The An & Schorfheide Dynamic Stochastic General Equilibrium Model

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1 Introduction

"Dynamic stochastic general equilibrium (DSGE) models are micro-founded optimization-based models that have become very popular in macroeconomics" [1, p.113]. They allow for a comprehensive quantitative evaluation of the economic impact of policies and shocks. Therefore, many economic frameworks have been developed by different authors. An and Schorfheide proposed a potent yet simple model that may serve as a benchmark for monetary policies. [1]

This paper provides a complete and formal description of the An and Schorfheide model. In the first part, we develop the formal framework of our model economy by giving a clear definition of the representative agents and their decision problems, as well as the imposed exogenous processes. On this basis, we step by step solve the agents decision problems and derive the models final equation system. We arranged the paper in the way that all assumptions of the model are formalised within the text while the mathematical deductions are conducted in the theorems. This way we ensure a sharp separation between assumptions and deductions.

2 Development of the An & Schorfheide Model

This section conducts a bottom-up development of the An and Schorfheide model. At first, we formalise the agents and their decision problems and define exogenous processes which allow to impose certain shocks to the model economy. Based on these model assumptions, the equilibrium conditions are derived by solving the agents decision problems. Furthermore, we shall proof that the model possesses a unique steady-state in terms of detrended variables. Finally, all previous results are combined to express the complete model in six equations.

2.1 The Agents and their Decision Problems

The model economy consists of a representative household, a continuum of intermediate goods producing firms, a final good producing firm, a fiscal authority, and a monetary authority. We will introduced these agents step by step.

2.1.1 Representative Household

The first agent in our economy is the representative household. As usual, its decision problem is to maximise its utility under a certain budget constraint. For that matter, we assume a utility function $U_t^{\mathcal{H}}$ in which the households utility derives from its relative consumption C_t/A_t, real money balances M_t/P_t, and hours of work H_t. The consumption C_t is considered with respect to a habit stock represented by the level of technology A_t.

These assumptions are reflected in the formal definition of the representative households utility function

$$U_t^{\mathcal{H}} = \frac{1}{1-\tau}\left(\left(\frac{C_t}{A_t}\right)^{1-\tau} - 1\right) + \chi_M \ln\frac{M_t}{P_t} - \chi_H H_t, \tag{1}$$

with the intertemporal elasticity of substitution $1/\tau$, and the scaling factors χ_M and χ_H. Nevertheless, we will restrict our subsequent analysis to $\chi_H = 1$.

Note that relative consumption and real money balances have a positive marginal utility while an increase in hours of work diminishes the households utility.

We shall now contemplate the households budget constraint.

The household takes the real wage W_t as given and supplies perfectly elastic labour service to the production sector. Additionally, it receives aggregated residual real profits D_t from the firms and pays a lump-sum tax T_t to the government. Furthermore, we assume a domestic bond market where the household has access to nominal government bonds B_t which pay gross interest R_t in the following period. Finally, the perfectly insured household receives net cash inflow from state-contingent securities SC_t.

In total, the household faces expenses consisting of the consumption expenses, purchase of bonds, change in money balance, and taxes, while it receives income from labour, bonds, residual profits, and state-contingent securities. This defines the households budget equation

$$P_tC_t + B_t + M_t - M_{t-1} + T_t = P_tW_tH_t + R_{t-1}B_{t-1} + P_tD_t + P_tSC_t, \tag{2}$$

where P_t represents the price level of the economy, i.e., the price of the final good.

Given the utility function (1) and budget constraint (2), the household maximises not only its present utility at time t, but the present value of its total future utility, i.e.,

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s U_{t+s}^{\mathcal{H}}\right],$$

with the discount factor β. This defines the optimisation problem of the household.

2.1.2 Production sector

The An and Schorfheide model uses a two-stage production sector. We assume a continuum of firms where each firm produces a unique intermediate good. These serve as input for a single new firm which processes all intermediate goods to one representative final good, which is available for consumption. The intermediate goods are produced by monopolist using labour input, while the final good is processed only from the intermediate goods. Moreover, the representative firm acts perfectly competitive.

Starting with the intermediate goods producing firms, we shall develop the formal framework and the optimization problem.

Consider the continuum of unique intermediate goods $[0, 1]$. Each good $j \in [0, 1]$ is produced by a monopolist using the production technology

$$Y_t(j) = A_t\, N_t(j), \tag{3}$$

where $Y_t(j)$ is the output of good j, $N_t(j)$ the labour input, and A_t the technology level serving as exogenous productivity process. Remember that labour supply is assumed to be perfectly elastic. Since the firms take the real wage W_t as given, they face the real labour costs $W_t N_t(j)$. Moreover, to account for nominal rigidities, we impose quadratic price adjustment costs

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \pi\right)^2 Y_t(j) \tag{4}$$

on each firm, where $P_t(j)$ is the price of intermediate good j, π the steady-state inflation, and parameter $\phi \in [0, \infty)$ represents the price stickiness in the economy.

On this basis, we can derive the intermediate goods producing firms decision problem. Analogously to the households decision problem, the firms strive to maximise the present value of future profits. Since each firm has a monopoly for its unique intermediate good, it chooses labour input $N_t(j)$ and price $P_t(j)$ to maximise

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}} Y_{t+s}(j) - W_{t+s}N_{t+s}(j) - AC_{t+s}(j)\right)\right], \tag{5}$$

where $Q_{t+s|t}$ is the households value at time t of a unit of the final good in period $t+s$. This defines the optimization problem of the production sector. However, we need the demand function of the intermediate goods in order to find a solution. This can be attained by considering the production of the final good and inducing one more assumption.

Although the final good producing firm is a monopoly, we impose a perfectly competitive behaviour. Hence, the firm takes input prices $P_t(j)$ and output prices P_t as given. Furthermore, the final consumption good is produced using the technology

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj\right)^{\frac{1}{1-\nu}}, \tag{6}$$

with the output Y_t, and the elasticity of demand for each intermediate good $1/\nu$. These assumptions, together with the profit maximization of the firm, allow us to derive the demand function for the intermediate goods as well as the relationship between the goods prices. The results are given in the following theorem.

Theorem 1.

(i) The demand for any intermediate good $j \in [0,1]$ is

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\frac{1}{\nu}} Y_t. \tag{7}$$

(ii) The relationship between the intermediate goods prices and the price of the final good is

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj\right)^{\frac{\nu}{\nu-1}} \tag{8}$$

Proof. (i) The final good producing firm takes input and output prices as given. Thus, it chooses the input of intermediate goods $Y_t(j)$ in a way that its profit Π_t is maximal.
Inserting the production technology (6) into the profit equation of the final good producing firm results in

$$\Pi_t = P_t Y_t - C_t = P_t \left(\int_0^1 Y_t(j)^{1-\nu} dj\right)^{\frac{1}{1-\nu}} - \int_0^1 P_t(j) Y_t(j) dj. \tag{9}$$

Hence, the first order condition is

$$\frac{d\Pi_t}{dY_t(j)} = P_t \left(\int_0^1 Y_t(j)^{1-\nu} dj\right)^{\frac{\nu}{1-\nu}} Y_t(j)^{-\nu} - P_t(j) = 0. \tag{10}$$

Considering the production technology (6), equation (10) can be written as

$$P_t \left(\frac{Y_t}{Y_t(j)}\right)^{\nu} - P_t(j) = 0. \tag{11}$$

Therefore, solving (11) for $Y_t(j)$ provides our first statement.

(ii) The second proposition can be proven by inserting equation (7) into (6),

$$Y_t = \left(\int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\frac{1-\nu}{\nu}} Y_t^{1-\nu} dj\right)^{\frac{1}{1-\nu}} = P_t^{\frac{1}{\nu}} \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj\right)^{\frac{1}{1-\nu}} Y_t. \tag{12}$$

Further manipulations result in the stated price relationship. □

Theorem 1 completes the formal description of the production sector.

2.1.3 Fiscal and Monetary Authority

The last two agents are the fiscal and the monetary authority. Unlike the previous agents, the authorities do not have a complex decision problem but simply follow some given behavioural rules.

On the one hand, we have the fiscal authority which consumes a fraction $\zeta_t \in [0, 1]$ of the final output Y_t. The fraction ζ_t follows an exogenous process and will be described in the following section. Hence, the governmental consumption is completely defined by

$$G_t = \zeta_t Y_t. \tag{13}$$

Although the fiscal authority has no decision problem, we define its budget constraint for the sake of completeness. With respect to the households budget constraint, it is given by

$$P_t G_t + R_{t-1} B_{t-1} = T_1 + B_t + M_t - M_{t-1}. \tag{14}$$

On the other hand, the monetary authority sets the nominal target rate R_t^* which will be implemented as the limit of the exogenous interest process R_t later on. An and Schorfheide account for two specifications of the central banks behaviour.

The first specification assumes that the central bank responds to inflation deviations as well as deviations of output Y_t from potential output Y_t^*, which is the output level in absence of nominal rigidities. In the second specification, the central bank does not react to the output gap, but to deviations of output growth from its equilibrium steady-state γ. Since only one specification can be used at the time, we shall distinguish both cases in the subsequent analysis.

The behaviour of the monetary authority is formally modelled by the following equations:

$$\text{a)} \quad R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{Y_t}{Y_t^*}\right)^{\psi_2} \qquad \text{(output gap rule)}, \tag{15}$$

$$\text{b)} \quad R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2} \qquad \text{(output growth rule)}. \tag{16}$$

Here r is the steady-state real interest rate, π^* is the target inflation rate, and $\pi_t = P_t/P_{t-1}$ is the gross inflation rate. We assume that in equilibrium the target inflation rate equals the steady-state inflation rate π. The parameters ψ_1 and ψ_2 serve as weights.

The single relationships between the variables will become clearer, after we introduced the exogenous processes in the following section

2.2 Exogenous Processes and Shocks

The An and Schorfheide model provides the possibility to analyse the impact of three different economic shocks: a technology shock $\epsilon_{z,t}$, a fiscal shock $\epsilon_{g,t}$, and a monetary policy shock $\epsilon_{R,t}$. Each shock is embedded into the corresponding exogenous process which ensures the convergence to a natural level in the absence of shocks.

At first, we consider the technology process $(A_t)_t$. The technology level A_t depends on the average technology growth rate γ, the previous technology level A_{t-1}, and the exogenous fluctuation of the technology growth rate z_t, which contains the technology shock $\epsilon_{z,t}$. Formally, the linearised process equations are

$$\ln A_t = \ln \gamma + \ln A_{t-1} + \ln z_t \tag{17}$$

and

$$\ln z_t = \rho_z \ln z_{t-1} + \epsilon_{z,t}, \tag{18}$$

with convergence parameter $\rho_z \in [0, 1)$.
Note that the fluctuation process $(z_t)_t$ converges to one in the absence of further shocks. Therefore, the growth of the technology level converges towards the average technology growth rate γ. Especially is γ the steady-state technology growth rate.

The next exogenous process describes the ratio of governmental consumption to the aggregated production. Since this ratio ζ_t is only defined for values between zero and one, we model the related process $(g_t)_t$ for $g_t = 1/(1 - \zeta_t) \in [1, \infty)$,

$$\ln g_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t}, \tag{19}$$

with parameter $\rho_g \in [0, 1)$ and the constant value g. Note that g_t is defined as the weighted sum of the previous value g_{t-1} and the constant value g as well as a shock $\epsilon_{g,t}$. We will proof later on that the process converges towards g in the absence of shocks. Hence, g is the steady-state value of the process.

Finally, the exogenous process of the interest rate $(R_t)_t$ is modelled similar to $(g_t)_t$ as

$$\ln R_t = (1 - \rho_R) \ln R_t^* + \rho_R \ln R_{t-1} + \epsilon_{R,t}, \tag{20}$$

where $\rho_R \in [0, 1)$ is the convergence parameter, and R_t^* is the target rate set by the monetary authority. Analogously to the process of the governmental consumption share, the interest rate converges to the target rate in absence of monetary policy shocks $\epsilon_{R,t}$ although the target rate is a process rather than a constant value.

2.3 Equilibrium Conditions

The previous two sections formulate a complete description of our model economy and allow to derive the equilibrium conditions by solving the agents decision problems. To simplify the analysis, we assume that all intermediate goods producing firms make identical choices. Hence, the price $P_t(j)$ and labour input $N_t(j)$ are equal for all firms and intermediate goods j. This allows us to omit the index j. We define this equilibrium as a symmetrical equilibrium. Furthermore, the market clearing condition for the goods and labour market are given by

$$Y_t = C_t + G_t + AC_t \quad \text{and} \quad H_t = N_t. \tag{21}$$

Solving the decision problems of the representative household and the intermediate goods producing firms define our two equilibrium conditions as shown in the following theorem.

Theorem 2. Let $c_t = \frac{C_t}{A_t}$ and $y_t = \frac{Y_t}{A_t}$ be the detrended processes of consumption and output. In the symmetrical equilibrium, output, consumption, interest rates, and inflation satisfy the conditions

$$\text{(i)} \quad 1 = \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right], \tag{22}$$

$$\begin{aligned} \text{(ii)} \quad 1 = \frac{1}{\nu}\left(1 - c_t^{\tau}\right) + \phi\left(\pi_t - \pi\right)\left(\left(1 - \frac{1}{2\nu}\right)\pi_t + \frac{\pi}{2\nu}\right) \\ - \beta\phi\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau} \frac{y_{t+1}}{y_t}\left(\pi_{t+1} - \pi\right)\pi_{t+1}\right]. \end{aligned} \tag{23}$$

Proof. The first condition results from the households maximisation of its utility. We assume that at each point of time the household maximises its expected total future utility. Hence, we have to solve the optimization problem

$$\max_{C_t, H_t, B_t} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s U_{t+s}^{\mathcal{H}} \right] \quad s.t. \quad (2),$$

which maximises the present value of future utility under the households budget constraint. This can be conducted by applying the method of Lagrange multipliers. Therefore, the corresponding Lagrange function is

$$\begin{aligned} \mathcal{L}_t^{\mathcal{H}} = \mathbb{E}_t \Bigg[\sum_{s=0}^{\infty} \beta^s \left(\frac{1}{1-\tau} \left(\left(\frac{C_{t+s}}{A_{t+s}} \right)^{1-\tau} - 1 \right) + \chi_M \ln \frac{M_{t+s}}{P_{t+s}} - \chi_H H_{t+s} \right) \\ - \lambda_s (P_{t+s} C_{t+s} + B_{t+s} + M_{t+s} - M_{t+s-1} + T_{t+s} \\ - P_{t+s} W_{t+s} H_{t+s} - R_{t+s-1} B_{t+s-1} - P_{t+s} D_{t+s} - P_{t+s} SC_{t+s}) \Bigg], \end{aligned}$$

giving us the following first order conditions

$$\frac{d\mathcal{L}_t^{\mathcal{H}}}{dC_{t+s}} = \mathbb{E}_t \left[\frac{\beta^s}{A_{t+s}} \left(\frac{C_{t+s}}{A_{t+s}} \right)^{-\tau} - \lambda_s P_{t+s} \right] = 0, \tag{24}$$

$$\frac{d\mathcal{L}_t^{\mathcal{H}}}{dH_{t+s}} = \mathbb{E}_t \left[-\beta^s \chi_H + \lambda_s P_{t+s} W_{t+s} \right] = 0, \tag{25}$$

$$\frac{d\mathcal{L}_t^{\mathcal{H}}}{dB_{t+s}} = \mathbb{E}_t \left[-\lambda_s + \lambda_{s+1} B_{t+s} \right] = 0. \tag{26}$$

Hence, solving the first order conditions for the Lagrange multiplier and setting $s = 0$ leads to

$$\lambda_0 = \mathbb{E}_t \left[\frac{\beta^0}{A_t} \left(\frac{C_t}{A_t} \right)^{-\tau} \right] \Big(\mathbb{E}_t \left[P_t \right] \Big)^{-1} = \frac{1}{A_t} \left(\frac{C_t}{A_t} \right)^{-\tau} \frac{1}{P_t}, \tag{27}$$

$$\lambda_0 = \frac{\beta^0 \chi_H}{\mathbb{E}_t \left[P_t W_t \right]} = \frac{\chi_H}{P_t W_t}, \tag{28}$$

$$\lambda_0 = \lambda_1 \mathbb{E}_t \big[R_t \big] = \lambda_1 R_t. \tag{29}$$

Note that the expected value $\mathbb{E}_t$ is a linear operator. Furthermore, the expected value of a process $(X_t)_t$ for time t at time t is its actual value, i.e., $\mathbb{E}_t \left[X_t \right] = X_t$, since X_t is known at time t.

Considering equation (24) for $s = 1$ and inserting (29) and (27) gives us our first equilibrium condition

$$\mathbb{E}_t \left[\frac{\beta}{A_{t+1}} \left(\frac{C_{t+1}}{A_{t+1}} \right)^{-\tau} - \frac{1}{A_t} \left(\frac{C_t}{A_t} \right)^{-\tau} \frac{P_{t+1}}{P_t} \frac{1}{R_t} \right] = \beta \mathbb{E}_t \left[\frac{c_{t+1}^{-\tau}}{A_{t+1}} - \frac{c_t^{-\tau}}{A_t} \frac{\pi_{t+1}}{R_t} \right] = 0$$

$$\iff \qquad \beta \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} \right)^{-\tau} \frac{A_t}{A_{t+1}} \frac{R_t}{\pi_{t+1}} \right] = 1 \tag{30}$$

for $c_t := \frac{C_t}{A_t}$ and $\pi_t = \frac{P_t}{P_{t-1}}$.

For the second equilibrium condition, we solve the optimisation problem for the intermediate goods producing firms. But first we derive another result from the households optimisation. We equate (27) and (28) for $\chi_H = 1$ and get

$$\frac{W_t}{A_t} = \left(\frac{C_t}{A_t} \right)^{\tau}. \tag{31}$$

Now consider the optimisation problem for the intermediate goods producing firms. As assumed, each firm produces its good j as a monopolist. Thus, the firms choose their labour input $N_t(j)$ and price $P_t(j)$ to maximises the present value of future profits under the restrictions of the production technology (3) and the demand function (7). Inserting these equations as well as the adjustment costs (4) into the profit function (5) leads to

$$\Pi^{\mathcal{I}} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s Q_{t+s|t} \left(\frac{P_{t+s}(j)}{P_{t+s}} \left(\frac{P_{t+s}(j)}{P_{t+s}} \right)^{-\frac{1}{\nu}} Y_t - W_{t+s} \left(\frac{P_{t+s}(j)}{P_{t+s}} \right)^{-\frac{1}{\nu}} \frac{Y_{t+s}}{A_{t+s}} \right.\right.$$
$$\left.\left. - \frac{\phi}{2} \left(\frac{P_{t+s}(s)}{P_{t+s-1}(j)} - \pi \right)^2 \left(\frac{P_{t+s}(j)}{P_{t+s}} \right)^{-\frac{1}{\nu}} Y_{t+s} \right) \right]. \tag{32}$$

Therefore, the optimisation problem of the good j producing firm is $\max_{P_t(j)} \Pi^{\mathcal{I}}$, leading to the first order condition

$$\frac{d\Pi^{\mathcal{I}}}{dP_t(j)} = \mathbb{E}_t\Bigg[Q_{t|t}\Bigg(\left(\left(1-\frac{1}{\nu}\right)\frac{Y_t}{P_t}\left(\frac{P_t(j)}{P_t}\right)^{-\frac{1}{\nu}} + \frac{1}{\nu}\frac{W_tY_t}{A_tP_t(j)}\left(\frac{P_t(j)}{P_t}\right)^{-\frac{1}{\nu}}\right.$$
$$+\phi\left(\frac{P_t(j)}{P_{t-1}(j)}-\pi\right)\left(\frac{P_t(j)}{P_t}\right)^{-\frac{1}{\nu}}\left(\frac{1}{2\nu}\left(\frac{P_t(j)}{P_{t-1}(j)}-\pi\right)\frac{Y_t}{P_t(j)}-\frac{Y_t}{P_{t-1}(j)}\right)\Bigg)$$
$$+\beta Q_{t+1|t}\phi\left(\frac{P_{t+1}(j)}{P_t(j)}-\pi\right)\frac{Y_{t+1}}{P_t(j)}\frac{P_{t+1}(j)}{P_t(j)}\left(\frac{P_{t+1}(j)}{P_{t+1}}\right)^{-\frac{1}{\nu}}\Bigg] = 0. \tag{33}$$

Since we assume a symmetric equilibrium, all firms make identical choices and set the same prices, i.e., $P_t(j) = P_t^{\mathcal{I}}\ \forall j$. Moreover, from the relationship between P_t and $P_t(j)$ in equation (8), we can derive that $P_t(j) = P_t^{\mathcal{I}} = P_t$. Hence, together with $\pi_t = \frac{P_t}{P_{t-1}}$ equation (33) can be expresst as

$$Q_{t|t}\left(\left(1-\frac{1}{\nu}\right)\frac{Y_t}{P_t}+\frac{1}{\nu}\frac{W_tY_t}{A_tP_t}+\phi\left(\pi_t-\pi\right)\left(\frac{1}{2\nu}\left(\pi_t-\pi\right)\frac{Y_t}{P_t}-\frac{Y_t}{P_{t-1}}\right)\right)$$
$$+\beta\phi\mathbb{E}_t\left[Q_{t+1|t}\left(\pi_{t+1}-\pi\right)\frac{Y_{t+1}}{P_t}\pi_{t+1}\right] = 0. \tag{34}$$

Further manipulations lead to

$$1 = \frac{1}{\nu}\left(1-\frac{W_t}{A_t}\right)+\phi\left(\pi_t-\pi\right)\left(\left(1-\frac{1}{2\nu}\right)\pi_t+\frac{\pi}{2\nu}\right)$$
$$-\beta\phi\mathbb{E}_t\left[\frac{Q_{t+1|t}}{Q_{t|t}}\frac{Y_{t+1}}{Y_t}\left(\pi_{t+1}-\pi\right)\pi_{t+1}\right]. \tag{35}$$

Since households can insure themselves perfectly, the intertemporal utility optimization provides

$$Q_{t+s|t} = \frac{\frac{dU^{\mathcal{H}}_{t+s}}{dC_{t+s}}}{\frac{dU^{\mathcal{H}}_t}{dC_t}} = \left(\frac{C_{t+s}}{C_t}\right)^{-\tau}\left(\frac{A_t}{A_{t+s}}\right)^{1-\tau} = \left(\frac{C_{t+s}A_t}{C_tA_{t+s}}\right)^{-\tau}\frac{A_t}{A_{t+s}}. \tag{36}$$

Finally, inserting (31) and (36) into (35) gives us the second equilibrium condition

$$1 = \frac{1}{\nu}\left(1-\left(\frac{C_t}{A_t}\right)^{\tau}\right)+\phi\left(\pi_t-\pi\right)\left(\left(1-\frac{1}{2\nu}\right)\pi_t+\frac{\pi}{2\nu}\right)$$
$$-\beta\phi\mathbb{E}_t\left[\left(\frac{C_{t+1}A_t}{C_tA_{t+1}}\right)^{-\tau}\frac{Y_{t+1}A_t}{Y_tA_{t+1}}\left(\pi_{t+1}-\pi\right)\pi_{t+1}\right]. \tag{37}$$

□

We can apply these results to derive the potential output level Y_t^*, i.e., the equilibrium output in the absence of nominal rigidities. Hence, the price stickiness ϕ is zero.

Theorem 3. The potential output level is

$$Y_t^* = (1-\nu)^{\frac{1}{\tau}}A_tg_t. \tag{38}$$

Proof. We determine the aggregate output level for $\phi = 0$. In the absence of nominal rigidities, the market clearing condition states $Y_t = C_t + G_t$. Inserting the governmental consumption $G = \zeta_t C_t$ and solving for Y_t provides

$$Y_t = \frac{1}{1-\zeta_t} C_t = g_t C_t. \tag{39}$$

Furthermore, considering the second equilibrium condition (23) of theorem 2 for $\phi = 0$ results in

$$1 = \frac{1}{\nu}\left(1 - \left(\frac{C_t}{A_t}\right)^{\tau}\right) \quad \Longleftrightarrow \quad C_t = (1-\nu)^{\frac{1}{\tau}} A_t. \tag{40}$$

Inserting (40) into (39) gives us the the output level in absence of nominal rigidities. □

2.4 Steady-State

An important property of a dynamic stochastic general equilibrium model is the convergence towards a steady-state after imposing an exogenous shock on the model economy. Since economic growth persists in the steady-state, An and Schorfheide used detrended variables to express the convergence. For that matter, the detrended variables are the ratio of the variable to the technology level, i.e., $c_t = C_t/A_t$ and $y_t = Y_t/A_t$.
We will show in the following theorem that in terms of these detrended variables a steady-state not only exists but is unique as well.

Theorem 4. The model economy has a unique steady-state, in terms of detrended variables, that is attained if all shocks $\epsilon_{R,t}$, $\epsilon_{z,t}$, and $\epsilon_{g,t}$ persist zero. Moreover, its values of the detrended processes are given by $\pi = \pi^*$, $z = 1$, and g, as well as

$$c = (1-\nu)^{\frac{1}{\tau}}, \quad y = g(1-\nu)^{\frac{1}{\tau}}, \quad R = r\pi^*, \quad r = \frac{\gamma}{\beta}.$$

Proof. To proof this theorem, we shall just derive the unique limits for the detrended processes, which existence already imply a unique steady-state.
The first claim, $\pi = \pi^*$, is true by definition. It is assumed that the steady-state inflation equals the targeted inflation rate. Although this is a intuitive assumption and simplifies the further results, it is not a necessary condition for a unique steady-state. In fact, the theorem holds for any unique limit of inflation rate.

The unique limits of the other detrended variables can be derived within the model.
In the absence of a technology shock $\epsilon_{z,t}$, the logarithmic exogenous fluctuations of the technology growth rate $(\ln z_t)_t$, as modelled in (18), converges to zero for $\rho_z \in (-1, 1)$. Hence, the

fluctuations of the technology growth rate equals one, i.e., $z = 1$.

Furthermore, the convergence of $(g_t)_t$ can be derived directly from its processes equation (19) as well. It is constructed in the way that, for $\epsilon_{g,t} = 0$ and $\rho_g \in (-1, 1)$, the process $(g_t)_t$ converges to g. This can be shown by considering

$$|\ln g_{t+s} - \ln g| = |\rho_g| \, |\ln g_{t+s-1} - \ln g| = |\rho_g|^s \, |\ln g_t - \ln g| < \epsilon, \tag{41}$$

for $\epsilon > 0$, some large $S > 0$ and every $s > S$. Hence, it is $\ln g_t \to \ln g$.

Since in the steady-state all processes persist in their steady-state value, equation (23) reduces itself to

$$1 = \frac{1}{\nu}\left(1 - c^{\tau}\right) \quad \Longleftrightarrow \quad c = (1-\nu)^{\frac{1}{\tau}} \tag{42}$$

giving us the steady-state value for the detrended consumption. Note that the second and third summand converge to zero for $\pi_t \to \pi$.

On this basis, we can derive the steady-state output. In equilibrium the market clearing condition holds, i.e., $Y_t = C_t + G_t + AC_t$. Furthermore, note that the adjustment costs AC_t are zero in the steady-state, as can be seen in equation (4) for $\frac{P_t(j)}{P_{t-1}(j)} \to \pi$. Thus, for $G_t = \zeta_t Y_t$ and $g_t = \frac{1}{1-\zeta_t}$, we gain

$$y = \frac{1}{1-\zeta} c = g(1-\nu)^{\frac{1}{\tau}} \tag{43}$$

for the detrended steady-state output.

In the next step, we shall contemplate the steady-state of the interest rate R. A glance at equation (20) shows that the interest process $(R_t)_t$ is similar modelled as the exogenous process $(g_t)_t$. Therefore, in the absence of a monetary policy shock $\epsilon_{R,t}$, the interest rate R_t converges towards the target rate R_t^*. Hence, in the steady-state the interest rate equals the target rate, i.e., $R = R^*$.

Since the model offers two specifications for the target rate, we have to examine them separately. Yet, as we will see below, both specifications converge towards the same value.

On the one hand, inserting equation (38) into the output gap specification of the target rate (15) gives us

$$R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{y_t}{(1-\nu)^{\frac{1}{\tau}} g_t}\right)^{\psi_2}. \tag{44}$$

From $\pi^* = \pi$ and equation (43) result that the fractions in the second and third factor converge to one. Thus, the target rate converges to $R^* = r\pi^*$.

On the other hand, the output growth specification (16) expressed in terms of the detrended output is

$$R_t^* = r\pi^* \left(\frac{\pi_t}{\pi^*}\right)^{\psi_1} \left(\frac{y_t}{\gamma y_{t-1}} \frac{A_t}{A_{t-1}}\right)^{\psi_2}. \tag{45}$$

In the steady-state, the fluctuations of the technology growth rate z_t are zero. Therefore, the technology growths at its average rate γ as can described in equation (17), i.e., $\frac{A_t}{A_{t-1}} = \gamma$. Hence, the second and third factor in (45) converge to one, leading again to $R^* = r\pi^*$.
These three results imply that the steady-state interest rate holds $R = r\pi^*$.

At last, the steady-state real interest rate r can be derived from the first equilibrium condition, equation (22). Due to the convergence of $c_t \to c$, $\pi_t \to \pi$, and $R_t \to R$, the right-hand side of the equation converges as well, and the steady-state holds

$$1 = \beta \frac{A_t}{A_{t+1}} \frac{R}{\pi} = \frac{\beta}{\gamma} \frac{R}{\pi}. \tag{46}$$

Inserting the previous result, $R = r\pi$, into equation (46) provides the final result $r = \frac{\gamma}{\beta}$. □

2.5 Final Model Equations

This section concentrates all previous results into six equation, which form a complete representation of the An and Schorfheide model. The equations contain both equilibrium conditions from theorem 2, the goods market clearing condition, the monetary policy equation, the governmental consumption, the adjustment costs, and the three exogenous processes. For simpler interpretation, the equations are denoted in relative variables, which describe the percentage deviation of a variable from its steady-state.

Theorem 5. Let $\hat{x}_t = \ln \frac{x_t}{x}$ for any variable $x \in \{c, y, R, z, \pi, g\}$. The model is completely defined by the following equations

$$\text{(i)} \quad 1 = \mathbb{E}_t \left[e^{-\tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{R}_t - \hat{z}_{t+1} - \hat{\pi}_{t+1}} \right] \tag{47}$$

$$\begin{aligned} \text{(ii)} \quad \frac{1-\nu}{\nu\phi\pi^2} \left(e^{\tau \hat{c}_t} - 1\right) = \left(e^{\hat{\pi}_t} - 1\right) \left[\left(1 - \frac{1}{2\nu}\right) e^{\hat{\pi}_t} + \frac{1}{2\nu} \right] \\ - \beta \mathbb{E}_t \left[\left(e^{\hat{\pi}_{t+1}} - 1\right) e^{-\tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{y}_{t+1} - \hat{y}_t + \hat{\pi}_{t+1}} \right] \end{aligned} \tag{48}$$

$$\text{(iii)} \quad e^{\hat{c}_t - \hat{y}_t} = e^{-\hat{g}_t} - \frac{\phi\pi^2 g}{2} \left(e^{\hat{\pi}_t} - 1\right)^2 \tag{49}$$

$$\text{(iv)} \quad \text{a)} \quad \hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t + (1-\rho_R)\psi_2 (\hat{y}_t - \hat{g}_t) + \epsilon_{R,t} \tag{50}$$

$$\text{b)} \quad \hat{R}_t = \rho_R \hat{R}_{t-1} + (1-\rho_R)\psi_1 \hat{\pi}_t + (1-\rho_R)\psi_2 (\Delta\hat{y}_t + \hat{z}_t) + \epsilon_{R,t} \tag{51}$$

$$\text{(v)} \quad \hat{g}_t = \rho_g \hat{g}_{t-1} + \epsilon_{g,t} \tag{52}$$

$$\text{(vi)} \quad \hat{z}_t = \rho_z \hat{z}_{t-1} + \epsilon_{z,t} \tag{53}$$

Proof. (i) At first, we consider the exogenous process of the technology level. Reversing the linearisation of equation (17) results in $A_{t+1} = \gamma A_t z_{t+1}$, which allows us to formulate

$$\frac{A_t}{A_{t+1}} = \frac{1}{\gamma z_{t+1}} = \frac{1}{\beta r z_{t+1}} = \frac{1}{\beta}\frac{\pi z}{R z_{t+1}} \tag{54}$$

by inserting $\gamma = \beta r$, $r = R/\pi$, and $z = 1$ from theorem 4.

On this basis, we can show the equivalence of equation (47) and the frist equilibrium condition in theorem 2. Inserting (54) into (22) provides

$$1 = \mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau}\frac{R_t}{R}\frac{z}{z_{t+1}}\frac{\pi}{\pi_{t-1}}\right], \tag{55}$$

which is equivalent to (47) in terms of $\hat{c}_t$, $\hat{R}_t$, $\hat{z}_t$, and $\hat{\pi}_t$.

(ii) The equation is equivalent to the second equilibrium condition. Straightforward manipulation of (23) gives us

$$\begin{aligned}\frac{1-\nu}{\nu\phi\pi^2}\left(\frac{c_t^\tau}{1-\nu}-1\right) &= \left(\frac{\pi_t}{\pi}-1\right)\left(\left(1-\frac{1}{2\nu}\right)\frac{\pi_t}{\pi}+\frac{1}{2\nu}\right)\\ &- \beta\mathbb{E}_t\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\tau}\frac{y_{t+1}}{y_t}\left(\frac{\pi_{t+1}}{\pi}-1\right)\frac{\pi_{t+1}}{\pi}\right].\end{aligned} \tag{56}$$

Theorem 4 gives us the steady-state value of the detrended consumption $c^\tau = 1-\nu$. Thus, inserting $\hat{c}_t$, $\hat{\pi}_t$, and $\hat{y}_t$ results in equation (48).

(iii) This is basically the clearing condition of the goods market. The market clearing condition states $Y_t = C_t + G_t + AC_t$. Inserting $G = \zeta_t Y_t$, $g_t = \frac{1}{1-\zeta_t}$, and the adjustment costs (4) leads to

$$\frac{y_t}{g_t} = c_t + \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)}-\pi\right)^2 y_t \tag{57}$$

in terms of detrended variables.

As we have seen before, the symmetric equilibrium implies $\frac{P_t(j)}{P_{t-1}(j)} = \pi_t$. Using this result along with some further manipulations gives us

$$\frac{g}{g_t} - \frac{\phi\pi^2 g}{2}\left(\frac{\pi_t}{\pi}-1\right)^2 = \frac{c_t}{y_t}g = \frac{c_t}{c}\left(\frac{y_t}{y}\right)^{-1}, \tag{58}$$

where the right equation follows from the steady-state result $y = gc$. Inserting $\hat{g}_t$, $\hat{\pi}_t$, $\hat{c}_t$, and $\hat{y}_t$ provides equation (49).

(iv) a) The equation results from combining the output gap rule specification (15) with the interest process (20). As a first result, inserting the equation of the steady-state output, i.e., $y = (1-\nu)^{\frac{1}{\tau}} g$ into the potential output (38) provides

$$Y_t^* = (1-\nu)^{\frac{1}{\tau}} A_t g_t = A_t y \frac{g_t}{g} \tag{59}$$

Furthermore, inserting (59) into (15) and logarithmise the equation gives us

$$\ln R_t^* = \ln r\pi^* + \psi_1 \ln \frac{\pi_t}{\pi^*} + \psi_2 \left(\ln \frac{y_t}{y} - \ln \frac{g_t}{g} \right) = \ln R + \psi_1 \hat{\pi}_t + \psi_2 \left(\hat{y}_t - \hat{g}_t \right), \tag{60}$$

with $R = r\pi^*$ and $\pi^* = \pi$ as in theorem 4.
We get the final equation by inserting (60) into the interest process equation (20). Note that $\hat{R} = \ln R_t - \ln R$.

b) Analogously, this equation is a combination of output growth rule specification (16) and the interest process (20). The equation of the technology process (17) gives us $A_t = \gamma A_{t-1} z_t$. Hence, we can state

$$\frac{Y_t}{\gamma Y_{t-1}} = \frac{y_t A_t}{\gamma y_{t-1} A_{t-1}} = \frac{y_t}{y_{t-1}} z. \tag{61}$$

Logarithmising equation (16) and inserting (61) provides

$$\ln R_t^* = \ln r\pi^* + \psi_1 \ln \frac{\pi_t}{\pi^*} + \psi_2 \ln \left(\frac{y_t}{y_{t-1}} z \right) = \ln R + \psi_1 \hat{\pi}_t + \psi_2 \left(\Delta \hat{y}_t + \hat{z}_t \right), \tag{62}$$

with $\Delta \hat{y}_t = \hat{y}_t - \hat{y}_{t-1}$ as well as $R = r\pi^*$, $\pi^* = \pi$, and $z = 1$.
Finally, inserting (62) into the interest process (20) results in equation (51).

(v) The equation is equivalent to the exogenous process related to the governmental consumption and is gained from inserting equation (19) into $\hat{g}_t = \ln g_t - \ln g$

$$\hat{g}_t = (1 - \rho_g) \ln g + \rho_g \ln g_{t-1} + \epsilon_{g,t} - \ln g = \rho_g \hat{g}_{t-1} + \epsilon_{z,t}. \tag{63}$$

(vi) The equation is equivalent to the fluctuation of the technology growth rate. From $z = 1$ follows $\hat{z}_t = \ln z_t - \ln z = \ln z_t$. Hence, equation (53) results directly from (18).

Since all assumed and deducted model equations are captured within equation (i)-(vi), they provide an equivalent definition of the model. □

Theorem 5 forms a non-linear expectation system which as to be solved in order to apply the model in further research. An and Schorfheide derived a linearised equation system to make the model accessible to there subsequent analysis.

Theorem 6. The linearised model is described by the following equations

$$\text{(i)} \quad \hat{y}_t = \mathbb{E}_t[\hat{y}_{t+1}] + \hat{g}_t - \mathbb{E}_t[\hat{g}_{t+1}] - \frac{1}{\tau}\left(\hat{R}_t - \mathbb{E}_t[\hat{\pi}_{t+1}] - \mathbb{E}_t[\hat{z}_{t+1}] \right) \tag{64}$$

$$\text{(ii)} \quad \hat{\pi}_t = \beta \mathbb{E}_t[\hat{\pi}_{t+1}] + \kappa(\hat{y}_t - \hat{g}_t) \tag{65}$$

$$\text{(iii)} \quad \hat{c}_t = \hat{y}_t - \hat{g}_t \tag{66}$$

for

$$\kappa = \tau \frac{1-\nu}{\nu \pi^2 \phi}, \tag{67}$$

as well as equation (iv) - (vi) of theorem 5.

Proof. All three equations are gained from linearisation and manipulation of the equations (i) - (iii) of theorem 5.

We will use two approximations to develop the linearised equations:

$$e^{\hat{x}_t + \alpha \hat{y}_t} \approx 1 + \hat{x}_t + \alpha \hat{y}_t, \tag{68}$$

$$\hat{x}_t \hat{y}_t \approx 0, \tag{69}$$

for any variables x, y with $\hat{x}_t = \ln \frac{x_t}{x}$ as above.

(iii) The linearisation of equation (49), i.e., theorem 5 (iii), results in

$$1 + \hat{c}_t - \hat{y}_t = 1 - \hat{g}_t, \tag{70}$$

leading directly towards equation (66). Note that

$$\left(e^{\hat{\pi}_t} - 1\right)^2 \approx \hat{\pi}_t^2 \approx 0. \tag{71}$$

(ii) Applying the approximation to equation (48), i.e., theorem 5 (ii), yields

$$\frac{1-\nu}{\nu \phi \pi^2} \tau \hat{c}_t = \hat{\pi}_t \left[1 + \hat{\pi}_t - \frac{\hat{\pi}_t}{2\nu}\right] - \beta \mathbb{E}_t \left[\hat{\pi}_{t+1}\right]. \tag{72}$$

Note that

$$\hat{\pi}_{t+1} e^{-\tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{y}_{t+1} - \hat{y}_t + \hat{\pi}_{t+1}} \approx \hat{\pi}_{t+1}(1 - \tau \hat{c}_{t+1})(1 + \tau \hat{c}_t)(1 + \hat{y}_{t+1} - \hat{y}_t)(1 + \hat{\pi}_{t+1}) \approx \hat{\pi}_{t+1}. \tag{73}$$

We define κ as in (67). Hence, further approximation and inserting κ lead to

$$\kappa \hat{c}_t = \hat{\pi}_t - \beta \mathbb{E}_t \left[\hat{\pi}_{t+1}\right]. \tag{74}$$

Thus, we get our second result by inserting the previously proven equation (66) into (74).

(i) Similar to the approximation in (73), equation (47), i.e., theorem 5 (i), can be approximated by

$$1 = \mathbb{E}_t \left[1 - \tau \hat{c}_{t+1} + \tau \hat{c}_t + \hat{R}_t - \hat{z}_{t+1} - \hat{\pi}_{t+1}\right]. \tag{75}$$

Inserting equation (66) along with some manipulations leads to equation (64). □

3 Comparison to the Smets & Wouters Model and Conclusion

As we have see in the last section, the model of An and Schorfheide abstracts several circumstances "to keep the model specification simple" [1, p. 115]. A more elaborate model was developed by Smets and Wouters [2]. This section targets to give a short comparison of both models.

The largest differences apply to the labour and capital market. Instead of a given exogenous real wage, Smets and Wouters modelled an endogenous labour market. Households perform as monopolists for labour service and set wages according to their decision problem. Furthermore, Smets and Wouters account for wage rigidities by using a Calvo rule, where only a certain proportion of households is able to adjust their wages at each period. Therefore, a continuum of households is used in place of a representative household.
In addition to governmental bonds, Smets and Wouters implemented the households ability to accumulated capital. For that matter, households choose between saving in bonds and investing in their capital stock, which is rented out to the intermediate goods producing firms. The household receive a return on their investment and the firms output increases according to their production function.
Although the final good sector is identical in both models, the intermediate sector differs in two details. On the hand, the firms production technology accounts for the capital stock as well as the labour. A Cobb–Douglas production function is used lessened by a fix cost. On the other hand, Smets and Wouters model price rigidities by a Calvo rule similar to the wage instead of quadratic price adjustment costs.
Furthermore, Smets and Wouters embedded various different shocks into the model equations allowing for more detailed analysis. For example, the households utility function contains a general shock to preferences, but as well a shock to the labour supply and a money demand shock.

In conclusion, the Smets and Wouters model is more elaborate, but also is significantly more complex. An and Schorfheide focus their model on monetary policy. Due to its simpler specifications, the model may be preferred to get a first impression on the effect of monetary policies and serves as a benchmark for their analysis.

References

[1] An, S. and Schorfheide, F. (2007). Bayesian Analysis of DSGE Models. *Economic Reviews*, **26(2-4)** 113–172.

[2] Smets, F. and Wouters, R. (2003). An Estimated Stochastic Dynamic General Equilibrium Model of the Euro Area. *Journal of Economic Association*, **1** 1123–1175.

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