Introduction to Communication Signal

1. Signals and System:-

Signals are time-varying quantities such as voltages or current.

A system is a combination of devices and networks (subsystems) chosen to perform a desired function .Because of the sophistication of modern communication systems , a great deal of analysis and experimentation with trial subsystems occurs before actual building of the desired system . Thus the communications engineer's tools are mathematical models for signals and systems.

2. Classification of Signals and System:-

2.1 Continuous-time and discrete-time signals

By the term **continuous-time** signal we mean a real or complex function of time $s(t)$, where the independent variable t is continuous.

If t is a discrete variable, i.e., s(t) is defined at discrete times, then the signal $s(t)$ is **a discrete-time** signal. A discrete-time signal is often identified as a sequence of numbers, denoted by {s(n)}, where n is an integer.

2.2 Analogue and digital signals:

If a continuous-time signal s(t) can take on any values in a continuous time interval, then s(t) is called an **analogue signal.**

If a discrete-time signal can take on only a finite number of distinct values, {s(n)}, then the signal is called **a digital signal.**

2.3 Deterministic and random signals:

Deterministic signals are those signals whose values are completely specified for any given time.

Random signals are those signals that take random values at any given times.

2.4 Periodic and nonperiodic signals:

A signal $s(t)$ is a **periodic signal** if $s(t) = s(t + nT_0)$, where T_0 is called the **period** and the integer $n > 0$.

If $s(t) \neq s(t + T_0)$ for all t and any T_0, then $s(t)$ is a **nonperiodic** or **a periodic** signal.

2.5 Power and energy signals:

A complex signal $s(t)$ is a power signal if the average normalized power **P** is finite, where

$$0 < P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s(t)s^*(t)dt < \infty$$

and **s*(t)** is the complex conjugate of $s(t)$.

A complex signal $s(t)$ is an energy signal if the normalized energy E is finite, where

$$0 < E = \int_{-\infty}^{\infty} s(t)s^*(t)\,dt = \int_{-\infty}^{\infty} |s(t)|^2\,dt < \infty$$

In communication systems, the received waveform is usually categorised into the desired part, containing the information signal, and the undesired part, called noise.

3. Some Useful Functions:

3.1 Unit impulse function:

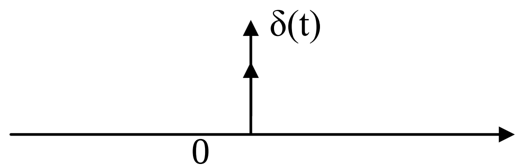
The unit impulse function, also known as the dirac delta function, ***δ(t),*** is defined by:

$$\int_{-\infty}^{\infty} s(t)\delta(t)dt = s(0)$$

An alternative definition is:

$$\int_{-\infty}^{\infty} \delta(t)dt = 1.........and$$

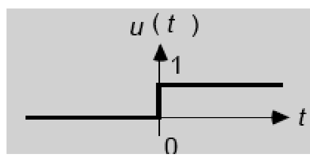
$$\delta(t) = \begin{cases} \infty,......t = 0 \\ 0,........t \neq 0 \end{cases}$$



3.2 Unit step function:

The unit step function ***u(t)*** is :

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

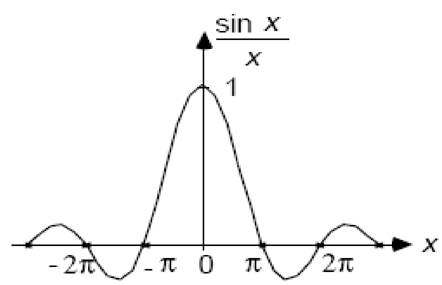


and the unit step function is related to the unit impulse function by **:**

$$u(t) = \int_{-\infty}^{\infty} \delta(t)\, d\, t$$ and $$\frac{du(t)}{dt} = \delta(t)$$

3.3 Sampling function:

A sampling function is denoted by: $\mathrm{Sa}(x) = \dfrac{\sin x}{x}$



3.4 Sinc function:

A sinc function is denoted by: $\mathrm{sinc}\, x = \dfrac{\sin \pi x}{\pi x}$

Hence, $\mathrm{Sa}(x) = \mathrm{sinc}\left(\dfrac{x}{\pi}\right)$

3.5 Rectangular function:

A single rectangular pulse is denoted by:

$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1, & |t| \leq \frac{T}{2} \\ 0, & |t| > \frac{T}{2} \end{cases}$$

3.6 Triangular function:

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \frac{|t|}{T}, & |t| \leq T \\ 0, & |t| > T \end{cases}$$

4. Other Useful Operations:

4.1 Cross-correlation:

The cross-correlation **of two real-valued power** waveforms ***$s_1(t)$*** and ***$s_2(t)$*** is defined by :

$$R_{12}(\tau) = \langle s_1(t)\, s_2(t+\tau)\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s_1(t)\, s_2(t+\tau)\, dt$$

If ***$s_1(t)$*** and ***$s_2(t)$*** are periodic with the same period ***T_0***, the

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s_1(t)\, s_2(t+\tau)\, dt$$

The cross-correlation of **two real-valued energy** waveforms ***$s_1(t)$*** and ***$s_2(t)$*** is defined by :

$$R_{12}(\tau) = \int_{-\infty}^{\infty} s_1(t)\, s_2(t+\tau)\, dt$$

Correlation: is a useful operation to measure the similarity between two waveforms. To compute the correlation between waveforms, it is necessary to specify which waveform is being shifted. In general,

$R_{12}(\tau)$ is not equal to ***$R_{21}(\tau)$***, where ***$R_{21}(\tau) = \langle s_2(t)\, s_1(t+\tau)\rangle$.***

The cross-correlation of two complex waveforms is: **$R_{12}(\tau) = \langle s_1^*(t)\, s_2(t+\tau)\rangle$.**

4.2 Auto-correlation:

The auto-correlation of a **real-valued power** waveform $s_1(t)$ is defined by:

$$R_{11}(\tau) = \langle s_1(t)\, s_1(t+\tau)\rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} s_1(t)\, s_1(t+\tau)\, dt$$

If $s_1(t)$ is **periodic** with fundamental period T_0, then

$$R_{11}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} s_1(t)\, s_1(t+\tau)\, dt$$

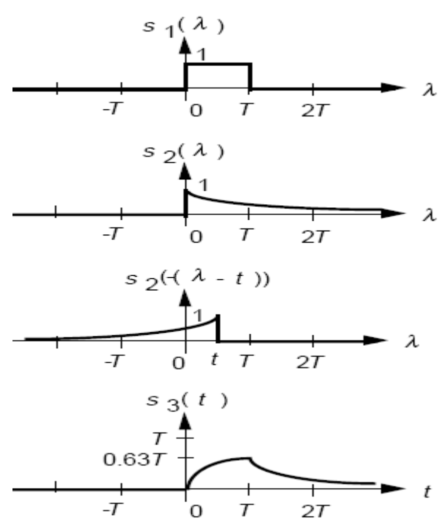
The auto-correlation of a real-valued energy waveform ***$s_1(t)$*** is defined by:

$$R_{11}(\tau) = \int_{-\infty}^{\infty} s_1(t)\, s_1(t+\tau)\, dt$$

The auto-correlation of a complex power waveform is:

$R_{11}(\tau) = \langle s_1^*(t)\, s_1(t+\tau)\rangle$.

4.3 Convolution:



The convolution of a waveform ***$s_1(t)$*** with a waveform ***$s_2(t)$*** is given by:

$$s_3(t) = s_1(t) * s_2(t) = \int_{-\infty}^{\infty} s_1(\lambda)\, s_2(t-\lambda)\, d\lambda$$

$$s_3(t) = s_1(t) * s_2(t) = \int_{-\infty}^{\infty} s_1(\lambda)\, s_2[-(\lambda-t)]\, d\lambda$$

Where * denotes the convolution operation. The above equation is obtained by:

1. Time reversal of $s_2(t)$ to obtain $s_2(-\lambda)$.
2. Time shifting of $s_2(-\lambda)$ to obtain $s_2[-(\lambda-t)]$.
3. Multiplying $s_1(\lambda)$ and $s_2[-(\lambda-t)]$ to form the integrand $s_1(\lambda)\, s_2[-(\lambda-t)]$.

Example 1: Convolution of a rectangular waveform

$$s_1(t) = \begin{cases} 1, & 0 < t < T \\ 0, & elsewhere \end{cases}$$

with an exponential waveform

$$s_2(t) = e^{-t/T} u(t).$$

Convolution of a rectangular waveform and an exponential waveform.

What is Digital Signal Processing?

A **signal** is a function of a set of independent variables, with time being perhaps the most prevalent single variable. The signal itself carries some kind of information available for observation. By **processing** we mean operating in some fashion on a signal to extract some useful information. In many cases this processing will be a nondestructive "transformation" of the given data signal; however some important processing methods turn out to be irreversible and thus destructive. The word **digital** shall mean that the processing is done with a digital computer or special purpose digital hardware.

Figure 1: A Typical DSP Schemel.

Why DSP?

- Rapid advances in integrated circuit design and manufacture are producing more powerful DSP system on a single chip at decreasing size and cost.
- Digital processing is inherently stable and reliable.
- In many cases DSP is used to process a number signals simultaneously. This may be done by using a technique known as "TDM" (time-division-multiplexing).
- Digital implementation permits easy adjustment of process characteristics during processing, such as that needed in implementing adaptive circuits.

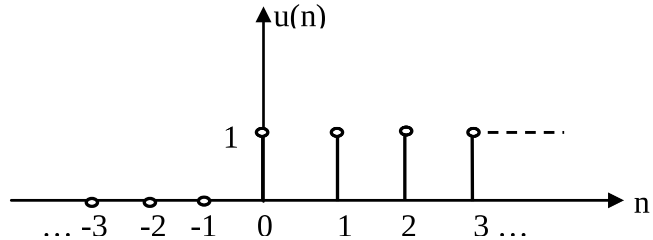
Applications (DSP)

- Spectral Analysis
- Speech Recognition
- Biomedical Signal Analysis
- Digital Filtering
- Digital Modems
- Data Encryption
- Image Enhancement and Compression………

Basic Types of Digital Signals

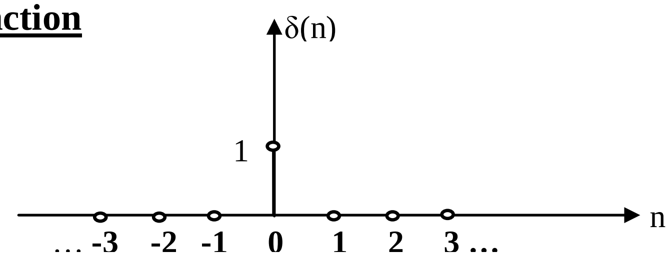
1. Step Function

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$



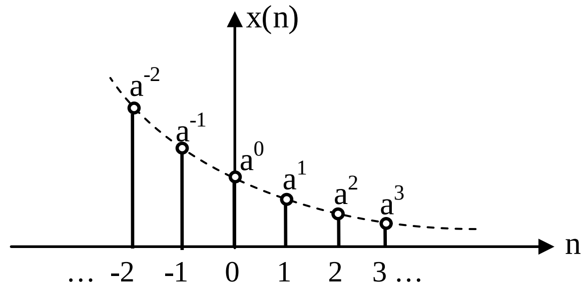
2. Unit Impulse (Sample) Function or Delta Function

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



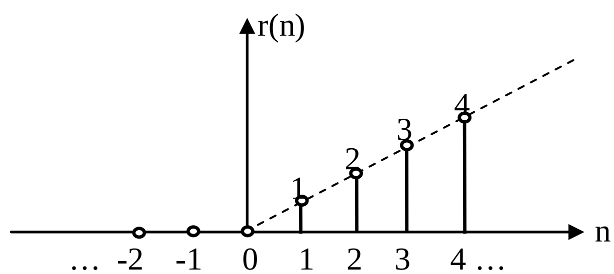
3. Exponential function

$$x(n) = a^n \qquad \text{for all n}$$



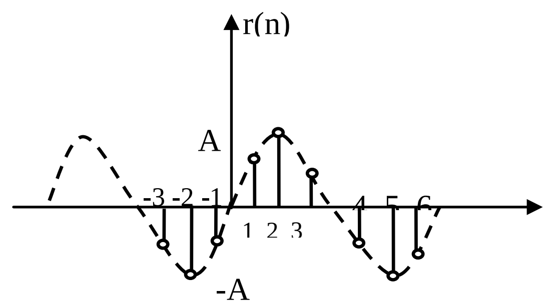
4. Ramp function

$$r(n) = \begin{cases} nu(n) & n > 0 \\ 0 & n \leq 0 \end{cases}$$

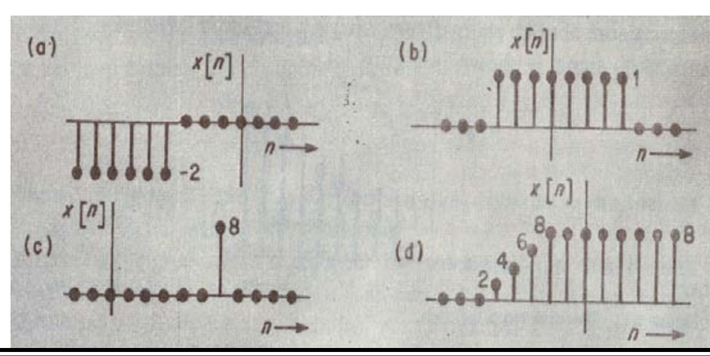


5. Sinusoidal function

$$x(n) = A \sin(w_0 n) \qquad \text{for all n}$$



Example: Find expressions for the various signals shown in figure below:



Solution:

(a) $x[n] = -2u[-n-4]$
(b) $x[n] = u[n+3] - u[n-5]$
(c) $x[n] = 8\delta[n-6]$
(d) $x[n] = 2r[n+6]-2r[n+2]$

Linear Time Invariant (LTI) Systems:

A linear system or processor may be defined as one which obeys the principle of superposition. The principle may be stated as follows:

If an input consisting of the sum of a number of signals is applied to a linear system, then the output is the sum, or superposition, of the system's responses to each signal considered separately.

Let us suppose than an input $\mathbf{x_1[n]}$, applied to a digital processor, produces the output $\mathbf{y_1[n]}$; and that input $\mathbf{x_2[n]}$ produces $\mathbf{y_2[n]}$. Then the processor is linear if its response to $\mathbf{\{ x_1[n] + x_2[n]\}}$ is $\mathbf{\{y_1[n] + y_2[n]\}}$. Furthermore, linearity implies that the response to an input $\mathbf{ax_1[n]}$ is $\mathbf{ay_1[n]}$**,** where **(a)** is a constant coefficient, or multiplier (also called a weighting factor). To generalize, the weighted sum of inputs: $\mathbf{ax_1[n] + bx_2[n] + cx_3[n] + \ldots\ldots}$

Must produce the corresponding weighted sum of outputs: $\mathbf{ay_1[n] + by_2[n] + cy_3[n] + \ldots\ldots}$

A non-linear system, which has not characteristics of superposition, for example a system which have square each sample value applied to its input: $\mathbf{y_1[n]=(x_1[n])^2}$ **&** $\mathbf{y_2[n]=(x_2[n])^2}$

The output is $\mathbf{y_3[n] = [x_1[n] + x_2[n]]^2 = (x_1[n])^2 + (x_2[n])^2 + 2x_1[n]x_2[n]}$

This is clearly not the sum of its responses to $\mathbf{x_1[n]}$ and $\mathbf{x_2[n]}$ applied separately.

In other words, property of linear systems is known as "**frequency-preservation**". It means that if we apply an input signal containing certain frequencies to a linear system, the output can contain only the same frequencies, and no others.

We next describe "**time invariance**". A time-invariant system is one whose properties do not vary with time. The only effect of a time-shift in an input signal to the system is a corresponding time-shift in its output.

$$\mathbf{x[n-k] = y[n-k]} \text{ for any value k}$$

More generally, digital LTI processors involve the following types of operation an input and/or output samples.

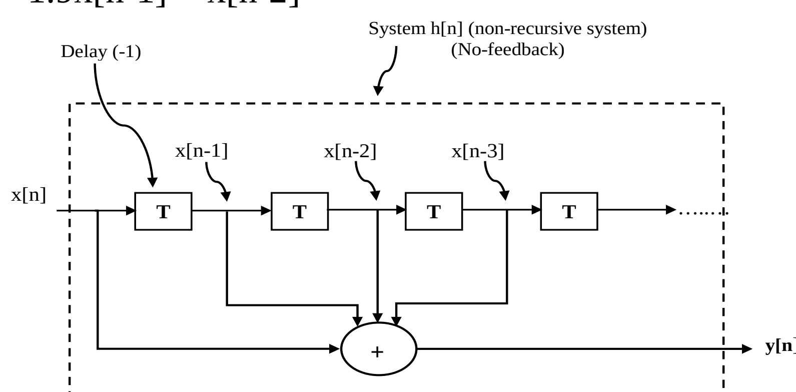
- ❖ Storage/delay
- ❖ Addition/subtraction
- ❖ Multiplication by constants

For example, consider the digital filters as:

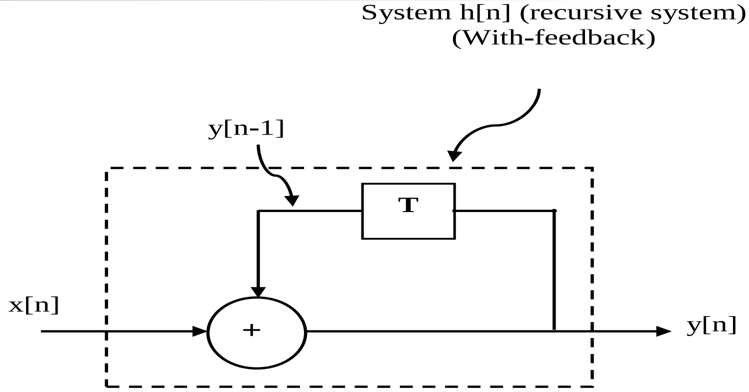
1) y[n] = x[n] + x[n-1] + x[n-2] + …..
2) y[n] = y[n-1] + x[n]
3) y[n] = 1.8y[n-1] – 0.9y[n-2] + x[n] – 1.9x[n-1] + x[n-2]

Solution

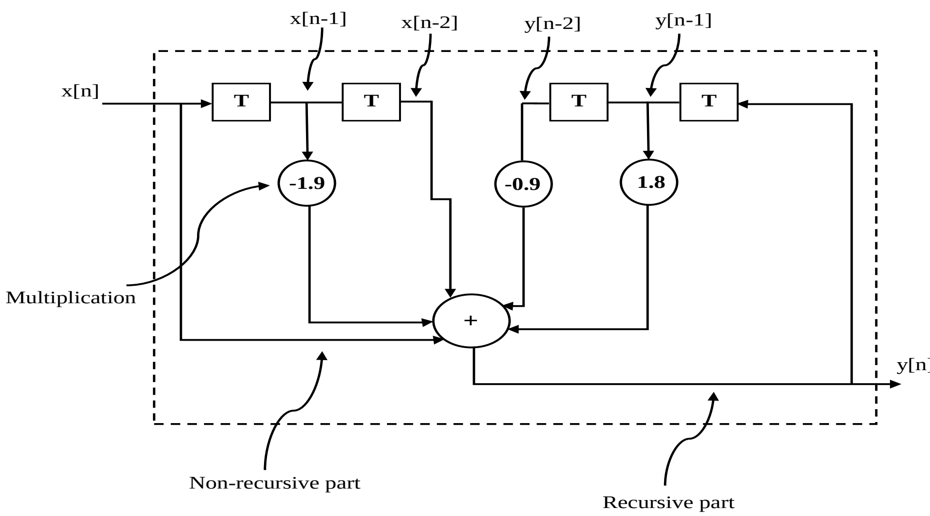
1) y[n] = x[n] + x[n-1] + x[n-2] + …..



2) y[n] = y[n-1] + x[n]



3) y[n] = 1.8y[n-1] – 0.9y[n-2] + x[n] – 1.9x[n-1] + x[n-2]



Other System Properties:

1) Causality

The output signal depends only on present and/or previous values of the input.

- Forward difference system (not causal (+))

$$y[n] = x[n+1] - x[n]$$

- backward difference system (causal (-))

$$y[n] = x[n] - x[n-1]$$

2) Stability (BIBO)

Is one which produces a finite, or bounded, output in response to a bounded input?

$$S = \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Stable if S is finite

3) Invertibility

If a digital processor with input x[n] gives an output y[n] then its inverse would produce x[n], if fed with y[n].

Example 1: y[n] = cx[n] $\longrightarrow$ $x[n] = \frac{1}{c} y[n]$ invertibility

Example 2: $y[n] = (x[n])^2$ $\longrightarrow$ $x[n] = \pm\sqrt{y[n]}$ non-invertibility

4) Memory

A processor possesses memory if its present output y[n] depends upon one or more previous input values x[n-1], x[n-2]… in other words it must contain storage/delay elements.

Example: **x[n]** and **y[n]** are the inputs and output signals of a DSP system. Determine which of the following properties are possessed by systems defined by the recurrence formulae (a) to (d) below: linearity, time-invariance, causality, stability, invertibility, and memory.

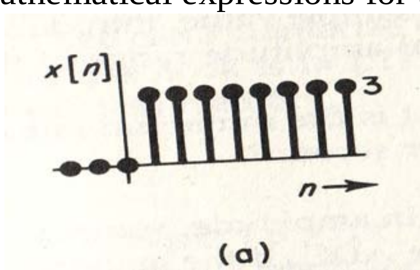
a) y[n] = 3x[n] – 4x[n-1]
b) y[n] = 2y[n-1] + x[n+2]
c) y[n] = nx[n]
d) y[n] = cos(x[n])

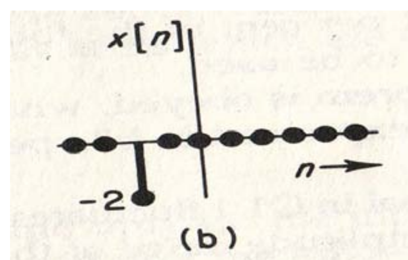
Solution:

a) Linear, time-invariant, causal, stable, invertible, and memory.
b) Not causal, unstable, linear, time-invariant, invertible, and memory.
c) No memory, linear, time-variant, causal, stable, and invertible.
d) Not invertible, no memory, not linear, time-invariant, causal, stable.

H.W. 1

Q1/ Find mathematical expressions for the signals below:





Q2/ Sketch the following signals:

a) – u[n-2]
b) u[n+1] + δ[n]
c) r[n] – 2r[n-3]

Q3/ x[n] and y[n] are the input and output signals of a digital processor. Determine which of the following properties are exhibited by each of the systems defined below: linearity, time-invariance, causality, stability, and memory.

a) y[n] = x[5-n]
b) y[n] = x[n] + x[n-1] + 3x[n-2]
c) y[n] = nx[n]

Q4/ draw a block diagram for a digital processor with the following recurrence formula. Distinguish clearly between its non-recursive and recursive parts.

y[n] = 1.625y[n-1] – 0.934y[n-2] + 0.5x[n] – 0.1x[n-2]

Review of Fourier Series:-

1. The Time-Frequency Concept:

Consider the following set of time functions $s(t) = \{3A \sin \omega_0 t, A \sin 2\omega_0 t\}$.

We can represent these functions in different ways by plotting the amplitude versus time t, amplitude versus angular frequency ω, or amplitude versus frequency f.

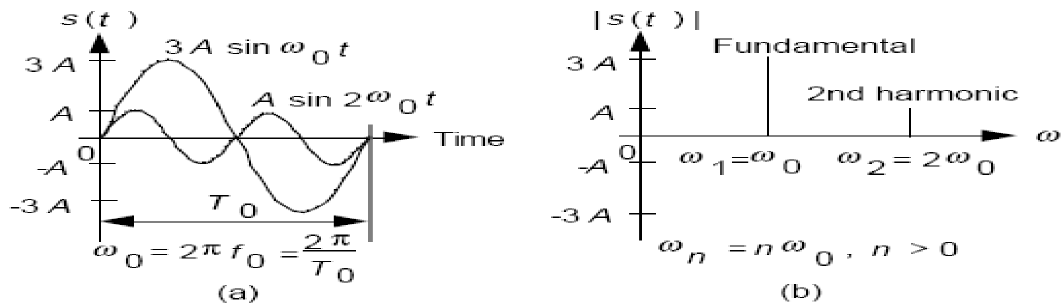


Figure 1: (a) Amplitude-time plot, (b) amplitude-angular frequency plot.

$\omega_0 = 2\pi/T_0$ is called the fundamental angular frequency and $\omega_2 = 2\omega_0$ is called the second harmonic of the fundamental. In general, $\omega_n = n\omega_0$ is said to be the nth harmonic of the fundamental, where $n > 1$.

In communication engineering we are interested in steady-state analysis much of the time.

The Fourier series provides a useful model for analysing the frequency content and the steady-state network response for periodic input signals.

2. Trigonometric (Quadrature) Fourier series:

A periodic time function $s(t)$ over the interval $a - \frac{T_0}{2} < t < a + \frac{T_0}{2}$

May be represented by an infinite sum of sinusoidal waveforms

$$s(t) = \frac{a_0}{T_0} + \frac{2}{T_0}\sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

Where T_0 is the period of the fundamental frequency f_0 and $f = 1/T_0$. This is called the trigonometric (quadrature) Fourier series representation of the time function $s(t)$. The coefficients a_n and b_n are given by:

$$a_n = \int_{a-T_0/2}^{a+T_0/2} s(t) \cos n\omega_0 t \, dt, \; n \geq 0$$

and

$$b_n = \int_{a-T_0/2}^{a+T_0/2} s(t) \sin n\omega_0 t \, dt, \; n > 0$$

The choice of a is arbitrary, and it is usually set to 0.

Many forms of the trigonometric Fourier series may be written.

For example: $s(t) = a'_0 + \sum_{n=1}^{\infty} (a'_n \cos n\omega_0 t + b'_n \sin n\omega_0 t)$ is commonly used.

The coefficients a'_n and b'_n are given by:

$$a'_n = \frac{2}{T_0}\int_{a-T_0/2}^{a+T_0/2} s(t) \cos n\omega_0 t \, dt, \; n \geq 0$$

and

$$b'_n = \frac{2}{T_0}\int_{a-T_0/2}^{a+T_0/2} s(t) \sin n\omega_0 t \, dt, \; n > 0$$

Example 2: Find the trigonometric Fourier series for the periodic time function ***s(t)*** shown in Figure .2.

$$a_n = \int_{-T_0/2}^{T_0/2} s(t)\cos n\omega_0 t\, dt$$

$$a_n = \int_{-\tau/2}^{\tau/2} A_m \cos n\omega_0 t\, dt$$

$$a_n = 2A_m \frac{\sin n\omega_0\tau/2}{n\omega_0} = A_m\tau\frac{\sin n\omega_0\tau/2}{n\omega_0\tau/2}$$

$$b_n = \int_{-T_0/2}^{T_0/2} s(t)\sin n\omega_0 t\, dt = 0$$

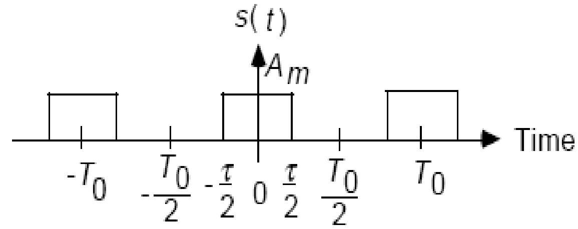


Figure 2: A periodic rectangular waveform.

Therefore,

$$s(t) = \frac{A_m\tau}{T_0} + \frac{2}{T_0}\sum_{n=1}^{\infty}\left(A_m\tau\frac{\sin n\omega_0\tau/2}{n\omega_0\tau/2}\right)\cos n\omega_0 t.$$

3. Exponential (Complex or Phasor) Fourier Series :

The time function ***s(t)*** may be represented over the interval $a - \frac{T_0}{2} < t < a + \frac{T_0}{2}$

By the equivalent exponential (complex or phasor) Fourier series $s(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

where the coefficients c_n are given by: $c_n = \int_{a-T_0/2}^{a+T_0/2} s(t)\, e^{-jn\omega_0 t}\, dt$

c_0 is equivalent to the dc value of the waveform s(t).

c_n is, in general, a complex number.

Furthermore, it is a phase since it is the coefficient of $e^{jn\omega_0 t}$.

The complex Fourier series is easier to use for analytical problems.

Many forms of the complex Fourier series may be written.

For example, $s(t) = \sum_{n=-\infty}^{\infty} c'_n e^{jn\omega_0 t}$ is commonly used.

The coefficients c'_n are given by: $c'_n = \frac{1}{T_0}\int_{a-T_0/2}^{a+T_0/2} s(t)\, e^{-jn\omega_0 t}\, dt$

Example 3: Find the complex Fourier series for the periodic time function ***s(t)*** shown in Figure 2.2.

$$c_n = \int_{-T_0/2}^{T_0/2} s(t)\, e^{-jn\omega_0 t}\, dt$$

$$c_n = \int_{-\tau/2}^{\tau/2} A_m\, e^{-jn\omega_0 t}\, dt$$

$$c_n = A_m \frac{e^{jn\omega_0\tau/2} - e^{-jn\omega_0\tau/2}}{jn\omega_0}$$

$$c_n = 2A_m\frac{\sin n\omega_0\tau/2}{n\omega_0} = A_m\tau\frac{\sin n\omega_0\tau/2}{n\omega_0\tau/2}$$

Therefore,

$$s(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} \left(A_m \tau \frac{\sin n\omega_0 \tau / 2}{n\omega_0 \tau / 2}\right) e^{jn\omega_0 t}$$

The frequency spectrum is shown in Figure .3.

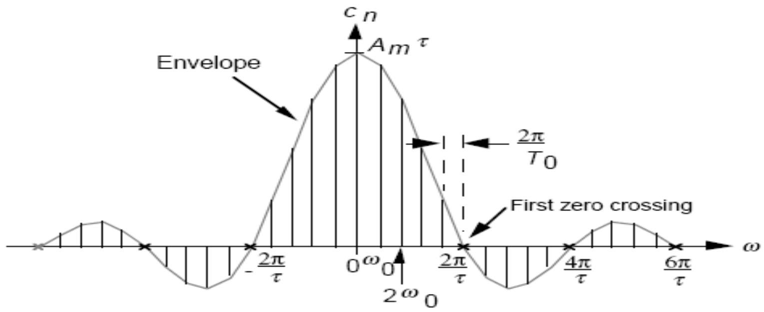


Figure 3: Frequency spectrum of a periodic rectangular waveform.

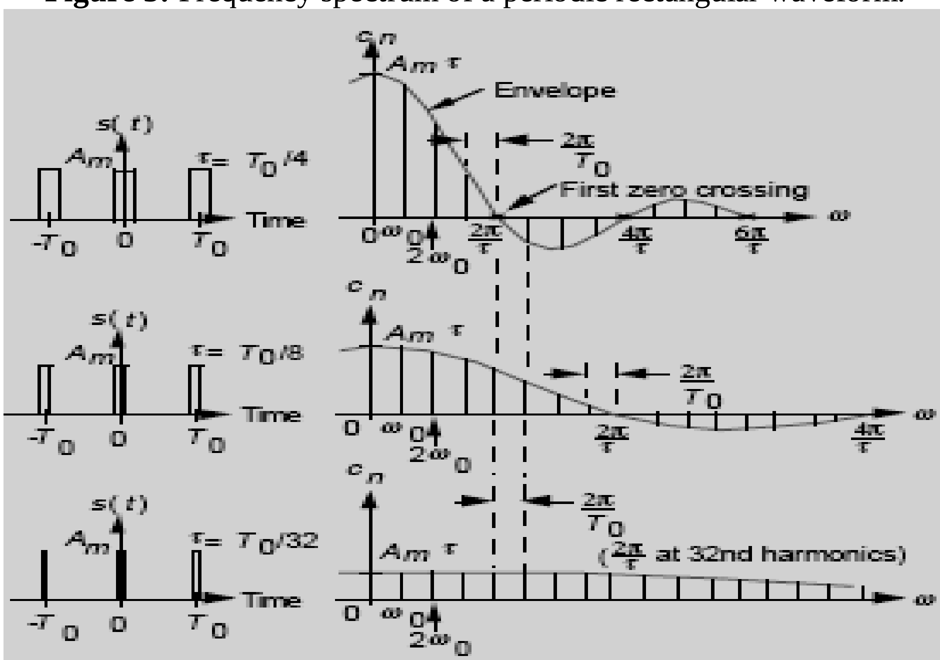


Figure 4: shows the effect on the frequency spectrum of smaller τ

If the bandwidth ***B*** is specified as the width of the frequency band of a waveform from zero frequency to the first zero crossing, then ***B* = *1/τ*** Hz.

If we let the pulse width τ in Figure 4, go to zero and the amplitude Am go to infinity with **Amτ = 1**, all spectral lines in the frequency domain have unity length. Figure .5 shows the periodic unit impulses and the frequency spectrum of the periodic unit impulses. The bandwidth becomes infinite.

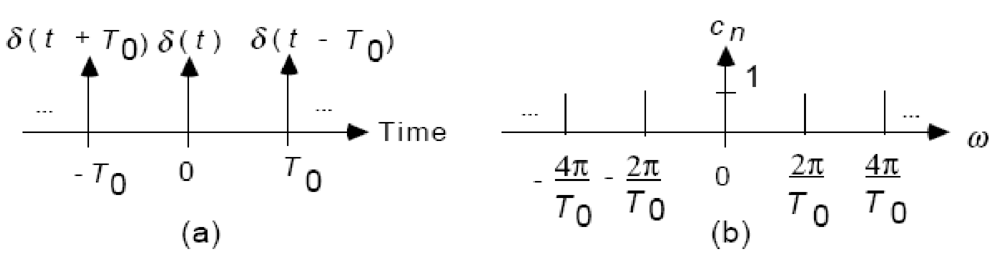


Figure 5: (a) Periodic unit impulses, and (b) frequency spectrum.

Properties of the Complex Fourier Series:

1. If s(t) is real, then $\mathbf{c_n = c^*_{-n}}$
2. If s(t) is real and even**, s(t) = s(-t)** , then **Img[c_n] = 0**
3. If s(t) is real and odd, **s(t) = - s(-t)** , then **Re[c_n]** = 0
4. The complex Fourier-series coefficients of a real waveform are related to the quadrature Fourier-series coefficients by:

$$c_n = \begin{cases} a_n - jb_n, & n > 0 \\ a_0, & n = 0 \\ a_{-n} + jb_{-n}, & n < 0 \end{cases}$$

$$|c_n| = \sqrt{a_n^2 + b_n^2}$$

represents the amplitude spectrum and $\angle c_n = \theta_n = \tan^{-1} \dfrac{-b_n}{a_n}$
represents the phase spectrum of the real waveform.

The equivalence between the Fourier series coefficients is demonstrated in Figure 6.

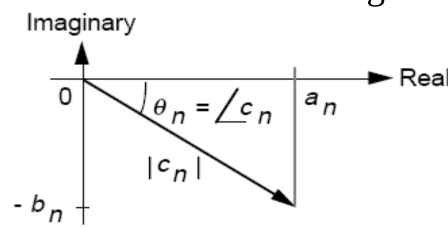


Figure 6: Fourier series coefficients, *n* > 1.

Parseval's Theorem for the Fourier Series :

Parseval's Theorem for the Fourier series states that, if ***s(t)*** is a periodic signal with period $\boldsymbol{T_0}$, then the average normalised power (across a 1Ω resistor) of ***s(t)*** is:

$$P = \frac{1}{T_0} \int_{a - T_0/2}^{a + T_0/2} |s(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} \left|\frac{c_n}{T_0}\right|^2$$

If ***s(t)*** is real, |***s(t)***| is simply replaced by ***s(t).***

Problems:

1- Find the **F.S**. expansion of the signal defined by:

$$f(t) = \begin{cases} V \ldots\ldots\ldots\ldots 0 \le \omega t \le \pi \\ -V \ldots\ldots\ldots \pi \le \omega t \le 2\pi \end{cases}$$

and plot its amplitude spectrum .

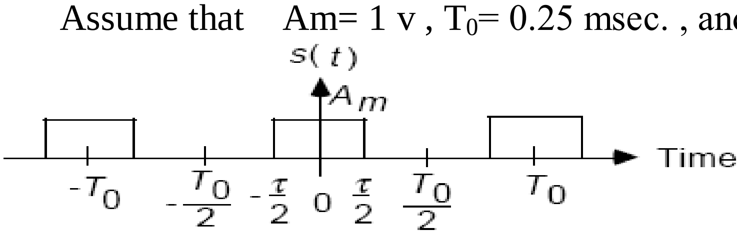
2-What is the **F.S.** expansion of the periodic signal whose definition in one period is:

$$s(t) = \begin{cases} 0 \ldots\ldots\ldots\ldots -\pi \le \omega t \le 0 \\ \sin \omega t \ldots\ldots. 0 \le \omega t \le \pi \end{cases}$$

Plot its amplitude spectrum.

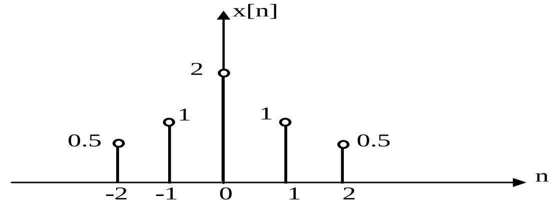
3-What percentage of the total power is contained within the first zero crossing of the spectrum envelope for ***s(t)*** as shown in the following figure.

Assume that Am= 1 v , T_0= 0.25 msec. , and τ = 0.05 msec



Time Domain Analysis

Consider **x[n]** as shown in figure below:



We can define the complete signal **x[n]** as:

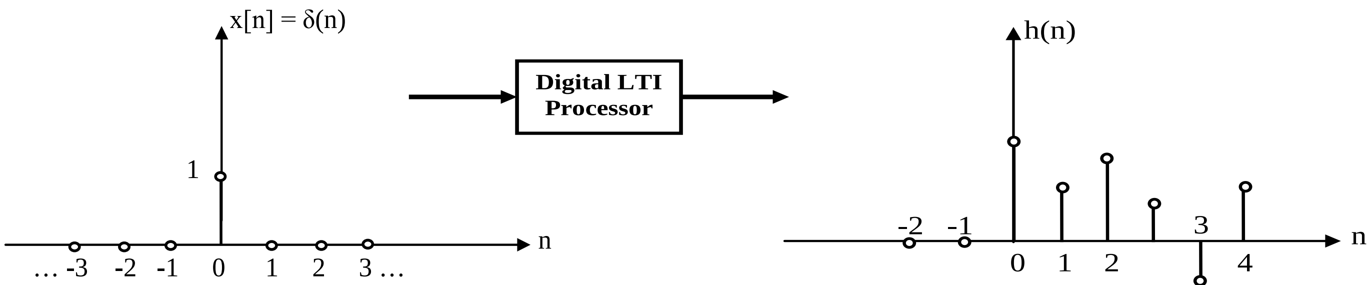
X[n] = x[-2]δ(n+2) + x[-1]δ(n+1) + x[0]δ(n) + x[1] δ(n-1) + x[2] δ(n-2)

or

$$x[n] = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Describing Digital LTI Processor

1. The Impulse Response

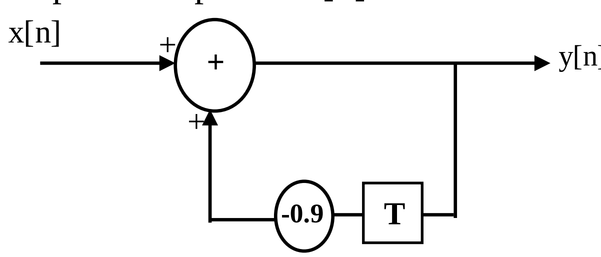


Any output signal observed after **n = 0** must be the characteristic of the processor itself. The response is called impulse response of the system and given the symbol **h[n].**

Example: find the first four sample values of the impulse response **h[n]** for each the following digital processors:

a) The system illustrated in figure below:

b) The system **y[n] = x[n] + x[n-1] + x[n-2] + …..**



Solution:

a) y[n] = -0.9y[n-1] + x[n]

The impulse response is

h[n] = -0.9h[n-1] + δ(n)

The system is clearly causal, so that

h[n] = 0 for n < 0, hence:

h[0] = -0.9h[-1] + δ(0) = 0+1 = 1

h[1] = -0.9h[0] + δ(1) = -0.9*1+0 = -0.9

h[2] = -0.9h[1] = 0.81

h[3] = -0.9h[2] = -0.729

b) h[n] = δ(n) + δ(n-1) + δ(n-2)+……

h[0] = δ(0) + δ(-1) + δ(-2) + …. = 1

h[1] = δ(1) + δ(0) + δ(-1) + …. = 1

h[2] = δ(2) + δ(1) + δ(0) + …. = 1

h[3] = δ(3) + δ(2) + δ(1) + …. = 1

2. The Step Response

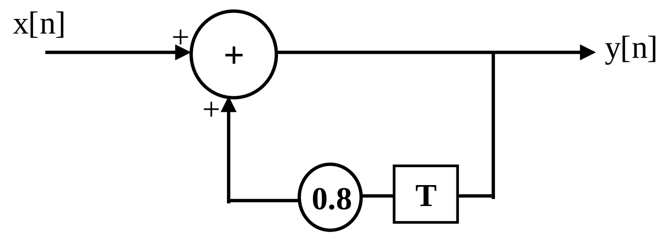
The step response of a **LTI** processor is the running sum of its impulse response and it is denoted as **s[n]:**

$$s(n) = \sum_{m=-\infty}^{n} h(m)$$

Alternatively, **h[n]** is the first-order difference of **s[n]:**

h[n] = s[n] – s[n-1]

Example: find and sketch the first few sample values of the impulse and step responses of the system given in figure below:

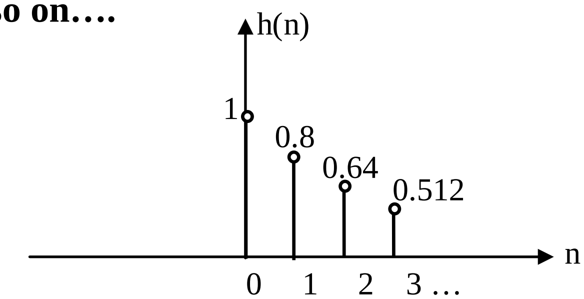


Solution:

1) **For impulse response**

h(n) = 0.8h(n-1) + δ(n)

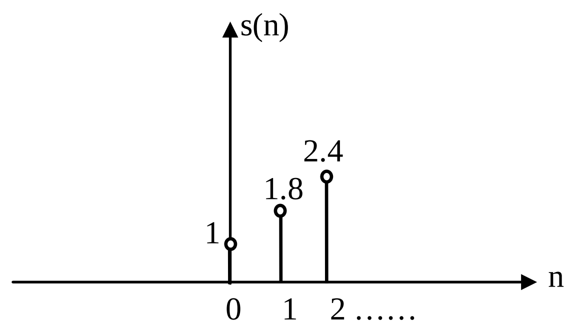
h(0) = 1, h(1) = 0.8, h(2) = 0.64, h(3) = 0.512 and so on….



2) **For step response**

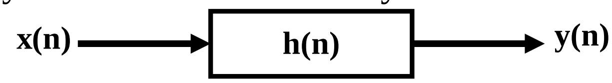
$$s(n) = \sum_{m=-\infty}^{n} h(m)$$

s(0) = h(0) = 1
s(1) = h(0) + h(1) = 1 + 0.8 = 1.8
s(2) = h(0) + h(1) + h(2) = 1 + 0.8 + 0.64 = 2.44
s(3) = h(0) + h(1) + h(2) + h(3) = 2.952
s(4) = s(3) + h(4) = 3.3616 and so on ….



Digital Convolution

The term convolution describes how the input to a system interacts with the system to produce the output.



y(n) = $x(n) \otimes h(n) = h(n) \otimes x(n)$

Time

Convolution

The convolution sum may be written as

$$y(n) = \sum_{m=-\infty}^{\infty} x(m).h(n-m) = x(n) \otimes h(n)$$

OR

$$y(n) = \sum_{m=-\infty}^{\infty} h(m).x(n-m) = h(n) \otimes x(n)$$

1. Circular Convolution

Circular convolution is used for periodic sequences and is given by:

$$y[n] = \sum_{m=0}^{N-1} h(m).x(n-m) = \sum_{m=0}^{N-1} x(m).h(n-m)$$

<u>Example:</u> Find the Circular convolution between

$x[n] = 1, 2, 3, 4 \quad n \geq 0$

$h[n] = 4, 3, 2, 1 \quad n \geq 0$

<u>Solution:</u>

N = 4

$$y[n] = \sum_{m=0}^{3} h(n-m).x(m)$$

$$y[0] = \sum_{m=0}^{3} h(-m).x(m) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3)$$

= 1*4 + 2*1 + 3*2 + 4*3 = 24

$$y[1] = \sum_{m=0}^{3} h(1-m).x(m) = x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2)$$

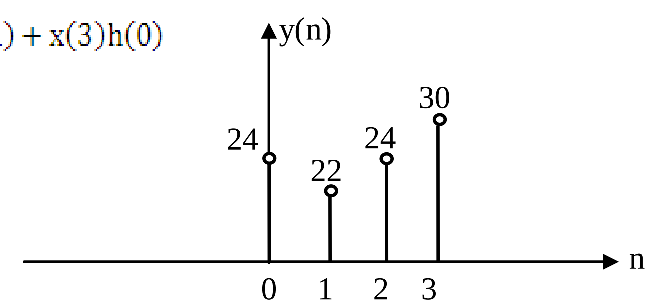
= 1*3 + 2*4 + 3*1 + 4*2 = 22

$$y[2] = \sum_{m=0}^{3} h(2-m).x(m) = x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(-1)$$

= 1*2 + 2*3 + 3*4 + 4*1 = 24

$$y[3] = \sum_{m=0}^{3} h(3-m).x(m) = x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$$

= 1*1 + 2*2 + 3*3 + 4*4 = 30



2. Linear Convolution

Linear convolution is used for energy sequences or a periodic sequence.

$$y(n) = \sum_{m=0}^{2N-1} x(m).h(n-m)$$

where; **N** = sequence length

If **x(n)** has length $\mathbf{N_1}$ and **h(n)** has length $\mathbf{N_2}$, then the upper limit of the convolution sum becomes $\mathbf{N_1 + N_2 - 1}$ (sequence length).

Example: Find the linear convolution between

x(n) = 1, 2, 3, 4 $n \geq 0$

h(n) = 4, 3, 2, 1 $n \geq 0$

Solution:

$N_1 = 4$, $N_2 = 4$, and $N_1 + N_2 - 1 = 7$ (y(0) to y(7))

$$y(n) = \sum_{m=0}^{7} x(m).h(n-m)$$

$$y(0) = \sum_{m=0}^{7} x(m).h(-m)$$

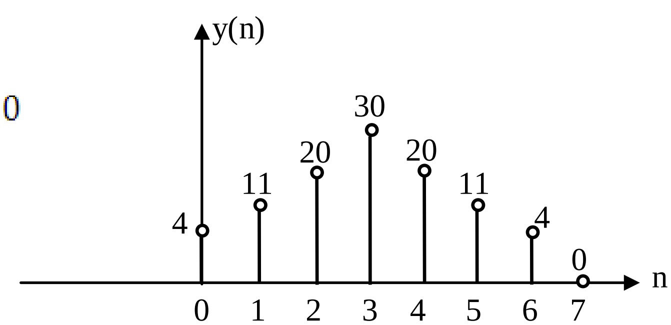
$$= x(0)h(0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3) + x(4)h(-4)$$
$$+ x(5)h(-5) + x(6)h(-6) + x(7)h(-7) = 1 * 4 + 0 = 4$$

$$y(1) = \sum_{m=0}^{7} x(m).h(1-m)$$

$$= x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2) + x(4)h(-3)$$
$$+ x(5)h(-4) + x(6)h(-5) + x(7)h(-6) = 1 * 3 + 2 * 4 + 0 = 11$$

$$y(2) = \sum_{m=0}^{7} x(m).h(2-m) = 20$$

$$y(3) = \sum_{m=0}^{7} x(m).h(3-m) = 1 + 4 + 9 + 16 = 30$$



$$y(4) = \sum_{m=0}^{7} x(m).h(4-m) = 2 + 6 + 12 = 20$$

$$y(5) = \sum_{m=0}^{7} x(m).h(5-m) = 1 * 0 + 2 * 0 + 3 * 1 + 4 * 2 = 11$$

$$y(6) = \sum_{m=0}^{7} x(m).h(6-m) = 4 * 1 = 4$$

$$y(7) = \sum_{m=0}^{7} x(m).h(7-m) = 0$$

y(n) = 4, 11, 20, 30, 20, 11, 4, 0

Linear Convolution (Method 2)

Example: Find the linear convolution between

x(n) = 1, 2, 3, 4 n ≥ 0
h(n) = 4, 3, 2, 1 n ≥ 0

Solution:

N_1 = 4, **N_2** = 4, and **$N_1 + N_2 – 1$** = 7 (y(0) to y(7))

				1	2	3	4					← x(n)
y(0)	1	2	3	4								← Reversed h(n)
y(1)		1	2	3	4							
y(2)			1	2	3	4						
y(3)				1	2	3	4					
y(4)					1	2	3	4				
y(5)						1	2	3	4			
y(6)							1	2	3	4		
y(7)								1	2	3	4	

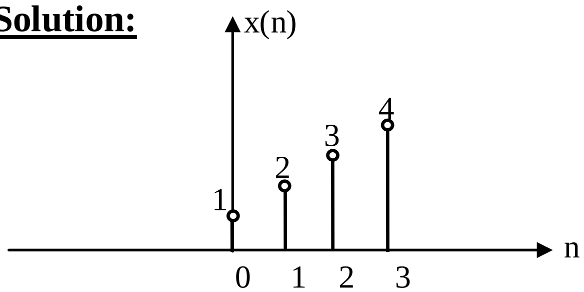
y(0) = 1*4 = 4, **y(1)** = 3*1 + 4*2 = 1, **y(2)** = 2*1 + 3*2 + 4*3 = 20
y(3) = 1*1 + 2*2 + 3*3 + 4*4 = 30, **y(4)** = 1*2 + 2*3 + 3*4 = 20
y(5) = 1*3 + 2*4 = 11, **y(6)** = 1*4 = 4, **y(7)** = 1*0 = 0

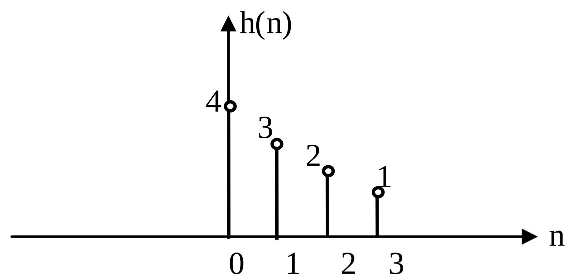
Linear Convolution (Method 3- Graphical)

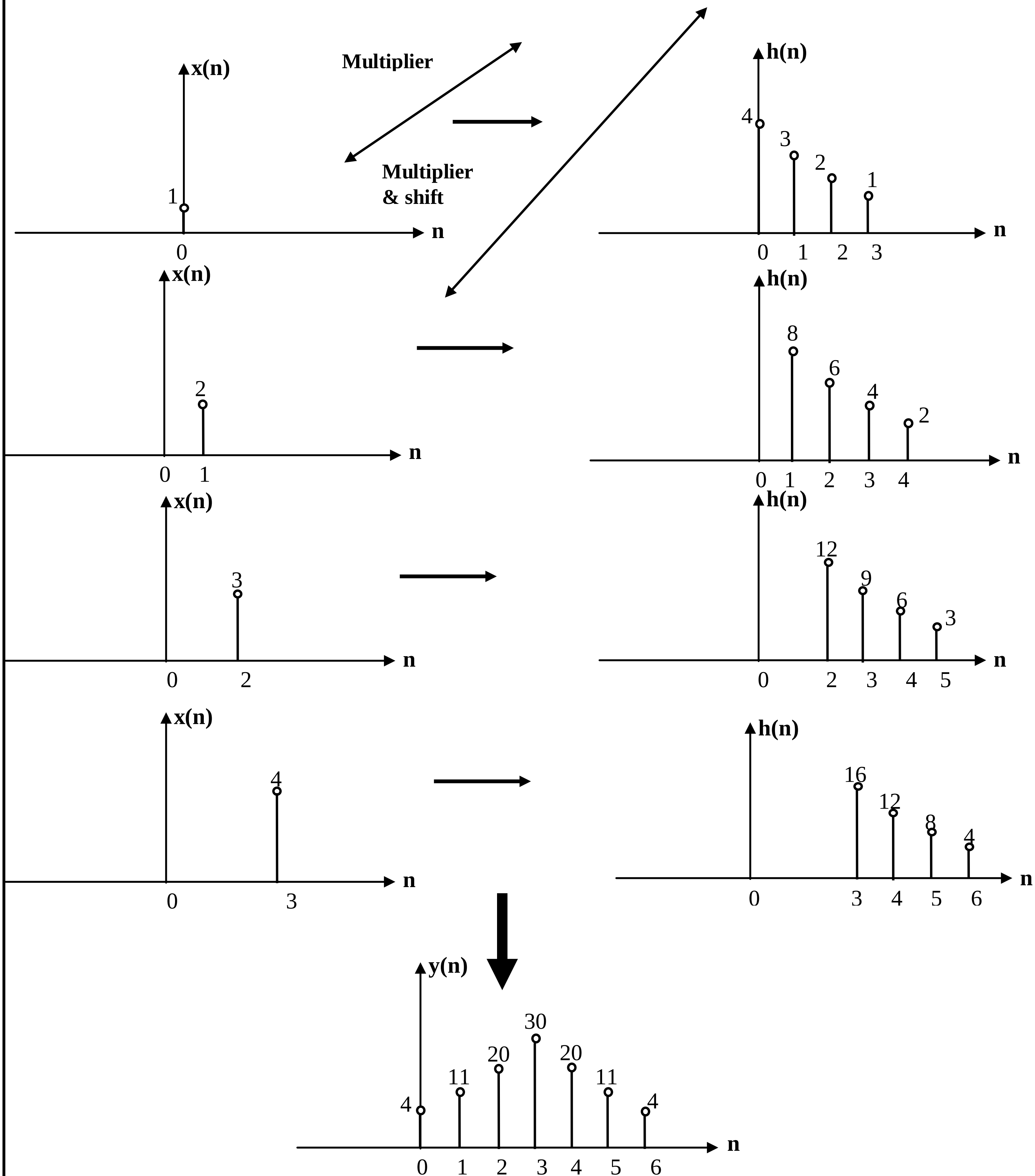
Example: Find the linear convolution between

x(n) = 1, 2, 3, 4 n ≥ 0
h(n) = 4, 3, 2, 1 n ≥ 0

Solution:






x(n)
1
0
n
Multiplier
Multiplier
& shift
h(n)
4
3
2
1
0 1 2 3
n
x(n)
2
0 1
n
h(n)
8
6
4
2
0 1 2 3 4
n
x(n)
3
0 2
n
h(n)
12
9
6
3
0 2 3 4 5
n
x(n)
4
0 3
n
h(n)
16
12
8
4
0 3 4 5 6
n
y(n)
4
11
20
30
20
11
4
0 1 2 3 4 5 6
n

H.W. 2

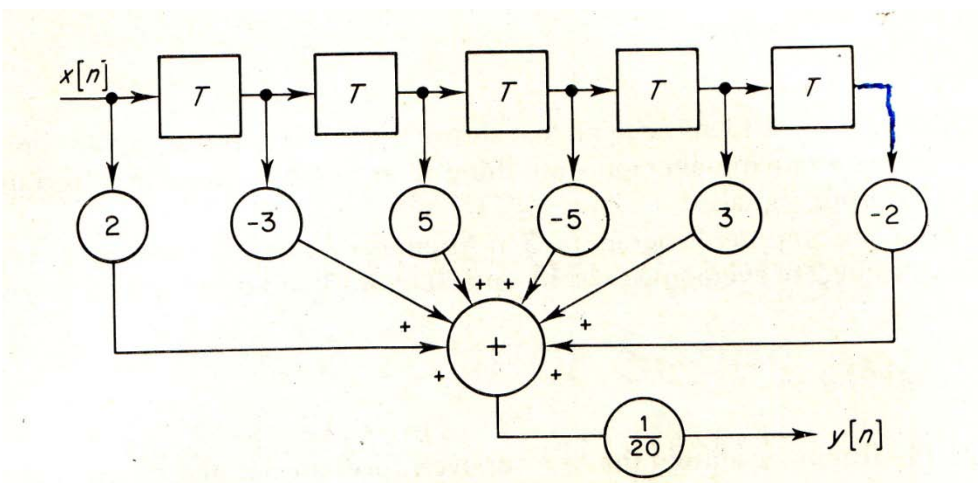
Q1/ sketch the first ten terms of the impulse responses of digital filter described by the following recurrence formulae;

a) y[n] = x[n] + x[n-4] + x[n-8]
b) y[n] = y[n-1] + x[n] – x[n-8]

Q2/ Sketch the step responses of the digital filters defined in **Q1**.

Q3/ Use the graphical interpretation of convolution to find the output **y[n]** for the input **x[n]** and impulse response h[n] given by:
x[n] = 0 0 1 1 1 1 1 1 1 0 0 0
h[n] = 1 1 1 0 0 0 0 0 0 0 0 0

Q4/ a block diagram of a non-recursive digital filter are shown in figure below. Find and sketch its impulse response and step response.



Classification of Signals

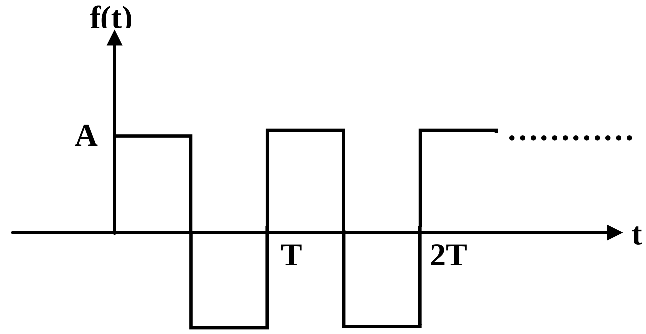
1. Power Signal (Periodic Signal)

The power signal can be:

a) **Periodic signal**

Ex: f(t) = $A\sin w_0 t$

Ex:

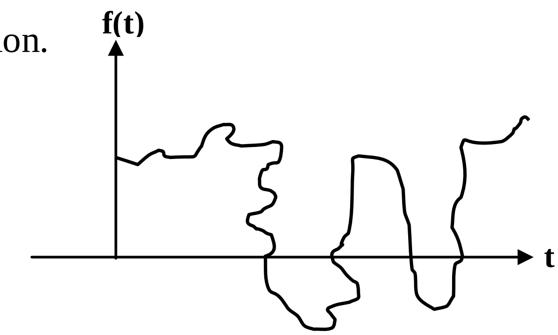


The periodic signal is defined as **f(t) = f(t+T)**, where **T**: is the periodic of the signal.

$$f = \frac{1}{T} \;\&\; w_0 = 2\Pi f = \frac{2\Pi}{T}$$

b) **Random Signal**

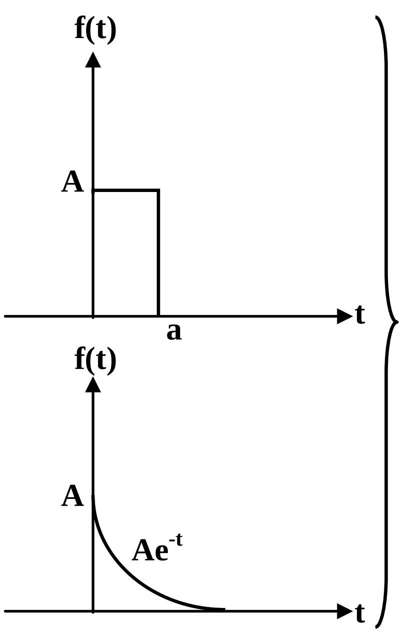
When the function f(t) cannot express by analytic function.



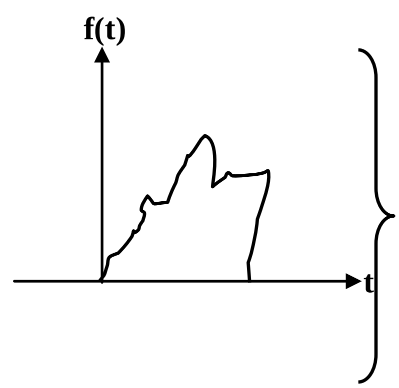
2. Energy Signal

The energy signal is that signal which exists for finite time. The energy signal is classified into:

a. **Deterministic**
b. **Random**



Deterministic



Random

Definition of Power

- If **f(t)** represent a current signal then the instantaneous power on a load resistance **R** is:

$\mathrm{P(t)} = |\mathrm{f(t)}|^2.\mathrm{R}$

- If **f(t)** represent a voltage signal then

$$\mathrm{P(t)} = \frac{|\mathrm{f(t)}|^2}{\mathrm{R}}$$

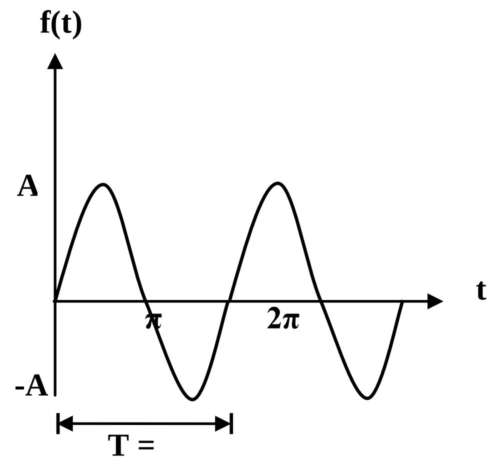
- Considering one ohm resistance (**R = 1Ω**) then, $\mathrm{P(t)} = |\mathrm{f(t)}|^2$ for current and voltage.

$P = \dfrac{1}{T}\displaystyle\int_0^T |f(t)|^2 dt$ Average power for periodic signal for 1 ohm resistance

$E = \displaystyle\int_{-\infty}^{\infty} |f(t)|^2 dt$ Energy signal (a periodic signal) for 1 ohm resistance

Example: classify the following signals if it is power or energy signal.

1.



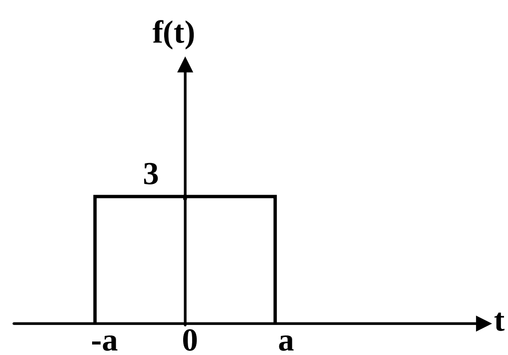
Solution:

$$P = \frac{1}{T}\int_0^T |f(t)|^2 dt$$

$$P = \frac{1}{2\pi}\int_0^{2\pi} |Asint|^2 dt$$

$$P = \frac{A^2}{2\pi}\int_0^{2\pi} \frac{1}{2}(1 - cos2t)\ \ dt = \frac{\mathrm{A}^2}{4\pi}\left[\mathrm{t}|_0^{2\pi} - \frac{\sin 2\mathrm{t}}{2}\Big|_0^{2\pi}\right] = \frac{\mathrm{A}^2}{2}$$

2.



Solution:

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt$$

$$E = \int_{-a}^{a} |3|^2 dt = 9t|_{-a}^{a} = 18a$$

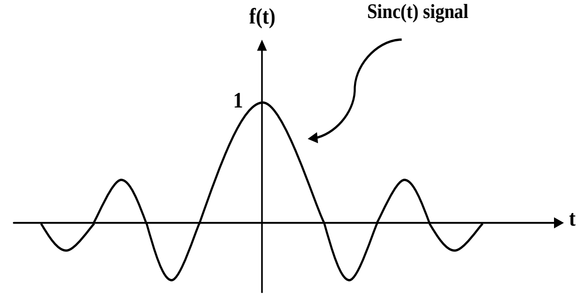
or

$$E = \int_{-a}^{0} |3|^2 dt + \int_0^{a} |3|^2 dt = 9a + 9a = 18a$$

Trigonometric Fourier Series (Periodic Signals)

$\sin x = \frac{e^{jx}-e^{-jx}}{2j}$ $\cos x = \frac{e^{jx}+e^{-jx}}{2}$ $\operatorname{sinc}(x) = \frac{\sin x}{x}$

$\operatorname{sinc}(0) = 1, \operatorname{sinc}(\pm\pi) = 0, \operatorname{sinc}(\pm 2\pi) = 0$



$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t).\cos(nw_0 t)dt$$

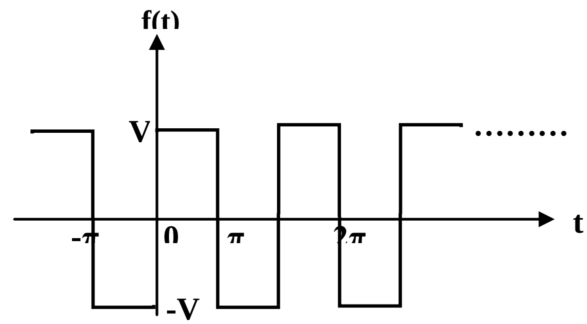
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t).\sin(nw_0 t)dt$$

$$f(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos(nw_0 t) + b_n \sin(nw_0 t))$$

Where, $w_0 = 2\pi f = \frac{2\pi}{T}$

- if the signal is even then **f(-t) = f(t)** $\Rightarrow$ $\mathbf{b_n = 0}$
- if the signal is odd then **f(-t) = -f(t)** $\Rightarrow$ $\mathbf{a_0 = a_n = 0}$

<u>Example:</u> find the Fourier series for the signal below:



<u>Solution:</u>

T = 2π

$$f(t) = \begin{cases} V & 0 < t < \pi \\ -V & \pi < t < 2\pi \end{cases}$$

The signal is odd $\mathbf{a_0 = a_n = 0}$

$$b_n = \frac{2}{T}\left[\int_0^{\pi} V\sin(nw_0 t) - \int_{\pi}^{2\pi} V\sin(nw_0 t)\right] = \frac{2}{2\pi}\left[-V\frac{\cos nt}{n}\Big|_0^{\pi} + V\frac{\cos nt}{n}\Big|_{\pi}^{2\pi}\right]$$

$$= \frac{2V}{2\pi}\left[-\frac{\cos nt}{n} + \frac{1}{n} + \frac{\cos 2\pi n}{n} - \frac{\cos \pi n}{n}\right]$$

$$= \frac{V}{\pi}\left[\frac{1}{n} + \frac{\cos 2\pi n}{n} - \frac{2\cos n\pi}{n}\right] = \frac{V}{n\pi}[2 - 2\cos n\pi] = \frac{2V}{n\pi}[1 - \cos n\pi]$$

- $\mathbf{\cos 2\pi n = 1}$

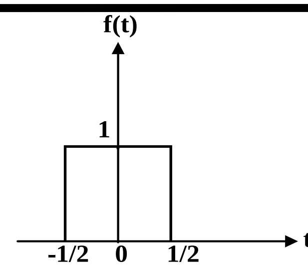
$$b_1 = \frac{2V}{\pi}[1 - \cos\pi] = \frac{4V}{\pi}, b_2 = 0, b_3 = \frac{4V}{3\pi}, b_4 = 0\ldots$$

$$f(t) = \frac{4V}{\pi}\sin t + \frac{4V}{3\pi}\sin 3t + \cdots = \frac{4V}{\pi}\left[\sin t + \frac{1}{3}\sin 3t + \cdots\right]$$

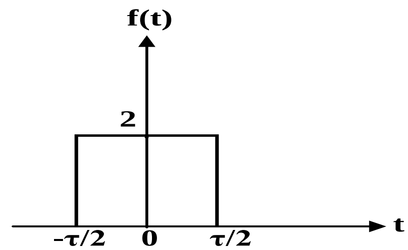
Notes:

❖ **Unit Gate Function**

$$rect(t)$$

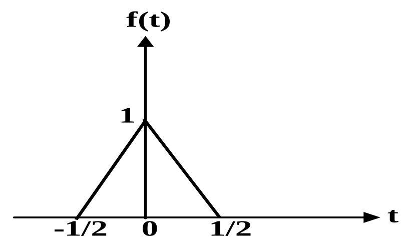


$$2rect(\frac{t}{\tau})$$

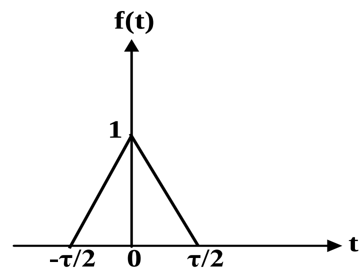


❖ **Unit Triangle Function**

$$\Delta(t)$$



$$\Delta(\frac{t}{\tau})$$



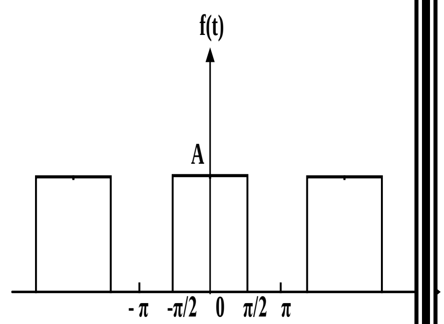
Fourier Complex Exponential

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jnw_0t}dt$$

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0t}$$

$$w_0 = \frac{2\pi}{T}$$

Example: Find the Fourier exponential for the signal as shown below:



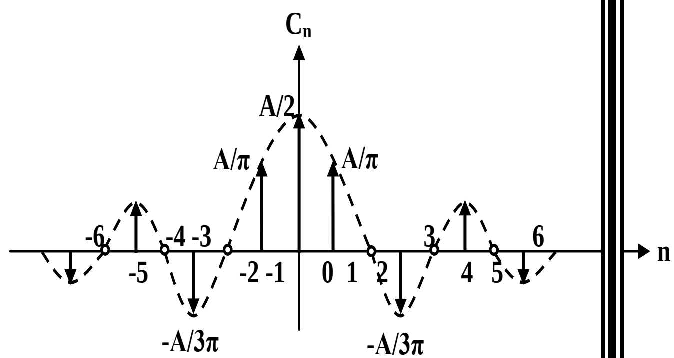
Solution:

$$f(t) = \begin{cases} A & -\pi/2 < t < \pi/2 \\ 0 & \pi/2 < t < \pi,\ -\pi < t < -\pi/2 \end{cases}$$

$$T = 2\pi, w_0 = \frac{2\pi}{T} = 1$$

$$C_n = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} Ae^{-jnt}dt = \frac{A}{2\pi}\left[-\frac{e^{-jnt}}{jn}\Big|_{-\pi/2}^{\pi/2}\right]$$

$$= -\frac{A}{jn2\pi}\left[e^{-jn\frac{\pi}{2}} - e^{jn\frac{\pi}{2}}\right] = \frac{A}{\pi n}\left[\frac{e^{jn\frac{\pi}{2}} - e^{-jn\frac{\pi}{2}}}{2j}\right] = \frac{A}{\pi n}sin\frac{n\pi}{2}$$



$$= \frac{\frac{A}{2}}{\frac{1}{2}\pi n} sin\frac{n\pi}{2}$$

$$C_n = \frac{A}{2}sinc(n\frac{\pi}{2})$$

$$C_0 = \frac{A}{2}, C_1 = \frac{A}{\pi}, C_2 = 0, C_3 = -\frac{A}{3\pi}, C_{-1} = \frac{A}{\pi}, C_{-3} = -\frac{A}{3\pi}$$

$$f(t) = \frac{A}{2} + \frac{A}{\pi}e^{jt} - \frac{A}{3\pi}e^{j3t} + \frac{A}{\pi}e^{-jt} - \frac{A}{3\pi}e^{-3jt}$$

$$= \frac{A}{2} + \frac{2A}{\pi}\left[\frac{e^{jt}+e^{-jt}}{2}\right] - \frac{2A}{3\pi}\left[\frac{e^{j3t}+e^{-j3t}}{2}\right] = \frac{A}{2} + \frac{2A}{\pi}cost - \frac{2A}{3\pi}cos3t = \frac{A}{2} + \frac{2A}{\pi}\left[cost - \frac{1}{3}cos3t + \cdots\right]$$

For sketch the spectrum of the signal above:

$$C_n = \frac{A}{2}sinc(\frac{n\pi}{2})$$

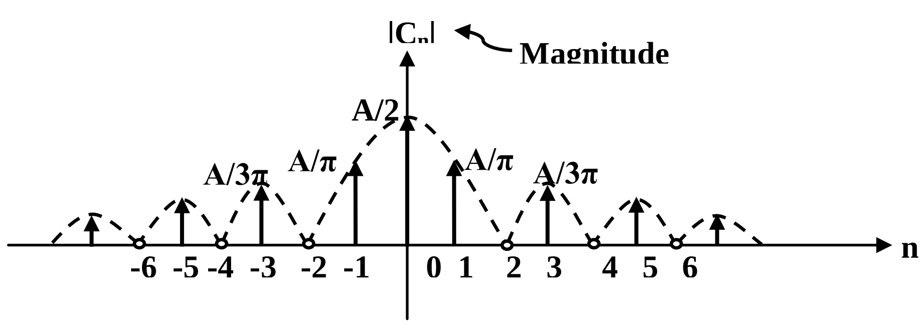
$$\frac{n\pi}{2} = k\pi$$

$$n = 2k$$

$$k = \pm1, \pm2, \pm3$$

$$n = \pm2, \pm4, \pm6$$

$$k = 0, n = 0, C_n = \frac{A}{2}$$



Parseval's Theorem for Power Signals

Parseval's theorem for periodic signal relates the power in the time domain to the power in the frequency domain.

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |C_n|^2$$

If we know the time function ***f(t)***, we can find the average power. Alternatively, if we know the Fourier coefficients, we can find the average power. The answer obtained in the time domain and in the frequency domain must agree.

Example: prove that parseval's theorem is satisfy on ***f(t) = 2sin(100t).***

Solution:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2 dt$$

$$P = \frac{1}{T}\int_{-T/2}^{T/2} 4sin^2 100t dt = 2\ W$$

The Fourier coefficients of ***f(t)*** are:

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$$

$$f(t) = 2sin100t = 2\,\frac{e^{j100t} - e^{-j100t}}{2j} = \frac{e^{j100t} - e^{-j100t}}{j} = \frac{e^{j100t}}{j} - \frac{e^{-j100t}}{j}$$

$$\therefore f(t) = -je^{j100t} + je^{-j100t}$$

$$\therefore C_1 = -j\ \&\ C_{-1} = j\quad for\ n = 1\& -1$$

$$\therefore C_n = 0\quad for\ all\ n \neq \pm 1$$

Then,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 = |j|^2 + |-j|^2 = 1 + 1 = 2W$$

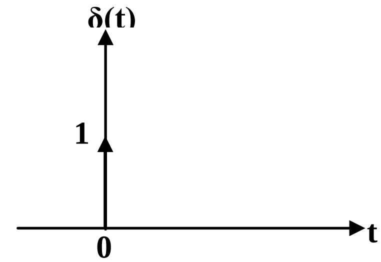
Fourier Transform (A Periodic Signal)

$$f(w) = \int_{-\infty}^{\infty} f(t) e^{-jwt} dt \qquad (F.T.)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(w) e^{jwt} dw \qquad (I.F.T.)$$

Example:

1): f(t) = δ(t)



Solution:

$$f(w) = \int_{-\infty}^{\infty} \delta(t) e^{-jwt} dt = e^{-jw.0} = e^{-j0} = 1$$

2): f(t) = δ(t+t_0)

Solution:

$t+t_0 = 0 \Longrightarrow t = -t_0$

$$f(w) = e^{jw t_0}$$

3): f(t) = $e^{-at}u(t)$ $\qquad a > 0$

Solution:

$$f(w) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-jwt} dt = \int_{0}^{\infty} e^{-(a+jw)t} dt = -\frac{1}{a+jw} e^{-(a+jw)t}\Big|_0^{\infty} = -\frac{1}{a+jw}[0-1] = \frac{1}{a+jw}$$

Notes:

$$F.T.\left[e^{\pm jw_0 t}\right] = 2\pi\delta(w \mp w_0)$$

Example: find the Fourier transform for the signal ***f(t) = cosw$_0$t***.

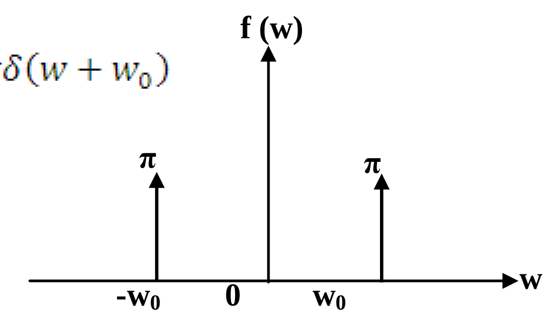
Solution:

$$f(w) = \int_{-\infty}^{\infty} cosw_0 t . e^{-jwt} dt = \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{jw_0 t} + e^{-jw_0 t}\right) e^{-jwt} dt$$

$$F.T.\left[e^{\pm jw_0 t}\right] = 2\pi\delta(w \mp w_0)$$

$$\therefore F.T.[cosw_0 t] = \frac{1}{2}\left(F.T.\left(e^{jw_0 t}\right) + F.T.\left(e^{-jw_0 t}\right)\right)$$

$$= \frac{1}{2}[2\pi\delta(w - w_0) + 2\pi\delta(w + w_0)] = \pi\delta(w - w_0) + \pi\delta(w + w_0)$$



Properties of Fourier Transform

1) Linearity : $F.T.[a_1 f_1(t) + a_2 f_2(t)] = a_1 f_1(w) + a_2 f_2(w)$

2) Time Shifting:

$$F.T.(f(t+t_0)) = f(w)e^{jwt_0}$$

$$F.T.(f(t-t_0)) = f(w)e^{-jwt_0}$$

3) Frequency Shifting:

$$F.T.(f(t)e^{jw_0 t}) = f(w-w_0)$$

$$F.T.(f(t)e^{-jw_0 t}) = f(w+w_0)$$

4) Scaling:

$$F.T.(f(\alpha t)) = \frac{1}{|\alpha|} f\left(\frac{w}{\alpha}\right)$$

5) Differentiation:

$$F.T.\left(f'(t)\right) = F.T.\left(\frac{\partial}{\partial t}(f(t))\right) = (jw)f(w)$$

$$F.T.\left(\frac{\partial^n}{\partial t^n}(f(t))\right) = (jw)^n f(w)$$

Example: Find the F.T. for the function $f(t) = e^{-at}u(t)$, then find the F.T. for:

1): $3f(t)$, **2):** $f(t+\frac{1}{3})$, **3):** $f(t)e^{jw_c t}$, **4):** $f(\frac{t}{2})$, 5): $f'(t)$

Solution:

$$F.T.\left(e^{-at}u(t)\right) = \frac{1}{a+jw}$$

1.

$$\frac{3}{a+jw}$$

2.

$$\frac{1}{a+jw} e^{jw\frac{1}{3}}$$

3.

$$\frac{1}{a+j(w-w_c)}$$

4.

$$\frac{1}{\left|\frac{1}{2}\right|} \frac{1}{a+j\left(\frac{w}{0.5}\right)} = \frac{2}{a+2jw}$$

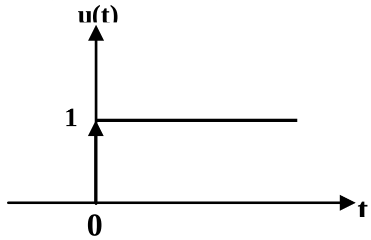
5.

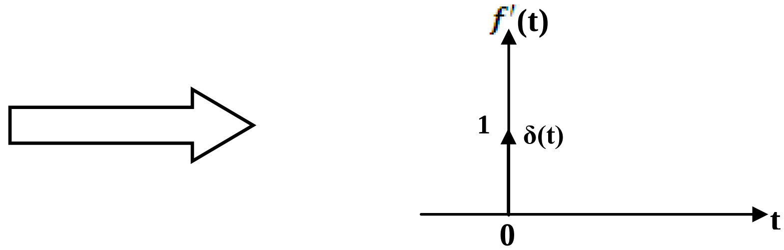
$$jwf(w) = \frac{1}{a+jw}$$

$$f(w) = \frac{1}{jw} \cdot \frac{1}{a+jw}$$

Example: Find **F.T.** of **u(t)** using differentiation property.

1) f(t) = u(t)

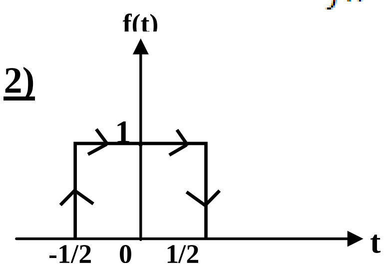


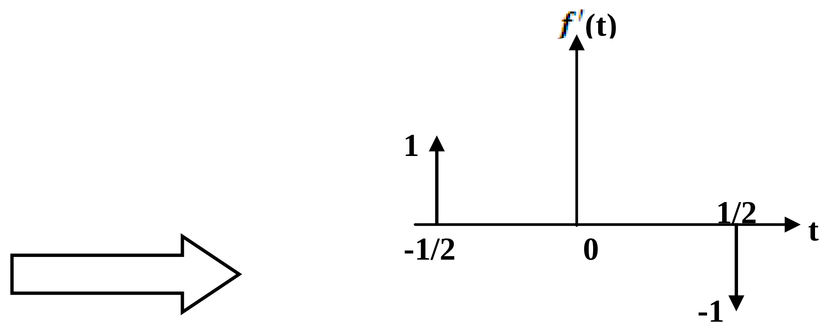


$$F.T.\left(f'(t)\right) = F.T.(\delta(t))$$

$$1 = jwf(w) \rightarrow f(w) = \frac{1}{jw}$$

2)





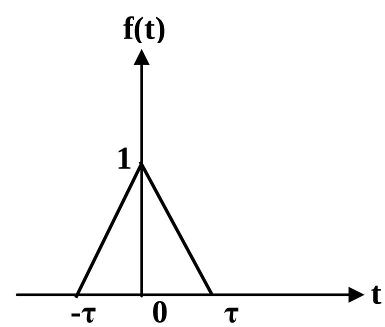
$$jwf(w) = F.T.(-\delta\left(t-\frac{1}{2}\right) + \delta\left(t+\frac{1}{2}\right))$$

$$jwf(w) = -e^{-j\frac{1}{2}w} + e^{j\frac{1}{2}w}$$

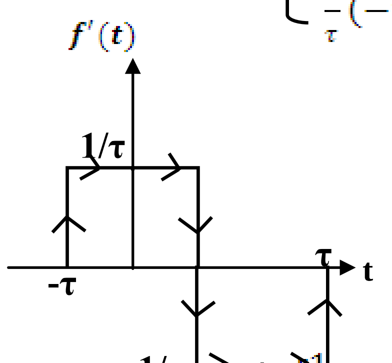
$$f(w) = \frac{2}{w}\frac{e^{j\frac{1}{2}w} - e^{-j\frac{1}{2}w}}{2j} = \frac{2}{w}sin\frac{1}{2}w = sinc\left(\frac{1}{2}w\right)$$

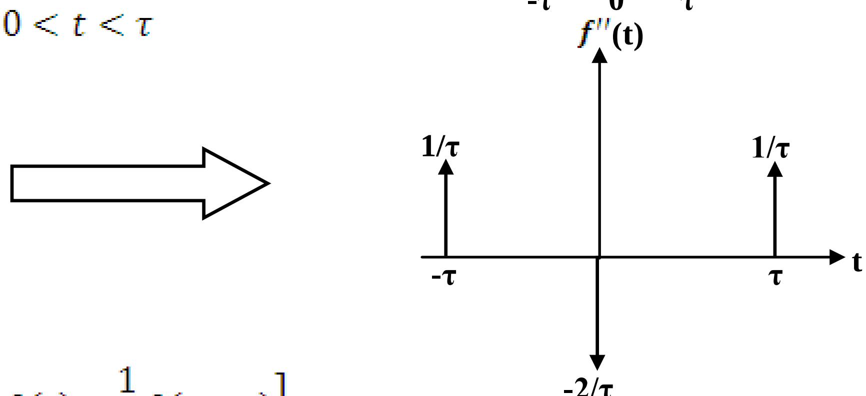
3)

Note: $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y \ - y_1}{x \ - x_1}$



$$f(t) = \begin{cases} \frac{1}{\tau}(t+\tau) & -\tau < t < 0 \\ \frac{1}{\tau}(-t+\tau) & 0 < t < \tau \end{cases}$$





$$(jw)^2 f(w) = F.T.\left[\frac{1}{\tau}\delta(t+\tau) - \frac{2}{\tau}\delta(t) + \frac{1}{\tau}\delta(t-\tau)\right]$$

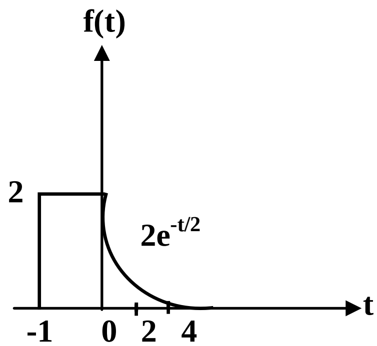
H.W. 3

Q1/ Find the Fourier transform for the signal $f(t) = \sin w_0 t$.

Q2/ Find the F.T. for the function $f(t) = e^{-at}u(t)$, then find the F.T. for: **a)** f(3t) **b)** $Af(t)\cos^2 w_0 t$.

Q3/ Find the **F.T.** of $f(t) = rect\left(\frac{t}{\tau}\right)$ then sketch the spectrum for signal above.

Q4/ Classify the following signal as shown below if it is power or energy signal.



Frequency Analysis of Discrete Time Signals

1. Discrete Time Fourier Series (DTFS)

If ***x[n] = x(n+N)*** for all n where N is the period of the sequence ***x[n].*** Then

$$x[n] = \sum_{k=0}^{N-1} X(k) e^{\frac{jk2\pi n}{N}}$$

Where the coefficients of the expansion ***X(k)*** and the fundamental **digital frequency** are given by:

$$X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-\frac{jk2\pi n}{N}}$$

k = 0, …. N-1

Example: Determine the spectra (**DFS**) of the signals below:

a- x[n] = $cos\sqrt{2}\pi n$

b- x[n] = 1, 1, 0, 0 n ≥ 0

Solution:

a. The signal is not periodic, and then the signal cannot expand in Fourier series.

b. $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-\frac{jk2\pi n}{N}}$

N = 4 (X(0) to X(3))

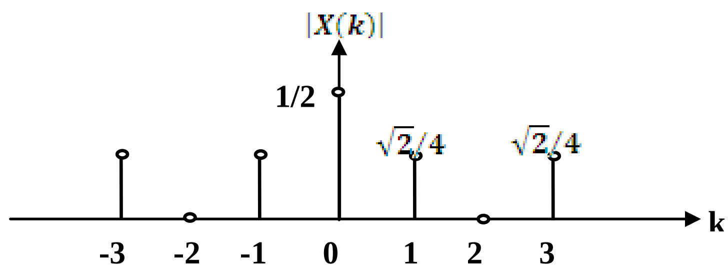
$$X[k] = \frac{1}{4}\sum_{n=0}^{3} x(n) e^{-\frac{jk\pi n}{2}} = \frac{1}{4}\left[x[0] + x[1]e^{-\frac{jk\pi}{2}} + x[2]e^{-jk\pi} + x[3]e^{-\frac{jk3\pi}{2}}\right]$$

$$X(0) = \frac{1}{4}[1+1] = \frac{1}{2}$$

$$X(1) = \frac{1}{4}\left[1 + e^{-\frac{j\pi}{2}}\right] = \frac{1}{4} - \frac{1}{4}j$$

$$X(2) = \frac{1}{4}\left[1 + e^{-j\pi}\right] = 0$$

$$X(3) = \frac{1}{4}\left[1 + e^{-\frac{j3\pi}{2}}\right] = \frac{1}{4} + \frac{1}{4}j$$



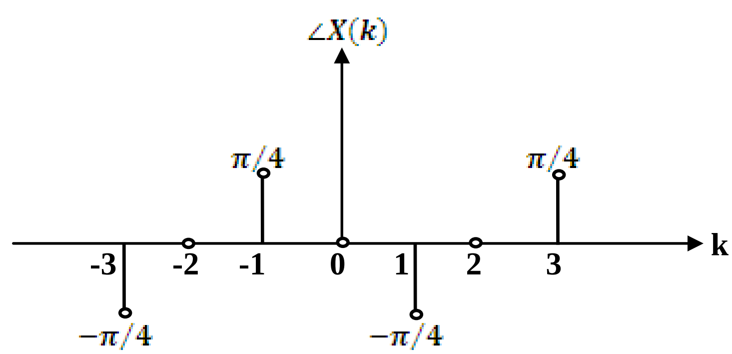
The magnitude and phase spectra are

$$|X(0)| = \frac{1}{2} \angle X(0) = 0$$

$$|X(1)| = \frac{\sqrt{2}}{4} \angle X(1) = -\frac{\pi}{4}$$

$$|X(2)| = 0$$

$$|X(3)| = \frac{\sqrt{2}}{4} \angle X(3) = \frac{\pi}{4}$$



Notes:

1. $| \quad | = \sqrt{x^2 + y^2} \& \angle = tan^{-1}\frac{y}{x}$

2. $e^{-jx} = cosx - jsinx$

2. The Discrete Fourier Transform (DFT)

$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-\frac{jk2\pi n}{N}} \qquad DFT$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{jk2\pi n}{N}} \qquad IDFT$$

$$W_N = e^{-\frac{j2\pi}{N}}$$

Properties of the DFT

1. Linearity

$x_1[n] \leftrightarrow X_1[k]$ & $x_2[n] \leftrightarrow X_2[k]$

Then,

$Ax_1[n] + B\, x_2[n] \leftrightarrow AX_1[k] + BX_2[k]$

2. Time-shifting

If

$x\ \ [n] \leftrightarrow X\ \ [k]$

Then,

$x[n - n_0] \leftrightarrow X[k] e^{-\frac{j2\pi k n_0}{N}}$

3. Convolution

If

$x_1[n] \leftrightarrow X_1[k]$ & $x_2[n] \leftrightarrow X_2[k]$

Then,

$$\sum_{m=0}^{N-1} x_1[n].x_2[m-n] \leftrightarrow X_1[k].X_2[k]$$

Example: evaluate the **DFT** of the sequence x[n] = [1, 0, 0, 1] n ≥ 0.

Solution:

N = 4 (k = 0 → N-1)

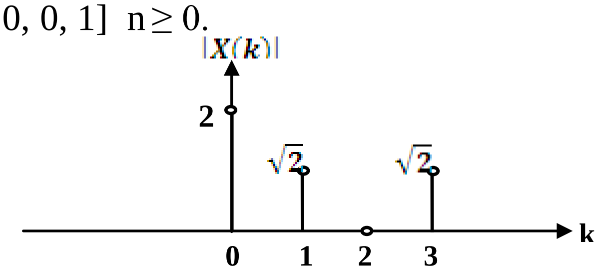
$$X[k] = \sum_{n=0}^{3} x(n) e^{-\frac{jk2\pi n}{4}}$$

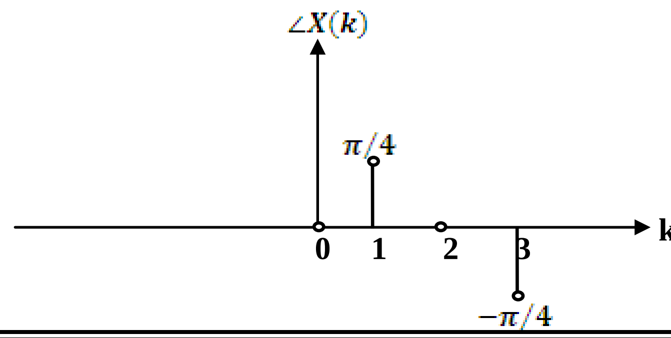
$X[0] = x[0] + x[1] + x[2] + x[3] = 2$

$X[1] = 1 + e^{-\frac{j3\pi}{2}} = 1 + j$

$X[2] = 0$

$X[3] = 1 - j$





The magnitude and phase spectra are

$|X(0)| = 2$

$|X(1)| = \sqrt{2}\angle 45^\circ$

$|X(2)| = 0$

$|X(3)| = \sqrt{2}\angle - 45^\circ$

Example: Evaluate the **IDFT** of the sequence **X[k] = [2, 1+j, 0, 1-j]**

Solution:

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\frac{jk2\pi n}{4}}$$

$$x[0] = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{\frac{jk2\pi 0}{4}} = 1$$

$$x[1] = 0, x[2] = 0, x[3] = 1$$

∴ x[n] = [1 0 0 1]

❖ We note that the computation of each point of the **DFT** can be accomplished by **N** complex multiplications and **(N-1)** complex additions. Hence the N-point DFT values can be computed in a total of $\mathbf{N^2}$ complex multiplications and **N(N-1)** complex additions.

3. The Fast Fourier Transform (FFT)

FFT is used to increase the computational speed of the **DFT**. There are two algorithms for FFT:-

a. Decimation in time algorithm (**DIT**)
b. Decimation in frequency algorithm (**DIF**)

1. DIT-FFT

$$X[k] = \sum_{n=0}^{N-1} x(n)e^{-\frac{jk2\pi n}{N}} \qquad for\ 0 \le k \le N-1$$

$$= \sum_{n=0}^{N-1} x(n)W_N^{nk}$$

$$= \underbrace{\sum_{n=0}^{N-1} x(n)W_N^{nk}}_{\textbf{Even part}} + \underbrace{\sum_{n=0}^{N-1} x(n)W_N^{nk}}_{\textbf{Odd part}}$$

Letting **n =2r for even part**

n = 2r+1 for odd part

$$X[k]=\sum_{r=0}^{\frac{N}{2}-1}x(2r)W_{N/2}^{rk}+W_N^k\sum_{r=0}^{\frac{N}{2}-1}x(2r+1)W_{N/2}^{rk}$$

We can demonstrate the decimation in process quite easily in the case of a short transform, for example **N = 8**. The signal x[n] is defined for: **n = [0, 1, 2, 3, 4, 5, 6, 7]**

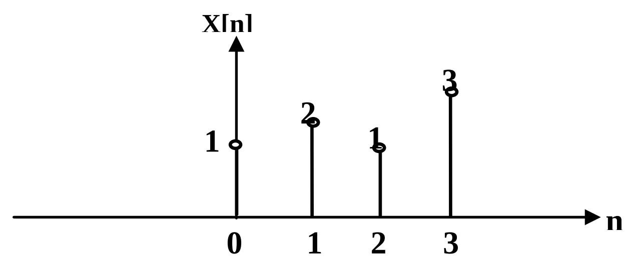
We form two subsequences of the even numbered and odd numbered points (**radix-2 FFT**)

n = [0, 2, 4, 6] & [1, 3, 5, 7]

Use of this modified module leads to a reduction in the total number of multiplication by **50%.**

Now, each subsequence can be further decimated giving: **n = [0, 4] [2, 6] [1, 5] [3, 7]** and the later is called **radix-4 FFT**.

Example: Determine **X[k]** for the input sequence shown below using **FFT** algorithm.



Solution:

x[n] = [1 2 1 3] & N = 4

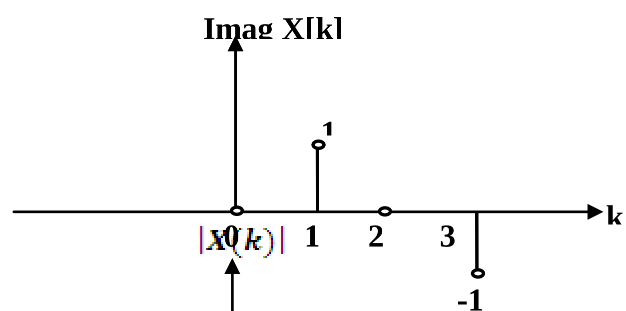
$$X[k]=\sum_{r=0}^{\frac{N}{2}-1}x(2r)W_{N/2}^{rk}+W_N^k\sum_{r=0}^{\frac{N}{2}-1}x(2r+1)W_{N/2}^{rk}$$

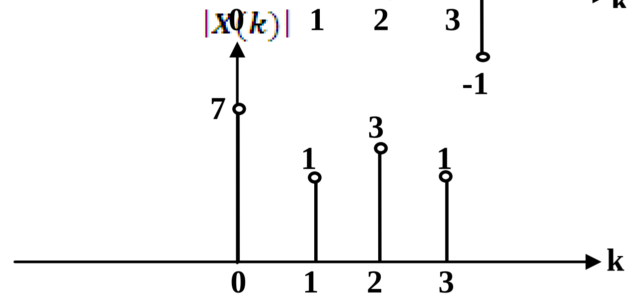
$$X[k]=\sum_{r=0}^{1}x(2r)W_2^{rk}$$

$$+W_4^k\sum_{r=0}^{1}x(2r+1)W_2^{rk}=x[0]+x[2]W_2^k+W_4^k\left[x[1]+x[3]W_2^k\right]$$

$$=1+W_2^k+2W_4^k+3W_2^kW_4^k=1+e^{-j\pi k}+2e^{-\frac{j\pi k}{2}}+3e^{-\frac{j3\pi k}{2}}$$

X[0] = 7, X[1] = j, X[2] = -3, X[3] = -j





Flow-Graph FFT

1. FFT of 2-Samples

X[n] = {x(0), x(1)}

By using DFT

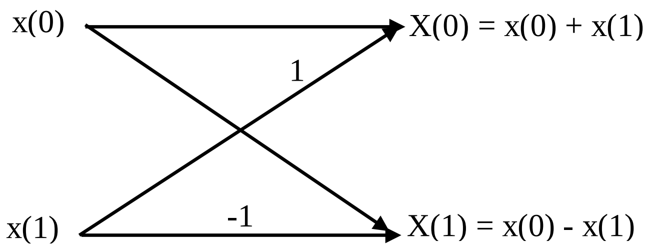
$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-\frac{jk2\pi n}{N}}$$

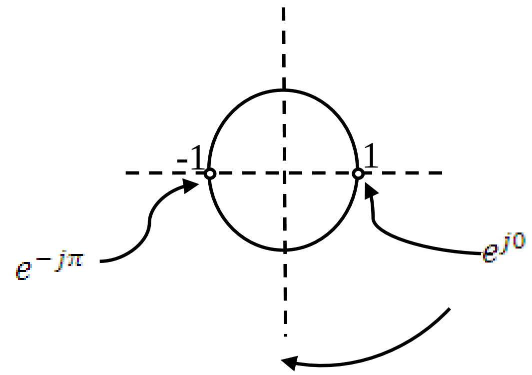
N = 2 (k =0 → N-1)

X[0] = X[0] + X[1]

X[1] = X[0] – X[1]

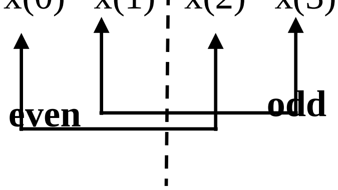
By using flow-graph **FFT**

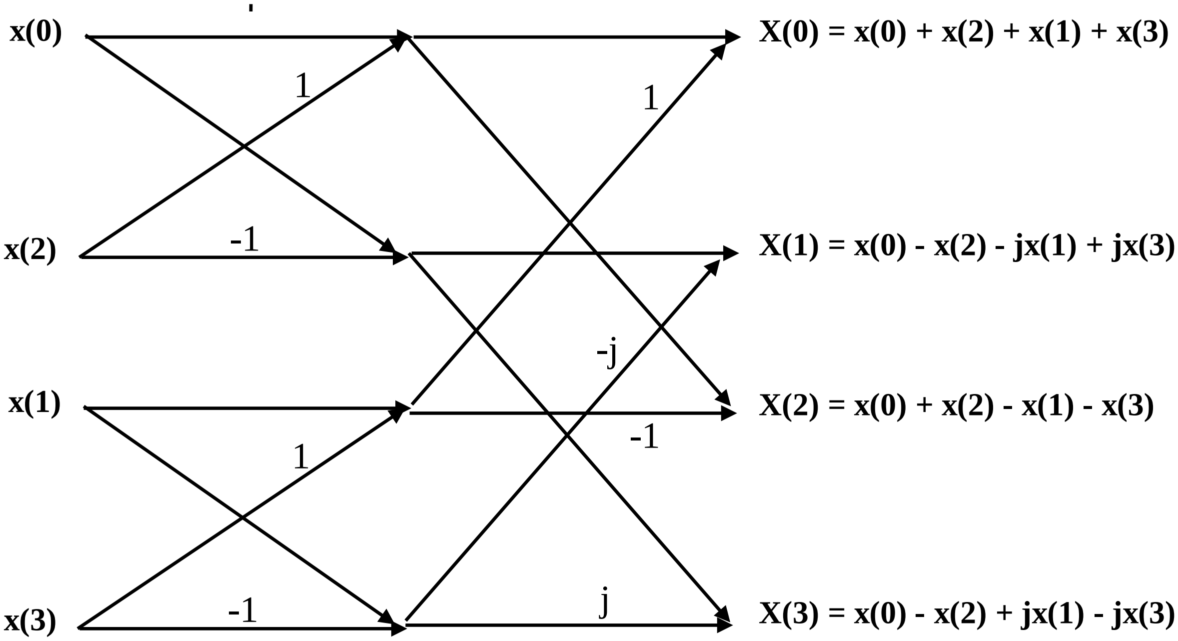


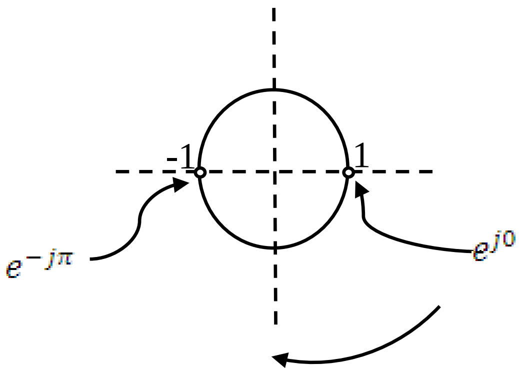


2. FFT of 4-Samples

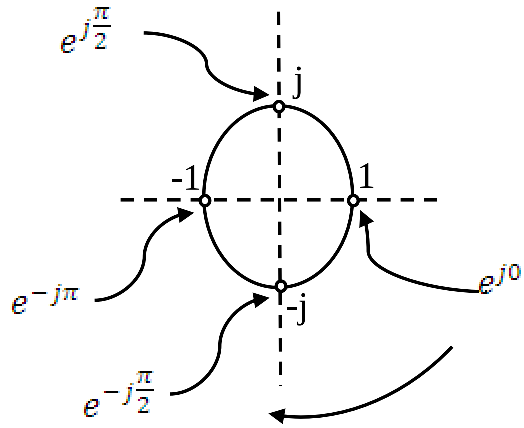
X[n] = {x(0) x(1) x(2) x(3)}







2-Samples



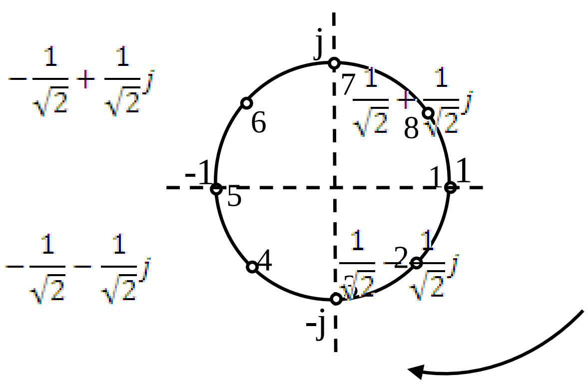
4-Samples

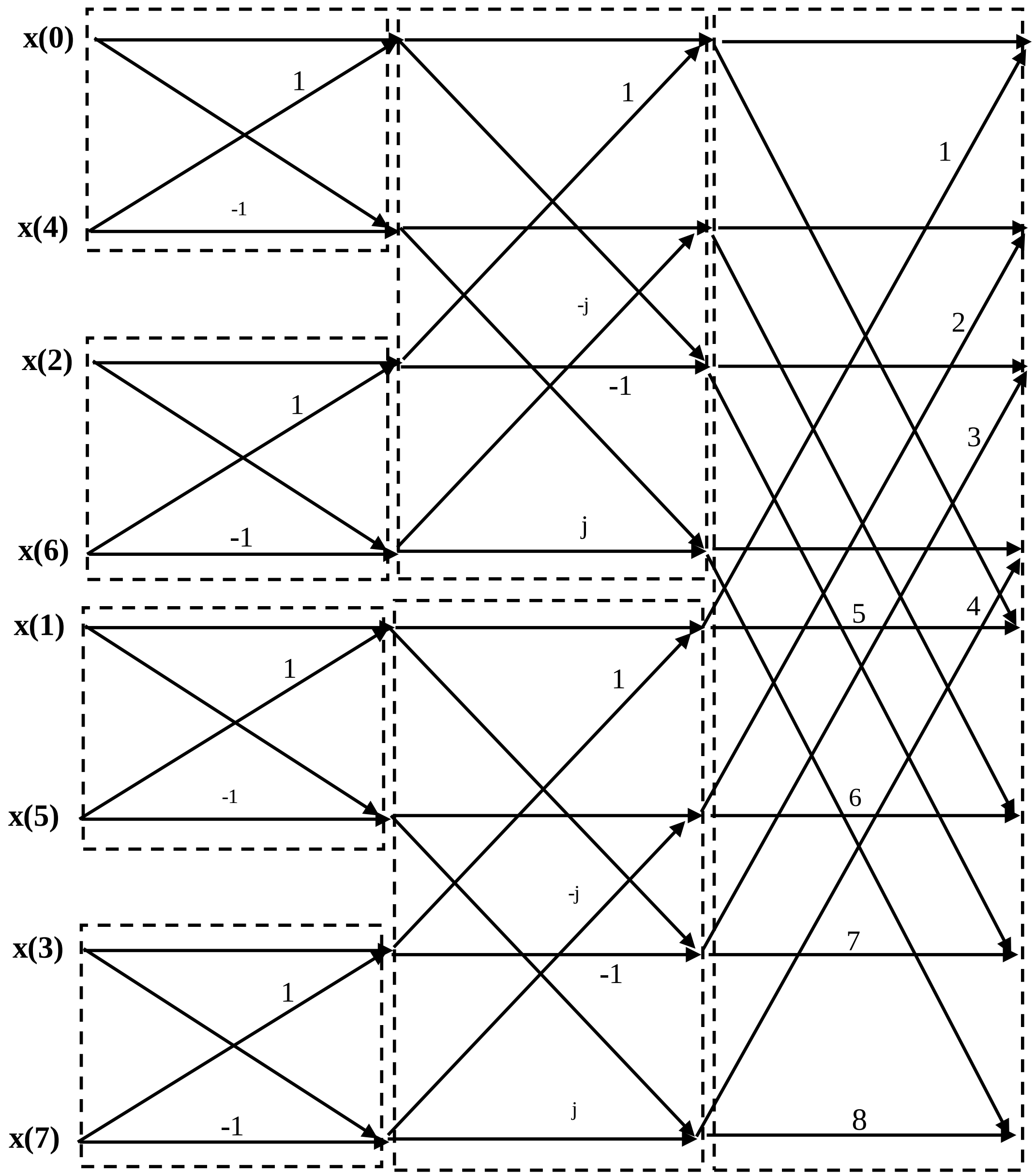
3. FFT of 8-Samples

X[n] = {x(0) x(1) x(2) x(3) x(4) x(5) x(6) x(7)}

8-Samples $\rightarrow \log_2 8 = 3$

0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	4
2	0	1	0	0	1	0	2
3	0	1	1	1	1	0	6
4	1	0	0	0	0	1	1
5	1	0	1	1	0	1	5
6	1	1	0	0	1	1	3
7	1	1	1	1	1	1	7





Example: Find the FFT of ***x(n) = 3cos(0.5πn)*** using flow-graph with **N = 4**.

Solution:

x(n) = 3cos(0.5πn)

x(0) = 3

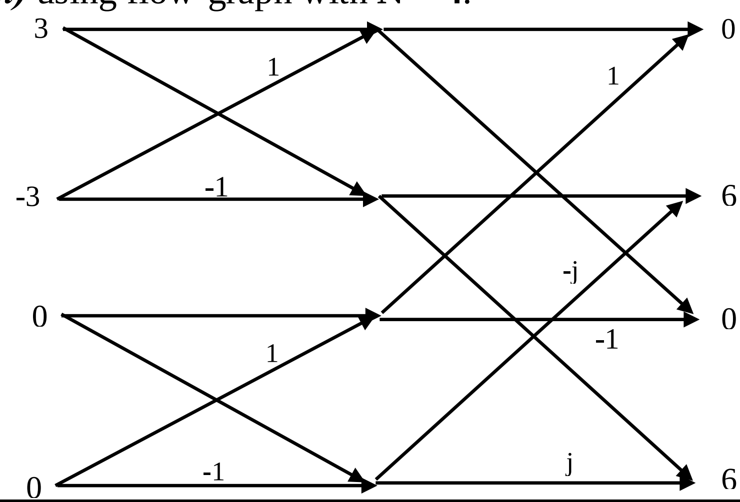
x(1) = 0

x(2) = -3

x(3) = 0

x(n) = [3 0 -3 0]

X[k] = [0 6 0 6 0]



Notes:

- If $\mathbf{N \neq 2^r}$, then **0's** are added to compute the sequence to the nearest $\mathbf{2^r}$ value (padding).

Ex: x(n) = [1 2 3] → x(n) = [1 2 3 0]

Ex: x(n) = [1 2 3 4 5] → x(n) = [1 2 3 4 5 0 0 0]

- The number of operation DFT = N^2
- The number of operation FFT = $N\log_2 N$

Ex: N = 4 → DFT = 4^2 = 16

FFT = 4 $\log_2 4$ = 8

- The flow-graph method know butterfly

2. DIF-FFT

Let **N** be a power of **2**, and we consider computing separately the even numbered frequency samples and the odd numbered frequency samples.

For even-number frequency samples **k=2r** and it follows:

$$X[k] = \sum_{n=0}^{\frac{N}{2}-1} [x(n) + x\left(n + \frac{N}{2}\right)] W_{\frac{N}{2}}^{nr}$$

For odd-number samples k=2r+1 and it follow:

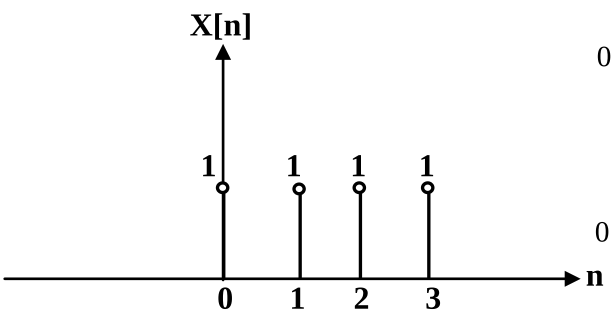
$$X[k] = W_N^n \sum_{n=0}^{\frac{N}{2}-1} [x(n) - x\left(n + \frac{N}{2}\right)] W_{\frac{N}{2}}^{nr}$$

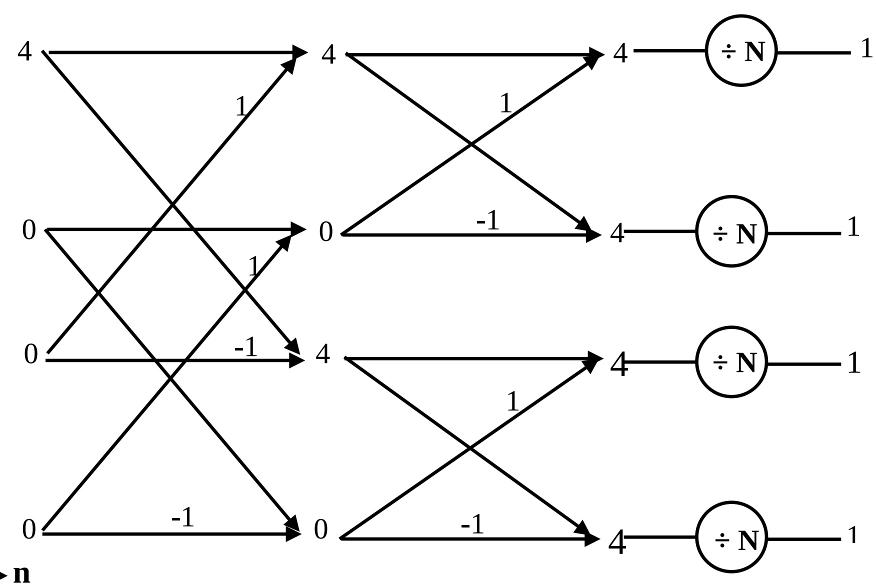
Example: Find the **DIF-FFT** for the X[k] = [4 0 0 0].

Solution:

N = 4

x[n] = [1 1 1 1]





H.W. 4

Q1/ Sketch the periodic digital signal $x[n] = 1 + sin\left(\frac{\pi n}{4}\right) + 2cos\left(\frac{\pi n}{2}\right)$ Find its Fourier series coefficients, and sketch their real and imaginary parts.

Q2/ The signal have the following sample value **1, 2, 1, 3** estimate the real and imaginary parts of their **DFT** coefficients **X[k]**.

Q3/ A signal $x[n] = cos\left(\frac{18\pi n}{40}\right)$ is applied to a 40-point DFT.

a) Which coefficients have the largest magnitudes?

Q4/ Find the **DIT-FFT** of the sequence $x[n] = cos\left(\frac{\pi n}{4}\right) + cos\left(\frac{\pi n}{2}\right)$ by using flow-graph method.

Q5/ Compute the N-point DFT for the sequence:

$$X[n] = \begin{cases} \frac{1}{3} & 0 \le n \le 2 \\ 0 & \text{otherwise} \end{cases}$$

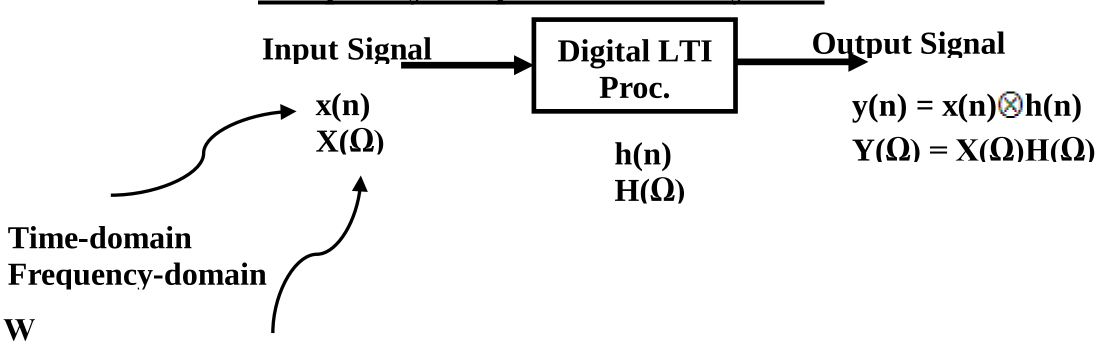
where N = 3, then sketch the magnitude and phase of the sequence above.

Q6/ find the **FFT** of **x(n) = [0 2 -1 1]** using signal flow graph.

Q7/ Perform the linear convolution with DFT given:

$$\mathbf{x[n]} = \begin{cases} 1 & n = 0 \\ 0.5 & n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad \mathbf{h[n]} = \begin{cases} 0.5 & n = 0 \\ 1 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Frequency Response of LTI System



Note: **Ω = W**

An alternative way of finding the frequency response of a digital processor is via its difference equation.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Taking the Fourier transform of both sides give:

$$\sum_{k=0}^{N} a_k Y(\Omega) e^{-jk\Omega} = \sum_{k=0}^{M} b_k X(\Omega)\, e^{-jk\Omega}$$

The result directly from the linearity and time-shifting properties of the transform.

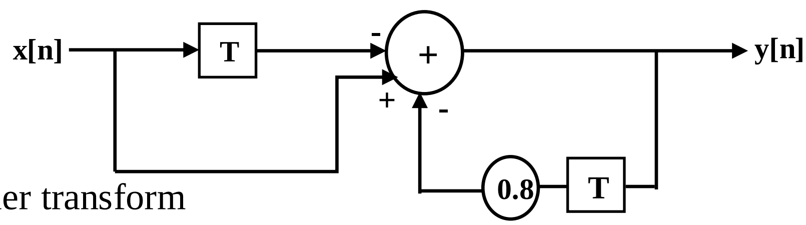
Now,

Y(Ω) = X(Ω)H(Ω)

$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\Omega}}{\sum_{k=0}^{N} a_k e^{-jk\Omega}}$$

$$H(\Omega) = \frac{b_0 + b_1 e^{-j\Omega} + b_2 e^{-2j\Omega} + b_3 \ldots}{a_0 + a_1 e^{-j\Omega} + a_2 e^{-2j\Omega} + a_3 \ldots}$$

Example: find the frequency response of the system shown in figure below and sketch its characteristics over the range $0 \le \Omega \le \pi$.



Solution:

y[n] = -x(n-1) + x(n) – 0.8y(n-1)

y(n) + 0.8y(n-1) = x(n) – x(n-1) ← Fourier transform

$y(\Omega) + 0.8y(\Omega)e^{-j\Omega} = x(\Omega) - x(\Omega)e^{-j\Omega}$

$y(\Omega)[1+0.8e^{-j\Omega}] = x(\Omega)[1- e^{-j\Omega}]$

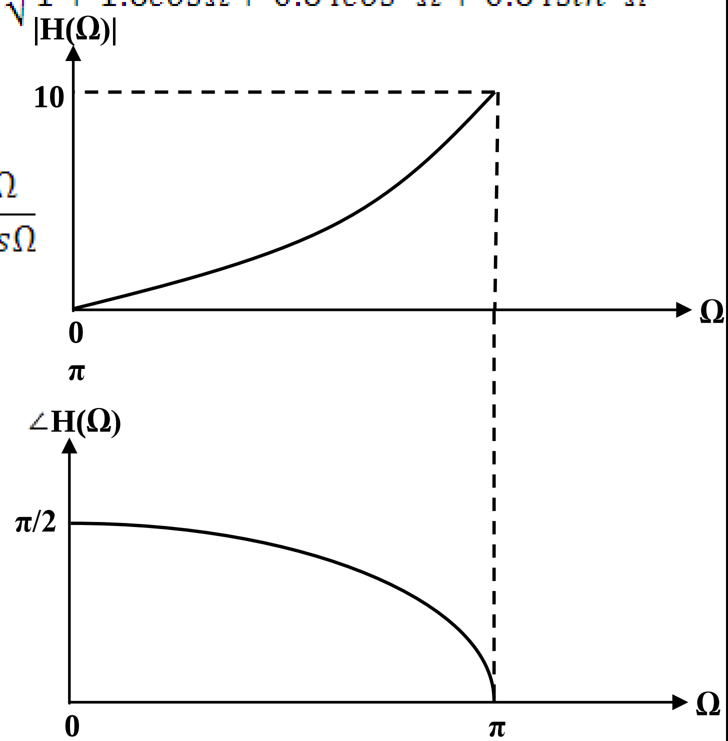
$$H(\Omega) = \frac{Y(\Omega)}{X(\Omega)} = \frac{1 - e^{-j\Omega}}{1 + 0.8e^{-j\Omega}}$$

$$H(\Omega) = \frac{1 - \cos\Omega + j\sin\Omega}{1 + 0.8\cos\Omega - 0.8j\sin\Omega}$$

$$|H(\Omega)| = \frac{\sqrt{(1-cos\Omega)^2 + sin^2\Omega}}{\sqrt{(1+0.8cos\Omega)^2 + (-0.8sin\Omega)^2}} = \sqrt{\frac{1 - 2cos\Omega + cos^2\Omega + sin^2\Omega}{1 + 1.6cos\Omega + 0.64cos^2\Omega + 0.64sin^2\Omega}}$$

$$= \sqrt{\frac{2 - 2cos\Omega}{1.64 + 1.6cos\Omega}}$$

$$\angle H(\Omega) = tan^{-1}\frac{sin\Omega}{1 - cos\Omega} - tan^{-1}\frac{-0.8sin\Omega}{1 + 0.8cos\Omega}$$



Frequency-Domain Analysis: The Z-Transform

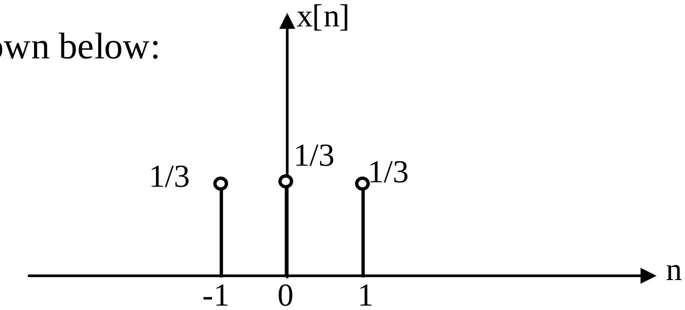
The Z-transform of a digital signal **x[n]** is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

The Z-transform and the DFT are closely related:

$X(z) = X(k)|_{e^{j2\pi k/N}=z}$

Example: Find the **Z-transform** of the signal shown below:



Solution:

$$X(z) = \sum_{n=-1}^{1} x[n]z^{-n}$$
$$= \frac{1}{3}z + \frac{1}{3} + \frac{1}{3}z^{-1}$$

Example: Find the **Z-transform** of exponentially signal given by: **x[n] = a^n u(n)**

Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$
$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Example: Find the Z-transform of the second order recursive filter, given:

$$h[n] = \begin{cases} r^n \cos(w_0 n) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$H(z) = \sum_{n=0}^{\infty} r^n \cos(w_0 n) z^{-n} \qquad = \sum_{n=0}^{\infty} r^n \frac{e^{jw_0 n} + e^{-jw_0 n}}{2} z^{-n}$$
$$= \frac{1}{2}\left[\sum_{n=0}^{\infty}\left(re^{jw_0}z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(re^{-jw_0}z^{-1}\right)^n\right] \qquad = \frac{1}{2}\left[\frac{1}{1-re^{jw_0}z^{-1}} + \frac{1}{1-re^{-jw_0}z^{-1}}\right]$$
$$= \frac{1}{2}\left[\frac{1-re^{-jw_0}z^{-1}+1+re^{jw_0}z^{-1}}{(1-re^{jw_0}z^{-1})(1-re^{-jw_0}z^{-1})}\right] \quad = \frac{1-r\cos(w_0)z^{-1}}{1-2r\cos(w_0)z^{-1}+r^2z^{-2}}$$

Note: Z-transform pairs of some important signals:

Table: (1) **x[n]** **X[z]**

x[n]	X[z]
δ[n]	1
δ[n-a]	z^{-a}
u[n]	$\frac{1}{1-z^{-1}} = \frac{z}{z-1}$
a^n u[n]	$\frac{z}{z-a}$
r[n]	$\frac{z}{(z-1)^2}$
x[n-a]	z^{-a} X(z)

Inverse Z-Transform

The inverse transform of a function **X(z)** is defined as:

$$x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

Where the circular symbol on the integral sign denotes a closed contour in the complex plane. Such integration is rather difficult and beyond our scope. Fortunately, several simpler approaches are available. Two simple methods for the inverse transform computation are reviewed in the next two examples.

Example: A signal has the **Z-transform**:

$$X(z) = \frac{1}{z(z-1)(2z-1)}$$

Use the method of partial fractions to recover the signal **x[n].**

Solution:

$$X(Z) = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{2z-1}$$

$$A = \lim_{z=0} \frac{1}{(z-1)(2z-1)} = 1$$

$$B = \lim_{z=1} \frac{1}{z(2z-1)} = 1$$

$$C = \lim_{z=0.5} \frac{1}{z(z-1)} = -4$$

$$\therefore X(z) = \frac{1}{z} + \frac{1}{z-1} + \frac{-4}{2z-1} \qquad = \left.\frac{1}{z} + \frac{1}{z-1} - \frac{2}{z-0.5}\right] \times (zz^{-1})$$

$$X(z) = z^{-1} + z^{-1}\frac{z}{z-1} - z^{-1}(2)\frac{z}{z-0.5}$$

From table (1)

x[n] = δ[n-1] + u[n-1] – (2) 0.5^{n-1} u[n-1].

__**Example:**__ Solve the previous example using long division.

$$X(z) = \frac{1}{z(z-1)(2z-1)}$$

$$X(z) = \frac{1}{2z^3 - 3z^2 + z}$$

$$\begin{array}{r|l}
 & 0.5z^{-3} + 0.75z^{-4} + 0.875z^{-5} + \cdots \\
\hline
2z^3 - 3z^2 + z & 1 \\
 & \underline{1 - 1.5z^{-1} + 0.5z^{-2}} \\
 & 1.5z^{-1} - 0.5z^{-2} \\
 & \quad 1.5z^{-1} - 2.25z^{-2} + 0.75z^{-3} \\
 & \quad \underline{1.75z^{-2} - 0.75z^{-3}} \\
 & \qquad \vdots
\end{array}$$

$$X(z) = 0.5z^{-3} + 0.75z^{-4} + 0.875z^{-5} + \cdots$$

$$x(n) = 0.5\delta(n-3) + 0.75\delta(n-4) + 0.875\delta(n-5) + \cdots$$

__**Z-Transform Properties**__

1) **Linearity**

$$Ax_1[n] + B\,x_2[n] \leftrightarrow AX_1[z] + BX_2[z]$$

2) **Time-shifting**

$$x[n-a] \leftrightarrow X(z)z^{-a}$$

3) **Convolution**

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1[z].X_2[z]$$

4) **Differentiation**

$$nx[n] \leftrightarrow (-z)\frac{dX(z)}{dz}$$

5) **Multiplication by an exponential sequence:**

$$a^n x[n] \leftrightarrow X\left(\frac{z}{a}\right)$$

Example: Determine the **Z-transform** of the sequence given by: **y[n] = (n+1) α^n u[n]**

Solution:

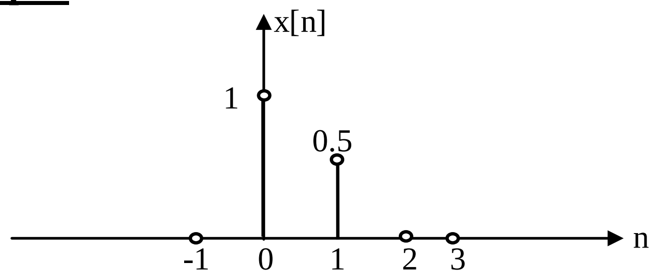
$y(n) = \alpha^n u[n] + n\alpha^n u[n]$

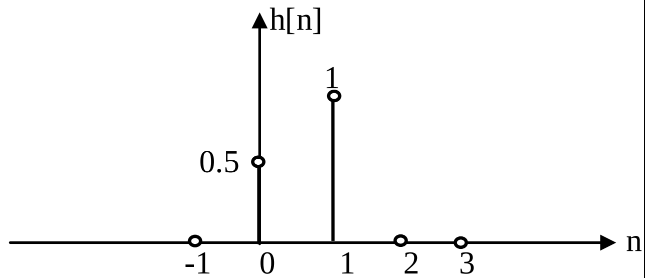
$$\because \alpha^n u[n] \Leftrightarrow \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}$$

$$Y(z) = \frac{1}{1-\alpha z^{-1}} + (-z)\frac{d}{dz}\frac{1}{1-\alpha z^{-1}}$$

$$= \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} = \frac{1}{(1-\alpha z^{-1})^2}$$

Example: Perform the linear convolution with **Z-transform**.



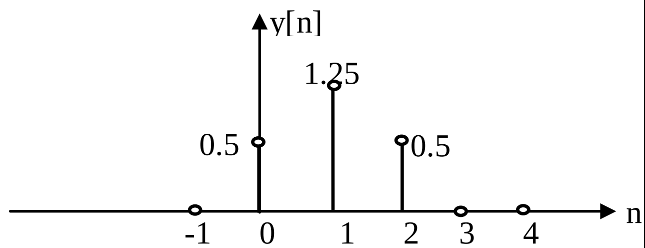


Solution:

$x[n] \otimes h[n] \leftrightarrow X[z].H[z]$

$$X(z) = \sum_{n=0}^{1} x[n]z^{-n} = 1 + 0.5z^{-1}$$

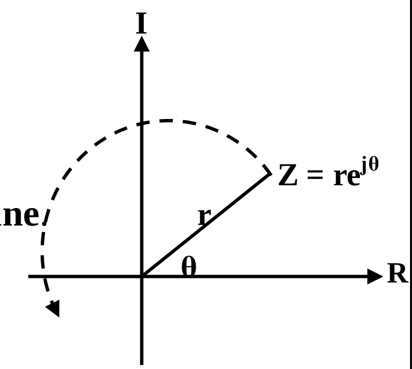
$$H(z) = \sum_{n=0}^{1} h[n]z^{-n} = 0.5 + z^{-1}$$



$$Y(z) = (1 + 0.5z^{-1})(0.5 + z^{-1}) = 0.5 + 1.25z^{-1} + 0.5z^{-2}$$

$$y[n] = 0.5\delta(n) + 1.25\delta(n-1) + 0.5\delta(n-2)$$

Z-Plane



The independent variable **(z)** is a complex variable.
Values of **(z)** can be associated with point in a plane called **the z-plane**.
The z-plane is an important graphical tool in the theory and application of the **z-transformation**.

Stability Determination Based Z-Transform

A digital signal or an LTI system can always be described using z-transform as the ratio is:

$$X(z) = \frac{N(z)}{D(z)} = \frac{k(z-z_1)(z-z_2)\ldots}{(z-p_1)(z-p_2)\ldots}$$

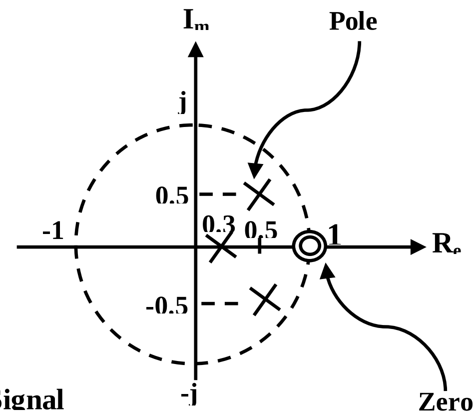
The constants **z_1, z_2, z_3 …** are called the "**zero's**" of **X(z),** because they are the values of **(z)** for which **X(z)** is **zero**. Conversely p_1, p_2, p_3 ... are known as the "**poles**" of **X(z).** The poles and zeros are either real or occur in complex conjugate pairs.

"The digital system is stable, if and only all the poles of the system lie inside the unit circle in the **z-plane**".

k: is the system gain

Example: Check the stability of the system given by:

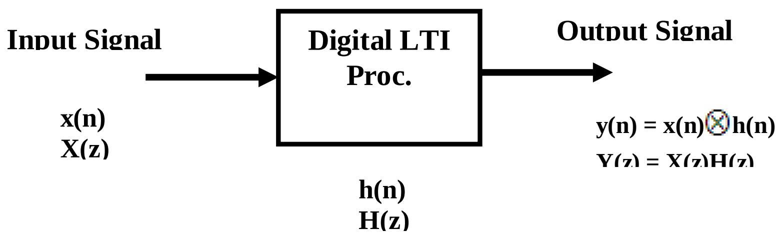
$$H(z) = \frac{k(z-1)^2}{(z-0.3)(z-0.5+j0.5)(z-0.5-j0.5)}$$



Solution:

The poles inside the unit circle, the system is stable

Evaluation of LTI System Response Using Z-Transform

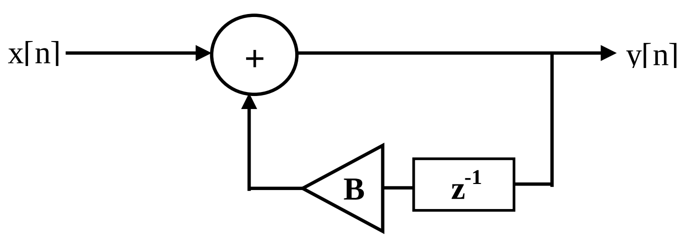


H(z) = H(e^jw^)

Where **h[n]** = impulse response of the system = $y[n]|_{x[n]=\delta[n]}$

H(z) = transfer function of the system = $\frac{Y(z)}{X(z)}$

Example: find the impulse response and the transfer function of the following system as shown below:



Solution:

$y[n] = x[n] + By[n-1]$

$Y(z) = X(z) + BY(z)z^{-1}$

For impulse response x[n] = δ[n]

X(z) = 1

$Y(z) = 1 + Bz^{-1}Y(z)$

$$Y(z) = \frac{1}{1-Bz^{-1}} = H(z)$$

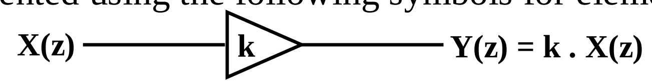
Digital System Implementation from Its Functions

Since the z-transform is a linear transformation, the system implementation procedure is similar to that in the time domain. The most convenient form for system synthesis is the **z-transform** of the general difference equation given by:

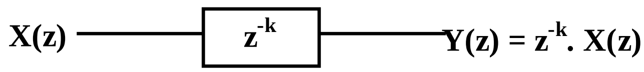
$$Y(z) = \sum_{k=1}^{M} a_k z^{-k} Y(z) + \sum_{k=a}^{b} b_k z^{-k} X(z)$$

This equation can be implemented using the following symbols for elementary **LTI** system.

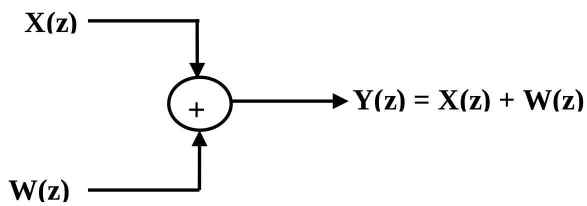
1. Gain (Amplification)



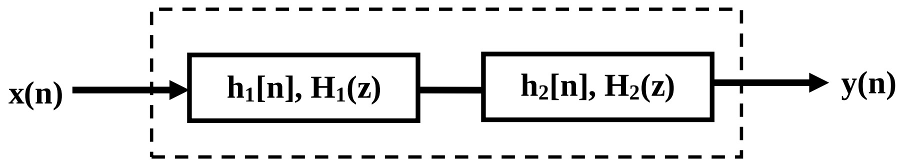
2. Delay



3. Addition



Note:



$h(n) = h_1(n) \otimes h_2(n)$

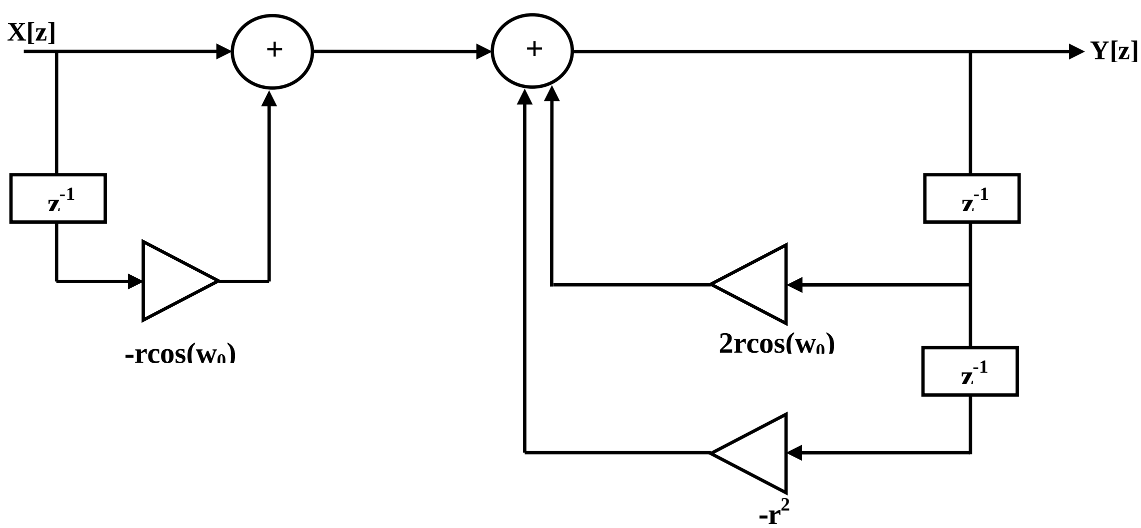
$H(z) = H_1(z).H_2(z)$

Example: Implement the 2^{nd} order recursive filter for:

$$y[n] = 2r\cos(w_0)y[n-1] - r^2y[n-2] + x[n] - r\cos(w_0)x[n-1]$$

Solution:

$$Y[z] = 2r\cos(w_0)Y[z]z^{-1} - r^2Y[z]z^{-2} + X[z] - r\cos(w_0)X[z]z^{-1}$$



H.W. 5

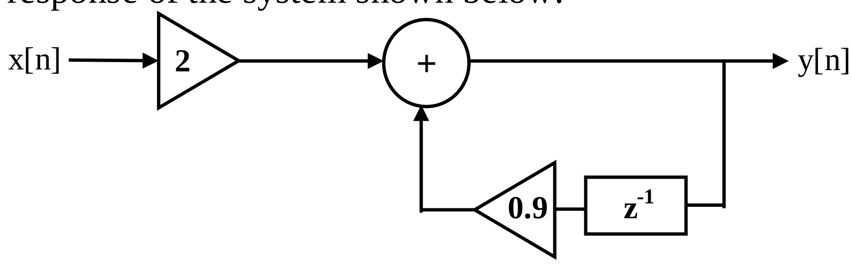
Q1/ Obtain the output for the following input **x[n]** and impulse response **h[n]** using z-transform:

x[n] = 5u[n]
h[n] = 2u[n]

Q2/ Obtain the inverse **z-transform** of the function:

$$H(z) = \frac{z+1}{(z+0.5)(z-0.2)}$$

Q3/ Find and sketch the frequency response of the system shown below:

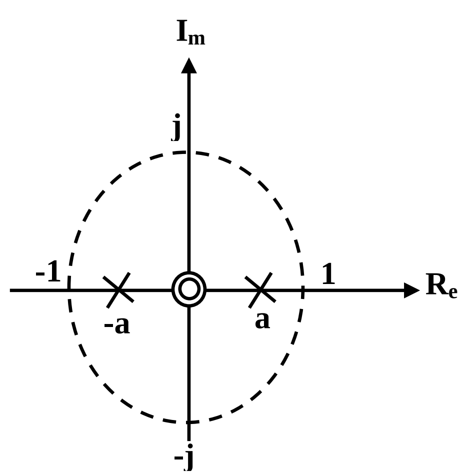


Q4/ Realize the digital system given by:

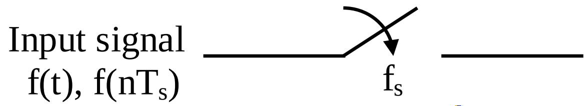
$$H(z) = \frac{z(z+1)}{(z^2 - z + 0.5)}$$

Q5/ A signal **x[n]** begins at **n = 0** and has six finite sample values **[1 2 3 1 -1 1]** it forms the input to an LTI processor whose impulse response h[n] begins at **n = 0** and has three finite sample values **[1 1 1]** convolve **x[n]** with **h[n]** to find the output signal **y[n].**

Q6/ For the z-plane pole/zero pattern shown in figure below, compute the **z-transform**. Sketch the filter block diagram, determine and sketch the impulse response sequence. Let the filter gain = 1.

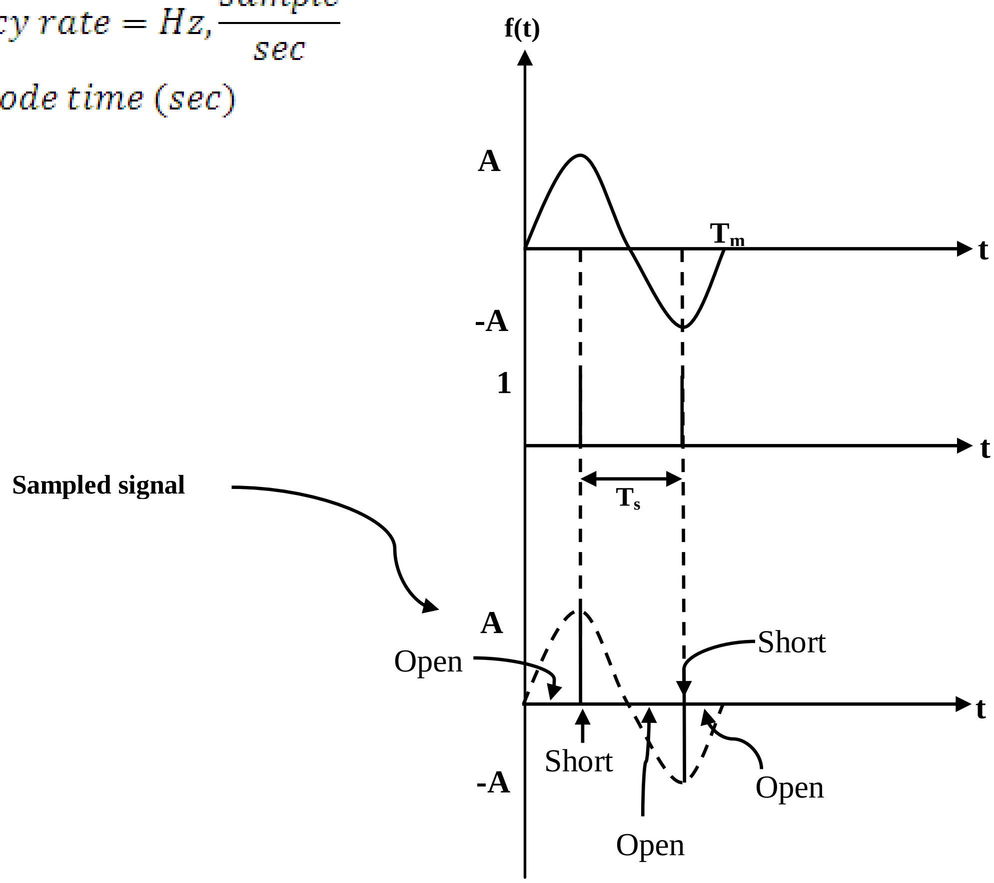


The Sampling Theorem



$f_s = sampling\ frequency\ rate = Hz, \frac{sample}{sec}$

$T_s = \frac{1}{f_s} = sampling\ periode\ time\ (sec)$



Nyquist Condition

$f_s \geq 2f_m$

$f_{s(min)} = 2f_m\ \ (Nyquist\ rate\ frequency, Hz, \frac{sample}{sec})$

Example: Calculate the sampling rate for signal f(t) = 2sin(2π*10^3t).

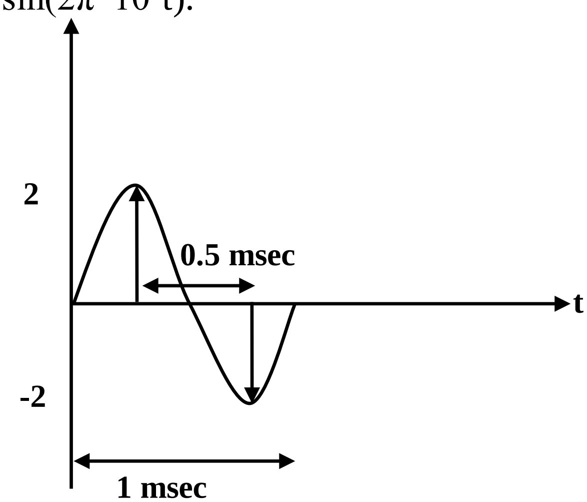
Solution:

$w_m = 2\pi \times 10^3 \frac{rad}{sec} \rightarrow w = 2\pi f_m$

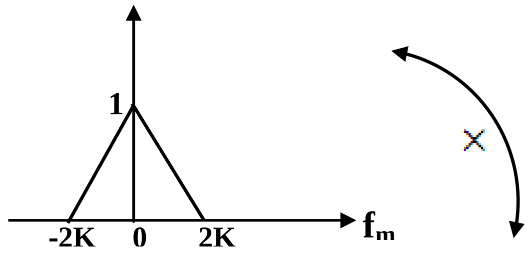
$f_m = 1\ KHz \rightarrow T_m = \frac{1}{f_m} = 1msec$

$f_s = 2f_m = 2 \times 1 = 2KHz$

$T_s = \frac{1}{f_s} = \frac{1}{2} = 0.5msec$

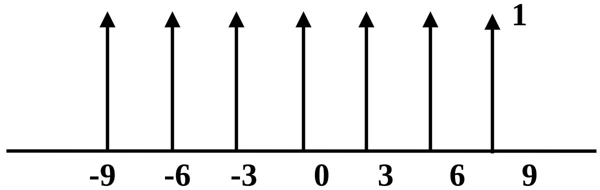


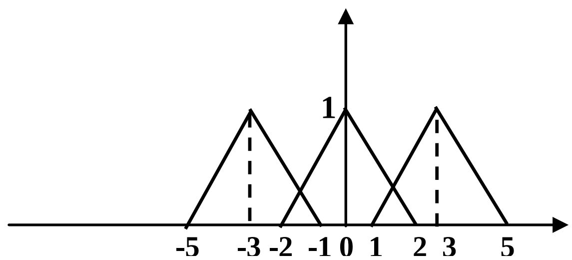
Example: If the spectrum of the signal shown below, find the output signal if
a) f_s = 3KHz, b) f_s = 4KHz, c) f_s = 6KHz



Solution:

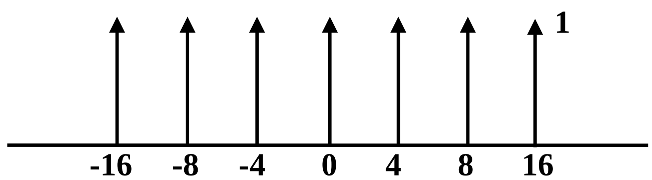
a) f_s = 3KHz

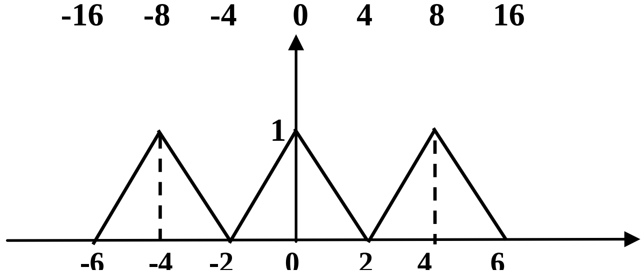




$f_s < 2f_m = 3 < 4$ *under sampling (aliasing)*

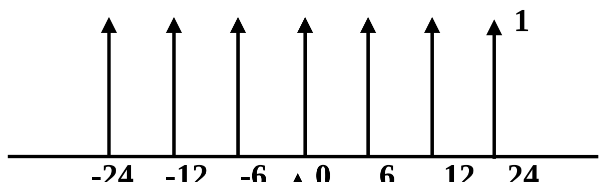
b) f_s = 4KHz

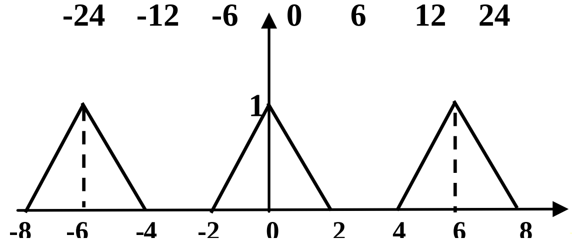




$f_s = 2f_m = 2 \times 2 = 4$ ***low bandwidth***

c) f_s = 6KHz





$f_s > 2f_m$ $6 > 4$ *high bandwidth*

- For convert the analog continuous signal to digital discrete signal by put (t) equal to (nT_s)

$\therefore t = nT_s$

Example: Sketch the following signals in discrete time with F_s = 4 Hz and N = 4.

a) x(t) = 3cos(2πt)
b) x(t) = 5sin(200πt) + 2cos(2πt)

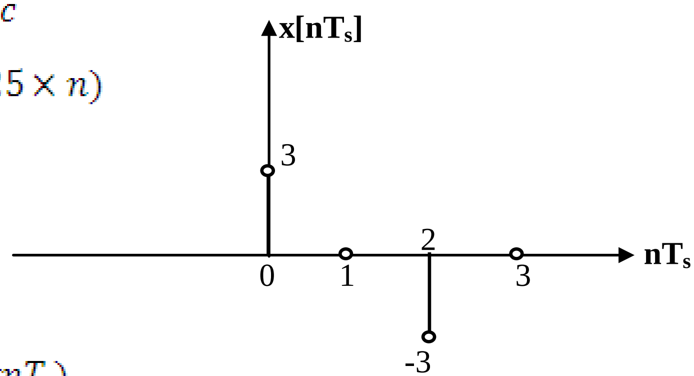
Solution:

a)

$$t = nT_s, f_s = 4 = \frac{1}{T_s} = \frac{1}{4} \Longrightarrow T_s = 0.25\ sec$$

$$x(nT_s) = 3\cos(2\pi nT_s) = 3\cos(2\pi \times 0.25 \times n)$$

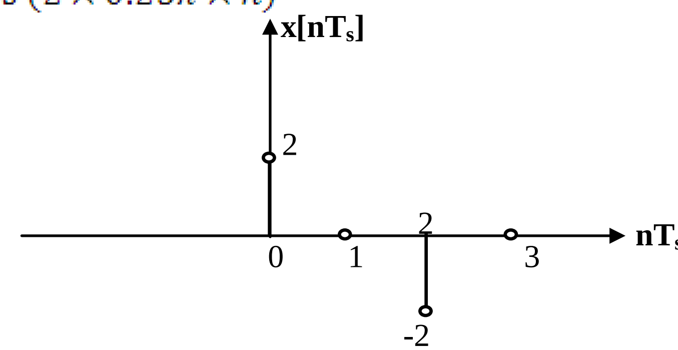
x(0) = 3, x(1) = 0, x(2) = -3, x(3) = 0



b) $x(nT_s) = 5\sin(200\pi nT_s) + 2\cos(2\pi nT_s)$

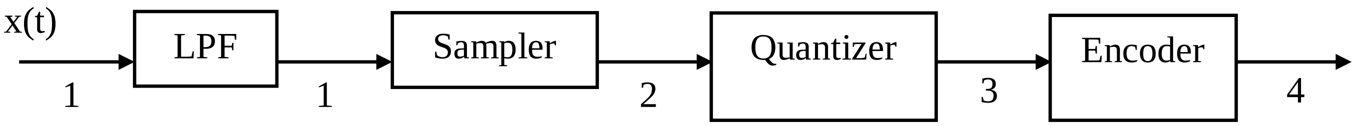
$$= 5\sin(200 \times 0.25\pi \times n) + 2\cos(2 \times 0.25\pi \times n)$$

x(0) = 2, x(1) = 0, x(2) = -2, x(3) = 0

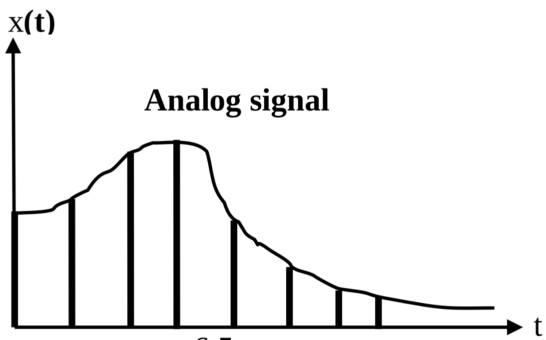


Analog to Digital (A/D)

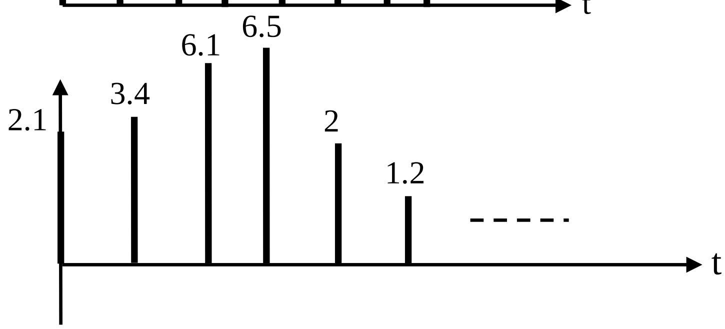
Pulse Code Modulation (PCM)



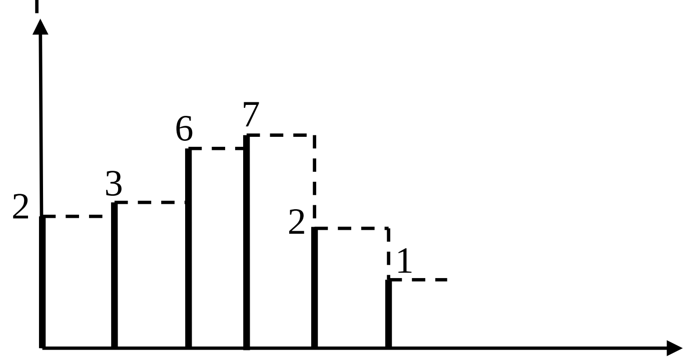
(1)



(2)

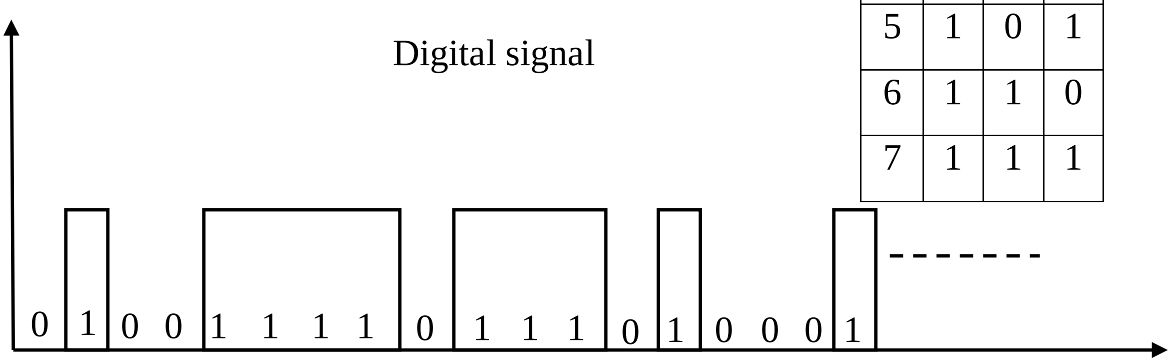


(3)



0	0	0	0
1	0	0	1
2	0	1	0
3	0	1	1
4	1	0	0
5	1	0	1
6	1	1	0
7	1	1	1

(4)



DSP System Implementation

We classified DSP system into two types:

1) Finite Impulse Response (FIR) System (Non-Recursive)

These have finite number of elements in h(n)

$h(n) = \{h(0), h(1), h(2),\ldots.. h(N-1)\}$

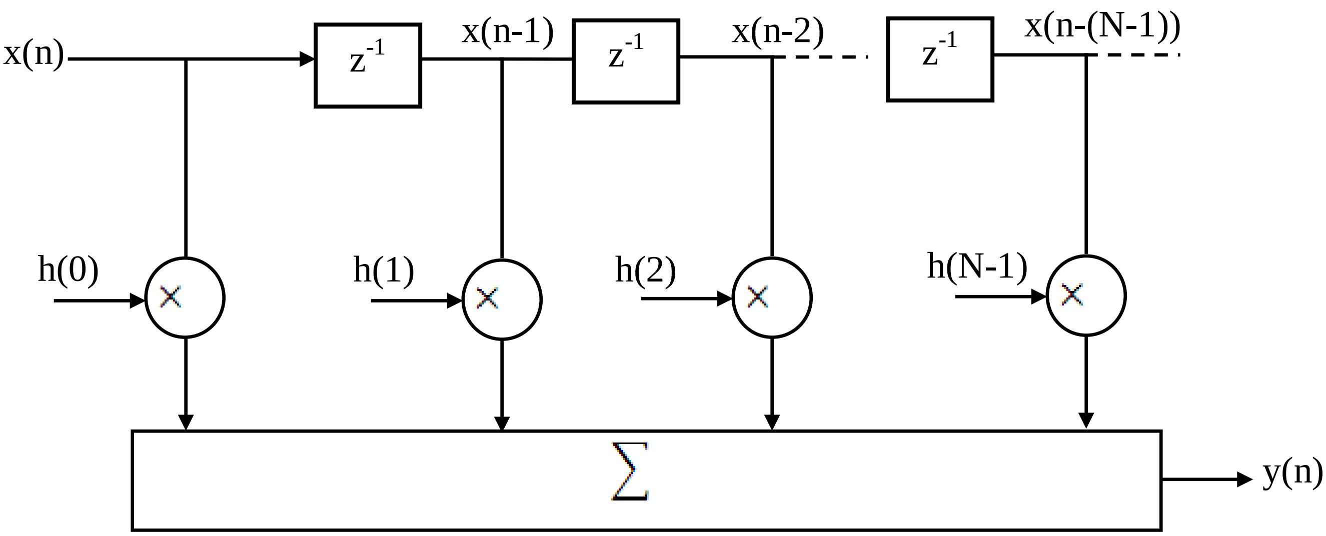
$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2}+\ldots. +h(N-1)z^{-(N-1)}$

$$H(z) = \sum_{k=0}^{N-1} h(k)z^{-k} = \frac{Y(z)}{X(z)}$$

$Y(z) = X(z)\,[h(0) + h(1)z^{-1} + h(2)z^{-2} +\ldots. + h(N-1)z^{-(N-1)}\,]$

$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \ldots..$

The system above is consisting only zero's and no has feedback.



The digital filter above consist three parts are:

1- Delay represented by z^{-1}

2- Multiplication

3- Addition

2) <u>Infinite Impulse Response (IIR) System (Recursive)</u>

$$H(z) = \frac{\sum_{k=0}^{m} a(k)z^{-k}}{\sum_{k=0}^{r} b(k)z^{-k}} = \frac{Y(z)}{X(z)}$$

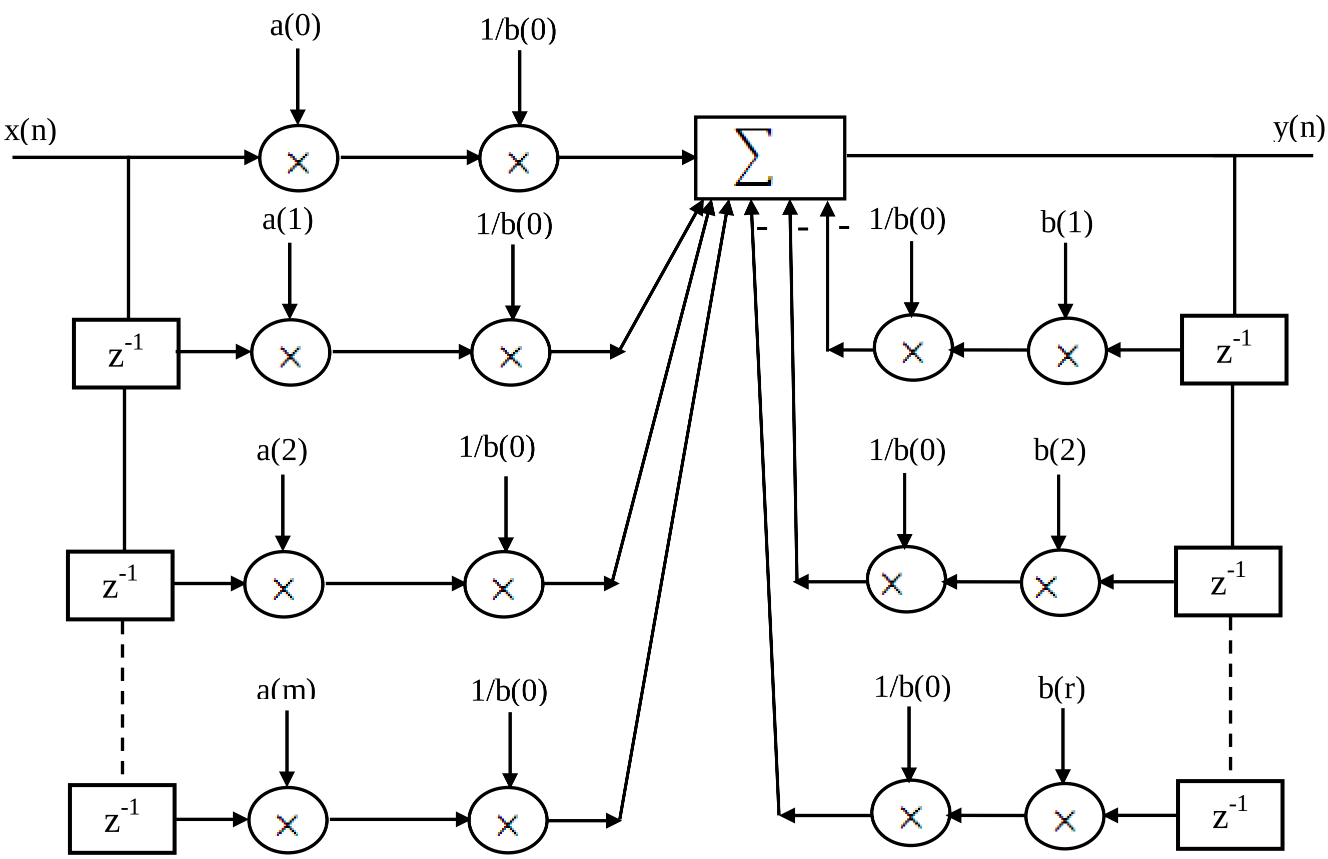
These systems have infinite number of elements limits impulse response.

The system contains poles and zeros and hence these are a possibility that the system will be unstable.

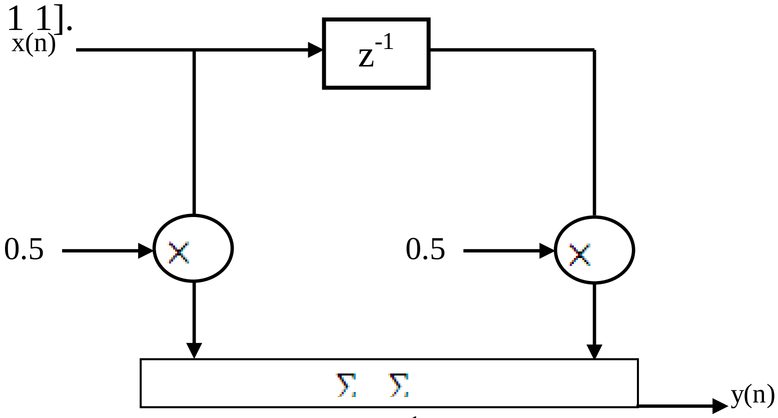
$$X(z)[\sum_{k=0}^{m} a(k)z^{-k}] = Y(z)[\sum_{k=0}^{r} b(k)z^{-k}]$$

$$X(z)[a(0) + a(1)z^{-1} + \cdots + a(m)z^{-m}$$
$$= Y(z)[b(0) + b(1)z^{-1} + b(2)z^{-2} \ldots b(r)z^{-r}]$$

$$y(n) = \frac{1}{b(0)}[a(0)x(n) + a(1)x(n-1) + \cdots + a(m)x(n-m) - b(1)y(n-1)$$
$$- b(2)y(n-2) - \cdots b(r)y(n-2)]$$

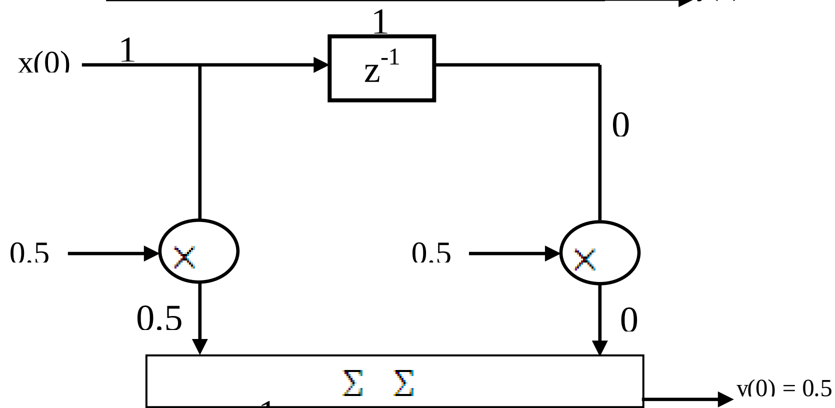


Example: Find the output samples values from the digital filter as shown below if the input signal are x[n] = [1 1 1 1].

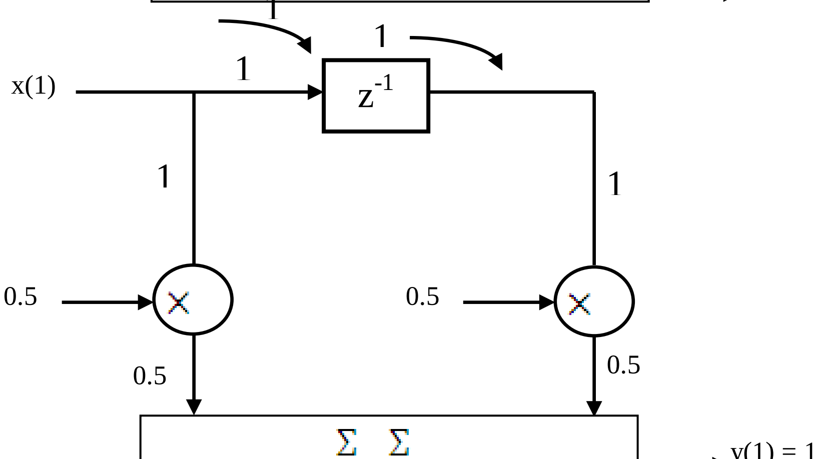


Solution:

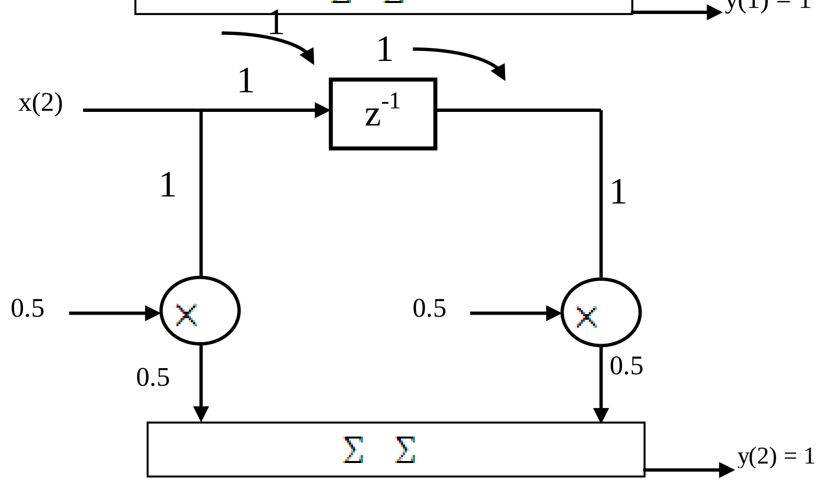
1) x(0)



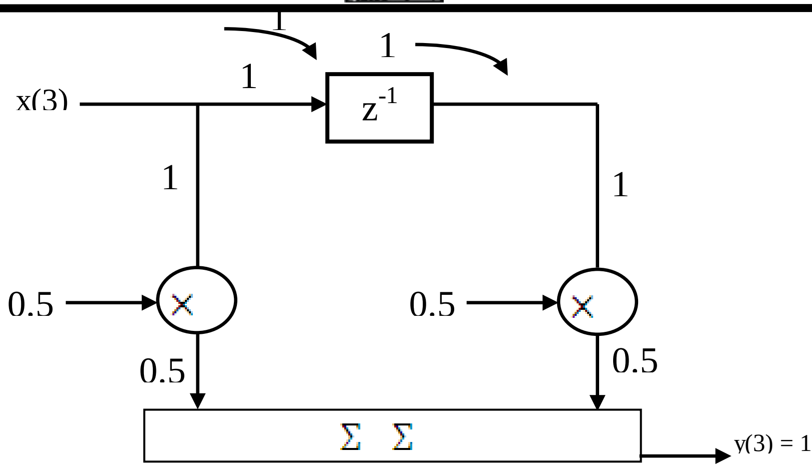
2) x(1)



3) x(2)

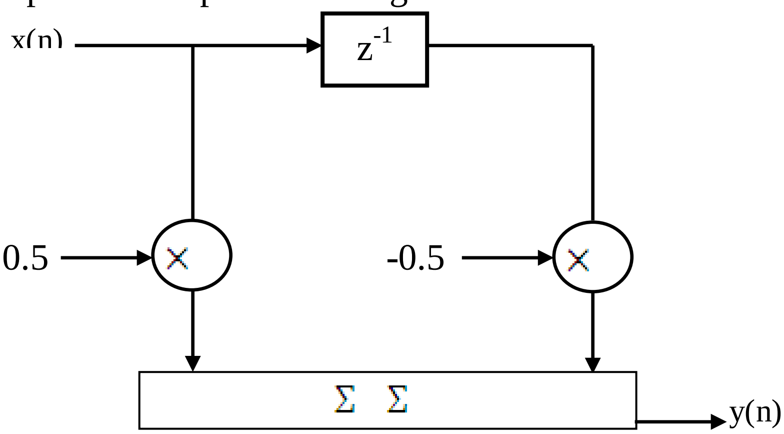


4) x(3)



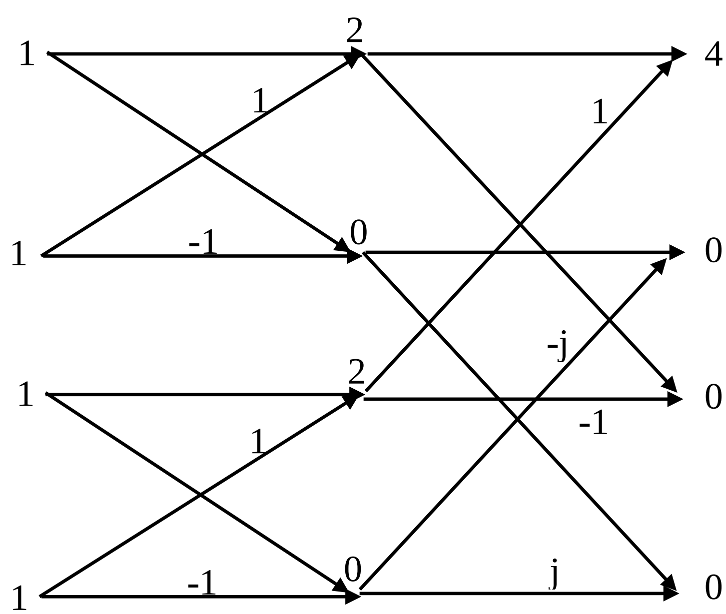
y(n) = [0.5 1 1 1]

Example: Find the FFT for the input and output of the digital filter as shown below for the input signal x(n) = [1 1 1 1].

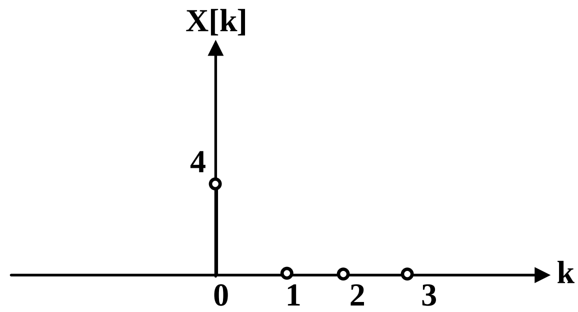


Solution:

1) FFT for x(n)



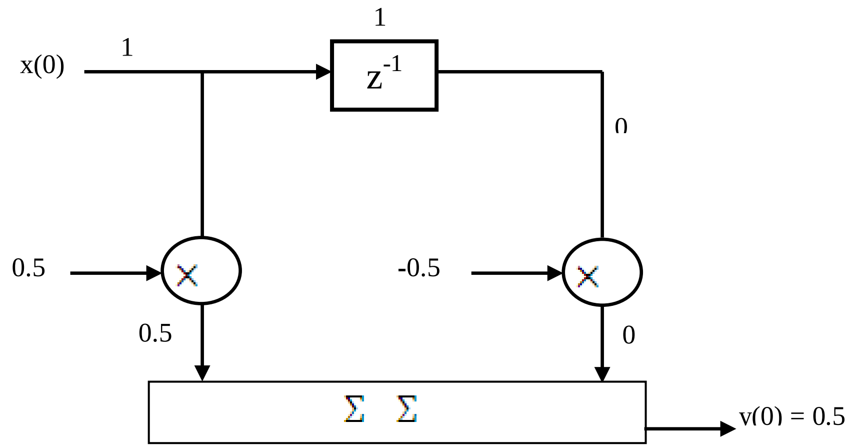
X(k) = [4 0 0 0]



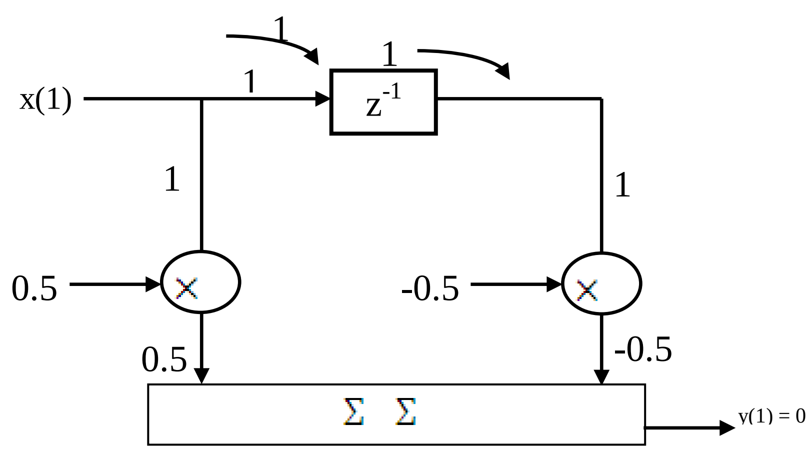
2) FFT for y(n)

Must find y(n) from digital filter

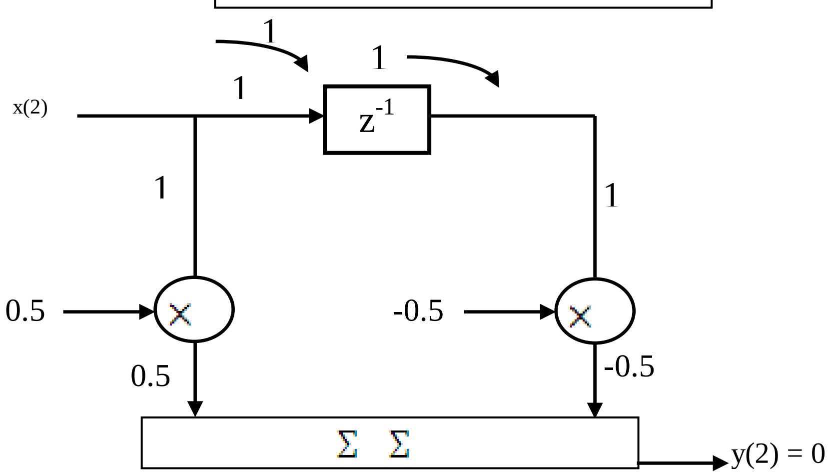
a) x(0)



b) x(1)

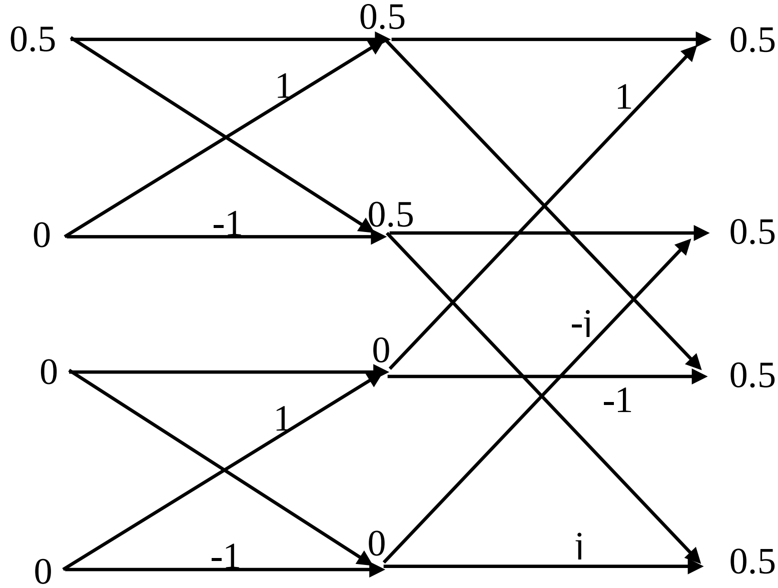


c) x(2)

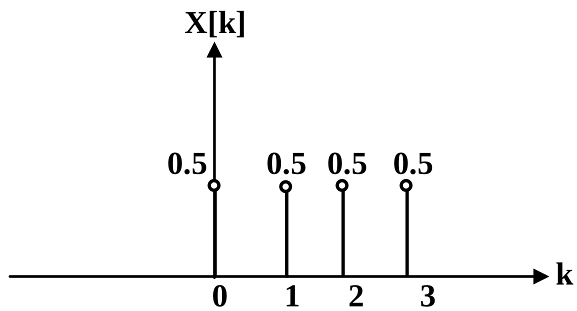


The same above for x(3) & y(3).

y(n) = [0.5 0 0 0]



X(k) = [0.5 0.5 0.5 0.5]



Digital Filter Design

These are classifying into IIR filters and FIR filters.

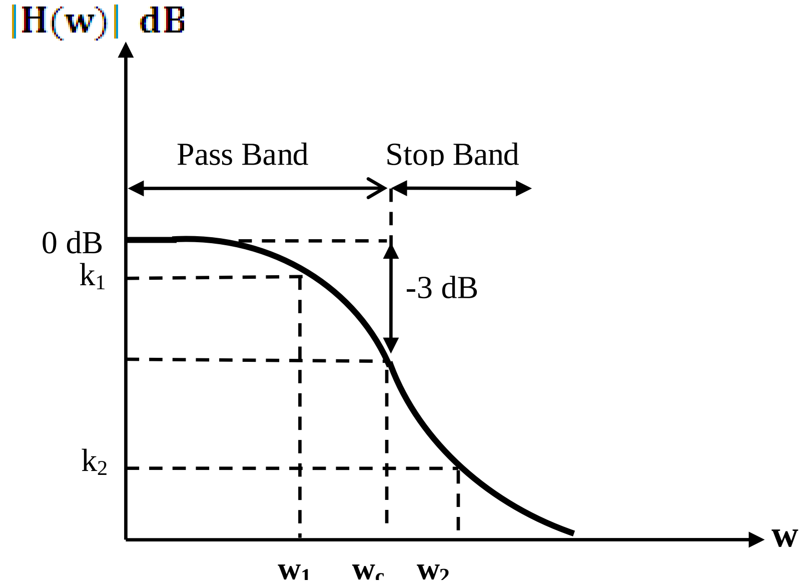
1) IIR Filter Design:

There are many methods to design IIR filter one of the simplest method is to use analogue filter design then by a suitable transformation the transfer function H(s) is transform into digital in z-domain H(z).

Analog Filter Design

These are classifying into Butterworth filter and Chebyshev filter.

1) **Butterworth Filter**
a) **LPF (Low Pass Filter)**



Step1 from the characteristics of the filter in frequency domain, you can find the order of the filter;

$$n = \frac{log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2\,log_{10}\frac{w_1}{w_2}}$$

Step2 from the table, find the function B(s) related to the order n;

n	B(s)
1	s+1
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7653s+1)(s^2+1.8477s+1)$

Step3 find the cutoff frequency;

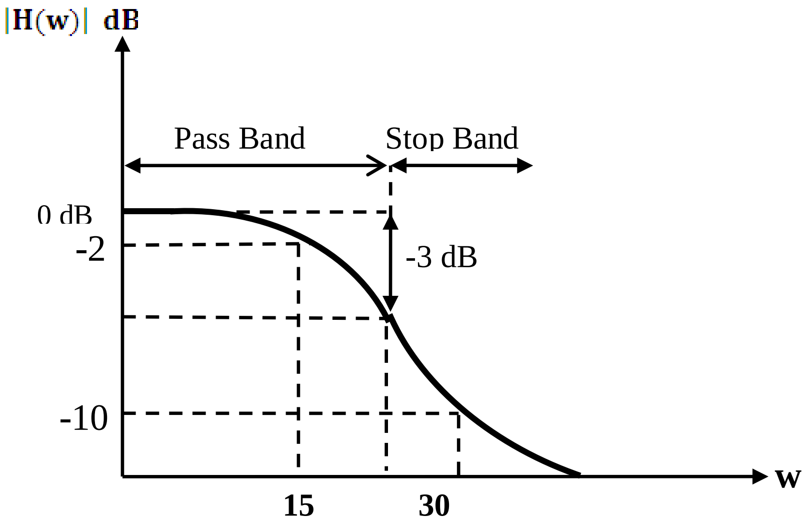
$$w_c = \frac{w_1}{(10^{-k_1/10}-1)^{1/2n}}$$

Step4 to find the transfer function in S-domain;

$$H(s) = \left.\frac{1}{B(s)}\right|_{s=\frac{s}{w_c}}$$

Example Design analog LPF that has pass band attenuation of 2dB at 15 rad/sec and stop band attenuation of 10 dB at 30 rad/sec.

Solution:



$$n = \frac{\log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2\log_{10}\frac{w_1}{w_2}} = 1.97 \cong 2$$

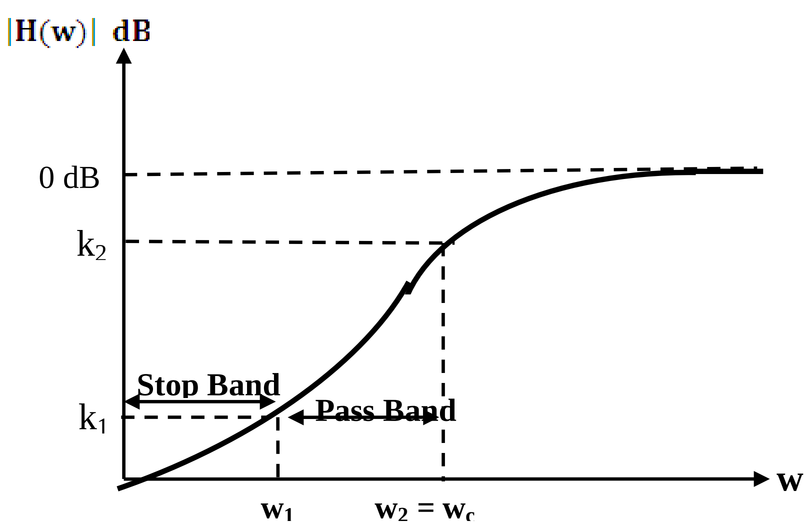
From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s=\frac{s}{w_c}}$$

$$w_c = \frac{w_1}{(10^{-k_1/10}-1)^{1/2n}} = \frac{15}{(10^{2/10}-1)^{1/4}} = 17.2\, rad/sec$$

$$\therefore H(s) = \frac{1}{\left(\frac{s}{17.2}\right)^2+\sqrt{2}\frac{s}{17.2}+1}$$

b) **HPF (high pass filter)**



Step1 from the characteristics of the filter in frequency domain, you can find the order of the filter;

$$n = \frac{log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2\,log_{10}\frac{w_2}{w_1}}$$

Step2 from the table, find the function B(s) related to the order n;

n	B(s)
1	s+1
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7653s+1)(s^2+1.8477s+1)$

Step3 find the cutoff frequency;

$w_c = w_2$

Step4 to find the transfer function in S-domain;

$$H(s) = \left.\frac{1}{B(s)}\right|_{s=\frac{w_c}{s}=\frac{w_2}{s}}$$

Example Design Butterworth HP analog filter having 3dB cutoff frequency of 5 rad/sec and an attenuation of 15 dB at frequency of 2 rad/sec.

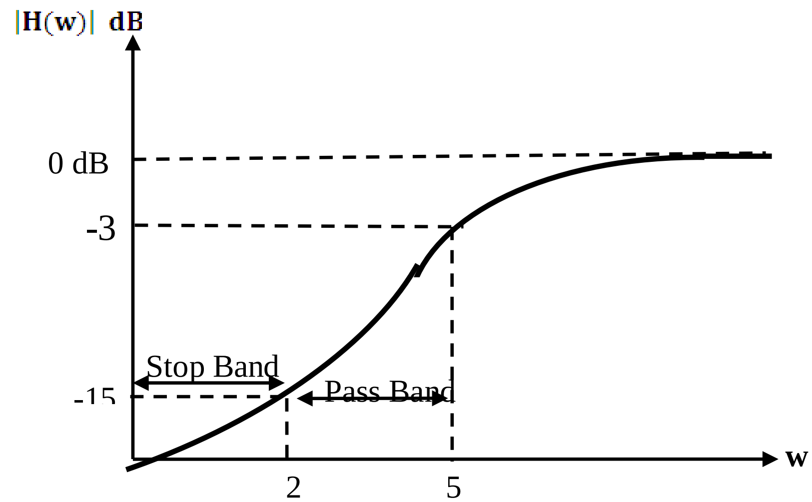
Solution:

$n = 1.87 \cong 2$

From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \left.\frac{1}{s^2+\sqrt{2}s+1}\right|_{s=\frac{w_c}{s}=\frac{w_2}{s}}$$

$$\therefore H(s) = \frac{1}{\left(\frac{5}{s}\right)^2+\sqrt{2}(\frac{5}{s})+1} = \frac{s^2}{s^2+5\sqrt{2}s+25}$$



c) BPF (band pass filter)

Step1 from the characteristics of the filter in frequency domain, you can find the order of the filter;

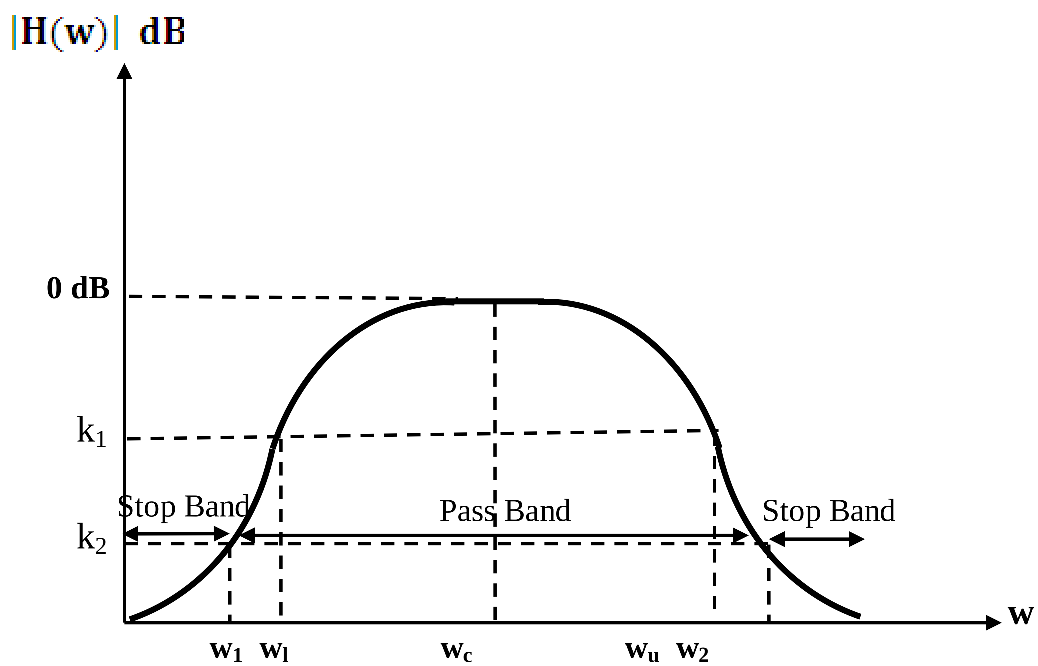
$$n = \frac{log_{10}\left[\frac{10^{-k_1/10}-1}{10^{-k_2/10}-1}\right]}{2log_{10}(P)}$$

$$P = \frac{1}{w_r}$$

$$w_r = min[|A| \; or \; |B|]$$

$$|A| = \frac{-w_1^2 + w_u w_l}{w_1(w_u - w_l)}$$

$$|B| = \frac{w_2^2 - w_u w_l}{w_2(w_u - w_l)}$$



Step2 from the table, find the function B(s) related to the order n;

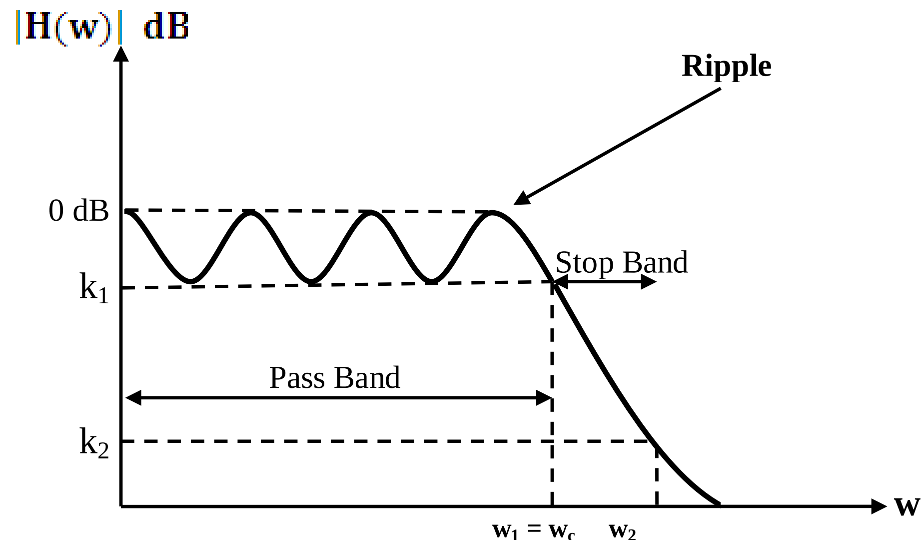
n	B(s)
1	s+1
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.7653s+1)(s^2+1.8477s+1)$

Step3 to find the transfer function in S-domain;

$$H(s) = \left.\frac{1}{B(s)}\right|_{s=\frac{s^2+w_l w_u}{s(w_u-w_l)}}$$

2. Chebyshev Filter

a) LPF (Low Pass Filter)



1) $k_1 = -10log_{10}(1+\epsilon^2)$

$k_2 = -20log_{10}A$

$g = \sqrt{\frac{A^2-1}{\epsilon^2}}$

$w_r = \frac{w_2}{w_1}$

2)

$$n = \frac{log_{10}(g+\sqrt{g^2-1})}{log_{10}(w_r+\sqrt{w_r^{\,2}-1})}$$

3) From table (3.4), we find the coefficients b_0, b_1, b_2….

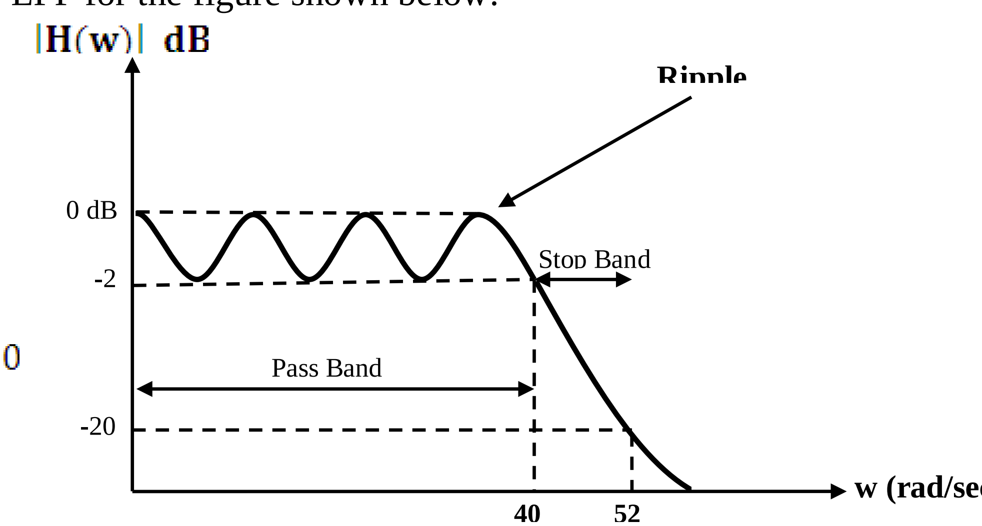
$$H(s) = \frac{k_n}{V_n(s)}$$

$$k_n = \begin{cases} \frac{b_0}{\sqrt{1+\epsilon^2}} & n\ even \\ b_0 & n\ odd \end{cases}$$

$$V_n(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots b_n s^n$$

$$H(s) = \left.\frac{k_n}{V_n(s)}\right|_{s=\frac{s}{w_c}}$$

Example: Design Chebyshev LPF for the figure shown below:



Solution:

1. $k_1 = -10log_{10}(1+\epsilon^2)$

$-2 = -10log_{10}(1+\epsilon^2)$

$\epsilon = 0.7647$

$-20 = -20log_{10}A \Rightarrow A = 10$

$g = \sqrt{\frac{A^2-1}{\epsilon^2}} = 13$

$$w_r = \frac{w_2}{w_1} = 1.3$$

$$n = 4.3 \cong 5$$

$$\because n = 5 \ odd \ \Rightarrow \ k_n = b_0$$

From table (3.4);

$$V_n(s) = b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \cdots b_5 s^5$$

$$= 0.1228 + 0.5805s + 0.97439s^2 + 1.6888s^3 + 0.9368s^4 + s^5$$

$$H(s) = \left.\frac{k_n}{V_n(s)}\right|_{s=\frac{s}{w_c}}$$

$$H(s) = \frac{0.1228}{0.1228 + 0.5805\left(\frac{s}{40}\right) + 0.97439\left(\frac{s}{40}\right)^2 + 1.6888\left(\frac{s}{40}\right)^3 + 0.9368\left(\frac{s}{40}\right)^4 + \left(\frac{s}{40}\right)^5}$$

b) HPF (High Pass Filter)

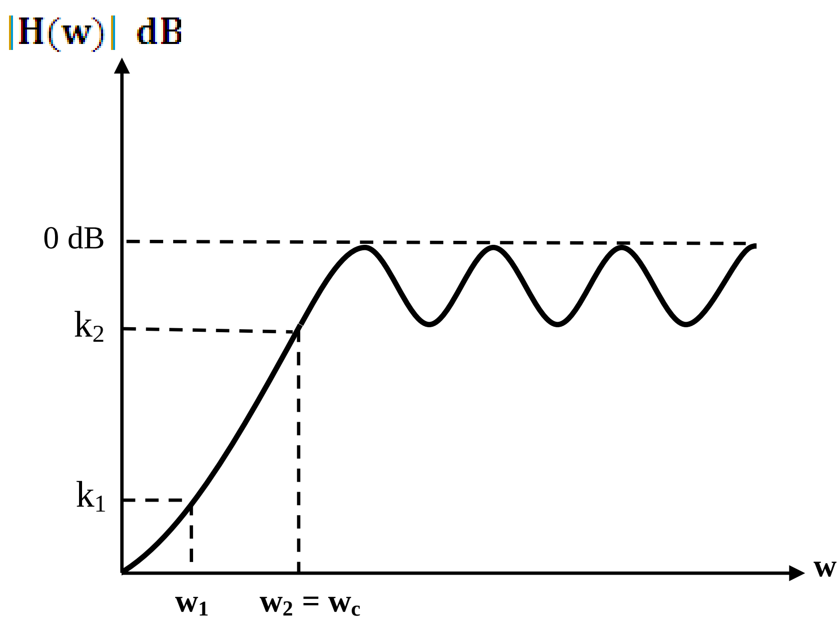
1)

$$k_2 = -10log_{10}(1+\epsilon^2)$$

$$k_1 = -20log_{10}A$$

$$g = \sqrt{\frac{A^2-1}{\epsilon^2}}$$

$$w_r = \frac{w_2}{w_1}$$



2)

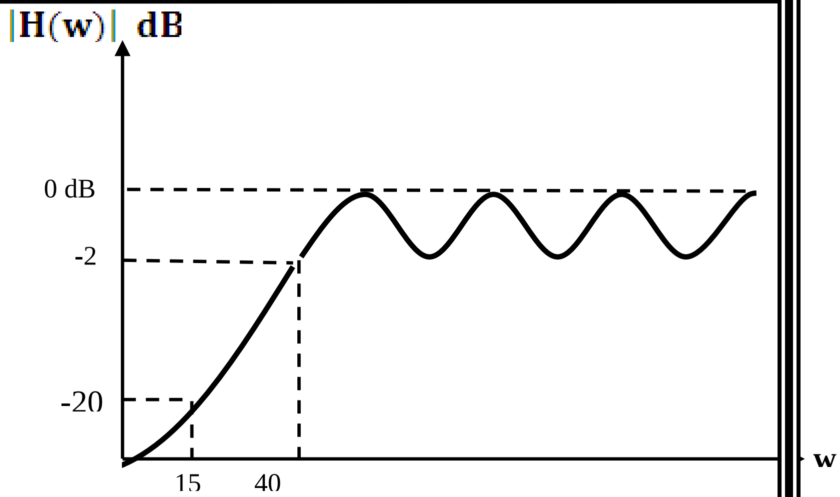
$$n = \frac{log_{10}(g + \sqrt{g^2-1})}{log_{10}(w_r + \sqrt{w_r^{\,2}-1})}$$

3)

$$H(s) = \left.\frac{k_n}{V_n(s)}\right|_{s=\frac{w_2}{s}}$$

Example: Design an analogue HPF having cutoff frequency of 40 rad/sec with permissible ripple of 2 dB. Stop bans attenuation of 20 dB at 15 rad/sec. Use Chebyshev approximation.

Solution:



$-2 = -10log_{10}(1+\epsilon^2) \Rightarrow \epsilon^2 = 0.58$
$-20 = -20log_{10}A \Rightarrow A = 10$

$$g = \sqrt{\frac{100-1}{0.58}} = 13$$

$$w_r = \frac{w_2}{w_1} = \frac{40}{15} = 2.6$$

$$n = 1.98 \cong 2$$

From table (3.4) $[\epsilon^2 = 0.58 \,\&\, n = 2] \Rightarrow b_0 = 0.63, b_1 = 0.8$

$$H(s) = \frac{\frac{b_0}{\sqrt{1+\epsilon^2}}}{0.63 + 0.8s + s^2} = \frac{\frac{0.63}{\sqrt{1+0.58}}}{0.63 + 0.8s + s^2}$$

$$H(s)|_{s=\frac{w_2}{s}=\frac{40}{s}} = \frac{\frac{0.63}{\sqrt{1+0.58}}}{0.63 + 0.8\left(\frac{40}{s}\right) + \left(\frac{40}{s}\right)^2}$$

c) BPF (Band Pass Filter)

1)

$w_r = min[|A| \; or \; |B|]$

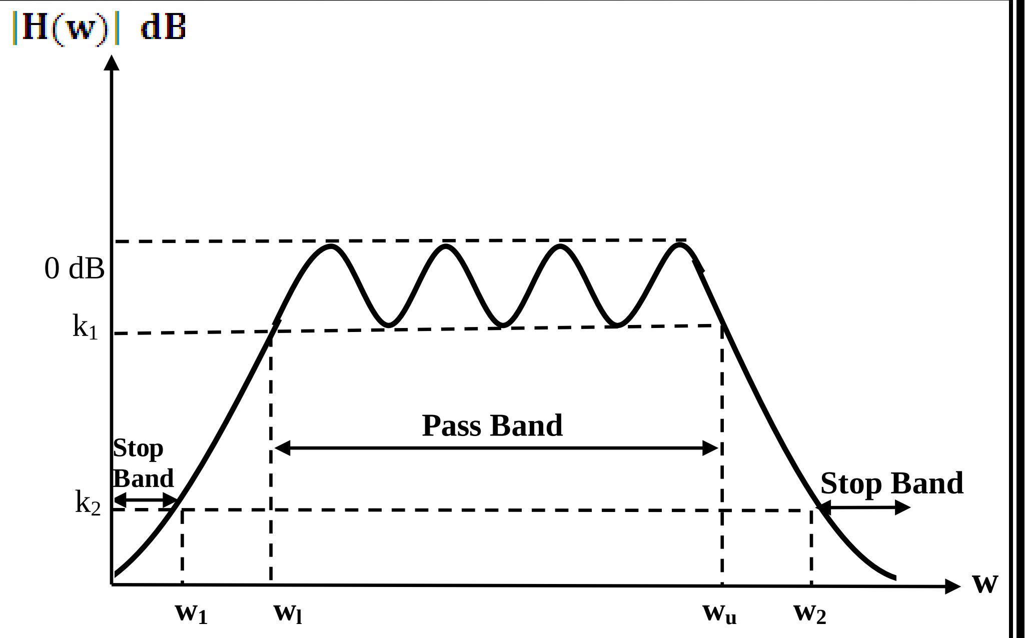
$|A| = \dfrac{-w_1^2 + w_u w_l}{w_1(w_u - w_l)}$

$|B| = \dfrac{w_2^2 - w_u w_l}{w_2(w_u - w_l)}$

2)

$k_1 = -10log_{10}(1+\epsilon^2)$
$k_2 = -20log_{10}A$

$$g = \sqrt{\frac{A^2 - 1}{\epsilon^2}}$$



3)

$$n = \frac{log_{10}(g + \sqrt{g^2 - 1})}{log_{10}(w_r + \sqrt{{w_r}^2 - 1})}$$

4)

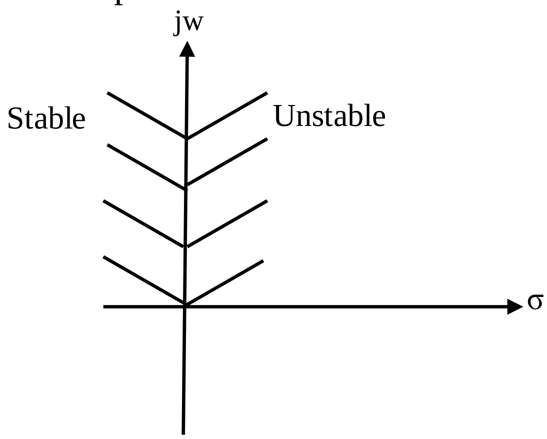
$$H(s) = \frac{k_n}{V_n(s)}\Bigg|_{s=\frac{s^2 + w_l w_u}{s(w_u - w_l)}}$$

<u>Example (H.W):</u> Design an analogue BPF having the following c/cs:
1) 3dB upper and lower cutoff frequency of 50 Hz and 20 Hz.
2) Stop band attenuation of 20 dB at frequency 20 Hz and 45 KHz.

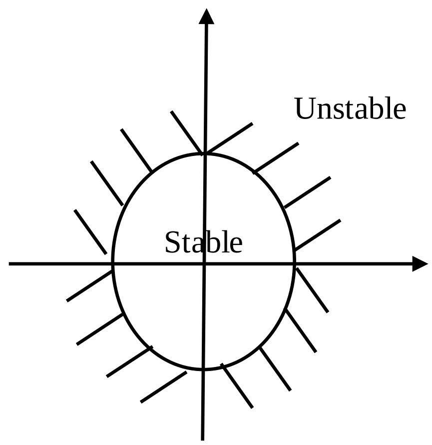
Bilinear-Transformation (IIR Digital Filter)

To design an IIR digital filter then a suitable transformation called bilinear transformation is used.

This transformation transforms the left hand side of the s-plane into interior inside of the unit circle in z-plane.



s-plane



z-plane

This bilinear transformation transforms the T.F H(s) of the analogue filter

$$s = 2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

f_s : sampling frequency
z^{-1}: delay element by T_s
if s = jw , $z^{-1} = e^{-j\lambda}$
where,
λ: Digital frequency (rad)
w: Analogue frequency (rad/sec)
To find a relationship between λ & w

$$s = jw = \frac{2}{T_s}\left(\frac{1-e^{-j\lambda}}{1+e^{-j\lambda}}\right) \cdot \frac{e^{j\lambda/2}}{e^{j\lambda/2}}$$

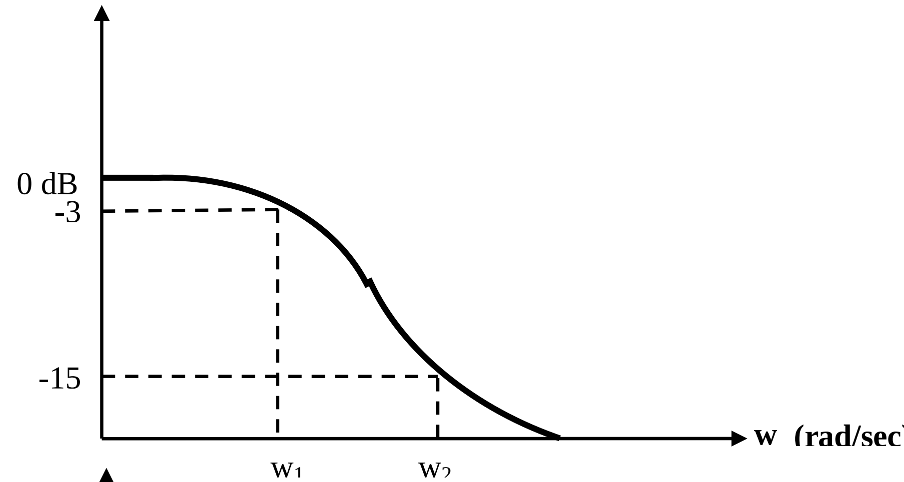
$$jw = \frac{2}{T_s}\left(\frac{e^{j\frac{\lambda}{2}} - e^{-j\frac{\lambda}{2}}}{e^{j\frac{\lambda}{2}} + e^{-j\frac{\lambda}{2}}}\right)$$

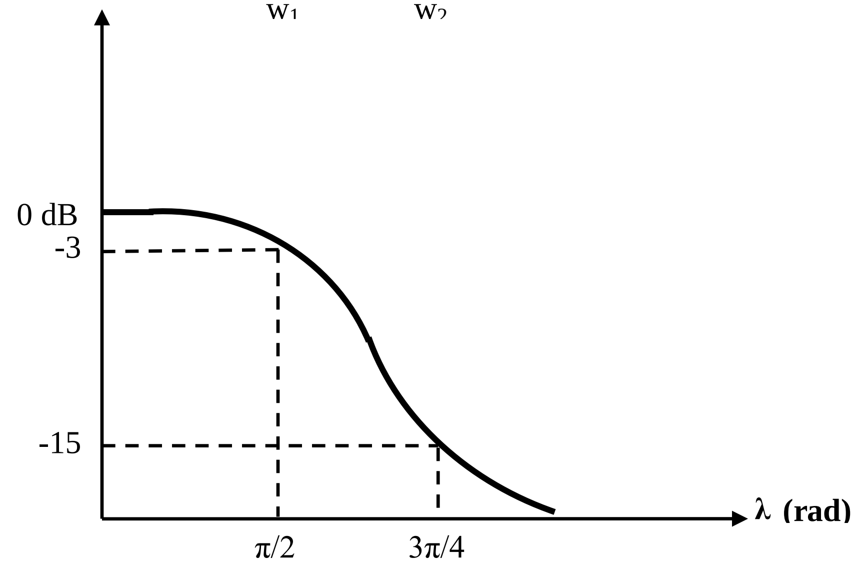
$$\therefore w = \frac{2}{T_s} tan\frac{\lambda}{2} \qquad \& \qquad \lambda = 2tan^{-1}\left(\frac{wT_s}{2}\right)$$

Example: using bilinear transformation design and realize a digital LPF having the following c/cs:

a) Monotone pass band
b) 3 dB cutoff frequency of π/2 rad.
c) Stop band attenuation of 15 dB at 3π/4 rad. (T_s = 1 sec).

Solution:





$$w = \frac{2}{T_s} tan\frac{\lambda}{2}$$

$$w_1 = 2tan\frac{\pi}{4} = 2\ rad/sec$$

$$w_2 = 2tan\frac{3\pi}{8} = 4.8\ rad/sec$$

Monotone →Butterworth

$$n = \frac{log_{10}\left[\frac{10^{3/10}-1}{10^{15/10}-1}\right]}{2log_{10}\frac{2}{4.8}} = 2$$

From table, $B(s) = s^2+\sqrt{2}s+1$

$$H(s) = \frac{1}{s^2+\sqrt{2}s+1}\bigg|_{s=\frac{s}{w_c}}$$

$$w_c = \frac{w_1}{(10^{-k_1/10}-1)^{1/2n}} = 2\ rad/sec$$

$$\therefore H(s) = \frac{1}{\left(\frac{s}{2}\right)^2+\sqrt{2}\frac{s}{2}+1}$$

To design digital filter

$$s = 2f_s\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$T_s = 1 \rightarrow f_s = 1$ Hz

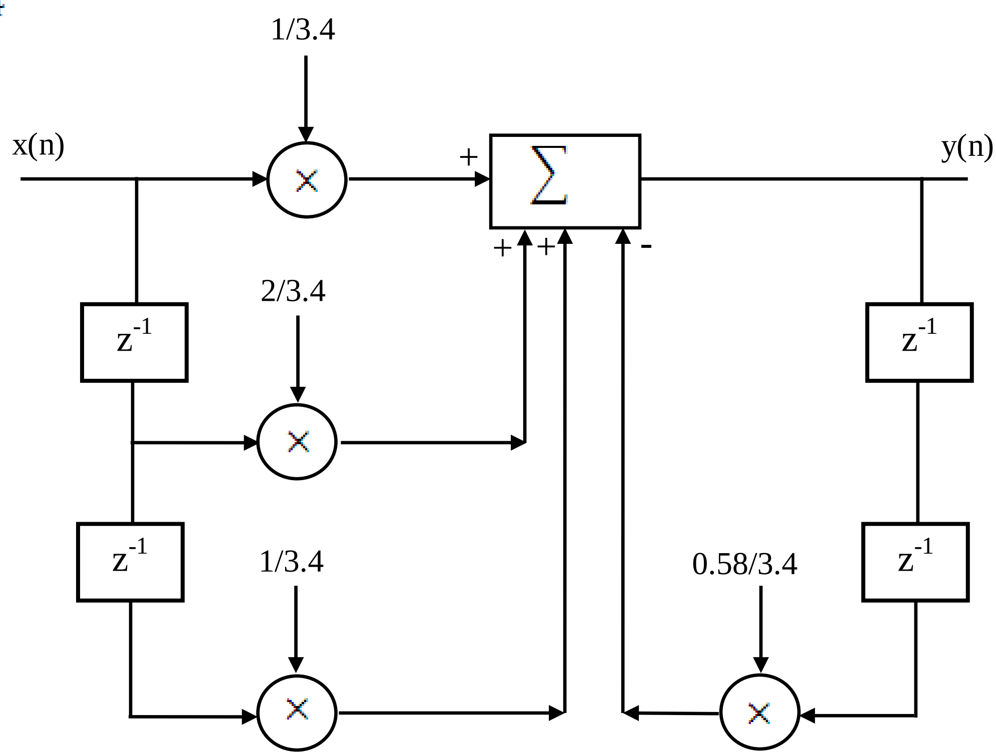
$$s = 2\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$\therefore H(z) = \frac{1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2+\sqrt{2}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1}$$

$$= \frac{1 + 2z^{-1} + z^{-2}}{1 - 2z^{-1} + z^{-2} + \sqrt{2}(1 - z^{-2}) + 1 + 2z^{-1} + z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 2z^{-1} + z^{-2}}{3.4 + 0.58z^{-2}}$$

$$\therefore y(n) = \frac{1}{3.4}[x(n) + 2x(n-1) + x(n-2) - 0.58y(n-2)]$$



H.W

Q1) design a LPF IIR digital filter having 3 dB cutoff frequency of 1 KHz and stop band attenuation of 28 dB at 2 KHz. Use f_s = 8 KHz and Chebyshev approximation.

Q2) design a HP IIR digital filter having monotone pass band and cutoff frequency of π/2 rad with attenuation of -3 dB and a stop band attenuation of -15 dB at a frequency of π/4. Use T_s = 1 sec.

FIR Filter Design (Linear Phase Condition)

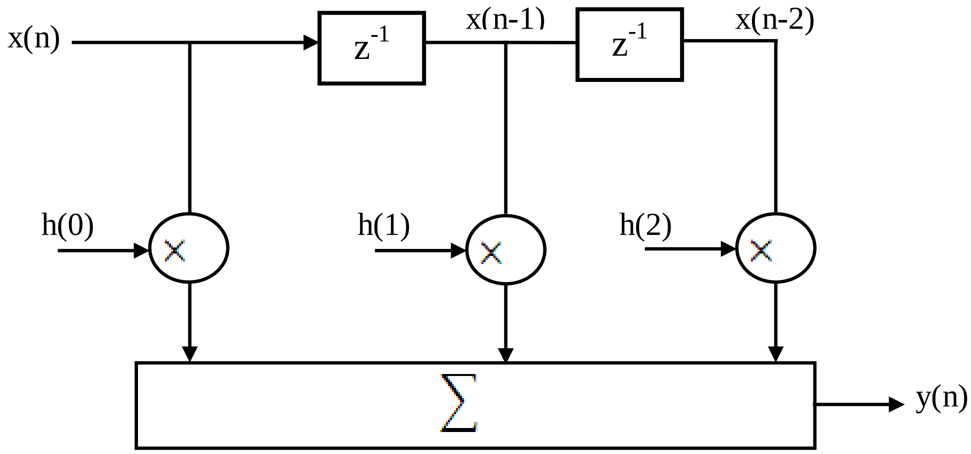
For a certain FIR filter to be linear phase, the following equations must be satisfied:

$h(n) = h(N-1-n) \qquad n = 0,1,2,\ldots N-1$

And N is odd (N: number of coefficient)

- For N = 3

h(n) = h(2-n)
h(0) = h(2)
h(1) = h(1)
h(2) = h(0)



Procedures:

1) Convert the units of w from rad/sec to rad by:

$$w_{1new} = w_1 \cdot T_s = w_1 \cdot \frac{1}{f_s}$$

$$w_{2new} = w_2 \cdot T_s = w_2 \cdot \frac{1}{f_s}$$

$$w_c = w_{1new}$$

2)

$$N = \frac{2\pi k}{w_{2new} - w_{1new}}$$

N: number of samples must be odd number

3)

$$\alpha = \frac{N-1}{2}$$

4)

$$h(n) = \frac{sin[w_c(n-\alpha)]}{\pi(n-\alpha)} \cdot w_n$$

5) We can find w_n from table (1)

Note:

- If N =2 become N = 3 (odd)
- If N = 53.2 become N = 55 (odd)

Table (1)

Window	**Factor (k)**	**w_n**	
Rectangular	2	1 0	$0 \le n \le N-1$ elsewhere
Barlet	4	$\frac{n}{\alpha}$ $2-\frac{n}{\alpha}$ 0	$0 \le n \le \alpha$ $\alpha < n \le N-1$ elsewhere
Hanning	4	$0.5-0.5cos\left(\frac{2\pi n}{N-1}\right)$ 0	$0 \le n \le N-1$ elsewhere
Hamming	4	$0.54-0.46cos\left(\frac{2\pi n}{N-1}\right)$ 0	$0 \le n \le N-1$ elsewhere
Blackman	6	$0.42-0.5cos\left(\frac{2\pi n}{N-1}\right)+0.08cos\left(\frac{4\pi n}{N-1}\right)$ 0	$0 \le n \le N-1$ elsewhere

<u>Example:</u> design LP digital filter having linear phase (FIR) with 3 dB cutoff frequency of 1 KHz and stop band attenuation of 28 dB at 2 KHz. Use f_s = 8 KHz and Hanning window.

<u>Solution:</u>

$$w_{1new} = 2\pi \times 10^3 \times \frac{1}{8 \times 10^3} = 0.7854 = 0.25\pi \quad rad$$

$$w_{2new} = 4\pi \times 10^3 \times \frac{1}{8 \times 10^3} = 1.57 = 0.5\pi \quad rad$$

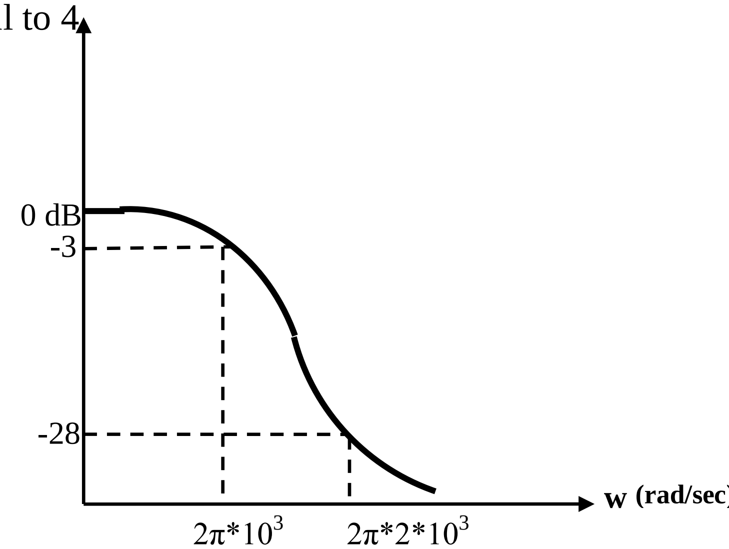
$$w_c = w_{1new} = 0.25\pi \quad rad$$

From table (1) the k for Hanning window equal to 4

$$N = \frac{2\pi \times 4}{0.5\pi - 0.25\pi} = 32 \cong 33\ (odd)$$

$$\alpha = \frac{N-1}{2} = \frac{33-1}{2} = 16$$

$$w(n) = \begin{cases} 0.5 - 0.5cos\left(\frac{\pi n}{16}\right) & 0 \le n \le 32 \\ 0 & \text{elsewhere} \end{cases}$$



$$h(n) = \frac{sin[w_c(n-\alpha)]}{\pi(n-\alpha)} \cdot w_n$$

$$h(n) = \frac{sin[0.25\pi(n-16)]}{\pi(n-16)} \cdot (0.5 - 0.5cos\left(\frac{\pi n}{16}\right))$$

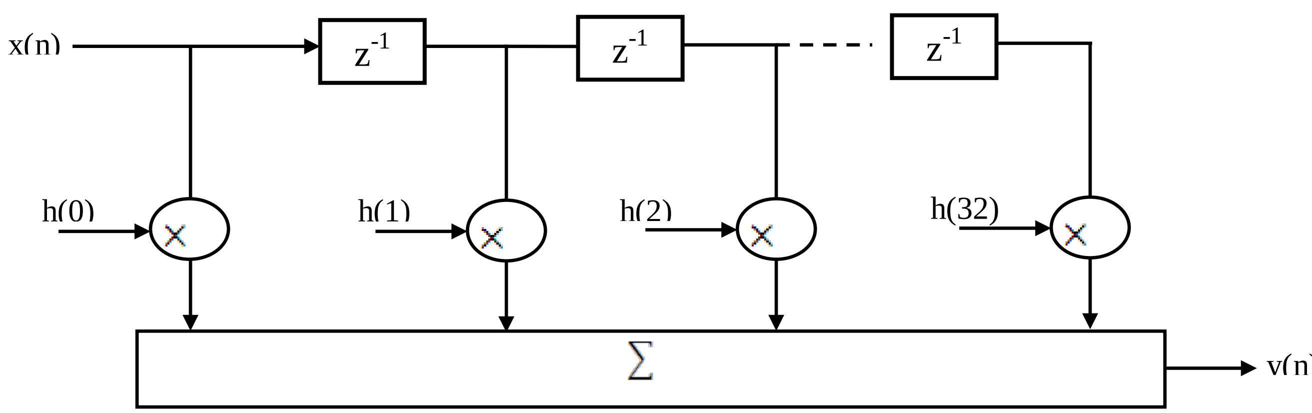
h(n) = h(N-1-n) = h(32-n)

h(0) = h(32), h(1) = h(31), h(2) = h(30), h(3) = h(29), h(4) = h(28), h(5) = h(27), h(6) = h(26), h(7) = h(25), h(8) = h(24), h(9) = h(23), h(10) = h(22), h(11) = h(21), h(12) = h(20), h(13) = h(19), h(14) = h(18), h(15) = h(17), h(16) = h(16), h(17) = h(15), h(18) = h(14), h(19) = h(13), h(20) = h(12), h(21) = h(11), h(22) = h(10), h(23) = h(9), h(24) = h(8), h(25) = h(7), h(26) = h(6), h(27) = h(5), h(28) = h(4), h(29) = h(3), h(30) = h(2), h(31) = h(1), h(32) = h(0).

n	0	1	2	3	4	5	6
h(n)	0	$-1.44*10^{-4}$	$-8.65*10^{-4}$	$-1.459*10^{-3}$	0	$4.547*10^{-3}$	$9.825*10^{-3}$

n	7	8	9	10	11	12	13
h(n)	$10*10^{-3}$	0	$-19*10^{-3}$	$-30*10^{-3}$	$-35*10^{-3}$	0	$63*10^{-3}$

n	14	15	16
h(n)	$153.1*10^{-3}$	$223*10^{-3}$	0.25



<u>Example (H.W):</u> design a linear phase digital LPF having 3 dB cutoff frequency of 1 KHz and stop band attenuation of 40 dB at 4 KHz. Use Hanning window and take f_s = 8 KHz.