left[a_t - (1-\alpha)k_t\right] \\
&= -R_* r_t + (R_* - (1-\delta))\left[a_t - (1-\alpha)k_t\right]
\end{aligned}
$$

So, $r_t \approx \left[1 - \frac{1-\delta}{R_*}\right]\left[a_t - (1-\alpha)k_t\right]$, *and using* $\beta R_* = 1$ *we obtain (r).*

7.2.1. *A qualitative analysis of how the model responds to a shock.* Consider the whole extended system, at all $t \geq 1$,

$$
\begin{aligned}
&\text{(y)} && y_t = a_t + \alpha k_t \\
&\text{(c)} && 0 = \mathbb{E}_t\left[\gamma(c_{t-1} - c_t) + r_t\right] \\
&\text{(k)} && k_{t+1} = -\frac{C_*}{K_*}c_t + R_* k_t + \frac{Y_*}{K_*}a_t \\
&\text{(i)} && k_{t+1} = k_t + (i_t - k_t)\delta \\
&\text{(a)} && a_{t+1} = \rho a_t + \varepsilon_{t+1} \\
&\text{(r)} && r_t = \pi\left[a_t - (1-\alpha)k_t\right], \quad \pi := \left[1 - \beta(1-\delta)\right]
\end{aligned}
$$

Assume, again, a one-time shock occurs at time one, while the system is at steady state,

$$(\varepsilon_0, \varepsilon_1, \varepsilon_2, ...) = (0, 1, 0, ...)$$

The shock is 'unespected', in the sense that the central planner expects a zero shock at all dates, and it produces the following effects:

- all variables $(y_t, c_t, i_t, k_{t+1}, r_t)$ increase in $t = 1$;
- consumption increases less than output;
- investment increases more than output;
- next period capital stock increases less than investments.

Variables, contemporaneous (date 1) response to the shock is captured by the following, date-one, multipliers.

$$
\begin{aligned}
&\frac{\partial y_1}{\partial a_1} = 1 \\
&\frac{\partial c_1}{\partial a_1} = \frac{1}{\gamma}\frac{\partial r_1}{\partial a_1} = \frac{\pi}{\gamma}, \quad \pi := 1 - \beta(1-\delta) \\
&\frac{\partial i_1}{\partial a_1} = \frac{1}{\delta}\frac{\partial k_2}{\partial a_1} \\
&\frac{\partial r_1}{\partial a_1} = \pi
\end{aligned}
$$

Thus, we are left to compute how capital stock k_2 changes with a_1. This is computed from (k),

$$\begin{aligned}
\frac{\partial k_2}{\partial a_1} &= -\frac{C_*}{K_*}\frac{\partial c_1}{\partial a_1} + \frac{Y_*}{K_*} \\
&= \frac{1}{K_*}\left(Y_* - C_*\frac{\pi}{\gamma}\right) \\
&= \frac{1}{K_*}\left(C_* + I_* - C_*\frac{\pi}{\gamma}\right) \\
&= \frac{1}{K_*}\left[C_*\left(1 - \frac{\pi}{\gamma}\right) + I_*\right] \\
&= \frac{1}{K_*}\left[C_*\left(1 - \frac{\pi}{\gamma}\right) + \delta K_*\right] \\
&= \frac{C_*}{K_*}\left(1 - \frac{\pi}{\gamma}\right) + \delta
\end{aligned}$$

Since, $C_*/K_* = K_*^{1-\alpha} - \delta$,

$$\frac{\partial k_2}{\partial a_1} = (K_*^{1-\alpha} - \delta)\left(1 - \frac{\pi}{\gamma}\right) + \delta = K_*^{1-\alpha}\left(1 - \frac{\pi}{\gamma}\right) + \delta\frac{\pi}{\gamma} > 0$$

From which,

$$\frac{\partial i_1}{\partial a_1} = \frac{1}{\delta}\frac{\partial k_2}{\partial a_1} = \left(\frac{K_*^{1-\alpha}}{\delta} - 1\right)\left(1 - \frac{\pi}{\gamma}\right) + 1 > 1$$

The inequalities hold, at least, for plausible parameter values (e.g., for quarterly US data, $\delta = 0.025, \beta = 0.98, \gamma \approx 1$, implying $0 \leq \pi \leq 1$, according to King and Rebelo, who also suggest $\alpha = 0.667$).

Therefore, upon a positive productivity shock, the productivity of capital and the income increase. As a response, it is optimal for the household to use a fraction of this additional income to raise consumption in the future (*i.e.* to engineer an intertemporal consumption smoothing). Contrary to the previous case with $\delta = 1$, the propensity to consume and the saving rate are not constant. Precisely, the positive income shock makes the propensity to consume fall and the propensity to save rise above theirs steady state levels. To see this, we can compute the deviation of the marginal propensity to consume from its steady state level; since, $\Gamma_t := C_t/Y_t$ has a steady state value of $\Gamma_* = 1 - \delta(K_*/Y_*)$, log-linearizing, at the steady state, we attain its local dynamics, $\gamma_t := d\Gamma_t/\Gamma_*$,

$$\gamma_t = \frac{Y_*}{1 - \delta(K_*/Y_*)}\left(\Gamma_* c_t - y_t\right)$$

which is obviously negative (the saving rate raises above steady state) whenever output increases more than consumption. From the second period onwards, as capital stock start increasing, the return to capital goes back to its normal level and so does the interest rate, dragging down al the other variables.

Saving decisions are very useful to understand the importance of investment and capital accumulation decisions to satisfactorily reproduce business cycle fluctuations. Thus, although our benchmark RBC model has little persistence and does not amplify much of the initial productivity shock, the endogenous, optimal responses of the household are central to capture actual, economic dynamics.

So far, there is an important aspect that is missing, a more realistic labor market, with elastic labor supply. This will turn out to be important to generate better predictions also on the other variables. Hence, before examining the model predictions further, we modify it assuming labor effort produces some disutility.

7.3. **The log-linear model with elastic labor supply.** Consider the 'log-log' case as above (see also example 4.1 above). Computing the balanced growth, $(Y_*, C_*, K_*, N_*, R_*, W_*)$, we find,

$$\text{(SS)}\qquad \begin{aligned} 1 &= \beta R_* \\ Y_* &= A_* K_*^{1-\alpha} N_*^{\alpha} \\ C_* &= Y_* - \delta K_* \\ N_* &= 1 - b\frac{C_*}{W_*} \\ A_* &= 1, \quad \text{this is a restriction we impose} \\ R_* &= F_{K*} + 1 - \delta = (1-\alpha)A_* \left(\frac{N_*}{K_*}\right)^{\alpha} + 1 - \delta = (1-\alpha)\frac{Y_*}{K_*} + 1 - \delta \\ W_* &= F_{N*} = \alpha A_* \left(\frac{K_*}{N_*}\right)^{1-\alpha} = \alpha \frac{Y_*}{N_*} \end{aligned}$$

We log-linearize the system around the balance growth and find that at all t in $\mathcal{T}$,

$$\text{(y)}\qquad y_t = a_t + (1-\alpha)k_t + \alpha n_t$$

$$\text{(c)}\qquad 0 = \mathbb{E}_{t-1}\left[c_{t-1} - c_t + r_t\right]$$

$$\text{(k)}\qquad c_t = -\frac{K_*}{C_*}k_{t+1} + \frac{K_*}{C_*}R_* k_t + \frac{N_*}{C_*}W_* n_t + \frac{Y_*}{C_*}a_t$$

$$\text{(n)}\qquad n_t = (1-N_*)(w_t - c_t)$$

$$\text{(a)}\qquad a_t = \rho a_{t-1} + \varepsilon_t$$

$$\text{(r)}\qquad r_t = \pi\left[a_t + \alpha(n_t - k_t)\right], \quad \pi := \left[1 - \beta(1-\delta)\right]$$

$$\text{(w)}\qquad w_t = a_t - (1-\alpha)(n_t - k_t)$$

As for the economy with inelastic labor supply, $I_* = \delta K_*$,

$$\text{(i)}\qquad i_t = \frac{1}{\delta}\left[k_{t+1} - (1-\delta)k_t\right]$$

We have learned above that the 'log-log' utility has intra-temporal income and substitution effects that cancel out. Hence, a productivity shock essentially produces the following response of labor supply. First, in the period when shock realizes, labor productivity jumps up and labor supply and demand follows (the change in the current interest rate places no role). From the second period onwards the intertemporal labor substitutions kick in. To explore this latter effect, observe that (by (n) and (c)) intertemporal labor supply is,

$$n_{t+1} - n_t = (1-N_*)\left(w_{t+1} - w_t - r_{t+1}\right), \quad \Delta n_{t+1} = (1-N_*)\left(\Delta w_{t+1} - r_{t+1}\right)$$

As in the inelastic labor supply economy, after a positive productivity shock, the marginal productivity of capital increases and so the rate of return to saving. However, now also the labor productivity raises, pushing up the real wage (labor demand). Thus, the representative household finds optimal to engineer some consumption smoothing, as before; except that now, this strategy can and will also be sustained increasing current labor supply (or work effort); implying that it does not necessarily and solely rely on an increase in the saving rate. In other words, total saving and capital can also be increased working and producing more output.

The labor response to a shock have several additional effects. First, it amplifies the consequences of a productivity shock on output, which now has a sharper change with respect to the inelastic labor-supply economy:

$$\frac{\partial y_1}{\partial a_1} = 1 + \frac{\partial n_1}{\partial a_1}$$

This does also amplify the response of consumption and investment. Second, elastic labor supply has important effects on the dynamics of the interest rate. You can see that its initial response is,

$$\frac{\partial r_1}{\partial a_1} = \pi + \frac{\partial n_1}{\partial a_1}$$

that is grater than π, namely of its response when labor supply is inelastic. Moreover, looking at (r) you can observe that the dynamics of r_t positively covariates with $(n_t - k_t)$, rather than with $-k_t$ only. In words, the increase in labor supply, following a positive productivity shock, pushes up interest rate in the first period (since the capital stock at t=1 is given at its steady-state, implying $k_1 = 0$), and later balances the effect of capital accumulation on the productivity of capital and interest rate. As the effects of the productivity shock vanishes, the latest effect prevails and the interest rate goes back to its steady state level. Thus, in the model with endogenous labor supply, interest rates fluctuations are more pronounced and long lasting.

7.4. **A quantitative analysis of how the model responds to a shock.** To compare the model dynamic predictions with data, we need to obtain some specific information, which may either be gathered directly from existent empirical studies or through the model *calibration*.[30] Oversimplifying, calibration consists in two phases, the choice of

(1) functional forms, for preferences, technologies and exogenous processes (*e.g.* technological progress, population dynamics etc.) and
(2) parameter values (*e.g.* capital depreciation δ, factor elasticities α, preference parameters γ, β, etc.).

As for functional forms, for example, we argued that some empirical studies support the choice of a log-log, per-period utility; Prescott did also suggested to use a Cobb-Douglas production function, based on the observation that this is consistent with the fact that the K/Y ratio is roughly constant over time. The choice of parameters of our benchmark RBC model exploits empirical evidence whenever possible. Those parameters which are not observable

[30]Read chapter 1, par. 4, in Cooley and Prescott (1995), *Economic Growth and Business Cycle*, for a sharper discussion on calibration.

are 'calibrated' by finding those 'admissible' figures which generate model steady-state values matching the evidence on *great ratios* (em e.g. capital share in output, investment-to-capital, investment-to-income, consumption-to-income, avg. hour of work per week). This matching test can be harder than it seems, as one has to reconcile measures obtained from an abstract model with those obtained from data. For example, US National Income and Product Accounts (NIPA) incompletely accounts for some measures of service from capital stock when it computes output or some measures of investment when it comes to compute capital stock; for instance, it does not impute to output the flow of services from consumer durables or from government expenditure (see the discussion in Cooley and Prescott, p. 17-20).

Next, using the outcome of calibration one can test if the dynamic predictions, resulting from a shock, fit the RBC evidence (according to some test comparing moments). If not, then one can go back to the model specification and calibration to try some alternative assumptions on its fundamentals.

As an illustration, usually the discount factor is taken to be the average return to investment in the stock market, an annual 6.5% in the US, $r_* = 1.625\%$ per quarter, based on Standard and Poor 500 Index between 1948 and 1986). The US labor/income share $(w_* N_*/Y_*)$ is about 2/3, hence we use this value for α. A_* does only affect the scale of the economy, hence we can normalize it to one. The rate of capital depreciation is taken to be $\delta = 2.5\%$ per quarter (10% annum). With this information, take the penultimate of the SS equations $(r_* = A_* F_{K_*} - \delta)$; then, in our model we find,

$$\frac{K_*}{N_*} = \frac{[(1-\alpha)A_*]^{1/\alpha}}{r_* + \delta}$$

Since,

$$\frac{Y_*}{N_*} = A_* \left(\frac{K_*}{N_*}\right)^{1-\alpha}$$

We can also retrieve $K_*/Y_* = (K/N)_*/(Y/N)_*$. All the other steady state values can be determined. Also $\beta = 1/(1 + r_* - \delta)$ is determined.

The process representing technological progress, in our specification, coincides with the 'Solow's residuals':

$$\ln A_t = \ln Y_t - \alpha \ln N_t - (1-\alpha) \ln K_t$$

Then, ρ is estimated using OLS, having assumed that A_t follows an AR(1),

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t$$

From the estimated residuals ε one also obtains an estimate of the standard deviation of technological progress. King and Rebelo (1999) found $\widehat{\rho} = 0.979$, $\widehat{\sigma} = 0.0072$.

We are now ready to present some quantitative results, which essentially use the above calibration procedure. The next two tables,[31] respectively, report the moments of some key macroeconomic variables obtained as predictions of the above RBC model and those computed with actual US data.

[31] Tables are from King and Rebelo (1999).

Table 3
Business Cycle Statistics for Basic RBC Model[35]

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.39	1.00	0.72	1.00
C	0.61	0.44	0.79	0.94
I	4.09	2.95	0.71	0.99
N	0.67	0.48	0.71	0.97
Y/N	0.75	0.54	0.76	0.98
w	0.75	0.54	0.76	0.98
r	0.05	0.04	0.71	0.95
A	0.94	0.68	0.72	1.00

Note: All variables have been logged (with the exception of the real interest rate)and detrended with the HP filter.

Figure 5. Model predictions - King and Rebelo (1999)

Table 1
Business Cycle Statistics for the U.S. Economy

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Figure 6. US actual data - King and Rebelo (1999)

Volatility. Volatility is measured by the standard deviation. The RBC model explains 77% of business fluctuations ($0.77 = 1.39/1.81$). Investment is about three times more volatile than output (see relative STDs) as for actual data. Consumption volatility is 1/3 than output, instead of 2/3 of actual data. The model explains only 1/6 of the actual RBC fluctuations of real interest rate, while it overestimate the wage-rate fluctuations by approximately 10 %. Employment is predicted to fluctuate less than a half of the actual data.

Comovement and persistence. One measure of persistence is the first-order autocorrelation. The RBC model generates less persistence than in the data. The model gives little amplification to productivity shocks: output is only 1.48 more volatile than productivity. Comovement is documented by contemporaneous correlation with output. The RBC model generates substantial comovement of output with employment, consumption, investment. These figures are a little higher than those computed with actual data.

We also report a few panels of the impulse response analysis presented in King and Rebelo (1999) for a 1% productivity shock (see figure 7).

Remark 7.2 (Dynamic multipliers)**.** *A business cycle is reproduced by introducing a small, transitory, shock in the factor productivity at the steady state. If the shock is $AR(1)$, with $0 \leq \rho < 1$, it will eventually vanish. Then, if the steady-state is asymptotically stable, the model will converge back to the original steady state. If the shock is a 1% one can think at the cumulated response of each variable, over the cycle, as a cyclical dynamic multiplier, measured with respect to technological progress. If we had to introduce fiscal (or monetary) variables, such as government expenditure, we could similar derive dynamic multipliers corresponding to alternative fiscal (or monetary) policies. These would be the dynamic analogue of the fiscal multipliers of the static macro models (e.g. IS-LM).*

7.5. **Main criticisms to the RBC methodology and early results.** First of all, recall that the RBC research program intended to deliver a theory which could simultaneously explain growth and business cycle evidence. A first glance to its early results says that it has been successful. Despite the goal was ambitious, a simple theory and methods delivered surprisingly realistic predictions, capturing several important, empirical regularities. The model is really basic and, by construction, looks only at Pareto efficient allocations; hence, it would deliver the same allocations of a general equilibrium, dynamic, Walrasian economy, with no market frictions. Moreover, the theory is purely constructed looking at the real economy, there are no issue related to malfunctioning credit or financial markets, nor of money and liquidity, which are instead central in the Keynesian explanations of economic cycles. A Keynesian economist, Rogoff (1986), wrote.

> The... real business cycle results... are certainly productive. It as been said that a brilliant theory is one which at first seems ridiculous and later seems obvious. There are many that feel that (RBC) research has passed the first test. But they should recognize the definite possibility that it may someday pass the second test as well.

There have been many criticisms raised by Keynesian economists which had not an important impact on RBC program, also because they could essentially be 'fixed' with minor changed of the benchmark model. These were mainly concerned with some parameters values used

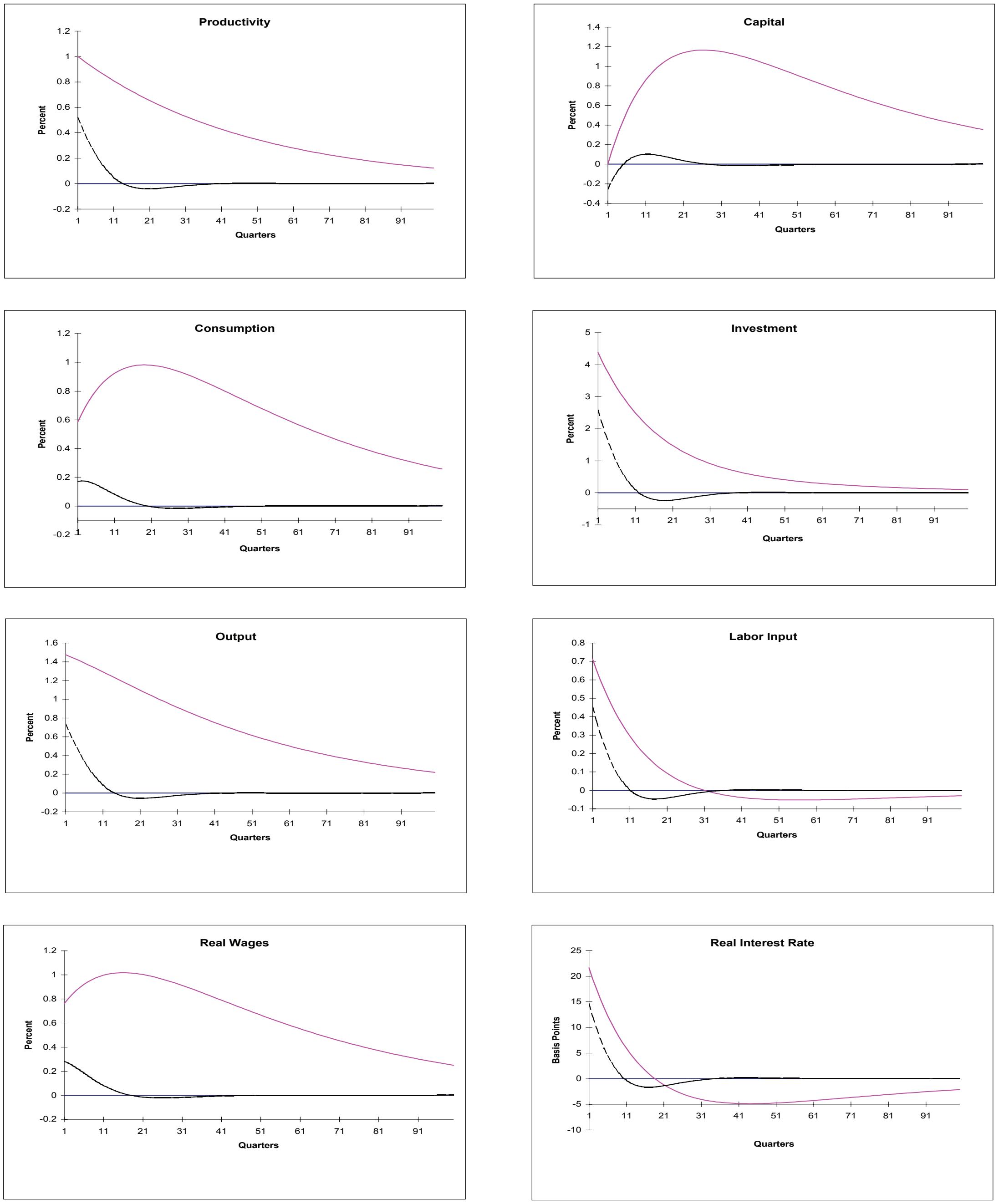

Figure 7. (King and Rebelo (1999) figure 10) smoother graphs are HP detrended series

in calibrations, such as an empirically unreasonably high intertemporal substitution of labor. Or with some predictions, such as wage rate being too strongly pro-cyclical with respect to data. Fair enough there is a main criticism that is still indisputable: the dominant role of technological shocks in explaining real business cycles and their measurability.

Methodologically, the measurement of technological shocks is based on Solow's residuals, which originally were defined to study long-run fluctuations, or growth cycles. At low frequencies, Solow's residuals aim to capture output short-run dynamics that are not explained by input dynamics, according to a specific technology (usually represented by a stationary, Cobb-Douglas production function). Indeed, Solow's residuals are more volatile than actual estimates of short-run technological shocks, suggesting they capture much more than just productivity fluctuations.[32] Someone argued that they may even capture the effects of unrepresented monetary and financial disturbance on real output. Alternatively, they may register increasing returns to scale, or changes in the factor intensity during cycles (recall that under CRTS the ratio of each factors to output is constant). For this reason Solow's residuals have sometime addressed as a 'measure of ignorance'.

Obviously, this criticism becomes unimportant if the Solow's residual explains very little of productivity changes over the business cycle. But for this to be true we both need good measures of factors' productivity and a model that is sufficiently reactive to more realistic, smaller and less persistent, exogenous shocks to the technology. Instead, we observed that, in order to match data, the productivity shocks used in the basic RBC model need to be large and persistent. Thus, in a sense, the RBC model relies on a phenomenon, technological shocks, that is largely unexplained.

These criticisms are key to understand the New-Keynesian contributions to business cycle theory. In extreme synthesis, these contributions, first, consider economies with market frictions that, because of price rigidities, may substantially amplify initial shocks, relative to Walrasian model. Second, because of such frictions, the role of monetary and financial disturbance, which is small in RBC theory, become a further, potentially important source of stochastic disturbances.

Problem 8. *Derive the steady state equilibrium of the model with elastic labor supply and its log-linearized first order conditions, from* (y) *to* (w) *and* (i)*. Then, compute the first-period response to a marginal productivity shock and compare them with the ones presented for the case of inelastic labor supply.*

[32] Except for the '970s oil shocks, there has been not much technological shocks. Studies estimate that even in the Great Depression or in the post world war 2 there have been a weak evidence of technology fluctuations of the magnitude which one would need to reproduce such cycles.