DSGE Methods

General framework and Solution Methods

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Repetition: Exercise

Consider the Clarida-Gali-Gertler model from the previous lecture.

1. Derive the optimality conditions for the nonlinear model with Calvo-price frictions as well as the efficient allocation of the model.
2. Write a mod-file for the nonlinear model (do not log-linearize!) and solve it using Dynare's `stoch_simul(order=1)`. Interpret Dynare's output.
3. Transform the variables in the model to logs. Hint: Use $x_t = log(X_t)$ and $X_t = e^{log(X_t)} = e^{x_t}$ in the equilibrium equations. Interpret Dynare's output. Compare it to the output of the cgg.mod file from the previous lecture given the same parameter values and variables.

DSGE framework

DSGE model consists of

- state variables (x_t), control variables (y_t), observable variables (d_t)
- set of optimality conditions f
- transition equations for structural shocks and measurement errors (ε_t)
- measurement equations with D selecting observable variables (d_t) out of controls and measurement errors
- vector of deep parameters θ

which can be cast into a nonlinear first-order system of expectational difference equations

$$\begin{aligned} 0 &= E_t f\left(x_{t+1}, y_{t+1}, x_t, y_t | \theta\right) \\ d_t &= D y_t + \eta_d(\theta)\varepsilon_t \end{aligned}$$

Solution is characterized by

- transition equations for state and control variables, so-called *policy-functions*, that solve (at least approximately) the system of equations f

$$\begin{aligned} x_{t+1} &= \tilde{h}(x_t|\theta) + \eta_x(\theta)\varepsilon_{t+1} \\ y_t &= \tilde{g}(x_t|\theta) \end{aligned}$$

DSGE solution

One distinguishes between linear and non-linear methods:

- Linear methods: Anderson/Moore (1983), Binder and Pesaran, Blanchard/Khan (1980), (1997), Christiano (2002), King and Watson (1998), Klein (2000), Sims (2001) and Uhlig (1999) (See Anderson (2008) for a comparison).
- Nonlinear methods: Projection methods, iteration procedures or perturbation approach (see DeJong/Dave (2011) for a comparison)

Our Focus will be on perturbation approach in the fashion of Schmitt-Grohé/Uribe (2004) and Gomme and Klein (2011):

- Perturbation approach finds a local approximation of the policy functions in a neighborhood of a particular point
- Mostly steady-state, since we can solve it analytically or numerically.

DSGE solution

Perturbation approach

- Problem: Analytical derivation of g and h most of the times impossible
- Solution: Approximate policy functions using a perturbation approach
- Idea: introduce perturbation parameter σ that captures stochastic nature of the model
  - We know f and the steady-state analytically
  - Thus, we know g and h at the non-stochastic steady-state ($\sigma = 0$)
  - How do the functions behave, if we „switch on stochastics“ ($\sigma > 0$)

  $\rightarrow$ Taylor-approximation of g and h around the steady-state!

DSGE solution

General perturbation framework (Schmitt-Grohé/Uribe (2004))

$$
\begin{aligned}
0 &= E_t f\left(x_{t+1}, y_{t+1}, x_t, y_t|\theta\right), \\
x_{t+1} &= h(x_t, \sigma|\theta) + \sigma\eta_x\varepsilon_{t+1}, \\
y_t &= g(x_t, \sigma|\theta), \\
d_t &= Dy_t + \eta_d\varepsilon_t.
\end{aligned}
$$

with $E(\varepsilon_t) = 0$, $E(\varepsilon_t\varepsilon_t') = \Sigma_\varepsilon$

Non-stochastic steady-state

$$
f\left(\bar{x}, \bar{y}, \bar{x}, \bar{y}|\theta\right) = 0,\ \bar{x} = h(\bar{x}, 0|\theta),\ \bar{y} = g(\bar{x}, 0|\theta),\ \bar{d} = D\bar{y}
$$

DSGE solution

Some remarks

- DSGE-models can be interpreted as *state-space-models*.
- Exogenous shocks and measurement errors are stacked into ε_t for convenience (η_x and η_d pick the relevant variables accordingly.
- Driving force of the model are exogenous shocks and innovations ε_t.
- Innovations can be mutually correlated as well as the conditional variance can change over time, this features are captured by Σ_ε.
- Flexible framework: you can add auxiliary variables and equations.

Equivalence to DYNARE notation

- SGU-framework (innovations enter linearly) includes Dynare-framework (innovations enter nonlinearly), if we extend the state vector by auxiliary variables for the shocks

DSGE solution

First-order approximation

- We are looking for approximations to g and h around the point $(x_t, \sigma) = (\bar{x}, 0)$ of the form

$$g(x_t, \sigma) = g(\bar{x}, 0) + g_x(\bar{x}, 0)(x_t - \bar{x}) + g_\sigma(\bar{x}, 0)(\sigma - 0)$$
$$h(x_t, \sigma) = h(\bar{x}, 0) + h_x(\bar{x}, 0)(x_t - \bar{x}) + h_\sigma(\bar{x}, 0)(\sigma - 0)$$

- We would like to know $g_x, h_x, g_\sigma, h_\sigma$.
- Substitute the solution into the model:

$$F(x_t, \sigma) := E_t f \left(\underbrace{h(x_t, \sigma) + \sigma\varepsilon_{t+1}}_{x_{t+1}}, \underbrace{g[\overbrace{h(x_t, \sigma) + \sigma\varepsilon_{t+1}}^{x_{t+1}}, \sigma]}_{y_{t+1}}, \underbrace{x_t}_{x_t}, \underbrace{g(x_t, \sigma)}_{y_t} \right)$$

- Insight: f as well as all derivatives of f are 0 when evaluated at the non-stochastic steady state $(\bar{x}, 0)$.

DSGE solution

First-order approximation, computing g_σ, h_σ

- Take derivative of F w.r.t. σ and evaluate at the non-stochastic steady-state

$$F_\sigma(\bar{x}, 0) = E_t f_1[h_\sigma + \varepsilon_{t+1}] + E_t f_2[g_x(h_\sigma + \varepsilon_{t+1}) + g_\sigma] + f_3 \cdot 0 + f_4 g_\sigma = 0$$

$$\Leftrightarrow \underset{(n \times n_x)}{\begin{bmatrix} f_1 + f_2 g_x \end{bmatrix}} \quad \underset{(n \times n_y)}{\begin{bmatrix} f_2 + f_4 \end{bmatrix}} \begin{bmatrix} \underset{(n_x \times 1)}{h_\sigma} \\ \underset{(n_y \times 1)}{g_\sigma} \end{bmatrix} = \underset{n \times 1}{0}$$

  with $f_1 = \partial f(\bar{z})/\partial x_{t+1}$, $f_2 = \partial f(\bar{z})/\partial y_{t+1}$, $f_3 = \partial f(\bar{z})/\partial x_t$, $f_4 = \partial f(\bar{z})/\partial y_t$

- We have n equations in n unknowns
- Notice that this is a linear and homogenous system, that is, if there is a unique solution, it must be

$$h_\sigma = \underset{n_x \times 1}{0} \text{ and } g_\sigma = \underset{n_y \times 1}{0}$$

DSGE solution

First-order approximation, Certainty-equivalence

Important theoretical result:

- Even though agents take the effect of future shocks into account when optimizing, in a linearization to the first-order they don't matter for the decision rule.

Certainty-equivalence-property

- In a first-order approximation the constant term needs not to be corrected for uncertainty (i.e. variance of shocks)
- Expectation of x_t and y_t is equal to its non-stochastic steady-state

- This is problematic when uncertainty does matter: risk premia, welfare evaluation, ...

DSGE solution

First-order approximation, computing g_x, h_x

- Take derivative of F w.r.t. x_t and evaluate at the non-stochastic steady-state

$$F_x(\overline{x}, 0) = f_1 h_x + f_2 g_x h_x + f_3 + f_4 g_x = 0$$

$$\underbrace{-\begin{bmatrix} f_1 & f_2 \\ (n\times n_x) & (n\times n_y) \end{bmatrix}}_{:=A} \begin{bmatrix} h_x \\ (n_x\times n_x) \\ g_x \\ (n_y\times n_x)(n_x\times n_x) \end{bmatrix} = \underbrace{\begin{bmatrix} f_3 & f_4 \\ (n\times n_x) & (n\times n_y) \end{bmatrix}}_{:=B} \begin{bmatrix} I \\ (n_x\times n_x) \\ g_x \\ (n_y\times n_x) \end{bmatrix}$$

- $n \times n_x$ equations for $n \times n_x$ unknown elements of h_x and g_x
- Postmultiply by $\widehat{x}_t := (x_t - \overline{x})$

$$A \begin{bmatrix} h_x \widehat{x}_t \\ g_x h_x \widehat{x}_t \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ g_x \widehat{x}_t \end{bmatrix}$$

- Notice that the coefficient matrices are equivalent to the first order approximation

$$A E_t \begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix} + \begin{bmatrix} \sigma \eta_x \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

DSGE solution

First-order approximation, computing g_x, h_x

- A and B are known matrices, inverting A yields solution for h_x and g_x
- **BUT:** In general A is singular and non-invertible.
- We follow Klein (2000)'s approach to approximate the first order solution using the generalized Schur decomposition
- Idea:
  - Uncouple system into a (block) triangular system of equations using a particular matrix decomposition method, and solve system recursively.
- We'll need some matrix theory about decompositions and eigenvalues for that...

Aside: Matrix theory

Matrix pencil

Let A and B be two $n \times n$ matrices. The set of all matrices of the form $A - \lambda B$ with $\lambda \in \mathbb{C}$ is said to be a *pencil*. The eigenvalues of the pencil are elements of the set $\lambda(A, B)$ defined by

$$\lambda(A, B) = \{z \in \mathbb{C} : det(A - zB) = 0\}$$

Generalized Eigenvalue problem

Let A and B be two $n \times n$ matrices. Then $\lambda \in \lambda(A, B)$ is called a generalized Eigenvalue if there exist a nonzero vector $q \in \mathbb{C}^n$ such that

$$Aq = \lambda Bq$$

- If $B = I$, then this simplifies to the ordinary Eigenvalue problem: $Aq = \lambda q$
- A always has n eigenvalues, which can be ordered (in more than one way) to form an $n \times n$ diagonal matrix Λ and a corresponding matrix of nonzero columns Q that satisfies the eigenvalue equation:

$$AQ = Q\Lambda$$

Aside: Matrix theory

Eigendecomposition (or spectral decomposition) of a matrix

Let A be a squared $(n \times n)$ matrix with n linearly independent columns, then A can be factorized as

$$AQ = Q\Lambda \Leftrightarrow A = Q\Lambda Q^{-1}$$

Λ is a diagonal matrix such that $\lambda_i = \Lambda_{ii}$ is the Eigenvalue of A associated to the eigenvector q_i stored in column i of the square $(n \times n)$ matrix Q.

- Not every matrix has a Eigendecomposition, since Q might not be invertible (sufficient: Eigenvalues must be distinct).
- Useful for solving differential or difference equations, computing powers of covariance matrices, etc., example:

$$x_t = Ax_{t-1} = A^t x_0 = Q\Lambda^t Q^{-1} x_0$$

- Λ^t is very easy to calculate, whereas A^t is computationally very demanding.

Aside: Matrix theory

Jordan decomposition

Let A be an $n \times n$ matrix. Denote by $J_k(\lambda)$ a $k \times k$ matrix of the form (a so-called Jordan-block)

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

where $J_1(\lambda) = \lambda$. Then there exists a non-singular $n \times n$ matrix T such that

$$T^{-1}AT = \begin{bmatrix} J_{k_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{k_2}(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & J_{k_r}(\lambda_r) \end{bmatrix}$$

with $k_1 + k_2 + \cdots + k_r = \text{n}$. The λ_i are the eigenvalues of A, not necessarily distinct.

- $T^{-1}AT$ is structured, i.e. upper triangular and diagonal elements are eigenvalues of A
- Special case: If A has distinct eigenvalues, then $T^{-1}AT = \Lambda$, with Λ diagonal.
- Useful for proofs due to its block structure, but numerically difficult due to instabilities.

Aside: Matrix theory

Schur decomposition (Complex version)

Let A be an $n \times n$ matrix. Then there exist a **unitary** $n \times n$ matrix S (that is, $S^*S = SS^* = S^{-1}S = I_n$) and an upper triangular matrix M whose diagonal elements are the eigenvalues of B, such that

$$S^*AS = M \Leftrightarrow A = SMS^*$$

Schur decomposition (Real version)

Let A be a real symmetric $n \times n$ matrix. Then there exist an **orthogonal** real $n \times n$ matrix S (that is, $S'S = SS' = S^{-1}S = I_n$), whose columns are eigenvectors of A and a diagonal matrix Λ whose diagonal elements are the eigenvalues of A, such that

$$S'AS = \Lambda \Leftrightarrow A = S\Lambda S'$$

- * denotes conjugate or Hermitian transpose, $'$ denotes the ordinary transpose.
- A complex matrix always has a complex Schur decomposition.
- A real matrix has a real Schur decomposition if and only if all eigenvalues are real.
- S is structured, i.e. unitary or orthogonal.
- Useful for proofs (e.g. Eigenvalues of Kronecker products, differentials,...) and numerically attractive.

Aside: Matrix theory

Generalized (complex) Schur decomposition or QZ decomposition

Let A and B be $n \times n$ matrices. Then there exist matrices Q, Z, T and S such that

$$Q^* A Z = S \Leftrightarrow A = QSZ^*$$
$$Q^* B Z = T \Leftrightarrow B = QTZ^*$$

1. Q and Z are unitary, i.e. $Q^*Q = QQ^* = I_n$ and $Z^*Z = ZZ^* = I_n$.
2. S and T are upper triangular.
3. pairs (s_{ii}, t_{ii}) can be arranged in any desired order.
4. If for some i, t_{ii} and s_{ii} are both zero, then $\lambda(A, B) = \mathbb{C}$. Otherwise: $\lambda(A, B) = \left\{ \frac{s_{ii}}{t_{ii}} : t_{ii} \neq 0 \right\}$

- There is also a real version.
- We will limit ourselves to the case $\lambda(A, B) \neq \mathbb{C}$ and rule-out unit roots, that is t_{ii} and s_{ii} are not both zero, and $|t_{ii}| \neq |s_{ii}|$.

Aside: Matrix theory

Singular value decomposition

Let A be a real $m \times n$ matrix with $rank(A) = r > 0$. Then there exist an $m \times r$ matrix S such that $S'S = I_r$, an $n \times r$ matrix T such that $T'T = I_r$ and an $r \times r$ diagonal matrix Λ with positive diagonal elements, such that

$$A = S\Lambda^{1/2}T'$$

- There is also a complex version using the conjugate transpose.
- Diagonal elements of $\Lambda^{1/2}$ are called singular values of A and m columns of S and the n columns of T are called left-singular vectors and right-singular vectors of A, respectively.
- Convention: List singular values in descending order.
- Λ contains the r non-zero eigenvalues of AA' (and of $A'A$) and S and T contain corresponding eigenvectors.
- Applications: Pseudoinverses, least squares, matrix approximations, rank, range, null space,...

DSGE solution

First-order approximation, continued, computing g_x and h_x

Going back to our problem of finding g_x and h_x:

$$AE_t \begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix}$$

The Schur decomposition of A and B are given by

$$Q^*A = SZ^*, \qquad Q^*B = TZ^*$$

where we choose the following order: the stable generalized eigenvalues ($|s_{ii}| > |t_{ii}|$) come first. Premultiplying by Q^* and using $\begin{bmatrix} s_t \\ n_x \times 1 \\ u_t \\ n_y \times 1 \end{bmatrix} := Z^* \begin{bmatrix} \widehat{x}_t \\ n_x \times 1 \\ \widehat{y}_t \\ n_y \times 1 \end{bmatrix}$, we get $S \begin{bmatrix} E_t s_{t+1} \\ E_t u_{t+1} \end{bmatrix} = T \begin{bmatrix} s_t \\ u_t \end{bmatrix}$

DSGE solution

First-order approximation, continued, computing g_x and h_x

S and T are upper triangular:

$$\begin{bmatrix} \underset{n_x \times n_x}{S_{11}} & \underset{n_x \times n_y}{S_{12}} \\ \underset{n_y \times n_x}{0} & \underset{n_y \times n_y}{S_{22}} \end{bmatrix} \begin{bmatrix} E_t s_{t+1} \\ E_t u_{t+1} \end{bmatrix} = \begin{bmatrix} \underset{n_x \times n_x}{T_{11}} & \underset{n_x \times n_y}{T_{12}} \\ \underset{n_y \times n_x}{0} & \underset{n_y \times n_y}{T_{22}} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix}$$

Let's solve the lower block first, that is

$$S_{22} E_t[u_{t+1}] = T_{22} u_t$$

and note that due to our ordering we have matrix pairs $|s_{ii}| < |t_{ii}|$ (or $|s_{ii}| \leq |t_{ii}|$). Using backward-substitution, it can be shown, that any solution with bounded mean and variance (we want that!) must satisfy $u_t = 0$ for all t (unless $\Sigma = 0$), otherwise we would have an exploding solution.

DSGE solution

First-order approximation, continued, computing g_x and h_x

Note: We need as many state variables as there are generalized eigenvalues with $|s_{ii}| > |t_{ii}|$! This property has a name:

Blanchard/Khan-conditions

The number of generalized Eigenvalues, that are in absolute terms greater than 1, must be equal to the number of state variables, in order to get a stable solution (saddle-path).

DSGE solution

First-order approximation, continued, computing g_x and h_x

Now given $u_t = 0$, let's back out x_t and y_t from the definition of s_t and u_t:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} \Rightarrow \begin{cases} x_t = Z_{11} s_t \\ y_t = Z_{21} s_t \end{cases}$$

If Z_{11} is invertible, then

$$y_t = \underbrace{Z_{21} Z_{11}^{-1}}_{=g_x} x_t$$

Thus, in order to compute g_x we need a nonsingular Z_{11}!

DSGE solution

First-order approximation, continued, computing g_x and h_x

Now given $u_t = 0$ and a nonsingular Z_{11}, let's solve the first block

$$E_t[s_{t+1}] = S_{11}^{-1} T_{11} s_t$$

assuming S_{11} is invertible. Plugging in $s_t = Z_{11}^{-1} x_t$ we get

$$E_t[x_{t+1}] = \underbrace{Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}}_{=h_x} x_t$$

Thus, in order to compute h_x we need nonsingular S_{11} and Z_{11}. However, S_{11} has full rank by construction.

DSGE solution

First-order approximation, continued, computing g_x and h_x

Summary Klein (2000)

- Compute generalized Schur decomposition of $A = -\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ and $B = \begin{bmatrix} f_3 & f_4 \end{bmatrix}$.
- Reorder generalized eigenvalues such that $|s_{ii}| > |t_{ii}|$ in the upper left.
- Check number of stable eigenvalues, i.e. Blanchard-Khan condition.
- Check if Z_{11} is invertible.
- Compute $h_x = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$ and $g_x = Z_{21} Z_{11}^{-1}$.

DSGE solution

First-order approximation, continued, example with CGG

Excercise

Consider the log-linearized version of the CGG-model:

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t \qquad \text{(Phillips curve)}$$

$$x_t = E_t x_{t+1} - (r_t - E_t \pi_{t+1} - r_t^{**}) \qquad \text{(IS equation)}$$

$$r_t = \alpha r_{t-1} + (1-\alpha)\left[\phi_\pi \pi_t + \phi_x x_t\right] \qquad \text{(baseline Taylor-rule)}$$

$$\Delta a_t = \rho_a \Delta a_{t-1} + \varepsilon_t^a \qquad \text{(technological shock)}$$

$$\tau_t = \rho_\tau \tau_{t-1} + \varepsilon_t^\tau \qquad \text{(preference shock)}$$

$$r_t^{**} = \rho_a \Delta a_t + \frac{1-\rho_\tau}{1+\varphi}\tau_t \qquad \text{(natural interest rate)}$$

$$\Delta y_t = x_t - x_{t-1} + \Delta a_t - \frac{\tau_t - \tau_{t-1}}{1+\phi} \qquad \text{(output growth)}$$

1. What are state variables, what are control variables?
2. Solve this model using Klein(2000)'s algorithm with Matlab.
3. Compare the solution to Dynare's policy functions.

DSGE solution

First-order approximation

Pros:

- Simple linear state-space representation of the model, which in many cases is sufficiently exact.
- One can use the Kalman-filter to empirically evaluate the system.

Cons:

- One looses important information during the linearization.
- Higher moments play an important role for analyzing markets, risk, welfare, etc.
- An approximation to, say, the second order can yield different results, because the variance of future shocks matters (risk premium).

DSGE solution

Second-order approximation

For the second-order approximations to g and h around the point $(\overline{x}, 0)$ we first need some useful tool in Matrix Calculus:

Magnus-Neudercker-definition of the Hessian

$$\mathcal{D}f(\bar{z}) := \begin{pmatrix} \frac{\partial f(\bar{z})}{\partial x'_{t+1}} & \frac{\partial f(\bar{z})}{\partial y'_{t+1}} & \frac{\partial f(\bar{z})}{\partial x'_t} & \frac{\partial f(\bar{z})}{\partial y'_t} \end{pmatrix} := \begin{pmatrix} f_1 & f_2 & f_3 & f_4 \end{pmatrix},$$

$$\mathcal{H}f(\bar{z}) := \mathcal{D}vec((\mathcal{D}f(\bar{z}))')$$

with Jacobian $\mathcal{D}f(\overline{z})$ and Hessian $\mathcal{H}f(\overline{z})$ of f evaluated at steady-state.

This definition simplifies the algorithm, since no tensor notation is needed and basic matrix algebra can be used. Main tool:

Chain-Rule

Let $f : R^n \mapsto R^m$ and $g : R^m \mapsto R^p$ be twice differentiable. Define $h(x) = g(f(x))$. Then letting $y = f(x)$,

$$\mathcal{H}h(x) = (I_p \otimes \mathcal{D}f(x))'(\mathcal{H}g(y)) * \mathcal{D}f(x) + (\mathcal{D}g(y) \otimes I_n)\mathcal{H}f(x)$$

DSGE solution

Second-order approximation

The second-order Taylor approximation at the non-stochastic steady-state is given by

Second order Taylor approximation

$$
\begin{aligned}
x_{t+1} &= \bar{x} + h_x(\bar{x},0)\cdot(x_t-\bar{x}) + \sigma\eta_x\varepsilon_{t+1} \\
&\quad + \frac{1}{2}\left[I_{n_x}\otimes(x_t-\bar{x})'\right] h_{xx}(\bar{x},0)(x_t-\bar{x}) + \frac{1}{2}\sigma^2 h_{\sigma\sigma}(\bar{x},0) \\
y_t &= \bar{y} + g_x(\bar{x},0)\cdot(x_t-\bar{x}) \\
&\quad + \frac{1}{2}\left[I_{n_y}\otimes(x_t-\bar{x})'\right] g_{xx}(\bar{x},0)(x_t-\bar{x}) + \frac{1}{2}\sigma^2 g_{\sigma\sigma}(\bar{x},0)
\end{aligned}
$$

- g_x and h_x are the gradients of g and h with respect to x (using e.g. Klein's algorithm), all evaluated at $(\bar{x},0)$
- g_{xx}, h_{xx} and $g_{\sigma\sigma}, h_{\sigma\sigma}$ the corresponding Magnus-Neudecker-Hessians, all evaluated at $(\bar{x},0)$
- Note: No cross terms involving x and σ, because approximation is around $\sigma = 0$

DSGE solution

Second-order approximation

- Use an algorithm to obtain $h_x(\bar{x}, 0)$ and $g_x(\bar{x}, 0)$, the coefficients of a first order approximation of the model
- The other matrices can be calculated by inserting the policy functions into f and differentiating it twice using the chain-rule of Magnus and Neudecker (1999).
- Evaluating the Jacobian $\mathcal{D}f = (f_1 \ f_2 \ f_3 \ f_4)$ and Hessian H of f at the non-stochastic steady-state, and setting it to 0 yields:

Second order solution matrices

$$\begin{bmatrix} vec(g_{xx}) \\ vec(h_{xx}) \end{bmatrix} = -Q^{-1} vec(A), \qquad \begin{bmatrix} h_{ss} \\ g_{ss} \end{bmatrix} = -B^{-1} C$$

$$Q = \begin{bmatrix} h_x' \otimes f_2 \otimes h_x' + I_{n_x} \otimes f_4 \otimes I_{n_x} & I_{n_x} \otimes (f_1 \otimes I_{n_x} + f_2 g_x \otimes I_{n_x}) \end{bmatrix}$$

$$A = (I_{n_x+n_y} \otimes M')HM,$$

$$B = \begin{bmatrix} f_1 + f_2 g_x & f_2 + f_4 \end{bmatrix}$$

$$C = f_2 trm[(I_{n_y} \otimes (\eta_x \eta_x'))g_{xx}] + trm[(I_{n_x+n_y} \otimes N')HN(\eta_x \eta_x')]$$

$$M = \begin{bmatrix} h_x \\ g_x h_x \\ I_{n_x} \\ g_x \end{bmatrix}, \quad N = \begin{bmatrix} I_{n_x} \\ g_x \\ 0_{(n_x+n_y)\times n_x} \end{bmatrix}$$

and *trm* defines the matrix trace of an $nm \times n$ matrix $[Y_1' \ Y_2' \ \dots \ Y_m']'$ as the $m \times 1$ vector $[tr(Y_1) \ tr(Y_2) \ \dots \ tr(Y_m)]'$.

DSGE solution

Excercise

Consider the An and Schorfheide (2007) model.

1. Write a mod-file for the nonlinear model. What are state variables, what are control variables?
2. Compare the solution of a first-order and a second-order approximation.

Unconditional Moments

First-order approximation

Given stationarity and the approximated solution, we can calculate the unconditional moments

Unconditional Moments for first-order approximation

$$
\begin{aligned}
E(x_t) = \bar{x};\, E(y_t) = \bar{y};\, E(d_t) &= \bar{d} = D\bar{y} \\
\Sigma_x &:= E(x_t - \bar{x})(x_t - \bar{x})' = h_x \Sigma_x h_x' + \sigma^2 \eta_x E(\varepsilon_t \varepsilon_t') \eta_x' \\
\Sigma_y &:= E(y_t - \bar{y})(y_t - \bar{y})' = g_x \Sigma_x g_x' \\
\Sigma_d &:= E(d_t - \bar{d})(d_t - \bar{d})' = D \Sigma_y D' + \eta_d E(\varepsilon_t \varepsilon_t') \eta_d' \\
\Sigma_x(t) &:= E[(x_t - \bar{x})(x_0 - \bar{x})'] = (h_x)^t \Sigma_x \\
\Sigma_y(t) &:= E[(y_t - \bar{y})(y_0 - \bar{y})'] = g_x (h_x)^t \Sigma_x g_x' \\
\Sigma_d(t) &:= E[(d_t - \bar{d})(d_0 - \bar{d})'] = D g_x (h_x)^t \Sigma_x g_x' D'
\end{aligned}
$$

Unconditional Moments

Second-order approximation

Unconditional mean for second-order approximation

$$E(x_t) := \bar{x} + (I_{n_x} - h_x)^{-1} \frac{\Lambda_x}{2},$$

$$E(y_t) := \bar{y} + g_x(\mu_x - \bar{x}) + \frac{\Lambda_y}{2},$$

$$E(d_t) := D\mu_y,$$

with

$$\Lambda_x = \left[I_{n_x} \otimes vec(\Sigma_x)'\right] vec(h'_{xx}) + \sigma^2 h_{\sigma\sigma},$$

$$\Lambda_y = \left[I_{n_y} \otimes vec(\Sigma_x)'\right] vec(g'_{xx}) + \sigma^2 g_{\sigma\sigma}.$$

Variances and autocovariances have additional higher order terms.

Unconditional Moments

Summary

- In a first-order approximation the unconditional mean is equal to the steady-state (certainty-equivalence!)
- Up to second order, the presence of uncertainty affects only the constant term of the decision rules

$\Rightarrow$ Unconditional mean can be significantly different from non-stochastic steady-state!

Further Topic: Indeterminacy

Further Topic: Indeterminacy

Toy model: Lubik and Schorfheide (2004)

Model

$$
\begin{aligned}
x_t &= E_t[x_{t+1}] - \tau(R_t E_t[\pi_{t+1}] + g_t) \\
\pi_t &= \beta E_t[\pi_{t+1}] + \kappa(x_t - z_t) \\
R_t &= \rho_R R_{t-1} + (1-\rho_R)(\psi_1 \pi_t + \psi_2[x_t - z_t]) + e_{R,t} \\
e_{R,t} &= \varepsilon_{R,t} \quad g_t = \rho_g g_{t-1} + \varepsilon_{g,t}, \quad z_t = \rho_z z_{t-1} + \varepsilon_{z,t}
\end{aligned}
$$

Definitions

$$
\begin{aligned}
\theta &= [\psi_1, \psi_2, \rho_R, \beta, \kappa, \tau, \rho_g, \rho_z, \rho_{gz}, \sigma_r, \sigma_g, \sigma_z]' \\
s_t &= [x_t, \pi_t, R_t, E_t[x_{t+1}], E_t[\pi_{t+1}], e_{R,t}, g_t, z_t]', \\
\varepsilon_t &= [\varepsilon_{R,t}, \varepsilon_{g,t}, \varepsilon_{z,t}]', \quad \varepsilon_{i,t} \sim N(0, \sigma_i), \\
\eta_t &= [(x_t - E_{t-1}[x_t]), (\pi_t) - E_{t-1}[\pi_t]]'
\end{aligned}
$$

Canonical form of Sims (2002)

$$
\begin{aligned}
\Gamma_0(\theta) s_t &= \Gamma_1(\theta) s_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t \\
0 &= \Psi^*(\theta)\varepsilon_t + \Pi^*(\theta)\eta_t \\
s_t &= F(\theta) s_{t-1} + G(\theta)\varepsilon_t
\end{aligned}
$$

Further Topic: Indeterminacy

Sunspots idea

Determinacy of DSGE models (usual approach)

- Focus on **one stable** solution, e.g. Taylor principle, $\eta_t = A_1(\theta)\varepsilon_t$
- Inherent in algorithms, e.g. MH parameter draws that yield indeterminacy are disregarded

Indeterminacy of DSGE models

- **multiple stable** solutions due to sunspots ξ_t
- sunspots can be interpreted as shocks to beliefs: $\eta_t = A_1^*\varepsilon_t + A_2^*\xi_t$
- Solution is now characterized by
  - No stable solution
  - Determinacy: $A_1^* = A_1(\theta), A_2^* = 0$, sunspot variance not identified
  - Indeterminacy: $A_1^* \neq A_1(\theta)$, $A_2^* \neq 0$
- Sunspots change propagation of shocks and induce self-fulfilling fluctuations in the economy

$$s_t = F(\theta)s_{t-1} + (G(\theta) + \Delta)\varepsilon_t + \Lambda\xi_t$$

Further Topic: Indeterminacy

Progress and Challenges

Progress

- New explanation of the business cycle: fluctuations in economic activity may be the result of variation in agent's beliefs.
- Sunspots and indeterminacy generate persistence without adding frictions, adjustment costs, indexation, etc.
- Data highly supports sunspots and indeterminacy, since model has more degrees of freedom

Challenges

- Mostly theoretical focus, empirical papers don't consider this!
- How do conclusions change if we include full set of solutions?