Introduction to Underwater Robotics, Modelling and Control

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MASTER ROC and MIR

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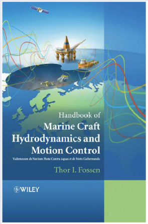
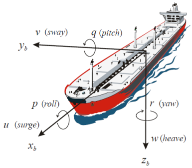


Handbook of Marine Craft Hydrodynamics and Motion Control. John Wiley & Sons Ltd.



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TECHNIQUES DE L'INGÉNIEUR

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Robots marins et sous-marins - Perception, modélisation, commande

Vehicle classification

- **Marine craft:** ocean and flight vehicles are usually called craft: ships, high-speed craft, semi-submersibles, floating rigs, submarines, remotely operated and autonomous underwater vehicles, torpedoes and other propelled/powered.
- **Vehicles:** travel only on land are usually called craft.
- **Vessel:** "hollow structure for purposes of transportation and navigation.

The words vessel, ship and boat are often used interchangeably.

- **Ship:** "any large floating vessel capable of crossing open waters, as opposed to a boat, which is generally a smaller craft. formerly sailing vessels having three or more masts. Now, vessel of more than 500 tons of displacement.
- **Submarine:** "any naval vessel that is capable of propelling itself under and on water
- **Underwater Vehicle:** "small vehicle that is capable of propelling itself beneath the water surface as well as on the water's surface. This includes unmanned underwater vehicles (UUV), remotely operated vehicles (ROV) and autonomous underwater vehicles (AUV).

ASV: Autonomous Surface Vehicles

Inspector, ECA

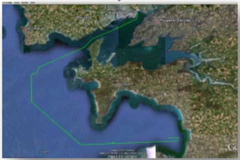
Vaimos, ENSTA Bretagne/Ifremer

PacX wave glider, Liquid Rbotics

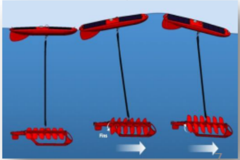
Defense

Oceanography

Climatology



Brest-Douarnenez, 2012



ROV: Remotely Operated Vehicles


Observer, Subsea Tech


L2ROV, LIRMM

H2000, ECA Hytec

Victor 6000, Ifremer

Applications:

- Offshore industry
- Manipulation
- Survey

K-Ster MineKiller, ECA

Crawler:

Definition:

Tracked vehicle / wheels and thrusters

AC-CELL 100, AC-CESS

LBC, Seabotix

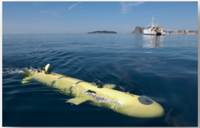
Roving BAT, ECA Hytec

Applications:

- Inspection / cleaning : pipe and hull
- Burying cables and mining

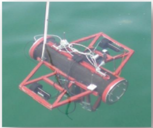
SeabedCrawler, Nautilus Minerals, Australie

AUV: Autonomous Underwater Vehicle


AstrX et IdefX, IFREMER


Daurade, ECA/GESMA


Remus, HYDROID


Sardine, Lab-STICC/ENSTA Bretagne

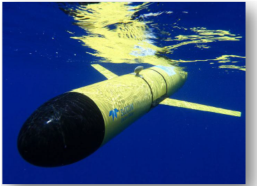
Aquatis, ESIEA

Lirmia 2, LAFMIA/LIRMM

Applications:

- Observation (video, sonar)
- Measure
- Oceanography
- Mine-clearing

Glider or Underwater Glider:



Slocum glider, Teledyne / WHOI
Record Atlantique, 2011

1KA Seaglider, iRobot

SeaExplorer, ACSA

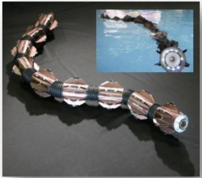
Applications:

- Increased autonomy and satellite communication on the surface
- Oceanography

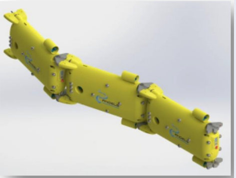
Inspired bio:

Jellyfish, fish, eels / snakes

Aquajelly, Festo



ACM-R5, Tokyo Inst. of Tech.



Angels, IRCCyN/Mines de Nantes

Applications:

- Military (land / water)
- Modular

Underwater robotic issues

Problematics:

- Water is an extremely absorbent and diffracting (composed of many layer) environment. Thus, the electromagnetic waves do not pass except slightly the light (several meters).
- Water has a very important mechanical action on moving objects (added mass and viscosity)

Communication:

- no fast and high-speed communication
- only by accoustic waves (sound speed $= 1.5 km.s^{-1}$ is 200,000 slower than light speed), thus a very slow flow.

Control:

- Thruster no effecient as a wheel
- added mass and frition forces are very important

Localisation

Base line systems:

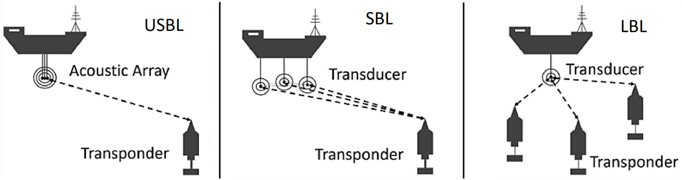
Operating principle:

- 3 or more acoustic baselines : exact absolute or relative positions known.
- Determined position by triangulation thanks to acoustic wave

Types of Acoustic Positioning Systems:

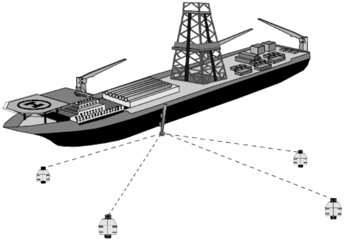
The distance between the active acoustic sensing elements is generally used to define 3 type:

- USBL: Ultra Short Baseline, <10cm
- SBL: Short Baseline, 20m to 50m
- LBL: Long Baseline, 100m to 10km



Localisation

Long Baseline (LBL)



Operating principle:

Long Baseline systems derive a position with respect to a seafloor deployed array (grid) of transponders (up to 10km). A LBL system does not require a IMU.

Localisation
Long Baseline (LBL)

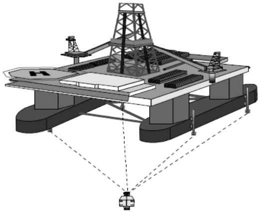
Advantages:

- Very good position accuracy independent of water depth.
- Observation redundancy.
- Can provide high relative accuracy positioning over large areas.
- Does not need a IMU
- Small transducer.

Disadvantages:

- Complex system requiring expert operators.
- Large arrays of expensive equipment.
- Operational time consumed for deployment/recovery.
- Conventional systems require comprehensive calibration at each deployment.

Localisation
Short Baseline (SBL)



Operating principle:

When the distances between the three (or more) transponders are much smaller than the area of operation and the acoustic devices are fixed under the hull of the vessel.
In this case, calibration is no longer a problem as it is only carried out once, but the accuracy of positioning by triangulation is less good than with a long base due to the proximity of the transponders. In addition, as the vessel carrying the SBL is not stationary, it is necessary to take into account its position (differential GPS) and its orientation (IMU) to correct the measurements.

Localisation
Short Baseline (SBL)

Advantages:

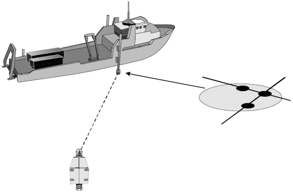
- Low system complexity makes SBL an easy tool to use.
- Good update rate when used with a pinger
- Good range accuracy.
- Spatial redundancy built in.
- Ship based system – no need to deploy transponders on the seafloor.
- Small transducers.

Disadvantages:

- System needs large baselines for accuracy in deep water (>30m).
- Very good dry dock/structure calibration required.
- Detailed offshore calibration of system required
- Absolute position accuracy depends on IMU and vertical reference unit.
- >3 transceiver deployment poles/machines needed.

Localisation

Ultra Short or Super Short Baseline (USBL or SSBL)



Operating principle:

The transponders of a short base line (SBL) can be brought together to form a single sensor, a small acoustic array (antenna) capable of sensing not only the distance of the vehicle (by measuring time of flight), but also the direction in which it is located (by measuring the phase shift of the various elements of the antenna). This type of device is called an Ultra Short BaseLine (USBL) or even a Super Short BaseLine (SSBL).
As with a SBL, it is necessary to take into account the position and orientation of the craft.

Localisation

Ultra Short or Super Short Baseline (USBL or SSBL)

Advantages:

- Low system complexity makes USBL an easy tool to use.
- Ship based system – no need to deploy a transponder array on the seafloor.
- Only a single transceiver at the surface – one pole/deployment machine.
- Good range accuracy.

Disadvantages:

- Detailed calibration of system required
- Accuracy depends on additional sensors - IMU and vertical reference unit.

Specifications:

- **Frame rate:** 1Hz max and more around 0.1Hz. no high dynamic estimation.
- **Price:** more than 2k€ up to 100k€ ore more.

Frequency Bands, Maximum Range and accuracy:

Frequency	Range	Maximum range	static accuracy
Low	8 kHz to 16 kHz	>10km	2m to 5m
Medium	18 kHz to 36 kHz	2km to 3.5km	0.25m to 1m
High	30 kHz o 60 kHz	1,500m	0.15m to 0.25m
Extra High	50 kHz to 110 kHz	<1,000m	<0.05m
Very High	200 kHz to 300 kHz	<100m	<0.01m

Maximum water depth:

- LF Operational to full ocean depth
- MF Problems beyond 3,500m
- EHF Problems beyond 800m to 1,000m
- VHF Problems beyond 100m

Localisation

Depth Measure:

Operating principle:

The depth measurement is obtained simply by measuring pressure. The relationship of proportionality that links pressure and depth depends on the density of the seawater and therefore mainly salinity. We note that in the sea, the pressure increases about 1025 hPa (1.025 bar) every 10 meters.

Positioning accuracy:

The choice of sensor will depend on the application. If we plan to control the immersion depth, it will be necessary to choose a sensor resolution smaller than the control resolution. Resolution and range of sensors being closely related, it is illusory to have centimeter accuracy at 6,000 meters depth. In this case, the distance between the vehicle and the bottom are controlled (by an optical or acoustic measuring device).

Frame rate:

Depends on the electronic system: Usually, between 10 and 100Hz

Localisation
DVL:

Definition

DVL for Doppler Velocity Log is an acoustic sensor that estimates velocity relative to the sea bottom or a water Layer.

Operating principle:

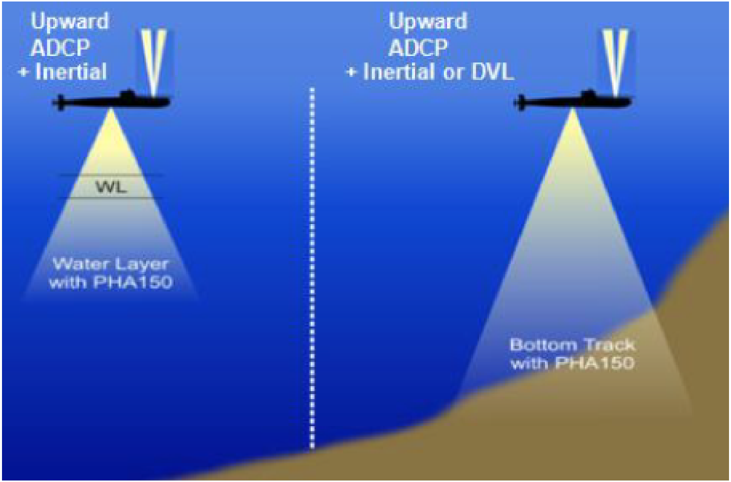
Sending a long pulse along a minimum of three acoustic beams, each pointing in a different direction.

Specifications:

- Speed relative to water or water layer,
- Velocity (up to 3 axes) into DVL's frame,
- $\pm 0.1 cm.s^{-1}$.
- frame rate between 1 and 10Hz
- cost: more than 2k€ up to 100k€.

Localisation

DVL:



Localisation

Inertial units:

IMU: Inertial Measurement Units

The objective of an IMU is to measure linear accelerations (accelerometer)and vehicle angular rate (gyrometer) in three dimensions.

AHRS: Attitude and Heading Reference Systems

AHRS is an IMU with a magnetometer and internal filtering (often a Kalman filter) to provide an estimate of the vehicle's attitude (Euler angles or quaternions).

INS: Inertial Navigation Systems

INS is bades on a AHRS and an algorithm allowing to compute the orientation of the vehicle.

Localisation

Inertial units:

Two technologies:

- Fiber Optic Gyroscopes (FOG):
  - very expensive (more than 5K€per axis),
  - small measurement bias (allow acceleration integration)
- Micro Electro-Mechanical Systems (MEMS):
  - very cheap (around 100/2000€),
  - important measurement bias

Conclusion:

- Frequent to associate an INS with a DVL
- On a ROV -> MEMS
- On a AUV -> FOG

Table of Content

Perception

Visual systems:

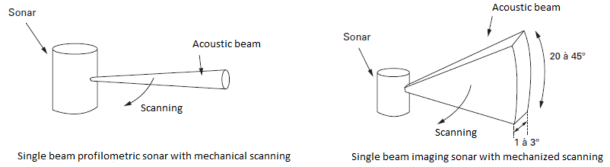
The most economical way to perceive the environment of a Underwater robot is the video. Most vehicles (especially ROV) are already equipped with one or more cameras. However, some phenomenons complicate the use of vision under the sea in relation to land use:

- High diffraction
- Turbidity: can be due to the propellants
- Low brightness

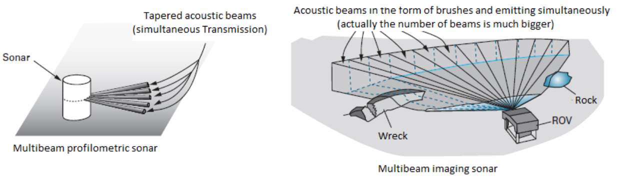
Perception

Acoustic systems:

Single-beam scanning sonar:



Multi-beam scanning sonar:



Perception
Acoustic systems:

Single-beam properties:

- Size: Micron de Tritech, 50mm diameter, 79mm height and 324g
- Less expensive than multi

Multi-beam properties:

- Size: BlueView M900-45, 10cm diameter, 20cm height and 2.2Kg
- Speed of acquisition (commonly between 10 and 20 frames per second)

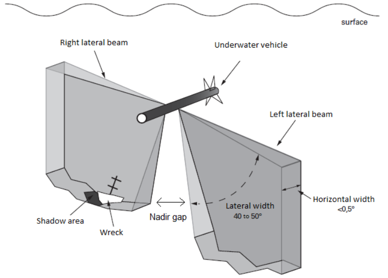
General properties:

- Acoustic frequencies between 300 kHz and 2 MHz
- Range of 300m
- from 5 k€up more than 100 k€

Perception

Acoustic systems:

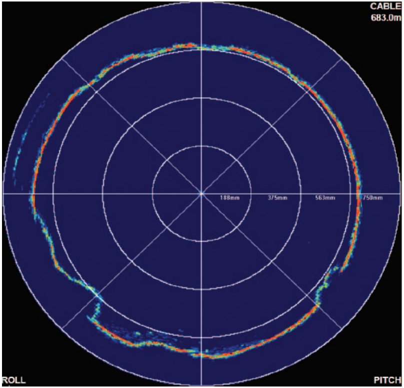
Lateral sonar:



The vehicle has to move with a constant attitude, depth and heading. It is used to obtain an image of the seabed over large areas on either side of the towing vehicle, mainly to search for and identify objects placed on the bottom (wrecks, mines, pipelines, etc.).

Perception

Single-beam sonar example:

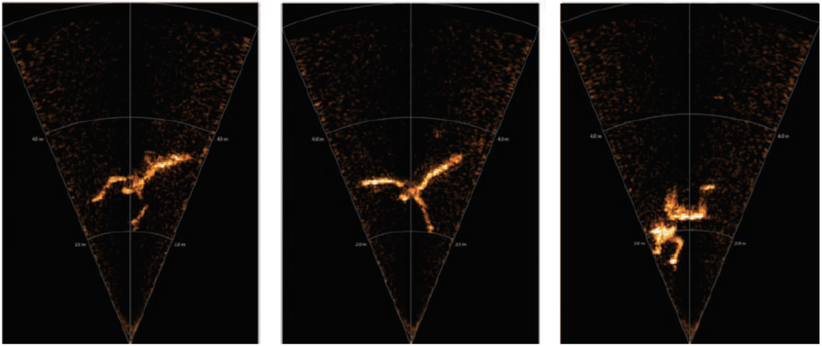


Example of an acoustic image obtained from a single beam profiling sonar : Mini Pipe Profiling Sonar - 2512 USB, Marine Electronics Ltd.

This is the interior of a 1.2 m diameter concrete pipe. There are fatty deposits at 8 and 4 o'clock and silt and gravel deposits at the bottom.

Perception

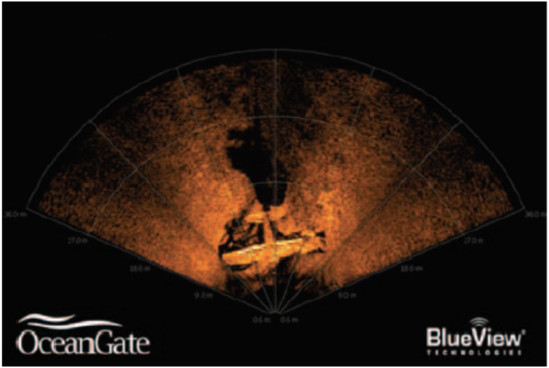
Multi-beam sonar example:



Example of a sequence of pelvis images taken using a BlueView P450-45 multi-beam sonar. We can clearly see a man swimming (credit: Teledyne BlueView)

Perception

Multi-beam sonar example:



Example of an acoustic image of an aircraft wreck lying on a sedimentary bottom, obtained by a BlueView M900 acoustic camera (credit: OceanGate / Teledyne BlueView)

Perception

Multi-beam lateral sonar example:

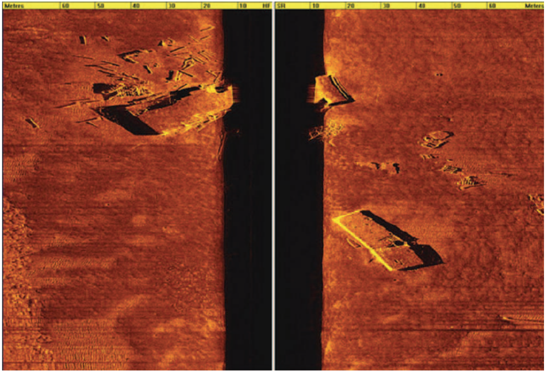
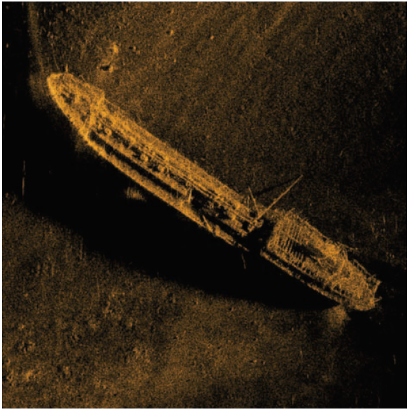


Image acquired by synthetic aperture sonar HISAS 1030 onboard the AUV HUGIN 1000. The length of the wreck is 68 m (Holmengraa tanker lying at 77 m bottom) (credit: Kongsberg Maritime)

Perception

Synthetic Multi-beam sonar example:



Example of an acoustic image of an aircraft wreck lying on a sedimentary bottom, obtained by a BlueView M900 acoustic camera (credit: OceanGate / Teledyne BlueView)

Perception and Localisation
SLAM:

Definition:

In robotic mapping and navigation, simultaneous localization and mapping (SLAM) is the computational problem of constructing or updating a map of an unknown environment while simultaneously keeping track of an agent's location within it.

SLAM mainly for small vehicles:

- Limit positioning errors
- Guarantee full sonar coverage of an area (mine-clearing)

Scientific and technical obstacles:

- Low resolution sonar images
- Few marks, different according to point of view

Communication
Problematic:

As for location, the non-penetration of electromagnetic waves into the water complicates communications between underwater vehicles or between an underwater vehicle and the surface. Several technological solutions exist. They are chosen essentially according to the desired flow and the range.

Communication
Acoustic systems:

- The phenomena of refraction, dispersion, reflection, absorption and diffraction, are all obstacles to sound propagation. They are at the origin of curved paths, multi-paths, attenuations, etc.
- As a result, the flow of acoustic communications is very limited and strongly related to the desired range and directivity. Thus, current commercial products can not exceed speeds of the order of thirty kilobits per second (38.4 kbps for LinkQuest High Speed modems and 31.2 kbps for EvoLogics modems) for ranges of less than 10 km. We will note these maximum values can not be reached simultaneously.
- In range limit, the flow rates are therefore often lower. Works research has resulted in higher flows in special conditions of use (e.g vertical transmission in deep waters), but these devices are not marketed.

Communication
Cables:

Electric cables:

- Simple, low cost
- Low data rate from 100m length

Optic Fiber cables:

- Complex, expensive
- High data rate

optical communication devices:

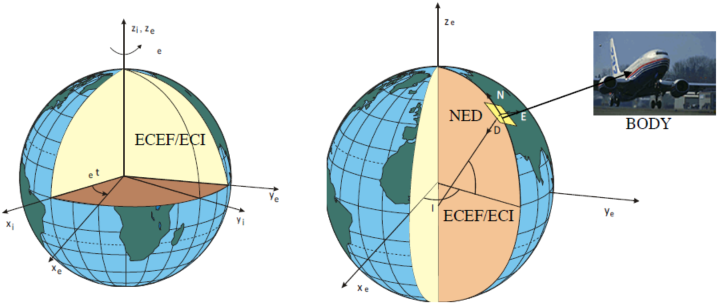
- Not mature technology
- Less than 10 meter

Table of Content

Introduction

"The study of dynamics can be divided into two parts: kinematics, which treats only geometrical aspects of motion, and kinetics, which is the analysis of the forces causing the motion"

Reference frames:



Reference frames:

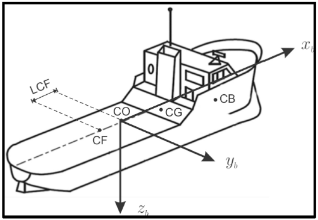
- **ECI** $\{i\}$**:** Earth centered inertial frame; non-accelerating frame (fixed in space) in which Newton's laws of motion apply.
- **ECEF** $\{e\}$**:** Earth-Centered Earth-Fixed frame; origin is fixed in the center of the Earth but the axes rotate relative to the inertial frame ECI.
- **NED** $\{n\}$**:** North-East-Down frame; defined relative to the Earth's reference ellipsoid.
- **BODY** $\{b\}$: Body frame; moving coordinate frame fixed to the vessel.
  - xb- longitudinal axis (directed from aft to fore)
  - yb- transversal axis (directed to starboard)
  - zb-normal axis (directed from top to bottom)

Reference Frames
Body-Fixed Reference Points

- CG - Center of gravity
- CB - Center of buoyancy
- CF - Center of flotation

CF is located a distance LCF from CO in the x-direction.

The center of flotation is the centroid of the water plane area Awp in calm water. The vessel will roll and pitch about this point.



Coordinate-free vector

$$\vec{u} = u_1^n \vec{n}_1 + u_2^n \vec{n}_2 + u_3^n \vec{n}_3$$

$\vec{n}_i$ $(i = 1, 2, 3)$ are the unit vectors that define $\{n\}$

Coordinate form of $\vec{u}$ in $\{n\}$

$$\mathbf{u}^n = [u_1^n, u_2^n, u_3^n]^\top$$

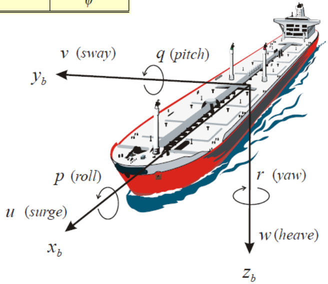
Reference frames

6-DOF motions

DOF		forces and moments	linear and angular velocities	positions and Euler angles
1	motions in the x-direction (surge)	X	u	x
2	motions in the y-direction (sway)	Y	v	y
3	motions in the z-direction (heave)	Z	w	z
4	rotation about the x-axis (roll, heel)	K	p	ϕ
5	rotation about the y-axis (pitch, trim)	M	q	θ
6	rotation about the z-axis (yaw)	N	r	ψ

The notation is adopted from:

SNAME (1950). Nomenclature for Treating the Motion of a Submerged Body Through a Fluid. *The Society of Naval Architects and Marine Engineers, Technical and Research Bulletin No. 1-5, April 1950, pp. 1-15.*



Reference Frames
Notation

ECEF position:	$\mathbf{p}^e_{b/e} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$	Longitude and latitude	$\boldsymbol{\Theta}_{en} = \begin{bmatrix} l \\ \mu \end{bmatrix} \in \mathcal{S}^2$
NED position:	$\mathbf{p}^n_{b/n} = \begin{bmatrix} N \\ E \\ D \end{bmatrix} \in \mathbb{R}^3$	Attitude (Euler angles)	$\boldsymbol{\Theta}_{nb} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \in \mathcal{S}^3$
Body-fixed linear velocity	$\mathbf{v}^b_{b/n} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \in \mathbb{R}^3$	Body-fixed angular velocity	$\boldsymbol{\omega}^b_{b/n} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} \in \mathbb{R}^3$
Body-fixed force:	$\mathbf{f}^b_b = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3$	Body-fixed moment	$\mathbf{m}^b_b = \begin{bmatrix} K \\ M \\ N \end{bmatrix} \in \mathbb{R}^3$

Generalized position, velocity and force

$$\boldsymbol{\eta} = \begin{bmatrix} \mathbf{p}^n_{b/n} \ (\text{or } \mathbf{p}^e_{b/n}) \\ \boldsymbol{\Theta}_{nb} \end{bmatrix}, \quad \boldsymbol{\nu} = \begin{bmatrix} \mathbf{v}^b_{b/n} \\ \boldsymbol{\omega}^b_{b/n} \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \mathbf{f}^b_b \\ \mathbf{m}^b_b \end{bmatrix}$$

Transformations between BODY and NED

Orthogonal matrices of order 3:

$$O(3) = \{\mathbf{R} | \mathbf{R} \in \mathbb{R}^{3\times3}, \quad \mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathbf{I}\}$$

Special orthogonal group of order 3:

$$SO(3) = \{\mathbf{R} | \mathbf{R} \in \mathbb{R}^{3\times3}, \quad \mathbf{R} \text{ is orthogonal and } \det\mathbf{R} = 1\}$$

Rotation matrix:

$$\mathbf{R}\mathbf{R}^\top = \mathbf{R}^\top\mathbf{R} = \mathbf{I}, \qquad \det\mathbf{R} = 1$$

Since **R** is orthogonal, $\mathbf{R}^{-1} = \mathbf{R}^\top$

Example:

$$\mathbf{v}^{\text{to}} = \mathbf{R}^{\text{to}}_{\text{from}}\mathbf{v}^{\text{from}}$$

Transformations between BODY and NED

Cross-product operator as matrix-vector multiplication:

$$\boldsymbol{\lambda} \times \mathbf{a} := \mathbf{S}(\boldsymbol{\lambda})\mathbf{a}$$

$$\mathbf{S}(\boldsymbol{\lambda}) = -\mathbf{S}^{\top}(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & -\lambda_3 & \lambda_2 \\ \lambda_3 & 0 & -\lambda_1 \\ -\lambda_2 & \lambda_1 & 0 \end{bmatrix}, \quad \boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix}$$

where $\mathbf{S} = -\mathbf{S}^{\top}$ is a skew-symmetric matrix

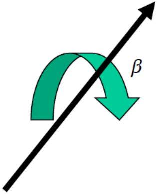
Transformations between BODY and NED

Eulers theorem on rotation:

$$\mathbf{R}_{\lambda,\beta} = \mathbf{I}_{3\times 3} + \sin\beta\ \mathbf{S}(\boldsymbol{\lambda}) + (1-\cos\beta)\ \mathbf{S}^2(\boldsymbol{\lambda})$$

$$\boldsymbol{\lambda} = [\lambda_1, \lambda_2, \lambda_3]^\top, \quad |\boldsymbol{\lambda}| = 1$$

$$\mathbf{v}^n_{b/n} = \mathbf{R}^n_b \mathbf{v}^b_{b/n}, \quad \mathbf{R}^n_b := \mathbf{R}_{\lambda,\beta}$$



where

$$\begin{aligned}
R_{11} &= (1-\cos\beta)\ \lambda_1^2 + \cos\beta \\
R_{22} &= (1-\cos\beta)\ \lambda_2^2 + \cos\beta \\
R_{33} &= (1-\cos\beta)\ \lambda_3^2 + \cos\beta \\
R_{12} &= (1-\cos\beta)\ \lambda_1\lambda_2 - \lambda_3\sin\beta \\
R_{21} &= (1-\cos\beta)\ \lambda_2\lambda_1 + \lambda_3\sin\beta \\
R_{23} &= (1-\cos\beta)\ \lambda_2\lambda_3 - \lambda_1\sin\beta \\
R_{32} &= (1-\cos\beta)\ \lambda_3\lambda_2 + \lambda_1\sin\beta \\
R_{31} &= (1-\cos\beta)\ \lambda_3\lambda_1 - \lambda_2\sin\beta \\
R_{13} &= (1-\cos\beta)\ \lambda_1\lambda_3 + \lambda_2\sin\beta
\end{aligned}$$

Transformations between BODY and NED

Euler Angle Transformation

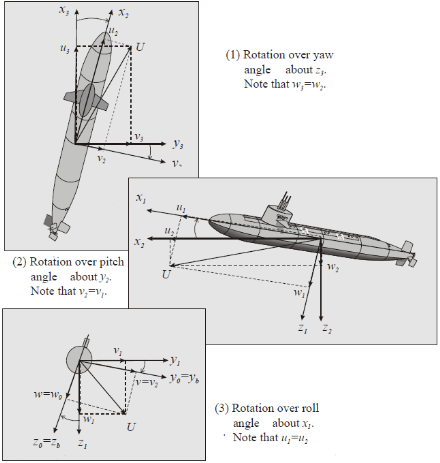
Three principal rotations:

$$\boldsymbol{\lambda} = [0, 0, 1]^\top \qquad \beta = \psi$$
$$\boldsymbol{\lambda} = [0, 1, 0]^\top \qquad \beta = \theta$$
$$\boldsymbol{\lambda} = [1, 0, 0]^\top \qquad \beta = \phi$$

$$\mathbf{R}_{z,\psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_{y,\theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$\mathbf{R}_{x,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$



(1) Rotation over yaw angle about z_3. Note that $w_3 = w_2$.

(2) Rotation over pitch angle about y_2. Note that $v_2 = v_1$.

(3) Rotation over roll angle about x_1. Note that $u_1 = u_2$

Transformations between BODY and NED

Euler Angle Transformation

Linear velocity transformation (*zyx*-convention):

$$\dot{\mathbf{p}}^n_{b/n} = \mathbf{R}^n_b(\boldsymbol{\Theta}_{nb})\mathbf{v}^b_{b/n}$$

where

$$\mathbf{R}^n_b(\boldsymbol{\Theta}_{nb}) := \mathbf{R}_{z,\psi}\mathbf{R}_{y,\theta}\mathbf{R}_{x,\phi} \qquad \mathbf{R}^n_b(\boldsymbol{\Theta}_{nb})^{-1} = \mathbf{R}^b_n(\boldsymbol{\Theta}_{nb}) = \mathbf{R}^\top_{x,\phi}\mathbf{R}^\top_{y,\theta}\mathbf{R}^\top_{z,\psi}$$

$$\mathbf{R}^n_b(\boldsymbol{\Theta}_{nb}) = \begin{bmatrix} c\psi c\theta & -s\psi c\phi + c\psi s\theta s\phi & s\psi s\phi + c\psi c\phi s\theta \\ s\psi c\theta & c\psi c\phi + s\phi s\theta s\psi & -c\psi s\phi + s\theta s\psi c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

Small angle approximation:

$$\mathbf{R}^n_b(\delta\boldsymbol{\Theta}_{nb}) \approx \mathbf{I}_{3\times 3} + \mathbf{S}(\delta\boldsymbol{\Theta}_{nb}) = \begin{bmatrix} 1 & -\delta\psi & \delta\theta \\ \delta\psi & 1 & -\delta\phi \\ -\delta\theta & \delta\phi & 1 \end{bmatrix}$$

Transformations between BODY and NED

Euler Angle Transformation

NED positions (continuous time and discrete time):

$$\dot{\mathbf{p}}^n_{b/n} = \mathbf{R}^n_b(\Theta_{nb})\mathbf{v}^b_{b/n} \quad \xrightarrow{\text{Euler integration}} \quad \mathbf{p}^n_{b/n}(k+1) = \mathbf{p}^n_{b/n}(k) + h\mathbf{R}^n_b(\Theta_{nb}(k))\mathbf{v}^b_{b/n}(k)$$

Component form:

$$\begin{aligned}
\dot{N} &= u\cos(\psi)\cos(\theta) + v(\cos(\psi)\sin(\theta)\sin(\phi) - \sin(\psi)\cos(\phi)) \\
&\quad + w(\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\phi)\sin(\theta)) \\
\dot{E} &= u\sin(\psi)\cos(\theta) + v(\cos(\psi)\cos(\phi) + \sin(\phi)\sin(\theta)\sin(\psi)) \\
&\quad + w(\sin(\theta)\sin(\psi)\cos(\phi) - \cos(\psi)\sin(\phi)) \\
\dot{D} &= -u\sin(\theta) + v\cos(\theta)\sin(\phi) + w\cos(\theta)\cos(\phi)
\end{aligned}$$

Transformations between BODY and NED

Euler Angle Transformation

Angular velocity transformation (*zyx*-convention):

$$\dot{\boldsymbol{\Theta}}_{nb} = \mathbf{T}_{\Theta}(\boldsymbol{\Theta}_{nb})\boldsymbol{\omega}^{b}_{b/n}$$

$$\boldsymbol{\omega}^{b}_{b/n} = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{R}^{\top}_{x,\phi}\begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{R}^{\top}_{x,\phi}\mathbf{R}^{\top}_{y,\theta}\begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} := \mathbf{T}^{-1}_{\Theta}(\boldsymbol{\Theta}_{nb})\dot{\boldsymbol{\Theta}}_{nb}$$

where

$$\mathbf{T}^{-1}_{\Theta}(\boldsymbol{\Theta}_{nb}) = \begin{bmatrix} 1 & 0 & -s\theta \\ 0 & c\phi & c\theta s\phi \\ 0 & -s\phi & c\theta c\phi \end{bmatrix} \quad \Rightarrow \quad \mathbf{T}_{\Theta}(\boldsymbol{\Theta}_{nb}) = \begin{bmatrix} 1 & s\phi t\theta & c\phi t\theta \\ 0 & c\phi & -s\phi \\ 0 & s\phi/c\theta & c\phi/c\theta \end{bmatrix}$$

Small angle approximation:

$$\mathbf{T}_{\Theta}(\delta\boldsymbol{\Theta}_{nb}) \approx \begin{bmatrix} 1 & 0 & \delta\theta \\ 0 & 1 & -\delta\phi \\ 0 & \delta\phi & 1 \end{bmatrix}$$

Notice that:

1. Singular point at $\theta = \pm 90^{o}$
   $\mathbf{T}^{-1}_{\Theta}(\boldsymbol{\Theta}_{nb}) \neq \mathbf{T}^{\top}_{\Theta}(\boldsymbol{\Theta}_{nb})$

Transformations between BODY and NED

Euler Angle Transformation

Euler angle attitude representations:

ODE for Euler angles:

$$\dot{\mathbf{\Theta}}_{nb} = \mathbf{T}_{\Theta}(\mathbf{\Theta}_{nb})\boldsymbol{\omega}^{b}_{b/n}$$

Component form:

$$\begin{aligned}
\dot{\theta} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\phi} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta}, \quad \theta \neq \pm 90^{o}
\end{aligned}$$

ODE for rotation matrix

$$\dot{\mathbf{R}}^{n}_{b} = \mathbf{R}^{n}_{b}\mathbf{S}(\boldsymbol{\omega}^{b}_{b/n})$$

where

$$\mathbf{S}(\boldsymbol{\omega}^{b}_{b/n}) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}$$

+ algorithm for computation of Euler angles $\mathbf{\Theta}_{nb} = [\phi, \theta, \psi]^{\top}$ from the rotation matrix $\mathbf{R}^{n}_{b}(\mathbf{\Theta}_{nb})$

Transformations between BODY and NED

Euler Angle Transformation

Summary: 6-DOF kinematic equations:

$$\dot{\boldsymbol{\eta}} = \mathbf{J}(\boldsymbol{\eta})\boldsymbol{\nu}$$

$$\Updownarrow$$

$$\begin{bmatrix} \dot{\mathbf{p}}^n_{b/n} \\ \dot{\boldsymbol{\Theta}}_{nb} \end{bmatrix} = \begin{bmatrix} \mathbf{R}^n_b(\boldsymbol{\Theta}_{nb}) & \mathbf{0}_{3\times 3} \\ \mathbf{0}_{3\times 3} & \mathbf{T}_\Theta(\boldsymbol{\Theta}_{nb}) \end{bmatrix} \begin{bmatrix} \mathbf{v}^b_{b/n} \\ \boldsymbol{\omega}^b_{b/n} \end{bmatrix}$$

3-parameter representation

$$\boldsymbol{\Theta}_{nb} = [\phi, \theta, \psi]^\top$$

with singularity at $\theta = \pm 90^o$

Component form:

$$\begin{aligned}
\dot{N} &= u\cos\psi\cos\theta + v(\cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi) \\
&\quad + w(\sin\psi\sin\phi + \cos\psi\cos\phi\sin\theta) \\
\dot{E} &= u\sin\psi\cos\theta + v(\cos\psi\cos\phi + \sin\phi\sin\theta\sin\psi) \\
&\quad + w(\sin\theta\sin\psi\cos\phi - \cos\psi\sin\phi) \\
\dot{D} &= -u\sin\theta + v\cos\theta\sin\phi + w\cos\theta\cos\phi
\end{aligned}$$

$$\begin{aligned}
\dot{\phi} &= p + q\sin\phi\tan\theta + r\cos\phi\tan\theta \\
\dot{\theta} &= q\cos\phi - r\sin\phi \\
\dot{\psi} &= q\frac{\sin\phi}{\cos\theta} + r\frac{\cos\phi}{\cos\theta}, \quad \theta \neq \pm 90^o
\end{aligned}$$

Table of Content

Newton–Euler Equations of Motion about CG

In order to derive the marine craft equations of motion, it is necessary to study of the motion of rigid bodies, hydrodynamics and hydrostatics.

The overall goal of Chapter 3 is to show that the rigid-body kinetics can be expressed in a vectorial setting according to:

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\mathbf{M}_{\mathrm{RB}}$ Rigid-body mass matrix
$\mathbf{C}_{\mathrm{RB}}$ Rigid-body Coriolis and centripetal matrix due to the rotation of {b} about {n}
$\boldsymbol{\nu} = [u,v,w,p,q,r]^T$ generalized velocity expressed in {b}
$\boldsymbol{\tau}_{\mathrm{RB}} = [X,Y,Z,K,M,N]^T$ generalized forces expressed in {b}

Newton–Euler Equations of Motion about CG

The equations of motion will be represented in two body-fixed reference points:

1) Center of gravity (CG), subscript *g*
2) Origin o_b of {b}, subscript *b*

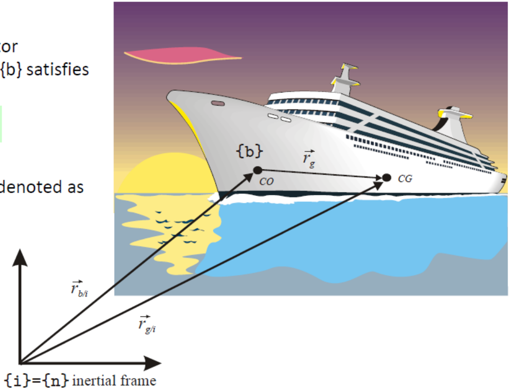
These points coincides if the vector $\vec{r}_g = \vec{0}$

Time differentiation of a vector in a moving reference frame {b} satisfies

$$\frac{{}^{i}d}{dt}\vec{a} = \frac{{}^{b}d}{dt}\vec{a} + \vec{\omega}_{b/i} \times \vec{a}$$

Time differentiation in {b} is denoted as

$$\dot{\vec{a}} := \frac{{}^{b}d}{dt}\vec{a}$$



Newton–Euler Equations of Motion about CG

Coordinate-free vector: A vector $\vec{v}_{b/n}$, velocity of {b} with respect to {n}, is defined by its magnitude and direction but without reference to a coordinate frame.
Coordinate vector: A vector $\vec{v}_{b/n}$ decomposed in the inertial reference frame is denoted by $\mathbf{v}^{i}_{b/n}$

Newton-Euler Formulation

Newton's Second Law relates mass *m*, acceleration $\dot{\vec{v}}_{g/i}$ and force $\vec{f}_g$ according to:

Isaac Newton (1642-1726)

$$m\dot{\vec{v}}_{g/i} = \vec{f}_g$$

where the subscript *g* denotes the center of gravity (CG).

Leonhard Euler (1707-1783)

Euler's First and Second Axioms
Euler suggested to express Newton's Second Law in terms of conservation of both linear momentum $\vec{p}_g$ and angular momentum $\vec{h}_g$ according to:

$$\frac{{}^{i}d}{dt}\vec{p}_g = \vec{f}_g \qquad \vec{p}_g = m\vec{v}_{g/i}$$

$$\frac{{}^{i}d}{dt}\vec{h}_g = \vec{m}_g \qquad \vec{h}_g = I_g\vec{\omega}_{b/i}$$

$\vec{f}_g$ and $\vec{m}_g$ are forces/moments about CG
$\vec{\omega}_{b/i}$ is the angular velocity of frame *b* relative frame *i*
I_g is the inertia dyadic about the body's CG

Newton–Euler Equations of Motion about CG

- Translational Motion:

When deriving the equations of motion it will be assumed that:

(1) The vessel is rigid
(2) The NED frame is inertial—that is, {n} ≈ {i}

The first assumption eliminates the consideration of forces acting between individual elements of mass while the second eliminates forces due to the Earth's motion relative to a star-fixed inertial reference system such that:

$$\vec{v}_{g/i} \approx \vec{v}_{g/n}$$

$$\vec{\omega}_{b/i} \approx \vec{\omega}_{b/n}$$

For guidance and navigation applications in space it is usual to use a star-fixed reference frame or a reference frame rotating with the Earth. Marine crafts are, on the other hand, usually related to the NED reference frame. This is a good assumption since forces on a marine craft due to the Earth's rotation:

$$\omega_{e/i} = 7.2921 \cdot 10^{-5} \text{ rad/s}$$

are quite small compared to the hydrodynamic forces.

Newton–Euler Equations of Motion about CG

- Translational Motion:

$$\vec{r}_{g/n} = \vec{r}_{b/n} + \vec{r}_g$$

{n} is inertial

$$\vec{r}_{g/i} = \vec{r}_{b/i} + \vec{r}_g$$

Time differentiation of $\vec{r}_{g/n}$ in a moving reference frame {b} gives

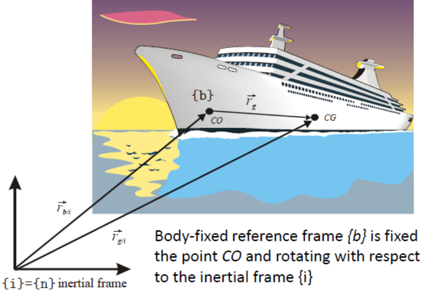
$$\vec{v}_{g/n} = \vec{v}_{b/n} + \left(\frac{{}^b d}{dt}\vec{r}_g + \vec{\omega}_{b/n} \times \vec{r}_g \right)$$

For a rigid body, CG satisfies

$$\frac{{}^b d}{dt}\vec{r}_g = \vec{0}$$

$$\vec{v}_{g/n} = \vec{v}_{b/n} + \vec{\omega}_{b/n} \times \vec{r}_g$$

$$\begin{aligned}
\vec{f}_g &= \frac{{}^i d}{dt}(m\vec{v}_{g/i}) \\
&= \frac{{}^i d}{dt}(m\vec{v}_{g/n}) \\
&= \frac{{}^b d}{dt}(m\vec{v}_{g/n}) + m\vec{\omega}_{b/n} \times (m\vec{v}_{g/n}) \\
&= m(\dot{\vec{v}}_{g/n} + \vec{\omega}_{b/n} \times \vec{v}_{g/n})
\end{aligned}$$



Body-fixed reference frame *{b}* is fixed in the point *CO* and rotating with respect to the inertial frame {i}

Translational Motion about CG Expressed in {b}

$$m[\dot{\mathbf{v}}^b_{g/n} + \mathbf{S}(\boldsymbol{\omega}^b_{b/n})\mathbf{v}^b_{g/n}] = \mathbf{f}^b_g$$

Newton–Euler Equations of Motion about CG

Rotational Motion:

The derivation starts with the Euler's 2nd axiom:

$$\begin{aligned}\vec{m}_g &= \frac{{}^i d}{dt}(I_g\vec{\omega}_{b/i}) \\ &= \frac{{}^i d}{dt}(I_g\vec{\omega}_{b/n}) \\ &= \frac{{}^b d}{dt}(I_g\vec{\omega}_{b/n}) + \vec{\omega}_{b/n} \times (I_g\vec{\omega}_{b/n}) \\ &= I_g\dot{\vec{\omega}}_{b/n} - (I_g\vec{\omega}_{b/n}) \times \vec{\omega}_{b/n}\end{aligned}$$

Rotational Motion about CG Expressed in {b}

$$\mathbf{I}_g\dot{\boldsymbol{\omega}}^b_{b/n} - \mathbf{S}(\mathbf{I}_g\boldsymbol{\omega}^b_{b/n})\boldsymbol{\omega}^b_{b/n} = \mathbf{m}^b_g$$

where $\mathbf{I}_g$ is the *inertia matrix*

where I_x, I_y, and I_z are the *moments of inertia* about {b} and $I_{xy}=I_{yx}$, $I_{xz}=I_{zx}$ and $I_{yz}=I_{zy}$ are the *products of inertia* defined as:

$$\mathbf{I}_g := \begin{bmatrix} I_x & -I_{xy} & -I_{xz} \\ -I_{yx} & I_y & -I_{yz} \\ -I_{zx} & -I_{zy} & I_z \end{bmatrix}, \qquad \mathbf{I}_g = \mathbf{I}_g^\top > 0$$

$$\begin{aligned} I_x &= \int_V (y^2 + z^2)\ \rho_m dV; & I_{xy} &= \int_V xy\ \rho_m dV = \int_V yx\ \rho_m dV = I_{yx} \\ I_y &= \int_V (x^2 + z^2)\ \rho_m dV; & I_{xz} &= \int_V xz\ \rho_m dV = \int_V zx\ \rho_m dV = I_{zx} \\ I_z &= \int_V (x^2 + y^2)\ \rho_m dV; & I_{yz} &= \int_V yz\ \rho_m dV = \int_V zy\ \rho_m dV = I_{zy} \end{aligned}$$

Newton–Euler Equations of Motion about CG

- Inertia of a solid:

Body inertia tensor:

$$I(G,S) = \begin{bmatrix} \int_S (y^2+z^2) & -\int_S xydm & -\int_S xzdm \\ -\int_S xydm & \int_S (x^2+z^2) & -\int_S yzdm \\ -\int_S xzdm & -\int_S yzdm & \int_S (x^2+y^2) \end{bmatrix}_{B_S} = \begin{bmatrix} A & -F & -E \\ -F & B & -D \\ -E & -D & C \end{bmatrix}_{B_S}$$

Huygens theory:

$$\overrightarrow{AG} = a\overrightarrow{x}_S + b\overrightarrow{y}_S + c\overrightarrow{z}_S$$

$$I(A,S) = I(G,S) + M \begin{bmatrix} b^2+c^2 & -ab & -ac \\ -ab & a^2+c^2 & -bc \\ -ac & -bc & a^2+b^2 \end{bmatrix}_{B_S}$$

Newton–Euler Equations of Motion about CG

- Inertia of a solid:

Symmetries and inertia tensor:

<table>
<tr><td>one plane of symmetry $(O, \vec{x}_s, \vec{y}_s)$</td><td>two planes of symmetry</td><td>revolution axe $(O, \vec{z}_s)$</td></tr>
<tr><td>$I(O,S) = \begin{bmatrix} A & -F & 0 \\ -F & B & 0 \\ 0 & 0 & C \end{bmatrix}_{B_S}$</td><td>$I(O,S) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}_{B_S}$</td><td rowspan="2">$I(O,S) = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & C \end{bmatrix}_{B_S}$
$A = \frac{C}{2} + \int_S z^2 dm$</td></tr>
<tr><td>Spherical bodies</td><td>$I(O,S) = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{bmatrix}_{B_S}$</td></tr>
</table>

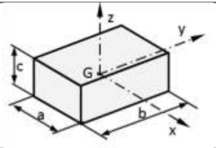
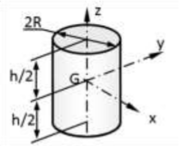
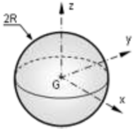
Inertia tensor of a multi body

$$I(A,S) = \sum_{i=1}^{N} I(A,S_i) = \sum_{i=1}^{N} \left[I(G_i,S_i) + m_i \begin{bmatrix} b_i^2 + c_i^2 & -a_i b_i & -a_i c_i \\ -a_i b_i & a_i^2 + c_i^2 & -b_i c_i \\ -a_i c_i & -b_i c_i & a_i^2 + b_i^2 \end{bmatrix}_{B_S} \right]$$

Newton–Euler Equations of Motion about CG

- Inertia of a solid:

Common inertia tensors:

	$I(G,S)=\begin{bmatrix}\frac{m}{12}(b^2+c^2) & 0 & 0\\ 0 & \frac{m}{12}(a^2+c^2) & 0\\ 0 & 0 & \frac{m}{12}(a^2+b^2)\end{bmatrix}_{\mathfrak{B}_S}$
	$I(G,S)=\begin{bmatrix}m\left(\frac{R^2}{4}+\frac{h^2}{12}\right) & 0 & 0\\ 0 & m\left(\frac{R^2}{4}+\frac{h^2}{12}\right) & 0\\ 0 & 0 & m\frac{R^2}{2}\end{bmatrix}_{\mathfrak{B}_S}$
	$I(G,S)=\begin{bmatrix}\frac{2}{5}mR^2 & 0 & 0\\ 0 & \frac{2}{5}mR^2 & 0\\ 0 & 0 & \frac{2}{5}mR^2\end{bmatrix}_{\mathfrak{B}_S}$

Newton–Euler Equations of Motion about CG

- Global Equations of Motion:

The Newton-Euler equations can be represented in matrix form according to:

$$\mathbf{M}_{RB}^{CG}\begin{bmatrix}\dot{\mathbf{v}}_{g/n}^{b}\\ \dot{\boldsymbol{\omega}}_{b/n}^{b}\end{bmatrix}+\mathbf{C}_{RB}^{CG}\begin{bmatrix}\mathbf{v}_{g/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\begin{bmatrix}\mathbf{f}_{g}^{b}\\ \mathbf{m}_{g}^{b}\end{bmatrix}$$

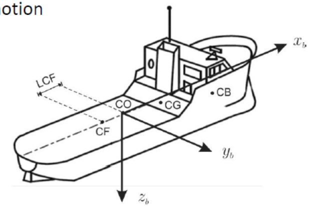
Isaac Newton (1642-1726)
Leonhard Euler (1707-1783)

Expanding the matrices give:

$$\underbrace{\begin{bmatrix}m\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{I}_{g}\end{bmatrix}}_{\mathbf{M}_{RB}^{CG}}\begin{bmatrix}\dot{\mathbf{v}}_{g/n}^{b}\\ \dot{\boldsymbol{\omega}}_{b/n}^{b}\end{bmatrix}+\underbrace{\begin{bmatrix}m\mathbf{S}(\boldsymbol{\omega}_{b/n}^{b}) & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & -\mathbf{S}(\mathbf{I}_{g}\boldsymbol{\omega}_{b/n}^{b})\end{bmatrix}}_{\mathbf{C}_{RB}^{CG}}\begin{bmatrix}\mathbf{v}_{g/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\begin{bmatrix}\mathbf{f}_{g}^{b}\\ \mathbf{m}_{g}^{b}\end{bmatrix}$$

Newton–Euler Equations of Motion about CO

For marine craft it is desirable to derive the equations of motion for an arbitrary origin CO to take advantage of the craft's geometric properties. Since the hydrodynamic forces and moments often are computed in CO, Newton's laws will be formulated in CO as well.



Transform the equations of motion from CG to CO using a coordinate transformation based on:

$$\begin{aligned} \mathbf{v}^b_{g/n} &= \mathbf{v}^b_{b/n} + \boldsymbol{\omega}^b_{b/n} \times \mathbf{r}^b_g \\ &= \mathbf{v}^b_{b/n} - \mathbf{r}^b_g \times \boldsymbol{\omega}^b_{b/n} \\ &= \mathbf{v}^b_{b/n} + \mathbf{S}^\top(\mathbf{r}^b_g)\boldsymbol{\omega}^b_{b/n} \end{aligned}$$

$$\Rightarrow \quad \begin{bmatrix} \mathbf{v}^b_{g/n} \\ \boldsymbol{\omega}^b_{b/n} \end{bmatrix} = \mathbf{H}(\mathbf{r}^b_g) \begin{bmatrix} \mathbf{v}^b_{b/n} \\ \boldsymbol{\omega}^b_{b/n} \end{bmatrix}$$

Transformation matrix:

$$\mathbf{H}(\mathbf{r}^b_g) := \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{S}^\top(\mathbf{r}^b_g) \\ \mathbf{0}_{3\times3} & \mathbf{I}_{3\times3} \end{bmatrix}, \qquad \mathbf{H}^\top(\mathbf{r}^b_g) = \begin{bmatrix} \mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{S}(\mathbf{r}^b_g) & \mathbf{I}_{3\times3} \end{bmatrix}$$

Newton–Euler Equations of Motion about CO

Newton-Euler equations in matrix form (about CG)

$$\mathbf{M}_{RB}^{CG}\begin{bmatrix}\dot{\mathbf{v}}_{g/n}^{b}\\ \dot{\boldsymbol{\omega}}_{b/n}^{b}\end{bmatrix}+\mathbf{C}_{RB}^{CG}\begin{bmatrix}\mathbf{v}_{g/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\begin{bmatrix}\mathbf{f}_{g}^{b}\\ \mathbf{m}_{g}^{b}\end{bmatrix}$$

$$\begin{bmatrix}\mathbf{v}_{g/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\mathbf{H}(\mathbf{r}_{g}^{b})\begin{bmatrix}\mathbf{v}_{b/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}$$

Expanding the matrices

$$\underbrace{\begin{bmatrix}m\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & \mathbf{I}_{g}\end{bmatrix}}_{\mathbf{M}_{RB}^{CG}}\begin{bmatrix}\dot{\mathbf{v}}_{g/n}^{b}\\ \dot{\boldsymbol{\omega}}_{b/n}^{b}\end{bmatrix}+\underbrace{\begin{bmatrix}m\mathbf{S}(\boldsymbol{\omega}_{b/n}^{b}) & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & -\mathbf{S}(\mathbf{I}_{g}\boldsymbol{\omega}_{b/n}^{b})\end{bmatrix}}_{\mathbf{C}_{RB}^{CG}}\begin{bmatrix}\mathbf{v}_{g/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\begin{bmatrix}\mathbf{f}_{g}^{b}\\ \mathbf{m}_{g}^{b}\end{bmatrix}$$

Newton-Euler equations in matrix form (about CO)

$$\mathbf{H}^{\top}(\mathbf{r}_{g}^{b})\mathbf{M}_{RB}^{CG}\mathbf{H}(\mathbf{r}_{g}^{b})\begin{bmatrix}\dot{\mathbf{v}}_{b/n}^{b}\\ \dot{\boldsymbol{\omega}}_{b/n}^{b}\end{bmatrix}+\mathbf{H}^{\top}(\mathbf{r}_{g}^{b})\mathbf{C}_{RB}^{CG}\mathbf{H}(\mathbf{r}_{g}^{b})\begin{bmatrix}\mathbf{v}_{b/n}^{b}\\ \boldsymbol{\omega}_{b/n}^{b}\end{bmatrix}=\mathbf{H}^{\top}(\mathbf{r}_{g}^{b})\begin{bmatrix}\mathbf{f}_{g}^{b}\\ \mathbf{m}_{g}^{b}\end{bmatrix}$$

Newton–Euler Equations of Motion about CO

$$\mathbf{H}^\top(\mathbf{r}_g^b)\mathbf{M}_{RB}^{CG}\mathbf{H}(\mathbf{r}_g^b)\begin{bmatrix}\dot{\mathbf{v}}_{b/n}^b \\ \dot{\boldsymbol{\omega}}_{b/n}^b\end{bmatrix} + \mathbf{H}^\top(\mathbf{r}_g^b)\mathbf{C}_{RB}^{CG}\mathbf{H}(\mathbf{r}_g^b)\begin{bmatrix}\mathbf{v}_{b/n}^b \\ \boldsymbol{\omega}_{b/n}^b\end{bmatrix} = \mathbf{H}^\top(\mathbf{r}_g^b)\begin{bmatrix}\mathbf{f}_g^b \\ \mathbf{m}_g^b\end{bmatrix}$$

The mass and Coriolis-centripetal matrices in CO are defined as:

$$\begin{aligned}\mathbf{M}_{RB}^{CO} &:= \mathbf{H}^\top(\mathbf{r}_g^b)\mathbf{M}_{RB}^{CG}\mathbf{H}(\mathbf{r}_g^b) \\ \mathbf{C}_{RB}^{CO} &:= \mathbf{H}^\top(\mathbf{r}_g^b)\mathbf{C}_{RB}^{CG}\mathbf{H}(\mathbf{r}_g^b)\end{aligned}$$

Expanding the matrices

$$\mathbf{M}_{RB}^{CO} = \begin{bmatrix} m\mathbf{I}_{3\times3} & -m\mathbf{S}(\mathbf{r}_g^b) \\ m\mathbf{S}(\mathbf{r}_g^b) & \mathbf{I}_g - m\mathbf{S}^2(\mathbf{r}_g^b)\end{bmatrix}$$

$$\mathbf{C}_{RB}^{CO} = \begin{bmatrix} m\mathbf{S}(\boldsymbol{\omega}_{b/n}^b) & -m\mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\mathbf{S}(\mathbf{r}_g^b) \\ m\mathbf{S}(\mathbf{r}_g^b)\mathbf{S}(\boldsymbol{\omega}_{b/n}^b) & -\mathbf{S}((\mathbf{I}_g - m\mathbf{S}^2(\mathbf{r}_g^b))\boldsymbol{\omega}_{b/n}^b)\end{bmatrix}$$

Newton–Euler Equations of Motion about CO

- Translational Motion about CO:

Translational Motion about CO Expressed in {b}

$$m[\dot{\mathbf{v}}_{b/n}^b + \mathbf{S}(\dot{\boldsymbol{\omega}}_{b/n}^b)\mathbf{r}_g^b + \mathbf{S}(\boldsymbol{\omega}_{b/n}^b)\mathbf{v}_{b/n}^b + \mathbf{S}^2(\boldsymbol{\omega}_{b/n}^b)\mathbf{r}_g^b] = \mathbf{f}_b^b$$

An alternative representation using vector cross products is:

$$m[\dot{\mathbf{v}}_{b/n}^b + \dot{\boldsymbol{\omega}}_{b/n}^b \times \mathbf{r}_g^b + \boldsymbol{\omega}_{b/n}^b \times \mathbf{v}_{b/n}^b + \boldsymbol{\omega}_{b/n}^b \times (\boldsymbol{\omega}_{b/n}^b \times \mathbf{r}_g^b)] = \mathbf{f}_b^b$$

Newton–Euler Equations of Motion about CO

- Rotational Motion about CO:

Rotational Motion about CO Expressed in {b}

$$\mathbf{I}_b\dot{\boldsymbol{\omega}}^b_{b/n} + \mathbf{S}(\boldsymbol{\omega}^b_{b/n})\mathbf{I}_b\boldsymbol{\omega}^b_{b/n} + m\mathbf{S}(\mathbf{r}^b_g)\dot{\mathbf{v}}^b_{b/n} + m\mathbf{S}(\mathbf{r}^b_g)\mathbf{S}(\boldsymbol{\omega}^b_{b/n})\mathbf{v}^b_{b/n} = \mathbf{m}^b_b$$

An alternative representation using vector cross products is:

$$\mathbf{I}_b\dot{\boldsymbol{\omega}}^b_{b/n} + \boldsymbol{\omega}^b_{b/n} \times \mathbf{I}_b\boldsymbol{\omega}^b_{b/n} + m\mathbf{r}^b_g \times (\dot{\mathbf{v}}^b_{b/n} + \boldsymbol{\omega}^b_{b/n} \times \mathbf{v}^b_{b/n}) = \mathbf{m}^b_b$$

Theorem 3.1 (Parallel Axes or Huygens-Steiner Theorem)

The inertia matrix I_b about an arbitrary origin o_b is given by:

$$\begin{aligned}\mathbf{I}_g + m\mathbf{S}(\mathbf{r}^b_g)\mathbf{S}^\top(\mathbf{r}^b_g) &= \mathbf{I}_g - m\mathbf{S}^2(\mathbf{r}^b_g)\\ &= \mathbf{I}_b\end{aligned}$$

where I_g is the inertia matrix about the body's center of gravity.


Christian Huygens (1629-1695)
Jakob Steiner (1796-1863)

Rigid-Body Equations of Motion

$$
\begin{array}{lcll}
\mathbf{f}_b^b & = & [X, Y, Z]^\top & \text{- force through } o_b \text{ expressed in } \{b\} \\
\mathbf{m}_b^b & = & [K, M, N]^\top & \text{- moment about } o_b \text{ expressed in } \{b\} \\
\mathbf{v}_{b/n}^b & = & [u, v, w]^\top & \text{- linear velocity of } o_b \text{ relative } o_n \text{ expressed in } \{b\} \\
\boldsymbol{\omega}_{b/n}^b & = & [p, q, r]^\top & \text{- angular velocity of } \{b\} \text{ relative to } \{n\} \text{ expressed in } \{b\} \\
\mathbf{r}_g^b & = & [x_g, y_g, z_g]^\top & \text{- vector from } o_b \text{ to } CG \text{ expressed in } \{b\}
\end{array}
$$

Component form (SNAME 1950)

$$
\begin{array}{rcl}
m[\dot{u} - vr + wq - x_g(q^2 + r^2) + y_g(pq - \dot{r}) + z_g(pr + \dot{q})] & = & X \\
m[\dot{v} - wp + ur - y_g(r^2 + p^2) + z_g(qr - \dot{p}) + x_g(qp + \dot{r})] & = & Y \\
m[\dot{w} - uq + vp - z_g(p^2 + q^2) + x_g(rp - \dot{q}) + y_g(rq + \dot{p})] & = & Z \\
I_x\dot{p} + (I_z - I_y)qr - (\dot{r} + pq)I_{xz} + (r^2 - q^2)I_{yz} + (pr - \dot{q})I_{xy} & & \\
+ m[y_g(\dot{w} - uq + vp) - z_g(\dot{v} - wp + ur)] & = & K \\
I_y\dot{q} + (I_x - I_z)rp - (\dot{p} + qr)I_{xy} + (p^2 - r^2)I_{zx} + (qp - \dot{r})I_{yz} & & \\
+ m[z_g(\dot{u} - vr + wq) - x_g(\dot{w} - uq + vp)] & = & M \\
I_z\dot{r} + (I_y - I_x)pq - (\dot{q} + rp)I_{yz} + (q^2 - p^2)I_{xy} + (rq - \dot{p})I_{zx} & & \\
+ m[x_g(\dot{v} - wp + ur) - y_g(\dot{u} - vr + wq)] & = & N
\end{array}
$$

Rigid-Body Equations of Motion

- Nonlinear 6-DOF Rigid-Body Equations of Motion:

Matrix-Vector Form (Fossen 1991)

$$\mathbf{M}_{RB}\dot{\boldsymbol{\nu}} + \mathbf{C}_{RB}(\boldsymbol{\nu})\boldsymbol{\nu} = \boldsymbol{\tau}_{RB}$$

$\boldsymbol{\nu} = [u, v, w, p, q, r]^\top$ generalized velocity

Property 3.1 (Rigid-Body System Inertia Matrix) $\mathbf{M}_{RB} = \mathbf{M}_{RB}^\top > 0, \quad \dot{\mathbf{M}}_{RB} = \mathbf{0}_{6\times 6}$

$$\mathbf{M}_{RB} = \begin{bmatrix} m\mathbf{I}_{3\times3} & -m\mathbf{S}(\mathbf{r}_g^b) \\ m\mathbf{S}(\mathbf{r}_g^b) & \mathbf{I}_o \end{bmatrix} = \begin{bmatrix} m & 0 & 0 & 0 & mz_g & -my_g \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & my_g & -mx_g & 0 \\ 0 & -mz_g & my_g & I_x & -I_{xy} & -I_{xz} \\ mz_g & 0 & -mx_g & -I_{yx} & I_y & -I_{yz} \\ -my_g & mx_g & 0 & -I_{zx} & -I_{zy} & I_z \end{bmatrix}$$

$\mathbf{I}_{3\times3}$ is the *identity matrix*

$\mathbf{I}_b = \mathbf{I}_b^\top > 0$ is the inertia matrix about CO

$\mathbf{S}(\mathbf{r}_g^b)$ is the matrix cross product operator

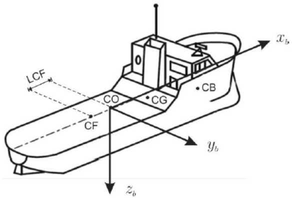
Rigid-Body Equations of Motion

Simplified 6-DOF Rigid-Body Equations of Motion

(1) Origin *CO* coincides with the *CG*

This implies that $\mathbf{r}_g^b = [0,0,0]^\top$, $\mathbf{I}_b = \mathbf{I}_g$ such that

$$\mathbf{M}_{RB} = \begin{bmatrix} m\mathbf{I}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \mathbf{I}_g \end{bmatrix}$$



A further simplification is obtained when the body axes (x_b, y_b, z_b) coincide with *the principal axes of inertia*. This implies that

$$\mathbf{I}_g = \text{diag}\{I_x^{cg}, I_y^{cg}, I_z^{cg}\}$$

The motions of a marine craft exposed to wind, waves and ocean currents are usually modeled in 6 DOF by applying Newton's 2nd law:

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}$$

$\mathbf{M} = \mathbf{M}_{RB} + \mathbf{M}_A$	- system inertia matrix (including added mass)
$\mathbf{C}(\boldsymbol{\nu}) = \mathbf{C}_{RB}(\boldsymbol{\nu}) + \mathbf{C}_A(\boldsymbol{\nu})$	- Coriolis-centripetal matrix (including added mass)
$\mathbf{D}(\boldsymbol{\nu})$	- damping matrix
$\mathbf{g}(\boldsymbol{\eta})$	- vector of gravitational/buoyancy forces and moments
$\mathbf{g}_o$	- vector used for pretrimming (ballast control)
$\boldsymbol{\tau}$	- vector of control inputs
$\boldsymbol{\tau}_{wind}$	- vector of wind loads
$\boldsymbol{\tau}_{wave}$	- vector of wave loads

Hydrostatics of Submerged Vehicles

Underwater Vehicles:

According to the SNAME (1950) it is standard to express the submerged *weight* of the body and *buoyancy force* as:

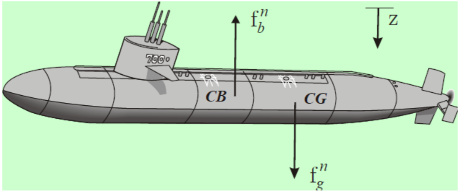
$$W = mg, \qquad B = \rho g \nabla$$

ρ = water density

∇ = volume of fluid displaced by the vehicle

m = mass of the vessel including water in free flooding space

g = acceleration of gravity



$$\mathbf{f}_g^n = \begin{bmatrix} 0 \\ 0 \\ W \end{bmatrix} \qquad \mathbf{f}_b^n = -\begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix}$$

The weight and buoyancy force can be transformed from NED to BODY by:

$$\mathbf{f}_g^b = \mathbf{R}_b^n(\Theta)^{-1}\mathbf{f}_g^n, \qquad \mathbf{f}_b^b = \mathbf{R}_b^n(\Theta)^{-1}\mathbf{f}_b^n$$

Hydrostatics of Submerged Vehicles

The sign of the restoring forces and moments $\mathbf{f}_i^b$ and $\mathbf{m}_i^b = \mathbf{r}_i^b \times \mathbf{f}_i^b$ must be changed when moving these terms to the left-hand side of Newton's 2nd law, e.g. ma = f $\Longrightarrow$ ma-f= 0:

We denote the generalized restoring forces $\mathbf{g}(\boldsymbol{\eta})$. Notice that the force and moment vectors are multiplied with -1.

Consequently, the generalized restoring force in BODY with coordinate origin CO becomes:

$$\mathbf{M}\dot{\boldsymbol{\nu}} + \mathbf{C}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{D}(\boldsymbol{\nu})\boldsymbol{\nu} + \mathbf{g}(\boldsymbol{\eta}) + \mathbf{g}_o = \boldsymbol{\tau} + \boldsymbol{\tau}_{wind} + \boldsymbol{\tau}_{wave}$$

$$\mathbf{g}(\boldsymbol{\eta}) = -\begin{bmatrix} \mathbf{f}_g^b + \mathbf{f}_b^b \\ \mathbf{r}_g^b \times \mathbf{f}_g^b + \mathbf{r}_b^b \times \mathbf{f}_b^b \end{bmatrix}$$

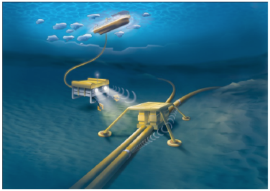
where

$\mathbf{r}_b^b = [x_b, y_b, z_b]^\top$ center of buoyancy with respect to CO

$\mathbf{r}_g^b = [x_g, y_g, z_g]^\top$ center of gravity with respect to CO

Hydrostatics of Submerged Vehicles

Main Result: Underwater Vehicles:



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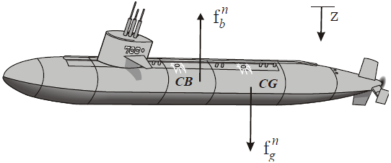


The 6-DOF gravity and buoyancy forces and moments about CO are given by:

$$\mathbf{g}(\boldsymbol{\eta}) = \begin{bmatrix} (W-B)\ \sin\theta \\ -\ (W-B)\ \cos\theta\sin\phi \\ -\ (W-B)\ \cos\theta\cos\phi \\ -\ (y_g W - y_b B)\ \cos\theta\cos\phi \ +\ (z_g W - z_b B)\ \cos\theta\sin\phi \\ (z_g W - z_b B)\ \sin\theta \ +\ (x_g W - x_b B)\ \cos\theta\cos\phi \\ -\ (x_g W - x_b B)\ \cos\theta\sin\phi \ -\ (y_g W - y_b B)\ \sin\theta \end{bmatrix}$$

Hydrostatics of Submerged Vehicles

Example 4.1: Neutrally Buoyant Underwater Vehicles:

Let the distance between the center of gravity *CG* and the center of buoyancy *CB* be defined by the vector:

$$\mathbf{BG} = [\overline{BG}_x,\ \overline{BG}_y,\ \overline{BG}_z]^\top = [x_g - x_b,\ y_g - y_b,\ z_g - z_b]^\top$$

For neutrally buoyant vehicles $W = B$, and this simplifies to:

$$\mathbf{g}(\boldsymbol{\eta}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\overline{BG}_y W \ \cos\theta\cos\phi + \overline{BG}_z W \ \cos\theta\sin\phi \\ \overline{BG}_z W \ \sin\theta + \overline{BG}_x W \ \cos\theta\cos\phi \\ -\overline{BG}_x W \ \cos\theta\sin\phi - \overline{BG}_y W \ \sin\theta \end{bmatrix}$$

An even simpler representation is obtained for vehicles where the *CG* and *CB* are located vertically on the *z*-axis, that is $x_b = x_g$ and $y_g = y_b$. This yields:

$$\mathbf{g}(\boldsymbol{\eta}) = \left[\ 0,\quad 0,\quad 0,\quad \overline{BG}_z W \cos\theta\sin\phi,\quad \overline{BG}_z W \sin\theta,\quad 0\ \right]^\top$$

Added Mass

Definition:

When a body moves in a fluid, some amount of fluid must move around it. When the body accelerates, so too must the fluid. Thus, more force is required to accelerate the body in the fluid than in a vacuum. Since force equals mass times acceleration, we can think of the additional force in terms of an imaginary added mass of the object in the fluid.

Expression:

This force is **applied on the CB**:

$$\tau_{addedmass} = -M_A^{CB} * \dot{v}$$

M_A^{CB} is added mass matrix with 36 coefficients m_{aij}. However, it has the same properties of the real mass matrix :

$$M_A > 0$$

And

$$M_a = M_a^T = \begin{bmatrix} M_{a11} & M_{a12} \\ M_{a21} & M_{a22} \end{bmatrix}_{Rb}^{CB} \text{ or } M_{a12} = M_{a21}^T \text{ or } m_{aij} = m_{aji}$$

This reduces the number of individual coefficients to 21.

Added Mass

Matrix coefficients

$$M_a = \begin{bmatrix} X_{\dot{u}} & X_{\dot{v}} & X_{\dot{w}} & X_{\dot{p}} & X_{\dot{q}} & X_{\dot{r}} \\ Y_{\dot{u}} & Y_{\dot{v}} & Y_{\dot{w}} & Y_{\dot{p}} & Y_{\dot{q}} & Y_{\dot{r}} \\ Z_{\dot{u}} & Z_{\dot{v}} & Z_{\dot{w}} & Z_{\dot{p}} & Z_{\dot{q}} & Z_{\dot{r}} \\ K_{\dot{u}} & K_{\dot{v}} & K_{\dot{w}} & K_{\dot{p}} & K_{\dot{q}} & K_{\dot{r}} \\ M_{\dot{u}} & M_{\dot{v}} & M_{\dot{w}} & M_{\dot{p}} & M_{\dot{q}} & M_{\dot{r}} \\ N_{\dot{u}} & N_{\dot{v}} & N_{\dot{w}} & N_{\dot{p}} & N_{\dot{q}} & N_{\dot{r}} \end{bmatrix}_{Rb}^{CB} = \begin{bmatrix} m_{11} & m_{a12} & m_{a13} & m_{a14} & m_{a15} & m_{a16} \\ m_{a21} & m_{a22} & m_{a23} & m_{a24} & m_{a25} & m_{a26} \\ m_{a31} & m_{a32} & m_{a33} & m_{a34} & m_{a35} & m_{a36} \\ m_{a41} & m_{a42} & m_{a43} & m_{a44} & m_{a45} & m_{a46} \\ m_{a51} & m_{a52} & m_{a53} & m_{a54} & m_{a55} & m_{a56} \\ m_{a61} & m_{a62} & m_{a63} & m_{a64} & m_{a65} & m_{a66} \end{bmatrix}_{Rb}^{CB}$$

- These coefficients represent the forces in six different degrees of freedom due to acceleration in each combination of degrees of freedom. For example, a force in the X-direction (1) due to an acceleration in the y-direction (2), $\dot{v}$, is represented by the term $X_{\dot{v}} = m_{a12}$.
- The **diagonal** elements of the matrix are the **primary coefficients**, relating movement in one direction to the force or moment in that same direction.
- The **non-diagonal** coefficients are **the coupled or secondary coefficients**.
- **All added mass coefficients depend entirely on the geometry of the vehicle**, together **with the density of the surrounding fluid**.

Added Mass

Computing global mass matrix:

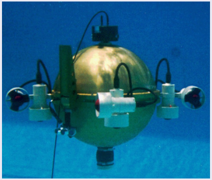
To add 2 mass matrices, they have to be computed at the same point :

$$M = M_{RB} + H^T(\overrightarrow{r_b^b - r_g^b})M_A H(\overrightarrow{r_b^b - r_g^b})$$

3 planes of symmetry

- If a body is symmetric in all three planes (XY, XZ, YZ), only the six coefficients on the diagonal are non-zero, and there is no coupling between different degrees of freedom :

$$M_a = \begin{bmatrix} m_{a11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{a22} & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{a33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{a44} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{a55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{a66} \end{bmatrix}_{Rb}^{CB}$$



- There is 2 coefficients to be estimated : with $m_{a11} = m_{a22} = m_{a33}$ and $m_{a44} = m_{a55} = m_{a66}$

Added Mass

Port/starboard symmetry

- A completely symmetric body is not realistic as a representation of an AUV (e.g. adding fins) although XZ-symmetry can usually be assumed :

$$M_a = \begin{bmatrix} m_{a11} & 0 & m_{a31} & 0 & m_{a15} & 0 \\ 0 & m_{a22} & 0 & m_{a24} & 0 & m_{a26} \\ m_{a31} & 0 & m_{a33} & 0 & m_{a35} & 0 \\ 0 & m_{a42} & 0 & m_{a44} & 0 & m_{a46} \\ m_{a51} & 0 & m_{a53} & 0 & m_{a55} & 0 \\ 0 & m_{a62} & 0 & m_{a64} & 0 & m_{a66} \end{bmatrix}_{Rb}^{CB}$$

- There is 12 coefficients to be estimated

Added Mass

Port/starboard and fore/aft symmetry

- Often, an AUV has a XZ and a YZ symmetry :

$$M_a = \begin{bmatrix} m_{a11} & 0 & 0 & 0 & m_{a15} & 0 \\ 0 & m_{a22} & 0 & m_{a24} & 0 & 0 \\ 0 & 0 & m_{a33} & 0 & 0 & 0 \\ 0 & m_{a42} & 0 & m_{a44} & 0 & 0 \\ m_{a51} & 0 & 0 & 0 & m_{a55} & 0 \\ 0 & 0 & 0 & 0 & 0 & m_{a66} \end{bmatrix}_{Rb}^{CB}$$

- There is 7 coefficients to be estimated : with $m_{a15} = m_{a24}$

Added Mass

Axisymmetric with respect to X-axis

- Like a torpedo :

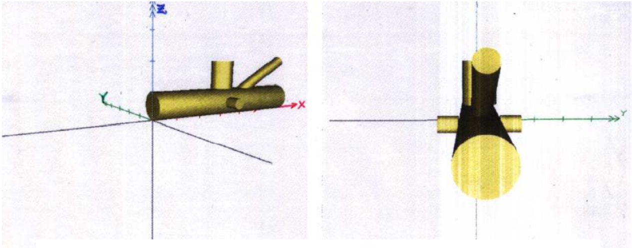
$$M_a = \begin{bmatrix} m_{a11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{a22} & 0 & 0 & 0 & m_{a26} \\ 0 & 0 & m_{a33} & 0 & m_{a35} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{a53} & 0 & m_{a55} & 0 \\ 0 & m_{a62} & 0 & 0 & 0 & m_{a66} \end{bmatrix}_{Rb}^{CB}$$

- There is 4 coefficients to be estimated : with $m_{a35} = m_{a26}$, $m_{a22} = m_{a33}$ and $m_{a55} = m_{a66}$

Added Mass

Exemple :

•Symmetry with respect to Y (= "X-Z" plane symmetry) **12** non-zero, independent coefficients

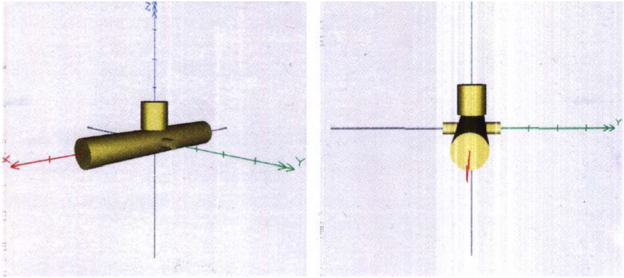


$$\frac{m_{ij}}{\rho} = \begin{vmatrix} 1.008 & 0.000 & -0.179 & 0.000 & 1.887 & 0.000 \\ 0.000 & 5.682 & 0.000 & -3.625 & 0.000 & 17.696 \\ -0.179 & 0.000 & 5.304 & 0.000 & -16.616 & 0.000 \\ 0.000 & -3.625 & 0.000 & 3.051 & 0.000 & -11.734 \\ 1.887 & 0.000 & -16.616 & 0.000 & 63.471 & 0.000 \\ 0.000 & 17.696 & 0.000 & -11.734 & 0.000 & 64.761 \end{vmatrix}$$

Added Mass

Exemple :

•Symmetry with respect to X and Y (= "Y-Z" and "X-Z" plane symmetry) **7** non-zero, independent coefficients



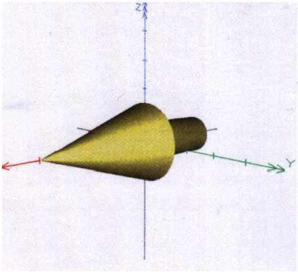
$$\frac{m_{ij}}{\rho} = \begin{bmatrix} 1.173 & 0.000 & 0.000 & 0.000 & 0.785 & 0.000 \\ 0.000 & 5.531 & 0.000 & -0.785 & 0.000 & 0.000 \\ 0.000 & 0.000 & 5.100 & 0.000 & 0.000 & 0.000 \\ 0.000 & -0.785 & 0.000 & 0.897 & 0.000 & 0.000 \\ 0.785 & 0.000 & 0.000 & 0.000 & 9.307 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 8.555 \end{bmatrix}$$

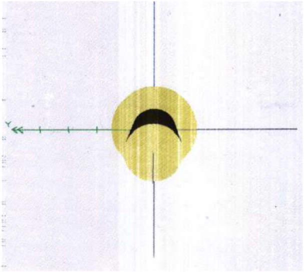
Added Mass

Exemple :

•Axisymmetric with respect to X-axis

4 non-zero, independent coefficients





$$\frac{m_{ij}}{\rho} = \begin{bmatrix} 4.418 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 19.439 & 0.000 & 0.000 & 0.000 & 1.473 \\ 0.000 & 0.000 & 19.439 & 0.000 & -1.473 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.473 & 0.000 & 29.394 & 0.000 \\ 0.000 & 1.473 & 0.000 & 0.000 & 0.000 & 29.394 \end{bmatrix}$$

Coriolis matrix with Added Mass

Mass Matrix:

Let $M = M_{RB} + H^T(\overrightarrow{r_b^b - r_g^b})M_A H(\overrightarrow{r_b^b - r_g^b})$ be a 6×6 system inertia matrix defined as:

$$M = M^T = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} > 0$$

with $M_{a12} = M_{a21}^T$ or $m_{aij} = m_{aji}$

Coriolis-centripetal matrix:

Then the Coriolis-centripetal matrix can always be parameterized such that :

$$\begin{aligned} C(v) &= -C^T(v) \\ &= \begin{bmatrix} 0_{3*3} & -S(M_{11}v_1 + M_{12}v_2) \\ -S(M_{11}v_1 + M_{12}v_2) & -S(M_{21}v_1 + M_{22}v_2) \end{bmatrix} \end{aligned}$$

where $v_1 = [u, v, w]^T$ and $v_2 = [p, q, r]^T$

Damping matrix

Dissipative Forces:

Hydrodynamic **viscous damping** for marine craft is mainly caused by :

- **Skin friction:** Linear skin friction is caused by laminar boundary layer theory and pressure variations.
- **Wave Drift Damping:** Wave drift damping can be interpreted as added resistance for surface craft advancing in waves.
- **Damping Due to Vortex Shedding:** This is commonly referred to as interference drag. It arises due to the shedding of vortex sheets (surface across which there is a discontinuity in fluid velocity) at sharp edges.
- **Lifting Forces** Hydrodynamic lift forces arise from two physical mechanisms. The first is due to the linear circulation of water around the hull. The second is a nonlinear effect, commonly called cross-flow drag, which acts from a momentum transfer from the body to the fluid.
- **. . . and other viscous effects**

  Unfortunately, it is very hard to model all these affects and collect the terms into a common nonlinear damping matrix D(n)

Damping matrix

Dissipative Forces:

Here we will only consider drag forces as they are predominant :

$$D(\upsilon) = D_{NL}|\upsilon|$$

Where D_{NL} is the drag matrix for quadratic friction, and υ is the velocity of the underwater system or the relative speed.

Expressions:

$$D_{NL} = {}^{CB}\begin{bmatrix} X_{uu} & X_{vv} & X_{ww} & X_{pp} & X_{qq} & X_{rr} \\ Y_{uu} & Y_{vv} & Y_{ww} & Y_{pp} & Y_{qq} & Y_{rr} \\ Z_{uu} & Z_{vv} & Z_{ww} & Z_{pp} & Z_{qq} & Z_{rr} \\ K_{uu} & K_{vv} & K_{ww} & K_{pp} & K_{qq} & K_{rr} \\ M_{uu} & M_{vv} & M_{ww} & M_{pp} & M_{qq} & M_{rr} \\ N_{uu} & N_{vv} & N_{ww} & N_{pp} & N_{qq} & N_{rr} \end{bmatrix}^{CB}_{Rb}$$

Note:

The same assumptions as for the added mass can be done with the symmetry planes.

Propulsions Forces:

Thrust Configuration Matrix:

The actuators are thrusters. They can be model as a force with one coordinate applied where the thruster is set up. Consequently, the actuator action τ can be model as follow:

$$\tau_g^b = \begin{bmatrix} f_g^b \\ m_g^b \end{bmatrix} = E_g^b F^b \qquad ou \qquad \tau_b^b = \begin{bmatrix} f_b^b \\ m_b^b \end{bmatrix} = E_b^b F^b$$

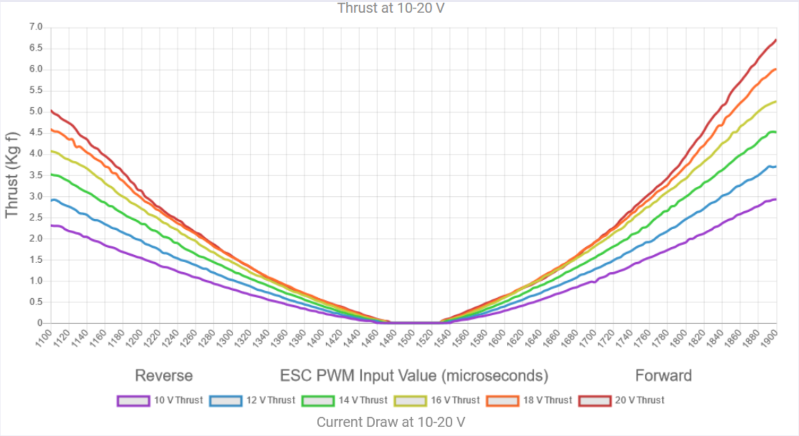
Where:

- $F^b \in \Re^{n,1}$: Represent a vector with the n forces produce by the thrusters : $\begin{bmatrix} F_1 \\ . \\ . \\ . \\ F_n \end{bmatrix}$
- $E^b_{g\ ou\ b} \in \Re^{6,n}$ is matrix depending of the position of the thrusters

Propulsions Forces:

Value of the thrust force :

- with an analytical model of the thruster not detailed here.
- with a curve regression of static experimentation :



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Problematic :

- The values of the added mass and drag coefficients depends of the pressure load along the shape of the body
- Their estimates require the solution of the Navier-Stokes equations :

Momentum Equations

$$\underbrace{\rho\frac{D\vec{V}}{Dt}}_{\text{Total derivative}} = \underbrace{-\nabla p}_{\text{Pressure gradient}} + \underbrace{\rho\vec{g}}_{\text{Body force term}} + \underbrace{\mu\nabla^2\vec{V}}_{\text{Diffusion term}}$$

Continuity Equation

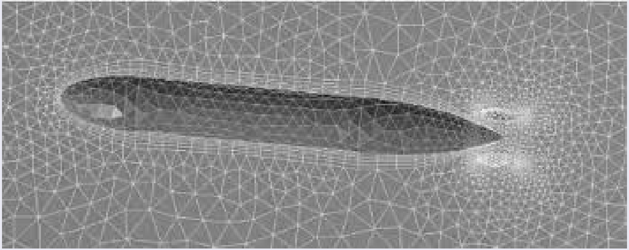
$$\nabla \cdot \vec{V} = 0$$

  - Complex with many simplifying assumptions
  - Possible in real conditions but only with the help of a computer with very long calculation times (e.g. 1 sec simulated need several minutes, hours or days depending of the precision required and the simulated volume)
- Thus, the parameters change greatly for the same shape as it moves through a liquid over time: $\pm 100\%$ average value or more.
- The idea is to estimate average values.

Numerically solutions :

Meshing software:

The objective is to solve the Navier-Stoke equations on a finite number of elementary volumes or surfaces. This means using a volume mesh for liquids and a surface mesh for solids.



Using this mesh and a solver of Navier-Stoke equation, it is possible to estimate the coefficients of drag and added mass forces using two different methods:

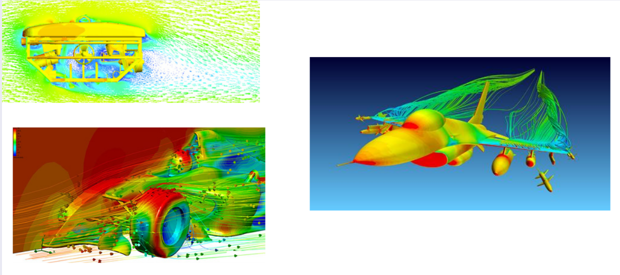
- Added Mass: **BEM software** for Boundary Element Method
- Drag coeff: **CFD software** for Computational Fluid Dynamic

Numerically solutions :

CFD software:

- Object is fixed in space
- Simulate a constant speed from 1 to 3 directions (no acceleration = no added mass)
- from the repartition of the pressure over the surface, we are able to compute the drag forces...

Ansys Fluent:



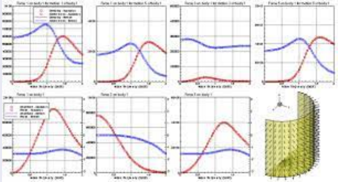
Numerically solutions :

BEM software:

- Computing Buoyancy and added mass
- Taking into account wave frequencies: added mass as a function of the wave frequencies.

Exemples:

- Nemoh : open source software developped by LHEEA laboratory (Nantes in France)



- Wamit : commercial software

WAMIT INC
The State of the Art in Wave Interaction Analysis

Experimentation method:

Principles:

- The objective is to impose a motion to the actual vehicle and measure during the experimentation the position, speed, acceleration, forces and moments.
  - using the thruster of the vehicle
  - using a mechanism equipped with a force sensor. e.g, a motorised linear guidance system.
- With these data and the dynamic model of the vehicle (link motions and forces), it is possible to obtain the hydrodynamics parameter thanks to a regression analysis algorithm like:
  - Least-squares function approximation
  - Gauss–Newton algorithm (non-linear least squares)
  - "fmincon" function from Matlab.

Classical Model of underwater vehicle in linear motion :

- Known by conception
- Measured (DVL, depth meter, IMU)
- To be estimated
- Computed by the algorithm

acceleration speed Restoring force

$$\text{mass} \rightarrow (M_A + M_{RB})\dot{v} + D_q v^2 + g(\eta) = \tau \leftarrow \text{Thrusters}$$

Added mass Drag coeff. Orientation

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Analytical solutions:

Assumptions :

If we consider simple shape (Cylinder, plate, sphere, torpedo, etc.), it is possible to solve the Navier-Stoke equations with some assumptions:

- **Deeply submerged body**: The body is assumed to be completely submerged and away from the surface.
- **No waves or currents**: No waves, currents or other disturbances are taken into account.
- **Two planes of symmetries** or an **axis symmetry** for the simple shape.

Methods :

- Slender Body Theory
- Lamb's k-factor for spheroid
- 3D coefficients

Added Mass by slender body

Assumptions and limitations:

- **Frictionless fluid and no circulation**: This enables the use of potential flow theory.
- **The characteristic length of the body**:
  - in the longitudinal direction is considerably larger than the body's characteristic length in the other two directions: the slenderness ratio $L/d >= 10$ (min 5).
  - The variations in y and z-direction are small along the x-axis.
- **The body has two planes of symmetry**: XY and ZX.

$$M_a = \begin{bmatrix} m_{a11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{a22} & 0 & 0 & 0 & m_{a26} \\ 0 & 0 & m_{a33} & 0 & m_{a35} & 0 \\ 0 & 0 & 0 & m_{a44} & 0 & 0 \\ 0 & 0 & m_{a53} & 0 & m_{a55} & 0 \\ 0 & m_{a62} & 0 & 0 & 0 & m_{a66} \end{bmatrix}_{Rb}^{CB}$$

- **Local coordinate system origin** along $\vec{x}$

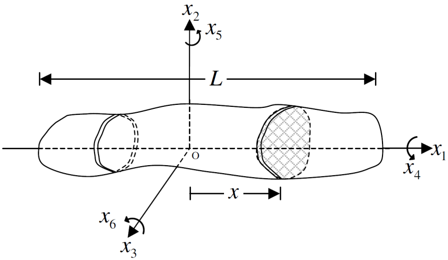
Added Mass by slender body

Principle of calculation

- **Idea:** Estimate m_{aij} of a slender 3D body using the known 2D sectional added mass coefficients a_{ij}.

$$m_{aij} = \sum [a_{ij}(x) \ contributions]$$

- **Discussion:** If the x-axis is the longitudinal axis of the slender body, then the 3D added mass coefficients m_{aij} are calculated by summing the added mass coefficients of all the thin slices which are perpendicular to the 1-axis. This means that forces in x-direction cannot be obtained by slender body theory.



Added Mass by slender body

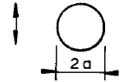
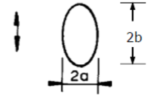
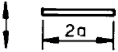
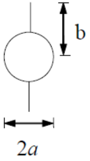
Principle of calculation

- m_{ij}: the 3D added mass coefficient in the ith direction due to a unit acceleration in the jth direction. $(i,j) \in [1:6]$
- a_{ij}: the 2D added mass coefficient in the ith direction due to a unit acceleration in the jth direction. $(i,j) \in [1:6]$
- the position of the origin along x axis defined where the mass matrix is computed.

	1	2	3	4	5	6
1						
2		$m_{22} = \int_L a_{22}\,dx$				$m_{26} = \int_L x\,a_{22}\,dx$
3			$m_{33} = \int_L a_{33}\,dx$		$m_{35} = -\int_L x\,a_{33}\,dx$	
4				$m_{44} = \int_L a_{44}\,dx$		
5			$m_{35} = -\int_L x\,a_{33}\,dx$		$m_{55} = \int_L x^2\,a_{33}\,dx$	
6		$m_{26} = \int_L x\,a_{22}\,dx$				$m_{66} = \int_L x^2\,a_{22}\,dx$

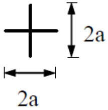
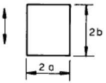
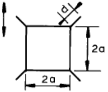
Added Mass by slender body

Table D-1 Analytical added mass coefficient for two-dimensional bodies, i.e. long cylinders in infinite fluid (far from boundaries). Added mass (per unit length) is $m_A = \rho C_A A_R$ [kg/m] where A_R [m^2] is the reference area.

Section through body	*Direction of motion*	C_A	A_R	*Added mass moment of inertia* [(kg/m) × m^2]
		1.0	πa^2	0
	Vertical	1.0	πa^2	$\rho \frac{\pi}{8}(b^2 - a^2)^2$
	Horizontal	1.0	πb^2	
	Vertical	1.0	πa^2	$\rho \frac{\pi}{8} a^4$
 Circular cylinder with two fins	Vertical	1.0	πa^2	$\rho a^4(\csc^4 \alpha\, f(\alpha) - \pi^2)/2\pi$ where $f(\alpha) = 2\alpha^2 - \alpha \sin 4\alpha$ $+ 0.5 \sin^2 2\alpha$ and $\sin\alpha = 2ab/(a^2 + b^2)$ $\pi/2 < \alpha < \pi$
	Horizontal	$1 - \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right)^4$	πb^2	

Added Mass by slender body

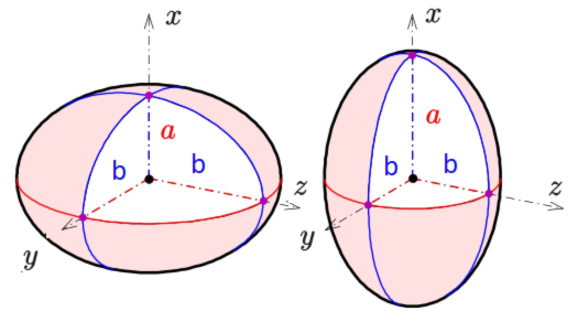
Table D-1 Analytical added mass coefficient for two-dimensional bodies, i.e. long cylinders in infinite fluid (far from boundaries). Added mass (per unit length) is $m_A = \rho C_A A_R$ [kg/m] where A_R [m^2] is the reference area.

<table>
<tr><th colspan="2">Section through body</th><th>Direction of motion</th><th>C_A</th><th>A_R</th><th>Added mass moment of inertia [(kg/m) × m^2]</th></tr>
<tr><td colspan="2"></td><td>Horizontal or Vertical</td><td>1.0</td><td>πa^2</td><td>$\frac{2}{\pi}\rho a^4$</td></tr>
<tr><td></td><td>a / b = ∞
a / b = 10
a / b = 5
a / b = 2
a / b = 1
a / b = 0.5
a / b = 0.2
a / b = 0.1</td><td>Vertical</td><td>1.0
1.14
1.21
1.36
1.51
1.70
1.98
2.23</td><td>πa^2</td><td>$\beta_1 \rho \pi a^4$ or $\beta_2 \rho \pi b^4$
<table>
<tr><th>a/b</th><th>β_1</th><th>β_2</th></tr>
<tr><td>0.1</td><td>-</td><td>0.147</td></tr>
<tr><td>0.2</td><td>-</td><td>0.15</td></tr>
<tr><td>0.5</td><td>-</td><td>0.15</td></tr>
<tr><td>1.0</td><td>0.234</td><td>0.234</td></tr>
<tr><td>2.0</td><td>0.15</td><td>-</td></tr>
<tr><td>5.0</td><td>0.15</td><td>-</td></tr>
<tr><td>∞</td><td>0.125</td><td>-</td></tr>
</table></td></tr>
<tr><td></td><td>d / a = 0.05
d / a = 0.10
d / a = 0.25</td><td>Vertical</td><td>1.61
1.72
2.19</td><td>πa^2</td><td>$\beta \rho \pi a^4$
<table>
<tr><th>$d\,/\,a$</th><th>β</th></tr>
<tr><td>0.05</td><td>0.31</td></tr>
<tr><td>0.10</td><td>0.40</td></tr>
<tr><td>0.10</td><td>0.69</td></tr>
</table></td></tr>
</table>

Added mass by Lamb's k-factors for spheroid:

Definition:

A spheroid, or ellipsoid of revolution, is a quadric surface obtained by rotating an ellipse about one of its principal axes. In other words, an ellipsoid with two equal semi-diameters. A spheroid has circular symmetry. There is two types of spheroid. If the ellipse is rotated about its major axis, the result is a prolate (elongated) spheroid. If the ellipse is rotated about its minor axis, the result is an oblate (flattened) spheroid. If the generating ellipse is a circle, the result is a sphere.



Added mass by Lamb's k-factors for spheroid:

Definitions:

- **Equation:** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$
- **Mass of the displaced volume:** $m_{df} = \frac{4}{3}\rho\pi ab^2$
- **Inertia of the displaced volume:**

$$I_{df} = \begin{bmatrix} I_{xdf} & 0 & 0 \\ 0 & I_{ydf} & 0 \\ 0 & 0 & I_{zdf} \end{bmatrix} = \begin{bmatrix} \frac{m_{adf}}{5}(2b^2) & 0 & 0 \\ 0 & \frac{m_{adf}}{5}(a^2+b^2) & 0 \\ 0 & 0 & \frac{m_{adf}}{5}(a^2+b^2) \end{bmatrix}$$

- **Added mass:**

$$\begin{aligned} m_{a11} &= k_1 * m_{df} \\ m_{a22} &= k_2 * m_{df} \\ m_{a33} &= k_2 * m_{df} \\ m_{a44} &= k_4 * I_{xdf} \\ m_{a55} &= k_5 * I_{ydf} \\ m_{a66} &= k_5 * I_{zdf} \end{aligned}$$

$$\begin{aligned} k_1 &= \frac{\alpha_0}{2-\alpha_0} \\ k_2 &= \frac{\beta_0}{2-\beta_0} \\ k_4 &= 0 \\ k_5 &= \frac{e^4(\beta_0-\alpha_0)}{(2-e^2)(2e^2-(2-e^2)(\beta_0-\alpha_0))} \end{aligned}$$

Added mass by Lamb's k-factors for spheroid:

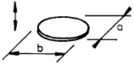
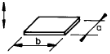
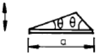
Definitions:

- **And with:**

$$\begin{aligned}
\alpha_0 &= 2\frac{1-e^2}{e^3}(\frac{1}{2}ln\left[\frac{1+e}{1-e}\right]-e) \\
\beta_0 &= \beta_0 = \frac{1}{e^2} - \frac{1-e^2}{2e^3}ln\frac{1+e}{1-e} \\
e &= \sqrt{1-\frac{b^2}{a^2}}
\end{aligned}$$

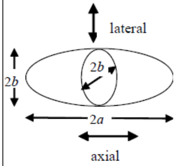
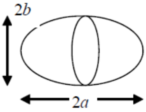
Added mass for 3D objects

Table D-2 Analytical added mass coefficient for three-dimensional bodies in infinite fluid (far from boundaries). Added mass is $m_A = \rho C_A V_R$ [kg] where V_R [m^3] is reference volume.

Body shape		*Direction of motion*	C_A				V_R
Flat plates	Circular disc 	Vertical	$2/\pi$				$\frac{4}{3}\pi a^3$
	Elliptical disc 	Vertical	b/a	C_A	b/a	C_A	$\frac{\pi}{6} a^2 b$
			∞	1.000	5.0	0.952	
			14.3	0.991	4.0	0.933	
			12.8	0.989	3.0	0.900	
			10.0	0.984	2.0	0.826	
			7.0	0.972	1.5	0.758	
			6.0	0.964	1.0	0.637	
	Rectangular plates 	Vertical	b/a	C_A	b/a	C_A	$\frac{\pi}{4} a^2 b$
			1.00	0.579	3.17	0.840	
			1.25	0.642	4.00	0.872	
			1.50	0.690	5.00	0.897	
			1.59	0.704	6.25	0.917	
			2.00	0.757	8.00	0.934	
			2.50	0.801	10.00	0.947	
			3.00	0.830	∞	1.000	
	Triangular plates 	Vertical	$\frac{1}{\pi}(\tan\theta)^{3/2}$				$\frac{a^3}{3}$

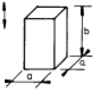
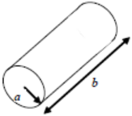
Added mass for 3D objects

Table D-2 Analytical added mass coefficient for three-dimensional bodies in infinite fluid (far from boundaries). Added mass is $m_A = \rho C_A V_R$ [kg] where V_R [m^3] is reference volume.

Body shape		*Direction of motion*	C_A			V_R
			a/b	C_A Axial	C_A Lateral	
	Spheres 	Any direction	½			$\frac{4}{3}\pi a^3$
Bodies of revolution	Spheroids 	Lateral or axial	1.0	0.500	0.500	$\frac{4}{3}\pi b^2 a$
			1.5	0.304	0.622	
			2.0	0.210	0.704	
			2.5	0.156	0.762	
			4.0	0.082	0.860	
			5.0	0.059	0.894	
			6.0	0.045	0.917	
			7.0	0.036	0.933	
			8.0	0.029	0.945	
Ellipsoid	Axis $a > b > c$ 	Axial	$C_A = \frac{\alpha_0}{2-\alpha_0}$ $\quad \varepsilon = b/a \quad \delta = c/a$ where $\alpha_0 = \varepsilon\delta \int_0^\infty (1+u)^{-3/2}(\varepsilon^2+u)^{-1/2}(\delta^2+u)^{-1/2}\,du$			$\frac{4}{3}\pi abc$

Added mass for 3D objects

Table D-2 Analytical added mass coefficient for three-dimensional bodies in infinite fluid (far from boundaries). Added mass is $m_A = \rho C_A V_R$ [kg] where V_R [m^3] is reference volume. (Continued)

Body shape		*Direction of motion*	C_A		V_R
Square prisms		Vertical	b/a 1.0 2.0 3.0 4.0 5.0 6.0 7.0 10.0	C_A 0.68 0.36 0.24 0.19 0.15 0.13 0.11 0.08	$a^2 b$
Right circular cylinder		Vertical	$b/2a$ 1.2 2.5 5.0 9.0 ∞	C_A 0.62 0.78 0.90 0.96 1.00	$\pi a^2 b$

Drag coefficients

Assumptions and limitations:

- **The body has three planes of symmetry**:

$$D_{NL} = \begin{bmatrix} X_{uu} & 0 & 0 & 0 & 0 & 0 \\ 0 & Y_{vv} & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_{ww} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{pp} & 0 & 0 \\ 0 & 0 & 0 & 0 & M_{qq} & 0 \\ 0 & 0 & 0 & 0 & 0 & N_{rr} \end{bmatrix}_{Rb}^{CB} = \begin{bmatrix} K_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & K_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & K_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & K_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & K_{66} \end{bmatrix}_{Rb}^{CB}$$

- **Local coordinate system origin** in the Buyoncy center of the body *cb* : in the middle of the geometry.
- with $v_{cb/n}^{b}$ the speed at the Buyoncy center, the global drag force is :

$$D = D_{NL} * |v_{cb/n}^{b}| * v_{cb/n}^{b}$$

Drag coefficients

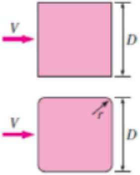
Drag coefficients:

- $K_{11} = \frac{1}{2}\rho.S_x.C_{D11}$, with S_x and C_{D11} the projected surface and the 3D drag coefficient in the $\vec{x}$ direction.
- $K_{22} = \frac{1}{2}\rho C_{D22}.D_y.L$ with D_y and C_{D22} the characteristic width and 2D drag coefficient along the length in the $\vec{y}$ direction.
- $K_{33} = \frac{1}{2}\rho C_{D33}.D_z.L$ with D_z and C_{D33} the characteristic width and 2D drag coefficient along the length in the $\vec{z}$ direction.
- $K_{44} = 0$
- $K_{55} = \frac{1}{64}\rho L^4 C_{D33}.D_z$
- $K_{66} = \frac{1}{64}\rho L^4 C_{D22}.D_y$

Drag coefficients

Drag coefficients C_D of various two-dimensional bodies for Re $> 10^4$ based on the frontal area $A = bD$, where b is the length in direction normal to the page (for use in the drag force relation $F_D = C_D A \rho V^2/2$ where V is the upstream velocity)

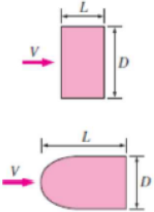
Square rod



Sharp corners: $C_D = 2.2$

Round corners $(r/D = 0.2)$: $C_D = 1.2$

Rectangular rod



Sharp corners:

L/D	C_D
0.0*	1.9
0.1	1.9
0.5	2.5
1.0	2.2
2.0	1.7
3.0	1.3

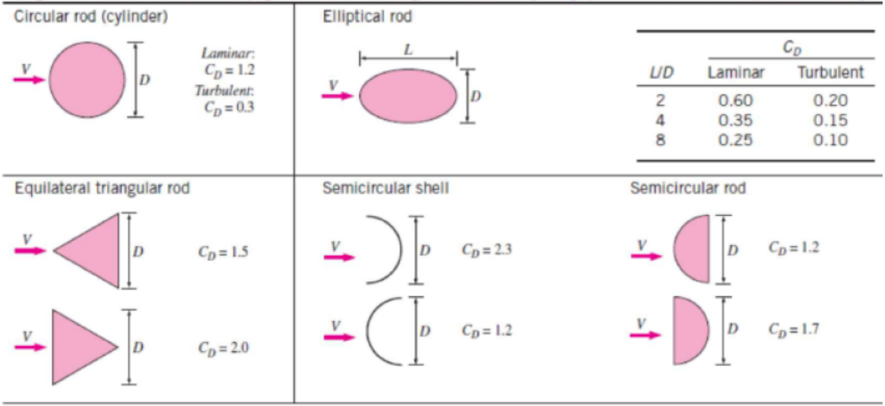
* Corresponds to thin plate

Round front edge:

L/D	C_D
0.5	1.2
1.0	0.9
2.0	0.7
4.0	0.7

Drag coefficients

Drag coefficients C_D of various two-dimensional bodies for Re $> 10^4$ based on the frontal area $A = bD$, where b is the length in direction normal to the page (for use in the drag force relation $F_D = C_D A_P V^2/2$ where V is the upstream velocity)



	C_D	
L/D	Laminar	Turbulent
2	0.60	0.20
4	0.35	0.15
8	0.25	0.10

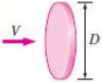
Drag coefficients

Representative drag coefficients C_D for various three-dimensional bodies based on the frontal area for Re > 10^4 unless stated otherwise (for use in the drag force relation $F_D = C_D A \rho V^2/2$ where V is the upstream velocity)

Cube, $A = D^2$

$C_D = 1.05$

Thin circular disk, $A = \pi D^2/4$



$C_D = 1.1$

Cone (for $\theta = 30°$), $A = \pi D^2/4$

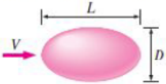
$C_D = 0.5$

Sphere, $A = \pi D^2/4$

Laminar:
$\mathrm{Re} \lesssim 2 \times 10^5$
$C_D = 0.5$
Turbulent:
$\mathrm{Re} \gtrsim 2 \times 10^6$
$C_D = 0.2$

See Fig. 11–36 for C_D vs. Re for smooth and rough spheres.

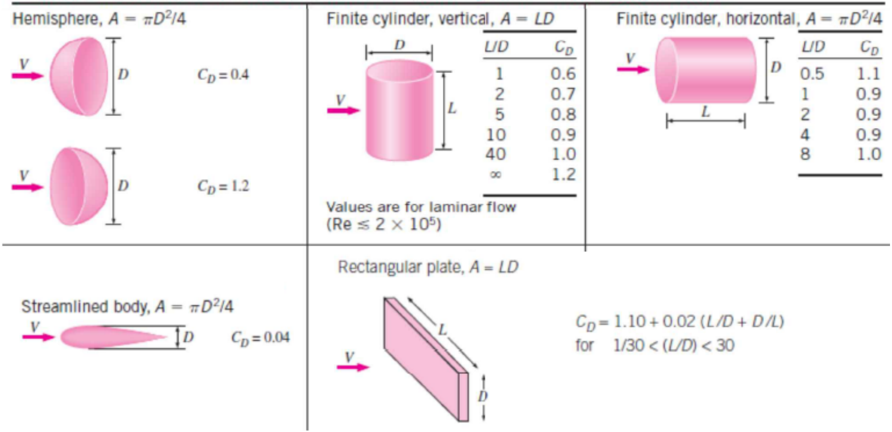
Ellipsoid, $A = \pi D^2/4$



	C_D	
L/D	Laminar $\mathrm{Re} \lesssim 2 \times 10^5$	Turbulent $\mathrm{Re} \gtrsim 2 \times 10^6$
0.75	0.5	0.2
1	0.5	0.2
2	0.3	0.1
4	0.3	0.1
8	0.2	0.1

Drag coefficients

Representative drag coefficients C_D for various three-dimensional bodies based on the frontal area for Re > 10^4 unless stated otherwise (for use in the drag force relation $F_D = C_D A \rho V^2/2$ where V is the upstream velocity)



Hemisphere, $A = \pi D^2/4$

$C_D = 0.4$

$C_D = 1.2$

Finite cylinder, vertical, $A = LD$

L/D	C_D
1	0.6
2	0.7
5	0.8
10	0.9
40	1.0
∞	1.2

Values are for laminar flow ($Re \lesssim 2 \times 10^5$)

Finite cylinder, horizontal, $A = \pi D^2/4$

L/D	C_D
0.5	1.1
1	0.9
2	0.9
4	0.9
8	1.0

Streamlined body, $A = \pi D^2/4$

$C_D = 0.04$

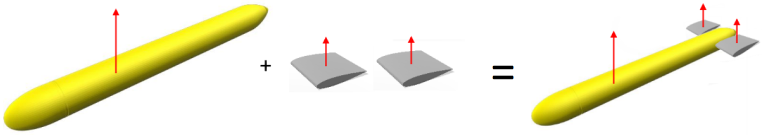
Rectangular plate, $A = LD$

$C_D = 1.10 + 0.02\,(L/D + D/L)$ for $1/30 < (L/D) < 30$

Principle:

Divide your vehicle into different simple solid with 2 symmetry planes (diagonal matrix):

- One of them will be the reference solid nb 0: the main solid of your vehicle.
- The others, nb i, are generally antennas, fins, motors, etc.
- Compute for each solid the matrices ($M_{rb,i}^{g_i}$, $M_{a,i}^{b_i}$, $K_{q,i}^{b_i}$). Their points of application are respectively the gravity center g_i and the center of buyoncy b_i



Global Mass matrix

- As the added mass and real mass has the same behavior, the formula to move a mass matrix between a point A to point B is :

$$M^{A}_{rb/a,i} = H^{T}(\overrightarrow{AB})M^{B}_{rb/a,i}H(\overrightarrow{AB})$$

- Move the added mass matrix to the gravity center of the solid. The global mass matrix of a solid is the sum of the body mass and the added mass matrix:

$$M^{g_i}_{i} = M^{g_i}_{rb,i} + M^{g_i}_{a,i}$$

- Move all global mass matrix of solids to the gravity center of the reference solid
- The general mass matrix is the sum of all global mass matrices:

$$M^{g_0} = M^{g_0}_{0} + \sum M^{g_0}_{i}$$

- Compute the general coriolis matrix

Global Drag matrices

As the drag matrix is multiply by the square of the speed, the same method cannot be used.

- Compute diagonal drag matrix $K_{q,i}^{b_i}$ for each body. Except the main body, the moment coefficients are neglected.
- Compute the speed at Buoyancy center of each body from the speed measure point (DVL position):

$$V_{b_i/n}^{b} = H(\overrightarrow{C_{DVL}b_i})V_{C_{DVL}/n}^{b}$$

- The global drag force of a body expressed on its buyoncy center b_i is:

$$\tau_{b_i,i}^{b} = K_{q,i}^{b_i} * |V_{b_i/n}^{b}| * V_{b_i/n}^{b}$$

- Move each global force $\tau_{b_i,i}^{b}$ to the gravity center of the ref body:

$$\tau_{g_0,i}^{b} = H^T(\overrightarrow{g_0 b_i})\tau_{b_i,i}^{b} = H^T(\overrightarrow{g_0 b_i})K_q^i * |V_{b_i/n}^{b}| * V_{b_i/n}^{b}$$

- The global drag force is the sum of all:

$$\tau^{g_0} = \tau_0^{g_0} + \sum \tau_i^{g_0}$$