Solution of DSGE models

Klein (2000) Algorithm

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Framework revisited

- Last lecture we started calculating the first order approximation
- Today we finish the first order solution step by step using the algorithm of Klein (JEDC, 2000)

→ We need proper notation, thus recall the linearized model

$$A \begin{bmatrix} \hat{\mathbf{w}}_{t+1} \\ \hat{\mathbf{y}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{\mathbf{w}}_{t} \\ \hat{\mathbf{y}}_{t} \end{bmatrix} + \begin{bmatrix} \sigma \boldsymbol{\epsilon}_{t+1} \\ \mathbf{0} \end{bmatrix}$$

where A, B are matrices and σ is the perturbation parameter. Further $\hat{\mathbf{w}}_t = (\hat{\mathbf{x}}_t \, \hat{\mathbf{z}}_t)' \in \mathbb{R}^{n_w}$ is a vector of predetermined variables $\hat{\mathbf{x}}_t \in \mathbb{R}^{n_x}$ at t and exogenous variables $\hat{\mathbf{z}}_t$,$\hat{\mathbf{y}}_t \in \mathbb{R}^{n_y}$ is a vector of not predetermined (jump) variables. $\boldsymbol{\epsilon}_{t+1} \in \mathbb{R}^{n}_{w}$ is a vector of iid normal shocks.

- We use notation from Gomme and Klein (JEDC 2011) and Heiberger, Klarl and Maußner (2012)

Framework revisited

- Taking conditional expectations we return to

$$A\mathbb{E}_t \begin{bmatrix} \hat{\mathbf{w}}_{t+1} \\ \hat{\mathbf{y}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{\mathbf{w}}_t \\ \hat{\mathbf{y}}_t \end{bmatrix},$$

$$\hat{\mathbf{w}}_t = \begin{bmatrix} \hat{\mathbf{x}}_t \\ \hat{\mathbf{z}}_t \end{bmatrix}, \hat{\mathbf{w}}_t \in \mathbb{R}^{n_w}$$

- The above representation arises when differentiating the system $F(\mathbf{w_t}, \sigma)$ with respect to $\mathbf{w}_t$
- $\hat{\mathbf{w}}_t$ and $\hat{\mathbf{y}}_t$ are deviations from steady state

Generalized Schur form (abbreviated)

For A and B $n \times n$ matrices and if there is a $z \in \mathbb{C}$ such that $|B - zA| \neq 0$, there exist matrices Q, Z, T and S such that:

$$Q^H S Z^H = A$$
$$Q^H T Z^H = B,$$

where

- Q and Z are Hermitian matrices ($Q^H Q = I$ and $Z^H Z = I$)
- S and T complex upper triangular matrices
- Eigenvalues t_{ii} and s_{ii} are ordered, such that unstable pairs ($\frac{|t_{ii}|}{|s_{ii}|} > 1$) are in the lower right blocks of S and T (S_{22} and T_{22})

Solution

- Define

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{w}}_t \\ \hat{\mathbf{y}}_t \end{bmatrix}$$

- Substitute the above equation in the system and left multiply by Q^H

$$\begin{bmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{y}}_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix}$$

- Given the ordering of the eigenvalues, the lower part of the system, $S_{22}\mathbb{E}_t\tilde{\mathbf{y}}_{t+1} = T_{22}\tilde{\mathbf{y}}_t$ is unstable (think of e.g. an AR(1) process)

$\rightarrow$ $\forall t \;\; \tilde{\mathbf{y}}_t = \mathbf{0}_{n_y}$

Solution

- Having determined $\tilde{\mathbf{y}}_t = \mathbf{0}_{n_y}$, the first line of the system yields

$$\mathbb{E}_t \tilde{\mathbf{w}}_{t+1} = S_{11}^{-1} T_{11} \tilde{\mathbf{w}}_t$$

- From the definition of $\tilde{\mathbf{w}}_t = Z_{11}^{-1} \hat{\mathbf{w}}_t$ we get

$$\begin{aligned} Z_{11}^{-1} \mathbb{E}_t \hat{\mathbf{w}}_{t+1} &= S_{11}^{-1} T_{11} Z_{11}^{-1} \hat{\mathbf{w}}_t \\ \mathbb{E}_t \hat{\mathbf{w}}_{t+1} &= Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} \hat{\mathbf{w}}_t \end{aligned}$$

- Dropping the expectation operator the transition law is

$$\mathbf{w}_{t+1} = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1} \mathbf{w}_t + \sigma \Sigma \boldsymbol{\epsilon}_{t+1}$$

Solution

- From the definitions of $\hat{\mathbf{y}}_t = Z_{21}\tilde{\mathbf{w}}_t$ and $\tilde{\mathbf{w}}_t = Z_{11}^{-1}\hat{\mathbf{w}}_t$ we get

$$\mathbf{y}_t = Z_{21}Z_{11}^{-1}\hat{\mathbf{w}}_t$$

- Without naming, we have used two Blanchard Kahn (1980) conditions:
  - Rank condition: Z_{11} is invertible $\leftrightarrow$ Z_{11} has full rank
  - Order condition: If we impose $\tilde{\mathbf{y}}_t = 0$ we rule out explosiveness $\leftrightarrow$ number of explosive eigenvalues equals number of jump variables

Solution

- Defining $L_w^w = Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}$ and $L_w^y = Z_{21} Z_{11}^{-1}$ and using the definition of $\mathbf{w}_t$ we denote

$$L_w^w = \begin{bmatrix} L_x^x & L_z^x \\ \mathbf{0}_{n_z \times n_x} & L_z^z \end{bmatrix}, \quad L_w^y = \begin{bmatrix} L_x^y & L_z^y \end{bmatrix}$$

- The sought solution in convenient notation is

$$\begin{aligned} \hat{\mathbf{x}}_{t+1} &= L_x^x \hat{\mathbf{x}}_t + L_z^x \hat{\mathbf{z}}_t \\ \hat{\mathbf{z}}_{t+1} &= L_z^z \hat{\mathbf{z}}_t + \sigma \Sigma \boldsymbol{\epsilon}_{t+1} \\ \hat{\mathbf{y}}_t &= L_y^x \hat{\mathbf{x}}_t + L_y^x \hat{\mathbf{z}}_t \end{aligned}$$

When calculating the second order derivative...

- We need to compute second order derivatives of the underlying system, thus we get three dimensional arrays containing the (cross)derivatives
- The variance (second derivatives with respect to the perturbation parameter) of the shock terms comes into play $\rightarrow$ this is were the perturbation parameter σ matters
- Gomme and Klein (2011) and Schmitt-Grohe and Uribe (2009) present detailed algorithms
- Requires to solve a *generalized Sylvester equation*

$\rightarrow$ Dynare provides second and third order approximations in no time!

Example

- The Equilibrium conditions of the stochastic growth model are:

$$
\begin{aligned}
c_t^{\gamma} &= \beta \mathbb{E}_t[c_{t+1}^{\gamma}(\alpha exp(a_{t+1})k_{t+1}^{\alpha-1} + (1-\delta)] \\
k_{t+1} &= exp(a_t)k_t^{\alpha} - c_t + (1-\delta)k_t \\
a_t &= \rho a_{t-1} + \epsilon_t \\
\forall\, t\, \epsilon_t &\sim N(0,1)
\end{aligned}
$$

- Execute the StochGrowth.mod file and save the returned transition matrix.

- If we want to reproduce the first-order solution of Dynare by hand, we need to add two more equations to our system with a slight adjustment of notation:

$$
\begin{aligned}
c_t^{\gamma} &= \beta \mathbb{E}_t[c_{t+1}^{\gamma}(\alpha exp(a_{t+1})k_{t+1}^{\alpha-1} + (1-\delta)] \\
k_{t+1} &= exp(a_t)k_t^{\alpha} - c_t + (1-\delta)k_t \\
a_t &= \rho \hat{a}_t + \epsilon_t \\
0 &= \mathbb{E}_t[\epsilon_{t+1}] \\
0 &= \hat{a}_{t+1} - a_t
\end{aligned}
$$

where $\hat{a}_t$ is defined as a_{t-1}. Note that this is **not necessary** in order to get a solution!

- Given the above equations we now have

$$
\mathbf{w}_t = \begin{bmatrix} k_t & \hat{a}_t & \epsilon_t \end{bmatrix}', \quad \mathbf{y}_t = \begin{bmatrix} c_t & a_t \end{bmatrix}'
$$

Example

- Given the equation system, we can take derivatives in order to linearize (first-order Taylor expansion) the model around the steady state.
- Given the Jacobian matrices we need some matrix calculus to solve for the policy and transition functions. A generally applicable approach is the algorithm by Klein (2000), which applies the QZ-decomposition.

→ Execute the Klein_StochGrowth.m file and track the algorithm step by step!