DSGE-Models

Linearization and Solution Methods

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Previously...

- Theory and intuition behind the Smets/Wouters' model as a prototype of current DSGE-models.
- Derivation of the structural form and log-linearization.

Insight

A DSGE-model consists of a set of expected, nonlinear optimality conditions and transition equations for stochastic processes, which one has to solve.

- A DSGE-model can in general be written as:

General form

$$\mathbf{\Gamma}\left(E_t\mathbf{x_{t+1}}, \mathbf{x_t}, \boldsymbol{\upsilon}_{\mathbf{t+1}}|\boldsymbol{\mu}\right) = \mathbf{\Gamma}\left(\mathbf{x_{t+1}}, \mathbf{x_t}, \boldsymbol{\upsilon}_{\mathbf{t+1}}, \boldsymbol{\eta}_{\mathbf{t+1}}|\boldsymbol{\mu}\right) = 0, \tag{1}$$

with $\mathbf{x_t}$: $(n \times 1)$-vector of stationary variables, $\boldsymbol{\upsilon}_{\mathbf{t}}$: $(m \times 1)$-vector of structural shocks, $\boldsymbol{\mu}$: $(k \times 1)$-vector of parameters.

General Form

- Rational Expectations: $\boldsymbol{\eta_{t+1}} = E_t \mathbf{x_{t+1}} - \mathbf{x_{t+1}}$.
- Non-predictable expectation error $\boldsymbol{\eta_{t+1}}$ occurs due to the realization of structural shocks: $\boldsymbol{\eta_t} = f(\boldsymbol{\upsilon_t})$.
- $\mathbf{x_t}$ consist of n_c control variables and n_s state variables.
- Control variables are denoted by $\mathbf{c_t}$: optimal behavior of the agents as a function of the current state of the economy.
- State variables are denoted by $\mathbf{s_t}$. They consist of exogenous - independent of the decisions of the agents - and endogenous state variables, that can be influenced by the agents.
- $\mathbf{s_t}$ is a function of previous states and current shocks.

Solution of a DSGE-model

- To solve such a rational expectations model means to find so-called *policy-functions* c and s, that solve (at least approximately) the system of equations $\mathbf{\Gamma}$:

Policy-functions

$$\mathbf{c_t} = c(\mathbf{s_t}), \qquad \mathbf{s_t} = s(\mathbf{s_{t-1}}, \boldsymbol{\upsilon}_\mathbf{t}).$$

- DSGE-models can be interpreted as *state-space-models*.
- One distinguishes between linear and non-linear methods:
  - Linear methods: Blanchard/Khan (1980), Binder and Pesaran (1997), Christiano (2002), King and Watson (1998), Klein (2000), Sims (2001) and Uhlig (1999).
  - Frequently used nonlinear method: Schmitt-Grohé/Uribe (2004). Excellent overview: Heer and Maussner (2009).

Repetition: Exercise 1

Assume the following stochastic growth-model:

$$\max_{c_t, k_{t+1}} E_0 \sum_{t=0}^{\infty} \beta^t U(c_t) \quad \text{with } U(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma},$$

$$c_t + k_{t+1} = z_t f(k_t) + (1-\delta) k_t \quad \text{with } f(k_t) = k_t^{\alpha},\ k_o \text{ given},$$

$$log(z_t) = \rho log(z_{t-1}) + \varepsilon_t \quad \text{with } \varepsilon_t \sim WN(0, \sigma_\varepsilon),\ 0 < \rho < 1.$$

(a) What is the unconditional expectation of $log(z_t)$, what of z_t? Find an expression for the unconditional variance of $log(z_t)$?

(b) Derive the first-order-conditions (FOC) by setting up the Bellman-equation and using methods of dynamic programming. What are state and control variables?

(c) Derive the first-order-conditions (FOC) by maximizing the lagrangian.

(d) Derive the *steady-state* and linearize the FOC.

Linear Solution Methods

Pros:

- Simple linear state-space representation of the model, which in many cases is sufficiently exact.
- One can use the Kalman-filter to empirically evaluate the system.

Cons:

- One looses important information during the linearization.
- Higher moments play an important role for analyzing markets, risk, welfare, etc.
- An approximation to, say, the second order can yield different results, because the variance of future shocks matters (risk premium).

Certainty-equivalence-property

In stochastic rational expectations models agents take the effect of future shocks into account. For a linearization to the first-order these expectations are zero, thus, they don't matter for the decision rules.

Linear Solution Methods

- First linearize or log-linearize the general form (1) around the deterministic *steady-state*.
- Together with the transition equation of the stochastic processes one gets the reduced-form model:

$$\mathbf{A}\mathbf{x}_{\mathbf{t+1}} = \mathbf{B}\mathbf{x}_{\mathbf{t}} + \mathbf{C}\boldsymbol{\upsilon}_{\mathbf{t+1}} + \mathbf{D}\boldsymbol{\eta}_{\mathbf{t+1}} + \mathbf{E}.$$

- The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$ are functions of the structural parameters $\boldsymbol{\mu}$; $\mathbf{E}$ is a vector of constants (mostly zero).
- The solution of this linear representation has a VAR(1)-form:

$$\mathbf{x}_{\mathbf{t+1}} = \mathbf{F}(\boldsymbol{\mu})\mathbf{x}_{\mathbf{t}} + \mathbf{G}(\boldsymbol{\mu})\boldsymbol{\upsilon}_{\mathbf{t+1}}, \tag{2}$$

  with $\mathbf{F}$ and $\mathbf{G}$ being functions of the parameters $\boldsymbol{\mu}$.
- The motor of the model is the vector of exogenous shocks $\boldsymbol{\upsilon}_{\mathbf{t}}$.
- (2) describes thus, the fluctuations around the steady-state as well as the decision rules given the stochastic innovations.

Notation

Predetermined variables: $E_t X_{1,t+1} = X_{1,t+1}$,
Non-predetermined variables: $E_t X_{2,t+1} = E_t X_{2,t+1}$.

Unitary matrices

$\mathbf{M}'\mathbf{M} = \mathbf{M}\mathbf{M}' = \mathbf{I}$ are the complex analogous to orthogonal matrices. They are diagonalizable.

- The method of Sims (2001) begins with a *QZ-factorization* (General Schur decomposition), in which the matrices **A** and **B** are transformed into unitary upper triangular matrices:

$$\mathbf{A} = \mathbf{Q}'\mathbf{\Lambda}\mathbf{Z}', \qquad \mathbf{B} = \mathbf{Q}'\mathbf{\Omega}\mathbf{Z}'.$$

- $\mathbf{\Lambda}$ and $\mathbf{\Omega}$ are upper triangular matrices with the generalized Eigenvalues of **A** and **B**, and they are sorted in increasing order from left to right.

The Sims (2001)-algorithm

- The Eigenvalues determine if a system of equations converges or explodes.

Blanchard/Khan-conditions

The number of Eigenvalues, that are in absolute terms greater than 1, must be equal to the number of non-predetermined variables, in order to get a stable solution (saddle-path).

- Let $\mathbf{z_{t+1}} = \mathbf{Z'x_{t+1}}$, then the system can be divided into a non-explosive (upper) and an explosive part (lower):

$$\begin{bmatrix} \mathbf{\Lambda_{11}} & \mathbf{\Lambda_{12}} \\ \mathbf{0} & \mathbf{\Lambda_{22}} \end{bmatrix} \begin{pmatrix} \mathbf{z_{1,t+1}} \\ \mathbf{z_{2,t+1}} \end{pmatrix} = \begin{bmatrix} \mathbf{\Omega_{11}} & \mathbf{\Omega_{12}} \\ \mathbf{0} & \mathbf{\Omega_{22}} \end{bmatrix} \begin{pmatrix} \mathbf{z_{1,t}} \\ \mathbf{z_{2,t}} \end{pmatrix} + \begin{pmatrix} \mathbf{Q_1} \\ \mathbf{Q_2} \end{pmatrix} \begin{bmatrix} \mathbf{E_1} + \mathbf{C_1} \boldsymbol{\upsilon}_{\mathbf{1,t+1}} + \mathbf{D_1} \boldsymbol{\eta}_{\mathbf{1,t+1}} \\ \mathbf{E_2} + \mathbf{C_2} \boldsymbol{\upsilon}_{\mathbf{2,t+1}} + \mathbf{D_2} \boldsymbol{\eta}_{\mathbf{2,t+1}} \end{bmatrix}. \tag{3}$$

- The difference equations belonging to the Eigenvalues greater than 1 are solved forwards.
- Remark: $\lim_{t\to\infty} \left(\mathbf{\Omega_{22}}^{-1}\mathbf{\Lambda_{22}}\right)^t \mathbf{z_{2,t}} = \mathbf{0}$ and for all $s > 0 : E_t \upsilon_{\mathbf{2,t+s}} = E_t \eta_{\mathbf{2,t+s}} = 0$ (Expectations don't matter)

$$\begin{aligned}
\mathbf{z_{2,t}} &= \mathbf{\Omega_{22}}^{-1}\mathbf{\Lambda_{22}}\mathbf{z_{2,t+1}} - \mathbf{\Omega_{22}}^{-1}\mathbf{Q_2}\left[\mathbf{E_2} + \mathbf{C_2}\upsilon_{\mathbf{2,t+1}} + \mathbf{D_2}\eta_{\mathbf{2,t+1}}\right] \\
&= -\sum_{i=0}^{\infty} \left(\mathbf{\Omega_{22}}^{-1}\mathbf{\Lambda_{22}}\right)^i \mathbf{\Omega_{22}}^{-1}\mathbf{Q_2}\left[\mathbf{E_2} + \mathbf{C_2}\upsilon_{\mathbf{2,t+1+i}} + \mathbf{D_2}\eta_{\mathbf{2,t+1+i}}\right] \\
&= -\sum_{i=0}^{\infty} \left(\mathbf{\Omega_{22}}^{-1}\mathbf{\Lambda_{22}}\right)^i \mathbf{\Omega_{22}}^{-1}\mathbf{Q_2}\mathbf{E_2} \\
&= -\left[\mathbf{I} - \mathbf{\Omega_{22}}^{-1}\mathbf{\Lambda_{22}}\right]^{-1} \mathbf{\Omega_{22}}^{-1}\mathbf{Q_2}\mathbf{E_2} = \left[\mathbf{\Lambda_{22}} - \mathbf{\Omega_{22}}\right]^{-1}\mathbf{Q_2}\mathbf{E_2}.
\end{aligned}$$

The Sims (2001)-algorithm

- The difference equations belonging to the Eigenvalues less or equal than 1 are solved backwards.
- Remark: Systematical relationship betweend the expectation errors $\eta_{1,t} \& \eta_{2,t}$.

Sufficient condition for a stable saddle-path

$$\mathbf{Q_1 D} = \mathbf{\Phi Q_2 D}.$$

- $\mathbf{\Phi}$ is of dimension $n_s \times n_c$ (with $\mathbf{z_{1,t}} : n_s \times 1$ and $\mathbf{z_{2,t}} : n_c \times 1$).
- (3) can be rewritten as:

$$\underbrace{\begin{bmatrix} \underset{n_s \times n_s}{\mathbf{I}} & \underset{n_s \times n_c}{-\mathbf{\Phi}} \end{bmatrix}}_{n_s \times (n_s + n_c)} \begin{bmatrix} \mathbf{\Lambda_{11}} & \mathbf{\Lambda_{12}} \\ \mathbf{0} & \mathbf{\Lambda_{22}} \end{bmatrix} \begin{pmatrix} \mathbf{z_{1,t}} \\ \mathbf{z_{2,t}} \end{pmatrix} =$$

$$\begin{bmatrix} \mathbf{I} & -\mathbf{\Phi} \end{bmatrix} \begin{bmatrix} \mathbf{\Omega_{11}} & \mathbf{\Omega_{12}} \\ \mathbf{0} & \mathbf{\Omega_{22}} \end{bmatrix} \begin{pmatrix} \mathbf{z_{1,t-1}} \\ \mathbf{z_{2,t-1}} \end{pmatrix} + \begin{bmatrix} \mathbf{I} & -\mathbf{\Phi} \end{bmatrix} \begin{pmatrix} \mathbf{Q_1} \\ \mathbf{Q_2} \end{pmatrix} \left[\mathbf{E} + \mathbf{C} \boldsymbol{\upsilon}_{\mathbf{t}} + \mathbf{D} \boldsymbol{\eta}_{\mathbf{t}} \right].$$

The Sims (2001)-algorithm

$$\Leftrightarrow \begin{bmatrix}\mathbf{\Lambda_{11}} & \mathbf{\Lambda_{12}-\Phi\Lambda_{22}}\end{bmatrix}\begin{pmatrix}\mathbf{z_{1,t}}\\ \mathbf{z_{2,t}}\end{pmatrix} = \begin{bmatrix}\mathbf{\Omega_{11}} & \mathbf{\Omega_{12}-\Phi\Omega_{22}}\end{bmatrix}\begin{pmatrix}\mathbf{z_{1,t-1}}\\ \mathbf{z_{2,t-1}}\end{pmatrix}$$
$$+ \begin{bmatrix}\mathbf{Q_1-\Phi Q_2}\end{bmatrix}\begin{bmatrix}\mathbf{E+C}\upsilon_\mathbf{t}\end{bmatrix} + \underbrace{\begin{bmatrix}\mathbf{(Q_1-\Phi Q_2)\,D}\eta_\mathbf{t}\end{bmatrix}}_{=\mathbf{0}}.$$

The algorithm

$$\text{QZ-factorization: } \mathbf{A=Q'\Lambda Z', B=Q'\Omega Z'},\ \mathbf{x_t=Z}\begin{pmatrix}\mathbf{z_{1,t}}\\ \mathbf{z_{2,t}}\end{pmatrix},$$

$$\mathbf{z_{1,t}} = -\mathbf{\Lambda_{11}}^{-1}\left(\mathbf{\Lambda_{12}-\Phi\Lambda_{22}}\right)\mathbf{z_{2,t}} + \mathbf{\Lambda_{11}}^{-1}\mathbf{\Omega_{11}z_{1,t-1}}$$
$$+\mathbf{\Lambda_{11}}^{-1}\left(\mathbf{\Omega_{12}-\Phi\Omega_{22}}\right)\mathbf{z_{2,t-1}} + \mathbf{\Lambda_{11}}^{-1}\left(\mathbf{Q_1-\Phi Q_2}\right)\left(\mathbf{E+C}\upsilon_\mathbf{t}\right),$$
$$\mathbf{z_{2,t}} = \left(\mathbf{\Lambda_{22}-\Omega_{22}}\right)^{-1}\mathbf{Q_2E}.$$

Exercise 1: continued

The Sims (2001)-algorithm is available for different program packages, and it is implemented in Dynare.

(e) Rewrite the linearized model into the form
$\mathbf{A x_{t+1}} = \mathbf{B x_t} + \mathbf{C} \boldsymbol{\upsilon}_{\mathbf{t+1}} + \mathbf{D} \boldsymbol{\eta}_{\mathbf{t+1}} + \mathbf{E}$

(f) Solve for the *policy-function* by using the Sims (2001)- Algorithm. Assume the following values for the parameters:
$\beta = 0.99, \quad \alpha = 0.36, \quad \sigma = 2, \quad \delta = 0.025, \quad \rho = 0.9.$

Homework

- Install the newest version of Matlab.
- https://zivdav.uni-muenster.de/ddfs/Soft.ZIV/TheMathWorks/ .
- Because of license issues you have to be connected to the university either via WLAN or VPN.
- Download all files from http://sims.princeton.edu/yftp/gensys/mfiles/ into a folder called gensys. Make this folder available to Matlab (`File-SetPath-Add Folder`).
- Install the newest version of Dynare (http://www.dynare.org/). Make the Dynare folder $c:\backslash dynare \backslash aktuelle\ Version \backslash matlab$) available to Matlab (`File-SetPath-Add Folder`).