

# Fundamentals of radio interferometry II

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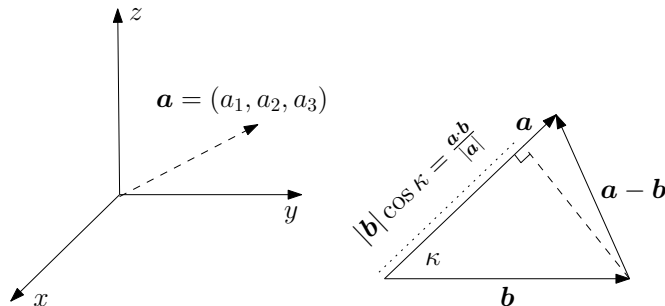
- ① **TCP:** G.B. Taylor et al., Synthesis imaging in radio astronomy II, 1998
- ② **CR:** J. J. Condon and S. M. Ransom, Essential radio astronomy, <http://www.cv.nrao.edu/course/astr534/ERA.shtml>

**PLEASE FIRST READ INTERFEROMETRY I AND II AVAILABLE  
AT: <http://www.cv.nrao.edu/course/astr534/ERA.shtml>  
BEFORE ATTEMPTING THE ASSIGNMENTS**

# Overview

- 1 Background
- 2 Flux density and brightness
- 3 Correlator, Fringes and Visibilities

# Vectors



- 1 Vector length:  $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$ .
- 2 Dot product:  $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = |\mathbf{a}| |\mathbf{b}| \cos \kappa$ .
- 3 Scalar projection of  $\mathbf{b}$  onto  $\mathbf{a}$ :  $\text{comp}_{\mathbf{a}} \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = |\mathbf{b}| \cos \kappa$ .
- 4 Commutativity:  $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

# Delayed Product Identities

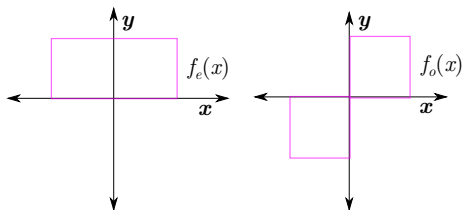
$$\cos(x - y) \cos(x) = \frac{1}{2} \cos(y) + \frac{1}{2} \cos(2x - y)$$

$$\sin(x - y) \sin(x) = \frac{1}{2} \cos(y) - \frac{1}{2} \cos(2x - y)$$

$$\sin(x - y) \cos(x) = -\frac{1}{2} \sin(y) + \frac{1}{2} \sin(2x - y)$$

$$\cos(x - y) \sin(x) = \frac{1}{2} \sin(y) + \frac{1}{2} \sin(2x - y)$$

# Even and Odd Functions



Any function  $f(x)$  can be uniquely written as

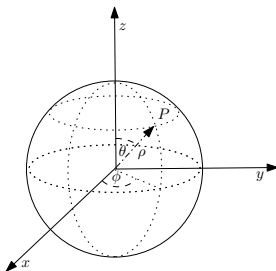
$$f(x) = f_e(x) + f_o(x)$$

where  $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$  and  $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$ .

# Even and Odd Functions: Properties

- ① The product of an even and an odd function is an odd function
- ② The product of two even functions is an even function
- ③ The product of two odd functions is an even function
- ④  $\int_{-a}^a f_o(x) dx = 0$

# Spherical Coordinates



## Relation between spherical and cartesian coordinates

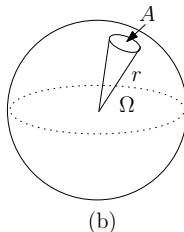
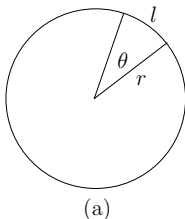
$$x = \rho \sin \theta \cos \phi$$

$$y = \rho \sin \theta \sin \phi$$

$$z = \rho \cos \theta$$



# Solid Angle



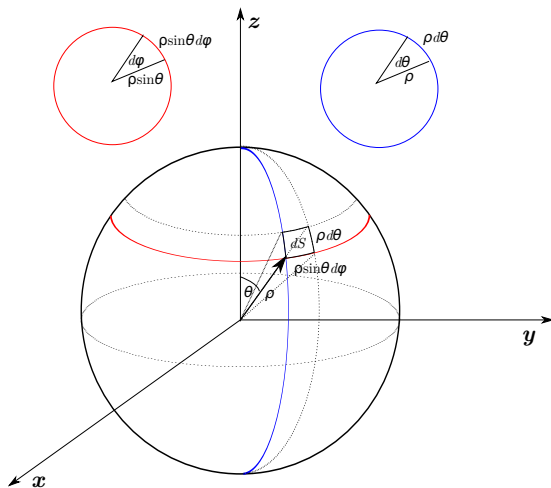
(a)  $\theta = \frac{l}{r}$  radians (rad) (b)  $\Omega = \frac{A}{r^2}$  steradians (sr)

$$\Omega_{\text{sphere}} = \frac{4\pi r^2}{r^2} = 4\pi \text{ sr}$$

## Conversion

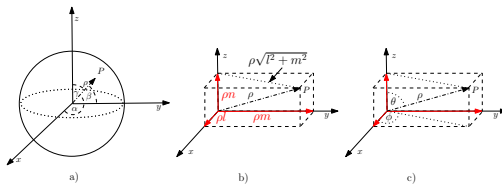
1 steradian (sr) =  $1 \text{ rad}^2 = 3282.8 \text{ deg}^2 = 4.25 \times 10^{10} \text{ arcsec}^2$ , where  $4\pi \text{ sr} = \text{sphere}$ .

# Infinitesimal Solid Angle $d\Omega$ : Spherical Coordinates



$$d\Omega = \frac{dS}{\rho^2} = \frac{\rho^2 \sin \theta d\theta d\phi}{\rho^2} = \sin \theta d\theta d\phi$$

# Infinitesimal Solid Angle $d\Omega$ : Direction Cosine Coordinates



$$\phi = \tan^{-1} \frac{l}{m} = f(l, m) \mid \theta = \sin^{-1} \sqrt{l^2 + m^2} = g(l, m)$$

$$\begin{aligned} d\Omega &= \sin \theta d\theta d\phi = \sin(g(l, m)) |J| dl dm \\ &= \frac{dl dm}{\sqrt{1 - l^2 - m^2}} = \frac{dl dm}{n} \quad \text{TCP (p. 21 Eq. 2.20)} \end{aligned}$$

$$|J| = \begin{vmatrix} \frac{d\theta}{dl} & \frac{d\theta}{dm} \\ \frac{d\phi}{dl} & \frac{d\phi}{dm} \end{vmatrix}$$

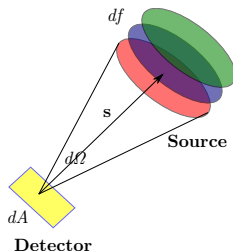
# Why interferometry?

- 1 Resolution with one dish:  $\frac{\lambda}{D}$ , where  $D$  is the diameter of the dish.
- 2 Resolution with an interferometer:  $\frac{\lambda}{|b|}$ , where  $|b|$  is the length of the longest baseline.

## Number of baselines

$$N_b = \frac{N_a^2 - N_a}{2}$$

# What are we trying to measure with an interferometer: Spectral radiance



- 1 Brightness or **spectral radiance**: is the total power received per unit area, per unit solid angle in the direction  $s$ , per unit frequency at frequency  $f$ .
- 2 The unit of spectral radiance is :  $\text{Wm}^{-2}\text{sr}^{-1}\text{Hz}^{-1}$ .
- 3 We denote the brightness in the direction  $s$  at frequency  $f$  with  $I(s)$ .
- 4 We denote the effective collecting area in the direction  $s$  at frequency  $f$  with  $A(s)$  [measured in  $\text{m}^2$ ].

# Flux density

- 1 The flux density  $F$  at frequency  $f$  is defined as

$$F = \int_S I(\mathbf{s}) d\Omega$$

The above integral is taken over the entire surface  $S$  of the celestial sphere, which subtends  $4\pi$  steradians.

- 2 The above integral is a **surface integral**.
- 3 The unit of flux is:  $\text{Wm}^{-2}\text{Hz}^{-1}$ .
- 4 Jansky:  $1 \text{ Jy} = 10^{-26} \text{ Wm}^{-2}\text{Hz}^{-1}$ .

Power in bandwidth  $\Delta f$  per  $d\Omega$

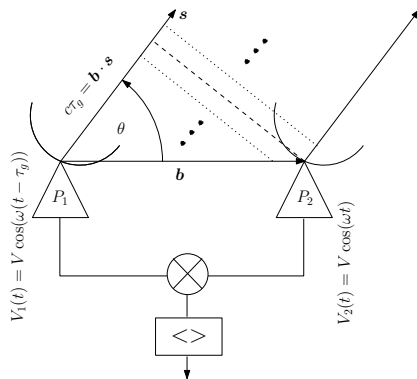
$$\Delta f A(\mathbf{s}) I(\mathbf{s}) d\Omega$$

# NB ASSUMPTIONS

- WE ASSUME A STATIONARY SKY AND INTERFEROMETRIC REFERENCE FRAME.  
Neither the sky nor the interferometer is moving.
- Quasi-monochromatic observation. Assume we are observing in a narrow bandwidth.
- Unpolarized
- Consider the electric fields from a solid angle  $d\Omega$  in the direction  $\mathbf{s}$ , in some small bandwidth  $\Delta f$  at  $f$ .
- We can express the temporal dependence of this field with

$$V(t) = V \cos(2\pi ft + \phi).$$

# Two element interferometer $d\Omega$



- 1  $c\tau_g = \Delta s$ :  $v \times \Delta t = \Delta s$ , where  $c$  is the speed of light and  $\tau_g$  is the difference in the arrival times of the plane waves.
- 2  $\mathbf{s} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{s} = \Delta s$ : the scalar projection of  $\mathbf{b}$  onto  $\mathbf{s}$ , with  $|\mathbf{s}| = 1$ .

$$c\tau_g = \mathbf{b} \cdot \mathbf{s} \quad \text{TCP (p. 12 Eq. 2.1)}$$



$$\begin{aligned}
 \langle V_1(t)V_2(t) \rangle &= \langle V^2 \cos(2\pi f(t - \tau_g)) \cos(2\pi ft) \rangle \\
 &= \underbrace{\left\langle \frac{V^2}{2} \cos(2\pi f\tau_g) \right\rangle}_{\text{slowly varying}} + \underbrace{\left\langle \frac{V^2}{2} \cos(4\pi ft - 2\pi f\tau_g) \right\rangle}_{\text{rapidly varying}} \\
 &= \left\langle \frac{V^2}{2} \cos\left(\frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) \right\rangle + \left\langle \frac{V^2}{2} \cos\left(4\pi ft - \frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) \right\rangle \\
 &\approx \frac{V^2}{2} \cos\left(\frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) + \left\langle \frac{V^2}{2} \cos\left(4\pi ft - \frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) \right\rangle \xrightarrow{0}
 \end{aligned}$$

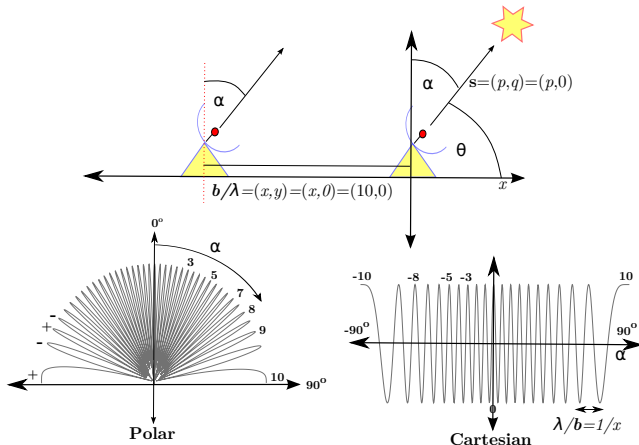
- ① The first term varies slowly as  $\mathbf{b} \cdot \mathbf{s}$  depends on the rotation speed of the earth (which is slow).
- ② The second term varies rapidly, since radio observations are high frequency observations – 1.4GHz.

based on TCP (p. 12 Eq. 2.4)

# Understanding the correlator output

- 1 The power in the signal  $A \cos(2\pi ft)$  is equal to  $\frac{A^2}{2}$ .
- 2 We can therefore say that  $V \propto \sqrt{P(\mathbf{s})}$ , where  $P(\mathbf{s})$  is the **power** received, per unit solid angle from **direction**  $\mathbf{s}$ , per unit frequency at frequency  $f$ .  $P(\mathbf{s})$  is measured in  $\text{Wsr}^{-1}\text{Hz}^{-1}$ .
- 3  $P(\mathbf{s}) = A(\mathbf{s})I(\mathbf{s})$ , where  $A$  is the **effective collecting area** of the two element interferometer and  $I$  is the **sky brightness**.
- 4  $\cos\left(\frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right)$  is a **modulation factor**. It changes sinusoidally with the change of source direction in the interferometer frame. These sinusoids are called **fringes**.

# Understanding 1D fringes



$$\begin{aligned} \cos\left(\frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) &= \cos(2\pi x p) = \cos(2\pi x \cos(\theta)) \\ &= \cos(2\pi x \sin(\alpha)) = \cos(2\pi(10) \sin(\alpha)) \end{aligned}$$

# The response from an extend source

The response from an **extended source** is obtained by **summing the responses** at each antenna to the emission over the **entire sky**, **multiplying** the two and **averaging**:

$$R_c(\mathbf{b}) = \left\langle \int_{S_1} V_1(t, \mathbf{s}_1, \mathbf{b}) d\Omega_1 \int_{S_2} V_2(t, \mathbf{s}_2, \mathbf{b}) d\Omega_2 \right\rangle$$

If we evaluate the above under the assumption that the emission is **spatially incoherent** we obtain:

Interferometer response  $R_c$

$$R_c^N(\mathbf{b}) = \int_S A_N(\mathbf{s}) I(\mathbf{s}) \cos\left(\frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda}\right) d\Omega$$

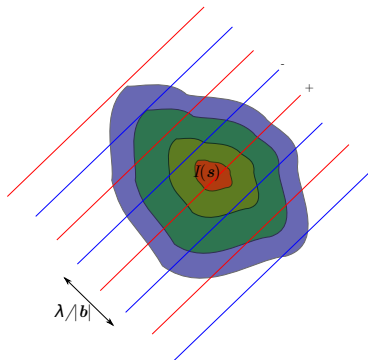
where  $A_N(\mathbf{s}) = \frac{A(\mathbf{s})}{A_0}$  is the normalized effective area. We now assume for the sake of simplicity that  $A_N(\mathbf{s}) = 1$ .

based on TCP (p. 13 Eq. 2.5)

# The response from an extended source: minor details

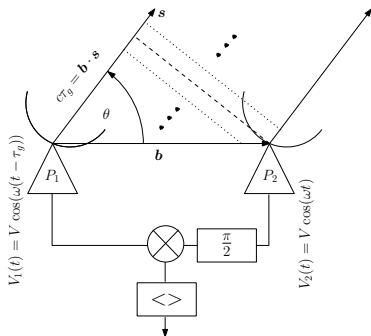
$$\begin{aligned} R_c(\mathbf{b}) &= \left\langle \int_{S_1} V_1(t, \mathbf{s}_1, \mathbf{b}) d\Omega_1 \int_{S_2} V_2(t, \mathbf{s}_2, \mathbf{b}) d\Omega_2 \right\rangle \\ &= \int_{S_1} \int_{S_2} \langle V_1(t, \mathbf{s}_1, \mathbf{b}) \cdot V_2(t, \mathbf{s}_2, \mathbf{b}) \rangle d\Omega_1 d\Omega_2 \\ &= \int \int_{\mathbf{s}_1=\mathbf{s}_2} \langle V_1(t, \mathbf{s}_1, \mathbf{b}) \cdot V_2(t, \mathbf{s}_2, \mathbf{b}) \rangle d\Omega_1 d\Omega_2 + \dots \\ &\dots \int \int_{\mathbf{s}_1 \neq \mathbf{s}_2} \langle V_1(t, \mathbf{s}_1, \mathbf{b}) \cdot V_2(t, \mathbf{s}_2, \mathbf{b}) \rangle d\Omega_1 d\Omega_2 \\ &= \int_S \langle V_1(t, \mathbf{s}, \mathbf{b}) \cdot V_2(t, \mathbf{s}, \mathbf{b}) \rangle d\Omega \\ &= \int_S P(\mathbf{s}) \cos(\omega \tau_g) d\Omega \\ &= \int_S A(\mathbf{s}) \cdot I(\mathbf{s}) \cos\left(\frac{2\pi f \mathbf{b} \cdot \mathbf{s}}{c}\right) d\Omega. \end{aligned}$$

# 2D fringes



- The correlator **casts** a **sinusoidal coherence pattern** with an angular scale  $\sim \frac{\lambda}{|b|}$  onto the sky and then **multiplies** it with the **sky brightness**.
- The correlator then **multiplies** this coherence pattern with the sky brightness and then **integrates** over the entire sky.

# SIN Correlator



$$\text{Output: } \frac{V^2}{2} \sin \left( \frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda} \right)$$

$$R_s^N(\mathbf{b}) = \int_S I(\mathbf{s}) \sin \left( \frac{2\pi \mathbf{b} \cdot \mathbf{s}}{\lambda} \right) d\Omega$$

# Why the SIN correlator?

- The cosine correlator can only detect the even part of  $I(\mathbf{s})$

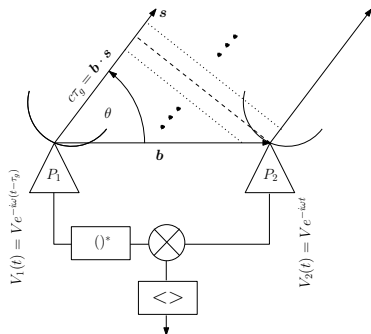
$$\begin{aligned} R_c^N(\mathbf{b}) &= \int_S I(\mathbf{s}) \cos\left(\frac{2\pi\mathbf{b} \cdot \mathbf{s}}{\lambda}\right) d\Omega \\ &= \int_S I_e(s) \cos\left(\frac{2\pi\mathbf{b} \cdot \mathbf{s}}{\lambda}\right) d\Omega \end{aligned}$$

- Similarly

$$\begin{aligned} R_s^N(\mathbf{b}) &= \int_S I(s) \sin\left(\frac{2\pi\mathbf{b} \cdot \mathbf{s}}{\lambda}\right) d\Omega \\ &= \int_S I_o(s) \sin\left(\frac{2\pi\mathbf{b} \cdot \mathbf{s}}{\lambda}\right) d\Omega \end{aligned}$$



# Complex correlator

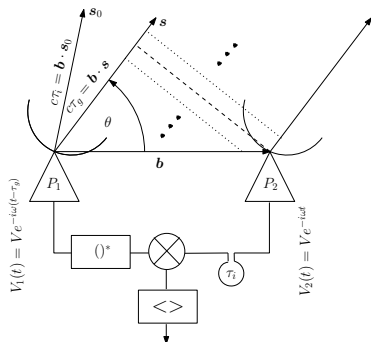


## Complex response

Output:  $V^2 e^{-2\pi i \frac{\mathbf{b} \cdot \mathbf{s}}{\lambda}}$

$$\tilde{V}(\mathbf{b}) = R_c^N - iR_s^N = \int_S I(\mathbf{s}) e^{-2\pi i \frac{\mathbf{b} \cdot \mathbf{s}}{\lambda}} d\Omega$$

# Fringe stopping/tracking

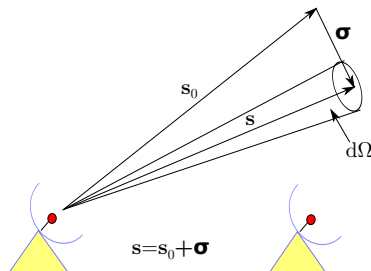


We can stop the fringes at one particular position in the sky  $\mathbf{s}_0$  by adding a time delay  $\tau_i$ .

## Complex response

$$\text{Output: } V^2 e^{-2\pi i \frac{\mathbf{b} \cdot (\mathbf{s} - \mathbf{s}_0)}{\lambda}}$$

# Complex Visibilities



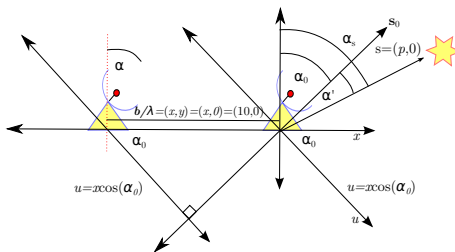
## Complex Visibilities

$$V(\mathbf{b}) = \int_S I(\sigma) e^{-2\pi i \frac{\mathbf{b} \cdot \sigma}{\lambda}} d\Omega$$

Units: Jy

TCP (p. 14 Eq. 2.7)

# Understanding fringe stopping/tracking



$$\begin{aligned}
 \cos\left(2\pi \frac{\mathbf{b} \cdot \boldsymbol{\sigma}}{\lambda}\right) &= \cos\left(2\pi \frac{\mathbf{b} \cdot (\mathbf{s} - \mathbf{s}_0)}{\lambda}\right) \\
 &= \cos(2\pi x (\sin \alpha_s - \sin \alpha_0)) = \cos(2\pi x (\sin(\alpha_0 + \alpha') - \sin \alpha_0)) \\
 &= \cos(2\pi x (\sin \alpha_0 \cos \alpha' + \cos \alpha_0 \sin \alpha' - \sin \alpha_0)) \\
 &\approx \cos(2\pi x \cos \alpha_0 \sin \alpha') = \cos(2\pi u \sin \alpha')
 \end{aligned}$$

We have therefore effectively shifted the fringe pattern to the field center.

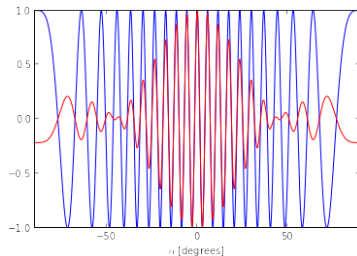
# The effect of bandwidth (no fringe stopping)

To find the finite bandwidth response we integrate our fundamental response over a frequency width  $\Delta f$ , centered at  $f_0$ :

$$\begin{aligned} V &= \int_S \frac{1}{\Delta f} \int_{f_0 - \frac{\Delta f}{2}}^{f_0 + \frac{\Delta f}{2}} I(\mathbf{s}) e^{-2\pi f \tau_g} df d\Omega \\ &= \int_S I(\mathbf{s}) \text{sinc}(\tau_g \Delta f) e^{-2\pi i f_0 \tau_g} d\Omega \end{aligned}$$

- 1  $\text{sinc}(\tau_g \Delta f)$  is known as the **fringe washing function** or the **fringe attenuation function**.
- 2 The fringe washing function attenuates the sources far from the meridional plane.
- 3  $\text{sinc}(x) = 0$  when  $x = 1$ .

# 1D Example



$$\text{sinc}\left(\frac{\Delta f}{c} b \sin(\alpha)\right) \cos(2\pi x \sin(\alpha))$$

$$x = 10 \quad b = 2\frac{1}{7} \quad \Delta f = 197.99 \text{ MHz} \quad f = 1.4 \text{ GHz}$$

First null

$$\frac{\Delta f}{c} b \sin(\alpha) = 1 \rightarrow \alpha_o \approx 45^\circ$$

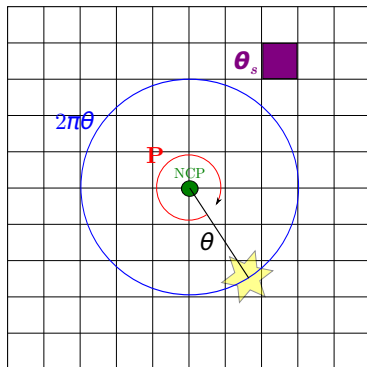
- 1 By fringe tracking we **shift the fringe pattern** to the field center, but what is even more important is that we **simultaneously** also **shift the fringe washing function** to the field center, which means we **do not** severely **attenuate** the **sources** close to the **field center**. **NB — Fringe stopping extremely important.**
- 2 The usable field of view  $2\Delta\theta$ :

$$\Delta\theta\Delta f \ll \theta_s f_0,$$

where  $\theta_s = \frac{\lambda}{|b|}$  is the angular resolution of interferometer.

CR (Int I Eq. 3F2)

# Time-smearing loss



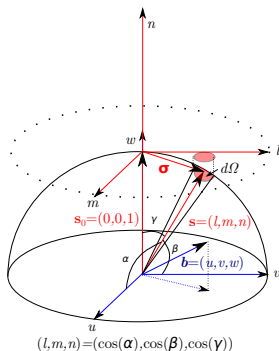
$$\frac{2\pi\Delta\theta}{P}\Delta t \ll \theta_s = \frac{\lambda}{|\mathbf{b}|}$$

$$P \approx 86164s \approx 23h56m04s$$

CR (Int I Eq. 3F3)



# Choosing an appropriate coordinate system

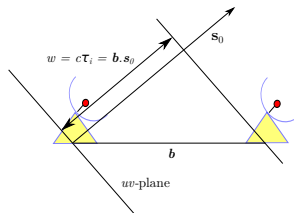


## Visibilities

$$V(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} l(l, m) e^{-2\pi i [ul + vm + w(\sqrt{1-l^2-m^2}-1)]} \frac{dl dm}{\sqrt{1-l^2-m^2}}$$

TCP (p. 21 Eq. 2.21)

# A closer look at the $w$ -term



$$w \approx \frac{|\mathbf{b}|}{\lambda} \approx \frac{1}{\theta_s} \mid \gamma = \frac{\Delta\theta_F}{2}$$

$$2\pi(\sqrt{1 - l^2 - m^2} - 1)w = 2\pi(\cos(\gamma) - 1)w \approx -\pi\gamma^2 w$$

$$\pi\gamma^2 w < 0.1 \rightarrow \frac{\pi\Delta\theta_F^2}{4\theta_s} < 0.1$$

When is  $w$  small enough to be negligible

$$\Delta\theta_F < \frac{1}{3}\sqrt{\theta_s}$$

## Fourier transform relationship

$$\begin{aligned}
 V(u, v) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(l, m) e^{-2\pi i [ul + vm + w(\sqrt{1-l^2-m^2}-1)]} \frac{dl dm}{\sqrt{1-l^2-m^2}} \\
 &\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I(l, m) e^{-2\pi i (ul + vm)} dl dm \\
 &= \mathcal{F}\{I(l, m)\} \\
 I(l, m) &\approx \mathcal{F}^{-1}\{V(u, v)\}
 \end{aligned}$$