Chapter 07.07
Integrating Improper Functions

After reading this chapter, you should be able to:

1. *integrate improper functions using methods such as the trapezoidal rule and Gaussian Quadrature schemes.*

What is integration?

Integration is the process of measuring the area under a function plotted on a graph. Why would we want to integrate a function? Among the most common examples are finding the velocity of a body from an acceleration function, and displacement of a body from a velocity function. Throughout many engineering fields, there are (what sometimes seems like) countless applications for integral calculus. You can read about some of these applications in Chapters 07.00A-07.00G.

Sometimes, the evaluation of expressions involving these integrals can become daunting, if not indeterminate. For this reason, a wide variety of numerical methods has been developed to simplify the integral.

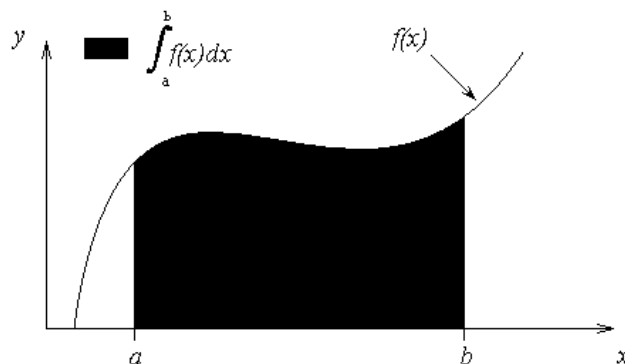Here, we will discuss the incorporation of these numerical methods into improper integrals.

Figure 1 Integration of a function

What is an improper integral?

An integral is improper if
   a) the integrand becomes infinite in the interval of integration (including end points) or/and
   b) the interval of integration has an infinite bound.

Example 1

Give some examples of improper integrals
Solution

The integral

$$I = \int_0^2 \frac{x}{\sqrt{4 - x^2}} dx$$

is improper because the integrand becomes infinite at $x = 2$.

The integral

$$I = \int_0^2 \frac{x}{\sqrt{1 - x}} dx$$

is improper because the integrand becomes infinite at $x = 1$.
The integral

$$I = \int_0^\infty e^{-t} t \, dt$$

is improper because the interval of integration has an infinite bound.
The integral

$$I = \int_0^\infty \frac{e^{-t}}{\sqrt{1 - t}} dt$$

is improper because the interval of integration has an infinite bound and the integrand is infinite at $t = 1$.

       If the integrand is undefined at a finite number of points, the value of the area under the curve does not change. Hence such integrals could theoretically be solved either by assuming any value of the integrand at such points. Also, methods such as Gauss quadrature rule do not use the value of the integrand at end points, and hence integrands that are undefined at end points can be integrated using such methods.
       For the case where there is an infinite interval of integration, one may make a change of variables that transforms the infinite range of integration to a finite one.
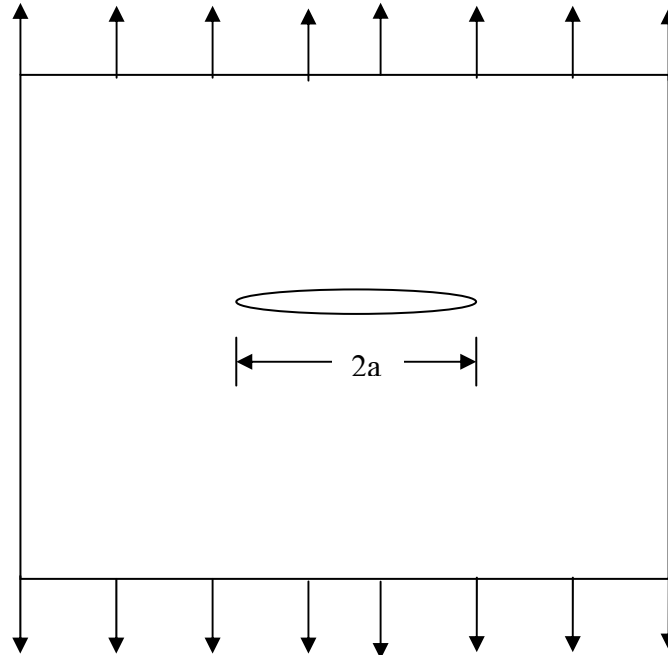   Let us illustrate these two cases with examples.

Figure 2 A plate with a crack under a uniform axial load

Example 2

In analyzing fracture of metals, one wants to know the opening displacement of cracks. In a large plate, if there is a crack length of $2a$ meters, then the maximum crack opening displacement (MCOD) is given by

$$\text{MCOD} = \frac{2\sigma}{E} \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \, dx$$

where

$\sigma =$ remote normal applied stress
$E =$ Young's modulus

Assume

$a = 0.02$ m
$E = 210$ GPa and
$\sigma = 70$ MPa .

Find the exact value of the maximum crack opening displacement.
Solution

The maximum crack opening displacement (MCOD) is given by

$$\text{MCOD} = \frac{2\sigma}{E} \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \, dx$$

Substituting $a = 0.02$ m , $E = 210$ GPa and $\sigma = 70$ MPa gives

$$\text{MCOD} = \frac{2\left(70\times10^6\right)^{0.02}}{210\times10^9}\int\limits_0^{0.02}\frac{x}{\sqrt{(0.02)^2-x^2}}\,dx$$

$$= \frac{1}{1500}\int\limits_0^{0.02}\frac{x}{\sqrt{0.0004-x^2}}\,dx$$

The exact value of the integral then is

$$MCOD = \frac{1}{1500}\left[-\sqrt{0.0004-x^2}\right]_0^{0.02}$$

$$= \frac{1}{1500}\left(-0+0.02\right)$$

$$= 1.3333\times10^{-5}\ \text{m}$$

Example 3

Any of the Newton-Cotes formulas, such as Trapezoidal rule and Simpson's 1/3 rule, cannot be used directly for integrals where the integrands become infinite at the ends of the intervals. Since Gauss quadrature rule does not require calculation of the integrand at the end points, it could be used directly to calculate such integrals. Knowing this, find the value of the integral

$$\frac{1}{1500}\int\limits_0^{0.02}\frac{x}{\sqrt{0.0004-x^2}}\,dx$$

from Example 2 by using two-point Gauss quadrature rule.

Solution

We will change the limits of integration from $[0,0.02]$ to $[-1,1]$, such that we may use the tabulated values of c_1, c_2, x_1, and x_2. Assigning

$$f(x) = \frac{x}{\sqrt{0.0004-x^2}}\,,$$

we get

$$\frac{1}{1500}\int\limits_0^{0.02}f(x)dx = \frac{1}{1500}\frac{0.02-0}{2}\int\limits_{-1}^1 f\left(\frac{0.02-0}{2}x+\frac{0.02+0}{2}\right)dx$$

$$= \frac{1}{150000}\int\limits_{-1}^1 f(0.01x+0.01)dx$$

The function arguments and weighting factors for two-point Gauss quadrature rule are

$$c_1 = 1.000000000$$
$$x_1 = -0.577350269$$
$$c_2 = 1.000000000$$
$$x_2 = 0.577350269$$

Giving us a formula of

$$\frac{1}{150000}\int_{-1}^{1}f(0.01x+0.01)dx \approx \frac{1}{150000}c_1 f(0.01x_1+0.01)+\frac{1}{150000}c_2 f(0.01x_2+0.01)$$

$$=\frac{1}{150000}f(0.01(-0.57735)+0.01)+\frac{1}{150000}f(0.01(0.57735)+0.01)$$

$$=\frac{1}{150000}f(0.0042265)+\frac{1}{150000}f(0.0157735)$$

$$=\frac{1}{150000}(0.21621)+\frac{1}{150000}(1.28279)$$

$$=9.9934\times10^{-6}\text{ m}$$

since

$$f(0.0042265)=\frac{0.0042265}{\sqrt{0.0004-(0.0042265)^2}}=0.21621$$

$$f(0.0157735)=\frac{0.0157735}{\sqrt{0.0004-(0.0157735)^2}}=1.28279$$

The absolute relative true error, $|\epsilon_t|$, is

$$|\epsilon_t|=\left|\frac{1.3333\times10^{-5}-9.9934\times10^{-6}}{1.3333\times10^{-5}}\right|\times100\%$$

$$=25.048\%$$

Example 4

The value of the integral

$$\frac{1}{1500}\int_{0}^{0.02}\frac{x}{\sqrt{0.0004-x^2}}dx$$

in Example 3 by using two-point Gauss quadrature rule has a large absolute relative true error of more than 25%. Use the double-segment two-point Gauss quadrature rule to find the value of the integral. Take the interval $[0,\,0.02]$ and split it into two equal segments of $[0,\,0.01]$ and $[0.01,\,0.02]$, and then apply the two-point Gauss quadrature rule over each segment.

Solution

Write the integral with interval of $[0,0.02]$ as sum of two integrals with intervals $[0,0.01]$ and $[0.01,0.02]$ gives

$$\frac{1}{1500}\int_{0}^{0.02}f(x)dx=\frac{1}{1500}\int_{0}^{0.01}f(x)dx+\frac{1}{1500}\int_{0.01}^{0.02}f(x)dx$$

$$=\frac{1}{1500}\frac{0.01-0}{2}\int_{-1}^{1}f\left(\frac{0.01-0}{2}x+\frac{0.01+0}{2}\right)dx$$

$$+\frac{1}{1500}\frac{0.02-0.01}{2}\int_{-1}^{1}f\left(\frac{0.02-0.01}{2}x+\frac{0.02+0.01}{2}\right)dx$$

$$= \frac{1}{300000} \int_{-1}^{1} f(0.005x + 0.005)dx + \frac{1}{300000} \int_{-1}^{1} f(0.005x + 0.015)dx$$

Using the two-point Gauss quadrature rule, this becomes

$$\frac{1}{1500} \int_{0}^{0.02} f(x)dx \approx \frac{1}{300000} c_1 f(0.005x_1 + 0.005) + \frac{1}{300000} c_2 f(0.005x_2 + 0.005)$$

$$+ \frac{1}{300000} c_1 f(0.005x_1 + 0.015) + \frac{1}{300000} c_2 f(0.005x_2 + 0.015)$$

Using the same arguments and weighting factors as before

$$\frac{1}{1500} \int_{0}^{0.02} f(x)dx \approx \frac{1}{300000} f(0.005(-0.57735) + 0.005) + \frac{1}{300000} f(0.005(0.57735) + 0.005)$$

$$+ \frac{1}{300000} f(0.005(-0.57735) + 0.015) + \frac{1}{300000} f(0.005(0.57735) + 0.015)$$

$$= \frac{1}{300000} f(0.0021132) + \frac{1}{300000} f(0.0078868)$$

$$+ \frac{1}{300000} f(0.0121132) + \frac{1}{300000} f(0.0178868)$$

$$= \frac{1}{300000} (0.10626 + 0.42911 + 0.76115 + 1.99900)$$

$$= 1.0985 \times 10^{-5} \text{ m}$$

since

$$f(0.0021133) = \frac{0.0021133}{\sqrt{0.0004 - (0.0021133)^2}} = 0.10626$$

$$f(0.0078868) = \frac{0.0078868}{\sqrt{0.0004 - (0.0078868)^2}} = 0.42911$$

$$f(0.0121133) = \frac{0.0121133}{\sqrt{0.0004 - (0.0121133)^2}} = 0.76115$$

$$f(0.0178868) = \frac{0.0178868}{\sqrt{0.0004 - (0.0178868)^2}} = 1.99900$$

The absolute relative true error, $|\epsilon_t|$, is

$$|\epsilon_t| = \left| \frac{1.3333 \times 10^{-5} - 1.0985 \times 10^{-5}}{1.3333 \times 10^{-5}} \right| \times 100\%$$

$$= 17.610\%$$

Repeating this process by splitting the interval into progressively more equal segments and applying the two-point Gaussian quadrature rule over each segment will obtain the data displayed in Table 1.

Table 1 Gauss quadrature rule on an improper integral

$$\left(\frac{1}{1500}\int\limits_{0}^{0.02}\frac{x}{\sqrt{0.0004-x^2}}\,dx\right)$$

Number of Segments	Value	$\left|\in_t\right|\%$
1	9.9934×10^{-6}	25.05
2	1.0985×10^{-5}	17.61
3	1.1420×10^{-5}	14.35
4	1.1679×10^{-5}	12.41
5	1.1855×10^{-5}	11.09
6	1.1984×10^{-5}	10.12
7	1.2085×10^{-5}	9.365
8	1.2166×10^{-5}	8.758

As evident from Table 1, the integral does not converge rapidly to the true value with an increase in number of quadrature points. Since the integrand becomes infinite at the end point $x=0.02$, its value changes rapidly near $x=0.02$. Since the multiple-segment two-point Gauss quadrature rule is non-adaptive, it will take a large number of segments to reach a converging value.

Example 5

Euler's constant in mathematics is defined as

$$\Gamma(x)=\int\limits_{0}^{\infty}e^{-t}t^{x-1}dt$$

Find $\Gamma(2.4)$ using two and three-point Gauss quadrature rules. Also, find the absolute relative true error for each case.
Solution

$$\Gamma(2.4)=\int\limits_{0}^{\infty}e^{-t}t^{2.4-1}dt$$

$$=\int\limits_{0}^{\infty}e^{-t}t^{1.4}dt$$

To solve the above improper integral, one may make a change of variables as

$$y=\frac{1}{1+t}$$

giving

$$t=\frac{1}{y}-1$$

$$dt=-\frac{1}{y^2}dy$$

At $t = 0, y = 1$, at $t = \infty, y = 0$. So the integral can be re-written as

$$\Gamma(2.4) = \int_1^0 e^{-\left(\frac{1}{y}-1\right)} \left(\frac{1}{y} - 1\right)^{1.4} \left(-\frac{1}{y^2}\right) dy$$

First, assigning

$$f(y) = e^{-\left(\frac{1}{y}-1\right)} \left(\frac{1}{y} - 1\right)^{1.4} \left(-\frac{1}{y^2}\right)$$

and then changing the limits of integration, we get

$$\Gamma(2.4) = \frac{0-1}{2} \int_{-1}^1 f\left(\frac{0-1}{2}y + \frac{0+1}{2}\right) dy$$

$$= -0.5 \int_{-1}^1 f(-0.5y + 0.5) dy$$

Now, one can use two-point Gauss Quadrature Rule to find the value of $\Gamma(2.4)$ with weighting factors and function arguments of

$c_1 = 1.000000000$

$y_1 = -0.577350269$

$c_2 = 1.000000000$

$y_2 = 0.577350269$

$$\Gamma(2.4) \approx -0.5c_1 f(-0.5y_1 + 0.5) - 0.5c_2 f(-0.5y_2 + 0.5)$$
$$= -0.5f(-0.5(-0.57735) + 0.5) - 0.5f(-0.5(0.57735) + 0.5)$$
$$= -0.5f(0.78868) - 0.5f(0.21133)$$
$$= -0.5(-0.19458) - 0.5(-3.38857)$$
$$= 1.7916$$

since

$$f(0.78868) = e^{-\left(\frac{1}{0.78868}-1\right)} \left(\frac{1}{0.78868} - 1\right)^{1.4} \left(-\frac{1}{(0.78868)^2}\right)$$

$$= -0.19458$$

$$f(0.21133) = e^{-\left(\frac{1}{0.21133}-1\right)} \left(\frac{1}{0.21133} - 1\right)^{1.4} \left(-\frac{1}{(0.21133)^2}\right)$$

$$= -3.38857$$

The true value of the integral

$$\Gamma(2.4) = \int_0^\infty e^{-t} t^{1.4} dt = 1.2422$$

so the absolute relative true error, $\left|\in_t\right|$, is

$$\left|\in_t\right| = \left|\frac{1.2422 - 1.7916}{1.2422}\right| \times 100\%$$

$$= 44.230\%$$

For three-point Gauss Quadrature Rule, the weighting factors and function arguments are

$$c_1 = 0.555555556$$
$$y_1 = -0.774596669$$
$$c_2 = 0.888888889$$
$$y_2 = 0.000000000$$
$$c_3 = 0.555555556$$
$$y_3 = 0.774596669$$

The limits of integration and $f(y)$ remain the same as for the two-point rule, so

$$\Gamma(2.4) \approx -0.5c_1 f(-0.5y_1 + 0.5) - 0.5c_2 f(-0.5y_2 + 0.5) - 0.5c_3 f(-0.5y_3 + 0.5)$$
$$= -0.5(0.55556)f(-0.5(-0.77460) + 0.5)$$
$$\quad - 0.5(0.88889)f(-0.5(0) + 0.5) - 0.5(0.55556)f(-0.5(0.77460) + 0.5)$$
$$= -0.27778 f(0.88730) - 0.44444 f(0.5) - 0.27778 f(0.11270)$$
$$= -0.27778(-0.06224) - 0.44444(-1.47152) - 0.27778(-0.53890)$$
$$= 0.82100$$

since

$$f(0.88730) = e^{-\left(\frac{1}{0.88730} - 1\right)} \left(\frac{1}{0.88730} - 1\right)^{1.4} \left(-\frac{1}{(0.88730)^2}\right)$$
$$= -0.06224$$

$$f(0.5) = e^{-\left(\frac{1}{0.5} - 1\right)} \left(\frac{1}{0.5} - 1\right)^{1.4} \left(-\frac{1}{(0.5)^2}\right)$$
$$= -1.47152$$

$$f(0.11270) = e^{-\left(\frac{1}{0.11270} - 1\right)} \left(\frac{1}{0.11270} - 1\right)^{1.4} \left(-\frac{1}{(0.11270)^2}\right)$$
$$= -0.53894$$

The absolute relative true error, $|\in_t|$, is

$$|\in_t| = \left|\frac{1.2422 - 0.82099}{1.2422}\right| \times 100\%$$
$$= 33.906\%$$

Example 6

As you can see from the plot given in Figure 3 for the integrand in $\int_0^\infty e^{-t}t^{1.4}\,dt$ of Example 5, once the value of t exceeds 10, the area under the curve looks insignificant. What would happen if you used the two-segment two-point Gauss quadrature rule within the significant range of $[0,10]$?
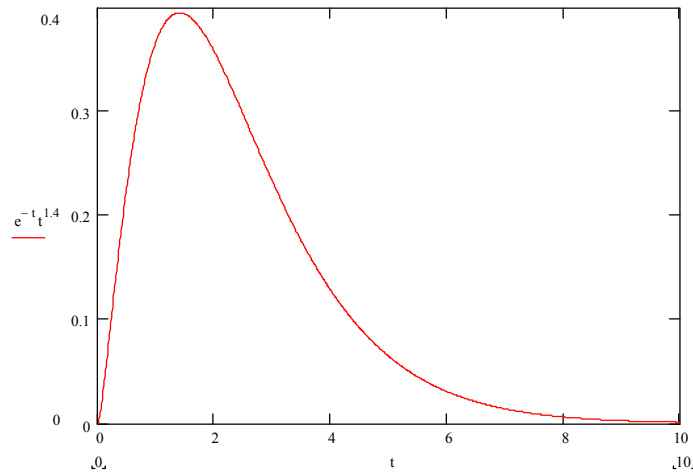
Figure 3 Plot of integrand

Solution

In doing this, no change of variables is necessary—only a change in the limits of each segment is needed to apply Gauss quadrature rule. Observe

$$\Gamma(2.4) = \int_0^\infty e^{-t} t^{1.4} dt$$

$$\approx \int_0^{10} e^{-t} t^{1.4} dt$$

$$= \int_0^{2.4} e^{-t} t^{1.4} dt + \int_{2.4}^{10} e^{-t} t^{1.4} dt$$

Setting $f(t) = e^{-t} t^{1.4}$ to make the change of variables, we get

$$\Gamma(2.4) \approx \frac{2.4 - 0}{2} \int_{-1}^1 f\left(\frac{2.4 - 0}{2} t + \frac{2.4 + 0}{2}\right) dt + \frac{10 - 2.4}{2} \int_{-1}^1 f\left(\frac{10 - 2.4}{2} t + \frac{10 + 2.4}{2}\right) dt$$

$$= 1.2 \int_{-1}^1 f(1.2t + 1.2) dt + 3.8 \int_{-1}^1 f(3.8t + 6.2) dt$$

Applying two-point Gauss quadrature rule gets

$$c_1 = 1.000000000$$
$$t_1 = -0.577350269$$
$$c_2 = 1.000000000$$
$$t_2 = 0.577350269$$

$$\Gamma(2.4) \approx 1.2 c_1 f(1.2t_1 + 1.2) + 1.2 c_2 f(1.2t_2 + 1.2) + 3.8 c_1 f(3.8t_1 + 6.2) + 3.8 c_2 f(3.8t_2 + 6.2)$$
$$= 1.2 f(1.2(-0.57735) + 1.2) + 1.2 f(1.2(0.57735) + 1.2)$$
$$+ 3.8 f(3.8(-0.57735) + 6.2) + 3.8 f(3.8(0.57735) + 6.2)$$
$$= 1.2 f(0.50718) + 1.2 f(1.89282) + 3.8 f(4.00607) + 3.8 f(8.39393)$$
$$= 1.2(0.23279) + 1.2(0.36805) + 3.8(0.12706) + 3.8(0.00445)$$
$$= 1.2207$$

since

$$f(0.50718) = e^{-0.50718}0.50718^{1.4} = 0.23279$$
$$f(1.89282) = e^{-1.89282}1.89282^{1.4} = 0.36805$$
$$f(4.00607) = e^{-4.00607}4.00607^{1.4} = 0.12706$$
$$f(8.39393) = e^{-8.39393}8.39393^{1.4} = 0.00445$$

The absolute relative true error, $|\epsilon_t|$, is

$$|\epsilon_t| = \left| \frac{1.2422 - 1.2207}{1.2422} \right| \times 100\%$$
$$= 1.731\%$$

INTEGRATION	
Topic	Integrating improper functions
Summary	These are textbook notes of integrating improper functions
Major	General Engineering
Authors	Autar Kaw, Michael Keteltas
Last Revised	December 23, 2009
Web Site	http://numericalmethods.eng.usf.edu