Advanced Macroeconomics (PhD) - DSGE methods

Solution methods

Willi Mutschler, M.Sc.

Institute of Econometrics and Economic Statistics
University of Münster
willi.mutschler@uni-muenster.de

Summer 2013

DSGE framework

DSGE model consists of

- set of Euler equations, i.e. first-order optimality conditions,
- transition equations for state (x_t) and control variables (y_t),
- transition equations for structural shocks and innovations (ε_t),
- measurement equation with D selecting observable variables (d_t) out of controls and measurement errors μ_t,
- vector of deep parameters θ,

which can be cast into a nonlinear first-order system of expectational difference equations

$$E_t f\left(x_{t+1}, y_{t+1}, x_t, y_t | \theta\right) = 0.$$

Perturbation approach

- introduce perturbation parameter σ that captures stochastic nature of the model
- solution is characterized by *policy-functions*, g and h, that solve (at least approximately) the system of equations f.

DSGE framework

General DSGE model, perturbation approach

$$
\begin{aligned}
0 &= E_t f\left(x_{t+1}, y_{t+1}, x_t, y_t | \theta\right), \\
x_{t+1} &= h(x_t, \sigma | \theta) + \sigma \varepsilon_{t+1}, \\
y_t &= g(x_t, \sigma | \theta), \\
d_t &= D y_t + \mu_t.
\end{aligned}
$$

with $E(\varepsilon_t) = 0$, $E(\varepsilon_t \varepsilon_t') = \Sigma_\varepsilon$ and $E(\mu_t) = 0$, $E(\mu_t \mu_t') = \Sigma_\mu$.

Non-stochastic steady-state

$$\bar{z} := (\bar{x}', \bar{y}')', \ \bar{x} = h(\bar{x}, 0 | \theta), \ \bar{y} = g(\bar{x}, 0 | \theta), \ \bar{d} = D\bar{y}$$

DSGE framework

Some remarks

- DSGE-models can be interpreted as *state-space-models*.
- Driving force of the model are exogenous shocks and innovations ε_t.
- Innovations can be mutually correlated as well as the conditional variance can change over time, this features are captured by Σ_ε.
- State vector x_t consists of n_{x_1} endogenous predetermined states and n_{x_2} exogenous predetermined states

$$x_{2,t+1} = h^2(x_{2,t}, \sigma|\theta) + \sigma\varepsilon_{t+1}, \qquad x_{1,t+1} = h^1(x_{2,t}, \sigma|\theta)$$

- Stability: Eigenvalues of Jacobian of $h^2(\cdot)$ w.r.t. $x_{2,t}$ lie inside the unit circle
- Flexible framework: you can add auxiliary variables and equations.
- Important property: x_t, y_t, d_t are weakly stationary (or ε_t is $i.i.d$).

DSGE solution

One distinguishes between linear and non-linear methods:

- Linear methods: Anderson/Moore (1983), Binder and Pesaran, Blanchard/Khan (1980), (1997), Christiano (2002), King and Watson (1998), Klein (2000), Sims (2001) and Uhlig (1999) (See Anderson (2008) for a comparison).
- Nonlinear methods: Projection methods, iteration procedures or perturbation approach (see DeJong/Dave (2011) for a comparison)

Our Focus will be on perturbation approach in the fashion of Schmitt-Grohé/Uribe (2004):

- Perturbation approach finds a local approximation of the policy functions in a neighborhood of a particular point
- Mostly steady-state, since we can solve it analytically or numerically.

DSGE solution

First-order approximation

- We are looking for approximations to g and h around the point $(x, \sigma) = (\bar{x}, 0)$ of the form

$$g(x, \sigma) = g(\overline{x}, 0) + g_x(\overline{x}, 0)(x - \overline{x}) + g_\sigma(\overline{x}, 0)(\sigma - 0)$$
$$h(x, \sigma) = h(\overline{x}, 0) + h_x(\overline{x}, 0)(x - \overline{x}) + h_\sigma(\overline{x}, 0)(\sigma - 0)$$

- We would like to know $g_x, h_x, g_\sigma, h_\sigma$.
- Idea:
  - Substitute the solution into the model,

$$F(x, \sigma) := E_t f \left(\underbrace{h(x_t, \sigma) + \sigma\varepsilon_{t+1}}_{x_{t+1}}, \underbrace{g[h(x_t, \sigma) + \sigma\varepsilon_{t+1}, \sigma]}_{y_{t+1}}, \underbrace{x_t}_{x_t}, \underbrace{g(x_t, \sigma)}_{y_t} \right)$$

  - Insight: f as well as all derivatives of f are 0 when evaluated at the non-stochastic steady state $(\overline{x}, 0)$.

DSGE solution

First-order approximation, computing g_σ, h_σ

- Take derivative of F w.r.t. σ and evaluate at the non-stochastic steady-state

$$F_\sigma(\overline{x}, 0) = E_t f_1[h_\sigma + \varepsilon_{t+1}] + E_t f_2[g_x(h_\sigma + \varepsilon_{t+1}) + g_\sigma] + f_3 \cdot 0 + f_4 g_\sigma = 0$$

$$\Leftrightarrow \underset{(n \times n_x)}{\begin{bmatrix} f_1 + f_2 g_x \end{bmatrix}} \quad \underset{(n \times n_y)}{\begin{bmatrix} f_2 + f_4 \end{bmatrix}} \begin{bmatrix} \underset{(n_x \times 1)}{h_\sigma} \\ \underset{(n_y \times 1)}{g_\sigma} \end{bmatrix} = \underset{n \times 1}{0}$$

  with $f_1 = \partial f(\overline{z})/\partial x_{t+1}$, $f_2 = \partial f(\overline{z})/\partial y_{t+1}$, $f_3 = \partial f(\overline{z})/\partial x_t$, $f_4 = \partial f(\overline{z})/\partial y_t$

- We have n equations in n unknowns
- Notice that this is a linear and homogenous system, that is, if there is a unique solution, it must be

$$h_\sigma = g_\sigma = 0$$

DSGE solution

First-order approximation, Certainty-equivalence

Important theoretical result:

- Even though agents take the effect of future shocks into account when optimizing, in a linearization to the first-order they don't matter for the decision rule.

Certainty-equivalence-property

- In a first-order approximation the constant term needs not to be corrected for uncertainty (i.e. variance of shocks)
- Expectation of x_t and y_t is equal to its non-stochastic steady-state

- This is problematic when uncertainty does matter: risk premia, welfare evaluation, ...

DSGE solution

First-order approximation, computing g_x, h_x

- Take derivative of F w.r.t. x_t and evaluate at the non-stochastic steady-state

$$F_x(\overline{x}, 0) = f_1 h_x + f_2 g_x h_x + f_3 + f_4 g_x = 0$$

$$\underbrace{-\begin{bmatrix} f_1 & f_2 \\ (n\times n_x) & (n\times n_y) \end{bmatrix}}_{:=A} \begin{bmatrix} h_x \\ (n_x \times n_x) \\ g_x \\ (n_y \times n_x)(n_x \times n_x) \end{bmatrix} = \underbrace{\begin{bmatrix} f_3 & f_4 \\ (n\times n_x) & (n\times n_y) \end{bmatrix}}_{:=B} \begin{bmatrix} I \\ (n_x \times n_x) \\ g_x \\ (n_y \times n_x) \end{bmatrix}$$

- $n \times n_x$ equations for $n \times n_x$ unknown elements of h_x and g_x
- Postmultiply by $\widehat{x}_t := (x_t - \overline{x})$

$$A \begin{bmatrix} h_x \widehat{x}_t \\ g_x h_x \widehat{x}_t \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ g_x \widehat{x}_t \end{bmatrix}$$

- Notice that this is equivalent to the first order approximation of the model

$$A \begin{bmatrix} \widehat{x}_{t+1} \\ E_t \widehat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix} + \begin{bmatrix} \sigma \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

DSGE solution

First-order approximation, computing g_x, h_x

- A and B are known matrices, inverting A yields solution for h_x and g_x
- **BUT:** In general A is singular and non-invertible.
- We follow Klein (2000)'s approach to approximate the first order solution using the generalized Schur decomposition
- Idea:
  - Uncouple system into a (block) triangular system of equations using a particular matrix decomposition method, and solve system recursively.
- We'll need some matrix theory about decompositions and eigenvalues for that...

Aside: Matrix theory

Matrix pencil

Let A and B be two $n \times n$ matrices. The set of all matrices of the form $A - \lambda B$ with $\lambda \in \mathbb{C}$ is said to be a *pencil*. The eigenvalues of the pencil are elements of the set $\lambda(A, B)$ defined by

$$\lambda(A, B) = \{z \in \mathbb{C} : det(A - zB) = 0\}$$

Generalized Eigenvalue problem

Let A and B be two $n \times n$ matrices. Then $\lambda \in \lambda(A, B)$ is called a generalized Eigenvalue if there exist a nonzero vector $q \in \mathbb{C}^n$ such that

$$Aq = \lambda Bq$$

- If $B = I$, then this simplifies to the ordinary Eigenvalue problem: $Aq = \lambda q$
- A always has n eigenvalues, which can be ordered (in more than one way) to form an $n \times n$ diagonal matrix Λ and a corresponding matrix of nonzero columns Q that satisfies the eigenvalue equation:

$$AQ = Q\Lambda$$

Aside: Matrix theory

Eigendecomposition (or spectral decomposition) of a matrix

Let A be a squared $(n \times n)$ matrix with n linearly independent columns, then A can be factorized as

$$AQ = Q\Lambda \Leftrightarrow A = Q\Lambda Q^{-1}$$

Λ is a diagonal matrix such that $\lambda_i = \Lambda_{ii}$ is the Eigenvalue of A associated to the eigenvector q_i stored in column i of the square $(n \times n)$ matrix Q.

- Not every matrix has a Eigendecomposition, since Q might not be invertible (sufficient: Eigenvalues must be distinct).
- Useful for solving differential or difference equations, computing powers of covariance matrices, etc., example:

$$x_t = Ax_{t-1} = A^t x_0 = Q\Lambda^t Q^{-1} x_0$$

- Λ^t is very easy to calculate, whereas A^t is computationally very demanding.

Aside: Matrix theory

Jordan decomposition

Let A be an $n \times n$ matrix. Denote by $J_k(\lambda)$ a $k \times k$ matrix of the form (a so-called Jordan-block)

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda \end{bmatrix}$$

where $J_1(\lambda) = \lambda$. Then there exists a non-singular $n \times n$ matrix T such that

$$T^{-1}AT = \begin{bmatrix} J_{k_1}(\lambda_1) & 0 & \dots & 0 \\ 0 & J_{k_2}(\lambda_2) & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & J_{k_r}(\lambda_r) \end{bmatrix}$$

with $k_1 + k_2 + \cdots + k_r = \text{n}$. The λ_i are the eigenvalues of A, not necessarily distinct.

- $T^{-1}AT$ is structured, i.e. upper triangular and diagonal elements are eigenvalues of A
- Special case: If A has distinct eigenvalues, then $T^{-1}AT = \Lambda$, with Λ diagonal.
- Useful for proofs due to its block structure, but numerically difficult due to instabilities.

Aside: Matrix theory

Schur decomposition (Complex version)

Let A be an $n \times n$ matrix. Then there exist a **unitary** $n \times n$ matrix S (that is, $S^*S = SS^* = S^{-1}S = I_n$) and an upper triangular matrix M whose diagonal elements are the eigenvalues of B, such that

$$S^*AS = M \Leftrightarrow A = SMS^*$$

Schur decomposition (Real version)

Let A be a real symmetric $n \times n$ matrix. Then there exist an **orthogonal** real $n \times n$ matrix S (that is, $S'S = SS' = S^{-1}S = I_n$), whose columns are eigenvectors of A and a diagonal matrix Λ whose diagonal elements are the eigenvalues of A, such that

$$S'AS = \Lambda \Leftrightarrow A = S\Lambda S'$$

- * denotes conjugate or Hermitian transpose, $'$ denotes the ordinary transpose.
- A complex matrix always has a complex Schur decomposition.
- A real matrix has a real Schur decomposition if and only if all eigenvalues are real.
- S is structured, i.e. unitary or orthogonal.
- Useful for proofs (e.g. Eigenvalues of Kronecker products, differentials,...) and numerically attractive.

Aside: Matrix theory

Generalized (complex) Schur decomposition or QZ decomposition

Let A and B be $n \times n$ matrices. Then there exist matrices Q, Z, T and S such that

$$Q^* AZ = S \Leftrightarrow A = QSZ^*$$
$$Q^* BZ = T \Leftrightarrow B = QTZ^*$$

1. Q and Z are unitary, i.e. $Q^*Q = QQ^* = I_n$ and $Z^*Z = ZZ^* = I_n$.
2. S and T are upper triangular.
3. pairs (s_{ii}, t_{ii}) can be arranged in any desired order.
4. If for some i, t_{ii} and s_{ii} are both zero, then $\lambda(A, B) = \mathbb{C}$. Otherwise: $\lambda(A, B) = \left\{ \frac{t_{ii}}{s_{ii}} : s_{ii} \neq 0 \right\}$

- There is also a real version.
- We will limit ourselves to the case $\lambda(A, B) \neq \mathbb{C}$ and rule-out unit roots, that is t_{ii} and s_{ii} are not both zero, and $|t_{ii}| \neq |s_{ii}|$.
- Eigenvalues
  - If A is singular, then there are some generalized eigenvalues missing, i.e. $s_{ii} = 0$ for some $i \Rightarrow$ call these *infinite*,
  - If $|\lambda_i| > 1 \Leftrightarrow |s_{ii}| < |t_{ii}| \Rightarrow$ call these finite and unstable,
  - If $|\lambda_i| < 1 \Leftrightarrow |s_{ii}| > |t_{ii}| \Rightarrow$ call these finite and stable.

Aside: Matrix theory

Singular value decomposition

Let A be a real $m \times n$ matrix with $rank(A) = r > 0$. Then there exist an $m \times r$ matrix S such that $S'S = I_r$, an $n \times r$ matrix T such that $T'T = I_r$ and an $r \times r$ diagonal matrix Λ with positive diagonal elements, such that

$$A = S\Lambda^{1/2}T'$$

- There is also a complex version using the conjugate transpose.
- Diagonal elements of $\Lambda^{1/2}$ are called singular values of A and m columns of S and the n columns of T are called left-singular vectors and right-singular vectors of A, respectively.
- Convention: List singular values in descending order.
- Λ contains the r non-zero eigenvalues of AA' (and of $A'A$) and S and T contain corresponding eigenvectors.
- Applications: Pseudoinverses, least squares, matrix approximations, rank, range, null space,...

DSGE solution

First-order approximation, continued, computing g_x and h_x

Going back to our problem of finding g_x and h_x:

$$AE_t \begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{y}_{t+1} \end{bmatrix} = B \begin{bmatrix} \widehat{x}_t \\ \widehat{y}_t \end{bmatrix}$$

The Schur decomposition of A and B are given by

$$Q^*A = SZ^*, \qquad Q^*B = TZ^*$$

where we choose the following order: the stable generalized eigenvalues ($|s_{ii}| > |t_{ii}|$) come first (ascending order of λ_i). Premultiplying by Q^* and using $\begin{bmatrix} s_t \\ n_x \times 1 \\ u_t \\ n_y \times 1 \end{bmatrix} := Z^* \begin{bmatrix} x_t \\ n_x \times 1 \\ y_t \\ n_y \times 1 \end{bmatrix}$, we get

$$SE_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = T \begin{bmatrix} s_t \\ u_t \end{bmatrix}$$

DSGE solution

First-order approximation, continued, computing g_x and h_x

S and T are upper triangular:

$$\begin{bmatrix} S_{11} & S_{12} \\ n_x \times n_x & n_x \times n_y \\ 0 & S_{22} \\ n_y \times n_x & n_y \times n_y \end{bmatrix} E_t \begin{bmatrix} s_{t+1} \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ n_x \times n_x & n_x \times n_y \\ 0 & T_{22} \\ n_y \times n_x & n_y \times n_y \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix}$$

Let's solve the lower block first, that is

$$S_{22} E_t[u_{t+1}] = T_{22} u_t$$

and note that due to our ordering we have unstable matrix pairs $|s_{ii}| < |t_{ii}|$ (or $|s_{ii}| \leq |t_{ii}|$). It can be shown that any solution with bounded mean and variance (we want that!) must satisfy $u_t = 0$ for all t (unless $\Sigma = 0$), otherwise we would have an exploding solution.
Note: We need as many state variables as there are stable generalized eigenvalues!

DSGE solution

First-order approximation, continued, computing g_x and h_x

Now given $u_t = 0$, let's back out x_t and y_t from the definition of s_t and u_t:

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} s_t \\ u_t \end{bmatrix} \Rightarrow \begin{cases} x_t = Z_{11} s_t \\ y_t = Z_{21} s_t \end{cases}$$

If Z_{11} is invertible, then

$$y_t = \underbrace{Z_{21} Z_{11}^{-1}}_{=g_x} x_t$$

Thus, in order to compute g_x we need a nonsingular Z_{11}!

DSGE solution

First-order approximation, continued, computing g_x and h_x

Now given $u_t = 0$ and a nonsingular Z_{11}, let's solve the first block

$$E_t[s_{t+1}] = S_{11}^{-1} T_{11} s_t$$

assuming S_{11} is invertible. Plugging in $s_t = Z_{11}^{-1} x_t$ we get

$$E_t[x_{t+1}] = \underbrace{Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}}_{=h_x} x_t$$

Thus, in order to compute h_x we need nonsingular S_{11} and Z_{11}. However, S_{11} has full rank by construction.

DSGE solution

First-order approximation, continued, computing g_x and h_x

Summary Klein (2000)

- Compute generalized Schur decomposition of $A = -\begin{bmatrix} f_1 & f_2 \end{bmatrix}$ and $B = \begin{bmatrix} f_3 & f_4 \end{bmatrix}$.
- Reorder generalized eigenvalues with the block inside the unit circle in the upper left.
- Check if Z_{11} is invertible.
- Check number of stable eigenvalues, i.e. Blanchard-Khan condition.
- Compute $h_x = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$ and $g_x = Z_{21} Z_{11}^{-1}$.

Blanchard/Khan-conditions

The number of Eigenvalues, that are in absolute terms greater than 1, must be equal to the number of state variables, in order to get a stable solution (saddle-path).

DSGE solution

First-order approximation

Pros:

- Simple linear state-space representation of the model, which in many cases is sufficiently exact.
- One can use the Kalman-filter to empirically evaluate the system.

Cons:

- One looses important information during the linearization.
- Higher moments play an important role for analyzing markets, risk, welfare, etc.
- An approximation to, say, the second order can yield different results, because the variance of future shocks matters (risk premium).

DSGE solution

Second-order approximation, Schmitt-Grohé/Uribe (2004)

The second-order approximations to g and h around the point $(\overline{x}, 0)$ are of the form

$$
\begin{aligned}
[g(x,\sigma)]^i &= [g(\overline{x},0)]^i + [g_x(\overline{x},0)]^i_a[(x-\overline{x})]_a + [g_\sigma(\overline{x},0)]^i[\sigma-0] \\
&+ \frac{1}{2}[g_{xx}(\overline{x},0)]^i_{ab}[(x-\overline{x})]_a[(x-\overline{x})]_b \\
&+ \frac{1}{2}[g_{x\sigma}(\overline{x},0)]^i_a[(x-\overline{x})]_a[\sigma-0] \\
&+ \frac{1}{2}[g_{\sigma x}(\overline{x},0)]^i_a[(x-\overline{x})]_a[\sigma-0] \\
&+ \frac{1}{2}[g_{\sigma\sigma}(\overline{x},0)]^i[\sigma-0][\sigma-0] \\
[h(x,\sigma)]^j &= [h(\overline{x},0)]^j + [h_x(\overline{x},0)]^j_a[(x-\overline{x})]_a + [h_\sigma(\overline{x},0)]^j[\sigma-0] \\
&+ \frac{1}{2}[h_{xx}(\overline{x},0)]^j_{ab}[(x-\overline{x})]_a[(x-\overline{x})]_b \\
&+ \frac{1}{2}[h_{x\sigma}(\overline{x},0)]^j_a[(x-\overline{x})]_a[\sigma-0] \\
&+ \frac{1}{2}[h_{\sigma x}(\overline{x},0)]^j_a[(x-\overline{x})]_a[\sigma-0] \\
&+ \frac{1}{2}[h_{\sigma\sigma}(\overline{x},0)]^j[\sigma-0][\sigma-0]
\end{aligned}
$$

DSGE solution

Second-order approximation, Schmitt-Grohé/Uribe (2004)

This is so-called tensor notation:

- $[f_2]^i_\alpha$ is the (i, α) element of the derivative of f with respect to y_{t+1}. It is of dimension $n \times n_y$.
- $[f_2]^i_\alpha [g_x]^\alpha_\beta [h_x]^\beta_j = \sum_{\alpha=1}^{n_y} \sum_{\beta=1}^{n_x} \frac{\partial f^i}{\partial y^\alpha_{t+1}} \frac{\partial g^\alpha}{\partial x^\beta_t} \frac{\partial h^\beta}{\partial x^j_t}$
- f_{11} is a three-dimensional array with n-rows, n_y columns and n_y pages. $[f_{11}]^i_{\alpha\gamma}$ denotes the element of f_{11} located at the intersection of row i, column α and page γ.
- Aside note: There is also a different (prettier) way to do this using the Magnus/Neudecker definition of the Hessian, see Gomme/Klein (2011).

DSGE solution

Second-order approximation, Schmitt-Grohé/Uribe (2004), computing g_{xx} and h_{xx}

We can use the derivative of $F_x = f_1 h_x + f_2 g_x h_x + f_3 + f_4 g_x$ w.r.t. x to identify g_{xx} and h_{xx}:

$$
\begin{aligned}
[F_{xx}(\overline{x},0)]^i_{jk} = &\left([f_{11}]^i_{\beta\delta}[h_x]^\delta_k + [f_{12}]^i_{\beta\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{13}]^i_{\beta k} + [f_{14}]^i_{\beta\gamma}[g_x]^\gamma_k\right)[h_x]^\beta_j \\
&+ [f_1]^i_\beta [h_{xx}]^\beta_{jk} \\
&+ \left([f_{21}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{22}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{23}]^i_{\alpha k} + [f_{24}]^i_{\alpha\gamma}[g_x]^\gamma_k\right)[g_x]^\alpha_\beta[h_x]^\beta_j \\
&+ [f_2]^i_\alpha [g_{xx}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j \\
&+ [f_2]^i_\alpha [g_x]^\alpha_\beta [h_{xx}]^\beta_{jk} \\
&+ [f_{31}]^i_{j\delta}[h_x]^\delta_k + [f_{32}]^i_{j\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{33}]^i_{jk} + [f_{34}]^i_{j\gamma}[g_x]^\gamma_k \\
&+ \left([f_{41}]^i_{\alpha\delta}[h_x]^\delta_k + [f_{42}]^i_{\alpha\gamma}[g_x]^\gamma_\delta[h_x]^\delta_k + [f_{43}]^i_{\alpha k} + [f_{44}]^i_{\alpha\gamma}[g_x]^\gamma_k\right)[g_x]^\alpha_j \\
&+ [f_4]^i_\alpha [g_{xx}]^\alpha_{jk} \\
&= 0, \qquad i = 1,\dots n, \qquad j,k,\beta,\delta = 1,\dots n_x, \qquad \alpha,\gamma = 1,\dots n_y
\end{aligned}
$$

This is a linear system of $n \times n_x \times n_x$ equations in $n \times n_x \times n_x$ unknowns given by the elements of g_{xx} and h_{xx} $\Rightarrow$ easy to solve using matrix algebra (and a computer).

DSGE solution

Second-order approximation, Schmitt-Grohé/Uribe (2004), computing g_{xx} and h_{xx}

Similarly, we can use the derivative of

$$F_\sigma = E_t f_1[h_\sigma + \varepsilon_{t+1}] + E_t f_2[g_x(h_\sigma + \sigma\varepsilon_{t+1}) + g_\sigma] + f_4 g_\sigma = 0$$

w.r.t. σ to identify $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$, we omit the expression here.
$\Rightarrow$ This yields a system of n linear equations in the n unknowns given by the elements of $g_{\sigma\sigma}$ and $h_{\sigma\sigma}$.

DSGE solution

Second-order approximation, Schmitt-Grohé/Uribe (2004), computing $g_{\sigma x}$ and $h_{\sigma x}$

Let's have a look at the cross-derivatives, i.e. $F_{\sigma,x}(\overline{x},0)=0$ using the fact that $g_\sigma = h_\sigma = 0$:

$$[F_{\sigma_x}]^i_j = [f_1]^i_\beta [h_{\sigma x}]^\beta_j + [f_2]^i_\alpha [g_x]^\alpha_\beta [h_{\sigma x}]^\beta_j + [f_2]^i_\alpha [g_{\sigma x}]^\alpha_\gamma [h_x]^\gamma_j + [f_4]^i_\alpha [g_{\sigma x}]^\alpha_j = 0$$

$$i = 1,\ldots,n; \alpha = 1,\ldots,n_y; \beta,\gamma,j = 1,\ldots,n_x$$

This is a system of $n \times n_x$ equations in $n \times n_x$ unknowns given by the elements of $g_{\sigma x}$ and $h_{\sigma x}$. Obviously, this is a linear **homogenous** system of equations, thus a unique solution implies $g_{\sigma x} = h_{\sigma x} = 0$

DSGE solution

Unconditional moments

- Given stationary and the approximated solution, we can calculate the unconditional moments.
- Note: we focus on second order accurate moments, that is to keep only first and second order terms in the formula to compute moments.
- Compute variance from first order approximation alone, since all other cross-product terms result in something of order higher than two.

Unconditional autocovariogram

$$\Sigma_x := E(x_t - \bar{x})(x_t - \bar{x})' = h_x \Sigma_x h_x' + \sigma^2 \Sigma_\varepsilon$$
$$\Sigma_y := E(y_t - \bar{y})(y_t - \bar{y})' = g_x \Sigma_x g_x'$$
$$\Sigma_d := E(d_t - \bar{d})(d_t - \bar{d})' = D \Sigma_y D' + \Sigma_\mu$$
$$\Sigma_x(t) := E[(x_t - \bar{x})(x_0 - \bar{x})'] = (h_x)^t \Sigma_x$$
$$\Sigma_y(t) := E[(y_t - \bar{y})(y_0 - \bar{y})'] = g_x (h_x)^t \Sigma_x g_x'$$
$$\Sigma_d(t) := E[(d_t - \bar{d})(d_0 - \bar{d})'] = D g_x (h_x)^t \Sigma_x g_x' D'$$

DSGE solution

Unconditional moments

Regarding the mean, we have

$$x_{t+1}^i \approx \overline{x}^i + h_x^i(x_t - \overline{x}) + \frac{1}{2}\underbrace{(x_t - \overline{x})' h_{xx}^i (x_t - \overline{x})}_{(x_t-\bar{x})'\otimes(x_t-\bar{x})' vec(h_{xx}^i)} + \frac{1}{2} h_{\sigma\sigma}^i + \sigma\varepsilon_{t+1}$$

Notice that $E\left[(x_t - \bar{x})' \otimes (x_t - \bar{x})'\right] = vec(\Sigma_x)'$

Unconditional mean

$$\mu_x := E(x_t) = \bar{x} + (I_{n_x} - h_x)^{-1} \frac{\Lambda_x}{2}, \quad \mu_y := E(y_t) = \bar{y} + g_x(\mu_x - \bar{x}) + \frac{\Lambda_y}{2},$$

with

$$\Lambda_x = \begin{bmatrix} vec(h_{xx}^1)' \\ vec(h_{xx}^2)' \\ \dots \\ vec(h_{xx}^{n_x})' \end{bmatrix} vec(\Sigma_x) + \sigma^2 h_{\sigma\sigma}, \quad \Lambda_y = \begin{bmatrix} vec(g_{xx}^1)' \\ vec(g_{xx}^2)' \\ \dots \\ vec(g_{xx}^{n_y})' \end{bmatrix} vec(\Sigma_x) + \sigma^2 g_{\sigma\sigma}.$$

First order approximation: $\Lambda_x = \Lambda_y = 0 \Rightarrow \mu_x = \bar{x}, \mu_y = \bar{y}$.

Summary

- In a first-order approximation the unconditional mean is equal to the steady-state.
- Up to second order, the presence of uncertainty affects only the constant term of the decision rules

$\Rightarrow$ Unconditional mean can be significantly different from non-stochastic steady-state!

Further topic: Indeterminacy

Lubik & Schorfheide (2004)

- Consider a different log-linear (first-order) canonical form of the DSGE model (e.g. Sims (2002))

$$\Gamma_0(\theta) z_t = \Gamma_1(\theta) z_{t-1} + \Psi(\theta)\varepsilon_t + \Pi(\theta)\eta_t$$

- z_t are all model variables, ε_t are model shocks and $\eta_t = \widetilde{x}_t - E_{t-1}[\widetilde{x}_{t-1}]$ are expectational errors for the non-predetermined variables $\widetilde{x}$.
- $\Gamma_0, \Gamma_1, \Psi, \Pi$ are parameter matrices dependent on deep parameters θ.

Further topic: Indeterminacy

Lubik & Schorfheide (2004)

- Assume that there is a sunspot shock ζ_t, i.e. the expectational errors are driven by

$$\eta_t = A_1 \varepsilon_t + A_2 \zeta_t$$

- Solution algorithms construct a mapping from the shocks to the expectation errors:
  1. nonexistence of a stable solution
  2. existence of a unique stable solution (determinacy), i.e. $A_1 = A_1(\theta)$ and $A_2 = 0$
  3. existence of multiple stable solutions (indeterminacy), i.e. A_1 is not uniquely determined by θ and $A_2 \neq 0$.
- Loosely speaking: there exists a unique stable solution, if the Taylor principle, that is the central bank raises the real interest rate in response to inflation more than one-to-one, and has multiple stable solutions otherwise (interesting in its own right)