Chapter 8
Random-Variate Generation

Banks, Carson, Nelson & Nicol

Discrete-Event System Simulation

Purpose & Overview

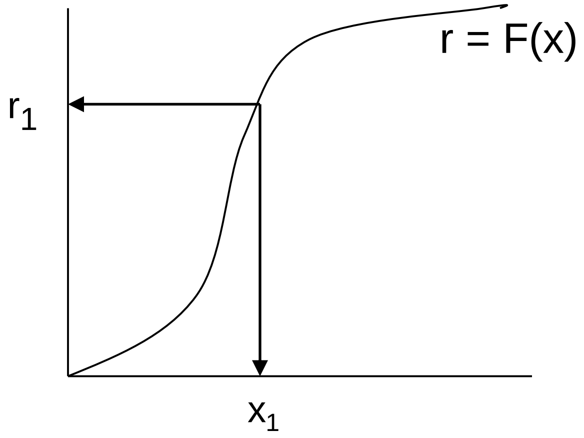
- Develop understanding of generating samples from a specified distribution as input to a simulation model.

- Illustrate some widely-used techniques for generating random variates.
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties

Inverse-transform Technique

- The concept:
  - For cdf function: $r = F(x)$
  - Generate r from uniform (0,1)
  - Find x:

$$\boxed{x = F^{-1}(r)}$$



Steps in inverse-transform technique

Step 1. Compute the cdf of the desired random variable X: $F(x) = 1 - e^{-\lambda x}$

$x \geq 0$

Step 2. Set $F(X) = R$ on the range of X

Step 3. Solve the equation $F(x) = R$ for X in terms of R.

$$1 - e^{-\lambda X} = R$$

$$e^{-\lambda X} = 1 - R$$

$$-\lambda X = \ln(1 - R)$$

$$X = -\frac{1}{\lambda}\ln(1 - R)$$

Step 4. Generate (as needed) uniform random numbers $R_1, R_2, R_3, \ldots$ and compute the desired random variates

See the next slide

Exponential Distribution [Inverse-transform]

- Exponential Distribution:
  - Exponential cdf:

$$r = F(x)$$
$$= 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

  - To generate $X_1, X_2, X_3 \ldots$

$$X_i = F^{-1}(R_i)$$
$$= -(1/\lambda) \ln(1-R_i) \quad \text{[Eq'n 8.3]}$$

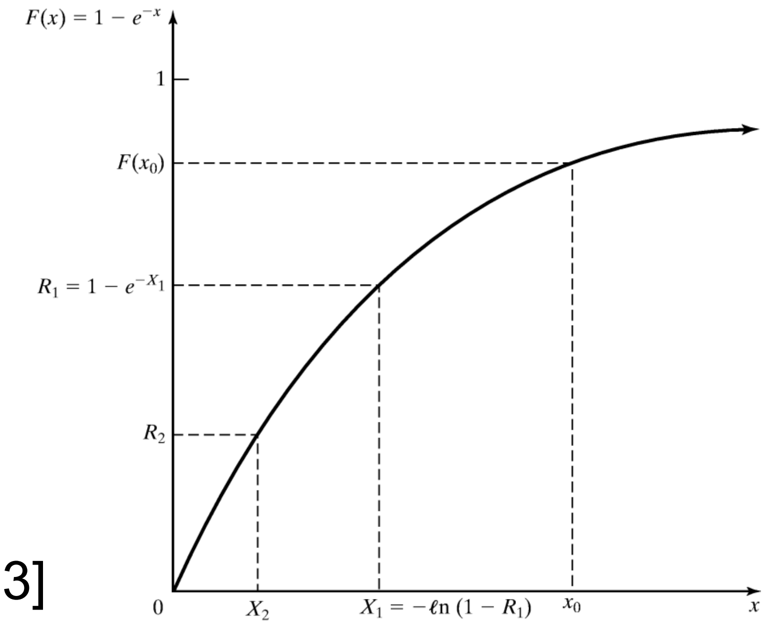
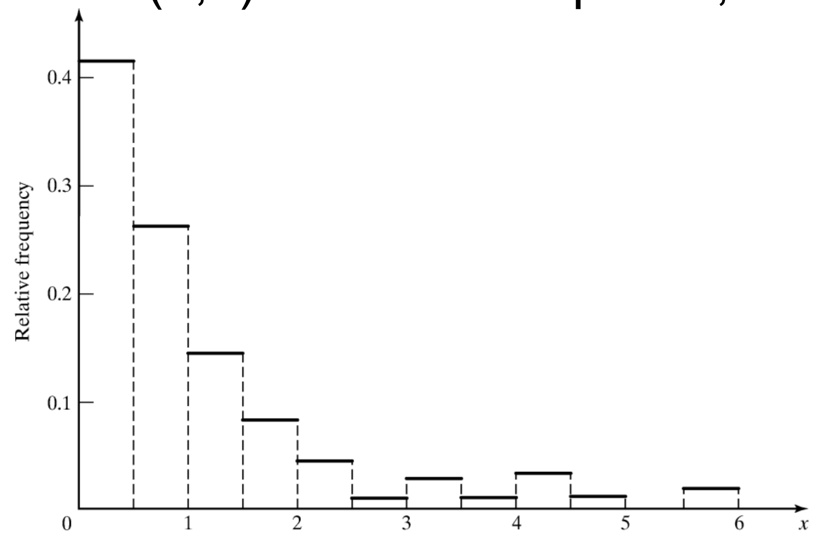


Figure: Inverse-transform technique for *exp(λ = 1)*

Exponential Distribution [Inverse-transform]

- Example: Generate *200* variates X_i with distribution *exp(λ= 1)*
  - Generate *200 Rs* with U(0,1) and utilize eq'n 8.3, the histogram of Xs become:



  - Check: Does the random variable X_1 have the desired distribution?

$$P(X_1 \le x_0) = P(R_1 \le F(x_0)) = F(x_0)$$

Does the random variable X_1 have the desired distribution?

- Pick a value of x_0 and compute the cummulative probability.

$$P(X_1 \leq x_0) = P(R_1 \leq F(x_0)) = F(x_0) \qquad (8.4)$$

- First equality: See figure 8.2 on slide 5.
- It can be seen that $X_1 \leq x_0$ when and only when $R_1 \leq F(x_0)$.
- Since $0 \leq F(x_0) \leq 1$, the second equality in the equation follows immediately from the fact that R_1 is uniformly distributed on [0,1].
- The equation shows that the cdf of X_1 is F;
  - hence X_1 has the desired distribution

Other Distributions

[Inverse-transform]

- Examples of other distributions for which inverse cdf works are:
  - Uniform distribution

    $X = a + (b - a)R$

  - Weibull distribution – time to failure – see steps on p278

    $X = \alpha[- \ln(1 - R)]^{1/\beta}$

  - Triangular distribution

$$X = \begin{cases} \sqrt{2R}, & 0 \le R \le 1/2 \\ 2 - \sqrt{2(1-R)}, & 1/2 < R \le 1 \end{cases}$$

Section 8.1.5 Empirical Continuous Distributions

- This is a worthwhile read (as is the whole chapter of course)
- Works on the question:
  - What do you do if you can't figure out what the distribution of the data is?
- The example starting on slide 13 is a good model to work from.

Empirical Continuous Dist'n [Inverse-transform]

- When theoretical distribution is not applicable
- To collect empirical data:
  - Resample the observed data (i.e. use the data for the distribution)
  - Interpolate between observed data points to fill in the gaps
- For a small sample set (size *n*):
  - Arrange the data from smallest to largest

$$\mathrm{x}_{(1)} \le \mathrm{x}_{(2)} \le \ldots \le \mathrm{x}_{(n)}$$

  - Assign the probability 1/n to each interval $\mathrm{x}_{(i-1)} \le \mathrm{x} \le \mathrm{x}_{(i)}$

$$\boxed{X = \hat{F}^{-1}(R) = x_{(i-1)} + a_i\left(R - \frac{(i-1)}{n}\right)}$$

where $a_i = \dfrac{x_{(i)} - x_{(i-1)}}{1/n - (i-1)/n} = \dfrac{x_{(i)} - x_{(i-1)}}{1/n}$

Empirical Continuous Dist'n [Inverse-transform]

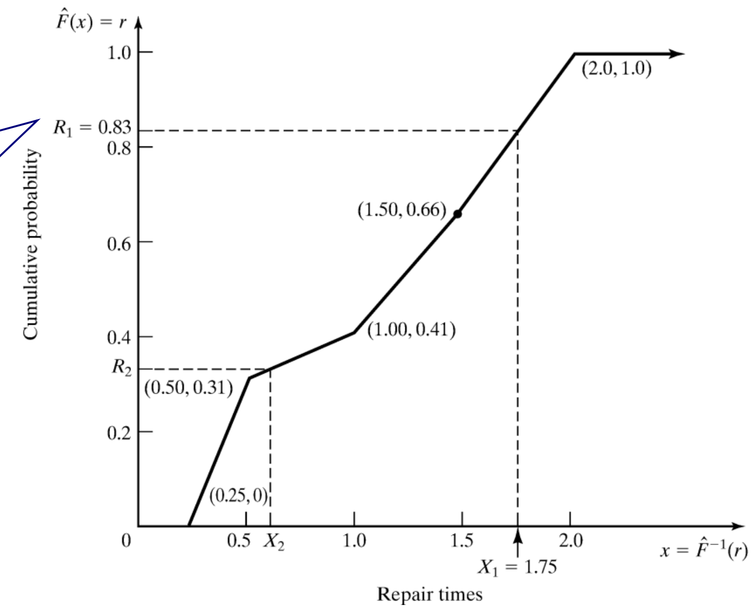
- Example: Suppose the data collected for100 broken-widget repair times are:

i	*Interval (Hours)*	*Frequency*	*Relative Frequency*	*Cumulative Frequency, c_i*	*Slope, a_i*
1	$0.25 \le x \le 0.5$	31	0.31	0.31	0.81
2	$0.5 \le x \le 1.0$	10	0.10	0.41	5.0
3	$1.0 \le x \le 1.5$	25	0.25	0.66	2.0
4	$1.5 \le x \le 2.0$	34	0.34	1.00	1.47

Consider $R_1 = 0.83$:

$c_3 = 0.66 < R_1 < c_4 = 1.00$

$$X_1 = x_{(4-1)} + a_4(R_1 - c_{(4-1)})$$
$$= 1.5 + 1.47(0.83-0.66)$$
$$= 1.75$$



8.1.6

- There are continuous distributions without a nice closed-form expression for their cdf or its inverse.
  - Normal distribution
  - Gamma
  - Beta

- Must approximate in these cases

Discrete Distribution [Inverse-transform]

- All discrete distributions can be generated via inverse-transform technique
- Method: numerically, table-lookup procedure, algebraically, or a formula
- Examples of application:
  - Empirical
  - Discrete uniform
  - Gamma

Example 8.4

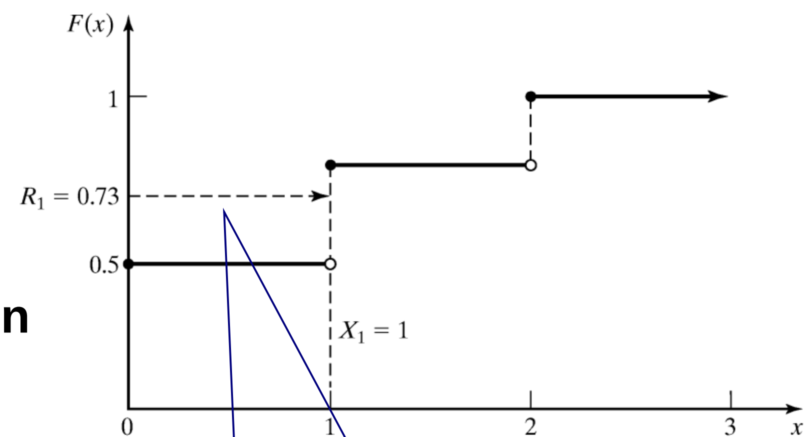
An Empirical Discrete Distribution [Inverse-transform]

- Example: Suppose the number of shipments, x, on the loading dock of IHW company is either *0*, *1*, or *2*
  - Data - Probability distribution:

x	p(x)	F(x)
0	0.50	0.50
1	0.30	0.80
2	0.20	1.00

  - Method - **Given R, the generation scheme becomes:**

$$x = \begin{cases} 0, & R \le 0.5 \\ 1, & 0.5 \prec R \le 0.8 \\ 2, & 0.8 \prec R \le 1.0 \end{cases}$$



Consider $R_1 = 0.73$:

$F(x_{i-1}) < R <= F(x_i)$

$F(x_0) < 0.73 <= F(x_1)$

Hence, $x_1 = 1$

Discrete distributions continued

- Example 8.5 concerns a Discrete Uniform Distribution

- Example 8.6 concerns the Geometric Distribution

8.2: Acceptance-Rejection technique

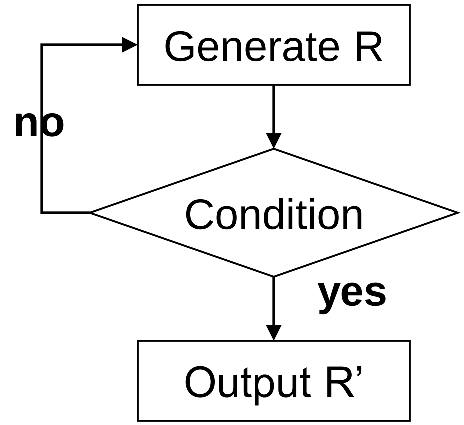
- Useful particularly when inverse cdf does not exist in closed form, a.k.a. thinning
- Illustration: To generate random variates, $X \sim U(1/4, 1)$

Procedures:

Step 1. Generate R ~ U[0,1]

Step 2a. If R >= ¼, accept X=R.

Step 2b. If R < ¼, reject R, return to Step 1

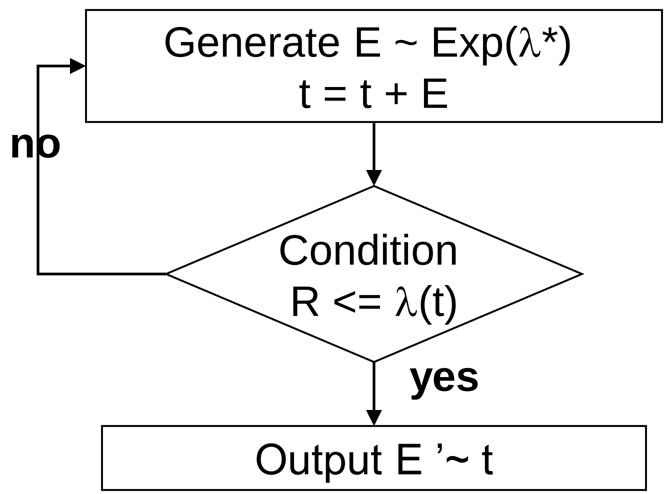


- *R* does not have the desired distribution, but *R* conditioned (R') on the event $\{R \geq 1/4\}$ does. (8.21, P. 289)
- Efficiency: Depends heavily on the ability to minimize the number of rejections.

NSPP

[Acceptance-Rejection]

- Non-stationary Poisson Process (NSPP): a Possion arrival process with an arrival rate that varies with time
- Idea behind thinning:
  - Generate a stationary Poisson arrival process at the fastest rate, $\lambda^* = \max \lambda(t)$
  - But "accept" only a portion of arrivals, thinning out just enough to get the desired time-varying rate



8.2 Acceptance – Rejection continued

- 8.2.1 Poisson Distribution
  - Step 1 set $n = 0$, $P = 1$
  - Step 2 generate a random number R_{n+1}

  And replace P by $P * R_{n+1}$
  - Step 3 if $P < e^{-\lambda}$, then accept, otherwise, reject the current n, increase n by 1 and return to step 2

Non-Stationary Poisson Process [Acceptance-Rejection]

- Example: Generate a random variate for a NSPP

Data: Arrival Rates

***t* (min)**	***Mean Time Between Arrivals* (min)**	***Arrival Rate $\lambda(t)$* (#/min)**
0	15	1/15
60	12	1/12
120	7	1/7
180	5	1/5
240	8	1/8
300	10	1/10
360	15	1/15
420	20	1/20
480	20	1/20

Procedures:

Step 1. $\lambda^* = max\ \lambda(t) = 1/5$, $t = 0$ and $i = 1$.

Step 2. For random number $R = 0.2130$,

$E = -5ln(0.213) = 13.13$

$t = 13.13$

Step 3. Generate $R = 0.8830$

$\lambda(13.13)/\lambda^*=(1/15)/(1/5)=1/3$

Since $R>1/3$, do not generate the arrival

Step 2. For random number $R = 0.5530$,

$E = -5ln(0.553) = 2.96$

$t = 13.13 + 2.96 = 16.09$

Step 3. Generate $R = 0.0240$

$\lambda(16.09)/\lambda^*=(1/15)/(1/5)=1/3$

Since $R<1/3$, $T_1 = t = 16.09$,

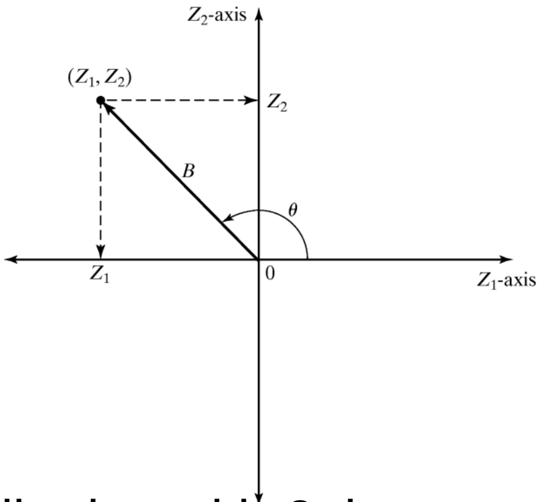
and $i = i + 1 = 2$

8.3: Special Properties

- Based on features of particular family of probability distributions
- For example:
  - Direct Transformation for normal and lognormal distributions
  - Convolution
  - Beta distribution (from gamma distribution)

Direct Transformation

- Approach for normal*(0,1)*:
  - Consider two standard normal random variables, Z_1 and Z_2, plotted as a point in the plane:



In polar coordinates:

$$Z_1 = B \cos \phi$$
$$Z_2 = B \sin \phi$$

  - $B^2 = Z^2_1 + Z^2_2$ ~ chi-square distribution with *2* degrees of freedom = $Exp(\lambda = 2)$. Hence, $B = (-2 \ln R)^{1/2}$
  - The radius B and angle ϕ are mutually independent.

$$\boxed{\begin{aligned} Z_1 &= (-2 \ln R)^{1/2} \cos(2\pi R_2) \\ Z_2 &= (-2 \ln R)^{1/2} \sin(2\pi R_2) \end{aligned}}$$

Direct Transformation [Special Properties]

- Approach for normal(μ, σ^2):
  - That is, with mean μ and variance σ^2
  - Generate $Z_i \sim N(0,1)$ *as above*

$$\boxed{X_i = \mu + \sigma Z_i}$$

- Approach for lognormal(μ, σ^2):
  - Generate $X \sim N((\mu, \sigma^2)$

$$\boxed{Y_i = e^{X_i}}$$

Summary

- Principles of random-variate generate via
  - Inverse-transform technique
  - Acceptance-rejection technique
  - Special properties
- Important for generating continuous and discrete distributions