Analytic solution for Solow model with Cobb Douglas production

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The Solow (1956) with Cobb-Douglas production is known to have a general analytic solution. The solution method presented here follows Chiang and Wainwright (2005) and is intentionally pedestrian. The basic idea is to use a clever change of variables to transform equation

$$\dot{k} = sk(t)^{\alpha} - (n+g+\delta)k(t),\ k(0) = k_0 \tag{0.0.1}$$

into a linear, first-order differential equation which can be solved using standard methods.

Start by defining a new variable, $z(t)$, as follows.[1]

$$z(t) = \frac{k(t)}{y(t)} = k(t)^{1-\alpha} \tag{0.0.2}$$

Next, differentiate equation 0.0.2 with respect to t to obtain the following relationship between $\dot{z}$ and $\dot{k}$

$$\dot{z} = (1-\alpha)k(t)^{-\alpha}\dot{k} \implies \dot{k} = \dot{z}(1-\alpha)^{-1}k(t)^{\alpha} \tag{0.0.3}$$

which can be used to substitute for $\dot{k}$ in equation 0.0.1 in order to yield the following linear, first-order differential equation

$$\dot{z} + (n+g+\delta)(1-\alpha)z(t) = s(1-\alpha) \tag{0.0.4}$$

with $z(0) = k_0^{1-\alpha}$.

The solution to equation 0.0.4, which is a non-homogenous, first-order linear differential equation with constant coefficient and constant term, will consist of the sum of two terms called the complementary function, z_c and the particular integral, z_p, both of which have significant economic interpretation.

Mathematically, the complementary function, z_c, is simply the general solution of the following reduced form, homogenous version of equation 0.0.4.

$$\dot{z} + (n+g+\delta)(1-\alpha)z(t) = 0 \tag{0.0.5}$$

[1] This clever change of variables was originally published in Sato (1963).

Standard techniques for solving homogenous, first-order linear differential equations demonstrate that the general solution of equation 0.0.5 must be of the form

$$z_c = Ce^{-(n+g+\delta)(1-\alpha)t} \tag{0.0.6}$$

where C is some, as yet unknown, constant.

The particular integral, z_p, is any particular solution of 0.0.4. Suppose that $z(t)$ is some constant function. In this case $\dot{z} = 0$ and equation 0.0.4 becomes

$$z_p = \frac{s}{n+g+\delta} \tag{0.0.7}$$

which is a valid solution so long as $n + g + \delta \neq 0$.

The sum of the complementary function and the particular integral constitutes the general solution to equation 0.0.4.

$$z(t) = z_c + z_p = Ce^{-(n+g+\delta)(1-\alpha)t} + \left(\frac{s}{n+g+\delta}\right) \tag{0.0.8}$$

Using the initial condition, $z(0) = k_0^{1-\alpha}$, to solve for the constant C yields

$$C = k_0^{1-\alpha} - \left(\frac{s}{n+g+\delta}\right) \tag{0.0.9}$$

which can be combined with equation 0.0.8 to give the closed for solution for the capital-output ratio, $z(t)$.

$$z(t) = \left(\frac{s}{n+g+\delta}\right)\left(1 - e^{-(n+g+\delta)(1-\alpha)t}\right) + k_0^{1-\alpha}e^{-(n+g+\delta)(1-\alpha)t} \tag{0.0.10}$$

At this point it is worth digressing slightly to discuss the economic interpretation of the complementary function and the particular integral. The particular integral, z_p, is the inter-temporal equilibrium value for the capital-output ratio, $z(t)$, whilst the complementary function, z_c, represents deviations from this long-run equilibrium. Dynamic stability of $z(t)$ requires that deviations from equilibrium described by z_c die out as $t \to \infty$. In order for $\lim_{t\to\infty} z_c = 0$, I require that $(n + g + \delta)(1 - \alpha) > 0$.

Finally, from equation 0.0.10 it is straightforward to obtain a closed form solution for the time path of $k(t)$ by substituting $z(t) = k(t)^{1-\alpha}$ and then solving for $k(t)$.

$$k(t) = \left[\left(\frac{s}{n+g+\delta}\right)\left(1 - e^{-(n+g+\delta)(1-\alpha)t}\right) + k_0^{1-\alpha}e^{-(n+g+\delta)(1-\alpha)t}\right]^{\frac{1}{1-\alpha}} \tag{0.0.11}$$

References

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