From Ross-Simulation

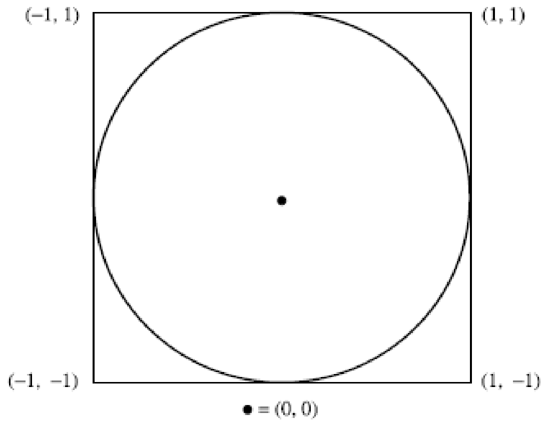

Figure 3.2 Circle within square.

Example 3a (The estimation of π). Suppose that the random vector (X, Y) is uniformly distributed in the square of area 4 centered at the origin. That is, it is a random point in the region specified in Fig. 3.1. Let us consider now the probability that this random point in the square is contained within the inscribed circle of radius 1 (see Fig. 3.2). Note that since (X, Y) is uniformly distributed in the square, it follows that

$$P\{(X, Y) \text{ is in the circle}\} = P\{X^2 + Y^2 \leqslant 1\} = \frac{\text{Area of the circle}}{\text{Area of the square}} = \frac{\pi}{4}$$

Hence, if we generate a large number of random points in the square, the proportion of points that fall within the circle will be approximately $\pi/4$. Now if X and Y were independent and both were uniformly distributed over $(-1, 1)$, their joint density would be

$$\begin{aligned} f(x, y) &= f(x)f(y) \\ &= \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{4}, \qquad -1 \leqslant x \leqslant 1, \quad -1 \leqslant y \leqslant 1 \end{aligned}$$

Since the density function of (X, Y) is constant in the square, it thus follows (by definition) that (X, Y) is uniformly distributed in the square. Now if U is uniform on $(0, 1)$, then $2U$ is uniform on $(0, 2)$, and so $2U - 1$ is uniform on $(-1, 1)$. Therefore, if we generate random numbers U_1 and U_2, set $X = 2U_1 - 1$ and $Y = 2U_2 - 1$, and define

$$I = \begin{cases} 1 & \text{if } X^2 + Y^2 \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

then

$$E[I] = P\{X^2 + Y^2 \leqslant 1\} = \frac{\pi}{4}$$

Hence we can estimate $\pi/4$ by generating a large number of pairs of random numbers u_1, u_2 and estimating $\pi/4$ by the fraction of pairs for which $(2u_1 - 1)^2 + (2u_2 - 1)^2 \leqslant 1$. □