DSGE methods

General Framework and First Order Approximation

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- Theory and intuition behind the Smets and Wouters (2002) model
- Derivation of the optimality conditions and log-linearization

Insight

DSGE model consists of

- set of Euler equations, i.e. first-order optimality conditions,
- transition equations for structural shocks and innovations,
- (observable variables and measurement errors)

which can be cast into a (non-linear) system of expectational difference equations.

To do ...

- Solution
- Filtering
- Estimation

DSGE framework

- DSGE models consists of
  - *Endogenous variables* $\mathbf{y}_t$, *exogenous state variables* $\mathbf{z}_t$ and *endogenous state variables* $\mathbf{x}_t$ with $\mathbf{w}_t = (\mathbf{x}_t, \mathbf{z}_t)'$
  - Transition equations for the exogenous processes $\mathbf{z}_{t+1} = h_z(\mathbf{z}_t, \epsilon^z_{t+1})$
  - Vector of deep parameters θ
  - Set of optimality conditions (FOCs)

  which can be summarized as a non-linear system of stochastic difference equations

$$\mathbf{0} = \mathbb{E}_t \left[f \left(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{w}_{t+1}, \mathbf{w}_t \right) \right]$$

- Searched-for solutions are *transition functions* and *policy-functions*

$$\mathbf{y}_t = g(\mathbf{w}_t)$$
$$\mathbf{w}_{t+1} = h(\mathbf{w}_t, \boldsymbol{\epsilon}_{t+1})$$

DSGE solution

- Linear methods: Anderson/Moore (1983), Binder and Pesaran, Blanchard/Khan (1980),(1997), Christiano (2002), King and Watson (1998), **Klein (2000)**, Sims (2001) and Uhlig (1999) (See Anderson (2008) for a comparison).
- Nonlinear methods: Projection methods, iteration procedures or higher-order **perturbations** (See DeJong and Dave (2011) for a comparison).

DSGE solution

Perturbation

- Problem: Closed form solutions for $g(\cdot)$ and $h(\cdot)$ are most of the times not available
- Solution: Approximate policy/transition functions using a perturbation approach
- Idea: Introduce a perturbation parameter σ that scales the underlying stochastics
  - Steady-state is known analytically
  - $g(\cdot)$ and $h(\cdot)$ at the deterministic steady-state $(\sigma = 0)$ are known

  $\rightarrow$ Taylor-approximation of $g(\cdot)$ and $h(\cdot)$ around the steady-state!
  - Assumption:

$$\begin{aligned} \mathbf{y}_t &= g(\mathbf{w}_t, \sigma) \\ \mathbf{w}_{t+1} &= h(\mathbf{w}_t, \sigma) + \sigma \boldsymbol{\epsilon}_{t+1}, \qquad \text{with } \boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \Sigma) \end{aligned}$$

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Taylor Expansion

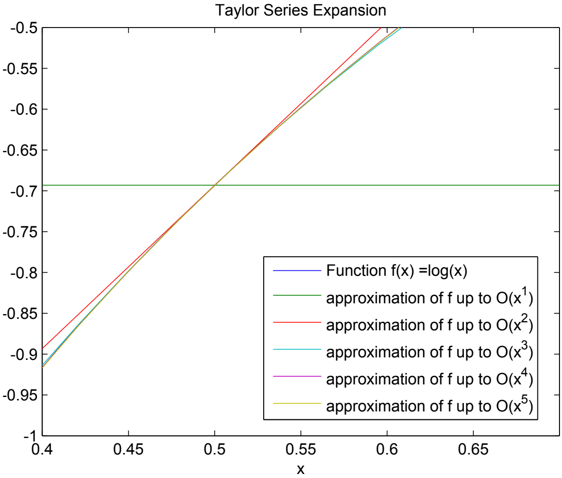
Representation of an analytic function $f(x)$ as an infinite sum of terms around an expansion point $x = a$:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$
$$= \sum_{m=0}^{\infty} \frac{f^{(m)}(a)}{m!}(x-a)^m$$

$\rightarrow$ If higher order terms $O(x^m)$ are neglectable, we can use a truncated Taylor expansion $\hat{f}(x) \approx f(x)$

Taylor Expansion

Example: $f(x) = log(x)$



Taylor Expansion

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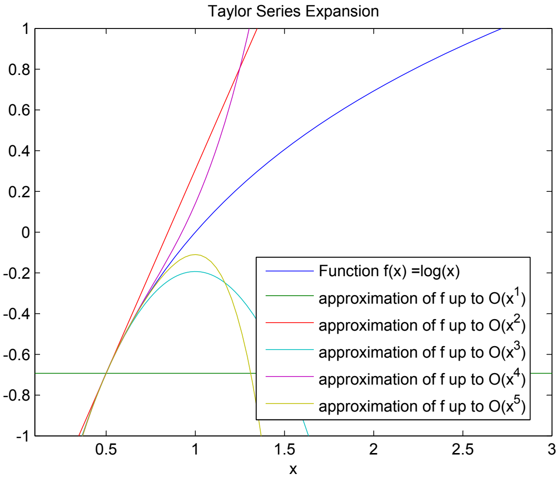

DSGE solution

- Substitution yields

$$\mathbf{y}_{t+1} = g(h(\mathbf{w}_t, \sigma) + \sigma \boldsymbol{\epsilon}_{t+1}, \sigma)$$

- System becomes

$$\mathbb{E}_t \left[f \left(g(h(\mathbf{w}_t, \sigma) + \sigma \boldsymbol{\epsilon}_{t+1}, \sigma), g(\mathbf{w}_t, \sigma), h(\mathbf{w}_t, \sigma) + \sigma \boldsymbol{\epsilon}_{t+1}, \mathbf{w}_t \right) \right]$$

Insight

The system of equations exclusively depends on *state variables* $\mathbf{w}_t$, i.e. we can write

$$F(\mathbf{w}_t, \sigma) = \mathbf{0}.$$

DSGE solution

We have

- System of equations $F(\mathbf{w}_t, \sigma)$
- Known point $\bar{\mathbf{w}}$

→ First-order Taylor expansion is straightforward

$$
\begin{aligned}
F(\mathbf{w}_t, \sigma) &\approx F(\bar{\mathbf{w}}, 0) + F_\sigma(\bar{\mathbf{w}}, 0)(\sigma - 0) + F_w(\bar{\mathbf{w}}, 0)(\mathbf{w}_t - \bar{\mathbf{w}}) \\
g(\mathbf{w}_t, \sigma) &\approx g(\bar{\mathbf{w}}, 0) + g_\sigma(\bar{\mathbf{w}}, 0)(\sigma - 0) + g_w(\bar{\mathbf{w}}, 0)(\mathbf{w}_t - \bar{\mathbf{w}}) \\
h(\mathbf{w}_t, \sigma) &\approx h(\bar{\mathbf{w}}, 0) + h_\sigma(\bar{\mathbf{w}}, 0)(\sigma - 0) + h_w(\bar{\mathbf{w}}, 0)(\mathbf{w}_t - \bar{\mathbf{w}})
\end{aligned}
$$

→ How to determine g_σ, h_σ, g_w and h_w?

Insight

$F(\mathbf{w}_t, \sigma)$ equals the zero vector for all $\mathbf{w}_t$, thus each derivative has to equal 0.

DSGE solution

$$
\begin{aligned}
F_{\sigma}(\bar{\mathbf{w}}, 0) &= \mathbb{E}_t \left[f_{y_{t+1}} [g_w (h_\sigma + \boldsymbol{\epsilon}_{t+1}) + g_\sigma] + f_{y_t} g_\sigma + f_{w_{t+1}} (h_\sigma + \boldsymbol{\epsilon}_{t+1}) \right] \\
&= f_{y_{t+1}} [g_w h_\sigma + g_\sigma] + f_{y_t} g_\sigma + f_{w_{t+1}} h_\sigma \\
&= \begin{bmatrix} f_{y_{t+1}} g_w + f_{w_{t+1}} & f_{y_t+1} + f_y \end{bmatrix} \begin{bmatrix} h_\sigma \\ g_\sigma \end{bmatrix} \\
&\stackrel{!}{=} \mathbf{0}
\end{aligned}
$$

$\rightarrow$ A linear and homogeneous equation in g_σ and h_σ, which is always solved by the zero vector

$\rightarrow$ If there exists a *unique* solution, it is $g_\sigma = \mathbf{0}$ and $h_\sigma = \mathbf{0}$

$\rightarrow$ Certainty equivalence

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$$\begin{aligned} F_w(\bar{\mathbf{w}}, 0) &= f_{y_{t+1}} g_w h_w + f_y g_w + f_{w_{t+1}} h_w + f_{w_t} \\ &= 0 \end{aligned}$$

$\rightarrow$ Rewriting the above equation yields

$$\begin{bmatrix} f_{w_{t+1}} & f_{y_{t+1}} \end{bmatrix} \begin{bmatrix} I \\ g_w \end{bmatrix} h_w = - \begin{bmatrix} f_{w_t} & f_{y_t} \end{bmatrix} \begin{bmatrix} I \\ g_w \end{bmatrix}$$

DSGE solution

Postmultiplying by $\hat{\mathbf{w}}_t = \mathbf{w}_t - \bar{\mathbf{w}}$ and making use of

- $\mathbb{E}_t[\hat{\mathbf{y}}_{t+1}] = g_w h_w \hat{\mathbf{w}}_t$
- $\mathbb{E}_t[\hat{\mathbf{w}}_{t+1}] = h_w \hat{\mathbf{w}}_t$

$$\begin{bmatrix} f_{w_{t+1}} & f_{y_{t+1}} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \hat{\mathbf{w}}_{t+1} \\ \hat{\mathbf{y}}_{t+1} \end{bmatrix} = - \begin{bmatrix} f_{w_t} & f_{y_t} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{w}}_t \\ \hat{\mathbf{y}}_t \end{bmatrix}$$

DSGE solution

Defining matrices A and B (both known!)

$$A = \begin{bmatrix} f_{w_{t+1}} & f_{y_{t+1}} \end{bmatrix}$$
$$B = -\begin{bmatrix} f_{w_t} & f_{y_t} \end{bmatrix}$$

the system is

$$A\mathbb{E}_t \begin{bmatrix} \hat{\mathbf{w}}_{t+1} \\ \hat{\mathbf{y}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \hat{\mathbf{w}}_t \\ \hat{\mathbf{y}}_t \end{bmatrix}$$

As $g_\sigma = h_\sigma = 0$, the approximative transitions/policies are

$$\hat{\mathbf{w}}_{t+1} = h_w(\bar{\mathbf{w}}, \sigma)\hat{\mathbf{w}}_t + \sigma\epsilon_{t+1},$$
$$\hat{\mathbf{y}}_t = g_w(\bar{\mathbf{w}}, \sigma)\hat{\mathbf{w}}_t.$$

$\rightarrow$ Klein (2000) shows how h_w and g_w can be determined from matrices A and B using a generalized Schur decomposition.