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C H A P T E R 12

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COMPLEX NUMBERS

In multi-phase generation of electricity, it is often convenient to express each phase as a complex number, Z, which has the form a + jb, where a and b are real numbers. This representation is known as Carterian form of Z.

12.1. Mathematical Representation of Vectors

There are various forms or methods of representing vector quantities, all of which enable those operations which are carried out graphically in a phasor diagram, to be performed analytically. The various methods are :

(i) Symbolic Notation. According to this method, a vector quantity is expressed algebraically in terms of its rectangular components. Hence, this form of representation is also known as Rectangular or Cartesian form of notation or representation.

(ii) Trigonometrical Form (iii) Exponential Form (iv) Polar Form.

12.2. Symbolic Notation

A vector can be specified in terms of its *X*-component and *Y*-component. For example, the vector OE_1 (Fig. 12.1) may be completely described by stating that its horizontal component is a_1 and vertical component is b_1. But instead of stating this verbally, we may express symbolically

$$\mathbf{E_1} = a_1 + jb_1$$

where symbol *j*, known as an operator, indicates that component b_1 is perpendicular to component a_1 and that the two terms are ***not*** to be treated like terms in any algebraic expression. The vector written in this way is said to be written in ***'complex form'***. In Mathematics, a_1 is known as real component and b_1 as imaginary component but in electrical engineering, these are known as ***in phase*** (or active) and ***quadrature*** (or reactive) components respectively.

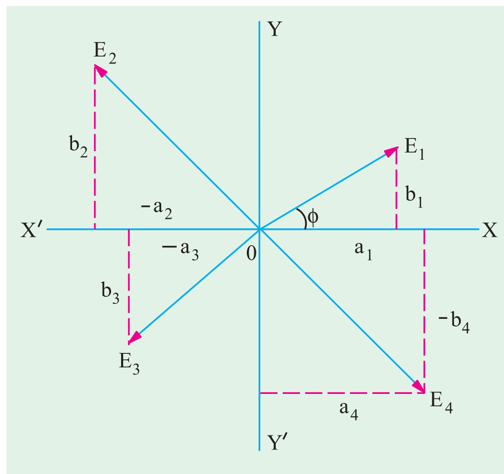

Fig. 12.1

The other vectors OE_2, OE_3 and OE_4 can similarly, be expressed in this form.

$$\mathbf{E_2} = -a_2 + jb_2\ ;\ \mathbf{E_3} = -a_3 - jb_3\ ;\ \mathbf{E_4} = +a_4 - jb_4$$

It should be noted that in this book, a vector quantity would be represented by letters in heavy type and its numerical or scalar value by the same letter in ordinary type.* Other method adopted for indicating a vector quantity is to put an arrow about the letter such as E .

The numerical value of vector E_1 is $\sqrt{a_1^2 + b_1^2}$. Its angle with *X*-axis is given by $\phi = \tan^{-1}(b_1/a_1)$.

12.3. Significance of Operator *j*

The letter *j* used in the above expression is a symbol of an operation. Just as symbols $\times, +, \sqrt{\ }, \int$ etc. are used with numbers for indicating certain operations to be performed on those numbers, similarly, symbol *j* is used to indicate the counter-clockwise rotation of a vector through 90º. It is assigned a value of $\sqrt{(\ 1)}$ **. The double operation of *j* on a vector rotates it counter-clockwise through 180º and hence reverses its sense because

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* The magnitude of a vector is sometimes called 'modulus' and is represented by | *E* | or *E*.

** In Mathematics, $\sqrt{(-1)}$ is denoted by *i* but in electrical engineering j is adopted because letter *i* is reserved for representing current. This helps to avoid confusion.

$$j\,j = j^2 = \sqrt{(-1)}^2 = -1$$

When operator j is operated on vector **E**, we get the new vector j**E** which is displaced by 90º in counter-clockwise direction from **E** (Fig. 12.2). Further application of j will give j^2**E** = −**E** as shown.

If the operator j is applied to the vector j^2**E**, the result is j^3**E** = −j**E**. The vector j^3**E** is 270º counter-clockwise from the reference axis and is directly opposite to j**E**. If the vector j^3**E** is, turn, operated on by j, the result will be

$$j^4\mathbf{E} = \left[\sqrt{(-1)}\right]^4 \mathbf{E} = \mathbf{E}$$

Hence, it is seen that successive applications of the operator j to the vector **E** produce successive 90º steps of rotation of the vector in the counter-clockwise direction without in anyway affecting the magnitude of the vector.

It will also be seen from Fig. 12.2 that the application of −j to **E** yields – j**E** which is a vector of identical magnitude but rotated 90º ***clockwise*** from **E**.

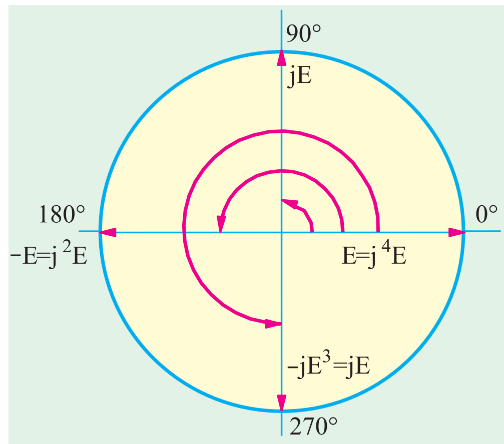

Fig. 12.2

Summarising the above, we have

$$j = 90^\circ \text{ ccw rotation} = \sqrt{(-1)}$$

$$j^2 = 180^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^2 = -1;$$

$$j^3 = 270^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^3 = -\sqrt{(-1)} = -j$$

$$j^4 = 360^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^4 = +1;$$

$$j^5 = 450^\circ \text{ ccw rotation} = [\sqrt{(-1)}]^5 = -\sqrt{(-1)} = j$$

It should also be noted that $\dfrac{1}{j} = \dfrac{j}{j^2} = \dfrac{j}{-1} = -j$

12.4. Conjugate Complex Numbers

Two complex numbers are said to be conjugate if they differ only in the algebraic sign of their quadrature components. Accordingly, the numbers ($a + jb$) and ($a - jb$) are conjugate. The sum of two conjugate numbers gives in-phase (or active) component and their difference gives quadrature (or reactive) component.

12.5. Trigonometrical Form of Vector

From Fig. 12.3, it is seen that ***X-component of* E** is $E \cos \theta$ and Y-component is $E \sin \theta$. Hence, we can represent the vector **E** in the form : **E** = $E(\cos \theta + j \sin \theta)$

This is equivalent to the rectangular form **E** = $a + jb$ because $a = E \cos \theta$ and $b = E \sin \theta$. In general, **E** = $E(\cos \theta \pm j \sin \theta)$.

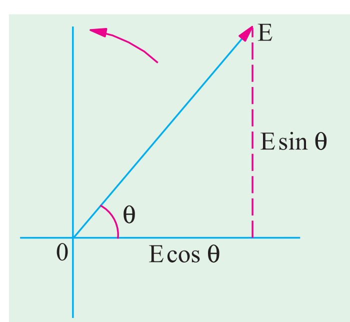

Fig. 12.3

12.6. Exponential Form of Vector

It can be proved that $e^{\pm j\theta} = (\cos \theta \pm j \sin \theta)$

This equation is known as Euler's equation after the famous mathematician of 18th century : Leonard Euler.

This equation follows directly from an inspection of Maclaurin* series expansions of sin θ, cos θ and $e^{j\theta}$.

When expanded into series form :

$$\cos\theta = 1 - \frac{\theta^2}{L\,2} + \frac{\theta^4}{L\,4} - \frac{\theta^6}{L\,6} + \ldots \text{ and } \sin\theta = \theta - \frac{\theta^3}{L\,3} + \frac{\theta^5}{L\,5} - \frac{\theta^7}{L\,7} + \ldots$$

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{L\,2} + \frac{(j\theta)^3}{L\,3} + \frac{(j\theta)^4}{L\,4} + \frac{(j\theta)^5}{L\,5} + \frac{(j\theta)^6}{L\,6} + \ldots$$

Keeping in mind that $j^2 = -1$, $j^3 = -j$, $j^4 = 1$, $j^5 = -j$, $j^6 = -1$, we get

$$e^{j\theta} = \left(1 - \frac{\theta^2}{L\,2} + \frac{\theta^4}{L\,4} - \frac{\theta^6}{L\,6} + \ldots\right) + j\left(\theta - \frac{\theta^3}{L\,3} + \frac{\theta^5}{L\,5} - \frac{\theta^7}{L\,7} + \ldots\right)$$

∴ $e^{j\theta} = \cos\theta + j\sin\theta$

Similarly, it can be shown that $e^{-j\theta} = \cos\theta - j\sin\theta$

Hence **E** = E (cos θ ± y´ sin θ) can be written as **E** = $Ee^{\pm j\theta}$. This is known as exponential form of representing vector quantities. It represents a vector of numerical value E and having phase angle of ± θ with the reference axis.

12.7. Polar Form of Vector Representation

The expression E (cos θ + j sin θ) is written in the simplified form of $E \angle\theta$. In this expression, E represents the magnitude of the vector and θ its inclination (in ccw direction) with the X-axis. For angles in clockwise direction the expression becomes $E \angle -\theta$. In general, the expression is written as $E \angle \pm\theta$. ***It may be pointed out here that E ∠± θ is simply a short-hand or symbolic style of writing $Ee^{\pm j\theta}$.*** Also, the form is purely conventional and does not possess the mathematical elegance of the various other forms of vector representation given above.

Summarizing, we have the following alternate ways of representing vector quantities

(i) Rectangular form (or complex form) **E** = $a + jb$

(ii) Trigonometrical form **E** = E (cos θ ± j sin θ)

(iii) Exponential form **E** = $Ee^{\pm j\theta}$

(iv) Polar form (conventional) **E** = $E \angle \pm\theta$.

Example 12.1. *Write the equivalent exponential and polar forms of vector 3 + j4. How will you illustrate the vector means of diagram ?*

Solution. With reference to Fig. 12.4., magnitude of the vector is = $\sqrt{3^2 + 4^2}$ = 5. tan θ = 4/3.

∴ θ = $\tan^{-1}$ (4/3) = 53.1°

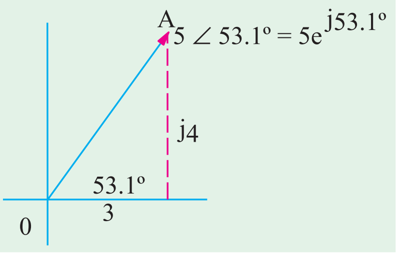

Fig. 12.4

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* Functions like cos θ, sin θ and $e^{j\theta}$ etc. can be expanded into series form with the help of Maclaurin's Theorem. The theorem states : $f(\theta) = f(0) + \frac{f'(0)\theta}{1} + \frac{f''(0)\theta^2}{\angle 2} + \frac{f'''(0)\theta^3}{\angle 3} + \ldots$ where $f(\theta)$ is function of θ which is to be expanded, $f(0)$ is the value of the function when θ = 0, $f'(0)$ is the value of first derivative of $f(\theta)$ when θ = 0, $f''(0)$ is the value of second derivative of function $f(\theta)$ when θ = 0 etc.

$\therefore$ Exponential form = $5\,e^{j53.1°}$

The angle may also be expressed in radians.

Polar form = $5\,\angle 53.1°$.

Example 12.2. *A vector is represented by $20\,e^{-j2\pi/3}$. Write the various equivalent forms of the vector and illustrate by means of a vector diagram, the magnitude and position of the above vector.*

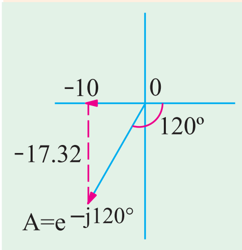

Fig. 12.5

Solution. The vector is drawn in a direction making an angle of $2\pi/3 = 120°$ in the clockwise direction (Fig. 12.5). The clockwise direction is taken because the angle is negative.

(*i*) **Rectangular Form** $a = 20\cos(-120°) = -10$;

$b = 20\sin(-120°) = -17.32$

$\therefore$ Expression is $= (-10 - j17.32)$

(*ii*) **Polar Form** is $20\angle -120°$

12.8. Addition and Subtraction of Vector Quantities

Rectangular form is best suited for addition and subtraction of vector quantities. Suppose we are given two vector quantities $\mathbf{E_1} = a_1 + jb_1$ and $\mathbf{E_2} = a_2 + jb_2$ and it is required to find their sum and difference.

Addition. $\mathbf{E} = \mathbf{E_1} + \mathbf{E_2} = a_1 + jb_1 + a_2 + jb_2 = (a_1 + a_2) + j(b_1 + b_2)$

The magnitude of resultant vector **E** is $\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$

The position of **E** with respect to X-axis is $\theta = \tan^{-1}\left(\dfrac{b_1 + b_2}{a_1 + a_2}\right)$

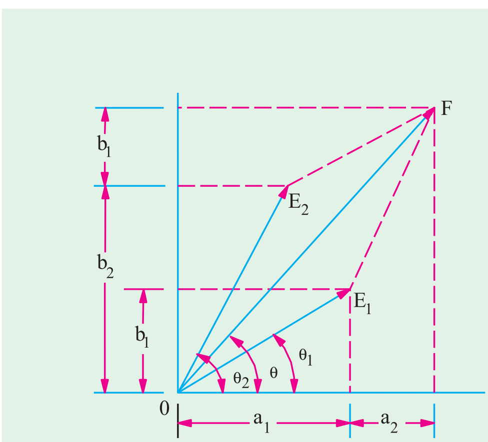

Fig. 12.6

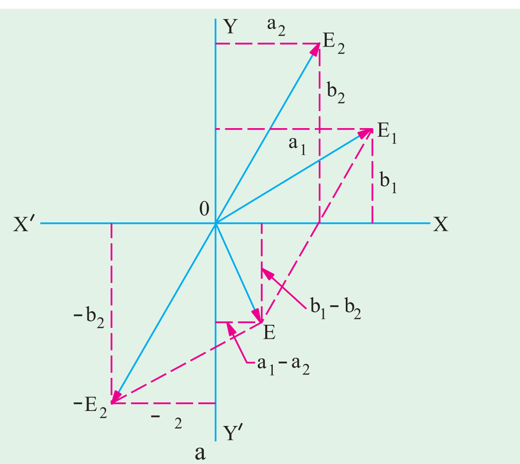

Fig. 12.7

A graphic representation of the addition process is shown in Fig. 12.6

Subtraction. $\mathbf{E} = \mathbf{E_1} - \mathbf{E_2} = (a_1 + jb_1) - (a_2 + jb_2) = (a_1 - a_2) + j(b_1 - b_2)$

Magnitude of $\mathbf{E} = \sqrt{(a_1 - a_2)^2 + (b_1 - b_2)^2}$

Its position with respect to x-axis is given by the angle $\theta = \tan^{-1}\left(\dfrac{b_1 - b_2}{a_1 - a_2}\right)$.

The graphic representation of the process of subtraction is shown in Fig. 12.7.

12.9. Multiplication and Division of Vector Quantities

Multiplication and division of vectors becomes very simple and easy ***if they are represented in the polar or exponential form***. As will be shown below, the rectangular form of representation is not well-suited for this process.

(i) Multiplication – *Rectangular form*

Let the two vectors be given by $\mathbf{A} = a_1 + jb_1$ and $\mathbf{B} = a_2 + jb_2$

$$\therefore \quad \mathbf{A} \times \mathbf{B} = \mathbf{C} = (a_1 + jb_1)(a_2 + jb_2) = a_1a_2 + j^2b_1b_2 + j(a_1b_2 + b_1a_2)$$

$$= (a_1a_2 - b_1b_2) + j(a_1b_2 + b_1a_2) \qquad (\because \; j^2 = -1)$$

The magnitude of $\mathbf{C} = \sqrt{[(a_1a_2 - b_1b_2)^2 + (a_1b_2 + b_1a_2)^2]}$

In angle with respect to X-axis is given by $\theta = \tan^{-1}\left(\dfrac{a_1b_2 + b_1a_2}{a_1a_2 - b_1b_2}\right)$

(ii) Division – *Rectangular Form :* $\dfrac{\mathbf{A}}{\mathbf{B}} = \dfrac{a_1 + jb_1}{a_2 + jb_2} = \dfrac{(a_1 + jb_1)(a_2 - jb_2)}{(a_2 + jb_2)(a_2 - jb_2)}$

Both the numerator and denominator have been multiplied by the conjugate of $(a_2 + jb_2)$ *i.e.* by $(a_2 - jb_2)$

$$\therefore \quad \frac{\mathbf{A}}{\mathbf{B}} = \frac{(a_1a_2 + b_1b_2) + j(b_1a_2 - a_1b_2)}{a_2^2 + b_2^2} = \frac{a_1a_2 + a_1b_2}{a_2^2 + b_2^2} + j\frac{b_1a_2 + a_1b_2}{a_2^2 + b_2^2}$$

The magnitude and the angle with respects X-axis can be found in the same way as given above.

As will be noted, both the results are somewhat awkward but unfortunately, there is no easier way to perform multiplication in rectangular form.

(iii) Multiplication – *Polar Form*

Let $\mathbf{A} = a_1 + jb_1 = A\angle\alpha = A\,e^{j\alpha}$ where $\alpha = \tan^{-1}(b_1/a_1)$

$\mathbf{B} = a_2 + jb_2 = B\angle\beta = B\,e^{j\beta}$ where $\beta = \tan^{-1}(b_2/a_2)$

$\therefore \quad \mathbf{AB} = A\angle\alpha \times B\angle\beta = AB\angle(\alpha + \beta)$* or $AB = Ae^{j\alpha} \times Be^{j\beta} = ABe^{j(\alpha+\beta)}$

Hence, product of any two vector **A** and **B** is given by another vector equal in length to **A** × **B** and having a phase angle equal to the sum of the angles of **A** and **B**.

$$\frac{\mathbf{A}}{\mathbf{B}} = \frac{A\angle\alpha}{B\angle\beta} = \frac{A}{B}\angle(\alpha - \beta)$$

Hence, the quotient **A** ÷ **B** is another vector having a magnitude of **A** ÷ **B** and phase angle equal to angle of **A** minus the angle of **B**.

Also $$\frac{\mathbf{A}}{\mathbf{B}} = \frac{Ae^{j\alpha}}{Be^{j\beta}} = \frac{A}{B}e^{j(\alpha-\beta)}$$

As seen, the division and multiplication become extremely simple if vectors are represented in their polar or exponential form.

Example 12.3. *Add the following vectors given in rectangular form and illustrate the process graphically.*

$$A = 16 + j\,12, \quad B = -6 + j\,10.4$$

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* $\mathbf{A} = A(\cos\alpha + j\sin\alpha)$ and $\mathbf{B} = B(\cos\beta + j\sin\beta)$

$\therefore \quad \mathbf{AB} = AB(\cos\alpha\cos\beta + j\sin\alpha\cos\beta + j\cos\alpha\sin\beta + j^2\sin\alpha\sin\beta)$

$= AB[\cos\alpha\cos\beta - \sin\alpha\sin\beta) + j(\sin\alpha\cos\beta + \cos\alpha\sin\beta)]$

$= AB[\cos(\alpha + \beta) + j\sin(\alpha + \beta)] = AB\angle(\alpha + \beta)$

Solution. $\mathbf{A + B} = C = (16 + j\,12) + (-6 + j\,10.4) = 10 + j\,22.4$

$\therefore$ Magnitude of $\mathbf{C} = \sqrt{(10^2 + 22.4^2)} = 25.5$ units

Slope of $\mathbf{C} = \theta = \tan^{-1}\left(\frac{22.4}{10}\right) = 65.95°$

The vector addition is shown in Fig. 12.8.

$\alpha = \tan^{-1}(12/16) = 36.9°$

$\beta = \tan^{-1}(-10.4/6) = -240°$ or $120°$

The resultant vector is found by using parallelogram law of vectors (Fig. 12.8).

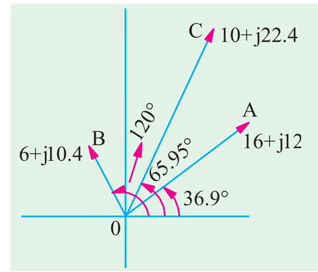

Fig. 12.8

Example 12.4. *Perform the following operation and express the final result in the polar form : $5 \angle 30° + 8 \angle -30°$.* **(Elect. Engg. & Electronics Bangalore Univ. 1989)**

Solution. $5 \angle 30° = 5(\cos 30° + j \sin 30°) = 4.33 + j\,2.5$

$8 \angle -30° = 8[\cos(-30°) + j \sin(-30°)] = 8(0.866 - 0.5) = 6.93 - j\,4$

$\therefore 5\angle 30° + 8\angle 30° = 4.33 + j2.5 + 6.93 - j4 = 11.26 - j1.5 = \sqrt{11.26^2 + 1.5^2}\ \angle \tan^{-1}(-1.5/11.26) = 11.35$

$\tan^{-1}(-0.1332) = 11.35 \angle 7.6°$

Example 12.5. *Subtract the following given vectors from one another :*

$A = 30 + j\,52$ and $B = -39.5 - j\,14.36$

Solution. $\mathbf{A - B} = \mathbf{C} = (30 + j\,52) - (-39.5 - j\,14.36) = 69.5 + j\,66.36$

$\therefore$ Magnitude of $C = \sqrt{(66.5^2 + 66.36^2)} = 96$

Slope of $C = \tan^{-1}(66.36/69.5) = 43.6°$ $\quad\therefore\ \mathbf{C} = 96 \angle 43.6°$.

Similarly $\mathbf{B - A} = -69.5 - j\,66.36 = 96 \angle 223.6°$* or $= 96 \angle -136.4°$

Example 12.6. *Given the following two vectors :*

$A = 20 \angle 60°$ and $B = 5 \angle 30°$

Perform the following indicated operations and illustrate graphically ***(i)*** *$A \times B$ and* ***(ii)*** *A/B.*

Solution. (*i*) $\mathbf{A \times B} = \mathbf{C} = 20 \angle 60° \times 5 \angle 30° = 100 \angle 90°$

Vectors are shown in Fig. 12.9.

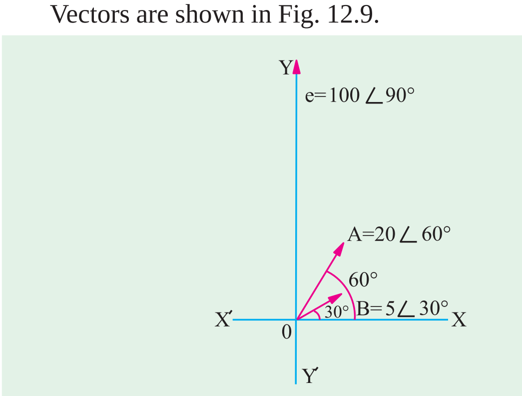

Fig. 12.9

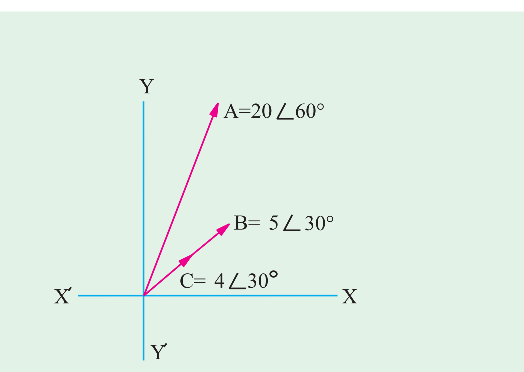

Fig. 12.10

* $\tan\theta = 66.36/69.5$ or $\theta = \tan^{-1}(66.36/69.5) = 43.6°$. Since both components are negative, the vector lies in third quadrant. Hence, the angle measured from + ve direction of X-axis and in the CCW directions is $= (180 + 43.6) = 223.6°$.

(ii) $\dfrac{\mathbf{A}}{\mathbf{B}} = \dfrac{20\angle 60°}{5\angle 30°} = 4\angle 30°$ —Fig. 12.10

Example 12.7. *Perform the following operation and the final result may be given in the polar form : (8 + j6) × (–10 –j 7.5)* **(Elect. Engg. & Electronics Bangalore Univ. 1990)**

Solution. We will use the following two methods to solve the above question.

Method No. 1

We know that multiplication of $(A + B)$ and $(C + D)$ can be found as under :

$$\begin{array}{llll} & A + B & & \\ \times & C + D & & \\ \hline & CA & + CB & \\ & & + DA + DB & \\ \hline & & CA + CD + DA + DB & \end{array}$$

Similarly, the required multiplication can be carried out as follows :

$$\begin{array}{l} 8 + j\,6 \\ \times -10 \;\; -j\,7.5 \\ \hline -80 \;\; -j\,60 \\ -j\,60 \; - j^2\,45 \\ \hline -80 \; -j120 + 45 \end{array}$$

or $-35 - j120 = \sqrt{(-35)^2 + (-120)^2}\;\underline{|\tan^{-1}(120/35)} = 125\tan^{-1} 3.42 = 125\,\underline{|73.8°}$

Since both the components of the vector are negative, it obviously lies in the third quadrant. As measured from the *X*-axis in the *CCW* direction, its angle is = 180° + 73.8° = 253.8°. Hence, the product vector can be written as 125 ∠ 53.8°.

Method No. 2 $8 + j\,6 = 10 \angle 36.9°$, $- 10 - j\,7.5 = 12.5 \tan^{-1} 0.75 = 12.5 \angle 36.9°$.

Again as explained in Method 1 above, the actual angle of the vector is 180° + 36.9° = 216.9°

∴ $-10 - j\,7.5 = 12.5 \angle 216.9$ ∴ $10 \angle 36.9° \times 12.5 \angle 216.9° = 125 < 253.8°$

Example 12.8. *The following three vectors are given :*

$$A = 20 + j20,\; B = 30 \angle -120 \text{ and } C = 10 + j0$$

Perform the following indicated operations :

(i) $\dfrac{AB}{C}$ and **(ii)** $\dfrac{BC}{A}$.

Solution. Rearranging all three vectors in polar form, we get

$$\mathbf{A} = 28.3 \angle 45°,\; \mathbf{B} = 30 \angle -120°,\; \mathbf{C} = 10 \angle 0°$$

(i) $\dfrac{\mathbf{AB}}{\mathbf{C}} = \dfrac{28.3\angle 45° \times 30\angle -120°}{10\angle 0°} = 84.9\angle -75°$

(ii) $\dfrac{\mathbf{BC}}{\mathbf{A}} = \dfrac{30\angle -120 \times 10\angle 0°}{28.3\angle 45°} = 10.6\angle -165°$

Example 12.9. *Given two current $i_1 = 10 \sin(\omega t + \pi/4)$ and $i_2 = 5\cos(\omega t - \pi/2)$, find the r.m.s. value of $i_1 + i_2$ using the complex number representation.* **[Elect. Circuit Theory, Kerala Univ.]**

Solution. The maximum value of first current is 10 A and it leads the reference quantity by 45°. The second current can be written as

$$i_2 = 5\cos(\omega t - \pi/2) = 5\sin[90 + (\omega t - \pi/2)] = 5\sin\omega t$$

Hence, its maximum value is 5 A and is in phase with the reference quantity.

∴ $\mathbf{I}_{m1} = 10(\cos 45° + j\sin 45°) = (7.07 + j\,7.07)$

$$\mathbf{I}_{m2} = 5\,(\cos 0° + j \sin 0°) = (5 + j\,0)$$

The maximum value of resultant current is

$$\mathbf{I}_m = (7.07 + j\,7.07) + (5 + j\,0) = 12.07 + j\,7.07 = 14 \angle 30.4°$$

$\therefore$ R.M.S. value $= 14/\sqrt{2} =$ **10 A**

12.10. Power and Roots of Vectors

(a) Powers

Suppose it is required to find the cube of the vector $3 \angle 15°$. For this purpose, the vector has to be multiplied by itself three times.

$\therefore$ $(3 \angle 15)^3 = 3 \times 3 \times 3.\angle(15° + 15° + 15°) = 27 \angle 45°$. In general, $\mathbf{A}^n = \mathbf{A}^n \angle n\alpha$

Hence, nth power of vector **A** is a vector whose magnitude is A^n and whose phase angle with respect to X-axis is $n\alpha$.

It is also clear that $\mathbf{A}^n\mathbf{B}^n = A^nB^n \angle (n\alpha + n\beta)$

(b) Roots

It is clear that $\sqrt[3]{(8 \angle 45°)} = 2 \angle 15°$

In general, $\sqrt[n]{\mathbf{A}} = \sqrt[n]{A} = \angle \alpha/n$

Hence, nth root of a vector **A** is a vector whose magnitude is $n\sqrt{A}$ and whose phase angle with respect to X-*axis is* α/n.

12.11. The 120° Operator

In three-phase work where voltage vectors are displaced from one another by 120°, it is convenient to employ an operator which rotates a vector through 120° forward or backwards without changing its length. This operator is 'a'. Any operator which is multiplied by 'a' remains unchanged in magnitude but is rotated by 120° in the counter-clockwise (ccw) direction.

$$\therefore \quad \alpha = 1 \angle 120°$$

This, when expressed in the cartesian form, becomes

$$a = \cos 120° + j \sin 120° = -0.5 + j\,0.866$$

Similarly, $a^2 = 1 \angle 120° \times 1 \angle 120° = 1 \angle 240° = \cos 240° + j \sin 240° = -0.5 - j\,0.866$

Hence, operator 'a^2' will rotate the vector in ccw by 240°. This is the same as rotating the vector in *clockwise* direction by 120°.

$\therefore$ $a^2 = 1 \angle -120°$. Similarly, $a^3 = 1 \angle 360° = 1$*

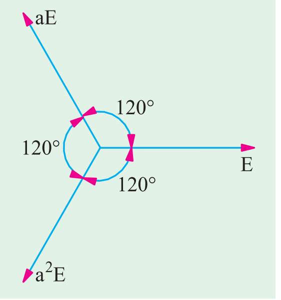

Fig. 12.11

As shown in Fig. 12.11, the 3-phase voltage vectors with standard phase sequence may be represented as E, a^2E and aE or as E, $E\,(-0.5 - j\,0.866)$ and $E\,(-0.5 + j\,0.866)$

It is easy to prove that

(*i*) $a^2 + a = -1$ (*ii*) $a^2 + a + 1 = 0$ (*iii*) $a^3 + a^2 + a = 0$

Note. We have seen in Art. 12.3 that operator $-j$ turns a vector through $-90°$ *i.e.* through 90° in *clockwise* direction. But it should be clearly noted that operator '$-a$' *does not* turn a vector through $-120°$. Rather '$-a$' turns a vector through $-60°$ as shown below.

Example 12.10. *Evaluate the following expressions in the polar form* (*i*) $a^2 - 1$, (*ii*) $1 - a - a^2$ (*iii*) $2a^2 + 3 + a$ (*iv*) ja. **[Elect. Meas and Meas. Inst., Madras Univ.]**

* Numerically, a is equivalent to the cube root of unity.

Solution. **(*i*)** $a^2 = a \times a = 1\angle 120° = 1\angle 240° = 1\angle -120° = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$

$$\therefore \quad a^2 - 1 = \quad \frac{1}{2} \quad j\frac{\sqrt{3}}{2} \quad 1 \quad j\frac{\sqrt{3}}{2}\sqrt{3} \quad \frac{\sqrt{3}}{2} \quad j\frac{1}{2} \quad \sqrt{3} \qquad 180 \quad \sqrt{\mathbf{3}} \quad \mathbf{210}$$

(*ii*)

$$a = 1\angle 120° = -\frac{1}{2} + j\frac{\sqrt{3}}{2};\ a^2 = 1\angle 240° = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$\therefore \quad 1 - a - a^2 = 1 \quad \frac{1}{2} \quad j\frac{\sqrt{3}}{2} \quad \frac{1}{2} \quad j\frac{\sqrt{3}}{2} \quad 2 \quad j\,0 \quad \mathbf{2} \quad \mathbf{0}$$

(*iii*)

$$2\,a^2 = 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = -1 - j\sqrt{3}$$

$$2\,a = 2 \quad \frac{1}{2} \quad j\frac{\sqrt{3}}{2} \qquad \mathbf{1} \quad \boldsymbol{j}\sqrt{\mathbf{3}}$$

$$\therefore \quad 2\,a^2 + 3 + 2\,a = 3 - 1 - j\sqrt{3} - 1 + j\sqrt{3} = 1\angle 0°$$

(*iv*)

$$ja = j \times a = 1\angle 90° \times 1\angle 120° = 1\angle 210°$$

Tutorial Problem No. 12.1

1. Perform the following indicated operations :
   (*a*) (60 + *j* 80) + (30 – *j* 40) (*b*) (12 – *j* 6) – (40 – *j* 20) (*c*) (6 + *j* 8) (3 – *j* 4) (*d*) 16 + *j*8) ÷ (3 – *j* 4)
   [(*a*) (90 + *j* 40) (*b*) (–28 + *j* 14) (*c*) (50 + *j* 0) (*d*) (–0.56 + *j* 1.92)]
2. Two impedances $\mathbf{Z}_1$ = 2 + *j* 6 Ω and $\mathbf{Z}_2$ = 6 – 12 Ω are connected in a circuit so that they are additive. Find the resultant impedance in the polar form. **[10 ∠ – 36.9°]**
3. Express in rectangular form and polar form a vector, the magnitude of which is 100 units and the phase of which with respect to reference axis is
   (*a*) + 30° (*b*) + 180° (*c*) –60° (*d*) + 120° (*e*) –120° (*f*) –210°.
   [(*a*) 86.6 + *j*50 ∠30° (*b*) (–100 + *j*0), 100 ∠180° (*c*) 50 –*j*86.6, 100 ∠–60° (*d*) (–50 + *j*86.6), 100 ∠–120° (*e*) (–50 – *j* 86.6), 100 ∠– 120° (*f*) (–50 + *j* 86.6), 100 ∠– 210°]
4. In the equation $\mathbf{V}_m = \mathbf{V} - \mathbf{ZI}$, **V** = 100 ∠ 0° volts, **Z** = 10 ∠ 60° Ω and **I** = 8 ∠– 30° amperes. Express $\mathbf{V}_m$ in polar form. **[50.5 ∠ – 52°]**
5. A voltage **V** = 150 + *j* 180 is applied across an impedance and the current flowing is found to be **1** = 5 – *j* 4. Determine (*i*) scalar impedance (*ii*) resistance (*iii*) reactance (*iv*) power consumed.
   [(*i*) 3.73 Ω (*ii*) 0.75 Ω (*iii*) 36.6 Ω (*iv*) 30 W]
6. Calculate the following in polar form :
   (*i*) Add (40 + j20) to (20 + j120) (*ii*) Subtract (10 + j30) from (20 – j20)
   (*iii*) Multiply (15 + j20) with (20 + j30) (*iv*) Devide (6 + j7) by (5 + j3)
   (Gujrat University, June/July 2003)
7. Why is impedance represented by a complex number? ***(RGPV Bhopal December 2002)***

OBJECTIVE TESTS – 12

1. The symbol *j* represents counterclockwise rotation of a vector through—degrees.
   (*a*) 180 (*b*) 90
   (*c*) 360 (*d*) 270
2. The operator *j* has a value of
   (*a*) + 1 (*b*) – 1
   (*c*) $\sqrt{-1}$ (*d*) $\sqrt{+1}$
3. The vector $j^5 E$ is the same as vector
   (*a*) *jE* (*b*) $j^2 E$
   (*c*) $j^3 E$ (*d*) $j^4 E$
4. The conjugate of (–*a* + *jb*) is
   (*a*) (*a* – *jb*) (*b*) (– *a* – *jb*)
   (*c*) (*a* + *jub*) (*d*) (*jb* – *a*)

ANSWERS

1. *b* **2.** *c* **3.** *a* **4.** *b*

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