Dynamic Stochastic General Equilibrium Models

Dr. Andrea Beccarini   Willi Mutschler

Summer 2012

Firms:

- The economy produces only one final product Y_t and a continuum of intermediates goods defined over $j : j\epsilon[0,1]$.
- Firm j produces the intermediate product y_t^j. Firms of the final product sector operate in a (perfect) competitive market, firms in the intermediate sector operate in a monopolistic competition and set their prices forward-looking.

Firms:
Final Product Sector

- For the production of the final product Y_t firm use the intermediate products y_t^j as Input.

- The aggregation is performed through the following technology:

$$Y_t = \left[\int_0^1 \left(y_t^j \right)^{\frac{1}{1+\lambda_{p,t}}} dj \right]^{1+\lambda_{p,t}} \tag{1}$$

- Define p_t^j as the price of the intermediate product y_t^j, then the representative firm minimizes the cost function $\int_0^1 p_t^j y_t^j \; dj$ of a particular combination of inputs observing the constraint of eq. (1).

- Define the Lagrange multiplier as P_t, hence the Lagrangian is:

$$L = \int_0^1 p_t^j y_t^j \; dj + P_t \left(Y_t - \left[\int_0^1 \left(y_t^j \right)^{\frac{1}{1+\lambda_{p,t}}} \; dj \right]^{1+\lambda_{p,t}} \right)$$

$$\frac{\partial L}{\partial y_t^j} = p_t^j - P_t \left((1+\lambda_{p,t}) \left[\int_0^1 \left(y_t^j \right)^{\frac{1}{1+\lambda_{p,t}}} \; dj \right]^{\lambda_{p,t}} \frac{1}{1+\lambda_{p,t}} \left(y_t^j \right)^{\frac{-\lambda_{p,t}}{1+\lambda_{p,t}}} \right) = 0$$

$$\Leftrightarrow p_t^j = P_t Y_t^{\frac{\lambda_{p,t}}{1+\lambda_{p,t}}} \left(y_t^j \right)^{\frac{-\lambda_{p,t}}{1+\lambda_{p,t}}}$$

$$\Leftrightarrow y_t^j = Y_t \left(\frac{p_t^j}{P_t} \right)^{\frac{-(1+\lambda_{p,t})}{\lambda_{p,t}}} \tag{2}$$

Eq. (2) is the optimal demand of good y_t^j. Substituting this in (1) yields:

$$Y_t = \left[\int_0^1 \left(\frac{p_t^j}{P_t} \right)^{\frac{-1}{\lambda_{p,t}}} Y_t^{\frac{1}{1+\lambda_{p,t}}} \, dj \right]^{1+\lambda_{p,t}}$$

$$\Leftrightarrow P_t = \left[\int_0^1 \left(p_t^j \right)^{\frac{-1}{\lambda_{p,t}}} \, dj \right]^{-\lambda_{p,t}} \tag{3}$$

- P_t can be interpreted as the price index of the final good sector.

- $\lambda_{p,t}$ is a stochastic parameter representing the mark-up of the good market. It holds:

$$\lambda_{p,t} = \lambda_p + \eta_t^p \qquad \text{with} \quad \eta_t^p \sim N(0, \sigma^2_{\eta_t^p})$$

Firms:

Intermediate sector

Each firms j of the intermediate good sector minimizes its labour and capital costs $W_t L_{j,t} + r_t^k \widetilde{K}_{j,t}$ observing its Cobb-Douglas production function:

$$y_t^j = \varepsilon_t^a \widetilde{K}_{j,t}^{\alpha} L_{j,t}^{1-\alpha} - \Phi$$

- $\widetilde{K}_{j,t}$ is the effective utilized stock of capital given by $\widetilde{K}_{j,t} = z_t K_{j,t-1}$.
- $L_{j,t}$ is the index of the differently utilized labour force.
- Φ are fix costs.
- ε_t^a is a productivity shock, that follows a AR(1)-Process: $\varepsilon_t^a = \rho_a \varepsilon_{t-1}^a + \eta_t^a$ with $\eta_t^a \sim N(0,1)$.

In order to find the optimal production quantity, it is convenient to show that marginal costs are constant.

One proceeds first by finding a functional relationship between $\widetilde{K}_{j,t}$ and $L_{j,t}$ derived by the minimization problem of $W_t L_{j,t} + r_t^k \widetilde{K}_{j,t}$ with respect to $L_{j,t}$ and $\widetilde{K}_{j,t}$ subject to $y_t^j = \varepsilon_t^a \widetilde{K}_{j,t}^{\alpha} L_{j,t}^{1-\alpha} - \Phi$:

$$\frac{W_t}{r_t^k} = \frac{(1-\alpha)\varepsilon_t^a \widetilde{K}_{j,t}^{\alpha} L_{j,t}^{-\alpha}}{\alpha \varepsilon_t^a \widetilde{K}_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha}} \Leftrightarrow \frac{W_t L_{j,t}}{r_t^k \widetilde{K}_{j,t}} = \frac{1-\alpha}{\alpha}$$

The utilization rate between capital and labour is identical for all firms and constant. This is the same of the economy as a whole.

The marginal costs of product j are derived as follows:

$$L_{j,t} = \frac{1-\alpha}{\alpha}\frac{r_t^k}{W_t}\widetilde{K}_{j,t} \quad \text{in} \quad y_t^j = \varepsilon_t^a \widetilde{K}_{j,t}^{\alpha} L_{j,t}^{1-\alpha} - \Phi$$

$$\Rightarrow \widetilde{K}_{j,t} = \left(y_t^j + \Phi\right)\frac{1}{\varepsilon_t^a}\left(\frac{\alpha}{1-\alpha}\frac{W_t}{r_t^k}\right)^{1-\alpha}$$

$$\text{and} \quad L_{j,t} = \left(y_t^j + \Phi\right)\frac{1}{\varepsilon_t^a}\left(\frac{\alpha}{1-\alpha}\frac{W_t}{r_t^k}\right)^{-\alpha} \tag{4}$$

$$\Rightarrow C_t = W_t L_{j,t} + r_t^k \widetilde{K}_{j,t} = \left(y_t^j + \Phi\right)\frac{1}{\varepsilon_t^a} W_t^{1-\alpha}\left(r_t^k\right)^{\alpha}\left(\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\right)$$

$$MC_t = \frac{\partial C_t}{\partial y_t^j} = \frac{1}{\varepsilon_t^a} W_t^{1-\alpha}\left(r_t^k\right)^{\alpha}\left(\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}\right) \tag{5}$$

The nominal profit for firm j is:

$$\Upsilon_t^j = \underbrace{p_t^j y_t^j}_{\text{Revenues}} - \underbrace{MC_t\left(y_t^j + \Phi\right)}_{\text{Costs}} = \left(p_t^j - MC_t\right)\underbrace{\left(\frac{p_t^j}{P_t}\right)^{\frac{-(1+\lambda_{p,t})}{\lambda_{p,t}}} Y_t}_{=y_t^j} - MC_t\Phi \tag{6}$$

Each firm j has a market power in the market of its good and hence it maximizes its expected profit.

In order to do this it uses a discount factor $(\beta^k \rho_{t+k})$ that is based on the fact that firms belong to households. From the Euler-equation of consumption, it holds:

$$\beta^k \rho_{t+k} = \beta^k \frac{\lambda_{t+k}}{\lambda_t P_{t+k}} \tag{7}$$

- The determination of prices also follows a Calvo-rule in order to model the nominal price rigidity.
- Each firm j is allowed in period t, with probability $1-\xi_p$, to peg its nominal price.
- Since firms are defined over a continuum between 0 and 1, each period, prices that are updated are $1-\xi_p$ and prices that are not updated amount to ξ_p.
- Prices of firms that was unable to re-set their prices are partially indexed by the past inflation rate Π_{t-1}:

$$p_t^j = \left(\Pi_{t-1}\right)^{\gamma_p} p_{t-1}^j = \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_p} p_{t-1}^j \tag{8}$$

where γ_p is the degree of price indexation: $\gamma_p = 0$ implies no indexation $\left(p_t^j = p_{t-1}^j\right)$ and $\gamma_p = 1$ a complete indexation to the past inflation rate $\left(p_t^j = \Pi_{t-1} p_{t-1}^j\right)$.

$\widetilde{p}_t$ denotes the nominal price of the firm that are able in period t to re-optimize., It holds:

$$p_t^j = \begin{cases} \widetilde{p}_t & \text{if } p_t^j \text{ in period } t \text{ is optimally set, with prob. } 1 - \xi_p \\ \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_p} P_{t-1}^j & \text{otherwise, with prob. } \xi_p \end{cases}$$

The equation of motion for the aggregated price index P_t can be obtained through eq. (3):

$$
\begin{aligned}
(P_t)^{\frac{-1}{\lambda_{p,t}}} &= \xi_p \cdot \int_0^1 \left(P_{t-1} \left(\frac{P_{t-1}}{P_{t-2}} \right)^{\gamma_p} \right)^{\frac{-1}{\lambda_{p,t}}} dj + (1-\xi_p) \cdot \int_0^1 \widetilde{p}_t^{\,\frac{-1}{\lambda_{p,t}}} dj \\
&= \xi_p \left[\left(\frac{P_{t-1}}{P_{t-2}} \right)^{\gamma_p} P_{t-1} \right]^{\frac{-1}{\lambda_{p,t}}} + (1-\xi_p)\, \widetilde{p}_t^{\,\frac{-1}{\lambda_{p,t}}} \qquad (9)
\end{aligned}
$$

- The optimal price set in t, $\widetilde{p}_t$, has a probability $(\xi_p)^i$ not to be changed until period i.
- Through the indexation of eq.(8) it holds for the not re-optimized price $t+i$:

$$
\begin{aligned}
p_t^j &= \widetilde{p}_t \\
p_{t+1}^j &= \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p} p_t^j = \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p} \widetilde{p}_t \\
p_{t+2}^j &= \left(\frac{P_{t+1}}{P_t}\right)^{\gamma_p} p_{t+1}^j = \left(\frac{P_{t+1}}{P_t}\right)^{\gamma_p} \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_p} \widetilde{p}_t = \left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p}_t \\
&\vdots \\
p_{t+i}^j &= \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p}_t
\end{aligned}
\tag{10}
$$

- Firms that are not allowed to change their prices in period t, maximize their profit function (6) under the constraint (10).

- Considering that wages hold until period i with probability $(\xi_p)^i$, the Lagrangian for period t is:

$$L^p = E_t \sum_{i=0}^{\infty} \xi_p^i \beta^i \rho_{t+i} \left\{ \begin{array}{c} \left[\left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_p} \widetilde{p}_t - MC_{t+i} \right] Y_{t+i} \cdot \\ \cdot \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_p} \widetilde{p}_t}{P_{t+i}} \right)^{\frac{-(1+\lambda_{p,t+i})}{\lambda_{p,t+i}}} \quad - \Phi MC_{t+i} \end{array} \right\}$$

The first order conditions are:

$$\frac{\partial L^p}{\partial \widetilde{p_t}} = E_t \sum_{i=0}^{\infty} \xi_p^i \beta^i \rho_{t+i} \left\{ \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} Y_{t+i} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p_t}}{P_{t+i}} \right)^{\frac{-(1+\lambda_{p,t+i})}{\lambda_{p,t+i}}} \right.$$

$$+ \left[\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p_t} - MC_{t+i} \right] \frac{-(1+\lambda_{p,t+i})}{\lambda_{p,t+i}} Y_{t+i} \cdot$$

$$\left. \cdot \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p_t}}{P_{t+i}} \right)^{\frac{-(1+\lambda_{p,t+i})}{\lambda_{p,t+i}} - 1} \frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p}}{P_{t+i}} \right\}$$

$$= E_t \sum_{i=0}^{\infty} \xi_p^i \beta^i \rho_{t+i} \left\{ \begin{array}{c} \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} y_{t+i}^j + \\ + \left[\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_p} \widetilde{p_t} - MC_{t+i} \right] \frac{-(1+\lambda_{p,t+i})}{\lambda_{p,t+i}} \frac{1}{\widetilde{P_t}} \end{array} \right\} = 0$$

Multiplying through the factor $-\widetilde{p}_t \lambda_{p,t+i}$ and considering eq (7) for the discount factor yields the following mark-up formula for the re-optimized price:

$$E_t \sum_{i=0}^{\infty} \beta^i \xi_p^i \frac{\lambda_{t+i}}{\lambda_t} y_{t+i}^j \left[\frac{\widetilde{p}_t}{P_t} \left(\frac{(P_{t+i-1}/P_{t-1})^{\gamma_p}}{P_{t+i}/P_t} \right) - (1+\lambda_{p,t+i}) \frac{MC_{t+i}}{P_{t+i}} \right] = 0 \tag{11}$$

With flexible prices ($\xi_p = 0$) the above equation becomes:

$$\widetilde{p}_t = (1+\lambda_{p,t}) \cdot MC_t$$

- The optimal price is set so that firms impose a premium over the average marginal costs.
- Through a log-linearization, the markup formula is easier to interpret:

$$\varphi_t = \mu + (1 - \beta\xi_p) E_t \sum_{i=0}^{\infty} \beta^i \xi_p^i \left[mc_{t+i} + p_{t+i} - \gamma_p (p_{t+i-1} - p_{t-1}) \right]$$

where $\varphi_t = \log(\widetilde{p}_t), \mu = \log(1 + \lambda_p), p_t = log(P_t)$ and $mc_t = log(MC_t / P_t)$.