LINEAR ALGEBRA

CHAPTER #4 Vector Space

→ Linear Combination:

$$w = k_1 v_1 + k_2 v_2 + \dots + k_n v_n$$

Steps:-

i- k_1, k_2, k_3 → form

Such that $k_1(x_1^2 + x_1 + 1) + k_2(x_2^2 + x_2 + 1) + \dots$

ii - Multiply k's values ~~all~~ such that

$k_1 x_1^2 + k_1 x_1 + k_1 + k_2 x_2^2 + k_2 x_2 + k_2 + \dots$

iii- Seperate the value of different degree's such that

$(k_1 x^2 + k_2 x^2 + k_3 x^2 + \dots) + (k_1 x + k_2 x + k_3 x \dots) + \dots$

iv- Put step #3 equals to given linear combination value w.r.t degree.

v- ~~Ata~~ Form Matrix

vi- Echelon Form

vii - Find values of k's.

viii- Put the values of k's in step #3

ix- If given linear combination value is equal to step # 2,

x- It's a linear Combination.

→ Span:-

$(u_1, u_2, u_3) = c_1(\quad) + c_2(\quad) + c_3(\quad)$

Steps:-

i- Form an equation, equals to u_1, u_2, u_3
ii- Matrix.
iii- Determinant.
iv- If determinant is equal to 0 linearly dependent means doesnot span, if determinant is not equal to 0 linearly independent means it's span.

→ Linear Independent.

$$k_1 v_1 + k_2 v_2 + k_3 v_3 = 0.$$

Steps:-

i- Make matrix form.
ii- Find determinant.
iii- If determinant is 0, it is linearly dependent else linearly independent.

→ Basis:-

~~Steps~~:-

i- Linearly independent.
ii- Span.
iii- If above both conditions satisfies, it means it form basis.

→ Dimension:-

→ if there're 4 variables, then it is 4 dimension.

- If no. of vector space > dimension #, then it is linearly dependent.
- If they're < dimension, it doesn't span vector space.

$R^n \rightarrow n$ dimension space

$R^3 \rightarrow 3$ dimension space.

→ Row space:-

Step #:-

i- After solving the matrix, through ~~Gauss elimination~~ row echelon method.

ii- Jis row mai leading 1 lie krta hai, wo row space hga.

→ Column space:-

Steps:

i- same as step 1 of row space.

ii- Jis column mai leading 1 lie krta hai, wo column space hga.

→ Row Vector & Column Vector:-

simple given matrix ky rows & columns, row ~~vect~~ and column vector respectively of those matrix.

→ Null Space:
solution space of homogeneous system of equations

$AX = 0$

matrix → x values → 0 matrix.

→ Rank (A):
→ Jitny vector space bnai ek matrix sy. if 3, then rank will be 3.
P.S: (Basis calculate krna lazmi hai, usky bad hi hoga)

→ Nullity (A): (leading variables).
→ Orginal variables bn rhy hai.
· if 2 then nullity is 2.

→ Important Theorem:
Rank (A) + Nullity (A) = n (no. of column in given matrix)

→ $Ax = b$ is consistent.

Steps:
i- Solve the matrix, using gaussian elimination.
ii- Find the value of x's.
iii- Put the values of x's in matrix.
iv- If it is equal to b, then it is consistent.

P.S: Solution exist krta hai tw matrix consistent hai.

→ Transition Matrix:

$P_{B \to B'}$, $P_{B' \to B}$

→ Steps | Procedure to compute $P_{B \to B'}$

i- Form the matrix $[B' | B]$.

ii- Use elementary row operation to reduce the matrix in step 1 in to reduced echelon form.

iii- The resulting matrix will be $[I | P_{B \to B'}]$.

iv- Extract the matrix from right side from Step #3 and it'll ~~be~~ form $P_{B \to B'}$.

Apply Row operation:-

$[B' | B] \to P_{B \to B'}$

$[B | B'] \to P_{B' \to B}$.

CHAPTER #5

Eigen Values & Eigen Vectors.

→ To find Eigen Values:- (λ)

Steps:-

i- $\det(A - \lambda I) = 0$

ii- You'll ~~find~~ find an equation put that equation equal to 0.

iii- ~~Find~~ Then, You'll find the value's of λ, that'll be eigen values of given matrix.

P.S: Eigen values must be different, ~~of course~~.

~~&~~ P.S.S: It can be same.

→ Eigen Vectors:

Steps:

i- Find the eigen values, ~~using eigen~~

ii- ~~E~~Eigen values must be different if 2 eigen values are same, you cannot find the eigen vector.

~~Because~~

iii- Put that eigen values linewise in a matrix, ~~in which~~ (that is of before find the determinant/eigen value, ~~in~~ which λ is included).

iv- You'll find a matrix, apply row echelon form on it.

v- See the pavit column. put 1 if zero after a value.

vi- That will be eigen vector.

→ Bases for eigen spaces of Matrix:

Steps:

i- Determinant of matrix.

ii- Put Determinat equal to 0.

iii- Find the eigen value.

iv- Those eigen values are the eigenspaces.

~~for each eig~~

vi- Then put values in $(\lambda I - A)x = 0$ $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

vii- Put λ values in matrix, You will find basis for eigen space.

Direct finding eigen values:-
Agr matrix diagonal Matrix hai, upper or lower triangular matrix hai, tw jo diagonals py values hngi wohi uski eigen values hngi.

→ Diagonalization:-

$$\rightarrow B = P^{-1} A P.$$

→ Diagonalizable:-

$$A = P D P^{-1}.$$

Steps:

i- Find eigen values. P.S: (if two values are same, matrix is not diagonalizable).
ii- Then find eigen vectors
iii- Put that eigen vector to form a matrix.
iv- That matrix will be P.
v- P^{-1} (inverse of P (matrix)).

CHAPTER # 6
INNER PRODUCT SPACES

→ 4 Axioms:

→ $u \cdot v = v \cdot u$
→ $u \cdot (v + w) = u \cdot v + u \cdot w$
→ $k(u \cdot v) = (ku) \cdot v$
→ $v \cdot v \geq 0$ & $v \cdot v = 0$ iff $v = 0$.

→ Norm:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\|v\| = \sqrt{\langle v, v \rangle} \Rightarrow \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$$

→ Distance:

$$d(u, v) = \|u - v\| \sqrt{\langle u - v, u - v \rangle}.$$

→ vector of norm 1 is unit vector.

→ To find $\langle u, v \rangle$ of a Matrix:-

Step:-

i- Multiply those two matrix with corresponding #'s.

→ Angle or orthogonality:-

$$\theta = \cos^{-1} \frac{\langle u, v \rangle}{\|u\| \|v\|}.$$

→ Orthogonality:

$\langle u, v \rangle = 0$ (product space is orthogal)

$\langle u \cdot v \rangle = 0.$

$u(1,1), v(1,-1) = (1)(1) + (1)(-1) = 0.$ ✓

→ Orthogonal:

$\langle v_1, v_2 \rangle = \langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0.$

→ Orthonormal:

$$u_1 = \frac{v_1}{\|v_1\|}, \quad u_2 = \frac{v_2}{\|v_2\|}, \quad u_3 = \frac{v_3}{\|v_3\|}$$

(HP#7)

→ Orthogonal Matrix:- $(\det(A)) = 1$ / $\det(A) = -1$

$A^{-1} = A^T$

$AA^T = A^T A = I.$

→ **Gram - Schmidt Process:**

steps:

i- $v_1 = u_1$

ii - $v_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

iii - $v_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

iv - $v_4 = u_4 - \dfrac{\langle u_4, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_4, v_2 \rangle}{\|v_2\|^2} v_2 - \dfrac{\langle u_4, v_3 \rangle}{\|v_3\|^2} v_3$

$\vdots$

→ **Ortogonal Complements:** $W^{\perp}$

steps:

i. Make Matrix.

ii- Apply row operation.

iii- Find the values, and Put the values.

CHAPTER #7

→ Same as chp#5, 6.

CHAPTER #8

→ **GENERAL LINEAR Transformations:**

$T: V \rightarrow W.$

→ **Linear Transformation:**

$T(u+v) = T(u) + T(v)$

$T(cu) = cT(u).$

$T \rightarrow$ linear operator.

→ **kernel Linear transformation:** $(\ker(T))$

$Tx = Ax.$

- Form homogeneous system.
- Reduced row echelon form.

→ **Inverse Transformation:**

- one to one.
- onto

Mahnoor

Subspace:

$\rightarrow u = \text{Proj}_W \, u + \text{Proj}_{W^\perp} \, u.$

Defination of projection:

$$\text{Proj}_W \, u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \dots + \frac{\langle u, v_n \rangle}{\|v_n\|^2} v_n.$$

~~Basis of ortho:~~

$\boxed{\|V_1\|^2}$

projection

projection

If A matrix is $n \times n$ matrix
then the
following statements are equivelent.

1. A is investible matrix.
2. $Ax=0$ has trivial solution.
3. The reduced echelon form of A is Idendity matrix.
4. $AX=b$ is consistance for every $n \times n$ matrix b.
5. $Ax=b$ has only ~~unique~~ one solution.
6. The column of A is linearly independent.
7. The row of A is linearly independent.
8. $\det |A| \neq 0$.
9. The columns of A spans R^n.
10. The rows of A spans R^n.
11. The columns of A forms a Basis for R^n.
12. The rows of A forms a Basis for R^n.
13. A has rank n.

(depe

(14) The nullity of A is zero.
(invertible matrix).

→ Fundamental spaces of A:

- Null space of A
- Column space of A
- Row space of A.

A^T

→ Rank (A) = Rank (A^T)

→ Orthogural Complement:- $(W^\perp)$

- If W is a subspace of R^n then the set of all vectors in R^n that are orthogural to every vector in W.

Called orthogural complement of W.

Note:

Determinant find kro agr zero ky equal ni ata tw Rank ki value no. of columns ky equal hg:
~~Aur~~ agr zero ho tw rank n

Jab vectors ky darmiyan angle 90° ka a jai tw wo orthogural hta hai.

$\theta = 90°$
$\boxed{a \cdot b = |a||b| \cos\theta}$

dot product / inner product is equal to zero.