Chapter 10.02
Parabolic Partial Differential Equations

After reading this chapter, you should be able to:

1. *Use numerical methods to solve parabolic partial differential equations by explicit, implicit, and Crank-Nicolson methods.*

The general second order linear PDE with two independent variables and one dependent variable is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D = 0 \tag{1}$$

where A, B, C are functions of the independent variables, x , y, and D can be a function of $x, y, u, \frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. If $B^2 - 4AC = 0$, Equation (1) is called a parabolic partial differential equation. One of the simple examples of a parabolic PDE is the heat-conduction equation for a metal rod (Figure 1)

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \tag{2}$$

where

$T =$ temperature as a function of location, x and time, t

in which the thermal diffusivity, α is given by

$$\alpha = \frac{k}{\rho C}$$

where

$k =$ thermal conductivity of rod material,

$\rho =$ density of rod material,

$C =$ specific heat of the rod material.

Figure 1: A metal rod

Explicit Method of Solving Parabolic PDEs

To numerically solve parabolic PDEs such as Equation (2), one can use finite difference approximations of the partial derivatives so that the dependent variable, T is now sought at particular nodes (x-location) and time (t) (Figure 2). The left hand side second derivative is approximated by the central divided difference approximation as

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \cong \frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} \tag{3}$$

where

$i =$ node number along the $x-$direction, $i = 0,1,....,n$,

$j =$ node number along the time,

$\Delta x =$ distance between nodes.

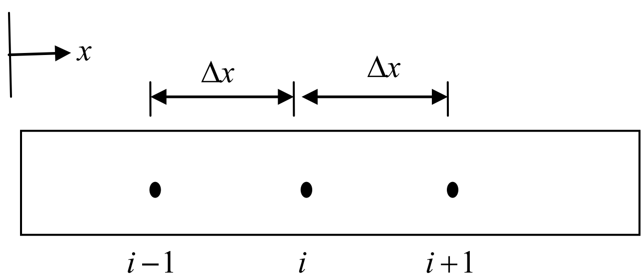

Figure 2: Schematic diagram showing the node representation in the model

For a rod of length L which is divided into $n+1$ nodes,

$$\Delta x = \frac{L}{n} \tag{4}$$

The time is similarly broken into time steps of Δt. Hence T_i^j corresponds to the temperature at node i, that is,

$$x = (i)(\Delta x)$$

and time,

$$t = (j)(\Delta t),$$

where

$\Delta t =$ time step.

The time derivative of the right hand side of Equation (2) is approximated by the forward divided difference approximation

$$\left.\frac{\partial T}{\partial t}\right|_{i,j} \cong \frac{T_i^{j+1}-T_i^j}{\Delta t} \tag{5}$$

Substituting the finite difference approximations given by Equations (3) and (5) in Equation (2) gives

$$\alpha\frac{T_{i+1}^j-2T_i^j+T_{i-1}^j}{(\Delta x)^2}=\frac{T_i^{j+1}-T_i^j}{\Delta t}$$

Solving for the temperature at the time node $j+1$, gives

$$T_i^{j+1}=T_i^j+\alpha\frac{\Delta t}{(\Delta x)^2}\left(T_{i+1}^j-2T_i^j+T_{i-1}^j\right)$$

Choosing

$$\lambda=\alpha\frac{\Delta t}{(\Delta x)^2} \tag{6}$$

$$T_i^{j+1}=T_i^j+\lambda\left(T_{i+1}^j-2T_i^j+T_{i-1}^j\right) \tag{7}$$

Equation (7) can be solved explicitly because it can be written for each internal location node of the rod for time node $j+1$ in terms of the temperature at time node j. In other words, if we know the temperature at node $j=0$, and knowing the boundary temperatures, which is the temperature at the external nodes, we can find the temperature at the next time step. We continue the process by first finding the temperature at all nodes $j=1$, and using these to find the temperature at the next time node, $j=2$. This process continues till we reach the time at which we are interested in finding the temperature.

Example 1

A rod of steel is subjected to a temperature of $100°C$ on the left end and $25°C$ on the right end. If the rod is of length $0.05m$, use the explicit method to find the temperature distribution in the rod from $t=0$ and $t=9$ seconds. Use $\Delta x=0.01m$, $\Delta t=3s$. Given:

$k=54\frac{W}{m-K}$, $\rho=7800\frac{kg}{m^3}$, $C=490\frac{J}{kg-K}$.

The initial temperature of the rod is $20°C$.

Solution

$$\alpha=\frac{k}{\rho C}$$

$$=\frac{54}{7800\times 490}$$

$$=1.4129\times 10^{-5}\ m^2/s$$

Then

$$\lambda=\alpha\frac{\Delta t}{(\Delta x)^2}$$

$$= 1.4129\times10^{-5}\frac{3}{(0.01)^2}$$
$$= 0.4239$$

Number of time steps$=\dfrac{t_{final}-t_{initial}}{\Delta t}$

$$=\frac{9-0}{3}$$
$$= 3$$

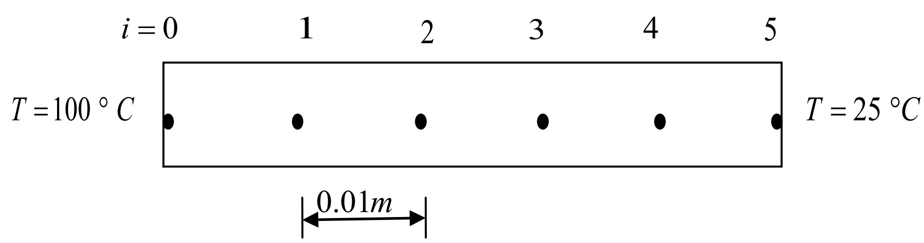

Figure 3: Schematic diagram showing the node distribution in the rod

The boundary conditions

$$\left.\begin{aligned} T_0^j &= 100°C \\ T_5^j &= 25°C \end{aligned}\right\} \text{for all } j=0,1,2,3 \qquad \text{(E1.1)}$$

The initial temperature of the rod is $20°C$, that is, all the temperatures of the nodes inside the rod are at $20°C$ when time, $t=0\,\text{sec}$ except for the boundary nodes as given by Equation (E1.1). This could be represented as

$$T_i^0 = 20°C, \text{ for all } i=1,2,3,4. \qquad \text{(E1.2)}$$

<u>Initial temperature at the nodes inside the rod (when t=0 sec)</u>

$$T_0^0 = 100°C \quad \text{from Equation (E1.1)}$$

$$\left.\begin{aligned} T_1^0 &= 20°C \\ T_2^0 &= 20°C \\ T_3^0 &= 20°C \\ T_4^0 &= 20°C \end{aligned}\right\} \text{from Equation (E1.2)}$$

$$T_5^0 = 25°C \quad \text{from Equation (E1.1)}$$

<u>Temperature at the nodes inside the rod when t=3 sec</u>

Setting $j=0$ and $i=0,1,2,3,4,5$ in Equation (7) gives the temperature of the nodes inside the rod when time, $t=3\,\text{sec}$.

$$T_0^1 = 100°C \quad \text{Boundary Condition (E1.1)}$$

$$\begin{aligned} T_1^1 &= T_1^0 + \lambda\left(T_2^0 - 2T_1^0 + T_0^0\right) \\ &= 20 + 0.4239\left(20 - 2(20) + 100\right) \\ &= 20 + 0.4239(80) \\ &= 20 + 33.912 \\ &= 53.912°C \end{aligned}$$

$$\begin{aligned} T_2^1 &= T_2^0 + \lambda\left(T_3^0 - 2T_2^0 + T_1^0\right) \\ &= 20 + 0.4239\left(20 - 2(20) + 20\right) \\ &= 20 + 0.4239(0) \\ &= 20 + 0 \\ &= 20°C \end{aligned}$$

$$\begin{aligned} T_3^1 &= T_3^0 + \lambda\left(T_4^0 - 2T_3^0 + T_2^0\right) \\ &= 20 + 0.4239\left(20 - 2(20) + 20\right) \\ &= 20 + 0.4239(0) \\ &= 20 + 0 \\ &= 20°C \end{aligned}$$

$$\begin{aligned} T_4^1 &= T_4^0 + \lambda\left(T_5^0 - 2T_4^0 + T_3^0\right) \\ &= 20 + 0.4239\left(25 - 2(20) + 20\right) \\ &= 20 + 0.4239(5) \\ &= 20 + 2.1195 \\ &= 22.120°C \end{aligned}$$

$$T_5^1 = 25°C \quad \text{Boundary Condition (E1.1)}$$

<u>Temperature at the nodes inside the rod when *t*=6 sec</u>

Setting $j = 1$ and $i = 0,1,2,3,4,5$ in Equation (7) gives the temperature of the nodes inside the rod when time, $t = 6\,\text{sec}$

$$T_0^2 = 100°C \quad \text{Boundary Condition (E1.1)}$$

$$
\begin{aligned}
T_1^2 &= T_1^1 + \lambda\left(T_2^1 - 2T_1^1 + T_0^1\right) \\
&= 53.912 + 0.4239\left(20 - 2(53.912) + 100\right) \\
&= 53.912 + 0.4239\left(12.176\right) \\
&= 53.912 + 5.1614 \\
&= 59.073°C
\end{aligned}
$$

$$
\begin{aligned}
T_2^2 &= T_2^1 + \lambda\left(T_3^1 - 2T_2^1 + T_1^1\right) \\
&= 20 + 0.4239\left(20 - 2(20) + 53.912\right) \\
&= 20 + 0.4239\left(33.912\right) \\
&= 20 + 14.375 \\
&= 34.375°C
\end{aligned}
$$

$$
\begin{aligned}
T_3^2 &= T_3^1 + \lambda\left(T_4^1 - 2T_3^1 + T_2^1\right) \\
&= 20 + 0.4239\left(22.120 - 2(20) + 20\right) \\
&= 20 + 0.4239\left(2.120\right) \\
&= 20 + 0.89867 \\
&= 20.899°C
\end{aligned}
$$

$$
\begin{aligned}
T_4^2 &= T_4^1 + \lambda\left(T_5^1 - 2T_4^1 + T_3^1\right) \\
&= 22.120 + 0.4239\left(25 - 2(22.120) + 20\right) \\
&= 22.120 + 0.4239\left(0.76\right) \\
&= 22.120 + 0.032220 \\
&= 22.442°C
\end{aligned}
$$

$T_5^2 = 25°C$ Boundary Condition (E1.1)

<u>Temperature at the nodes inside the rod when *t*=9 sec</u>

Setting $j = 2$ and $i = 0,1,2,3,4,5$ in Equation (7) gives the temperature of the nodes inside the rod when time, $t = 9\,\text{sec}$

$T_0^3 = 100°C$ Boundary Condition (E1.1)

$$
\begin{aligned}
T_1^3 &= T_1^2 + \lambda\left(T_2^2 - 2T_1^2 + T_0^2\right) \\
&= 59.073 + 0.4239\left(34.375 - 2(59.073) + 100\right) \\
&= 59.073 + 0.4239\left(16.229\right) \\
&= 59.073 + 6.8795 \\
&= 65.953°C
\end{aligned}
$$

$$
\begin{aligned}
T_2^3 &= T_2^2 + \lambda\left(T_3^2 - 2T_2^2 + T_1^2\right) \\
&= 34.375 + 0.4239\left(20.899 - 2(34.375) + 59.073\right) \\
&= 34.375 + 0.4239\left(11.222\right) \\
&= 34.375 + 4.7570 \\
&= 39.132^\circ C
\end{aligned}
$$

$$
\begin{aligned}
T_3^3 &= T_3^2 + \lambda\left(T_4^2 - 2T_3^2 + T_2^2\right) \\
&= 20.899 + 0.4239\left(22.442 - 2(20.899) + 34.375\right) \\
&= 20.899 + 0.4239\left(15.019\right) \\
&= 20.899 + 6.367 \\
&= 27.266^\circ C
\end{aligned}
$$

$$
\begin{aligned}
T_4^3 &= T_4^2 + \lambda\left(T_5^2 - 2T_4^2 + T_3^2\right) \\
&= 22.442 + 0.4239\left(25 - 2(22.442) + 20.899\right) \\
&= 22.442 + 0.4239\left(1.0150\right) \\
&= 22.442 + 0.4303 \\
&= 22.872^\circ C
\end{aligned}
$$

$$T_5^3 = 25^\circ C \quad \text{Boundary Condition (E1.1)}$$

To better visualize the temperature variation at different locations at different times, temperature distribution along the length of the rod at different times is plotted in the Figure 4.

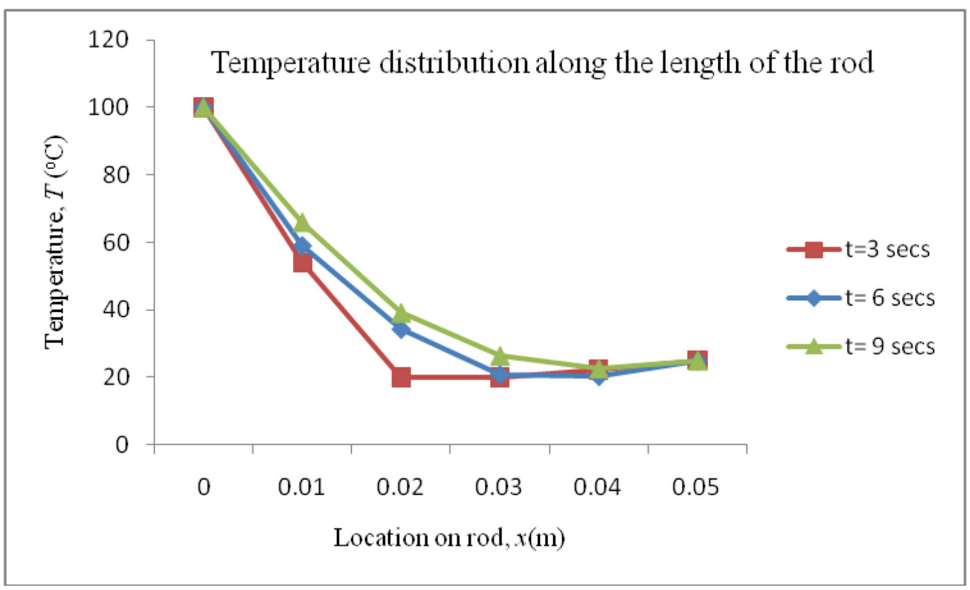

Figure 4: Temperature distribution from explicit method

Implicit Method for Solving Parabolic PDEs

In the explicit method, one is able to find the solution at each node, one equation at a time. However, the solution at a particular node is dependent only on temperature from neighboring nodes from the previous time step. For example, in the solution of Example 1, the temperatures at node 2 and 3 artificially stay at the initial temperature at $t=3$ seconds. This is contrary to what we would expect physically from the problem.

Also the explicit method does not guarantee stability which depends on the value of the time step, location step and the parameters of the elliptic equation. For the PDE

$$\alpha\frac{\partial^2 T}{\partial x^2}=\frac{\partial T}{\partial t},$$

the explicit method is convergent and stable for

$$\frac{\alpha\Delta t}{(\Delta x)^2}\le\frac{1}{2} \tag{8}$$

These issues are addressed by using the implicit method. Instead of the temperature being found one node at a time, the implicit method results in simultaneous linear equations for the temperature at all interior nodes for a particular time.

The implicit method to solve the parabolic PDE given by equation (2) is as follows. The second derivative on the left hand side of the equation is approximated by the central divided difference scheme at time level $j+1$ at node i as

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j+1}\approx\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2} \tag{9}$$

The first derivative on the right hand side of the equation is approximated by backward divided difference approximation at time level $j+1$ and node i as

$$\left.\frac{\partial T}{\partial t}\right|_{i,j+1}\approx\frac{T_i^{j+1}-T_i^j}{\Delta t} \tag{10}$$

Substituting Equations (9) and (10) in Equation (2) gives

$$\alpha\frac{T_{i+1}^{j+1}-2T_i^{j+1}+T_{i-1}^{j+1}}{(\Delta x)^2}=\frac{T_i^{j+1}-T_i^j}{\Delta t}$$

giving

$$-\lambda T_{i-1}^{j+1}+(1+2\lambda)T_i^{j+1}-\lambda T_{i+1}^{j+1}=T_i^j \tag{11}$$

where

$$\lambda=\alpha\frac{\Delta t}{(\Delta x)^2}$$

Now Equation (11) can be written for all nodes (except the external nodes), at a particular time level. This results in simultaneous linear equations which can be solved to find the nodal temperature at a particular time.

Example 2

A rod of steel is subjected to a temperature of $100°C$ on the left end and $25°C$ on the right end. If the rod is of length $0.05m$, use the implicit method to find the temperature distribution in the rod from $t=0$ to $t=9$ seconds. Use $\Delta x=0.01m$ and $\Delta t=3s$.

Given

$$k = 54\frac{W}{m-K},\ \rho = 7800\frac{kg}{m^3},\ C = 490\frac{J}{kg-K}.$$

The initial temperature of the rod is $20°C$.

Solution

$$\alpha = \frac{k}{\rho C}$$
$$= \frac{54}{7800 \times 490}$$
$$= 1.4129 \times 10^{-5} \quad m^2/s$$

Then

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$
$$= 1.412 \times 10^{-5} \frac{3}{(0.01)^2}$$
$$= 0.4239$$

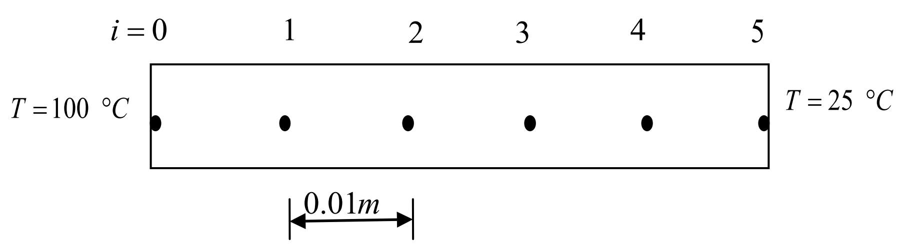

Figure 5: Schematic diagram showing the node representation in the model

The boundary conditions

$$\left.\begin{aligned} T_0^j &= 100°C \\ T_5^j &= 25°C \end{aligned}\right\} \text{ for } j = 0,1,2,3 \qquad \text{(E2.1)}$$

The initial temperature of the rod is $20°C$, that is, the temperatures of all the nodes inside the rod are at $20°C$ when time, $t = 0$ except for the boundary nodes where the temperatures are given by satisfying the Equation (E2.1). This could be represented as

$$T_i^0 = 20°C, \text{ for } i = 1,2,3,4. \qquad \text{(E2.2)}$$

<u>Initial temperature at the nodes inside the rod (when *t*=0 sec)</u>

$$T_0^0 = 100°C \quad \text{from Equation (E2.1)}$$

$$\left.\begin{aligned} T_1^0 &= 20^\circ C \\ T_2^0 &= 20^\circ C \\ T_3^0 &= 20^\circ C \\ T_4^0 &= 20^\circ C \end{aligned}\right\} \text{ from Equation (E2.2)}$$

$$T_5^0 = 25^\circ C \qquad \text{from Equation (E2.1)}$$

<u>Temperature at the nodes inside the rod when t=3 sec</u>

$$\left.\begin{aligned} T_0^1 &= 100^\circ C \\ T_5^1 &= 25^\circ C \end{aligned}\right\} \text{Boundary Condition (E2.1)}$$

For all the interior nodes, putting $j = 0$ and $i = 1, 2, 3, 4$ in Equation (11) gives the following equations

<u>*i*=1</u>

$$-\lambda T_0^1 + (1+2\lambda)T_1^1 - \lambda T_2^1 = T_1^0$$

$$(-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^1 - (0.4239 T_2^1) = 20$$

$$-42.39 + 1.8478 T_1^1 - 0.4239 T_2^1 = 20$$

$$1.8478 T_1^1 - 0.4239 T_2^1 = 62.390 \qquad \text{(E2.3)}$$

<u>*i*=2</u>

$$-\lambda T_1^1 + (1+2\lambda)T_2^1 - \lambda T_3^1 = T_2^0$$

$$-0.4239 T_1^1 + 1.8478 T_2^1 - 0.4239 T_3^1 = 20 \qquad \text{(E2.4)}$$

<u>*i*=3</u>

$$-\lambda T_2^1 + (1+2\lambda)T_3^1 - \lambda T_4^1 = T_3^0$$

$$-0.4239 T_2^1 + 1.8478 T_3^1 - 0.4239 T_4^1 = 20 \qquad \text{(E2.5)}$$

<u>*i*=4</u>

$$-\lambda T_3^1 + (1+2\lambda)T_4^1 - \lambda T_5^1 = T_4^0$$

$$-0.4239 T_3^1 + 1.8478 T_4^1 - (0.4239 \times 25) = 20$$

$$-0.4239 T_3^1 + 1.8478 T_4^1 - 10.598 = 20$$

$$-0.4239 T_3^1 + 1.8478 T_4^1 = 30.598 \qquad \text{(E2.6)}$$

The simultaneous linear equations (E2.3) – (E2.6) can be written in matrix form as

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 62.390 \\ 20 \\ 20 \\ 30.598 \end{bmatrix}$$

The above coefficient matrix is tri-diagonal. Special algorithms such as Thomas' algorithm can be used to solve simultaneous linear equation with tri-diagonal coefficient matrices. The solution is given by

$$\begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 39.451 \\ 24.792 \\ 21.438 \\ 21.477 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 3\,\text{sec}$ is

$$\begin{bmatrix} T_0^1 \\ T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \\ T_5^1 \end{bmatrix} = \begin{bmatrix} 100 \\ 39.451 \\ 24.792 \\ 21.438 \\ 21.477 \\ 25 \end{bmatrix}$$

<u>Temperature at the nodes inside the rod when t=6 sec</u>

$$\left.\begin{aligned} T_0^2 &= 100°C \\ T_5^2 &= 25°C \end{aligned}\right\}\text{Boundary Condition (E2.1)}$$

For all the interior nodes, putting $j = 1$ and $i = 1,2,3,4$ in Equation (11) gives the following equations

<u>i=1</u>

$$-\lambda T_0^2 + (1+2\lambda)T_1^2 - \lambda T_2^2 = T_1^1$$

$$(-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^2 - 0.4239T_2^2 = 39.451$$

$$-42.39 + 1.8478T_1^2 - 0.4239T_2^2 = 39.451$$

$$1.8478T_1^2 - 0.4239T_2^2 = 81.841 \tag{E2.7}$$

<u>i=2</u>

$$-\lambda T_1^2 + (1+2\lambda)T_2^2 - \lambda T_3^2 = T_2^1$$

$$-0.4239T_1^2 + 1.8478T_2^2 - 0.4239T_3^2 = 24.792 \tag{E2.8}$$

<u>i=3</u>

$$-\lambda T_2^2 + (1+2\lambda)T_3^2 - \lambda T_4^2 = T_3^1$$

$$-0.4239T_2^2 + 1.8478T_3^2 - 0.4239T_4^2 = 21.438 \tag{E2.9}$$

<u>i=4</u>

$$-\lambda T_3^2 + (1+2\lambda)T_4^2 - \lambda T_5^2 = T_4^1$$

$$-0.4239T_3^2 + 1.8478T_4^2 - (0.4239 \times 25) = 21.477$$

$$-0.4239T_3^2 + 1.8478T_4^2 - 10.598 = 21.477$$

$$-0.4239T_3^2 + 1.8478T_4^2 = 32.075 \tag{E2.10}$$

The simultaneous linear equations (E2.7) – (E2.10) can be written in matrix form as

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 81.841 \\ 24.792 \\ 21.438 \\ 32.075 \end{bmatrix}$$

The solution of the above set of simultaneous linear equation is

$$\begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 51.326 \\ 30.669 \\ 23.876 \\ 22.836 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 6\,\text{sec}$ is

$$\begin{bmatrix} T_0^2 \\ T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \\ T_5^2 \end{bmatrix} = \begin{bmatrix} 100 \\ 51.326 \\ 30.669 \\ 23.876 \\ 22.836 \\ 25 \end{bmatrix}$$

<u>Temperature at the nodes inside the rod when *t*=9 sec</u>

$$\left.\begin{aligned} T_0^3 &= 100°C \\ T_5^3 &= 25°C \end{aligned}\right\} \text{Boundary Condition (E2.1)}$$

For all the interior nodes, setting $j = 2$ and $i = 1,2,3,4$ in Equation (11) gives the following equations

<u>*i*=1</u>

$$-\lambda T_0^3 + (1+2\lambda)T_1^3 - \lambda T_2^3 = T_1^2$$

$$(-0.4239 \times 100) + (1 + 2 \times 0.4239)T_1^3 - (0.4239 T_2^3) = 51.326$$

$$-42.39 + 1.8478 T_1^3 - 0.4239 T_2^3 = 51.326$$

$$1.8478 T_1^3 - 0.4239 T_2^3 = 93.716 \qquad \text{(E2.11)}$$

<u>*i*=2</u>

$$-\lambda T_1^3 + (1+2\lambda)T_2^3 - \lambda T_3^3 = T_2^2$$

$$-0.4239 T_1^3 + 1.8478 T_2^3 - 0.4239 T_3^3 = 30.669 \qquad \text{(E2.12)}$$

<u>*i*=3</u>

$$-\lambda T_2^3 + (1+2\lambda)T_3^3 - \lambda T_4^3 = T_3^2$$

$$-0.4239 T_2^3 + 1.8478 T_3^3 - 0.4239 T_4^3 = 23.876 \qquad \text{(E2.13)}$$

<u>*i*=4</u>

$$-\lambda T_3^3 + (1+2\lambda)T_4^3 - \lambda T_5^3 = T_4^2$$
$$-0.4239T_3^3 + 1.8478T_4^3 - (0.4239 \times 25) = 22.836$$
$$-0.4239T_3^3 + 1.8478T_4^3 - 10.598 = 22.836$$
$$-0.4239T_3^3 + 1.8478T_4^3 = 33.434 \quad \text{(E2.14)}$$

The simultaneous linear equations (E2.11) – (E2.14) can be written in matrix form as

$$\begin{bmatrix} 1.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 1.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 1.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 1.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 93.716 \\ 30.669 \\ 23.876 \\ 33.434 \end{bmatrix}$$

The solution of the above set of simultaneous linear equation is

$$\begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 59.043 \\ 36.292 \\ 26.809 \\ 24.243 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 9 \text{ sec}$ is

$$\begin{bmatrix} T_0^3 \\ T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \\ T_5^3 \end{bmatrix} = \begin{bmatrix} 100 \\ 59.043 \\ 36.292 \\ 26.809 \\ 24.243 \\ 25 \end{bmatrix}$$

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted in Figure 6.

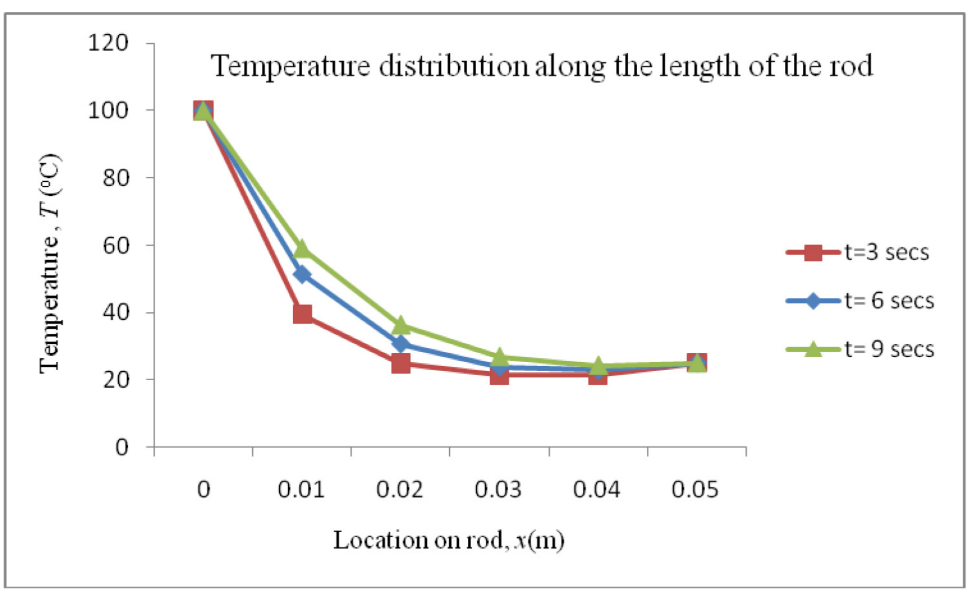

Figure 6: Temperature distribution in rod from implicit method

Crank-Nicolson Method

The Crank-Nicolson method provides an alternative scheme to implicit method. The accuracy of Crank-Nicolson method is same in both space and time. In the implicit method, the approximation of $\frac{\partial^2 T}{\partial x^2}$ is of $O(\Delta x)^2$ accuracy, while the approximation for $\frac{\partial T}{\partial t}$ is of (Δt) accuracy. The accuracy in the Crank-Nicolson method is achieved by approximating the derivative at the mid point of time step. To numerically solve PDEs such as Equation (2), one can use finite difference approximations of the partial derivatives. The left hand side of the second derivative is approximated at node i as the average value of the central divided difference approximation at time level $j+1$ and time level j.

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \approx \frac{1}{2}\left[\frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}\right] \tag{12}$$

The first derivative on the right side of Equation (2) is approximated using forward divided difference approximation at time level $j+1$ and node i as

$$\left.\frac{\partial T}{\partial t}\right|_{i,j} \approx \frac{T_i^{j+1} - T_i^{j}}{\Delta t} \tag{13}$$

Substituting Equations (12) and (13) in Equation (2) gives

$$\alpha \bullet \frac{1}{2}\left[\frac{T_{i+1}^{j} - 2T_i^{j} + T_{i-1}^{j}}{(\Delta x)^2} + \frac{T_{i+1}^{j+1} - 2T_i^{j+1} + T_{i-1}^{j+1}}{(\Delta x)^2}\right] = \frac{T_i^{j+1} - T_i^{j}}{\Delta t}$$

(14)
giving

$$-\lambda T_{i-1}^{j+1} + 2(1+\lambda)T_i^{j+1} - \lambda T_{i+1}^{j+1} = \lambda T_{i-1}^{j} + 2(1-\lambda)T_i^{j} + \lambda T_{i+1}^{j} \tag{15}$$

where

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$

Now Equation (15) is written for all nodes (except the external nodes). This will result in simultaneous linear equations that can be solved to find the temperature at a particular time.

Example 3

A rod of steel is subjected to a temperature of $100°C$ on the left end and $25°C$ on the right end. If the rod is of length $0.05m$, use Crank-Nicolson method to find the temperature distribution in the rod from $t=0$ to $t=9$ seconds. Use $\Delta x = 0.01m$, $\Delta t = 3s$.
Given

$$k = 54\frac{W}{m-K},\ \rho = 7800\frac{kg}{m^3},\ C = 490\frac{J}{kg-K}.$$

The initial temperature of the rod is $20°C$.

Solution

$$\alpha = \frac{k}{\rho C}$$
$$= \frac{54}{7800 \times 490}$$
$$= 1.4129 \times 10^{-5}\ m^2/s$$

Then

$$\lambda = \alpha \frac{\Delta t}{(\Delta x)^2}$$
$$= 1.412 \times 10^{-5} \frac{3}{(0.01)^2}$$
$$= 0.4239$$

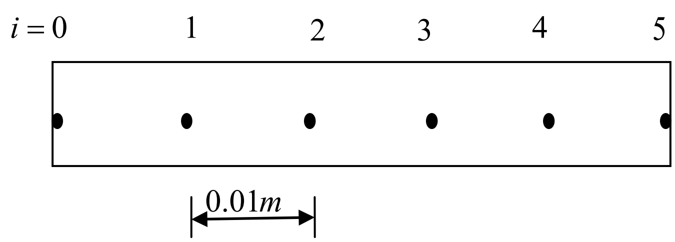

Figure 7: Schematic diagram showing the node representation in the model

The boundary conditions are

$$\left.\begin{aligned} T_0^j &= 100^\circ C \\ T_5^j &= 25^\circ C \end{aligned}\right\} \text{ for } j = 0,1,2,3 \tag{E3.1}$$

The initial temperature of the rod is $20^\circ C$, that is, all the temperatures of the nodes inside the rod are at $20^\circ C$ at, $t = 0$ except for the boundary nodes given by Equation (E3.1). This could be represented as

$$T_i^0 = 20^\circ C, \text{ for } i = 1,2,3,4. \tag{E3.2}$$

<u>Initial temperature at the nodes inside the rod (when t=0 sec)</u>

$$T_0^0 = 100^\circ C \quad \text{from Equation (E3.1)}$$

$$\left.\begin{aligned} T_1^0 &= 20^\circ C \\ T_2^0 &= 20^\circ C \\ T_3^0 &= 20^\circ C \\ T_4^0 &= 20^\circ C \end{aligned}\right\} \text{from Equation (E3.2)}$$

$$T_5^0 = 25°C \qquad \text{from Equation (E3.1)}$$

<u>Temperature at the nodes inside the rod when t=3 sec</u>

$$\left.\begin{aligned} T_0^1 &= 100°C \\ T_5^1 &= 25°C \end{aligned}\right\} \text{Boundary Condition (E3.1)}$$

For all the interior nodes, setting $j = 0$ and $i = 1,2,3,4$ in Equation (15) gives the following equations

<u>i=1</u>

$$-\lambda T_0^1 + 2(1+\lambda)T_1^1 - \lambda T_2^1 = \lambda T_0^0 + 2(1-\lambda)T_1^0 + \lambda T_2^0$$

$$(-0.4239 \times 100) + 2(1+0.4239)T_1^1 - 0.4239T_2^1 = (0.4239)100 + 2(1-0.4239)20 + (0.4239)20$$

$$-42.39 + 2.8478T_1^1 - 0.4239T_2^1 = 42.39 + 23.044 + 8.478$$

$$2.8478T_1^1 - 0.4239T_2^1 = 116.30 \qquad \text{(E3.3)}$$

<u>i=2</u>

$$-\lambda T_1^1 + 2(1+\lambda)T_2^1 - \lambda T_3^1 = \lambda T_1^0 + 2(1-\lambda)T_2^0 + \lambda T_3^0$$

$$-0.4239T_1^1 + 2(1+0.4239)T_2^1 - 0.4239T_3^1 = (0.4239)20 + 2(1-0.4239)20 + (0.4239)20$$

$$-0.4239T_1^1 + 2.8478T_2^1 - 0.4239T_3^1 = 40.000 \qquad \text{(E3.4)}$$

<u>i=3</u>

$$-\lambda T_2^1 + 2(1+\lambda)T_3^1 - \lambda T_4^1 = \lambda T_2^0 + 2(1-\lambda)T_3^0 + \lambda T_4^0$$

$$-0.4239T_2^1 + 2(1+0.4239)T_3^1 - 0.4239T_4^1 = (0.4239)20 + 2(1-0.4239)20 + (0.4239)20$$

$$-0.4239T_2^1 + 2.8478T_3^1 - 0.4239T_4^1 = 40.000 \qquad \text{(E3.5)}$$

<u>i=4</u>

$$-\lambda T_3^1 + 2(1+\lambda)T_4^1 - \lambda T_5^1 = \lambda T_3^0 + 2(1-\lambda)T_4^0 + \lambda T_5^0$$

$$-0.4239T_3^1 + 2(1+0.4239)T_4^1 - (0.4239)25 = (0.4239)20 + 2(1-0.4239)20 + (0.4239)25$$

$$-0.4239T_3^1 + 2.8478T_4^1 - 10.598 = 8.478 + 23.044 + 10.598$$

$$-0.4239T_3^1 + 2.8478T_4^1 = 52.718 \qquad \text{(E3.6)}$$

The coefficient matrix in the above set of equations is tridiagonal. Special algorithms such as Thomas' algorithm are used to solve equation with tridiagonal coefficient matrices

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 116.30 \\ 40.000 \\ 40.000 \\ 52.718 \end{bmatrix}$$

The above matrix is tridiagonal. Solving the above matrix we get

$$\begin{bmatrix} T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \end{bmatrix} = \begin{bmatrix} 44.372 \\ 23.746 \\ 20.797 \\ 21.607 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 3$ sec is

$$\begin{bmatrix} T_0^1 \\ T_1^1 \\ T_2^1 \\ T_3^1 \\ T_4^1 \\ T_5^1 \end{bmatrix} = \begin{bmatrix} 100 \\ 44.372 \\ 23.746 \\ 20.797 \\ 21.607 \\ 25 \end{bmatrix}$$

Temperature at the nodes inside the rod when t=6 sec

$$\left.\begin{aligned} T_0^2 &= 100°C \\ T_5^2 &= 25°C \end{aligned}\right\}\text{Boundary Condition (E3.1)}$$

For all the interior nodes, putting $j = 1$ and $i = 1,2,3,4$ in Equation (15) gives the following equations

i=1

$$-\lambda T_0^2 + 2(1+\lambda)T_1^2 - \lambda T_2^2 = \lambda T_0^1 + 2(1-\lambda)T_1^1 + \lambda T_2^1$$

$$(-0.4239 \times 100) + 2(1+0.4239)T_1^2 - 0.4239T_2^2 =$$
$$(0.4239)100 + 2(1-0.4239)44.372 + (0.4239)23.746$$

$$-42.39 + 2.8478T_1^2 - 0.4239T_2^2 = 42.39 + 51.125 + 10.066$$

$$2.8478T_1^2 - 0.4239T_2^2 = 145.971 \tag{E3.7}$$

i=2

$$-\lambda T_1^2 + 2(1+\lambda)T_2^2 - \lambda T_3^2 = \lambda T_1^1 + 2(1-\lambda)T_2^1 + \lambda T_3^1$$

$$-0.4239T_1^2 + 2(1+0.4239)T_2^2 - 0.4239T_3^2 =$$
$$(0.4239)44.372 + 2(1-0.4239)23.746 + (0.4239)20.797$$

$$-0.4239T_1^2 + 2.8478T_2^2 - 0.4239T_3^2 = 18.809 + 27.360 + 8.8158$$

$$-0.4239T_1^2 + 2.8478T_2^2 - 0.4239T_3^2 = 54.985 \tag{E3.8}$$

i=3

$$-\lambda T_2^2 + 2(1+\lambda)T_3^2 - \lambda T_4^2 = \lambda T_2^1 + 2(1-\lambda)T_3^1 + \lambda T_4^1$$

$$-0.4239T_2^2 + 2(1+0.4239)T_3^2 - 0.4239T_4^2 =$$
$$(0.4239)23.746 + 2(1-0.4239)20.797 + (0.4239)21.607$$

$$-0.4239T_2^2 + 2.8478T_3^2 - 0.4239T_4^2 = 10.066 + 23.962 + 9.1592$$

$$-0.4239T_2^2 + 2.8478T_3^2 - 0.4239T_4^2 = 43.187 \tag{E3.9}$$

i=4

$$-\lambda T_3^2 + 2(1+\lambda)T_4^2 - \lambda T_5^2 = \lambda T_3^1 + 2(1-\lambda)T_4^1 + \lambda T_5^1$$

$$-0.4239T_3^2 + 2(1+0.4239)T_4^2 - (0.4239)25 =$$
$$(0.4239)20.797 + 2(1-0.4239)21.607 + (0.4239)25$$

$$-0.4239T_3^2 + 2.8478T_4^2 - 10.598 = 8.8158 + 24.896 + 10.598$$

$$-0.4239T_3^2 + 2.8478T_4^2 = 54.908 \quad \text{(E3.10)}$$

The simultaneous linear equations (E3.7) – (E3.10) can be written in matrix form as

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 145.971 \\ 54.985 \\ 43.187 \\ 54.908 \end{bmatrix}$$

Solving the above set of equations, we get

$$\begin{bmatrix} T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \end{bmatrix} = \begin{bmatrix} 55.883 \\ 31.075 \\ 23.174 \\ 22.730 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 6$ sec is

$$\begin{bmatrix} T_0^2 \\ T_1^2 \\ T_2^2 \\ T_3^2 \\ T_4^2 \\ T_5^2 \end{bmatrix} = \begin{bmatrix} 100 \\ 55.883 \\ 31.075 \\ 23.174 \\ 22.730 \\ 25 \end{bmatrix}$$

<u>Temperature at the nodes inside the rod when *t*=9 sec</u>

$$\left.\begin{aligned} T_0^3 &= 100^\circ C \\ T_5^3 &= 25^\circ C \end{aligned}\right\} \text{Boundary Condition (E3.1)}$$

For all the interior nodes, setting $j = 2$ and $i = 1, 2, 3, 4$ in Equation (15) gives the following equations

<u>*i*=1</u>

$$-\lambda T_0^3 + 2(1+\lambda)T_1^3 - \lambda T_2^3 = \lambda T_0^2 + 2(1-\lambda)T_1^2 + \lambda T_2^2$$

$$(-0.4239 \times 100) + 2(1+0.4239)T_1^3 - 0.4239T_2^3 =$$
$$(0.4239)100 + 2(1-0.4239)55.883 + (0.4239)31.075$$

$$-42.39 + 2.8478T_1^3 - 0.4239T_2^3 = 42.39 + 64.388 + 13.173$$

$$2.8478T_1^3 - 0.4239T_2^3 = 162.34 \quad \text{(E3.11)}$$

<u>*i*=2</u>

$$-\lambda T_1^3 + 2(1+\lambda)T_2^3 - \lambda T_3^3 = \lambda T_1^2 + 2(1-\lambda)T_2^2 + \lambda T_3^2$$

$$-0.4239T_1^3 + 2(1+0.4239)T_2^3 - 0.4239T_3^3 =$$
$$(0.4239)55.883 + 2(1-0.4239)31.075 + (0.4239)23.174$$

$$-0.4239T_1^3 + 2.8478T_2^3 - 0.4239T_3^3 = 23.689 + 35.805 + 9.8235$$

$$-0.4239T_1^3 + 2.8478T_2^3 - 0.4239T_3^3 = 69.318 \quad \text{(E3.12)}$$

<u>*i*=3</u>

$$-\lambda T_2^3 + 2(1+\lambda)T_3^3 - \lambda T_4^3 = \lambda T_2^2 + 2(1-\lambda)T_3^2 + \lambda T_4^2$$

$$-0.4239T_2^3 + 2(1+0.4239)T_3^3 - 0.4239T_4^3 =$$
$$(0.4239)31.075 + 2(1-0.4239)23.174 + (0.4239)22.730$$

$$-0.4239T_2^3 + 2.8478T_3^3 - 0.4239T_4^3 = 13.173 + 26.701 + 9.635$$

$$-0.4239T_2^3 + 2.8478T_3^3 - 0.4239T_4^3 = 49.509 \quad \text{(E3.13)}$$

<u>*i*=4</u>

$$-\lambda T_3^3 + 2(1+\lambda)T_4^3 - \lambda T_5^3 = \lambda T_3^2 + 2(1-\lambda)T_4^2 + \lambda T_5^2$$

$$-0.4239T_3^3 + 2(1+0.4239)T_4^3 - (0.4239)25 =$$
$$(0.4239)23.174 + 2(1-0.4239)22.730 + (0.4239)25$$

$$-0.4239T_3^3 + 2.8478T_4^3 - 10.598 = 9.8235 + 26.190 + 10.598$$

$$-0.4239T_3^3 + 2.8478T_4^3 = 57.210 \quad \text{(E3.14)}$$

The simultaneous linear equations (E3.11) – (E3.14) can be written in matrix form as

$$\begin{bmatrix} 2.8478 & -0.4239 & 0 & 0 \\ -0.4239 & 2.8478 & -0.4239 & 0 \\ 0 & -0.4239 & 2.8478 & -0.4239 \\ 0 & 0 & -0.4239 & 2.8478 \end{bmatrix} \begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 162.34 \\ 69.318 \\ 49.509 \\ 57.210 \end{bmatrix}$$

Solving the above set of equations, we get

$$\begin{bmatrix} T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \end{bmatrix} = \begin{bmatrix} 62.604 \\ 37.613 \\ 26.562 \\ 24.042 \end{bmatrix}$$

Hence, the temperature at all the nodes at time, $t = 9\,\sec$ is

$$\begin{bmatrix} T_0^3 \\ T_1^3 \\ T_2^3 \\ T_3^3 \\ T_4^3 \\ T_5^3 \end{bmatrix} = \begin{bmatrix} 100 \\ 62.604 \\ 37.613 \\ 26.562 \\ 24.042 \\ 25 \end{bmatrix}$$

To better visualize the temperature variation at different locations at different times, the temperature distribution along the length of the rod at different times is plotted in Figure 8.

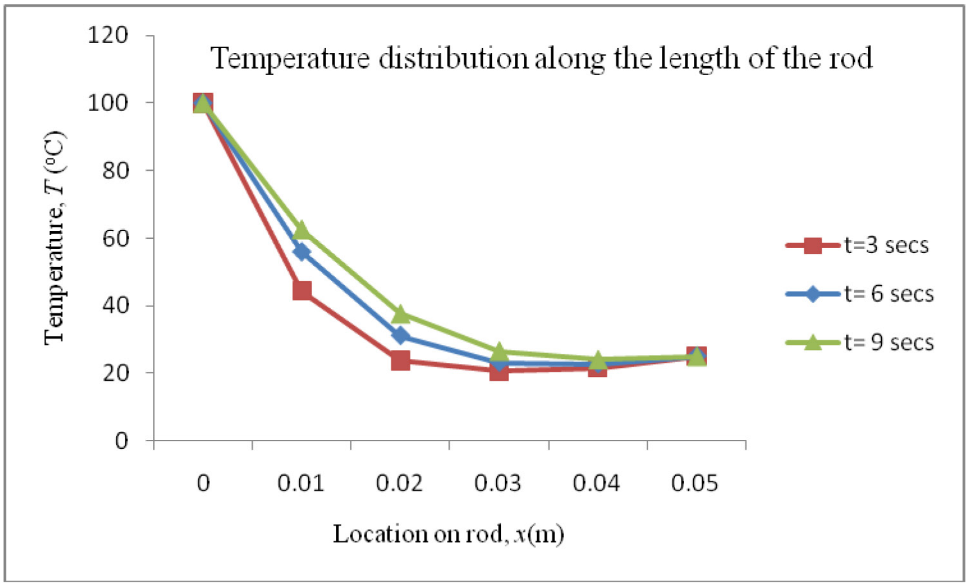

Figure 8: Temperature distribution in rod from Crank-Nicolson method

Analytical Method

Appendix A

The parabolic heat conduction equation given by Equation (2) is formulated as

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad 0 < x < 0.05,\ t > 0$$

with boundary conditions

$$T = 100°C \text{ at } x = 0,\ t > 0 \tag{16}$$

$$T = 25°C \text{ at } x = 0.05,\ t > 0 \tag{17}$$

and initial conditions

$$T = 20°C \text{ at } t = 0,\ 0 < x < 0.05 \tag{18}$$

We split the problem into a steady state problem and a transient (homogeneous) problem. The solutions of the steady state problem and transient problem are found separately and by applying the principle of superposition, the final solution would be obtained. This formulation can be represented as

$$T(x,t) = T_s(x) + T_h(x,t) \tag{19}$$

where

T_s = solution for steady state problem,

T_h = solution for transient problem.

<u>Steady State Solution</u>

Since the temperature at steady state is not changing, $\frac{\partial T}{\partial t} = 0$, the steady state problem is formulated as

$$\frac{d^2 T_s}{dx^2} = 0, \qquad 0 < x < 0.05 \tag{20}$$

with boundary conditions

$$T_s = 100°C \text{ at } x = 0 \tag{21}$$

$$T_s = 25°C \text{ at } x = 0.05 \tag{22}$$

The solution to Equation (20) is given by integrating it on both sides to give

$$\frac{dT_s}{dx} = A$$

where A is a constant of integration and by integrating again to give

$$T_s = Ax + B \tag{23}$$

where B is another constant of integration. By substituting the boundary condition (21), we obtain

$$A(0) + B = 100$$

$$B = 100$$

By substituting the boundary condition (22), we obtain

$$A(0.05)+100=25$$

$$A=\frac{-75}{0.05}$$

$$=-1500$$

Plugging back the values of A and B in Equation (23), we get the steady state solution as

$$T_s=-1500x+100 \tag{24}$$

<u>Transient Solution</u>

The transient problem is formulated as

$$\alpha\frac{\partial^2 T_h}{\partial x^2}=\frac{\partial T_h}{\partial t}, \qquad 0<x<0.05 \tag{25}$$

with boundary conditions

$$T_h=0°C \text{ at } x=0 \tag{26}$$

$$T_h=0°C \text{ at } x=0.05 \tag{27}$$

Note: from Equation (19),

$$T(x,t)=T_s(x)+T_h(x,t)$$

and by substituting Equations (21) and (22), the boundary conditions of T_h are obtained.

Initial conditions for the transient problem are hence given by

$$T_h=20-T_s, \quad t=0,\ 0<x<0.05$$

$$=20-(-1500x+100)$$

$$=20+1500x-100$$

$$=1500x-80,\ t=0,\ 0<x<0.05 \tag{28}$$

To obtain solution for the transient problem, let us assume $T_h(x,t)$ is function of the product of a spatial function and a temperature function. That is

$$T_h(x,t)=X(x).\tau(t) \tag{29}$$

Substituting Equation (29) in Equation (25), we get

$$\alpha\tau\frac{d^2X}{dx^2}=X\frac{d\tau}{dt}$$

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{\alpha\tau}\frac{d\tau}{dt} \tag{30}$$

The left hand side of Equation (30) represents the spatial term and the right hand side represents the temporal (time) term. We will attempt to find the solutions of the spatial and temporal term independently. To do so, let us assume that both the left hand side and the right hand side of the Equation (30) is equal to a constant $-\beta^2$ (say)

$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{\alpha\tau}\frac{d\tau}{dt}=-\beta^2 \tag{31}$$

Spatial solution

Taking just the spatial term from Equation (31), we have

$$\frac{1}{X}\frac{d^2X}{dx^2}=-\beta^2$$

$$\frac{d^2X}{dx^2}+\beta^2X=0 \tag{32}$$

The Equation (32) is a homogeneous second order ordinary differential equation. These type of equations have the solution of the form $X(x)=e^{mx}$. Substituting $X(x)=e^{mx}$ in Equation (32) we get,

$$m^2e^{mx}+\beta^2e^{mx}=0$$

$$e^{mx}(m^2+\beta^2)=0$$

$$m^2+\beta^2=0$$

$$m_1,\ m_2=i\beta,\ -i\beta$$

From the values of m_1 and m_2, the solution of $X(x)$ is written of the form

$$X(x)=C\cos(\beta x)+D\sin(\beta x) \tag{33}$$

Temporal solution

Taking just the temporal term from Equation (31), we have

$$\frac{1}{\alpha\tau}\frac{d\tau}{dt}=-\beta^2$$

$$\frac{d\tau}{dt}+\alpha\tau\beta^2=0 \tag{34}$$

The above equation is a homogeneous first order ordinary differential equation. These type of equations have the solution of the form $\tau(t)=e^{mt}$. Substituting $\tau(t)=e^{mt}$ in Equation (34) we get

$$me^{mt}+\alpha\beta^2e^{mt}=0$$

$$e^{mt}(m+\alpha\beta^2)=0$$

$$m+\alpha\beta^2=0$$

$$m=-\alpha\beta^2$$

From the value of m, the solution of $\tau(t)$ is written as

$$\tau(t)=Ee^{-\alpha\beta^2t} \tag{35}$$

Substituting Equations (33) and (35) in Equation (29), we have

$$T_h(x,t)=Ee^{-\alpha\beta^2t}\left[C\cos(\beta x)+D\sin(\beta x)\right]$$

$$T_h(x,t)=e^{-\alpha\beta^2t}\left[F\cos(\beta x)+G\sin(\beta x)\right] \tag{36}$$

Substituting boundary condition represented by Equation (26) in Equation (36) gives

$$e^{-\alpha\beta^2t}\left[F\cos(\beta.0)+G\sin(\beta.0)\right]=0$$

$$e^{-\alpha\beta^2t}\left[F.1+G.0\right]=0$$

$$e^{-\alpha\beta^2t}[F]=0$$

Since, $e^{-\alpha\beta^2t}$ cannot be zero, $F=0$. Now substituting $F=0$ in Equation (36) gives

$$T_h(x,t)=Ge^{-\alpha\beta^2t}\sin(\beta x) \tag{37}$$

Substituting boundary condition represented by Equation (27) in Equation (37) gives

$$Ge^{-\alpha\beta^2 t}\sin(0.4\beta) = 0$$
$$\sin(0.05\beta) = 0$$
$$0.05\beta = n\pi$$
$$\beta = \frac{n\pi}{0.05}$$
$$= 20n\pi$$

Substituting the value of β in Equation (37) gives

$$T_h(x,t) = Ge^{-\alpha(20n\pi)^2 t}\sin(20n\pi x)$$

As the general solution can have any value of n,

$$T_h(x,t) = \sum_{n=1}^{\infty} G_n e^{-\alpha(20n\pi)^2 t}\sin(20n\pi x) \quad (38)$$

Substituting the initial condition

$$T_h(x,0) = (1500x - 80)°C$$

from Equation (28) in Equation (38)

$$\sum_{n=1}^{\infty} G_n \sin(20n\pi x) = 1500x - 80$$

Multiplying both sides by $\sin(20m\pi x)$ and integrating from 0 to 0.05 gives

$$\sum_{n=1}^{\infty}\int_0^{0.05} G_n \sin(20n\pi x)\sin(20m\pi x)dx = \int_0^{0.05}(1500x - 80)\sin(20m\pi x)dx$$

$$\sum_{n=1}^{\infty}\frac{G_n}{2}\int_0^{0.05} 2\sin(20n\pi x)\sin(20m\pi x)dx = \int_0^{0.05}(1500x - 80)\sin(20m\pi x)dx$$

$$\sum_{n=1}^{\infty}\frac{G_n}{2}\left[\int_0^{0.05}\cos(20(m-n)\pi x)dx - \int_0^{0.05}\cos(20(m+n)\pi x)dx\right] =$$
$$1500\int_0^{0.05} x\sin(20m\pi x)dx - \int_0^{0.05} 80\sin(20m\pi x)dx$$

Substituting the following in the above equation,

$$\int_0^{0.05}\cos(20(m-n)\pi x)dx = 0, \quad m \neq n$$

$$\int_0^{0.05}\cos(20(m-n)\pi x)dx = 0.05, \quad m = n$$

$$\int_0^{0.05}\cos(20(m+n)\pi x)dx = 0, \quad \text{for any } m$$

we get

$$\frac{G_m}{2}0.05 = 1500\int_0^{0.05} x\sin(20m\pi x)dx - \int_0^{0.05} 80\sin(20m\pi x)dx$$

$$=1500\left[\left[\frac{-x\cos(20m\pi)}{20m\pi}\right]_0^{0.05}+\frac{1}{20m\pi}\int_0^{0.05}\cos(20m\pi)dx\right]-80\int_0^{0.05}\sin(20m\pi)dx$$

$$=1500\left[\left[\frac{-x\cos(20m\pi)}{20m\pi}\right]_0^{0.05}+\frac{1}{20m\pi}\int_0^{0.05}\cos(20m\pi)dx\right]-80\int_0^{0.05}\sin(20m\pi)dx$$

$$=1500\left[\frac{-0.05\cos(m\pi)+0}{20m\pi}\right]+80\left[\frac{(-1)^m-1}{20m\pi}\right]$$

$$G_m=\frac{-150(-1)^m}{m\pi}+\frac{160}{m\pi}\left[(-1)^m-1\right]$$

$$=\frac{10(-1)^m-160}{m\pi} \tag{39}$$

Substituting Equation (39) in Equation (38), we get

$$T_h(x,t)=\sum_{m=1}^{\infty}\left\{\frac{10(-1)^m-160}{m\pi}\right\}e^{-\alpha[2.5m\pi]^2t}\sin(2.5m\pi x) \tag{40}$$

Substituting Equations (40) and (24) in Equation (19) we have

$$T(x,t)=-1500x+100+\sum_{m=1}^{\infty}\left\{\frac{10(-1)^m-160}{m\pi}\right\}e^{-\alpha[20m\pi]^2t}\sin(20m\pi x) \tag{41}$$

Now

$$\alpha=\frac{k}{\rho C}$$

$$=\frac{54}{7800\times 490}$$

$$=1.4129\times10^{-5}\ m^2/s$$

and substituting the value of α in Equation (40) gives

$$T(x,t)=-1500x+100+\sum_{m=1}^{\infty}\left\{\frac{10(-1)^m-160}{m\pi}\right\}e^{-1.4129\times10^{-5}[20m\pi]^2t}\sin(20m\pi x) \tag{42}$$

Equation (42) is the analytical solution of the problem. Substituting the values of x and t gives the temperature inside the rod at a particular location and time. For example using the analytical solution, we will find the temperature of the rod at the first node, that is, $x=0.01m$ when $t=9$ secs.

$$T(0.01,9)=-1500(0.01)+100+\sum_{m=1}^{\infty}\left\{\frac{10(-1)^m-160}{m\pi}\right\}e^{-1.4129\times10^{-5}[20m\pi]^2\times9}\sin(0.2m\pi)$$

$$=62.510^{\circ}C$$

Similarly using Equation (42), the temperature of the rod at any location at any time can be found by substituting the corresponding values of x and t.

Comparison of the three numerical methods

To compare all three numerical methods with the analytical solution, the temperature values obtained at all the interior nodes at time, $t = 9$ sec are presented in the Table 1. From Table 1, it is clear that among the numerical methods used to solve partial differential equations, Crank-Nicolson method provides better accuracy compared to the other two numerical methods (Explicit Method and Implicit Method) explained in this chapter.

Table 1: Comparison of temperature obtained at interior nodes using different methods discussed in this chapter (absolute true error is given in parenthesis)

Temperature at Nodes	Explicit Method ($°C$)	Implicit Method ($°C$)	Crank-Nicolson Method ($°C$)	Analytical Solution ($°C$)
T_1^3	65.953(3.443)	59.043(3.467)	62.604(0.094)	62.510
T_2^3	39.132(2.048)	36.292(0.792)	37.613(0.529)	37.084
T_3^3	27.266(1.422)	26.809(0.965)	26.562(0.282)	25.844
T_4^3	22.872(0.738)	24.243(0.633)	24.042(0.432)	23.610

PARTIAL DIFFERENTIAL EQUATIONS

Topic	Parabolic Differential Equations
Summary	Textbook notes for the parabolic partial differential equations
Major	All engineering majors
Authors	Autar Kaw, Sri Harsha Garapati
Date	March 17, 2011
Web Site	http://numericalmethods.eng.usf.edu