Chapter 9
Input Modeling

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Discrete-Event System Simulation

Purpose & Overview

- Input models provide the driving force for a simulation model.
- The quality of the output is no better than the quality of inputs.
- In this chapter, we will discuss the 4 steps of input model development:
  - Collect data from the real system
  - Identify a probability distribution to represent the input process
  - Choose parameters for the distribution
  - Evaluate the chosen distribution and parameters for goodness of fit.

9.1 Data Collection

- One of the biggest tasks in solving a real problem. GIGO – garbage-in-garbage-out
- Examples
  - Stale data – different methods of collection
  - Unexpected data – metal detector delay
  - Time-varying data – camel hump call center
  - Dependent data – drinks in season

Data Collection - suggestions start on page 340

- Suggestions that may enhance and facilitate data collection:
  - Plan ahead: begin by a practice or pre-observing session, watch for unusual circumstances
  - Analyze the data as it is being collected: check adequacy
  - Combine homogeneous data sets, e.g. successive time periods, during the same time period on successive days
  - Be aware of data censoring: the quantity is not observed in its entirety, danger of leaving out long process times
  - Check for relationship between variables, e.g. build scatter diagram
  - Check for autocorrelation
  - Collect input data, not performance data

9.2 Identifying the Distribution

- Histograms
- Selecting families of distribution
- Parameter estimation
- Goodness-of-fit tests
- Fitting a non-stationary process

Histograms [Identifying the distribution]

- A frequency distribution or histogram is useful in determining the shape of a distribution

1. Divide the range of data into intervals.

   (Usually of equal length)
2. Label the horizontal axis to conform to the intervals selected.
3. Find the frequency of occurrences within each interval.
4. Label the vertical axis so that the total occurrences can be plotted for each interval.
5. Plot the frequencies on the vertical axis.

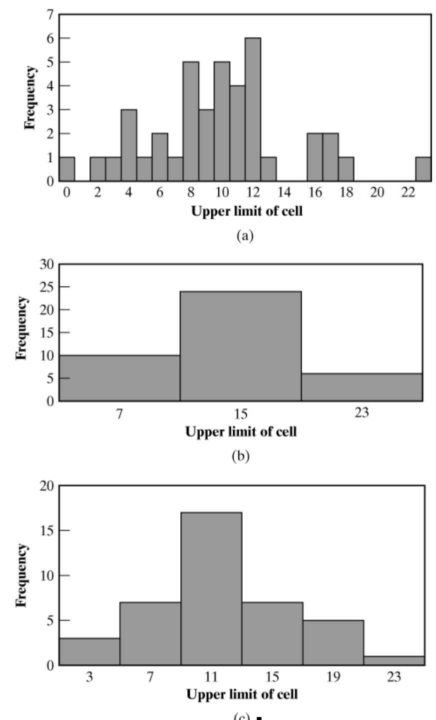
Histograms continued

- The number of class intervals depends on:
  - The number of observations
  - The dispersion of the data
  - Suggested: the square root of the sample size
- For continuous data:
  - Corresponds to the probability density function of a theoretical distribution
- For discrete data:
  - Corresponds to the probability mass function
- If few data points are available: combine adjacent cells to eliminate the ragged appearance of the histogram
  - Does this differ from homogeneity in Section 9.1?

Histograms Ex 9.5 [Identifying the distribution]

- Vehicle Arrival Example: # of vehicles arriving at an intersection between 7 am and 7:05 am was monitored for *100* random workdays.

Arrivals per Period	Frequency
0	12
1	10
2	19
3	17
4	10
5	8
6	7
7	5
8	5
9	3
10	3
11	1



Same data with different interval sizes

- There are ample data, so the histogram may have a cell for each possible value in the data range

9.2.2 Selecting the Family of Distributions

- A family of distributions is selected based on:
  - The context of the input variable
  - Shape of the histogram
- Frequently encountered distributions:
  - Easier to analyze: exponential, normal and Poisson
  - Harder to analyze: beta, gamma and Weibull

Selecting the Family of Distributions

[Identifying the distribution]

- Page 346: Use the physical basis of the distribution as a guide, for example:
  - Binomial: # of successes in *n* trials
  - Poisson: # of independent events that occur in a fixed amount of time or space
  - Normal: dist'n of a process that is the sum of a number of component processes
  - Exponential: time between independent events, or a process time that is memoryless
  - Weibull: time to failure for components
  - Discrete or continuous uniform: models complete uncertainty
  - Triangular: a process for which only the minimum, most likely, and maximum values are known
  - Empirical: resamples from the actual data collected

Selecting the Family of Distributions

[Identifying the distribution]

- Remember the physical characteristics of the process
  - Is the process naturally discrete or continuous valued?
  - Is it bounded?
- There is no “true” distribution for any stochastic input process
- Goal: obtain a good approximation
- See exercises 6 – 12 for work with shapes

Quantile-Quantile Plots [Identifying the distribution]

- *Q-Q* plot is a useful tool for evaluating the fit of the chosen distribution
- If X is a random variable with cdf F, then the q-quantile of X is the γ such that

$$F(\gamma) = P(X \le \gamma) = q, \qquad \text{for } 0 < q < 1$$

  - When F has an inverse, $\gamma = F^{-1}(q)$

- Let $\{x_i, i = 1,2, \ldots, n\}$ be a sample of data from X and $\{y_j, j = 1,2, \ldots, n\}$ be the observations in ascending order:

$$y_j \text{ is approximately } F^{-1}\left(\frac{j - 0.5}{n}\right)$$

where j is the ranking or order number

Quantile-Quantile Plots [Identifying the distribution]

- The plot of y_j versus $F^{-1}((j-0.5)/n)$ is
  - Approximately a straight line if F is a member of an appropriate family of distributions
  - The line has slope 1 if F is a member of an appropriate family of distributions with appropriate parameter values

Quantile-Quantile Plots Ex 9.4 [Identifying the distribution]

- Example: Check whether the door installation times follows a normal distribution.
  - The observations are now ordered from smallest to largest:

j	Value	j	Value	j	Value
1	99.55	6	99.98	11	100.26
2	99.56	7	100.02	12	100.27
3	99.62	8	100.06	13	100.33
4	99.65	9	100.17	14	100.41
5	99.79	10	100.23	15	100.47

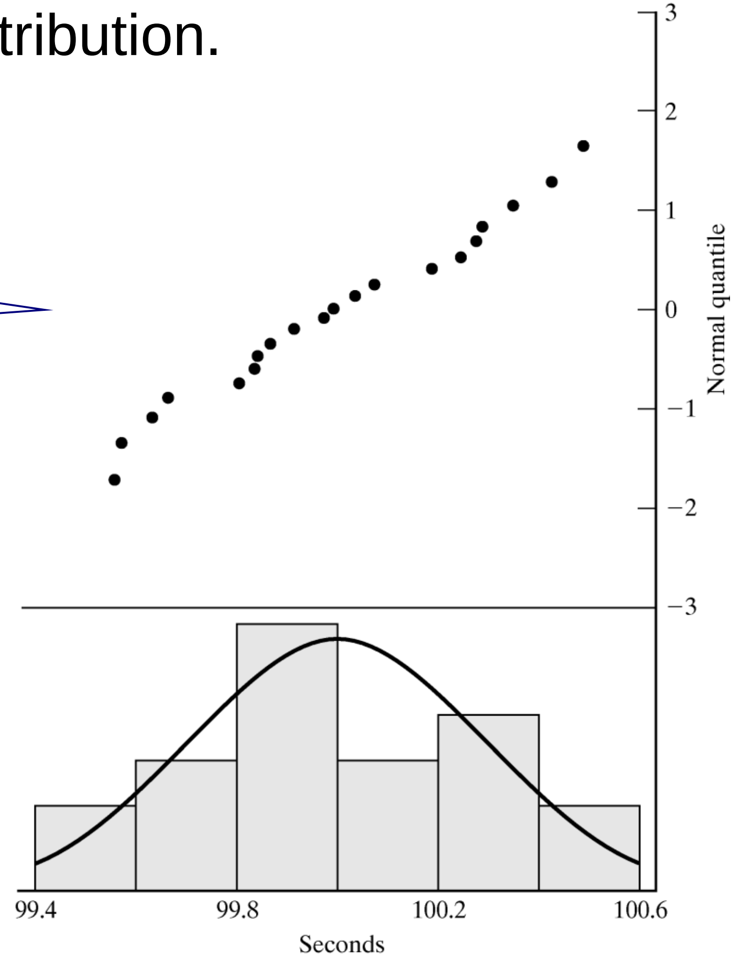
  - y_j are plotted versus $F^{-1}((j-0.5)/n)$ where F has a normal distribution with the sample mean *(99.99 sec)* and sample variance ($0.2832^2\ sec^2$)

Quantile-Quantile Plots [Identifying the distribution]

- Example (continued): Check whether the door installation times follow a normal distribution.

Straight line, supporting the hypothesis of a normal distribution

Superimposed density function of the normal distribution



Quantile-Quantile Plots [Identifying the distribution]

- Consider the following while evaluating the linearity of a *q-q* plot:
  - The observed values never fall exactly on a straight line
  - The ordered values are ranked and hence not independent, unlikely for the points to be scattered about the line
  - Variance of the extremes is higher than the middle. Linearity of the points in the middle of the plot is more important.
- *Q-Q* plot can also be used to check homogeneity
  - Check whether a single distribution can represent both sample sets
  - Plotting the order values of the two data samples against each other

9.3 Parameter Estimation [Identifying the distribution]

- This is the next step after selecting a family of distributions
- If observations in a sample of size *n* are $X_1, X_2, \ldots, X_n$ (discrete or continuous), the sample mean and variance are:

$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$

- If the data are discrete and have been grouped in a frequency distribution:

$$\bar{X} = \frac{\sum_{j=1}^{n} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j X_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of value X_j

Grouped data – example 9.8 on page 350

[Parameter estimation - Identifying the distribution]

- Vehicle Arrival Example (continued): Table in the histogram example on slide 6 (Table 9.1 in book) can be analyzed to obtain:

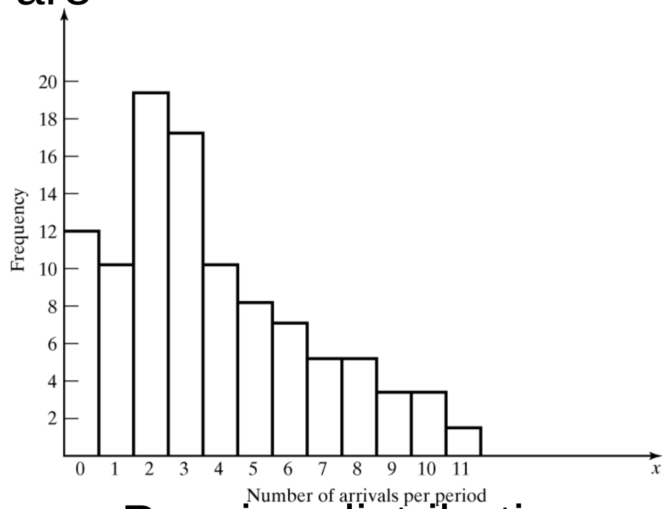
$$n = 100,\ f_1 = 12,\ X_1 = 0,\ f_2 = 10,\ X_2 = 1, \ldots,$$

$$\text{and } \sum_{j=1}^{k} f_j X_j = 364, \text{ and } \sum_{j=1}^{k} f_j X_j^2 = 2080$$

  - The sample mean and variance are

$$\bar{X} = \frac{364}{100} = 3.64$$

$$S^2 = \frac{2080 - 100 * (3.64)^2}{99} = 7.63$$



  - The histogram suggests *X* to have a Possion distribution
    - However, note that sample mean is not equal to sample variance.
    - Reason: each estimator is a random variable, is not perfect.

Parameter Estimation [Identifying the distribution]

- When raw data are unavailable (data are grouped into class intervals), the approximate sample mean and variance are:

$$\bar{X} = \frac{\sum_{j=1}^{c} f_j X_j}{n} \qquad S^2 = \frac{\sum_{j=1}^{n} f_j m_j^2 - n\bar{X}^2}{n-1}$$

where f_j is the observed frequency of in the jth class interval
m_j is the midpoint of the jth interval, and c is the number of class intervals

- A parameter is an unknown constant, but an estimator is a statistic.

9.3.2 on page 352

- Numerical estimates of the distribution's parameters are needed to
  - reduce the family of distributions to a specific distribution
  - test the resulting hypothesis

- Suggested estimators (Ch. 5) – see table 9.3

- Examples 9.10 through 9.16
  - α Parameter is an unknown constant, but
  - $\hat{\alpha}$ Eestimator is a statistic (or random variable), because it depends on the sample values

9.4 Goodness-of-Fit Tests

[Identifying the distribution]

- Conduct hypothesis testing on input data distribution using:
  - Kolmogorov-Smirnov test
  - Chi-square test
- No single correct distribution in a real application exists.
  - If very little data are available, it is unlikely to reject any candidate distributions
  - If a lot of data are available, it is likely to reject all candidate distributions

Chi-Square test

- Intuition: comparing the histogram of the data to the shape of the candidate density or mass function
- Valid for **large** sample sizes when parameters are estimated by maximum likelihood
- By arranging the *n* observations into a set of *k* class intervals or cells, the test statistics is:

$$\chi_0^2 = \sum_{i=1}^{k} \frac{(O_i - E_i)^2}{E_i}$$

Observed Frequency

Expected Frequency

$E_i = n * p_i$

where p_i is the theoretical prob. of the *i*th interval.

Suggested Minimum = 5

which **approximately** follows the chi-square distribution with *k-s-1* degrees of freedom, where s = # of parameters of the hypothesized distribution estimated by the sample statistics.

Chi-Square test

- The hypothesis of a chi-square test is:

  H_0: The random variable, X, conforms to the distributional assumption with the parameter(s) given by the estimate(s).

  H_1: The random variable X does not conform.

- If the distribution tested is discrete and combining adjacent cell is not required (so that E_i > minimum requirement):
  - Each value of the random variable should be a class interval, unless combining is necessary, and

$$p_i = p(x_i) = P(X = x_i)$$

Chi-Square test

[Goodness-of-Fit Tests]

- If the distribution tested is continuous:

$$p_i = \int_{a_{i-1}}^{a_i} f(x)\,dx = F(a_i) - F(a_{i-1})$$

where a_i-1 and a_i are the endpoints of the i^{th} class interval and $f(x)$ is the assumed pdf, $F(x)$ is the assumed cdf.

  - Recommended number of class intervals (k):

Sample Size, n	Number of Class Intervals, k
20	Do not use the chi-square test
50	5 to 10
100	10 to 20
> 100	$n^{1/2}$ to n/5

  - Caution: Different grouping of data (i.e., k) can affect the hypothesis testing result.

Chi-Square test Ex 9.5,9.12.9.17

- Vehicle Arrival :

  H_0: the random variable is Poisson distributed.

  H_1: the random variable is not Poisson distributed.

x_i	Observed Frequency, O_i	Expected Frequency, E_i	$(O_i - E_i)^2/E_i$
0	12	2.6	7.87
1	10	9.6	
2	19	17.4	0.15
3	17	21.1	0.8
4	19	19.2	4.41
5	6	14.0	2.57
6	7	8.5	0.26
7	5	4.4	11.62
8	5	2.0	
9	3	0.8	
10	3	0.3	
> 11	1	0.1	
	100	100.0	27.68

$$E_i = np(x) = n\frac{e^{-\alpha}\alpha^x}{x!}$$

Combined because of min E_i

- Degree of freedom is $k-s-1 = 7-1-1 = 5$, hence, the hypothesis is rejected at the 0.05 level of significance.

$$\chi_0^2 = 27.68 > \chi_{0.05,5}^2 = 11.1$$

Kolmogorov-Smirnov Test

- Intuition: formalize the idea behind examining a *q-q* plot
- Recall from Chapter 7.4.1:
  - The test compares the **continuous** cdf, $F(x)$, of the hypothesized distribution with the empirical cdf, $S_N(x)$, of the N sample observations.
  - Based on the maximum difference statistics (Tabulated in A.8):

$$D = max|\ F(x) - S_N(x)|$$

- A more powerful test, particularly useful when:
  - Sample sizes are small,
  - No parameters have been estimated from the data.
- When parameter estimates have been made:
  - Critical values in Table A.8 are biased, too large.
  - More conservative, i.e., smaller Type I error than specified.

p-Values and "Best Fits"

[Goodness-of-Fit Tests]

- *p-value* for the test statistics
  - The significance level at which one would just reject H_0 for the given test statistic value.
  - A measure of fit, the larger the better
  - Large *p-value*: good fit
  - Small *p-value*: poor fit

- Vehicle Arrival Example (cont.):
  - H_0: data is Possion
  - Test statistics: $\chi_0^2 = 27.68$, with *5* degrees of freedom
  - *p-value = 0.00004*, meaning we would reject H_0 with *0.00004* significance level, hence Poisson is a poor fit.

p-Values and "Best Fits"

- Many software use *p-value* as the ranking measure to automatically determine the "best fit". Things to be cautious about:
  - Software may not know about the physical basis of the data, distribution families it suggests may be inappropriate.
  - Close conformance to the data does not always lead to the most appropriate input model.
  - *p-value* does not say much about where the lack of fit occurs
- Recommended: always inspect the automatic selection using graphical methods.

9.5 Fitting a Non-stationary Poisson Process

- Fitting a NSPP to arrival data is difficult, possible approaches:
  - Fit a very flexible model with lots of parameters or
  - Approximate constant arrival rate over some basic interval of time, but vary it from time interval to time interval.

Our focus

- Suppose we need to model arrivals over time [0,T], our approach is the most appropriate when we can:
  - Observe the time period repeatedly and
  - Count arrivals / record arrival times.

Fitting a Non-stationary Poisson Process

- The estimated arrival rate during the ith time period is:

$$\hat{\lambda}(t) = \frac{1}{n\Delta t}\sum_{j=1}^{n} C_{ij}$$

where n = # of observation periods, Δt = *time interval length*

C_{ij} = # of arrivals during the i^{th} time interval on the j^{th} observation period

- Example: Divide a *10*-hour business day [*8am,6pm*] into equal intervals $k = 20$ whose length $\Delta t = ½$, and observe over n =3 days

Time Period	Number of Arrivals Day 1	Day 2	Day 3	Estimated Arrival Rate (arrivals/hr)
8:00 - 8:00	12	14	10	24
8:30 - 9:00	23	26	32	54
9:00 - 9:30	27	18	32	52
9:30 - 10:00	20	13	12	30

For instance,
1/3(0.5)*(23+26+32)
= 54 arrivals/hour

9.6 Selecting Model without Data

- If data is not available, some possible sources to obtain information about the process are:
  - Engineering data: often product or process has performance ratings provided by the manufacturer or company rules specify time or production standards.
  - Expert option: people who are experienced with the process or similar processes, often, they can provide optimistic, pessimistic and most-likely times, and they may know the variability as well.
  - Physical or conventional limitations: physical limits on performance, limits or bounds that narrow the range of the input process.
  - The nature of the process.
- The uniform, triangular, and beta distributions are often used as input models.

Selecting Model without Data

- Example 9.20: Production planning simulation.
  - Input of sales volume of various products is required, salesperson of product XYZ says that:
    - No fewer than *1,000* units and no more than *5,000* units will be sold.
    - Given her experience, she believes there is a *90%* chance of selling more than *2,000* units, a *25%* chance of selling more than *2,500* units, and only a *1%* chance of selling more than *4,500* units.
  - Translating these information into a cumulative probability of being less than or equal to those goals for simulation input:

i	Interval (Sales)	Cumulative Frequency, c_i
1	$1000 \leq x \leq 2000$	0.10
2	$2000 < x \leq 3000$	0.75
3	$3000 < x \leq 4000$	0.99
4	$4000 < x \leq 5000$	1.00

9.7 Multivariate and Time-Series Input Models

- Multivariate:
  - For example, lead time and annual demand for an inventory model, increase in demand results in lead time increase, hence variables are dependent.
- Time-series:
  - For example, time between arrivals of orders to buy and sell stocks, buy and sell orders tend to arrive in bursts, hence, times between arrivals are dependent.

Covariance and Correlation

[Multivariate/Time Series]

- Consider the model that describes relationship between X_1 and X_2:

$$(X_1 - \mu_1) = \beta(X_2 - \mu_2) + \varepsilon$$

ε is a random variable with mean *0* and is independent of X_2

  - $\beta = 0$, X_1 and X_2 are statistically independent
  - $\beta > 0$, X_1 and X_2 tend to be above or below their means together
  - $\beta < 0$, X_1 and X_2 tend to be on opposite sides of their means

- Covariance between X_1 and X_2 :

$$\mathrm{cov}(X_1, X_2) = E[(X_1 - \mu_1)(X_2 - \mu_2)] = E(X_1 X_2) - \mu_1 \mu_2$$

  - where $\mathrm{cov}(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$

Covariance and Correlation

[Multivariate/Time Series]

- Correlation between X_1 and X_2 (values between *-1* and *1*):

$$\rho = \operatorname{corr}(X_1, X_2) = \frac{\operatorname{cov}(X_1, X_2)}{\sigma_1 \sigma_2}$$

  - where $corr(X_1, X_2) \begin{cases} = 0, \\ < 0, \\ > 0, \end{cases}$ then $\beta \begin{cases} = 0 \\ < 0 \\ > 0 \end{cases}$
  - The closer ρ is to *-1* or *1*, the stronger the linear relationship is between X_1 and X_2.

Covariance and Correlation

[Multivariate/Time Series]

- A time series is a sequence of random variables X_1, X_2, X_3, … , are identically distributed (same mean and variance) but dependent.
  - cov(X_t, X_{t+h}) is the *lag-h autocovariance*
  - corr(X_t, X_{t+h}) is the *lag-h autocorrelation*
  - If the autocovariance value depends only on h and not on t, the time series is covariance stationary

Multivariate Input Models

[Multivariate/Time Series]

- If X_1 and X_2 are normally distributed, dependence between them can be modeled by the bivariate normal distribution with μ_1, μ_2, σ_1^2, σ_2^2 and correlation ρ
  - To Estimate μ_1, μ_2, σ_1^2, σ_2^2, see "Parameter Estimation" (Section 9.3.2)
  - To Estimate ρ, suppose we have n independent and identically distributed pairs (X_{11}, X_{21}), (X_{12}, X_{22}), … (X_{1n}, X_{2n}), then:

$$\hat{\text{cov}}(X_1, X_2) = \frac{1}{n-1}\sum_{j=1}^{n}(X_{1j} - \hat{X}_1)(X_{2j} - \hat{X}_2)$$

$$= \frac{1}{n-1}\left(\sum_{j=1}^{n} X_{1j}X_{2j} - n\hat{X}_1\hat{X}_2\right)$$

$$\hat{\rho} = \frac{\hat{\text{cov}}(X_1, X_2)}{\hat{\sigma}_1\hat{\sigma}_2}$$

Sample deviation

Time-Series Input Models

[Multivariate/Time Series]

- If X_1, X_2, X_3, … is a sequence of identically distributed, but dependent and covariance-stationary random variables, then we can represent the process as follows:
  - Autoregressive order-1 model, AR(1)
  - Exponential autoregressive order-1 model, EAR(1)
    - Both have the characteristics that:

$$\rho_h = corr(X_t, X_{t+h}) = \rho^h, \quad \text{for } h = 1,2,...$$

    - *Lag-h* autocorrelation decreases geometrically as the lag increases, hence, observations far apart in time are nearly independent

AR(1) Time-Series Input Models

[Multivariate/Time Series]

- Consider the time-series model:

$$X_t = \mu + \phi(X_{t-1} - \mu) + \varepsilon_t, \quad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. normally distributed with $\mu_\varepsilon = 0$ and variance σ_ε^2

- If X_1 is chosen appropriately, then
  - $X_1, X_2, \ldots$ are normally distributed with *mean* = μ, and *variance* = $\sigma^2/(1-\phi^2)$
  - Autocorrelation $\rho_h = \phi^h$
- To estimate ϕ, μ, σ_ε^2 :

$$\hat{\mu} = \bar{X}, \qquad \hat{\sigma}_\varepsilon^2 = \hat{\sigma}^2(1 - \hat{\phi}^2), \qquad \hat{\phi} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

EAR(1) Time-Series Input Models

[Multivariate/Time Series]

- Consider the time-series model:

$$X_t = \begin{cases} \phi X_{t-1}, & \text{with probability } \phi \\ \phi X_{t-1} + \varepsilon_t, & \text{with probability } 1\text{-}\varphi \end{cases} \qquad \text{for } t = 2,3,\ldots$$

where $\varepsilon_2, \varepsilon_3, \ldots$ are i.i.d. exponentially distributed with $\mu_\varepsilon = 1/\lambda$, and $0 \le \phi < 1$

- If X_1 is chosen appropriately, then
  - $X_1, X_2, \ldots$ are exponentially distributed with *mean* = $1/\lambda$
  - Autocorrelation $\rho_h = \phi^h$, and only positive correlation is allowed.
- To estimate ϕ, λ :

$$\hat{\lambda} = 1/\bar{X} \ , \qquad \hat{\phi} = \hat{\rho} = \frac{\hat{\text{cov}}(X_t, X_{t+1})}{\hat{\sigma}^2}$$

where $\hat{\text{cov}}(X_t, X_{t+1})$ is the *lag*-1 autocovariance

Normal to Anything

- The NORTA method on page 375

Summary

- In this chapter, we described the 4 steps in developing input data models:
  - Collecting the raw data
  - Identifying the underlying statistical distribution
  - Estimating the parameters
  - Testing for goodness of fit