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Accuracy of stochastic perturbation methods: The case of asset pricing models[☆]

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Abstract

This paper investigates the accuracy of a perturbation method in approximating the solution to stochastic equilibrium models under rational expectations. As a benchmark model, we use a version of asset pricing models proposed by Burnside (1998, Journal of Economic Dynamics and Control 22, 329–340) which admits a closed-form solution while not making the assumption of certainty equivalence. We then check the accuracy of perturbation methods — extended to a stochastic environment — against the closed form solution. Second- and especially fourth-order expansions are then found to be more efficient than standard linear approximation, as they are able to account for higher-order moments of the distribution — which constitutes a major improvement of this stochastic approach to approximation compared to other methods that assume certainty equivalence. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

Modern macroeconomics essentially relies on computable model economies that can be used as small laboratories to evaluate the effects of economic policies or changes in the agents' environment such as changes in the market structure, shocks to fundamentals. Unfortunately, most of the models economists develop do not admit any closed-form solutions for the dynamic decision problems that arise once the non-linear functional forms have been stated for agent's behaviors. One way to circumvent this difficulty is to use numerical methods to approximate the 'true solution' to the model. Much effort has been put in this area in these recent years. There now exists a wide range of methods to approximate the agent's decision rules: LQ approximation (Kydland and Prescott, 1982; Cooley and Prescott, 1995), parameterized expectation algorithm (Marcet, 1988), Gaussian quadrature (Tauchen and Hussey, 1991), minimum weighted residual methods (Judd, 1992). In this paper we focus on a smooth approximation method advocated by Judd (1998), that relies on the Taylor series expansion of the functional equation characterizing the agent's behavior.

We propose a systematic exploration of this method and provide a comparison with two of the most widely used approximation method in macroeconomics: linearization and the method of weighted residuals. This is undertaken within an asset pricing model à la Lucas (1978). The version of the model we consider has been recently proposed by Burnside (1998). Because the marginal intertemporal rate of substitution is an exponential function of the rate of growth of consumption and the endowment is a Gaussian exogenous process, the model admits a closed-form solution that can be used to check the accuracy of the approximation.

Our results show the potential of Taylor series expansion methods in approximating the solution. In particular, they show that, while being as simple as standard linearization, perturbation methods can accommodate the major drawback of linearization around the deterministic steady state in face of a high volatility of shocks to fundamentals or a large curvature of the utility function. Indeed, one of the merits of higher-orders perturbation methods is their ability to exploit the information contained in the higher-order moments of distribution of the shocks that hit the economy. Taking higher-order moments into account is then found to alter the benchmark point around which the model is approximated — thus leading to the abandonment of the certainty equivalence hypothesis that standard linearization approximation procedures impose — and the form of the decision rule. We then explore the implications of the different methods under study for the approximate distribution of the variable of interest. We find that higher orders in the Taylor expansion are often needed to obtain a good approximation of the distribution, thus advocating in favor of higher-order expansions.

The paper is organized as follows. The first section presents our benchmark model. The second section discusses the perturbation method and proposes an extension to account for a pseudo stochastic steady state. The third section presents accuracy results. A last section offers some concluding remarks.

1. A benchmark asset pricing model

This section describes the basic theoretical framework that we use to evaluate the accuracy of Taylor series expansion approximation method. The model builds heavily on previous work by Burnside (1998). The model is a standard asset pricing model for which (i) the marginal intertemporal rate of substitution is an exponential function of the rate of growth of consumption and (ii) the endowment is a Gaussian exogenous process. As shown by Burnside (1998), this setting permits to obtain a closed-form solution to the problem. We consider a frictionless pure exchange economy à la Mehra and Prescott (1985) and Rietz (1988) with a single household and a unique perishable consumption good produced by a single 'tree'. The household can hold equity shares to transfer wealth from one period to another. The problem of a single agent is then to choose consumption and equity holdings to maximize her expected discounted stream of utility, given by[2]

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{c_{t+\tau}^{\theta}}{\theta} \quad \text{with } \theta \in (-\infty, 0) \cup (0, 1] \tag{1}$$

subject to the budget constraint

$$p_t e_{t+1} + c_t = (p_t + d_t) e_t \tag{2}$$

where $\beta \in (0,1)$ is the agent's subjective discount factor, c_t is household's consumption of a single perishable good at date t, p_t denotes the price of the equity in period t and e_t is the household's equity holdings in period t. Finally, d_t is the tree's dividend in period t. Dividends are assumed to grow at rate x_t such that

$$d_t = \exp(x_t) d_{t-1} \tag{3}$$

where x_t, the rate of growth of dividends, is assumed to be a Gaussian stationary AR(1) process

$$x_t = (1-\rho)\bar{x} + \rho x_{t-1} + \varepsilon_t \tag{4}$$

[2] Hereafter, $E_t(\cdot)$ denotes the mathematical expectation operator conditional on information in period t, such that $E_t(x_{t+j}) = \int_S x_{t+j}(s)\eta(s|s_t)\,\mathrm{d}s$ where $\eta(s|s_t)$ is the pdf of $s_{t+j} \in S$ conditional on $s_t \in S$.

where ε is i.i.d. $\mathcal{N}(0,\sigma^2)$ with $|\rho|<1$. The first order condition for the household's problem is then given by

$$p_t c_t^{\theta-1} = \beta E_t[c_{t+1}^{\theta-1}(p_{t+1} + d_{t+1})]. \tag{5}$$

Market clearing requires that $e_t = 1$ so that $c_t = d_t$ in equilibrium. This yields to

$$p_t = \beta E_t\left[\left(\frac{d_{t+1}}{d_t}\right)^{\theta-1}(p_{t+1} + d_{t+1})\right]. \tag{6}$$

Like in Burnside (1998), let y_t denote the price-dividend ratio (hereafter PDR), $y_t = p_t/d_t$. Then, plugging (3) in (6) we get

$$y_t = \beta E_t[\exp(\theta x_{t+1})(1 + y_{t+1})]. \tag{7}$$

Burnside (1998) shows that the above equation admits an exact solution of the form[3]

$$y_t = \sum_{i=1}^{\infty} \beta^i \exp[a_i + b_i(x_t - \bar{x})] \tag{8}$$

where

$$a_i = \theta\bar{x}i + \frac{\theta^2\sigma^2}{2(1-\rho)^2}\left[i - \frac{2\rho(1-\rho^i)}{1-\rho} + \frac{\rho^2(1-\rho^{2i})}{1-\rho^2}\right]$$

and

$$b_i = \frac{\theta\rho(1-\rho^i)}{1-\rho}$$

Simple algebra shows that the expected value of the PDR is then given by

$$E_{t-1}[y_t] = \sum_{i=1}^{\infty} \beta^i \exp\left(a_i + \frac{b_i^2\sigma^2}{2} + b_i\rho(x_{t-1} - \bar{x})\right) \tag{9}$$

Therefore, the innovation to the PDR is

$$\begin{aligned}\zeta_t &= y_t - E_{t-1}[y_t]\\ &= \sum_{i=1}^{\infty} \beta^i \exp(a_i + b_i\rho(x_{t-1} - \bar{x}))\left(\exp(b_i\varepsilon_t) - \exp\left(\frac{b_i^2\sigma^2}{2}\right)\right).\end{aligned} \tag{10}$$

This fully worked out analytical solution thus provides a benchmark against which approximated solutions can be checked. It is worth noting that contrary to other models admitting an analytical solution (see, e.g. McCallum (1989) (in its simplest version) or Hercowitz and Sampson (1991)), Burnside's (1998) model

[3] See Burnside (1998) for a detailed exposition of the solution method.

does not make the certainty equivalent hypothesis, thus implying that higher-order moments (here variance) affects the decision rules. Therefore, as a benchmark model, it allows us to check the ability of a given approximation method to capture higher-order phenomena.[4] Further, as can be seen from Eq. (10), the innovation to the PDR process displays heteroskedasticity. Therefore, to be considered as accurate, any approximation should mimic this feature, and this will be a dimension along which the approximation method will be tested.

2. Perturbation methods

This section presents the perturbation method advocated by Judd (1996), and proposes an extension of the method to accommodate stochastic rational expectations models.

2.1. The perturbation method

Many economists facing the previous problem, without any knowledge of the true solution would probably use a linearization method to get insights on the properties of the solution. This method would amount actually to taking a first-order Taylor series expansion of the model around the deterministic steady state. To do that let us rewrite the model as

$$x_t = h(x_{t-1}, \varepsilon_t), \tag{11}$$

$$y_t = E_t[g(y_{t+1}, x_{t+1})], \tag{12}$$

where $g(y,x) = \beta \exp(\theta x)(1+y)$ and $h(x,\varepsilon) = \rho x + (1-\rho)\bar{x} + \varepsilon$. In this setting, the deterministic steady state is given by

$$\begin{aligned} &x^* = \rho x^* + (1-\rho)\bar{x} \Leftrightarrow x^* = \bar{x}, \\ &y^* = g(y^*, x^*) \Leftrightarrow y^* = \beta \exp(\theta\bar{x})/(1-\beta\exp(\theta\bar{x})). \end{aligned} \tag{13}$$

The solution to (12) is a function $f(\cdot)$ such that $y_t = f(x_t)$. Then (12) can be rewritten as

$$f(x_t) = E_t[g(f(h(x_t, \varepsilon_{t+1})), \quad h(x_t, \varepsilon_{t+1}))] = E_t[G(x_t, \varepsilon_{t+1})]. \tag{14}$$

Let us take a first-order Taylor series expansion to the previous equation, in order to compute the integral involved by the rational expectations hypothesis

$$f_0 + f_1 \hat{x}_t = E_t[G_{0,0} + G_{1,0}\hat{x}_t + G_{0,1}\hat{\varepsilon}_{t+1}]$$

where $f_k, k = 0, 1$, denotes $(\mathrm{d}^k f(x_t)/\mathrm{d}x_t^k)|_{x_t = x^*}$ and $G_{i,j} = (\partial^{i+j} G(x,\varepsilon)/\partial x^i \partial \varepsilon^j)|_{x=x^*,\, \varepsilon=0}$.

[4] We will essentially restrict ourselves to solution for which $\theta \neq 0$, as in the latter case the solution is simply given by $y_t = \beta/(1-\beta)$.

As ε_{t+1} is assumed to be a zero mean i.i.d. process, $E_t\hat{\varepsilon}_{t+1} = 0$, so that the previous relation reduces to

$$f_0 + f_1\hat{x}_t = [G_{0,0} + G_{1,0}\hat{x}_t].$$

Solving for f_0 and f_1 we get[5]

$$f_0 = G_{0,0} = g(y^*, x^*) = y^* \quad \text{and} \quad f_1 = G_{1,0}.$$

As we have

$$\begin{aligned} G_{1,0} &= f_1 \left.\frac{\partial g(y,x)}{\partial y}\right|_{\substack{y=y^*\\x=x^*}} \left.\frac{\partial h(x,\varepsilon)}{\partial x}\right|_{\substack{x=x^*\\ \varepsilon=0}} + \left.\frac{\partial g(y,x)}{\partial x}\right|_{\substack{y=y^*\\x=x^*}} \left.\frac{\partial h(x,\varepsilon)}{\partial x}\right|_{\substack{x=x^*\\ \varepsilon=0}} \\ &= f_1\rho\beta\exp(\theta\bar{x}) + \rho\theta\beta\exp(\theta\bar{x})(1 + f_0) \end{aligned}$$

we get

$$f_1 = \frac{\rho\theta\beta\exp(\theta\bar{x})(1 + y^*)}{1 - \rho\beta\exp(\theta\bar{x})}.$$

This first-order perturbation method exactly corresponds to the solution obtained by standard linearization of first-order conditions. This simple example thus highlights that the common local approach to solving dynamic system using linearization is just a particular case of a perturbation method. However, one well known drawback of such a solution, especially in the case of asset pricing models, is that it does not take advantage of any piece of information contained in the distribution of the shocks. In particular, assuming certainty equivalence, this solution leads to a solution that ignores risk. Therefore, following Judd (1996), we propose to extend the perturbation method and propose a modification to account for higher moments of the distribution, thus allowing us to obtain a solution that fundamentally depends on the level of risk the household faces.

Before presenting the general case, we first propose to expose the higher-order stochastic perturbation method at order 2. It should be clear to the reader that, as higher-order moments will be taken into account, values for f_0 and f_1 will be affected. More particularly, both of them will now depend on volatilities.

The method we now present essentially amounts to taking a second-order Taylor series expansion of (14) around $x_t = x^* = \bar{x}$ and $\varepsilon_{t+1} = 0$. This yields

$$\begin{aligned} &f_0 + f_1\hat{x}_t + \tfrac{1}{2}f_2\hat{x}_t^2 \\ &\quad = E_t[G_{0,0} + G_{1,0}\hat{x}_t + G_{0,1}\hat{\varepsilon}_{t+1} + \tfrac{1}{2}G_{2,0}\hat{x}_t^2 \\ &\qquad + \tfrac{1}{2}G_{0,2}\hat{\varepsilon}_{t+1}^2 + G_{1,1}\hat{\varepsilon}_{t+1}\hat{x}_t]. \end{aligned}$$

[5] It is worth noting that the problem is considerably simplified in our case, as the only state variable is exogenous. Nevertheless, if the state variable were endogenous, the problem would only amount to solve a Riccati equation.

As ε_t is a Gaussian white noise process, $E_t\hat{\varepsilon}_{t+1} = E_t\hat{x}_t\hat{\varepsilon}_{t+1} = 0$ and $E_t\hat{\varepsilon}^2_{t+1} = \sigma^2$, such that the preceding equation reduces to

$$f_0 + f_1\hat{x}_t + \tfrac{1}{2}f_2\hat{x}_t^2 = G_{0,0} + (\sigma^2/2)G_{0,2} + G_{1,0}\hat{x}_t + \tfrac{1}{2}G_{2,0}\hat{x}_t^2.$$

Plugging the expressions of $G_{i,j}$ in the above equation and identifying term by term, we end up solving the following linear system for $f_k, k = 0, 1, 2$:

$$f_0 = \beta\exp(\theta\bar{x})\left[\left(1 + \frac{(\theta\rho\sigma)^2}{2}\right)(1 + f_0) + \frac{(\rho\sigma)^2}{2}(2\theta f_1 + f_2)\right],$$

$$f_1 = \rho\beta\exp(\theta\bar{x})(f_1 + \theta(1 + f_0)),$$

$$f_2 = \rho^2\beta\exp(\rho\bar{x})(\theta^2(1 + f_0) + 2\theta f_1 + f_2).$$

Two facts are worth noting. The first one is that extending the order of the Taylor approximation implies that the solution can now better handle the curvature of the true decision rule, through the quadratic term. This turns out to be of importance as soon as the curvature of the utility function is large or whenever the individual faces highly volatile shocks. But more important is the fact that the level of the rule now fundamentally depends on the risk, as the constant term in the solution, f_0, now takes the volatility of the shocks that hit the economy into account (first equation of the above system). This essentially reflects the abandonment of the certainty equivalence principle in approximating the solution, which implies that even when the shock is set to its mean, the level of the PDR is not equal to its steady-state level but rather depends on the volatility of the shock, and therefore takes the risk into account. In a sense, the solution takes the stochastic dimension of the problem into account. Obviously, higher-order moments, such as skewness or kurtosis, may matter for obtaining of an accurate solution — especially in finance — we therefore propose to extend the method to higher-order terms. This can be easily achieved in this specific problem, as the following section demonstrates.[6]

2.2. *A higher-order stochastic perturbation method*

We now propose to extend the perturbation method to take advantage of some information carried by the higher moments of the distribution. This actually amounts to expanding the first-order condition of the model with respect to both x_t and ε_t and exploiting the higher moments of ε_t. Then, the

[6] The interested reader can refer to Collard and Juillard (1999) for the general case.

nth-order approximation is then determined by

$$\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(x_t)|_{x_t=x}\ \hat{x}_t^k = E_t\left[\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}G_{k-j,j}\ \hat{x}_t^{k-j}\hat{\varepsilon}_{t+1}^j\right] \tag{15}$$

$$=\left[\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}G_{k-j,j}\ \hat{x}_t^{k-j}E_t\hat{\varepsilon}_{t+1}^j\right] \tag{16}$$

$$=\left[\sum_{k=0}^{n}\frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j}G_{k-j,j}\ \hat{x}_t^{k-j}\mu_j\right] \tag{17}$$

where $G_{k-j,j}$ denotes $(\partial^k G(x_t,\varepsilon_{t+1})/\partial x_t^{k-j}\partial\varepsilon_{t+1}^j)|_{x_t=x,\,\varepsilon_{t+1}=0}$ and $\mu_j = E_t\hat{\varepsilon}_{t+1}^j$.

Identifying terms by terms, and after some tedious accounting, we obtain a system of n equations:

$$\frac{1}{k!}f_k = \sum_{j=0}^{n-k}\frac{1}{(j+k)!}\binom{k}{j}G_{k,j}\mu_j \quad \text{for } k=0,\ldots,n \tag{18}$$

It shall be clear that each $G_{k,j}$ is a function of f_k, $k=0,\ldots,n$, as in our case

$$G_{k,j} = \beta\exp(\theta x)\rho^k\left[\theta^{j+k} + \sum_{\ell=0}^{j+k}\binom{j+k}{\ell}\theta^{j+k-\ell}f_\ell\right] \tag{19}$$

where $x_{t+1} = h(x_t,\varepsilon_{t+1})$. Therefore, (18) together with (19) defines a linear system[7] that should be solved for f_k, $k=0,\ldots,n$:

$$f_0 = \beta\exp(\theta x)\sum_{j=0}^{n}\frac{1}{j!}\left[\theta^j + \sum_{\ell=0}^{j}\binom{j}{\ell}\theta^{j-\ell}f_\ell\right]\mu_j\,,$$

$$\vdots$$

$$\frac{1}{k!}f_k = \sum_{j=0}^{n-k}\frac{1}{(j+k)!}\binom{k}{j}\beta\exp(\theta x)\rho^k\left[\theta^{j+k} + \sum_{\ell=0}^{j+k}\binom{j+k}{\ell}\theta^{j+k-\ell}f_\ell\right]\mu j,$$

$$\vdots$$

$$f_n = \beta\exp(\theta x)\rho^n\left[\theta^n + \sum_{\ell=0}^{n}\binom{n}{\ell}\theta^{n-\ell}f_\ell\right]. \tag{20}$$

It is important to note that contrary to the certainty equivalent solution, the solution of this system does depend on higher moments of the distribution of the exogenous shocks, μ_j. Therefore, the properties of the decision rule (slope and

[7] It is worth noting that this approximation is particularly simple in our showcase economy, as the solution only depends on an exogenous variable. In the general case, the solution will depend on endogenous variables and will involve solving a Riccati equation for the first-order approximation. In this 'stochastic approximation method', the linear coefficients depend on higher-orders

curvature) will depend on these moments. But more importantly, the level of the rule will differ from the certainty equivalent solution, such that the steady-state level, y^*, is not given anymore by (13) but will be affected by the higher moments of the distribution of the shocks (essentially the volatility for our model as the shocks are assumed to be normally distributed).

Likewise for the true solution, we can compute the expected value of the price–dividend ratio and its innovation:

$$E_{t-1}[\tilde{y}_t^n] = \sum_{k=0}^{n} (\rho\hat{x}_{t-1})^k \sum_{i=k+1}^{n} \frac{1}{i!}\binom{k}{i} f_i E_{t-1}[\hat{\varepsilon}_t^{i-k}], \tag{21}$$

$$\tilde{\zeta}_t^n = \tilde{y}_t^n - \mathrm{E}_{t-1}[\tilde{y}_t^n] = \sum_{k=0}^{n} (\rho\hat{x}_{t-1})^k \sum_{i=k+1}^{n} \frac{1}{i!}\binom{k}{i} f_i\, (\hat{\varepsilon}_t^{i-k} - \mu_{i-k}). \tag{22}$$

$\tilde{\zeta}_t^n$ denotes the innovation of the approximated PDR. It is worth noting that likewise in the theoretical model, this innovation displays strong heteroskedasticity, that essentially comes from the fact that this method takes information carried by higher-order moments of the distributions of the exogenous shocks into account.

3. Accuracy check

This section checks the accuracy of the method presented above. As the model possesses a closed-form solution, we can directly check the accuracy of each solution we propose against this 'true' solution. We first present the evaluation criteria that will be used to check the accuracy. We then conduct the evaluation of the approximation methods under study. For benchmark purposes, we also

Footnote 7 continued

coefficients, which complicates the resolution. We thus propose a fixed-point algorithm for the general case (see Collard and Juillard (1999) for a detailed exposition)

1. solve the certainty equivalent problem, which is fully recursive;
2. solve the Riccati equation associated to the stochastic problem, using the certainty equivalent solution for the f_k's, $k > 1$;
3. solve for the f_k, $k \neq 1$
4. if the f_k do not change from one iteration to the other then stop; else go back to step 2, using the new solution as an initial condition.

This procedure may be viewed as a generalization of Hall and Stephenson (1990) procedure for optimal control problems in that it can be applied to higher-order moments. Moreover, it is important to note that one may not be interested in evaluating the model around steady state (when estimating the model for example), such that the procedure may be simplified (we are thankful to a referee for mentioning this point).

report results obtained using the spectral Galerkin approach, using fourth-order Chebychev polynomials (see Judd (1992, 1996) for a detailed exposition of the Galerkin method).[8] The merit of this method essentially stems from its global feature, and thus allow to evaluate the loss of only relying on a local approach.

3.1. Criteria

Several criteria are considered to tackle the question of the accuracy of the different approximation methods under study. As the model admits a closed-form solution, the accuracy of the approximation method can be directly checked against the 'true' decision rule. This is undertaken relying on the two following criteria:

$$E_1 = 100 \times \frac{1}{N} \sum_{t=1}^{N} \left| \frac{y_t - \tilde{y}_t}{y_t} \right|$$

and

$$E_\infty = 100 \times \max \left\{ \left| \frac{y_t - \tilde{y}_t}{y_t} \right| \right\}$$

where y_t denotes the true solution to price-dividend ratio and $\tilde{y}_t$ is the approximation of the true solution by the method under study. E_1 represents the average relative error an agent makes using the approximation rather than the true solution, while E_∞ is the maximal relative error made using the approximation rather than the true solution. These criteria are evaluated over the interval $x_t \in [\bar{x} - \Delta\sigma_x, \bar{x} + \Delta\sigma_x]$ where Δ is selected such that we explore 99.99% of the distribution of x.

However, one may be more concerned — especially in finance — with the ability of the approximation method to account for the distribution of the PDR, and therefore the moments of the distribution. We first compute the mean of y_t for different calibration and different approximation methods. Further, we explore the stochastic properties of the innovation of y_t: $\zeta_t = y_t - E_{t-1}(y_t)$, in order to assess the ability of each approximation method to account for the internal stochastic properties of the model. We thus report standard deviation, skewness and kurtosis of ζ_t, which provides information on the ability of the model to capture the heteroskedasticity of the innovation and more importantly the potential non-linearities the model can generate.[9]

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[8] This choice was made in order to make things comparable. Higher accuracy may obviously be obtained for higher-order polynomials in some cases.

[9] The cdf of ζ_t is computed using 20 000 replications of Monte–Carlo simulations of 1000 observations for ζ_t.

3.2. Approximation errors

Table 1 reports E_1 and E_∞ for the different cases and different approximation methods under study. Hereafter, these methods will be noted as follows:

- O1 denotes the approximation of order 1 (linear);
- O2 denotes the stochastic approximation of order 2;
- O4 denotes the stochastic approximation of order 4;
- G4 denotes the Galerkin approximation, where fourth-order Chebychev polynomials are used, with 100 nodes. The integral involved by the rational expectations hypothesis is approximated using a Gauss–Hermite quadrature, with 20 nodes.

We also consider different cases. Our benchmark experiment amounts to considering the Mehra and Prescott's (1985) parameterization of the asset pricing model. We therefore set the mean of the rate of growth of dividend to $\bar{x} = 0.0179$, its persistence to $\rho = -0.139$ and the volatility of the innovations to $\sigma = 0.0348$. These values are consistent with the properties of consumption growth in annual data from 1889 to 1979. θ was set to -1.5, the value widely used in the literature, and β to 0.95, which is standard for annual frequency. We then investigate the implications of changes in these parameters in terms of accuracy. In particular, we study the implications of larger and lower impatience, higher volatility, larger curvature of the utility function and more persistence in the rate of growth of dividends.

Table 1
Accuracy check[a]

	O1		O2		O4		G4	
Case	E_1	E_∞	E_1	E_∞	E_1	E_∞	E_1	E_∞
Benchmark	1.43	1.46	0.00	0.00	0.00	0.00	0.00	0.00
$\beta = 0.5$	0.24	0.26	0.00	0.00	0.00	0.00	0.00	0.00
$\beta = 0.99$	2.92	2.94	0.00	0.00	0.00	0.00	0.00	0.00
$\theta = -10$	23.53	24.47	0.49	1.11	0.01	0.02	0.00	0.00
$\theta = -5$	8.57	8.85	0.06	0.14	0.00	0.00	0.00	0.00
$\theta = 0$	0.50	0.51	0.00	0.00	0.00	0.00	0.00	0.00
$\theta = 0.5$	0.29	0.29	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.001$	0.01	0.03	0.00	0.00	0.00	0.00	0.00	0.00
$\sigma = 0.1$	11.70	11.72	0.03	0.05	0.00	0.00	0.00	0.00
$\rho = 0$	1.57	1.57	0.00	0.00	0.00	0.00	0.00	0.00
$\rho = 0.5$	5.52	6.76	0.11	0.27	0.00	0.00	0.00	0.00
$\rho = 0.9$	37.50	118.94	15.88	73.52	2.16	14.24	0.73	1.24

[a] The series defining the true solution was truncated after 800 terms, as no significant improvement was found adding additional terms at the machine accuracy. When exploring variations in ρ, the overall volatility of the rate of growth of dividends was maintained to its benchmark level.

At a first glance at Table 1, it appears that linear approximation can only accommodate situations where the economy does not experiment high volatility or large persistence of the growth of dividends, or where the utility of individuals does not exhibit much curvature. This is for instance the case in the Mehra and Prescott's (1985) parameterization (benchmark) case as both the average and maximal approximation error lie around 1.5%. But, as is nowadays well-known, increases along one of the aforementioned dimension yields lower accuracy of the linear approximation. For instance, increasing the volatility of the innovations of the rate of growth of dividends to $\sigma = 0.1$ yields approximation errors of almost 12% both in average and at the maximum, thus indicating that the approximation performs particularly badly in this case. This is even worse when the persistence of the exogenous process increases, as $\rho = 0.9$ yields an average approximation error of about 40% and a maximal approximation of about 120%. This is also true for increases in the curvature of the utility function (see row 4 and 5 of Table 1).

If we now apply higher-order Taylor expansions as proposed in Section 2.2, the gains in accuracy are particularly significant. In most of the cases under study, the approximation error is reduced to less than 0.01%. For instance, in the benchmark case, the linear approximation led to a 1.5% error, the O2 approximation leads to a less than 0.001% error and is essentially zero in the O4 approximation. This error is even lower for low volatility economies, and is only 0.03% in the case of $\sigma = 0.1$ for O2 and 0 for O4 to be compared to more than 10% in the linear case. The gain is also important for low degrees of intertemporal substitution as the error is less than 1% for $\theta = -10$ compared to the 25% previously obtained. Hence, the O2 procedure can reduce the error by 98% (97%) in terms of average (maximal) approximation error compared to the linear approximation while the O4 procedure yields a reduction of 99.9% (99.9%).

The potential gains of taking higher-order approximations are perfectly illustrated considering the case $\rho = 0.9$, as the average (and maximal) approximation errors drop from 16% (74%) to 2% (14%) moving from a second-order approximation to a fourth-order approximation. Indeed, for highly persistent shocks, not only does the second-order approximation perform poorly in terms of curvature approximation, but the higher-order moments of the distribution (such as kurtosis) are not taken into account when computing the steady state.[10]

Fig. 1 sheds light on these results, as it reports the decision rules for the PDR when $\theta = -5$ and $p = 0.5$. In this graph, DO2 and DO4 correspond to situations where we did not take the moments of the distribution into account. This essentially amounts to setting the moments of the distribution of ε to zero.

[10] This will be illustrated in the next section where we investigate the properties of the approximation methods in terms of distribution of the PDR.

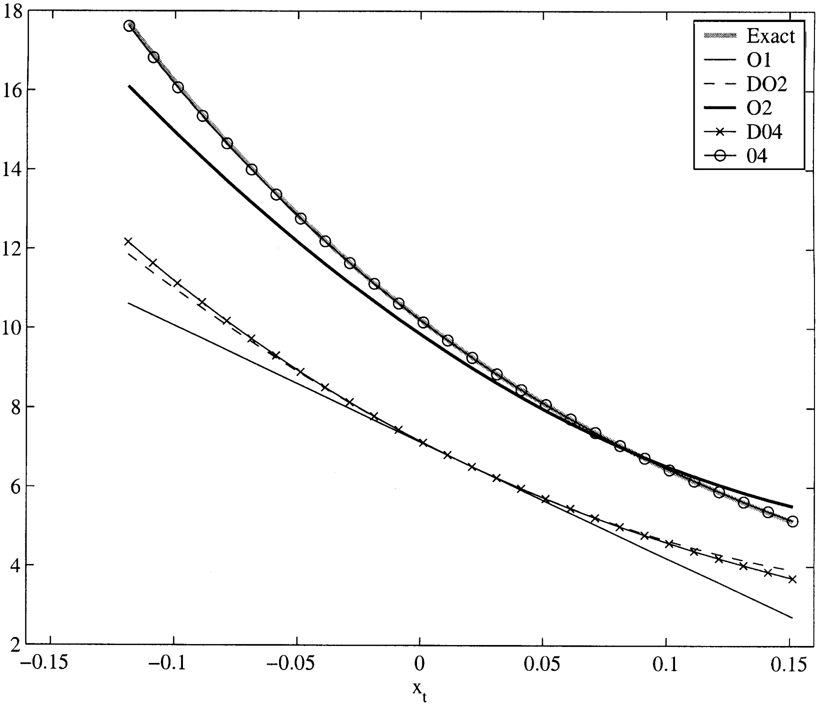

Fig. 1.

In such a case, the first-order moment of the PDR distribution corresponds to the deterministic steady-state value of the problem. Therefore, DO2 and DO4 both assume that certainty equivalence holds. As can be seen from Fig. 1, the major gains from moving from a linear to a 'non-stochastic' higher-order Taylor expansion (DO2 and DO4) is found in the ability of the latter to take care of the curvature of the decision rule. But this is not a sufficient improvement as far as accuracy of the solution is concerned. This is confirmed by Tables 2 and 3 which report the decision rules and the accuracy measures for the benchmark, $\theta = -10$ and $\rho = 0.9$ cases. Both tables indicate that the average (maximal) approximation error is of 23% (23%) in the DO2 and DO4 cases to be compared to the 24% (24%) associated to the linear approximation. In others, increasing the order of the Taylor approximation, without taking into account the information carried by the moments of the distribution — ignoring the stochastic dimension of the problem — does not add that much in terms of accuracy. The main improvement can be precisely found in adding the higher-order moments, as it yields a correction in the level of the rule, as can be seen from Fig. 1. As soon as the volatility of the shock explicitly appears in the rule

Table 2
Decision rules of the PDR (Order 2)

Case	Approx.	f_0	f_1	f_2	E_1	E_∞
Benchmark	O2	12.48	2.30	0.43	0.00	0.00
	DO2	12.30	2.27	0.42	1.42	1.42
$\theta = -10$	O2	5.00	5.97	7.50	0.49	1.11
	DO2	3.86	4.83	6.07	23.14	23.16
$\rho = 0.9$	O2	14.50	− 115.40	1137.81	15.88	73.52
	DO2	12.30	− 99.07	976.84	20.73	46.03

Table 3
Decision rules of the PDR (Order 4)

Case	Approx.	f_0	f_1	f_2	f_3	f_4	E_1	E_∞
Benchmark	O4	12.48	2.31	0.43	0.08	0.01	0.00	0.00
	DO4	12.30	2.27	0.42	0.08	0.01	1.42	1.42
$\theta = -10$	O4	5.02	6.25	7.81	9.33	11.73	0.01	0.02
	DO4	3.86	4.83	6.07	7.65	9.67	23.14	23.20
$\rho = 0.9$	O4	14.94	− 131.55	1383.07	− 14 828.50	168 970.45	2.16	14.24
	DO4	12.30	− 99.07	976.84	− 10 573.35	120 413.03	17.59	27.94

(O2) then the rule shifts upward, thus getting closer to the true solution of the problem. Then, the average (maximal) approximation error decreases to 0.5% (1.1%) to be compared to the earlier 23% (23%). Therefore, most of the approximation error essentially lies in the level of the rule rather than in its curvature. The gain is even greater whenever skewness and kurtosis are taken into account by the O4 approximation, as the average and maximal approximation errors reduce to respectively 0.01% and 0.02%. Therefore, a higher curvature in the utility function necessitates an increase in the order of the moments of the distribution that should be taken into account in the approximate solution.

There may however be situations where increasing the curvature constitutes per se a real improvement that contributes to reduce the approximation error. This is illustrated by considering the high persistence case ($\rho = 0.9$). Indeed, as shown in Tables 2 and 3, increasing the order of the Taylor series expansion from 2 to 4 — although not taking into account the stochastic component (DO2 and DO4) — leads to a decrease in the average (maximal) approximation error of about 15% (40%). In others, for highly persistent shocks, the curvature of the rule plays an important rule. However, the gains are even greater in the 'stochastic' approximation, as moving from O2 to O4 yields an average (maximal) accuracy gain of 86% (81%).

However, Tables 2 and 3 also reveal another interesting feature. Not only the constant term of the rule is affected by taking the higher order moments of the distribution of exogenous shocks, but it also affects the slope and the curvature of the rule. For instance, moving from order 2 to order 4 leads to a 0.5% increase in f_0 in the case $\theta = -10$, and yields a 5% increase in f_1 and a 4.5% in f_2. Changes are greater for the highly persistent case as f_0 and f_2 increase by respectively 3 and 22% while f_1 decreases by 14%. In others, skewness and kurtosis affect not only the level but also the form of the decision rule. Conversely, such changes do not take place when the certainty equivalence is assumed. Therefore, changes are not triggered by increasing the order of the approximation per se, but rather by taking into account higher-order moments of the distribution of exogenous shocks. This may be understood by noting that in system (20) every moment of the distribution potentially affects the determination of each f_k. In our case, and because of the Gaussian assumption, this is the standard deviation of the shock that will affect each f_k (as any moment of order greater than 2 can be written as a function of σ).

To sum up, it appears that increasing the order of approximation and further considering information carried by the higher-order moments of the distribution yields great improvements in the accuracy of the approximation, mainly because higher-order moments enhance the ability of the approximation to match the level of the decision rule. In such a situation, the gains from using a global method such as spectral Galerkin are not so clear. This is true for example for highly volatile economies. This would be even more pronounced if the state space was to be extended, as the Galerkin method would dramatically run into the curse of dimensionality problem. Further, the spectral Galerkin is extremely sensitive to initial conditions problem, as they rely on a non-linear equation solver. Extending the state space would thus require us to find a way of feeding the algorithm with good enough initial conditions — which is not always possible. However, problems still remain to be solved as the errors are still large for highly persistent economies. This can be easily explained if we consider the Taylor series expansion to the true decision rule, given by

$$y_t \simeq \sum_{i=0}^{\infty} \beta^i \exp(a_i) \left[\sum_{k=0}^{p} \frac{b_i^k}{k!} (x_t - \bar{x})^k \right].$$

Table 4 then reports the average and maximal errors induced by the Taylor expansion to the true rule. We only report these errors for cases where the previous analysis indicated a error greater than 1% for the O2 method. Table 4 clearly shows that approximation errors are large (greater than 1%) when a second-order Taylor series expansion to the true rule is already not sufficient to produce an accurate representation to the analytical solution. For instance, in the $\theta = -10$ case, a 'good' approximation of the true rule can be obtained only after a third-order Taylor series expansion. This indicates that we should use at

Table 4
Taylor series expansion to the true solution

		Order of Taylor series expansion					
Case	Crit.	1	2	3	4	10	12
Benchmark	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
$\beta = 0.99$	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
$\theta = -10$	$E_1(v)$	0.49	0.02	0.00	0.00	0.00	0.00
	$E_\infty(v)$	1.60	0.09	0.00	0.00	0.00	0.00
$\theta = -5$	$E_1(v)$	0.12	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.37	0.01	0.00	0.00	0.00	0.00
$\sigma = 0.1$	$E_1(v)$	0.01	0.00	0.00	0.00	0.00	0.00
	$E_\infty(v)$	0.03	0.00	0.00	0.00	0.00	0.00
$\rho = 0.5$	$E_1(v)$	0.63	0.03	0.00	0.00	0.00	0.00
	$E_\infty(v)$	2.05	0.14	0.01	0.00	0.00	0.00
$\rho = 0.9$	$E_1(v)$	32.30	13.25	4.49	1.29	0.00	0.00
	$E_\infty(v)$	154.99	87.24	37.15	12.75	0.00	0.00

least a third-order Taylor series expansion to increase significantly the accuracy of the approximation. This phenomenon is even more pronounced as we consider persistence. Let us focus on the case $\rho = 0.9$ for which approximation errors (see Table 1) are huge — more than 15% in the O2 approximation. As can be seen from Table 4, the second-order Taylor series expansion to the true rule is very inaccurate as it produces maximal errors around 87%. It is worth noting that the fourth-order Taylor approximation of the true rule yields very similar approximation errors to the ones we obtained relying on the O4 approximation method. This, in fact, highlights that the method is relevant as the approximation error is of the same order (and the same kind) as the one we would get approximating the 'true' rule at the same order. An order of 12 is actually needed to generate an accurate approximation of the true rule. This hence explains the poor approximation, even using a global approach, we obtained in the $\rho = 0.9$ case, and indicates that a much higher corrected Taylor series expansion is required to increase accuracy. This, thus does not invalidate the method but underlines one of its features, large orders can be necessary to get an accurate approximation. This is however a shared feature of most of approximation methods, as, if one is willing to get a level of accuracy of less than 0.1% both in average and the maximal error, Chebychev polynomials of order 11 are needed in a Galerkin approximation ($E_1 = 0.01\%$ and $E_\infty = 0.03\%$).

Thus, error approximation results indicate that this perturbation procedure is accurate for most of the cases under study. But they also provide an explanation, and thus a solution, when the approximation is not accurate. However, economists are more often interested in implications of the model rather than the true solution. We now investigate the errors on selected moments we generate using the proposed approximating procedure.

3.3. Matching the true distribution

In order to locate the source of deviations from the true distribution of the PDR, we report in Tables 5–8 moments of order 1 to 4 for the 'true' and approximated distributions.

A first striking result from Table 5 is that, as expected, all certainty equivalent methods fail to account for the mean of the PDR distribution. However, as soon as one increases the order of the approximation, even to order 2, then the ability to match the mean of the distribution is greatly enhanced. For instance, in the case $\theta = -10$, O2 underestimates the mean by only 0.41%, and the bias is null in the cases of O4 or G4. Going to order 4 with the perturbation method even outperforms the Galerkin method for $\rho = 0.9$ as the O4 method underestimates the mean by 0.39% compared to 12.20% for the fourth-order Galerkin approximation (G4). From a general point of view, higher-order stochastic perturbation methods seem to perform better in terms of moment matching than the Galerkin approach. The explanation for this result may lie in the fact that, in this particular model, we do not need to rely to numerical integration when computing the solution by the perturbation method (as we only have to compute the expectations of powers of $\varepsilon_t \rightsquigarrow \mathcal{N}(0,\sigma)$), whereas we need to use Gaussian quadrature in the Galerkin method.

As can be seen from Table 6, results are fairly similar for the standard deviation of the distribution of the innovation of the rule ζ_t. Let us first recall that, as Section 1 has showed, this innovation is fundamentally heteroskedastic,

Table 5
Mean of the distribution

	True	O1	O2	O4	G4
Benchmark	12.4816	12.3036	12.4815	12.4816	12.4816
$\beta = 0.5$	0.9506	0.9485	0.9506	0.9506	0.9506
$\beta = 0.99$	27.4052	26.6069	27.4050	27.4052	27.4052
$\theta = -10$	5.0290	3.8614	5.0084	5.0288	5.0291
$\theta = -5$	7.2798	6.6627	7.2767	7.2798	7.2798
$\theta = 0$	19.0000	19.0955	18.9996	19.0000	19.0000
$\theta = 0.5$	23.1860	23.1200	23.1860	23.1860	23.1860
$\sigma = 0.001$	12.3037	12.3043	12.3037	12.3037	12.3037
$\sigma = 0.1$	13.9349	12.3037	13.9307	13.9349	13.9349
$\rho = 0$	12.5350	12.3386	12.5348	12.5350	12.5350
$\rho = 0.5$	12.9556	12.3024	12.9503	12.9555	12.9555
$\rho = 0.9$	15.8715	12.8822	15.2225	15.8092	13.9351

Table 6
Standard deviation of the distribution[a]

	True	O1	O2	O4	G4
Benchmark	0.0802	0.0791	0.0801	0.0802	0.0810
$\beta = 0.5$	0.0065	0.0064	0.0065	0.0065	0.0065
$\beta = 0.99$	0.1753	0.1702	0.1751	0.1753	0.1770
$\theta = -10$	0.2181	0.1681	0.2078	0.2178	0.2202
$\theta = -5$	0.1569	0.1443	0.1551	0.1568	0.1584
$\theta = 0$	—	—	—	—	—
$\theta = 0.5$	0.0495	0.0493	0.0495	0.0495	0.0499
$\sigma = 0.001$	0.0023	0.0023	0.0023	0.0023	0.0023
$\sigma = 0.1$	0.2571	0.2272	0.2550	0.2571	0.2596
$\rho = 0$	—	—	—	—	—
$\rho = 0.5$	0.5522	0.5222	0.5474	0.5522	0.6371
$\rho = 0.9$	2.3338	1.5972	1.8777	2.2876	1.6008

[a] $\theta = 0$ and $\rho = 0$ cases yield degenerated distributions.

such that its distribution may display strong non-linearities and may be difficult to account for. Significant enhancements of the ability of the approximation method to account for the standard deviation of ζ_t can only be obtained when the stochastic features of the model are taken into account. The case $\theta = -10$ illustrates the potential gains from using higher-orders stochastic perturbation methods, as the O2 method underestimates the standard deviation of the PDR by 5%, whereas the O4 method generates a bias of only 0.1%. This gain is even greater in the case of a highly persistent economy, as for $\rho = 0.9$, the O2 method underestimates the volatility by almost 20% while the bias associated to the O4 method drops to less than 2% (to be compared to the 31% of the G4 method). A potential explanation for such a result may be found in the ability of the methods to handle higher-order moments, as the main additional feature of O4 compared to the O2 method lies in the exploitation of the higher-order moments of the innovations in particular skewness and kurtosis. As aforementioned, this may be of importance to account for the heterosckedasticity of ζ_t.

A first glance at Table 7 indicates that the linear approximation obviously performs very badly in terms of skewness (this will also be the case for kurtosis as Table 8 will show) as this model does not accommodate any non-linearity. But, a striking result that emerges from Table 7 is that the perturbation method always outperforms the Galerkin method (G4) even for lower orders. For instance, the Galerkin method over-estimates the skewness of the distribution of ζ_t by 2% in the benchmark case whereas the O2 method generates the right skewness. Once again, a potential explanation for this results may lie in the fact that we do not need to rely on numerical integration in the stochastic

Table 7
Skewness of the distribution[a]

	True	O1	O2	O4	G4
Benchmark	0.0210	0.0018	0.0210	0.0210	0.0214
$\beta = 0.5$	0.0221	0.0018	0.0221	0.0221	0.0226
$\beta = 0.99$	0.0209	0.0018	0.0209	0.0209	0.0213
$\theta = -10$	0.1316	0.0018	0.1319	0.1316	0.1331
$\theta = -5$	0.0662	0.0018	0.0661	0.0662	0.0671
$\theta = 0$	—	—	—	—	—
$\theta = 0.5$	0.0046	− 0.0018	0.0046	0.0046	0.0044
$\sigma = 0.001$	0.0023	0.0018	0.0023	0.0023	0.0026
$\sigma = 0.1$	0.0569	0.0018	0.0569	0.0569	0.0577
$\rho = 0$	—	—	—	—	—
$\rho = 0.5$	0.1283	− 0.0018	0.1276	0.1283	0.1491
$\rho = 0.9$	0.5888	− 0.0018	0.4234	0.5605	− 0.7585

[a] $\theta = 0$ and $\rho = 0$ cases yield degenerated distributions.

Table 8
Kurtosis of the distribution[a]

	True	O1	O2	O4	G4
Benchmark	2.9913	2.9902	2.9911	2.9913	2.9900
$\beta = 0.5$	2.9914	2.9902	2.9912	2.9914	2.9902
$\beta = 0.99$	2.9913	2.9902	2.9911	2.9913	2.9900
$\theta = -10$	3.0236	2.9902	3.0162	3.0233	3.0227
$\theta = -5$	2.9992	2.9902	2.9973	2.9991	2.9980
$\theta = 0$	—	—	—	—	—
$\theta = 0.5$	2.9901	2.9902	2.9901	2.9901	2.9888
$\sigma = 0.001$	2.9902	2.9902	2.9902	2.9902	2.9889
$\sigma = 0.1$	2.9969	2.9902	2.9956	2.9969	2.9957
$\rho = 0$	—	—	—	—	—
$\rho = 0.5$	3.0256	2.9902	3.0176	3.0255	3.0321
$\rho = 0.9$	5.4110	2.9902	4.2099	5.1628	2.9101

[a] $\theta = 0$ and $\rho = 0$ cases yield degenerated distributions.

perturbation method, whereas we did in the Galerkin. An interesting feature that Table 7 reveals is that for high persistence ($\rho = 0.9$), increases in the order of the approximation yield significant gains. For instance, the bias associated to the skewness computed from the O4 approximation is only 4.8% compared to the 28% generated by the O2 method. In others, higher orders approximations reveal useful information on the distribution of the innovation of the rule (ζ_t). Another interesting feature is that the O4 method does a far better job than the G4 method, which underestimates the skewness of the distribution by 230%.[11]

The analysis of the kurtosis, reported in Table 8 lead to similar conclusions. Increasing the order of the approximation yield, as can be expected, strong bias reductions in the kurtosis compared to lower-order approximations. This implies that if one is interested in the *n*th-order moments of the distribution of the PDR, an, at least, *n*th order stochastic perturbation approximation is required to obtain accurate results. Further as before, the O4 method outperforms the G4 method, revealing that, in a sense, Taylor series expansions exploits the information contained in the successive derivatives of the Euler equation in a better way than the Galerkin approach — at least from a stochastic point of view.

A last point worth noting, that arises from these different experiments, is that, at least for the O4 case, the maximal bias generated on each moment is always lower than the maximal approximation error (Table 1), and is even lower than the average approximation error in the case of the mean and standard deviation. For instance, for a highly persistent economy ($\rho = 0.9$), the average and maximal approximation errors associated to the O4 method were 2% and 14%, while it yields to underestimate the mean and the standard deviation of the distribution by only 0.4% and 2%. Therefore, it appears that, contrary to the decision rule, high approximation orders are not necessarily needed to get accurate approximation of the distribution.

4. Concluding remarks

This paper has investigated the accuracy of a stochastic perturbation method to solve stochastic equilibrium models under rational expectations. As a benchmark model, we use Burnside's (1998) version of the asset-pricing model, which admits an closed-form solution. This allow us to check the accuracy of the method under study. We first show that standard linearization of first-order conditions is a particular case of general perturbation methods advocated by Judd (1996), and can thus be generalized to higher orders. The accuracy of the approximation is then checked against the closed-form solution. We show that the accuracy of second- and fourth-order Taylor series expansions, while being satisfactory in most of the cases, can largely be increased by explicitly exploiting the information contained in the distribution of the shocks. While being simple to implement, this modification of the perturbation method has proved to be efficient both in terms of accuracy and moment matching.

[11] In this case, only much higher Galerkin approximation yield a bias of order less than 1%.

References

Burnside, C., 1998. Solving asset pricing models with Gaussian shocks. Journal of Economic Dynamics and Control 22, 329–340.

Collard, F., Juillard, M., 1999. A higher-order Taylor expansion approach to simulation of stochastic forward-looking models with an application to a nonlinear Phillips curve model. Working Paper 99–19, Cepremap; Computational Economics, forthcoming.

Cooley, T.F., Prescott, E.C., 1995. Economic growth and the business cycle. In: Cooley, T. (Ed.), Frontiers of Business Cycle Research, Princeton University Press, Princeton, NJ, pp. 1–38 (Chapter 1).

Hall, S.G., Stephenson, M.J., 1990. An algorithm for the solution of stochastic optimal control problems for large nonlinear econometric models. Journal of Applied Econometrics 5, 393–399.

Hercowitz, Z., Sampson, M., 1991. Output growth, the real wage, and employment fluctuations. American Economic Review 81 (5), 1215–1237.

Judd, K.L., 1992. Projection methods for solving aggregate growth models. Journal of Economic Theory 58, 410–452.

Judd, K.L., 1996. Approximation, perturbation, and projection methods in economic analysis. In: Amman, H.M., Kendrick, D.A., Rust, J. (Eds.), Handbook of Computational Economics. North-Holland, Amsterdam, pp. 511–585 (Chapter 12).

Judd, K.L., 1998. Numerical Methods in Economics. MIT Press, Cambridge, MA.

Kydland, F.E., Prescott, E.C., 1982. Time-to-build and aggregrate fluctuations. Econometrica 50, 1345–1370.

Lucas, R.E., 1978. Asset prices in an exchange economy. Econometrica 46, 1429–1445.

Marcet, A., 1988. Solving nonlinear stochastic models by parameterizing expectations. Working paper, Carnegie Mellon University.

McCallum, B., 1989. Real business cycles models. In: Barro, R. (Ed.), Modern Business Cycle Theories. Harvard University Press, Boston, pp. 16–50.

Mehra, R., Prescott, E.C., 1985. The equity premium: a puzzle. Journal of Monetary Economics 15, 145–161.

Rietz, T.A., 1988. The equity risk premium: a solution. Journal of Monetary Economics 22, 117–131.

Tauchen, G., Hussey, R., 1991. Quadrature-based methods for obtaining approximate solutions to nonlinear asset pricing models. Econometrica 59, 371–396.