Techniques of Robust Control

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1 Introduction

The aim of the lecture is to introduce participants to the central techniques and methods used in applying robust control to economic problems. The motivation for doing so is to provide a disciplined response to economic agents fearing that the model they are using to make decisions may be misspecified. The theory has obvious attraction for macroeconomic policymakers concerned that they may not perfectly understand the workings of the economy, but can equally well be applied to consumers, firms or unions making decisions in an environment of unstructured uncertainty. By the end of the lecture participants will be able to derive robustly optimal policies in a wide variety of environments.

2 Key readings

The primary reading is the textbook *Robustness* by Lars Peter Hansen and Thomas J. Sargent, Princeton University Press, 2008. We will cover the most important parts of Chapters 2, 3 and 9, although Chapter 1 is also highly recommended as an introduction to many of the concepts covered.

3 Key concepts

Detection error probabilities, Entropy, Epstein-Zin preferences, Optimal control, Robust control

4 Optimal linear-quadratic control

It is useful to begin with a quick recap of the optimal linear regulator problem without a preference for robustness:

$$\begin{aligned} -y_0 P y_0 - p &= \max_{\{u_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t r(y_t, u_t) \\ & \text{s.t.} \\ y_{t+1} &= A y_t + B u_t + C \check{\epsilon}_{t+1} \end{aligned}$$

where $r(y, u) = -(y'Qy) - u'Ru$ is a quadratic objective function and the expectation E_0 is evaluated with respect to the distribution of $\check{\epsilon}$. The solution is a policy rule $u_t = -F y_t$ defining the control variable as a function of the predetermined variables y_t. The matrix F is defined by $F = \beta(R + \beta B'PB)^{-1} B'PA$, where the matrix P satisfies the Riccati equation:

$$P = Q + \beta A'PA - \beta^2 A'PB(R + \beta B'PB)^{-1} B'PA$$

and the constant p in the value function satisfies:

$$p = \frac{\beta}{1 - \beta} trace(PCC')$$

The optimal linear regulator problem exhibits certainty equivalence, in the sense that the matrix F is independent of the variance-covariance matrix CC' of the disturbances. In other words, in this framework the policymaker chooses the same policy irrespective of the amount of noise in the system. This is a useful property as it enables the policymaker to solve for optimal policy by setting $CC' = \underline{0}$ and working instead with a purely deterministic model. The presence of noise does enter the value function through the constant term p, although since this enters in an additive fashion independent of policy there is no need for the policymaker to internalise its effects when deciding on policy. The key assumptions that ensure that certainty equivalence holds are a quadratic objective function and a linear model for the state transition.

5 Robust control in a linear-quadratic framework

The aim of robust control techniques is to enable the policymaker to design policies that work well even if they distrust their view of how the economy works. We assume that the policymaker believes an approximating model describes the transition of the state variables. In a linear quadratic framework this is:

$$y_{t+1} = A y_t + B u_t + C \check{\epsilon}_{t+1}$$

The policymaker is concerned that the model is only an approximation, so is prepared to entertain alternative models that are "close" to the approximating model. A suitable distorted model is:

$$y_{t+1} = Ay_t + Bu_t + C\left(\epsilon_{t+1} + w_{t+1}\right)$$

where w_{t+1} is an (as yet) unspecified distortion to the conditional mean of the process for y_{t+1}. The conditional distributions under the approximating and distorted model are:

$$f_0(y^* \,|y) \sim N(Ay + Bu, CC')$$
$$f(y^* \,|y) \sim N(Ay + Bu + Cw, CC')$$

so the distortion is to the conditional mean of y^*. To economise on notation we have dropped the t subscripts and use the $*$ superscript to denote next period's values. The policymaker is interested in distorted models that are "close" to the approximating model, which we interpret here as distorted models that have a conditional distribution similar to that of the approximating model. To define what is meant by closeness we need a simple univariate measure of the distance between two distributions. There are many candidate measures, for example the famous GINI coefficient is a measure of how close the income distribution of a country is to a uniform distribution with perfect equality.

6 Conditional relative entropy

In robust control it is convenient to work with *conditional relative entropy* as a measure of the closeness of the conditional distributions of the approximating and distorted models. Conditional relative entropy is defined mathematically as follows:

$$I(f_0, f)(y) = \int \log\left(\frac{f(y^* \,|y)}{f_0(y^* \,|y)}\right) f(y^* \,|y) dy^*$$

If the two conditional distributions are the same then $f(y^* \,|y) = f_0(y^* \,|y)$ and $I(f_0, f)(y) = 0$. As the two conditional distributions start to drift apart the value of $I(f_0, f)(y)$ increases and so is positive. The conditional relative entropy measure defined in this way leads to elegant and intuitive results later in this lecture. It is also used to measure the distance between two distributions in econometrics, where conditional relative entropy is referred to as the Kullback-Leibler distance or KLIC (Kullback-Leibler Information Criterion). It additionally rears its head in the rational inattention literature, where it is typically assumed that agents are restricted in their ability to reduce conditional relative entropy in their view of the world. In what follows it is convenient to rewrite the definition of conditional relative entropy as:

$$I(f_0, f)(y) = \int \log\left(\frac{f(y^* \,|y)}{f_0(y^* \,|y)}\right)\frac{f(y^* \,|y)}{f_0(y^* \,|y)} f_0(y^* \,|y)dy^*$$

by taking the integral with respect to the conditional distribution of the approximating rather than the distorted model. The expression $\frac{f(y^*|y)}{f_0(y^*|y)}$ is the likelihood ratio which we denote $m(f(y^* \,|y))$, in which case conditional relative entropy can be expressed as:

$$\begin{aligned} I(f_0, f)(y) &= \int (m \log m)\, f_0(y^* \,|y)dy^* \\ &= E_{f_0}\left(m \log m \,|y\right) \end{aligned}$$

We will use $I(f_0, f)(y)$ as a measure of the closeness of the approximating and distorted distributions in a period. Since we are interested in dynamic models, we define an intertemporal measure:

$$R(w) = 2E_0 \sum_{t=0}^{\infty} \beta^t I(w_{t+1})$$

that measures the difference between the approximating and distorted models.

7 Robust control problems

The idea of robust control is that the policymaker wants to design policy that is robust within a set of models that are close to its approximating model. Using the definition of closeness provided by the measure of conditional relative entropy, we define close to mean models that satisfy:

$$R(w) \leq \eta_0$$

where $0 \leq \eta_0 < \bar{\eta}$ is an exogenous constraint measuring how much the policymaker wishes to guard against distorted models. The larger is η_0 the larger the set of alternative models under consideration. η_0 is restricted to be less than a strict upper bound $\bar{\eta}$ so that the problem is well defined. The problem if η_0 is not restricted is that the policymaker would be attempting to guard against very large model misspecifications. In extreme situations the policymaker then loses control over economic outcomes and there is no way for the policymaker to avoid a return of $-\infty$. This problem is well-known in the robust control literature, with the value of η_0 for which the problem breaks down known alternatively as the "point of neurotic breakdown" or "point of utter psychotic despair". Engineers devising systems with the maximum robustness typically want η_0 as high as possible - in their terminology they work under the H^∞ norm. In Section 11 we will examine ways economists can discipline the choice of η_0 made by the policymaker in terms of detection error probabilities. For now we simply take it as given exogenously. The question is how a policymaker can design a policy that is robust to distortions in its approximating model, subject to the constraint that the distortions satisfy a conditional relative entropy constraint. One way to think about this is to assume that the policymaker maximises their return, subject to the worst possible outcome for $\{w_{t+1}\}$ that can occur given the conditional relative entropy constraint. To operationalise this consider a max-min control problem:

$$\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t r(y_t, u_t)$$

$$\text{s.t.}$$

$$y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + w_{t+1})$$

$$R(w) \leq \eta_0$$

In this *constraint problem* the policymaker chooses actions $\{u_t\}_{t=0}^{\infty}$ that maximise a payoff function $\sum_{t=0}^{\infty} \beta^t r(y_t, u_t)$, subject to their approximating model and a view that the distortions $\{w_{t+1}\}_{t=0}^{\infty}$ between the conditional distributions of the approximating and distorted models will cause the maximum harm possible whilst still respecting conditional relative entropy. The min in the optimisation problem captures the idea that the policymaker wants to guard against the worst-case distortions that can occur given the set of distorted models under consideration. One way to think about this is to imagine that the policymaker constructs a hypothetical evil agent who chooses the distortions $\{w_{t+1}\}_{t=0}^{\infty}$ to create the most damage possible. If the policymaker maximises a payoff subject to these minimising "evil actions" then they will have a policy that guards against the worst case scenario that can occur within the set of models being considered. It is important to stress that the evil agent is purely a theoretical construct that the policymaker uses to help them design a policy robust to possible misspecification of its approximating model. The evil agent is the policymaker's friend.

It is also possible to define a *multiplier problem* by placing the conditional relative entropy constraint $R(w) = 2E_0 \sum_{t=0}^{\infty} \beta^t I(w_{t+1}) \leq \eta_0$ in the objective directly.

$$\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \{r(y_t, u_t) + \beta\theta I(w_{t+1})\}$$

$$\text{s.t.}$$

$$y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + w_{t+1})$$

Chapters 7 and 8 of Sargent and Ljunqvist's *Robustness* book describe in detail the conditions under which solving the constraint and multiplier problems leads to equivalent policies. The parameter $\theta \in (\bar{\theta}, +\infty]$ is intimately related to η_0 and how tightly the entropy constraint binds. $\bar{\theta}$ is the neurotic breakdown point corresponding to $\bar{\eta}$. Under certain conditions it can be considered as the Lagrange multiplier on the entropy constraint.

8 Linear-quadratic-Gaussian robust control problems

We begin by stating that the measure of conditional relative entropy in linear-quadratic Gaussian robust control problems is given by $I(w_{t+1}) = .5w_{t+1}'w_{t+1}$. We will later verify that this is in fact the case, but for now note that given this form the multiplier problem becomes:

$$\max_{\{u_t\}_{t=0}^{\infty}} \min_{\{w_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{r(y_t, u_t) + \beta\theta w_{t+1}'w_{t+1}\right\}$$

$$\text{s.t.}$$

$$y_{t+1} = Ay_t + Bu_t + C(\epsilon_{t+1} + w_{t+1})$$

If the return function $r(y_t, u_t)$ is quadratic then this is a linear-quadratic max-min problem. There are many solution techniques to solve such a problem, all of which result in the policy rules derived by solving the stochastic problem with ϵ_{t+1} a random variable being identical to those derived by solving the deterministic problem with $\epsilon_{t+1} = 0$. A modified certainty equivalence therefore holds and we can work with only the deterministic problem. Note though that unlike the standard certainty equivalence result this does not mean that policy rules are independent of the variance-covariance matrix CC' of the shocks to the system. Even after setting $\epsilon_{t+1} = 0$ we still have the distortions w_{t+1} premultiplied by the C matrix and so C will in general impact on policy rules. Intuitively, the amount of noise in the system affects the ability of the evil agent to distort the approximating model for a given conditional relative entropy constraint. A large part of the robustness literature makes use of this modified certainty equivalence result. The resulting robust linear regulator-multiplier problem is deterministic:

$$-y'Py = \max_u \min_w \{r(y,u) + \beta\theta w'w - \beta y^{*\prime}Py^*\}$$
$$\text{s.t.}$$
$$y^* = Ay + Bu + Cw$$

where $p = 0$ in the value function $-y'Py + p$ because there is no stochastic element. The solution to the inner minimisation problem for a given P is a value function $-y^{*\prime}D(P)y^*$ where:

$$\begin{aligned} -y^{*\prime}D(P)y^* &= \min_w \{\theta w'w - y^{*\prime}Py^*\} \\ &\text{s.t.} \\ y^* &= Ay + Bu + Cw \end{aligned}$$

Given that $-y^{*\prime}D(P)y^* = -(Ay + Bu + Cw)'D(P)(Ay + Bu + Cw)$, this can be solved by standard linear-quadratic techniques for $D(P)$:

$$D(P) = P + \theta^{-1}PC(I - \theta^{-1}C'PC)^{-1}C'P$$

We return to the outer maximisation problem:

$$\begin{aligned} -y^{*\prime}Py^* &= \max_u \{r(y,u) - \beta y^{*\prime}D(P)y^*\} \\ &\text{s.t} \\ y^* &= Ay + Bu \end{aligned}$$

which has solution $P = Q + \beta A'D(P)A - \beta^2 A'D(P)B(R + \beta B'D(P)B)^{-1}B'D(P)A$. Iterating this expression and the one for $D(P)$ solves the max-min problem for P and $D(P)$:

$$\begin{aligned} u &= -Fy \\ w &= Ky \\ F &= \beta(R + \beta B'D(P)B)^{-1}B'D(P)A \\ K &= \theta^{-1}(I - \theta^{-1}C'PC)^{-1}C'P(A - BF) \end{aligned}$$

9 General model

In the general case we are interested in a policymaker who wishes to guard against distortions to an approximating density $\pi(\varepsilon)$, where ε is a dummy variable with the same number of elements as in the sequence of shocks $\{\epsilon_t\}$ that enter the transition equation of the approximating model. We allow the distorted density to potentially condition on an initial value y_0 and to be evaluated for the history of shocks ϵ^t, thereby obtaining $\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)$. We proceed as before by requiring the policymaker to derive a policy that performs well when $\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)$ is close to $\pi(\varepsilon)$. The likelihood ratio is define by:

$$m_{t+1} = \frac{\hat{\pi}(\epsilon_{t+1}\,|\epsilon^t, y_0)}{\pi(\epsilon_{t+1})}$$

If we take expectations of the likelihood ratio with respect to the approximating model then we obtain:

$$E(m_{t+1}\left|\epsilon^t, y_0\right.) = \int \frac{\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)}{\pi(\varepsilon)}\pi(\varepsilon)d\varepsilon = 1$$

because $\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)$ is a density function. For what follows it is useful to define a variable $M_0 = 1$ and recursively construct a series $\{M_t\}$ according to $M_{t+1} = m_{t+1}M_t$. The random variable M_t is then a ratio of joint densities of ϵ^t conditioned on y_0 evaluated for the history ϵ^t. Rolling the recursion forward we can also write M_t as the factorisation of the joint density:

$$M_t = \prod_{j=0}^{t} m_j$$

We observe that M_{t+1} satisfies:

$$E(M_{t+1}\left|\epsilon^t, y_0\right.) = E(m_{t+1}M_t\left|\epsilon^t, y_0\right.) = E(m_{t+1}\left|\epsilon^t, y_0\right.)M_t = M_t$$

so M_t is a Martingale relative to sequence of information sets generated by the shocks. As before, the entropy of the distortion associated with M_t is defined as the expectation of the log-likelihood ratio with respect to the distorted distribution.

$$\int \log\left(\frac{\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)}{\pi(\varepsilon)}\frac{\hat{\pi}(\varepsilon\,|\epsilon^{t-1}, y_0)}{\pi(\varepsilon)}\cdots\frac{\hat{\pi}(\varepsilon\,|\epsilon^0, y_0)}{\pi(\varepsilon)}\right)\times$$
$$\hat{\pi}(\varepsilon\left|\epsilon^t, y_0\right.)\hat{\pi}(\varepsilon\left|\epsilon^{t-1}, y_0\right.)\ldots\hat{\pi}(\varepsilon\left|\epsilon^0, y_0\right.)d\varepsilon d\varepsilon\ldots d\varepsilon = E(M_t \log M_t\,|y_0)$$

Note that $\frac{d^2}{dM^2}M_t \log M_t = 1/M_t > 0$ and the function $M_t \log M_t$ is convex in M_t. By definition $E(M_t \log M_t\,|y_0) \geq 0$ and it is possible to write:

$$E(M_t \log M_t\,|y_0) = \sum_{j=0}^{t-1} E\left[M_j E\left(m_{j+1}\log m_{j+1}\left|\epsilon^j, y_0\right.\right)|y_0\right]$$

where $m_{j+1}\log m_{j+1}\,|\epsilon^j, y_0$ is the conditional relative entropy of the perturbation to the one-step ahead transition density we used earlier in the definition of conditional relative entropy.

This definition of entropy is potentially problematic for economists as it implies that distortions in the distant future receive the same weight in calculating entropy as distortions in the near future, i.e. there is no discounting of conditional relative entropy. It is reasonable to assume that a policymaker is more worried about distortions that sooner, so Hansen and Sargent (2007) suggest working with a measure of discounted entropy of the form:

$$(1-\beta)\sum_{j=0}^{\infty}\beta^{j}E(M_j\log M_j\,|y_0)=\sum_{j=0}^{t-1}\beta^{j}E\left[M_jE\left(m_{j+1}\log m_{j+1}\left|\epsilon^{j},y_0\right.\right)|y_0\right]$$

We are now ready to define the general stochastic robust control problem:

$$\max_{\{u_t\}}\min_{\{m_{t+1}\}}\sum_{t=0}^{\infty}E\left[\beta^{t}M_t\left\{r(y_t,u_t)+\alpha\beta E\left(m_{t+1}\log m_{t+1}\left|\epsilon^{t},y_0\right.\right)\right\}|y_0\right]$$

s.t.

$$y_{t+1}=\varpi(y_t,u_t,\epsilon_{t+1})$$
$$M_{t+1}=m_{t+1}M_t$$

where $y_{t+1}=\varpi(y_t,u_t,\epsilon_{t+1})$ is the equation for the evolution of the predetermined variables and the control process $\{u_t\}$ is function of ϵ^t,y_0. $r(y_t,u_t)$ is the return function and y_0 is an initial condition. The likelihood ratio m_{t+1} is defined as the ratio of the approximating and distorted densities as before, so it is a function of ϵ^{t+1},y_0 and we have that $E(m_{t+1}\,|\epsilon^t,y_0)=1$ when the expectation is taken under the distribution of the approximating model. α is penalty parameter $\in[\underline{\alpha},+\infty]$. The lower bound $\underline{\alpha}$ is the familiar neurotic breakdown point. The general stochastic robust control problem is difficult to solve at present - what is needed is a recursive formulation. The predetermined variables in the system are M_t and y_t, so it is sensible to guess that the value function $W(M,y)$ has a multiplicative form $MV(y)$. Now consider the following Bellman equation:

$$MV(y)=\max_{u}\min_{m(\varepsilon)}M\left\{r(y,u)+\beta\int\left(m(\varepsilon)V(\varpi(y,u,\varepsilon))+\alpha m(\varepsilon)\log m(\varepsilon)\right)\pi(\varepsilon)d\varepsilon\right\}$$

s.t.

$$\int m(\varepsilon)\pi(\varepsilon)d\varepsilon=1$$

which has a useful recursive structure. The problem is linear in the scaling variable M so it was correct to guess that the value function has a multiplicative form and it is equivalent to work with:

$$V(y) = \max_{u} \min_{m(\varepsilon)} \left\{ r(y,u) + \beta \int \left(m(\varepsilon) V(\varpi(y,u,\varepsilon)) + \alpha m(\varepsilon) \log m(\varepsilon) \right) \pi(\varepsilon) d\varepsilon \right\}$$

$$\text{s.t.}$$

$$\int m(\varepsilon)\pi(\varepsilon)d\varepsilon = 1$$

The inner minimisation problem $\mathcal{R}$ is:

$$\mathcal{R}(V)(y,u) = \min_{m(\varepsilon)} \int \left(m(\varepsilon) V(\varpi(y,u,\varepsilon)) + \alpha m(\varepsilon) \log m(\varepsilon) \right) \pi(\varepsilon) d\varepsilon$$

$$\text{s.t.}$$

$$\int m(\varepsilon)\pi(\varepsilon)d\varepsilon = 1$$

which has a convex objective and linear constraint so can be solved by defining a Lagrangean:

$$\mathcal{L} = \min_{m(\varepsilon)} \int \left[m(\varepsilon) V(\varpi(y,u,\varepsilon)) + \alpha m(\varepsilon) \log m(\varepsilon) \right] \pi(\varepsilon) d\varepsilon + \alpha(1+\lambda) \left(1 - \int m(\varepsilon)\pi(\varepsilon)d\varepsilon \right)$$

The Lagrange multiplier is defined as $\alpha(1+\lambda)$ rather than the more usual λ to make mathematical expressions more convenient below. The first order condition with respect to $m(\varepsilon)$ is:

$$V(\varpi(y,u,\varepsilon)) + \alpha(1 + \log m(\varepsilon)) - \alpha(1+\lambda) = 0$$

from which it follows that:

$$m(\varepsilon) = e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}} e^{\lambda}$$

Since $\int m(\varepsilon)\pi(\varepsilon)d\varepsilon = 1$ we can integrate this to solve for λ

$$\int m(\varepsilon)\pi(\varepsilon)d\varepsilon = e^{\lambda} \int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}} \pi(\varepsilon)d\varepsilon = 1$$

and define the worst-case distortion as $m(\varepsilon)$:

$$m(\varepsilon) = \frac{e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}}{\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}} \pi(\varepsilon)d\varepsilon}$$

Substituting for $m(\varepsilon)$ back in the continuation value of the original maximisation problem gives:

$$\begin{aligned}
&\int \left(m(\varepsilon)V(\varpi(y,u,\varepsilon)) + \alpha m(\varepsilon)\log m(\varepsilon)\right)\pi(\varepsilon)d\varepsilon \\
&= \frac{1}{\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon}\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\left(V(\varpi(y,u,\varepsilon)) + \alpha\log\frac{e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}}{\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon}\right)\pi(\varepsilon)d\varepsilon \\
&= \frac{1}{\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon}\int -\alpha e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\log\left(\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon\right)\pi(\varepsilon)d\varepsilon \\
&= \frac{-\alpha}{\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon}\log\left(\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon\right)\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon \\
&= -\alpha\log\left(\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon\right)
\end{aligned}$$

We are now ready to state the full recursive problem:

$$\max_u \left[r(y,u) - \alpha\beta\log\left(\int e^{\frac{-V(\varpi(y,u,\varepsilon))}{\alpha}}\pi(\varepsilon)d\varepsilon\right)\right]$$

This is a beautiful expression. It shows that solving a robust control problem by imposing a constraint on conditional relative entropy is mathematically equivalent to the policymaker having preferences that apply an exponential transform to the continuation value. This type of preference can be written as:

$$V(x) = \max_{x'}\left[r(x) - \alpha\beta\log E(\exp(-V(x')/\alpha))\right]$$

and have already received considerable attention in the literature. They are perhaps best known as Epstein-Zin preferences, following the desire of Epstein and Zin to describe preferences in which the intertemporal elasticity of substitution is not constrained to be equal to the coefficient of risk aversion. We will use this interpretation of a preference for robustness in our discussions of asset market implications in later lectures. The preferences also map into the work of Whittle and his focus on risk sensitivity.

10 The linear-quadratic-Guassian case

We earlier claimed that conditional relative entropy is given by $I(w_{t+1}) = .5w'_{t+1}w_{t+1}$ in the linear-quadratic-Gaussian case. To see why this is consistent with our calculations, note first that the approximating density $\pi(\varepsilon)$ is by definition Gaussian. With the modified certainty equivalence principle holding, the distorted density is only distorted with respect to its conditional mean. In other words, it too will have a Gaussian distribution and the likelihood ratio is the ratio of two Gaussian densities. Assume without loss of generality that $\pi(\varepsilon) \sim N(0, I)$ is a standard normal distribution. The distorted distribution $\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)$ has conditional mean w and variance-covariance matrix I. From the definition of a multivariate normal density, it follows that the log-likelihood ratio is given by:

$$\log\left(\frac{\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)}{\pi(\varepsilon)}\right) = \frac{1}{2}\left(-(\varepsilon - w)'(\varepsilon - w) + \varepsilon'\varepsilon\right)$$

The definition of conditional relative entropy requires calculating the log-likelihood ratio expected under the distorted model:

$$\begin{aligned} &\int \log\left(\frac{\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)}{\pi(\varepsilon)}\right)\hat{\pi}(\varepsilon\,\big|\epsilon^t, y_0)d\varepsilon \\ = &\frac{1}{2}\int\left(-(\varepsilon - w)'(\varepsilon - w) + \varepsilon'\varepsilon\right)\hat{\pi}(\varepsilon\,\big|\epsilon^t, y_0)d\varepsilon \end{aligned}$$

The first term is the integral of a multivariate normal distribution because the distorted distribution is Gaussian:

$$-\int\frac{1}{2}(\varepsilon - w)'(\varepsilon - w)\hat{\pi}(\varepsilon\,\big|\epsilon^t, y_0)d\varepsilon = -\frac{1}{2}trace(I)$$

To calculate the second term, write $\varepsilon = w + (\varepsilon - w)$ such that

$$\begin{aligned} \frac{1}{2}\varepsilon'\varepsilon &= \frac{1}{2}w'w + \frac{1}{2}(\varepsilon - w)'(\varepsilon - w) + w'(\varepsilon - w) \\ \frac{1}{2}\int\varepsilon'\varepsilon d\varepsilon &= \frac{1}{2}w'w + \frac{1}{2}trace(I) \end{aligned}$$

The final expression for conditional relative entropy is:

$$\int \log\left(\frac{\hat{\pi}(\varepsilon\,|\epsilon^t, y_0)}{\pi(\varepsilon)}\right)\hat{\pi}(\varepsilon\,\big|\epsilon^t, y_0)d\varepsilon = \frac{1}{2}w'w$$

which is of the form originally assumed.

11 Detection error probabilities

Robert Lucas reportedly warned that we should "beware of economists bearing free parameters", because of his desire to find appropriate microfoundations for macroeconomic phenomena. So far in this lecture we have been guilty of bearing one free parameter, the η_0 in the constraint problem or the equivalent θ in the multiplier problem. We therefore need some way of disciplining our choice. Anderson, Hansen and Sargent (2003) suggest calibrating θ in such a way that it is difficult to distinguish between the approximating and distorted models in finite data samples. Their idea is to use a statistical theory of model selection, asking with what probability it would be possible to distinguish between the two distributions given a data sample of given length. This is the "detection error probability". It will be context-specific, depending on both the approximating model at hand and the objective function of the policymaker. To make things more formal, consider the linear-quadratic-Gaussian robust control problem with decision rules $u_t = -F(\theta)x_t$ for the policymaker and $w_{t+1} = \kappa(\theta)x_t$ for the hypothetical evil agent. We can write the approximating and distorted models as:

$$\begin{aligned} x_{t+1} &= A_0 x_t + C\check{\epsilon}_{t+1} \\ x_{t+1} &= \underbrace{(A_0 + C\kappa(\theta))}_{\hat{A}} x_t + C\epsilon_{t+1} \end{aligned}$$

Both conditional distributions are absolutely continuous, putting some positive probability mass on the same events. This makes the calculation of detection error probabilities straightforward, since if it was not the case then the occurrence of some events would completely reveal which distribution was determining the data.

At the heart of the calculation of detection error probabilities is a likelihood ratio test between the approximating and distorted models. Defining L_A as the likelihood under the approximating model and L_B as the likelihood under the distorted model, the log-likelihood ratio is $\log(L_A/L_B)$. If it greater than zero then the test selects the approximating model, if it is less than zero then the test selects the distorted model. The power of this likelihood test to discriminate between two distributions depends on (i) how different the distributions are and (ii) how much data is available. When calculating detection error probabilities we fix the amount of data but vary how different the distributions are by changing the value of the multiplier θ. Start by considering what happens if the approximating model is generating the data. We are interested in the ability of the likelihood ratio test to detect this, i.e. we are interested in how often the likelihood ratio test selects the *wrong* model:

$$P_A = Prob\left(\log(L_A/L_B) < 0 \middle| \text{approximating model generates data}\right)$$

It not usually possible to calculate this probability for a finite data series, so Anderson et al. (2003) recommend approximating P_A by simulation, which can be done at relatively low cost by simulating from $x_{t+1} = A_0 x_t + C\check{\epsilon}_{t+1}$ and then constructing the likelihoods according to $x_{t+1} \sim N(A_0 x_t, CC')$ and $x_{t+1} \sim N(\hat{A}x_t, CC')$. 500 simulations is usually adequate in simple models. The proportion of simulations for which the likelihood ratio test incorrectly selects the distorted model gives a measure of P_A. As θ increases the policymaker is concerned with a large set of misspecified models, which drives an increasingly larger wedge between the approximating and distorted models and makes P_A fall. To complete the calculation of detection error probabilities, consider the complementary calculation of distinguishing between the approximating and distorted models if the distorted model is generating the data. The probability of interest is:

$$P_B = Prob\left(\log(L_A/L_B) > 0 | \text{ distorted model generates data}\right)$$

The probability in this case measures the probability of incorrectly selecting the *wrong* approximating model when the distorted model is correct. It can again be approximated by simulation. A final measure capturing the combined detection error probability is:

$$P(\theta) = \frac{1}{2}\left(P_A + P_B\right)$$

A sensible detection error probability would be around 20-25%.