Dynamic Stochastic General Equilibrium Models

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- Labour is a differentiated good which is supplied by Households in a monopolistic competition to the production sector. It follows that firms consider labour supplies l_t^τ as imperfect substitutes.

- The aggregate labour demand, L_t , and the aggregate nominal wage, W_t , are given by the following Dixit-Stiglitz type aggregator where $\lambda_{w,t}$ is the substitution parameter:

$$L_t = \left[\int_0^1 (l_t^\tau)^{\frac{1}{1+\lambda_{w,t}}} \quad d\tau \right]^{1+\lambda w,t} \qquad \text{with } \lambda_{w,t} > 0 \tag{1}$$

- The condition $\lambda_{w,t} = 0$ corresponds to the perfect substitution of labour.

Firms minimize the cost function $\int\limits_0^1 W_t^\tau l_t^\tau \; d\tau$ under the aggregate labour demand.

- Define W_t as the Lagrange multiplier, the cost minimization function is:

$$L = \int_0^1 W_t^\tau l_t^\tau \, d\tau + W_t \left(L_t - \left[\int_0^1 \left(l_t^\tau\right)^{\frac{1}{1+\lambda_{w,t}}} \, d\tau \right]^{1+\lambda_{w,t}} \right)$$

$$\frac{\partial L}{\partial l_t^\tau} = W_t^\tau - W_t \left((1+\lambda_{w,t}) \left[\int_0^1 \left(l_t^\tau\right)^{\frac{1}{1+\lambda_{w,t}}} \, d\tau \right]^{\lambda_{w,t}} \frac{1}{1+\lambda_{w,t}} \left(l_t^\tau\right)^{\frac{-\lambda_{w,t}}{1+\lambda_{w,t}}} \right) =$$

$$\Leftrightarrow W_t^\tau = W_t L_t^{\frac{\lambda_{w,t}}{1+\lambda_{w,t}}} \left(l_t^\tau\right)^{\frac{-\lambda_{w,t}}{1+\lambda_{w,t}}}$$

$$\Leftrightarrow l_t^\tau = L_t \left(\frac{W_t^\tau}{W_t} \right)^{\frac{-(1+\lambda_{w,t})}{\lambda_{w,t}}} \tag{2}$$

Eq. ((2)) is the optimal demand function of labour. Substituting this in eq. (1) yields:

$$L_t = \left[\int_0^1 \left(\frac{W_t^\tau}{W_t} \right)^{\frac{-1}{\lambda_{w,t}}} L_t^{\frac{1}{1+\lambda_{w,t}}} \, d\tau \right]^{1+\lambda_{w,t}} \tag{3}$$

$$\Leftrightarrow W_t = \left[\int_0^1 \left(W_t^\tau \right)^{\frac{-1}{\lambda_{w,t}}} \, d\tau \right]^{-\lambda_{w,t}} \tag{4}$$

- The Lagrange multiplier W_t (which in general represents the the marginal value of relaxing the constraint) can be interpreted here as the price of a working hour and hence a wage index.

- The determination of wages follows a Calvo rule.
- Each household τ has a (constant) probability of $1 - \xi_w$ in every Period t to be able to set its nominal wage.
- Since households are a continuum between 0 and 1, in each Period a fraction of households equal to $1 - \xi_w$ adapts its wage.

- Wages of the fraction of households ξ_w that cannot reoptimize are partially anchored to the inflation rate:

$$W_t^\tau = (\Pi_{t-1})^{\gamma_w} W_{t-1}^\tau = \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} W_{t-1}^\tau \tag{5}$$

where γ_w is the Degree of the wage indexation. $\gamma_w = 0$ implies no indexation $\left(W_t^\tau = W_{t-1}^\tau\right)$ and $\gamma_w = 1$ implies a complete indexation $\left(W_t^\tau = \Pi_{t-1} W_{t-1}^\tau\right)$.

Define $\widetilde{W}_t$ as the wage of the household that in period t can reoptimize. it follows:

$$W_t^\tau = \begin{cases} \widetilde{W}_t & \text{with probability } 1 - \xi_w \\ \left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} W_{t-1}^\tau & \text{with probability } \quad \xi_w \end{cases}$$

- The equation of motion for the aggregate wage index W_t can be obtained through (3):

$$
\begin{aligned}
(W_t)^{\frac{-1}{\lambda_{w,t}}} &= \xi_w \cdot \int_0^1 \left(W_{t-1} \left(\frac{P_{t-1}}{P_{t-2}} \right)^{\gamma_w} \right)^{\frac{-1}{\lambda_{w,t}}} d\tau + (1-\xi_w) \cdot \int_0^1 \widetilde{W}_t^{\frac{-1}{\lambda_{w,t}}} d\tau \\
&= \xi_w \left[\left(\frac{P_{t-1}}{P_{t-2}} \right)^{\gamma_w} W_{t-1} \right]^{\frac{-1}{\lambda_{w,t}}} + (1-\xi_w) \widetilde{W}_t^{\frac{-1}{\lambda_{w,t}}} \qquad (6)
\end{aligned}
$$

- The probability for $\widetilde{W}_t$ not to be re-set until period i is $(\xi_w)^i$. Through the indexation (5) follows that the not reoptmized wage in $t+i$, is

$$
\begin{aligned}
W_t^\tau &= \widetilde{W}_t \\
W_{t+1}^\tau &= \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w} W_t^\tau = \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t \\
W_{t+2}^\tau &= \left(\frac{P_{t+1}}{P_t}\right)^{\gamma_w} W_{t+1}^\tau = \left(\frac{P_{t+1}}{P_t}\right)^{\gamma_w} \left(\frac{P_t}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t = \left(\frac{P_{t+1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t \\
&\vdots \\
W_{t+i}^\tau &= \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t \qquad (7)
\end{aligned}
$$

- Households that can re-set their wages maximize their objective function subject to their budget constraint (see last slides) and the demand of labour (2). and considering the fact that wages remain fixed until period i with a probability $(\xi_w)^i$.

The Lagragean function in t is:

$$L_t = E_t \sum_{i=0}^{\infty} \xi_w^i \beta^i \Bigg[\underbrace{U(C_{t+i}^{\tau}, l_{t+i}^{\tau}, M_{t+i}^{\tau})}_{\text{Objective function}} - \lambda_{t+i} \underbrace{\left(\cdots - \frac{W_{t+i}^{\tau}}{P_{t+i}} l_{t+i}^{\tau} + \dots \right)}_{\text{Budget constraint}}$$
$$- \mu_{t+i} \underbrace{\left(l_{t+i}^{\tau} - L_{t+i} \left(\frac{W_{t+i}^{\tau}}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}} \right)}_{\text{Labour demand}} \Bigg] \tag{8}$$

Substituting the demand of labour (2) in the objective function and in the budget constraint and considering (7), one obtains

$$
\begin{aligned}
L \;=\; & E_t \sum_{i=0}^{\infty} \xi_w^i \beta^i U \left(C_{t+i}^{\tau},\; L_{t+i} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}},\; M_{t+i}^{\tau} \right) \\
& + \lambda_{t+i} \left(\begin{array}{c} \cdots + \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \frac{\widetilde{W}_t}{P_{t+i}} L_{t+i} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}} \\ + \ldots \end{array} \right) \qquad (9)
\end{aligned}
$$

The first order conditions are the following:

$$\frac{\partial L}{\partial \widetilde{W}_t} =$$

$$E_t \sum_{i=0}^{\infty} \xi_w^i \beta^i \left\{ U^L_{t+i} \frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}} L_{t+i} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}-1} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w}}{W_{t+i}} \right) \right.$$

$$+ \lambda_{t+i} \left[\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \frac{1}{P_{t+i}} L_{t+i} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}} + \right.$$

$$+ \left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \frac{\widetilde{W}_t}{P_{t+i}} L_{t+i} \cdot$$

$$\left. \cdot \frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w} \widetilde{W}_t}{W_{t+i}} \right)^{\frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}}-1} \left(\frac{\left(\frac{P_{t+i-1}}{P_{t-1}}\right)^{\gamma_w}}{W_{t+i}} \right) \right]$$

$$= E_t \sum_{i=0}^{\infty} \xi_w^i \beta^i \left\{ \begin{array}{c} U^L_{t+i} \frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}} \frac{l^\tau_{t+i}}{\widetilde{W}_t} + \lambda_{t+i} \cdot \\ \cdot \left[\left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_w} \frac{1}{P_{t+i}} l^\tau_{t+i} + \frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}} \left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_w} \frac{1}{P_{t+i}} l^\tau_{t+i} \right] \end{array} \right\} \tag{10}$$

$$= E_t \sum_{i=0}^{\infty} \xi_w^i \beta^i \left[\begin{array}{c} U^L_{t+i} \frac{-(1+\lambda_{w,t+i})}{\lambda_{w,t+i}} \frac{l^\tau_{t+i}}{\widetilde{W}_t} + \\ +\lambda_{t+i} \left(\frac{-1}{\lambda_{w,t+i}} \left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_w} \frac{1}{P_{t+i}} l^\tau_{t+i} \right) \end{array} \right] = 0$$

Multiplying by the factor $\frac{-\lambda_{w,t+i}}{1+\lambda_{w,t+i}}$ and considering that $\lambda_t = U_t^c$ for λ_{t+i} one obtains

$$\frac{\widetilde{W}_t}{P_t} E_t \left\{ \sum_{i=0}^{\infty} \beta^i \xi_w^i \left(\frac{P_{t+i-1}}{P_{t-1}} \right)^{\gamma_w} \frac{P_t}{P_{t+i}} \left(\frac{l_{t+i}^{\tau} U_{t+i}^{C}}{1+\lambda_{w,t+i}} \right) \right\} \tag{11}$$
$$= \quad E_t \left\{ \sum_{i=0}^{\infty} \beta^i \xi_w^i l_{t+i}^{\tau} U_{t+i}^{l} \right\}$$

Assuming perfect flexibility of wages ($\xi_w = 0$) eq. (11) becomes:

$$\frac{\widetilde{W}_t}{P_t} = (1+\lambda_{w,t}) \frac{U_t^L}{U_t^c}$$

It will be assumed that $\lambda_{w,t}$ is affected by a n.i.d shock.