Second Order Approximation Methods for DSGE Models

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Contents

1 Introduction

Numerical Methods and Systems of Partial Differential Equations

- A model is a system of partial differential equations describing the evolution of (economic) variables over time (dynamic approach)
- Different models are judged by their implications (variances, covariances, impulse responses etc.)
- "Good" models (models whose implications match empirical facts under study) are used for normative analysis

Economic Research

Identify interesting empirical facts

$\downarrow$

Write down and solve (derive first order conditions) the model that describe these empirical facts

$\downarrow$

Solve the system of differential equations (global solutions or local solutions like Taylor expansion)

$\downarrow$

Study the implications of the model (variances and variance decomposition, impulse response functions)

$\downarrow$

Perform Normative Analysis

Today's Lecture

We will study numerical methods to implement the last three steps and (some) theory behind these methods

Our road map for today is:

1. Present two simple examples to get basic ideas
2. Algebraic and economic preliminary definitions
3. First Order Taylor Approximation (recap of the generalized Schur decomposition)
4. Higher Order Taylor Approximations (the Schmitt-Grohé-Uribe and Lombardo Sutherland approaches)
5. RBC application
6. Normative Analysis

2 Starting Examples

2.1 A Simple Economic Example

A simple economic example: A Phillips Curve and the Quantity Theory of Money

Suppose that inflation and aggregate demand can be described:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa \triangle y_t \text{ "Phillips Curve"} \\
\triangle m_t &= \pi_t + \triangle y_t \text{ Agg. Demand Function (quantity Theory of Money)}
\end{aligned}$$

This is a system of two equations in three variables (describing dynamic behavior rather than "static" values)

$\triangle m_t$ is a policy variable.

IF this model were true, what are the implications?

A simple economic example: solve the system

This system implies:

$$\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa \left[\triangle m_t - \pi_t\right] \Rightarrow \pi_t = \frac{\beta}{1+\kappa} E_t \pi_{t+1} + \frac{\kappa}{1+\kappa} \triangle m_t \Rightarrow \\
\pi_t &= \frac{\beta}{1+\kappa} E_t \left\{ \frac{\beta}{1+\kappa} E_{t+1} \pi_{t+2} + \frac{\kappa}{1+\kappa} \triangle m_{t+1} \right\} + \frac{\kappa}{1+\kappa} \triangle m_t ... \Rightarrow \\
\pi_t &= \frac{\kappa}{1+\kappa} \sum_{i=0}^{\infty} E_t \left[\left(\frac{\beta}{1+\kappa} \right)^i \triangle m_{t+i} \right] \\
\triangle y_t &= \triangle m_t - \pi_t
\end{aligned}$$

A simple economic example: implication of the model

1. Constant Money Supply: $\triangle m_{t+i} = m \ \forall i$

$$\begin{aligned}
\pi_t &= \frac{\kappa}{1+\kappa} \sum_{i=0}^{\infty} E_t \left[\left(\frac{\beta}{1+\kappa} \right)^i \triangle m_{t+i} \right] = \frac{\kappa}{1+\kappa} \sum_{i=0}^{\infty} \left[\left(\frac{\beta}{1+\kappa} \right)^i m \right] = \frac{\kappa}{1+\kappa-\beta} m \\
\triangle y_t &= \triangle m_t - \pi_t = \frac{1-\beta}{1+\kappa-\beta} m
\end{aligned}$$

2. AR(1) Money Supply $\triangle m_{t+1} = \rho \triangle m_t + \varepsilon_{t+1}$

   ($E_t \varepsilon_{t+1} = 0$, ≡unpredictable, from the private sector, component of monetary policy):

$$\begin{aligned}
\pi_t &= \frac{\kappa}{1+\kappa} \sum_{i=0}^{\infty} E_t \left[\left(\frac{\beta\rho}{1+\kappa} \right)^i \triangle m_t \right] = \frac{\kappa}{1+\kappa-\rho\beta} m \\
\triangle m_{t+1} &= \rho \triangle m_t + \varepsilon_{t+1} \Rightarrow \pi_{t+1} = \rho \pi_t + \frac{1+\kappa-\rho\beta}{\kappa} \varepsilon_{t+1}
\end{aligned}$$

   What you "feed in" as a description of the policy is also the description of the evolution of inflation. Match evidence?!

2.2 A Simple Numerical Example

A simple numerical example: a monetary model with two period stickiness

Consider the following system of equations (this model comes from the overlapping contract model à la Taylor 1979 1980):

$$
\begin{aligned}
\text{Optimal Price Equation} &: \quad x_t = \frac{1}{2}x_{t-1} + \frac{1}{2}E_{t-1}x_{t+1} + ... \\
&\quad\ ... + \frac{\kappa}{2}E_{t-1}y_t + \frac{\kappa}{2}E_{t-1}y_{t+1} \\
\text{The price Level} &: \quad p_t = \frac{1}{2}x_t + \frac{1}{2}x_{t-1} \\
\text{Agg. Demand} &: \quad y_t = m_t - p_t \\
\text{Policy Rule} &: \quad \triangle m_t = g\triangle p_t + \varepsilon_t
\end{aligned}
$$

A simple numerical example: the solution

Note that the policy implies:

$$m_t = gp_t - gp_{t-1} + m_{t-1} + \varepsilon_t$$

Substitute into the Agg Demand to get $E_{t-1}y_t$ and $E_{t-1}y_{t+1}$(note the implicit assumption of rational expectations):

$$E_{t-1}y_t = -(1-g)E_{t-1}p_t - gp_{t-1} + m_{t-1}$$

Substituting in the Optimal Price Equation:

$$
\begin{aligned}
-\kappa(m_{t-1} - gp_{t-1}) &= \frac{(1-h)}{2}x_{t-1} - (1+h)E_{t-1}x_t + \frac{(1-h)}{2}E_{t-1}x_{t+1} \\
h &= \frac{(1-g)\kappa}{2}
\end{aligned}
$$

The Key Question of solving dynamic equations: for what *law of motion* this equation is *always* true (whatever is the value of m_{t-1}, p_{t-1} and x_{t-1})? Solving a system of dynamic equations "boils down" to answer this question

A simple numerical example: the solution I

"Guess and Verify" Method

Note that we can (wisely) guess that the solution has the form:

$$x_t = dx_{t-1} + b(m_{t-1} - gp_{t-1})$$

where d and b are coefficients to be determined. Substituting this equation we obtain:

$$\frac{(1-h)}{2}\left[d^2x_{t-1} + (d+1)b(m_{t-1} - gp_{t-1})\right] + ...$$

$$... - (1+h)\left[dx_{t-1} + b(m_{t-1} - gp_{t-1})\right] + \frac{(1-h)}{2}x_{t-1} = -\kappa(m_{t-1} - gp_{t-1})$$

A simple numerical example: the solution II

"Guess and Verify" Method
For this equation to be true:

$$\begin{aligned}\frac{(1-h)}{2}d^2-(1+h)\,d+\frac{(1-h)}{2} &= 0\\ \frac{(1-h)\,(d+1)\,b}{2}-(1+h)\,b &= -\kappa\end{aligned}$$

Which gives us the solution:

$$d=\frac{1-\sqrt{h}}{1+\sqrt{h}} \quad ; \quad d=\frac{1-d}{1+g}$$

This method is clearly unfeasible and/or inefficient in general! However the methods we see/saw in this course rely on the same ideas: solving backward or forward some equations (first example) to find a low of motion for endogenous variable (second example).

3 Algebraic and Economic Preliminary Definitions

3.1 Algebraic Tools

Algebraic Definitions

Definition 1 *Matrix Pencil: let $A_0, A_1, ..., A_l$ be nxn complex matrices s.t. $A_l \neq [0]$ than the matrix valued function:*

$$P(\lambda)=\sum_{i=0}^{l}\lambda^i A_i$$

is called a matrix pencil, written as $(A_0, A_1, ..., A_l)$.
When l=1 we have a linear matrix pencil.

Definition 2 *A linear matrix pencil is said to be regular: if $\exists\lambda$ s.t.* $\det(A_0, A_1) \neq 0$

Corollary 3 *The eigenvalues of $P(\lambda)$ are the set of $\lambda' s$ s.t.* $\det(A_0, A_1) = 0$

Algebraic Definitions

Definition 4 *Unitary Matrix: A square complex matrix Q is said to be unitary if it satisfies*:

$$QQ^* = I_n$$

where Q^* is its conjugate transpose (take the transpose and than the complex conjugate of each element).
When l=1 we have a linear matrix pencil.

Corollary 5 *$Q^* = Q^{-1}$, i.e. (roughly speaking Q is made of independent lines)*

Corollary 6 *If Q is real, it is said to be orthogonal (made of orthonormal vectors)*

Algebraic Definitions

Definition 7 *Generalized Schur Decomposition (GSD): let $P(\lambda) = (A, B)$ be a regular pencil matrix. Then $\exists$ unitary matrices Q and Z such that:*

$$\begin{aligned} QAZ &= S \text{ is upper triangular} \\ QBZ &= T \text{ is upper triangular} \\ \lambda(A, B) &= \left\{ \frac{t_{ii}}{s_{ii}} : s_{ii} \neq 0 \right\} \end{aligned}$$

Corollary 8 *A GSD always exists (given regularity)*

Corollary 9 *If $t_{ii} = s_{ii} = 0$ than $\exists$ a linear combination of lines (rows) in each matrix that gives 0 at the same line (row).*

Algebraic Operators

Definition 10 *Let X be a general nxm matrix. Than vec(X) is a mnx1 vector obtained stacking all the columns of X one under each other.*

Corollary 11 *Let A, B, C matrices such that the product ABC exists, then $vec(ABC) = (C' \otimes A)\, vec(B)$*

Definition 12 *Let X be a general square nxn matrix. Than vech(X) is a $\frac{n(n+1)}{2}$x1 vector obtained stacking by columns the lower-triangular part of X.*

Corollary 13 *For any symmetric square matrix X there exist a unique matrix L such that $vec(X) = L vech(X)$*

Furthermore $L^h vec(X) = vech(X)$ where $L^+ = (L'L)^{-1} L'$ and $L^+ L = I$

3.2 Economic Tools

Economic Definition

Definition 14 *Martingale Difference Process: A process ε_t is said to be a martingale difference process if $E_t(\varepsilon_t) = 0\ \forall t$*

Definition 15 *Backward-looking (state) variables: a variable y_t is said to be a backward-looking (state) variable if:*

- *$y_{t+1} - E_t[y_{t+1}]$ is martingale difference process;*
- *y_0 is given*

Economic Tools: obtaining a VAR(1) representation

We are interested in studying a system of expectational difference equation of the form:

$$\underset{nxn}{A}\underset{nx1}{E_t[x_{t+1}]} = Bx_t + \underset{nxm}{\eta}\underset{mx1}{\xi_t}$$

where x_t is a vector of correlated endogenous variables and ξ_t is a set of exogenous variables.

Most, if not all, models can be written into this form. We now briefly discuss how to accomplish this task.

More than one lag

If the model has more than one lag:

$$E_t[x_{t+1}] = Bx_t + Cx_{t-1}$$

introduce an auxiliary variable to write:

$$\begin{aligned} E_t[x_{t+1}] &= Bx_t + C\tilde{x}_t \\ \tilde{x}_{t+1} &= x_t \end{aligned}$$

to obtain an equivalent VAR(1) system.

More than one expectation

If the model has more than one lag:

$$E_t[x_{t+1}] = Bx_t + CE_{t-1}[x_t]$$

introduce an auxiliary variable to write:

$$\begin{aligned} E_t[x_{t+1}] &= Bx_t + C\tilde{x}_t \\ x_{t+1} &= \tilde{x}_t + \varepsilon_t \end{aligned}$$

where $E(\varepsilon_t) = 0$ to accommodate the hypotesis of rational expectation.

This two cases are enough to handle probably the 99% of macro models nowadays used...

4 First Order Approximation

Solving a System of Partial Differential Equations up to the first order: The Generalized Schur Decomposition

The Problem

With a small loss of generality we are interested in studying a system like:

$$\underset{nxn}{A}\underset{nx1}{E_t[x_{t+1}]} = Bx_t + \underset{nxm}{\eta}\underset{mx1}{z_t}$$

It's not uncommon that the matrix A is not invertible, so that pre-multiplying by A^{-1} and apply the Jordan decomposition is unfeasible

The Generalized Schur Decomposition: a quick review

Transformation of Variables

Re-order the variables in the system, states must come first:

$$x_t \equiv \left[\begin{array}{c} x_t^1 \\ x_t^2 \end{array} \right]$$

label all non-states variables controls.

Consider the following transformation of variables:

$$\underbrace{\left[\begin{array}{c} s_t \\ u_t \end{array} \right]}_{y_t} = Z^H \left[\begin{array}{c} x_t^1 \\ x_t^2 \end{array} \right]$$

where Z^H is the conjugate transpose of the Z matrix in the GSD.

The Generalized Schur Decomposition: a quick review

Transformation of Variables

Note that:

$$\begin{aligned} AE_t\left[x_{t+1}\right] &= Bx_t + \eta\xi_t \Rightarrow Q^{-1}SZ^{-1}E_t\left[x_{t+1}\right] = Q^{-1}TZ^{-1}x_t + \eta z_t \\ &\Rightarrow Q^{-1}SZ^H E_t\left[x_{t+1}\right] = Q^{-1}TZ^H x_t + \eta z_t \\ &\Rightarrow Q^{-1}SE_t\left[y_{t+1}\right] = Q^{-1}Ty_t + \eta z_t \\ &\Rightarrow SE_t\left[y_{t+1}\right] = Ty_t + Q\eta z_t \end{aligned}$$

Partitioned as:

$$\left[\begin{array}{cc} S_{11} & S_{21} \\ 0 & S_{22} \end{array} \right] E_t \left[\begin{array}{c} s_{t+1} \\ u_{t+1} \end{array} \right] = \left[\begin{array}{cc} T_{11} & T_{21} \\ 0 & T_{22} \end{array} \right] \left[\begin{array}{c} s_t \\ u_t \end{array} \right] + \left[\begin{array}{c} Q_1 \\ Q_2 \end{array} \right] \eta z_t$$

This system is clearly equivalent to the original one.

The Generalized Schur Decomposition: a quick review

Solving for controls

All controls are solved forward:

$$S_{22}E_t\left[u_{t+1}\right] = T_{22}u_t + Q_2\eta z_t$$

The solution is:

$$u_t = -T_{22}\sum_{k=0}^{\infty}\left[T_{22}^{-1}S_{22}\right]^k Q_2\eta E_t\left[z_{t+k}\right]$$

that can be easily solved depending on the structure of the endogenous processes.

The Generalized Schur Decomposition: a quick review

Solving for states

Solved for controls, we can treat present controls and their expectations as given and solve backward for states:

$$S_{22}E_t\left[u_{t+1}\right] = T_{22}u_t + Q_2\eta z_t$$

The solution is:

$$u_t = -T_{22} \sum_{k=0}^{\infty} \left[T_{22}^{-1} S_{22} \right]^k Q_2 \eta E_t \left[\xi_{t+k} \right]$$

that can be easily solved depending on the structure of the endogenous processes.

5 Higher Order Approximations

5.1 A General Approach

Higher order approximations: why we need them?

First order approximation techniques are useful when studying impulse response functions (dynamic response to exogenous shocks)

However they have been proved to be ill suited to develop welfare analysis: second order terms (related to the variances of endogenous variables) are generally important for the equilibrium welfare functions (Kim et Kim (JIE 2005), see Woodford (2002) for a discussion of the sufficiency of the first order approximation for welfare analysis)

A general approach to approximation methods.

For simplicity let me change the notation: x are states in period t, x' states in period t+1, y are controls, y' controls at period t+1.

Clearly we can write the system characterizing the model as:

$$E_t \left[\underset{nx1}{f} \left(y', y, x', x \right) \right] = 0 \tag{1}$$

higher order approximations are based on this simple representation

A general approach to approximation methods

Define implicitly:

$$\begin{aligned} y &= g\left(x, \sigma\right) & (2) \\ x' &= h\left(x, \sigma\right) + \varphi \sigma \varepsilon_t & (3) \end{aligned}$$

where σ is a variance (scalar) parameter.

Than an n-th order Taylor approximation of these functions around the non-stochastic steady-state are:

$$\begin{aligned} y &= g\left(x, \sigma\right) = g\left(\overline{x}, 0\right) + \sum_{i=1}^{n} \frac{1}{i!} \left[\frac{\partial^i g_x \left(\overline{x^1}, 0 \right)}{\partial^i \overline{x}} \left(x - \overline{x} \right)^i + \frac{\partial^i g_\sigma \left(\overline{x}, 0 \right)}{\partial^i \overline{x}} \sigma^i \right] \\ x' &= h\left(x, \sigma\right) + \varphi \sigma \varepsilon_t = h\left(\overline{x}, 0\right) + \sum_{i=1}^{n} \frac{1}{i!} \left[\frac{\partial^i h_x \left(\overline{x}, 0 \right)}{\partial^i \overline{x}} \left(x - \overline{x} \right)^i + \frac{\partial^i h_\sigma \left(\overline{x}, 0 \right)}{\partial^i \overline{x}} \sigma^i \right] \end{aligned}$$

Objective: find higher order approximations of g and h around $\overline{x}$, $\overline{y}$ and $\sigma = 0$;

Observations:

By definition

$$\begin{aligned}\bar{y} &= g(\bar{x}, 0) \\ \bar{x} &= h(\bar{x}, 0)\end{aligned}$$

and:

$$\begin{aligned}E_t \left[\underset{nx1}{f}(y', y, x', x) \right] &= 0 \text{ for ANY value of } y', y, x', x \text{ and } \sigma \\ & \quad SO \\ \frac{\partial f}{\partial^i x \partial^j y \partial^s \sigma} &= 0 \text{ for } \forall i, j, s, t, x, y\end{aligned}$$

A direct Taylor approximation: state variables

- Substitute eqs.(2)-(3) in (1):

$$E_t \underset{nx1}{f}\left(g\left(h(x,\sigma) + \varphi\sigma\varepsilon_t, \sigma\right), g(x,\sigma), h(x,\sigma) + \varphi\sigma\varepsilon_t, x\right) = 0 \qquad (4)$$

- and take the total derivative w.r.t. x :

$$\begin{aligned}\frac{\partial f}{\partial x} &= 0 \Rightarrow \\ & \underset{n\text{x}n^y}{\frac{\partial f}{\partial y'}_{|(\bar{x},0)}} \underset{n^y\text{x}n^x}{\frac{\partial g}{\partial x'}_{|(\bar{x},0)}} \underset{n^x\text{x}n^x}{\frac{\partial h}{\partial x}_{|(\bar{x},0)}} + \underset{n\text{x}n^y}{\frac{\partial f}{\partial y}_{|(\bar{x},0)}} \underset{n^y\text{x}n^x}{\frac{\partial g}{\partial x}_{|(\bar{x},0)}} + \ldots \\ & + \underset{n\text{x}n^x}{\frac{\partial f}{\partial x'}_{|(\bar{x},0)}} \underset{n^x\text{x}n^x}{\frac{\partial h}{\partial x}_{|(\bar{x},0)}} + \underset{n\text{x}n^x}{\frac{\partial f}{\partial x}_{|(\bar{x},0)}} = \underset{n\text{x}n^x}{0}\end{aligned}$$

- System of $n\text{x}n^x$ equations in $n\text{x}n^x$ variables.

A direct Taylor approximation: stochastic parameter

- Take the total derivative w.r.t. σ :

$$\begin{aligned}\frac{\partial f}{\partial \sigma} &= 0 \Rightarrow \\ & \frac{\partial f}{\partial y'}_{|(\bar{x},0)} \frac{\partial g}{\partial x'}_{|(\bar{x},0)} \frac{\partial h}{\partial \sigma}_{|(\bar{x},0)} + \frac{\partial f}{\partial y'}_{|(\bar{x},0)} \frac{\partial g}{\partial \sigma}_{|(\bar{x},0)} + \ldots \\ & + \frac{\partial f}{\partial y}_{|(\bar{x},0)} \frac{\partial g}{\partial \sigma}_{|(\bar{x},0)} + \frac{\partial f}{\partial x'}_{|(\bar{x},0)} \frac{\partial h}{\partial \sigma}_{|(\bar{x},0)} = 0\end{aligned}$$

- this is a linear homogenous equation in $\frac{\partial h}{\partial \sigma}_{|(\bar{x},0)}$ and $\frac{\partial g}{\partial \sigma}_{|(\bar{x},0)}$ so if a unique solution exist it must be $\frac{\partial h}{\partial \sigma}_{|(\bar{x},0)} = [0]$ and $\frac{\partial g}{\partial \sigma}_{|(\bar{x},0)} = [0]$

- the same method shows that $\frac{\partial h}{\partial\sigma\partial x_t^1}_{|(\bar{x},0)} = [0]$ and $\frac{\partial g}{\partial\sigma\partial x_t^1}_{|(\bar{x},0)} = [0]$

A direct Second Order Taylor approximation: state variables

$$\begin{aligned}[0] \quad = \quad & f_{y'y'}g_xh_x + f_{y'y}g_x + f_{y'x'}h_x + f_{y'x}g_xh_x + f_{y'}g_{xx}h_xh_x + f_{y'}g_xh_{xx} + ... = \\ & \left(f_{yy'}g_xh_x + f_{yy}g_x + f_{yx'}h_x + f_{yx}\right)g_x + f_yg_{xx} + \left(f_{x'y'}g_xh_x + f_{x'y}g_x + f_{x'x'}h_x + f_{x'x}\right)h_x + f_{x'}h_{xx} \\ & f_{xy'}g_xh_x + f_{xy}g_x + f_{xx'}h_x + f_{xx}\end{aligned}$$

General Results

Theorem 16 *Consider the model whose equations can be written as (1), then:*

- $\frac{\partial h}{\partial\sigma}_{|(\bar{x},0)} = [0]$
- $\frac{\partial g}{\partial\sigma}_{|(\bar{x},0)} = [0]$
- $\frac{\partial h}{\partial\sigma\partial x}_{|(\bar{x},0)} = [0]$
- $\frac{\partial g}{\partial\sigma\partial x}_{|(\bar{x},0)} = [0]$

Corollary 17 *The low of motion of the linearized system is independent of the volatility of the shocks*

A second order approximation to the policy function of a stochastic model differ from the non-stochastic version only by a constant parameter of the control vector.

5.2 Using first order methods to solve up to the second order

Finding second order solutions using first order methods

A second-order approximation of model can be written as:

$$\underset{nxn}{A_1}\underset{nx1}{E_t[x_{t+1}]} = \underset{nxnnx1}{A_2\ x_t} + \underset{nxmmx1}{A_3\ z_t} + \underset{nx\frac{n(n+1)}{2}}{A_4}\ \underset{\frac{n(n+1)}{2}x1}{\Lambda_t} + \underset{nx\frac{n(n+1)}{2}}{A_5}\ \underset{\frac{n(n+1)}{2}x1}{E_t\left[\Lambda_{t+1}\right]}$$

where:

$$\Lambda_t = vech\left(\tilde{x}_t\tilde{x}_t'\right)$$

where:

$$\tilde{x}_t \equiv \begin{bmatrix} z_t \\ x_t^1 \\ x_t^2 \end{bmatrix}$$

Finding second order solutions using first order methods

Suppose we have applied the Schur decomposition to solve the first order approximation obtaining the state-space form:

$$\begin{aligned} {}_f x^1_{t+1} &= F_1 z_t + F_2 {}_f x^1_t \\ {}_f x^2_t &= P_1 {}_f x^1_t + P_2 {}_f x_t \end{aligned}$$

and we can compactly write the solution as:

$$\begin{aligned} \begin{bmatrix} z_t \\ {}_f x^1_t \\ {}_f x^2_t \end{bmatrix} &= \Omega \begin{bmatrix} z_t \\ {}_f x^1_t \end{bmatrix} \\ \underbrace{\begin{bmatrix} z_t \\ {}_f x^1_t \end{bmatrix}}_{s_t} &= \Phi \begin{bmatrix} z_{t-1} \\ {}_f x^1_{t-1} \end{bmatrix} + \Gamma \varepsilon_t \end{aligned}$$

where ${}_f x^1_t$ is the first order approximation of the states, Ω and Φ are appropriate matrices constructed using F_1, F_2, P_1, P_2 and I

Finding second order solutions using first order methods

Since we are already dealing with an approximation, we can substitute x^1_t with its first order approximation.

Note that:

$$\begin{aligned} \Lambda_t &= vech\left(\tilde{x}_t \tilde{x}'_t\right) = Lvec\left(\Omega \begin{bmatrix} z_t \\ x^1_t \end{bmatrix} \begin{bmatrix} z_t \\ x^1_t \end{bmatrix}' \Omega'\right) \\ &= L\left(\Omega \otimes \Omega\right) vec\left(\begin{bmatrix} z_t \\ x^1_t \end{bmatrix} \begin{bmatrix} z_t \\ x^1_t \end{bmatrix}'\right) \\ &= L\left(\Omega \otimes \Omega\right) L^h vech\left(\begin{bmatrix} z_t \\ {}_f x^1_t \end{bmatrix} \begin{bmatrix} z_t \\ {}_f x^1_t \end{bmatrix}'\right) \\ &= RV_t \end{aligned}$$

Finding second order solutions using first order methods

Second:

$$\begin{aligned} V_t &= vech\left(s_t s'_t\right) = vech\left(\left[\Phi s_{t-1} + \Gamma \varepsilon_t\right]\left[\Phi s_{t-1} + \Gamma \varepsilon_t\right]'\right) \\ &= vech\left(\Phi s_{t-1} s'_{t-1} \Phi' + \Phi s_{t-1} \Gamma' \varepsilon'_t + \Gamma \varepsilon_t s'_{t-1} \Phi' + \Gamma \varepsilon_t \Gamma' \varepsilon'_t\right) \\ &= Lvec\left(\Phi s_{t-1} s'_{t-1} \Phi'\right) + Lvec\left(\Phi s_{t-1} \varepsilon'_t \Gamma'\right) + Lvec\left(\Gamma \varepsilon_t s'_{t-1} \Phi'\right) + Lvec\left(\Gamma \varepsilon_t \varepsilon'_t \Gamma'\right) \\ &= L\left(\Phi \otimes \Phi\right) L^h vech\left(s_{t-1} s'_{t-1}\right) + L\left(\Gamma \otimes \Gamma\right) L^h vech\left(\varepsilon_t \varepsilon'_t\right) + \\ &\quad + L\left(\Gamma \otimes \Phi\right) L^h vech\left(s_{t-1} \varepsilon'_t\right) + L\left(\Phi \otimes \Gamma\right) L^h vech\left(\varepsilon_t s'_{t-1}\right) \\ &= \tilde{\Phi} V_{t-1} + \tilde{\Gamma} \tilde{\varepsilon}_t + \tilde{\Psi} \tilde{\xi}_t \end{aligned}$$

where $\tilde{\Psi} = L\left(\Gamma \otimes \Phi\right) L^h + L\left(\Phi \otimes \Gamma\right) L^h P$ and P is such that $Pvech\left(X'\right) = vech\left(X\right)$

Therefore the second order elements of the system are themselves an independent system.

Finding second order solutions using first order methods

Hence we have now an augmented system:

$$\begin{aligned} A_1 E_t\left[x_{t+1}\right] &= A_2 x_t + A_3 z_t + G\Lambda_t + H\Sigma \\ V_t &= \tilde{\Phi} V_{t-1} + \tilde{\Gamma}\tilde{\varepsilon}_t + \tilde{\Psi}\tilde{\xi}_t \\ z_t &= N z_{t-1} + \varepsilon_t \end{aligned}$$

where $G = A_4 R + A_5 R\tilde{\Phi}$ $H = A_5 R\tilde{\Gamma}$ and $\Sigma = E_t \tilde{\varepsilon}_t$.

We can than solve backward for V_t and than treating it and $H\Sigma$ as exogenous variables in the main representation.

5.3 A Simple RBC application

Finding second order solutions: A Simple RBC application

We now consider a simple application. The Ramsey model is:

$$\max E_0 \sum_{t=0}^{\infty} \frac{C_t^{1-\gamma}}{1-\gamma}$$

$$\begin{aligned} s.t. K_{t+1} &= (1-\delta) K_t + I_t \\ I_t + C_t &= A_t K_t^{\alpha} \\ \log A_{t+1} &= \rho \log A_t + \varepsilon_{t+1} \end{aligned}$$

we assume $\varepsilon_t \sim N(0,1)$ and, for simplicity, $\delta = 1$ and $\rho = 0$.

Finding second order solutions: A Simple RBC application

FOC's imply:

$$\begin{aligned} C_t^{-\gamma} &= \beta\alpha E_t\left[C_{t+1}^{-\gamma} A_{t+1} K_{t+1}^{\alpha-1}\right] \\ K_{t+1} &= A_t K_t^{\alpha} - C_t \\ \log A_{t+1} &= \varepsilon_{t+1} \end{aligned}$$

which implies:

$$\begin{aligned} -\gamma \log C_t &= \log(\beta\alpha) - \gamma E_t\left[\log C_{t+1}\right] + E_t\left[\log A_{t+1}\right] + (\alpha-1) E_t\left[\log K_{t+1}\right] \\ \log\left(K_{t+1} + C_t\right) &= \log\left[A_t K_t^{\alpha}\right] \\ \log A_{t+1} &= \varepsilon_{t+1} \end{aligned}$$

So:

$$\begin{aligned} E_t\left[\underset{nx1}{f}\left(x_{t+1}^2, x_t^2, x_{t+1}^1, x_t^1\right)\right] &\equiv E_t\left[\begin{array}{c} \gamma \log C_t + \log(\beta\alpha) - \gamma E_t\left[\log C_{t+1}\right] + E_t\left[\log A_{t+1}\right] + (\alpha-1) E_t\left[\log K_{t+1}\right] \\ \log\left(K_{t+1} + C_t\right) - \log\left[A_t K_t^{\alpha}\right] \end{array}\right. \\ &= 0 \end{aligned}$$

$x_t^1 = K_t, x_t^2 = C_t$ and $z_t = a_t$

Finding second order solutions: taking the first order approximation

Recall that in this case we have one state and one control.

For the SGU Method:

$$\begin{aligned} f_{y'}gh + f_y g + f_{x'}h + f_x &= [0] \\ \begin{bmatrix} -\gamma \\ 0 \end{bmatrix} gh + \begin{bmatrix} \gamma \\ \frac{C_{ss}}{K_{ss}+C_{ss}} \end{bmatrix} g + \begin{bmatrix} (\alpha-1) \\ \frac{K_{ss}}{K_{ss}+C_{ss}} \end{bmatrix} h + \begin{bmatrix} 0 \\ \alpha \end{bmatrix} &= [0] \end{aligned}$$

Note that we are looking for the roots of this system of equations (see example at the beginning of the class).

For the GSD:

$$\begin{aligned} -\gamma E_t \hat{c}_{t+1} + (\alpha-1)\hat{k}_{t+1} &= -\gamma \hat{c}_t \\ \frac{K_{ss}}{K_{ss}+C_{ss}}\hat{k}_{t+1} &= -\frac{C_{ss}}{K_{ss}+C_{ss}}\hat{c}_t + \alpha\hat{k}_t + \hat{a}_t \end{aligned}$$

In Matrix notation:

$$\begin{bmatrix} (\alpha-1) & -\gamma \\ \frac{K_{ss}}{K_{ss}+C_{ss}} & 0 \end{bmatrix} \begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} + \begin{bmatrix} 0 & \gamma \\ -\alpha & \frac{C_{ss}}{K_{ss}+C_{ss}} \end{bmatrix} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \hat{a}_t = [0]$$

Finding second order solutions: the second order approximation for SGU Method

For the second order we have:

$$\begin{aligned} [0] = \; & f_{y'y'}g_x h_x + f_{y'y}g_x + f_{y'x'}h_x + f_{y'x}g_x h_x + f_{y'}g_{xx}h_x h_x + f_{y'}g_x h_{xx} + \ldots = \\ & (f_{yy'}g_x h_x + f_{yy}g_x + f_{yx'}h_x + f_{yx})\, g_x + f_y g_{xx} + (f_{x'y'}g_x h_x + f_{x'y}g_x + f_{x'x'}h_x + f_{x'x})\, h_x + f_{x'}h_{xx} \\ & f_{xy'}g_x h_x + f_{xy}g_x + f_{xx'}h_x + f_{xx} \end{aligned}$$

which is a system of two equations in two unknowns.

Finding second order solutions: the second order approximation for the Lombardo Sutherland Method

For the second order we have:

$$\begin{aligned} -\gamma E_t\hat{c}_{t+1} + (\alpha-1)\hat{k}_{t+1} + \frac{1}{2}E_t\left[\left(\hat{a}_{t+1} + \gamma E_t\hat{c}_{t+1} + (\alpha-1)\hat{k}_{t+1}\right)^2\right] &= -\gamma\hat{c}_t + \frac{\gamma^2}{2}\hat{c}_t^2 \\ -\frac{C_{ss}}{K_{ss}+C_{ss}}\hat{c}_t + \alpha\hat{k}_t + \hat{a}_t - \frac{1}{2}\frac{C_{ss}}{K_{ss}+C_{ss}}\hat{c}_t^2 + \frac{\alpha^2}{2}\hat{k}_t^2 + \frac{1}{2}\hat{a}_t^2 + \alpha\hat{k}_t\hat{a}_t &= \frac{K_{ss}}{K_{ss}+C_{ss}}\hat{k}_{t+1} + \frac{1}{2}\frac{K_{ss}}{K_{ss}+C_{ss}}\hat{k}_{t+1}^2 \end{aligned}$$

In matrix notation:

$$A_1 \begin{bmatrix} \hat{k}_{t+1} \\ E_t\hat{c}_{t+1} \end{bmatrix} = A_2 \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + A_3\hat{a}_t + A_4\Lambda_t + A_5 E_t[\Lambda_{t+1}]$$

where:

$$
\begin{aligned}
A_1 &= \begin{bmatrix} (1-\alpha) & -\gamma \\ \frac{K_{ss}}{K_{ss}+C_{ss}} & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & \gamma \\ \alpha & -\frac{C_{ss}}{K_{ss}+C_{ss}} \end{bmatrix}, A_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\frac{\gamma^2}{2} \\ \frac{1}{2} & \alpha & \frac{\alpha^2}{2} & 0 & 0 & -\frac{1}{2}\frac{C_{ss}}{K_{ss}+C_{ss}} \end{bmatrix} \\
A_5 &= \begin{bmatrix} \frac{1}{2} & (\alpha-1) & \frac{(\alpha-1)^2}{2} & \gamma & \gamma(\alpha-1) & \frac{\gamma^2}{2} \\ 0 & 0 & -\frac{1}{2}\frac{K_{ss}}{K_{ss}+C_{ss}} & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

trough which we can construct the auxiliary system.

6 Normative Analysis

Normative Analysis

We distinguish between two cases:

- policies NOT affecting the steady state, stochastic environment
- policies affecting the steady state, non stochastic environment

The two cases can be combined in a conceptually easy (tough not always easy to implement) way.

6.1 Policies not affecting the steady state

6.1.1 An RBC application

Policies not affecting the steady state

The standard Ramsey model augmented with consumption taxes is:

$$\max_{c_t,k_{t+1},i_t} E_0\left[\sum_{t=0}^{\infty}\beta^t \frac{c_t^{1-\gamma}}{1-\gamma}\right]$$

$$
\begin{aligned}
s.t. k_{t+1} &= (1-\delta)k_t + i_t \\
(1+\tau_t)c_t + i_i &= a_t k_t^{\alpha} \\
y_t &= a_t k_t^{\alpha} \\
a_{t+1} &= \rho a_t + \varepsilon_t
\end{aligned}
$$

Consider an exogenous set of policies:

$$\hat{\tau}_t = \phi_k E_t\left[\hat{y}_{t+1}\right]$$

where the hat denote deviations from equilibrium level.

Taxes on consumption are set as a (linear) function function of expected level of production.

Policies not affecting the steady state

FOC's imply the following system of 3 equations in 3 variables:

$$\begin{aligned} c_t^{-\gamma} &= \beta E_t \left\{ c_{t+1}^{-\gamma} \left[\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta) \right] \right\} \\ k_{t+1} &= (1-\delta) k_t + a_t k_t^{\alpha} - (1+\tau_t) c_t \\ \hat{\tau}_t &= \phi_k E_t \left[\hat{y}_{t+1} \right] \end{aligned}$$

Plus:

$$\begin{aligned} c_t^{-\gamma} &= (1+\tau_t) \lambda_t \\ i_i &= a_t k_t^{\alpha} - (1+\tau_t) c_t \end{aligned}$$

Policies not affecting the steady state

This is the basic RBC model with an additional variable and an additional equation.

Type of questions we can address:

- What is the optimal taxation policy in this class?
- What is the optimal taxation within linear policies? i.e. would it be better to respond to past or current level of production?
- What are the implication of the policy that maximizes private welfare?

Tomorrow we will discuss how to code and solve this problem.

6.2 Policies affecting the steady state

6.2.1 An RBC application

Policies affecting the steady state: the problem

Consider the following problem:

$$\max_{c_t, k_{t+1}, i_t, \tau_t} \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\gamma}}{1-\gamma} \right]$$

$$\begin{aligned} s.t. k_{t+1} &= (1-\delta-\tau_t) k_t + i_t \\ c_t + i_t &= a_t k_t^{\alpha} x_t^{1-\alpha-\theta} \\ x_t &= \tau_t k_t \\ \log a_{t+1} &= \rho \log a_t \end{aligned}$$

That is the government can impose wealth taxes to provide services that enter in the production function.

In this problem the level of taxes affects the steady state.

Policies affecting the steady state: analytical solution

Write the lagragian:

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^{t}\left\{\frac{c_{t}^{1-\gamma}}{1-\gamma}+\underset{1x2}{\Lambda_{t}}\left[\begin{array}{c}(1-\delta-\tau_{t})\,k_{t}+a_{t}k_{t}^{1-\theta}\,(\tau_{t})^{1-\alpha-\theta}-c_{t}-k_{t+1}\\ a_{t}k_{t}^{1-\theta}\,(\tau_{t})^{1-\alpha-\theta}-c_{t}-i_{t}\\ \log a_{t+1}=\rho\log a_{t}\end{array}\right]\right\}$$

FOC's imply:

$$\begin{aligned}c_{t}^{-\gamma} &= \lambda_{t}^{1}\\ \lambda_{t}^{1} &= \beta\left\{\lambda_{t+1}^{1}\left[(1-\theta)\,a_{t+1}\,(k_{t+1})^{-\theta}\,(\tau_{t+1})^{1-\alpha-\theta}+1-\delta-\tau_{t+1}\right]\right\}\\ k_{t} &= (1-\alpha-\theta)\,a_{t}k_{t}^{1-\theta}\,(\tau_{t})^{-\alpha-\theta}\\ &\quad+\text{ the constraints}\end{aligned}$$

Policies affecting the steady state: analytical solution
In steady state the FOC's imply:

$$\begin{aligned}c^{-\gamma} &= \lambda^{1}\\ k &= \left[\frac{1-\beta\,(1-\delta-\tau)}{\beta\,(1-\theta)\,\tau^{1-\alpha-\theta}}\right]^{-\frac{1}{\theta}}\\ 1 &= (1-\alpha-\theta)\,k^{-\theta}\,(\tau)^{-\alpha-\theta}\\ c &= (-\delta-\tau)\,k+k^{1-\theta}\,(\tau)^{1-\alpha-\theta}\end{aligned}$$

which implies:

$$1=(1-\alpha-\theta)\,\frac{1-\beta\,(1-\delta-\tau)}{\beta\,(1-\theta)\,\tau}$$

which implies:

$$\tau=\frac{(1-\alpha-\theta)\,[1-\beta\,(1-\delta)]}{[\alpha\beta]}$$

Policies affecting the steady state: numerical solution
This problem is simple, so we can use it to test a numerical approach.
Consider the problem written in matrix notation:

$$\mathcal{L}=\sum_{t=0}^{\infty}\beta^{t}\left[O_{t}+\underset{1xn^{c}}{\Lambda_{t}}\underset{n^{c}x1}{L_{t}}\right]$$

we have two sets of first order conditions, those w.r.t. endogenous variables and those w.r.t. the lagrange multipliers.

The latter are the constraints, that we can use to solve for the steady state, given a value of the policy variable.

Policies affecting the steady state
First order conditions w.r.t endogenous variables imply:

$$\frac{\partial O_{t}}{\partial x_{t}}+\Lambda_{t}\frac{\partial L_{t}}{\partial x_{t}}=[0]$$

Note that this is a system of n variables in n^c unknowns (the lagrange multipliers) and therefore can not be solved exactly.

Furthermore $\frac{\partial L_t}{\partial x_t}$ is not a square matrix and therefore can not be inverted.

However we can, given a value of the policy parameter that determine the value of endogenous variables, solve this system by OLS (i.e. the values of the lagrange multipliers that minimize the sum of squared residuals).

Than we can check if this is a solution.

Tomorrow we will see how to implement this.

6.3 Exercises policy not affecting the steady state: an RBC model with Money and Optimal Policy

$$\max_{c_t,k_{t+1},i_t,m_t,b_t} \sum_{t=0}^{\infty} \beta^t \frac{\left[\chi c_t^{1-\theta} + (1-\chi) m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}}}{1-\gamma}$$

$$\begin{aligned} s.t. k_{t+1} &= (1-\delta) k_t + i_t \\ c_t + i_t + \frac{B_t}{P_t} + \frac{M_t}{P_t} &= a_t k_t^{\alpha} + (1+R_t) \frac{B_{t-1}}{P_t} + \frac{M_{t-1}}{P_t} \\ \log a_{t+1} &= \rho \log a_t \end{aligned}$$

which implies:

$$c_t + k_{t+1} + b_t + m_t = a_t k_t^{\alpha} + (1-\delta) k_t + (1+R_t) \frac{b_{t-1}}{\pi_t} + \frac{m_{t-1}}{\pi_t}$$

Lagrangian and First Order Conditions are:

$$\max_{c_t,k_{t+1},m_t,b_t} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \frac{\left[\chi c_t^{1-\theta} + (1-\chi) m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}}}{1-\gamma} + \\ +\lambda_t \left[a_t k_t^{\alpha} + (1-\delta) k_t + (1+R_t) \frac{b_{t-1}}{\pi_t} + \frac{m_{t-1}}{\pi_t} - c_t - k_{t+1} - b_t - m_t\right] \end{array} \right\}$$

$$\left[\chi c_t^{1-\theta} + (1-\chi) m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} \chi (1-\theta) c_t^{-\theta} = \lambda_t$$

$$\lambda_t = \beta \lambda_{t+1} \left[\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

$$\lambda_t = \beta \lambda_{t+1} \left[\frac{1+R_{t+1}}{\pi_{t+1}}\right]$$

$$\left[\chi c_t^{1-\theta} + (1-\chi) m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} (1-\chi)(1-\theta) m_t^{-\theta} = \lambda_t - \beta \frac{\lambda_{t+1}}{\pi_{t+1}}$$

Couple of notes:

This is a system of 7 variables $c_t, m_t, k_t, b_t, \pi_t, a_t, R_t$ in 4 equations, the government budget constraint, the exogenous low of motion and the policy rule:

$$\left[\chi c_t^{1-\theta} + (1-\chi) m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} c_t^{-\theta} = \beta \left[\chi c_{t+1}^{1-\theta} + (1-\chi) m_{t+1}^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} c_{t+1}^{-\theta} \left[\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

$$\left[\chi c_t^{1-\theta} + (1-\chi)\, m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} c_t^{-\theta} = \beta \left[\chi c_{t+1}^{1-\theta} + (1-\chi)\, m_{t+1}^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} c_{t+1}^{-\theta} \left[\frac{1+R_{t+1}}{\pi_t}\right]$$

$$\begin{aligned}\left[\chi c_t^{1-\theta} + (1-\chi)\, m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} (1-\chi)(1-\theta)\, m_t^{-\theta} &= \left[\chi c_t^{1-\theta} + (1-\chi)\, m_t^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} \chi (1-\theta)\, c_t^{-\theta} + \\ &\quad -\beta \frac{\left[\chi c_{t+1}^{1-\theta} + (1-\chi)\, m_{t+1}^{1-\theta}\right]^{\frac{1-\gamma}{1-\theta}-1} (1-\theta)\, c_{t+1}^{-\theta}}{\pi_{t+1}}\end{aligned}$$

$$c_t + k_{t+1} + b_t + m_t = a_t k_t^{\alpha} + (1-\delta)\, k_t + (1+R_t)\, \frac{b_{t-1}}{\pi_t} + \frac{m_{t-1}}{\pi_t}$$

$$\log a_{t+1} = \rho \log a_t$$

$$\begin{aligned}B_t - B_{t-1} + M_t - M_{t-1} &= R_{t-1} B_{t-1} \\ P_t b_t + P_t m_t - P_{t-1} m_{t-1} &= (1+R_{t-1})\, P_{t-1} b_{t-1} \\ \pi_t b_t + \pi_t m_t - m_{t-1} &= (1+R_{t-1})\, b_{t-1}\end{aligned}$$

Steady state: $a = 1$ than

$$k = \left[\frac{1-\beta(1-\delta)}{\beta\alpha}\right]^{\frac{1}{\alpha-1}}$$

$$R = \frac{\pi}{\beta} - 1$$

$$\begin{aligned}(1-\chi)\, m_t^{-\theta} &= \chi c_t^{-\theta} - \beta \frac{\chi c_{t+1}^{-\theta}}{\pi_{t+1}} \\ m &= \underbrace{\left[\frac{\chi - \chi\frac{\beta}{\pi}}{(1-\chi)}\right]^{-\frac{1}{\theta}}}_{AA} c\end{aligned}$$

$$\begin{aligned}b &= \frac{\pi - 1}{1+R-\pi} m \\ b &= \underbrace{\left[\frac{\pi-1}{1+R-\pi}\right]\left[\frac{\chi - \beta\frac{\chi}{\pi}}{(1-\chi)}\right]^{-\frac{1}{\theta}}}_{BB} c\end{aligned}$$

$$
\begin{aligned}
c + BBc + AAc &= k_t^{\alpha} - \delta k + (1+R)\,\frac{BB}{\pi}c + \frac{AA}{\pi}c \\
\left[1 + AA\left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1+R}{\pi}\right)BB\right]c &= k_t^{\alpha} - \delta k \\
c &= \frac{k^{\alpha} - \delta k}{\left[1 + A\left(1 - \frac{1}{\pi}\right) + \left(1 - \frac{1+R}{\pi}\right)B\right]}
\end{aligned}
$$

Than we are ready to solve the model.

6.4 Exercise with Money policy affecting the steady state: optimal level of inflation in an RBC model with Money

Consider the same model with money, but with instantaneous utility function:

$$\log c_t + m_t \exp\left(-\gamma m_t\right)$$

Lagrangian and First Order Conditions are:

$$\max_{c_t, k_{t+1}, m_t, b_t} \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \log c_t + m_t \exp\left(-\gamma m_t\right) + \\ +\lambda_t \left[a_t k_t^{\alpha} + (1-\delta)\,k_t + (1+R_t)\,\frac{b_{t-1}}{\pi_t} + \frac{m_{t-1}}{\pi_t} - c_t - k_{t+1} - b_t - m_t\right] \end{array} \right\}$$

$$\frac{1}{c_t} = \lambda_t$$

$$\lambda_t = \beta\lambda_{t+1}\left[\alpha a_{t+1} k_{t+1}^{\alpha-1} + (1-\delta)\right]$$

$$\lambda_t = \beta\lambda_{t+1}\left[\frac{1+R_{t+1}}{\pi_{t+1}}\right]$$

$$(1-\gamma m_t)\exp\left(-\gamma m_t\right) = \frac{1}{c_t} + \frac{\beta}{\pi_{t+1}}\frac{1}{c_{t+1}}$$

The last four equation constitute our model.
In steady state:

$$k = \left[\frac{1-\beta\,(1-\delta)}{\beta\alpha}\right]^{\frac{1}{\alpha-1}}$$

$$R = \frac{\pi}{\beta} - 1$$

$$c\,(1-\gamma m)\exp\left(-\gamma m\right) = \left[1 - \frac{\beta}{\pi}\right]$$

$$
\begin{aligned}
b &= \frac{\pi-1}{1+R-\pi}m \\
b &= \frac{\pi-1}{\frac{\pi}{\beta}-\pi}m \\
b &= \underbrace{\frac{\pi-1}{\pi\left(\frac{1-\beta}{\beta}\right)}}_{BB}m \\
b &= \underbrace{\frac{\beta\left(\pi-1\right)}{\pi\left(1-\beta\right)}}_{BB}m
\end{aligned}
$$

$$
\begin{aligned}
k^{\alpha}-\delta k+\left(1+R\right)\frac{b}{\pi}+\frac{m}{\pi} &= c+b+m \\
k^{\alpha}-\delta k &= c+\left(1-\frac{\left(1+R\right)}{\pi}\right)b+\left(1-\frac{1}{\pi}\right)m \\
k^{\alpha}-\delta k &= c+\left(1-\frac{1}{\beta}\right)b+\left(1-\frac{1}{\pi}\right)m \\
k^{\alpha}-\delta k &= c+\left(1-\frac{1}{\beta}\right)\frac{\beta\left(\pi-1\right)}{\pi\left(1-\beta\right)}m+\left(1-\frac{1}{\pi}\right)m \\
k^{\alpha}-\delta k &= c+\left(\frac{\beta-1}{\beta}\right)\frac{\beta\left(\pi-1\right)}{\pi\left(1-\beta\right)}m+\left(1-\frac{1}{\pi}\right)m \\
k^{\alpha}-\delta k &= c-\frac{\left(\pi-1\right)}{\pi}m+\left(1-\frac{1}{\pi}\right)m \\
k^{\alpha}-\delta k &= c
\end{aligned}
$$

Since steady state consumption is independent of money, we need to set an inflation rate to maximize utility derived by money holdings. Than maximization of $m_t \exp\left(-\gamma m_t\right)$ w.r.t m_t implies $m=\frac{1}{\gamma}$,which is true if $\pi=\beta$.

We are ready to solve the model and check the numerical solution.