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A Higher-Order Taylor Expansion Approach to Simulation of Stochastic Forward-Looking Models with an Application to a Nonlinear Phillips Curve Model

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Abstract. We propose to apply to the simulation of general nonlinear rational-expectation models a method where the expectation functions are approximated through a higher-order Taylor expansion. This method has been advocated by Judd (1998) and others for the simulation of stochastic optimal-control problems and we extend its application to more general cases. The coefficients for the first-order approximation of the expectation function are obtained using a generalized eigenvalue decomposition as it is usual for the simulation of linear rational-expectation models. Coefficients for higher-order terms in the Taylor expansion are then obtained by solving a succession of linear systems. In addition, we provide a method to reduce a bias in the computation of the stochastic equilibrium of such models. These procedures are made available in DYNARE, a MATLAB and GAUSS based simulation program. This method is then applied to the simulation of a macroeconomic model embodying a nonlinear Phillips curve. We show that in this case a quadratic approximation is sufficient, but different in important ways from the simulation of a linearized version of the model.

Key words: stochastic simulations, nonlinear models, rational-expectations

1. Introduction

Nonlinear stochastic models have unique features that make them important tools for economic analysis and policy evaluation. Whenever agents take risk into account or when one is interested in measuring welfare effects of policy change, nonlinearities arise (see, for example, Kim and Kim, 1999). Furthermore, models theoretically derived from intertemporal optimization behavior or macroeconmic models using the rational-expectation hypothesis make explicit use of future values of some variables.

However, the stochastic simulation of nonlinear models with forward-looking agents remains difficult in practice. Important contributions have been made for small models (see, for example, Marimon and Scott, 1999), but much remains to be done for medium to large ones. In this paper, we explore the possibility of

using a higher-order Taylor expansion to approximate the expectation function that would be used by rational agents. Such a perturbation method has been proposed by Gaspar and Judd (1997), Judd and Guu (1997), and Judd (1998) for perfect foresight and stochastic control problems. We extend this method to a general class of nonlinear rational-expectation models. The system of first-order conditions derived from a stochastic optimal-control problem is a particular case in the class of models that we study in this paper.

Although this technique involves tedious algebra, it has the advantage of requiring mostly standard mathematical operations. It is therefore well suited for a general computer program for the stochastic simulation of different types of nonlinear models, and we use it in the program DYNARE.[1]

As an illustration, we present the simulation of a macroeconomic model embodying the effects of a nonlinear Phillips curve. It is not a large model, but, with two stochastic shocks and six state variables, the simulation is not trivial to compute.

In the Section 2, we state the problem to be solved, fix notation, and outline our methodology. In the Section 3, we present the usual linearization of the problem through a first-order Taylor expansion. In Section 4, we extend the expansion to a quadratic approximation and discuss means for reducing a forecasting bias that arises in nonlinear stochastic models. In Section 5, we outline the steps needed to generalize the method to a Taylor expansion to any order. In Section 6, we present numerical results, and, in Section 7, we discuss problems that remain to be solved.

2. Rational Expectation Models and Rational Expectation Functions

Consider the following rational-expectation model:

$$E_t[f(y_{t+1}, y_t, x_t, x_{t-1}, u_t, u_{t+1})] = 0\,,$$

where f is a set of n equations, y is a vector of n_y variables about which agents form expectations, x is a vector of n_x predetermined variables, and u is a vector of s stochastic shocks. Furthermore, there must be sufficient equations to determine all x_t and y_t variables in the current period: $n = n_x + n_y$. In some models, as, for example, the stochastic growth model, future shocks appear explicitly. In applied macroeconomic models, this is usually not the case, but rather, it is the expectation of future values that appears.

In what follows, we assume that the stochastic shocks are normal i.i.d. with zero mean and variance/covariance matrix Σ. The assumption of normality is not necessary but makes computations easier. However, the assumption that the shocks are independently distributed through time is essential. This is not as constraining as it may seem. It merely means that any autocorrelation in the shocks must be explicitly modeled through the inclusion of appropriate predetermined variables.

At time t, it is assumed that agents know the value of the predetermined variables x_{t-1} in the previous period, and have observed the realization of the current

shocks u_t. Their decisions and actions are based on what they believe to be the future values of y_{t+1}, and they set the current values y_t and x_t. Of course, future values y_{t+1} are unknown at time t. The rational-expectation hypothesis requires only that no systematic error should be made or, equivalently, that the expectation of $f(\,)$ conditional on what is known at time t should be null.

This formulation is quite general provided that a model with leads and lags of more than one period be transformed into one with lags and leads of only one period through the use of auxiliary variables and corresponding equations. A variable appearing both with a lead and a lag should become two using an auxiliary variable and equation.

2.1. AN ILLUSTRATIVE EXAMPLE

Let's consider a small macroeconomic model:

$$v_t = 0.88v_{t-1} + 0.07r_{t-1} + \epsilon_{ut}$$

$$p_t = 0.5E_t(\pi_{t+1}) + 0.5p_{t-1} - 4.5\frac{v_t}{4 - v_t} + \epsilon_{p_t}$$

$$\pi_t = 100\left[\left(1 + \frac{p_t}{100}\right)\left(1 + \frac{p_{t-1}}{100}\right)\left(1 + \frac{p_{t-2}}{100}\right)\left(1 + \frac{p_{t-3}}{100}\right)\right]^{0.25} - 100$$

$$r_t = 0.5(\pi_t - \bar{\pi}),$$

where v_t is the difference between the observed unemployment rate and the natural rate, p_t is the inflation rate in each quarter and π_t its annual average, $\bar{\pi}$ is the annual inflation target pursued by the monetary authorities, and r_t is the difference between the rate of interest and an equilibrium rate compatible with full capacity utilization. For the simulation, we set the inflation target to 2% per year.

In this model, unemployment has strong inertia and increases when the interest rate rises. Inflation depends on past inflation, expectations about the future, and unemployment. The effect of unemployment is nonlinear: the lower the unemployment rate, the higher its effect on inflation. This effect becomes indefinitely large when the unemployment rate nears four percentage points below the natural rate. Annual inflation is the harmonic mean of quarterly inflation rates. Finally, monetary authorities 'lean against the wind'; they raise interest rates when inflation is above target and decrease them in the opposite case. There are two stochastic shocks: one in the price equation and one in the unemployment equation. The shape of the nonlinear Phillips curve is inspired by Debelle and Laxton (1997), but the exact specification and calibration of the model is for illustration only.

The model needs two auxiliary variables to account for p_{t-2} and p_{t-3}, and the main technical problem facing us is dealing with the expectation of future inflation. We assume in this economy that agents form their expectations on the basis of

past inflation, interest rates, unemployment and observed current shocks and do so without systematic error.

2.2. THE EXPECTATION FUNCTION AND TAYLOR EXPANSION

The problem in computing simulations for this type of model is finding an expectation function that provides us with the conditional expectation of y_{t+1} as a function of the observable state of the system at time t. To do this, we postulate the existence of two auxiliary functions describing the evolution of the system,

$$y_t = g(x_{t-1}, u_t)$$
$$x_x = h(x_{t-1}, u_t)\,.$$

From there it follows that

$$E_t y_{t+1} = E_t[g(h(x_{t-1}, u_t), u_{t+1})]\,.$$

The implementation of this method as suggested by Judd (1998) approximates these two functions through a Taylor series. The successive partial derivatives of functions $g(\,)$ and $h(\,)$, necessary for computing their Taylor expansion, can easily be retrieved from those of the original $f(\,)$ functions.

Writing the Taylor expansion of a multidimensional system is very demanding on notation. We therefore adopt the following representation for partial derivatives, loosely adapted from tensor notation:

$$[f_x]^i_j = \frac{\partial f_i}{\partial x_j}$$

$$[f_{x^2}]^i_{jk} = \frac{\partial^2 f_i}{\partial x_j \partial x_k}$$

$$[f_{xy}]^i_{jk} = \frac{\partial^2 f_i}{\partial x_j \partial y_k}\,.$$

The same index in both superscript and subscript positions indicates a summation. For example,

$$[f_{x^2}]^i_{\alpha\beta}[h_x]^\alpha_j[h_x]^\beta_k = \sum_\alpha \sum_\beta \frac{\partial^2 f_i}{\partial x_\alpha \partial x_\beta}\frac{\partial h_\alpha}{\partial x_j}\frac{\partial h_\beta}{\partial x_k}\,.$$

Then, we can write the Taylor expansion at order p of one of the expectation functions $g_i(x_{t-1}, u_t)$ about the point $x_{t-1} = \bar{x}$ and $u_t = 0$ as

$$g_i(x, u) \approx g_i(\bar{x}, 0)$$
$$+ \sum_{j=1}^{p} \frac{1}{j!} \sum_{k=0}^{j} [g_{x^k u^{j-k}}]^i_{\alpha_1 \ldots a_k \beta_1 \ldots \beta_{j-k}} [\hat{x}]^{\alpha_1} \ldots [\hat{x}]^{\alpha_k} [u]^{\beta_1} \ldots [u]^{\beta_{j-k}}\,,$$

where $\hat{x} = x - \bar{x}$.

Define $\bar{y}$ and $\bar{x}$ as the equilibrium of the model in the absence of stochastic shocks ($u_t = u_{t+1} = 0$), i.e.,

$$f(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0) = 0\,.$$

In general, for nonlinear models, this equilibrium is different from the expectation of the variables ($\bar{y} \neq E(y_t)$).

In what follows, we approximate $g(\,)$ by its Taylor expansion about the equilibrium defined above (see Judd, 1998). To do so, it is also necessary to compute the approximation of $h(\,)$, and we rely extensively on the properties of implicit functions. Let's adopt the following notation: $y' = y_{t+1}$, $x' = x_t$, $y = y_t$, $x = x_{t-1}$, $u' = u_{t+1}$ and $u = u_t$. The model can then be written as

$$\begin{aligned}\bar{F}(x,u) &= E_t[F(x,u,u')] \\ &= E_t[f(g(h(x,u),u'),h(x,u),g(x,u),x,u,u')] = 0\,.\end{aligned}$$

In this expression, the expectation is taken conditionally on the state of the system at period t, and it is supposed that the random shocks of this period, $u = u_t$, are already observed. The only stochastic term is therefore u'. As $F(x,u,u')$ is a system of nonlinear equations, $\bar{F}(x,u)$, the conditional expectations, will in general be a function of all the moments of the distribution of the stochastic shocks u'.

The partial derivatives of $g(\,)$ and $h(\,)$ can be retrieved from the fact that all the partial derivatives of $\bar{F}(x,u)$, at any order, must be null.

3. Linear Approximation of the Expectation Function

We start by taking the first-order approximation of $\bar{F}$, noted $\bar{F}^{(1)}$

$$\begin{aligned}\bar{F}_i^{(1)}(x,u) &= E_t[F_i^{(1)}(x,u,u')] \\ &= E_t[F_i(\bar{x},0,0) + [F_x]_\alpha^i[\hat{x}]^\alpha + [F_u]_\alpha^i[u]^\alpha + [F_{u'}]_\alpha^i[u']^\alpha] \\ &= E_t[F_i(\bar{x},0,0) + ([f_{y'}]_\alpha^i[g_{u'}]_\gamma^\alpha + [f_{u'}]_\gamma^i)[u']^\gamma \\ &\quad + ([f_{y'}]_\alpha^i[g_{x'}]_\beta^\alpha[h_x]_\gamma^\beta + [f_{x'}]_\beta^i[h_x]_\gamma^\beta + [f_y]_\alpha^i[g_x]_\gamma^\alpha + [f_x]_\gamma^i)[\hat{x}]^\gamma \\ &\quad + ([f_{y'}]_\alpha^i[g_{x'}]_\beta^\alpha[h_u]_\gamma^\beta + [f_{x'}]_\beta^i[h_u]_\gamma^\beta + [f_y]_\alpha^i[g_u]_\gamma^\alpha + [f_u]_\gamma^i)[u]^\gamma] \\ &= 0\,.\end{aligned}$$

All the partial derivatives are taken around the deterministic equilibrium in the absence of shocks and are therefore non-stochastic. Furthermore, in the above expression $E_t(u') = 0$, and we get

$$\begin{aligned}\bar{F}_i^{(1)}(x,u) &= F_i(\bar{x},0,0) \\ &\quad + ([f_{y'}]_\alpha^i[g_{x'}]_\beta^\alpha[h_x]_\gamma^\beta + [f_{x'}]_\beta^i[h_x]_\gamma^\beta + [f_y]_\alpha^i[g_x]_\gamma^\alpha + [f_x]_\gamma^i)[\hat{x}]^\gamma \\ &\quad + ([f_{y'}]_\alpha^i[g_{x'}]_\beta^\alpha[h_u]_\gamma^\beta + [f_{x'}]_\beta^i[h_u]_\gamma^\beta + [f_y]_\alpha^i[g_u]_\gamma^\alpha + [f_u]_\gamma^i)[u]^\gamma \\ &= 0\,.\end{aligned}$$

This is verified as long as $F(\bar{x}, 0, 0)$ and the partial derivatives of F with respect to each of the x and the u are null for any value of x and u.

It follows from their definition that $\bar{y}$ and $\bar{x}$ insure that

$$F(\bar{x}, 0, 0) = f(\bar{y}, \bar{y}, \bar{x}, \bar{x}, 0, 0) = 0\,,$$

and therefore will be the constant terms of the linear approximation of $g(x_{t-1}, u_t)$ and $h(x_{t-1}, u_t)$, respectively, that is,

$$\begin{aligned}\bar{y} &= g(\bar{x}, 0)\\ \bar{x} &= h(\bar{x}, 0)\,.\end{aligned}$$

We then compute the partial derivatives $[g_{x'}]^{\alpha}_{\beta}$, $[g_x]^{\alpha}_{\beta}$, $[g_u]^{\alpha}_{\beta}$, $[h_x]^{\alpha}_{\beta}$, and $[h_u]^{\alpha}_{\beta}$ insuring that the partial derivatives of F with respect to each of the x and u are null.

Taking the first partial derivatives of F with respect to x, we get

$$\begin{aligned}[F_x]^i_j &= [f_{y'}]^i_\alpha [g_{x'}]^\alpha_\beta [h_x]^\beta_j + [f_{x'}]^i_\beta [h_x]^\beta_j + [f_y]^i_\alpha [g_x]^\alpha_j + [f_x]^i_j\\ &= 0\,.\end{aligned}$$

As we are computing an approximation about the deterministic equilibrium, the derivatives are evaluated at that point. So, we have $[g_{x'}]^{\alpha}_{\beta} = [g_x]^{\alpha}_{\beta} = [g_{\bar{x}}]^{\alpha}_{\beta}$.

This defines $(n_y + n_x) \times n_x$ equations, sufficient to determine $n_y \times n_x$ partial derivatives of y with respect to x and $n_x \times n_x$ partial derivatives of x' with respect to x. This system of equations is a matrix polynomial equation $[g_{x'}]^{\alpha}_{\beta}$ and is solved under the requirement that the system returns asymptotically to equilibrium in the absence of other future shocks (condition of transversality). To do so, we use the generalized Schur decomposition (see Sims, 1995; Klein, 1997; Söderlind, 1999; Juillard, 1999), but one could also use the AIM algorithm developed by Anderson and Moore (1985) or the one proposed by Zadrozny (1998).

We form the system

$$[f_{y'} \;\; f_{x'}] \begin{bmatrix} y_{t+1} - \bar{y} \\ x_t - \bar{x} \end{bmatrix} = -[f_y \;\; f_x] \begin{bmatrix} y_t - \bar{y} \\ x_{t-1} - \bar{x} \end{bmatrix}$$

or

$$Dw_{t+1} = Ew_t\,.$$

It is possible to get a generalized Schur decomposition in the form

$$\begin{aligned}D &= QTZ^H\\ E &= QSZ^H\end{aligned}$$

with

$$\begin{aligned}QQ^H &= Q^HQ = I\\ ZZ^H &= Z^HZ = I\,,\end{aligned}$$

and S and T are upper triangular matrices and S_{ii}/T_{ii} is a generalized eigenvalue of the pencil $\langle D, E\rangle$. When the D matrix is singular, some of the T_{ii} are necessarily null, but we exclude the case where both T_{ii} and S_{ii} are simultaneously null as it appears only in degenerate models (see Juillard, 1999). After decomposition, the original system can be written in partitioned form

$$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_{t+1} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ 0 & S_{22} & S_{23} \\ 0 & 0 & S_{33} \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \,.$$

This partition is done so that the generalized eigenvalues corresponding to the first block (T_{11} and S_{11}) – the ratio of the diagonal elements – be smaller than one in modules. For the second block, the corresponding generalized eigenvalues are larger than one.[2] For the third block, the diagonal elements of T_{33} are null and S_{33} is necessarily of full rank, as its diagonal elements must be different from zero if the model is not degenerate.

If this is the case, then[3]

$$Z_3^H w_t = Z_3^H w_{t+1} = 0\,,$$

and there exists a static relationship between the elements of w_t. The dynamics of the system are then given by the equivalent model

$$\begin{aligned}
\begin{bmatrix} T_{11} & T_{12} & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_{t+1} &= \begin{bmatrix} S_{11} & S_{12} & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \\
\begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_{t+1} &= \begin{bmatrix} T_{11} & T_{12} & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} S_{11} & S_{12} & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \\
&= \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22}0 & \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \\
w_{t+1} &= [Z_1 \;\; Z_2 \;\; Z_3] \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \,.
\end{aligned}$$

The diagonal elements of matrix A are the finite generalized eigenvalues of the original system – smaller than one for A_{11}, larger than one for A_{22}.

This system can easily be used to compute w_{t+k} for $k > 1$ as

$$w_{t+k} = [Z_1 \;\; Z_2 \;\; Z_3] \begin{bmatrix} A_{11} & A_{12} & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}^k \begin{bmatrix} Z_1^H \\ Z_2^H \\ Z_3^H \end{bmatrix} w_t \,,$$

which can tend towards a finite limiting value as k goes to infinity only if $Z_2^H w_t = 0$.

Since $w_t = \begin{bmatrix} g_x \hat{x}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix}$, we have two conditions to identify g_x

$$\begin{aligned} 0 &= \begin{bmatrix} Z_2^H \\ Z_3^H \end{bmatrix} \begin{bmatrix} g_x \hat{x}_{t-1} \\ \hat{x}_{t-1} \end{bmatrix} \\ g_x &= -\begin{bmatrix} Z_{1,2}^H \\ Z_{1,3}^H \end{bmatrix}^{-1} \begin{bmatrix} Z_{2,2}^H \\ Z_{2,3}^H \end{bmatrix}. \end{aligned}$$

A unique solution exists for g_x only if there are as many forward-looking variables as there are generalized eigenvalues (including infinite ones) larger than one and that the rank requirement is satisfied. The first condition is the well-known identification condition stated in Blanchard and Kahn (1981) and the second is given in Anderson and Moore (1985).

Knowing $\bar{y}$ and g_x is all that is needed to run a simulation of the model with a linear approximation for the expectation function, as we shortly make clear. But first, we show how to recover the other partial derivatives that are necessary in turn to obtain the partial derivatives of the second order.

h_x can then be retrieved directly as

$$h_x = [T_{11}(Z_{1,1}^H g_x + Z_{2,1}^H)]^{-1} S_{11}(Z_{1,1}^H g_x + Z_{2,1}^H).$$

Next, the partial derivatives of the original system with respect to u provide us with g_u and h_u:

$$[F_u]_j^i = [f_{y'}]_\alpha^i [g_{x'}]_\beta^\alpha [h_u]_j^\beta + [f_{x'}]_\beta^i [h_u]_j^\beta + [f_y]_\alpha^i [g_u]_j^\alpha + [f_u]_j^i = 0.$$

This is a system of linear equations in the unknowns $[g_u]_j^\alpha$ and $[h_u]_j^\beta$, which can be rewritten in matrix form as

$$\begin{bmatrix} g_u \\ h_u \end{bmatrix} = -[f_y \; f_{y'} g_{x'} + f_{x'}]^{-1} f_u .$$

The linear approximations to the unknown functions are given by

$$\begin{aligned} \tilde{g}_\alpha^{(1)}(x,u) &= \bar{y}_\alpha + [g_x]_\beta^\alpha [\hat{x}]^\beta + [g_u]_\beta^\alpha [u]^\beta \\ \tilde{h}_\alpha^{(1)}(x,u) &= \bar{x}_\alpha + [h_x]_\beta^\alpha [\hat{x}]^\beta + [h_u]_\beta^\alpha [u]^\beta . \end{aligned}$$

Let us denote by $\tilde{F}^{(1)}(x,u)$ the model using the linear approximations of $g(\,)$ and $h(\,)$:

$$\tilde{F}^{(1)}(x,u) = f(\tilde{g}^{(1)}(\tilde{h}^{(1)}(x,u),0), \tilde{h}^{(1)}(x,u)\tilde{g}^{(1)}(x,u), x, u).$$

This approximated model will be useful to compute the second derivatives of $g(\,)$ and $h(\,)$.

It should be emphasized here that, for the purpose of simulation, one needs only the expectation function $g(\,)$. Further, using the linear approximation for $g(\,)$ in the

original model captures, although imperfectly, some of the nonlinear features of the model but is essentially different from simulating the linearized version of the original model.

With a linear expectation function, we suggest simulating the model

$$f(\bar{y} + g_x x_t, y_t, x_t, x_{t-1}, u_t, 0) = 0\,,$$

solving for x_t and y_t in each period and using the fact that $E(u_{t+1}) = 0$. In general, such a model, contrarily to the linearized version of the original model, does not have certainty-equivalence properties. It does provide systematically biased expectations for y_{t+1} when the true expectation function is poorly approximated by a linear form. In this case, artificial series obtained by stochastic simulations would normally fail tests of the rational-expectation hypothesis and would indicate that a higher-order approximation to the expectation function is indeed necessary.

4. The Quadratic Approximation

To obtain a second-order approximation to the expectation function $g(x, u)$, we must start with a second-order Taylor expansion of the original model:

$$\begin{aligned}
\bar{F}_i^{(2)}(x,u) &= E_t[F_i^{(2)}(x,u',u)] \\
&= E_t[F_i^{(1)}(x,u',u) + \frac{1}{2}[F_{x^2}]^i_{\alpha\beta}[x]^\alpha[x]^\beta + [F_{xu'}]^i_{\alpha\beta}[x]^\alpha[u']^\beta \\
&\quad + [F_{xu}]^i_{\alpha\beta}[x]^\alpha[u]^\beta + \frac{1}{2}[F_{u'^2}]^i_{\alpha\beta}[u']^\alpha[u']^\beta + [F_{u'u}]^i_{\alpha\beta}[u']^\alpha[u]^\beta \\
&\quad + \frac{1}{2}[F_{u^2}]^i_{\alpha\beta}[u]^\alpha[u]^\beta] \\
&= \bar{F}_i^{(1)}(x,u) + \frac{1}{2}[F_{x^2}]^i_{\alpha\beta}[x]^\alpha[x]^\beta + [F_{xu}]^i_{\alpha\beta}[x]^\alpha[u]^\beta \\
&\quad + \frac{1}{2}[F_{u'^2}]^i_{\alpha\beta}[\Sigma]^{\alpha\beta} + \frac{1}{2}[F_{u^2}]^i_{\alpha\beta}[u]^\alpha[u]^\beta \\
&= 0\,.
\end{aligned}$$

Again, the expectations of the terms linear in u' are null. The covariances between u and u' are also null from the hypothesis that the shocks are independently distributed through time, but there is now a constant term involving Σ, the variance/covariance matrix of future shocks. We will discuss shortly the appropriate treatment of this constant term, but, first, we deal with the computation of the second-order derivatives of $g(\,)$ and $h(\,)$.[4]

The second-order partial derivatives are easily determined, resulting from a large linear problem. The development of the second partial derivatives of the original model with respect to x gives

$$\begin{aligned}
[F_{x^2}]^i_{jk} &= ([f_{y'^2}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k \\
&\quad + [f_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{y'x}]^i_{\alpha k})[g_{x'}]^\alpha_\beta[h_x]^\beta_j \\
&\quad + [f_{y'}]^i_\alpha[g_{x'^2}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j + [f_{y'}]^i_\alpha[g_{x'}]^\alpha_\beta[h_{x^2}]^\beta_{jk} \\
&\quad + ([f_{x'y'}]^i_{\beta\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{x'^2}]^i_{\beta\delta}[h_x]^\delta_k \\
&\quad + [f_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [f_{x'x}]^i_{\beta k})[h_x]^\beta_j + [f_{x'}]^i_\beta[h_{x^2}]^\beta_{jk} \\
&\quad + ([f_{yy'}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k \\
&\quad + [f_{y^2}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{yx}]^i_{\alpha k})[g_x]^\alpha_j \\
&\quad + [f_y]^i_\alpha[g_{x^2}]^\alpha_{jk} + [f_{xy'}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k \\
&\quad + [f_{xx'}]^i_{j\delta}[h_x]^\delta_k + [f_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [f_{x^2}]^i_{jk} \\
&= 0\,.
\end{aligned}$$

By contrast, the second-order partial derivatives of the model using the linear approximations for the expectation and dynamic functions are

$$\begin{aligned}
[\tilde{F}^{(1)}_{x^2}]^i_{jk} &= ([f_{y^2}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{y'x'}]^i_{\alpha\delta}[h_x]^\delta_k \\
&\quad + [f_{y'y}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{y'x}]^i_{\alpha k})[g_{x'}]^\alpha_\beta[h_x]^\beta_j \\
&\quad + ([f_{x'y'}]^i_{\beta\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{x'^2}]^i_{\beta\delta}[h_x]^\delta_k \\
&\quad + [f_{x'y}]^i_{\beta\gamma}[g_x]^\gamma_k + [f_{x'x}]^i_{\beta k})[h_x]^\beta_j \\
&\quad + ([f_{yy'}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k + [f_{yx'}]^i_{\alpha\delta}[h_x]^\delta_k \\
&\quad + [f_{y^2}]^i_{\alpha\gamma}[g_x]^\gamma_k + [f_{yx}]^i_{\alpha k})[g_x]^\alpha_j \\
&\quad + [f_{xy'}]^i_{\alpha\gamma}[g_{x'}]^\gamma_\delta[h_x]^\delta_k \\
&\quad + [f_{xx'}]^i_{j\delta}[h_x]^\delta_k + [f_{xy}]^i_{j\gamma}[g_x]^\gamma_k + [f_{x^2}]^i_{jk}\,.
\end{aligned}$$

It is therefore possible to write the second derivatives of the original model as

$$\begin{aligned}
[F_{x^2}]^i_{jk} &= [\tilde{F}^{(1)}_{x^2}]^i_{jk} + [f_{y'}]^i_\alpha[g_{x'^2}]^\alpha_{\beta\delta}[h_x]^\delta_k[h_x]^\beta_j \\
&\quad + [f_{y'}]^i_\alpha[g_{x'}]^\alpha_\beta[h_{x^2}]^\beta_{jk} + [f_{x'}]^i_\beta[h_{x^2}]^\beta_{jk} \\
&\quad + [f_y]^i_\alpha[g_{x^2}]^\alpha_{jk} \\
&= 0\,,
\end{aligned}$$

and again, given that the partial derivatives are evaluated about the deterministic equilibrium, we obtain

$$[f_{y'}]^i_\alpha [h_x]^\delta_k [h_x]^\beta_j [g_{x^2}]^\alpha_{\beta\delta} + [f_y]^i_\alpha [g_{x^2}]^\alpha_{jk} + ([f_{y'}]^i_\alpha [g_{x'}]^\alpha_\beta + [f_{x'}]^i_\beta)[h_{x^2}]^\beta_{jk} = -[\tilde{F}^{(1)}_{x^2}]^i_{jk},$$

which is readily solved for $[g_{x^2}]^\alpha_{jk}$ and $[h_{x^2}]^\beta_{jk}$.

Similarly, we get

$$\begin{aligned}
&[f_y]^i_\alpha [g_{xu}]^\alpha_{jk} + ([f_{y'}]^i_\alpha [g_x]^\alpha_\beta + [f_{x'}]^i_\beta [h_{xu}]^\beta_{jk} \\
&\quad = -[\tilde{F}^{(1)}_{xu}]^i_{jk} - [f_{y'}]^i_\alpha [h_u]^\delta_k [h_x]^\beta_j [g_{x'^2}]^\alpha_{\beta\delta} \\
&[f_y]^i_\alpha [g_{u^2}]^\alpha_{jk} + ([f_{y'}]^i_\alpha [g_{x'}]^\alpha_\beta + [f_{x'}]^i_\beta)[h_{u^2}]^\beta_{jk} \\
&\quad = -[\tilde{F}^{(1)}_{u^2}]^i_{jk} - [f_{y'}]^i_\alpha [h_u]^\delta_k [h_u]^\beta_j [g_{x'^2}]^\alpha_{\beta\delta}.
\end{aligned}$$

4.1. A BIAS REDUCTION PROCEDURE

The presence in the second-order Taylor expansion of the model of a constant term involving Σ, the variance/covariance of future shocks, means that $\bar{y}$ and $\bar{x}$, the value of the variables at the deterministic equilibrium, do not, in general, form a solution for $\bar{F}^{(2)}(x, 0) = 0$. New values for the constant of the second-order approximation of $g(x_{t-1}, u_t)$ and $h(x_{t-1}, u_+t)$, denoted $\tilde{x}$ and $\tilde{y}$, must be computed by solving the equation that enforces the rational-expectation hypothesis when the observed shock is null

$$\begin{aligned}
\bar{F}^{(2)}_i(\tilde{x}, 0) &= F^i(\tilde{x}, 0, 0) + \frac{1}{2}[F_{u'^2}]^i_{\alpha\beta}[\Sigma]^{\alpha\beta} \\
&= f_i(\tilde{y}, \tilde{y}, \tilde{x}, \tilde{x}, 0, 0) + \frac{1}{2}[F_{u'^2}|_{\tilde{y},\tilde{x}}]^i_{\alpha\beta}[\Sigma]^{\alpha\beta} \\
&= 0.
\end{aligned}$$

Note that, in implementating this, care must be taken that the derivatives are indeed computed at the new point defined by $\tilde{x}$ and $\tilde{y}$.

5. Higher-Order Approximation

It is possible to use basically the same technique to compute approximations of higher-orders. The basic steps are as follows.

The Taylor expansion of the model of order p is written

$$
\begin{aligned}
\bar{F}_i^{(p)}(x,u) &= E_t[F_i^{(p)}(x,u,u')] \\
&= E_t\Bigg[F^i(\tilde{x},0,0) \\
&\quad + \sum_{j=1}^{p}\frac{1}{j!}\sum_{k=0}^{j}\sum_{l=0}^{j-k}[F^i_{x^k u^l u'^{j-k-l}}]_{\alpha_1\ldots\alpha_k\beta_1\ldots\beta_l\delta_1\ldots\delta_{j-k-l}} \\
&\quad [\hat{x}]^{\alpha_1}\ldots[\hat{x}]^{\alpha_k}[u]^{\beta_1}\ldots[u]^{\beta_l}[u']^{\delta_1}\ldots[u']^{\delta_{j-k-l}}\Bigg] \\
&= F_i(\tilde{x},0,0) + \sum_{m=2}^{p}\frac{1}{m!}[F_{u'^m}]^i_{\delta_1\ldots\delta_m}[\Sigma]^{\delta_1\ldots\delta_m} \\
&\quad + \sum_{n=1}^{p}\frac{1}{n!}\sum_{j=0}^{n}\Bigg([F_{x^j u^{n-j}}]^i_{\alpha_1\ldots\alpha_j\beta_1\ldots\beta_{n-j}} \\
&\quad + \sum_{m=2}^{p-n}[F_{x^j u^{n-j}u'^m}]^i_{\alpha_1\ldots\alpha_j\beta_1\ldots\beta_{n-j}\delta_1\ldots\delta_m}[\Sigma]^{\delta_1\ldots\delta_m}\Bigg) \\
&\quad [\hat{x}]^{\alpha_1}\ldots[\hat{x}]^{\alpha_j}[u]^{\beta_1}\ldots[u]^{\beta_{n-j}}.
\end{aligned}
$$

There are now additional terms made necessary by the higher moments of future shocks $[\Sigma]^{\delta_1\ldots\delta_m}$ for all derivatives until order $p-2$.

The partial derivatives without the correction factor can be computed recursively. For example, one obtains $[g_{x^p}]^{\alpha}_{j_1\ldots j^p}$ and $[h_{x^p}]^{\alpha}_{j_1\ldots j^p}$ by solving

$$
\begin{aligned}
&[f_{y'}]^i_\alpha[h_x]^{\beta_1}_{j_1}\ldots[h_x]^{\beta_p}_{j_p}[g_{x^p}]^{\alpha}_{\beta_1\ldots\beta_p} + [f_y]^i_\alpha[g_{x^p}]^{\alpha}_{j_1\ldots j_p} \\
&\quad + ([f_{y'}]^i_\alpha[g_{x'}]^{\alpha}_{\beta} + [f_{x'}]^i_\beta)[h_{x^p}]^{\beta}_{j_1\ldots j_p} = -[\tilde{F}^{(p-1)}_{x^p}]^i_{j_1\ldots j_p},
\end{aligned}
$$

where $\tilde{F}^{(p-1)}$ is the original model containing the $p-1$ order approximations for $g(\,)$ and $h(\,)$.

The other partial derivatives are computed in similar manner. Then, the correction factor is computed by successive iterations of the whole procedure. It should be highlighted that, after the linearization step, which requires solving a matrix polynomial equation, the calculation of higher-order approximations involves nothing more complex than solving a (very) large linear system.

Table I. Moments of the simulated series.

Variable	p_t	π_t	$E_t\pi_{t+1}$	r_t	v_t
Linear approximation					
Mean	2.919	2.901	2.844	0.451	0.284
Variance	50.323	46.705	44.499	11.676	1.386
Skewness	369.438	326.415	302.299	40.802	1.529
Quadratic approximation					
Mean	3.055	3.038	3.037	0.519	0.324
Variance	52.336	48.836	47.601	12.209	1.521
Skewness	423.428	385.334	398.544	48.167	2.397

6. Simulating a Nonlinear Phillips Curve Model

We use the procedure described above to compute both linear and second-order approximattions to the expectation function for the Phillips curve model described in Section 2 and use them to simulate the model over 10,000 periods. For the stochastic simulation, we set $\sigma^2_{\epsilon_p} = 0.4$ and $\sigma^2_{\epsilon_u} = 0.15$.

Although this model is strongly nonlinear, the expectation function is not; the largest term of $[g_{x^2}]$ is the second derivative of annual inflation with respect to the interest rate and equals only 0.023. The stochastic shock on unemployment has a slightly stronger effect, since the second derivative of annual inflation with respect to that shock is 0.17.

In Table I, we report the first three moments of the series obtained by simulation. As expected the results display a strong skewness in the distributions and the means of the variables are quite different from the values at the deterministic equilibrium, where the inflation variables would be at 2%, and $r_t = v_t = 0$. Because inflation increases more quickly when there is an improvement in unemployment than it decreases when there is a worsening, the stochastic equilibrium demonstrates more inflation and more unemployment than does the deterministic equilibrium. In contrast, simulating the linearized version of the same model results in variables which have symmetric distributions and in means for these variables that are equal to those of the deterministic equilibrium.

When using the quadratic approximation to the expectation function, the mean of expected inflation matches almost perfectly the mean of annual inflation itself, we can therefore assume that the rational-expectation restriction is satisfied. This is not exactly the case when a linear expectation function is used. Note, however, that the main features of the model are well described if we use the linear approximation to expectations in the nonlinear model, as opposed to simulating a linearized version of the whole model.

7. Conclusion

In this paper, we extend to a general class of rational-expectation models a Taylor expansion method proposed by Judd and others for solving stochastic optimal control problems. This is a rather simple method that involves solving first a matrix polynomial equation and then successive linear systems of larger dimensions depending on the order of expansion desired. We are able to implement it easily in DYNARE, our simulation program. The algorithm is sufficiently general that the user needs only specify the equations of the model and the variances of the stochastic shocks.

Our illustration with a nonlinear Phillips curve model indicates that, in simple cases, a quadratic approximation to the expectation function may be sufficient, and, even a linear approximation produces the main qualitative features as long as it is used in the original nonlinear model.

One of the main difficulties with the simulation of nonlinear rational-expectation models is the curse of dimension when trying to deal with a large number of state variables and stochastic shocks. While the Taylor expansion method seems able to handle larger problems, it is clear that the efficiency of the computer implementation becomes more important. Sparse matrix techniques would seem to be essential. The computation of higher-order derivatives is a more difficult problem. To our knowledge, there exist no libraries containing algorithms for the numerical computation of third and higher derivatives of multivariate systems, and indeed, it is not even clear that, were such algorithms available, acceptable accuracy could be obtained with a double precision floating point representation. It is therefore certainly worth exploring the feasibility of automatic differentiation techniques.

Notes

[1] DYNARE is a front-end and a collection of routines for GAUSS and MATLAB freely available at http://www.cepremap.cnrs.fr/∼michel/dynare

[2] We do not discuss in this paper cases where there are unit roots in the dynamic of the system.

[3] See Juillard (1999).

[4] The second-order derivatives of $h(\,)$ are only needed if one is to compute the third derivatives of $g(\,)$.

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