CHAPTER – 8

CONTOUR INTEGRATION

In this chapter we will evalute a varity types of real definite integrals with the help of the Cauchy's residue theorem using a suitable types of closed path or contour. For this reason, the process is called contour integration. Now we will discuss it dividing several forms.

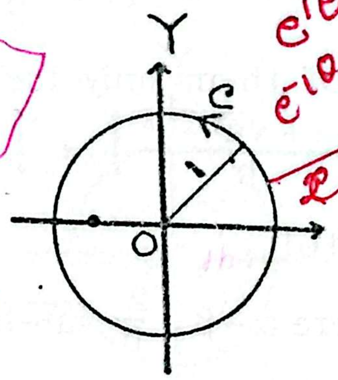
234. (Form 1). Integrals of the form :

$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$, **where** $f(\cos\theta, \sin\theta)$ **is a rational function of** $\cos\theta$ **and** $\sin\theta$ **:**

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ ... (1), where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$. Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z}$,

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$ and $dz = ie^{i\theta}\, d\theta$

$= iz\, d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these (1) $\Rightarrow I = \oint_C g(z)\, dz$,



$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$
$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

where C is the unit circle $|z| = 1$, whose centre is at the origin and radius is equal to 1.

Example 270 : Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ if $a > |b|$.

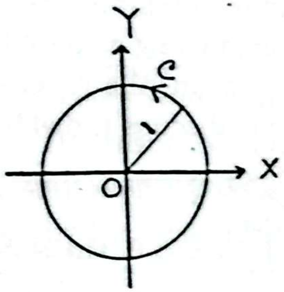
D. U. H. T. '75, 77, 87; D. U. H. '87.

$e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z}$

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ ... (1)

Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}$

and $dz = ie^{iz}\, d\theta = iz\, d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these



$$(1) \Rightarrow \oint_C \frac{1}{a + b\frac{z^2+1}{2z}} \frac{dz}{iz}$$

$$= \frac{2}{i} \oint_C \frac{dz}{bz^2 + 2az + b} \text{ ... (2).}$$

where C is the unit circle $|z| = 1$ whose radius is 1 and centre at the origin. The poles of $\frac{1}{bz^2 + 2az + b}$ are obtained by solving $bz^2 + 2az + b = 0$. If α and β are the poles, then we suppose $z = \alpha = \frac{-a + \sqrt{a^2 - b^2}}{b}$ and $z = \beta = \frac{-a - \sqrt{a^2 - b^2}}{b}$.

Of them only the pole α lies inside C since $|\alpha|$ $= \left|\frac{-a + \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{a - \sqrt{a^2 - b^2}}{b}\right| = \left|\frac{b}{a + \sqrt{a^2 - b^2}}\right| < 1$

if $a > |b|$.

Here $\alpha - \beta = \frac{2}{b}\sqrt{a^2 - b^2}$ ... (3) Now

$$\text{Res}(\alpha) = \lim_{z \to \alpha} \left\{ (z - \alpha) \frac{1}{bz^2 + 2az + b} \right\}$$

$$= \lim_{z \to \alpha} \frac{(z - \alpha)}{b(z - \alpha)(z - \beta)} = \frac{1}{b(\alpha - \beta)} = \frac{1}{2\sqrt{a^2 - b^2}}, \text{ by ... (3)}$$

Now by the Cauchy's residue theorem

$(2) \Rightarrow I = \frac{2}{i} \cdot 2\pi i \cdot \text{Res}(\alpha) = \frac{2\pi}{\sqrt{a^2 - b^2}}$ and the required result is obtained.

Example - 271 : Show that $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$ $= \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ if $a > |b|$.

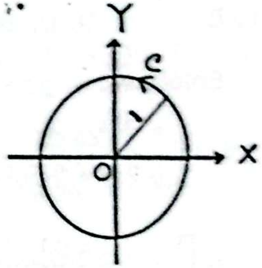
Solution : Try yourself.

Example - 272 : Show that $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta} = \frac{\pi}{2}$.

Solution : Try yourself.

Example - 273 : Show that $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta} = \frac{2\pi}{1 - a^2}$, $0 < a < 1$. **D. U. H. '84; R. U. '83; R. U. H. '73, 75, 81.**

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a\cos\theta}$ ... (1).



Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2 + 1}{2z}$

and $dz = ie^{i\theta}\, d\theta = iz d\theta$ or $d\theta = \frac{dz}{iz}$.

Using these (1) $\Rightarrow I = \oint_C \frac{1}{1 + a^2 - a\frac{z^2+1}{z}} \frac{dz}{iz}$

$$= -\frac{1}{ai} \oint_C \frac{dz}{z^2 - (a + a^{-1})z + 1}$$

$$= -\frac{1}{ai}\oint_C \frac{dz}{(z-a)(z-a^{-1})} \quad \dots (2)$$

where C is the circle $|z| = 1$.

The poles of $\frac{1}{(z-a)(z-a^{-1})}$ are the simple poles $z = a$, a^{-1}. Of them only $z = a$ lies inside C since $|z| < 1$ as $a < 1$.

$$\text{Now Res }(a) = \lim_{z \to a} \left\{ (z-a) \frac{1}{(z-a)(z-a^{-1})} \right\}$$

$$= \frac{1}{a-a^{-1}} = \frac{a}{a^2-1}$$

Now by the Cauchy's residue theorem

$$(2) \Rightarrow I = -\frac{1}{ai} \cdot 2\pi i.\ \text{Res }(a) = \frac{2\pi}{1-a^2}$$ and the required result is proved.

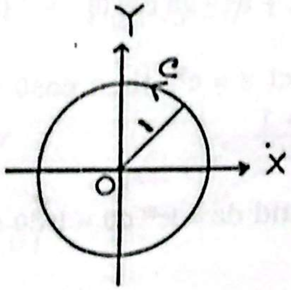
Example – 274 : Show that $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ where $a > 0$.

D. U. '84; D. U. H. T. '86; D. U. H. '86; R. U. '83.

Solution : Suppose I =

$$\int_0^{\pi} \frac{2a\,d\theta}{2a^2+2\sin^2\theta}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta} \dots (1).$$



Let $t = 2\theta$,

then $dt = 2d\theta$ and the limit of t is from o to 2π. Using these $(1) \Rightarrow I = \int_0^{2\pi} \frac{a\,dt}{2a^2+1-\cos t} \dots (2)$.

Now putting $z = e^{it}$, then $dz = ie^{it}\,dt = iz dt$ or $dt = \frac{dz}{iz}$

and $\cos t = \frac{e^{it}+e^{-it}}{2} = \frac{z^2+1}{2z}$.

$$\text{Now using these } (2) \Rightarrow I = \oint_C \frac{a}{2a^2+1-\frac{z^2+1}{2z}} \frac{dz}{iz}$$

$$= \frac{-2a}{i} \oint_C \frac{dz}{z^2-2(2a^2+1)z+1} \dots (3).$$

where C is the circle $|z| = 1$. The poles of $\frac{1}{z^2-2(2a^2+1)z+1}$ are obtained by solving $z^2-2(2a^2+1)z+1 = 0$. If α and β are these poles, then

$$z = \frac{2(2a^2+1) \pm \sqrt{4(2a^2+1)^2-4}}{2} = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

where $\alpha = 2a^2+1+2a\sqrt{a^2+1} = (\sqrt{a^2+1}+a)^2$ and

$\beta = 2a^2+1-2a\sqrt{a^2+1} = (\sqrt{a^2+1}-a)^2$. Here $\alpha\beta = 1$

where $\alpha > 1$, $\beta < 1$ and $\beta - \alpha = -4a\sqrt{1+a^2} \dots (4)$

Only the pole β lies inside C and it is a simple pole.

$$\text{Now Res }(\beta) = \lim_{z \to \beta} \left\{ (z-\beta) \frac{1}{z^2-2(a^2+1)z+1} \right\}$$

$$= \lim_{z \to \beta} \frac{z-\beta}{(z-\alpha)(z-\beta)} = \frac{1}{\beta-\alpha} = \frac{-1}{4a\sqrt{1+a^2}}, \text{ by } (4)$$

Hence by the Cauchy's residue theorem $(3) \Rightarrow$

$$I = \frac{-2a}{i} \cdot 2\pi i.\ \text{Res }(\beta) = -4a\pi \cdot \frac{-1}{4a\sqrt{1+a^2}} = \frac{\pi}{\sqrt{1+a^2}}$$

and we obtained the required result.

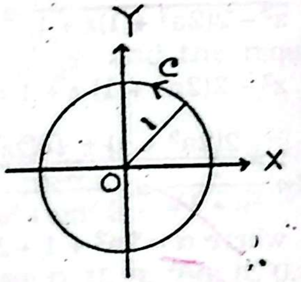
Example -275 : Show that $\int_0^{\pi} \frac{a\,d\theta}{a^2+\cos^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$ where $a > 0$.

Solution : Try yourself.

Example – 276 : Show that $\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = \pi$.

Solution : Suppose $I = \int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$ ... (1).

Let $z = e^{i\theta}$, then $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$ $= \frac{z^2-1}{2iz}$, $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ $= \frac{z^2+1}{2z}$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these (1) $\Rightarrow I =$



$\oint_C \frac{2dz}{(1-2i)z^2+6iz-1-2i}$ ...(2), where C is the circle $|z| = 1$.

The poles of $\frac{2}{(1-2i)z^2+6iz-1-2i}$ are obtained by solving $(1-2i)z^2+6iz-1-2i = 0 \Rightarrow z = \frac{-6i \pm 4i}{2(1-2i)}$

$= \frac{-2i}{10}(1+2i), \frac{-10i}{10}(1+2i) \Rightarrow z = \frac{1}{5}(2-i), 2-i$ and they are simple. Of them only the pole $\frac{1}{5}(2-i)$ lies inside C. Now

$\text{Res}\left\{\frac{1}{5}(2-i)\right\}$

$= \lim\limits_{z \to \frac{1}{5}(2-i)} \left[\left\{z - \frac{1}{5}(2-i)\right\} \frac{2}{(1-2i)z^2+6iz-1-2i}\right]$

[by L. Hospital's rule]

$= \lim\limits_{z \to \frac{1}{5}(2-i)} \frac{2}{2(1-2i)z+6i}$

$= \frac{5}{2-2-i-4i+15i} = \frac{1}{2i}$

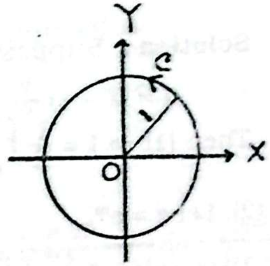
Now by the Cauchy's residue theorem (2) $\Rightarrow$

$I = 2\pi i.\ \text{Res}\left\{\frac{1}{5}(2-i)\right\} = 2\pi i.\ \frac{1}{2i} = \pi$ and the required value is obtained.

Example – 277 : Show that $\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta} = \frac{\pi}{6}$.

Solution : Suppose $I = \int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{5+4\cos\theta}$

$=$ Real part of $\int_0^{2\pi} \frac{e^{2i\theta}\,d\theta}{5+4\cos\theta}$ ...



(1).

Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ $= \frac{z^2+1}{2z}$ and $dz = ie^{i\theta}d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$.

Now using these (1) $\Rightarrow I =$ Real part of $\frac{1}{i}\oint_C \frac{z^2 dz}{2z^2+5z+2}$ ... (2) where C is the unit circle $|z| = 1$. The poles of $\frac{z^2}{2z^2+5z+2}$ are obtained by solving $2z^2+5z+z = 0$

$\Rightarrow (z+2)(2z+1) = 0 \Rightarrow z = -2, \frac{-1}{2}$ and they are simple poles. Of them only $z = -\frac{1}{2}$ lies inside C. Now

$$\operatorname{Res}\left(\frac{-1}{2}\right) = \lim_{z \to -\frac{1}{2}} \left\{\left(z+\frac{1}{2}\right) \frac{z^2}{(z+2)(2z+1)}\right\}$$

$$= \frac{\left(\frac{-1}{2}\right)^2}{2\left(\frac{-1}{2}+2\right)} = \frac{1}{12}$$

Now by the Cauchy's residue theorem,

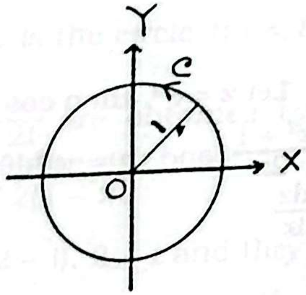
(2) $\Rightarrow I =$ Real part of $\frac{1}{i}\left\{2\pi i. \operatorname{Res}\left(\frac{-1}{2}\right)\right\}$

$\Rightarrow I = 2\pi.\left(\frac{1}{12}\right) = \frac{\pi}{6}$ and the result is proved.

Example – 278 : Show that $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0.$

Solution : Suppose $I = \int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0.$ ..... (1)

Then (1) $\Rightarrow I = \frac{1}{2}\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta$



... (2). Let $z = e^{i\theta}$,

then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$ and

$dz = ie^{i\theta}d\theta$

$= izd\theta$ or $d\theta = \frac{dz}{iz}$. Now using

these (2) $\Rightarrow$

$$I = \frac{1}{2i}\oint_C \frac{z^2+z+1}{z(2z^2+5z+2)} = \frac{1}{4i}\oint_C \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)} dz \; ... (3)$$

where C is the circle $|z| = 1$.

The poles of $\frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}$ are obtained by solving

$z\left(z+\frac{1}{2}\right)(z+2) = 0 \Rightarrow z = 0, -\frac{1}{2}, -2$

and they are simple. Of them the poles $z = 0, -\frac{1}{2}$ lie inside C. Now

$$\operatorname{Res}(0) = \lim_{z\to 0}\left\{z \frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\right\} = 1 \text{ and } \operatorname{Res}\left(-\frac{1}{2}\right)$$

$$= \lim_{z\to -\frac{1}{2}}\left\{\left(z+\frac{1}{2}\right)\frac{z^2+z+1}{z\left(z+\frac{1}{2}\right)(z+2)}\right\} = \frac{\frac{1}{4}-\frac{1}{2}+1}{\left(\frac{-1}{2}\right)\left(\frac{-1}{2}+2\right)}$$

$= -1$

Hence by the Cauchy's residue theorem (3) $\Rightarrow$

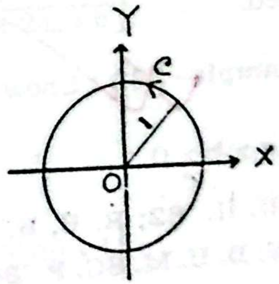
$$I = \frac{1}{4i}.2\pi i\left[\operatorname{Res}(0) + \operatorname{Res}\left(-\frac{1}{2}\right)\right] = \frac{\pi}{2}(1-1) = 0$$

Hence the result is proved.

Example – 279 : Show that $\int_0^{\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta = \frac{\pi a^2}{1-a^2}$

if $a^2 < 1$. **D. U. H. T. '76, '86; C. U. H. '90.**

Solution : Suppose I =



$$\int_0^{\pi} \frac{\cos 2\theta \, d\theta}{1-2a\cos\theta+a^2}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{\cos 2\theta}{1-2a\cos\theta+a^2} d\theta$$

$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{e^{2i\theta}\,d\theta}{1 - a(e^{i\theta} + e^{-i\theta}) + a^2}$ ... (1). Let $z = e^{i\theta}$

then

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z} \text{ and } dz = ie^{i\theta}\,d\theta = iz\,d\theta \text{ or } d\theta = \frac{dz}{iz}$$

Now using these (1) $\Rightarrow$

$$I = \text{Real part of } \frac{1}{2}\oint_C \frac{z^2}{1 - a\frac{z^2+1}{z} + a^2}\,\frac{dz}{iz}$$

$$= \text{Real part } \frac{-1}{2ai}\oint_C \frac{z^2\,dz}{z^2 - (a + a^{-1})z + 1} \quad \text{... (2)}$$

where C is the circle $|z| = 1$. Here the poles of

$\frac{z^2}{z^2 - (a + a^{-1})z + 1} = \frac{z^2}{(z-a)(z-a^{-1})}$ are the simple poles $z = a, a^{-1}$. Of which only $z = a$ lies inside C since $|z| < 1$ as $a < 1$. Now

$$\text{Res}(a) = \lim_{z\to a}\left\{(z-a)\frac{z^2}{(z-a)(z-a^{-1})}\right\} = \frac{a^3}{a^2-1}.$$

Hence by the Cauchy's residue theorem.

$$(2) \Rightarrow I = \text{Real part of } \frac{-1}{2ai}\left\{2\pi i.\ \text{Res}(a)\right\}.$$

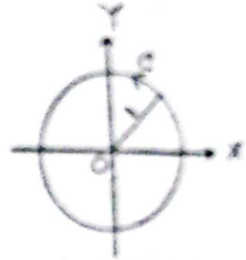
$\Rightarrow I = \frac{-1}{a}.\ \pi.\ \frac{a^3}{a^2-1} = \frac{\pi a^2}{1-a^2}$ and the required result is obtained.

Example- 280 : Show that $\int_0^{2\pi} \frac{\sin^2\theta\,d\theta}{a + b\cos\theta} = \frac{2\pi}{b^2}(a - \sqrt{a^2-b^2})$ where $a > b > 0$.

C. U. H. '82; R. U. H. '74; D. U. H. '90; D. U. H. T. '77, '83, '87; D. U. M. SC. P. '84.

Solution : Suppose $I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\,d\theta$

$= \int_0^{2\pi} \frac{1 - \cos 2\theta}{2a + 2b\cos\theta}\,d\theta = \text{Real part of } \int_0^{2\pi} \frac{1 - e^{2i\theta}}{2a + b(e^{i\theta} + e^{-i\theta})}\,d\theta$ ... (1).



Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z}$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ or $d\theta = \frac{dz}{iz}$.

Using these (1) $\Rightarrow I = \text{Real part of } \int_C \frac{(1-z^2)}{2a + b\frac{(z^2+1)}{z}}\,\frac{dz}{iz}$

$= \text{Real part of } \frac{1}{i}\oint_C \frac{(1-z^2)\,dz}{bz^2 + 2az + b}$ ... (2). where C is the circle $|z| = 1$. The poles of $\frac{1-z^2}{bz^2+2az+b}$ are obtained by solving $bz^2 + 2az + b = 0$.

Now if α and β are the poles, then $z = \alpha = \frac{-a + \sqrt{a^2-b^2}}{b}$ and $z = \beta = \frac{-a - \sqrt{a^2-b^2}}{b}$.

Of them α lies inside C, which is simple.

$$\text{Now Res}(\alpha) = \lim_{z\to\alpha}\left\{(z-\alpha)\frac{(1-z^2)}{bz^2+2az+b}\right\}$$

$$= \lim_{z\to\alpha}\left\{(z-\alpha)\frac{1-z^2}{b(z-\alpha)(z-\beta)}\right\}$$

$$= \frac{1-\alpha^2}{b(\alpha-\beta)} = \frac{\alpha\beta-\alpha^2}{b(\alpha-\beta)} = -\frac{\alpha}{b} \qquad [\because \alpha\beta = 1]$$

$$\Rightarrow \text{Res}\,(\alpha) = \frac{a-\sqrt{a^2-b^2}}{b^2}$$

Now by the Cauchy's residue theorem

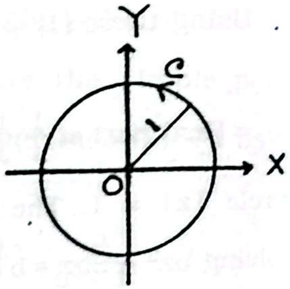
$$(2) \Rightarrow I = \text{Real Part of } \frac{1}{i}\left\{2\pi i.\ \text{Res}\,(\alpha)\right\}$$

$\Rightarrow I = 2\pi.\ \text{Res}\,(\alpha) = \dfrac{2\pi}{b^2}\left(a-\sqrt{a^2-b^2}\right)$ and the required result is obtained.

Example – 281 : Show that $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}$.

D. U. M. SC. P. T. '90.

Solution : Suppose $I = \displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta$ ... (1)



Let $z = e^{i\theta}$, then $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z^2+1}{2z}$.

$\therefore \cos 3\theta = \dfrac{e^{3i\theta}+e^{-3i\theta}}{2} = \dfrac{z^6+1}{2z^3}$ and

$dz = ie^{i\theta}\,d\theta = iz\,d\theta$

or $d\theta = \dfrac{dz}{iz}$. Using these (1) $\Rightarrow$

$$I = \oint_C \frac{(z^6+1)/2z^3}{5-4(z^2+1)/2z}\,\frac{dz}{iz} = -\frac{1}{2i}\oint_C \frac{z^6+1}{z^3(2z-1)(z-2)}\,dz$$

... (2), where C is the circle $|z| = 1$.

The poles of $\dfrac{z^6+1}{z^3(2z-1)(z-2)}$ are obtained by solving

$z^3(2z-1)(z-2) = 0 \Rightarrow z = 0$ is a pole of order 3 and $z = \dfrac{1}{2}$, 2 are simple poles.

Of them $z = 0$ and $z = \dfrac{1}{2}$ lie inside C. Now

$$\text{Res}\,(0) = \lim_{z\to 0} \frac{1}{2!}\frac{d^2}{dz^2}\left\{z^3\frac{(z^6+1)}{z^3(2z-1)(z-2)}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{z^6+1}{(2z-1)(z-2)}\right\}$$

$$= \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{z^6}{(2z-1)(z-2)}\right\} + \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{1}{(2z-1)(z-2)}\right\}$$

$$= 0 + \frac{1}{2}\lim_{z\to 0}\frac{d^2}{dz^2}\left\{\frac{1/3}{z-2} - \frac{2/3}{2z-1}\right\} \quad [\text{ by partial fractions}]$$

$$= \frac{1}{2}\lim_{z\to 0}\left\{\frac{1}{3}(-1)^2\,2!\,(z-2)^{-3} - \frac{2}{3}(-1)^2\,2!\,(2z-1)^{-3}\,2^2\right\}$$

$$= -\frac{1}{24} + \frac{8}{3} = \frac{63}{24} = \frac{21}{8}$$

$$\text{Again, Res}\left(\frac{1}{2}\right) = \lim_{z\to\frac{1}{2}}\left\{\left(z-\frac{1}{2}\right)\frac{z^6+1}{z^3(2z-1)(z-2)}\right\}$$

$$= \frac{\frac{1}{64}+1}{\left(\frac{1}{8}\right)(2)\left(\frac{-3}{2}\right)} = -\frac{65}{24}$$

Now by the Cauchy's residue theorem, (2) $\Rightarrow$

$$I = -\frac{1}{2i}.\ 2\pi i\left[\text{Res}\,(0) + \text{Res}\left(\frac{1}{2}\right)\right]$$

$= -\pi\left[\dfrac{21}{8} - \dfrac{65}{24}\right] = \dfrac{\pi}{12}$ and the required result is proved.

Of them only $z = \frac{i}{3}$ lies inside C. Now

$$\text{Res }(i/3) = \lim_{z \to i/3} \frac{d}{dz}\left\{(z - i/3)^2 \frac{z}{(3z^2 - 10iz - 3)^2}\right\}$$

$$= \lim_{z \to i/3} \frac{d}{dz}\left\{\frac{z}{9(z - 3i)^2}\right\}$$

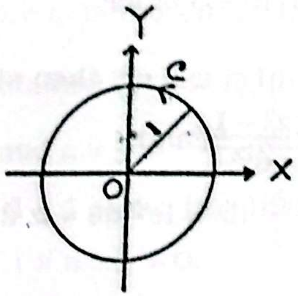
$$= \lim_{z \to i/3} \frac{(z - 3i)^2 - 2z\,(z - 3i)}{9(z - 3i)^4}$$

$$= \frac{(i/3 - 3i) - 2.i/3}{9(i/3 - 3i)^3} = -\frac{5}{256}$$

Now by the Cauchy's residue theorem

$(2) \Rightarrow I = -\frac{4}{i}.\ 2\pi i.\ \text{Res }(i/3) = (-8\pi)\left(-\frac{5}{256}\right) = \frac{5\pi}{32}$ and the required result is obtained.

Example – 285 : Show that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{1 - 2p\cos 2\theta + p^2} = \frac{\pi(1 - p + p^2)}{1 - p}$ where $0 < p < 1$.



Solution : Suppose $I = \displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2\,p\cos 2\theta + p^2}\, d\theta$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta$$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1 + e^{6i\theta}}{1 - p\,(e^{2i\theta} + e^{-2i\theta}) + p^2}\, d\theta \quad \dots (1).$$

Let $z = e^{i\theta}$, then $dz = ie^{i\theta}\, d\theta = iz\, d\theta$ or $d\theta = \frac{dz}{iz}$.

Using these $(1) \Rightarrow I = \text{Real part of } \dfrac{-1}{2ip}\displaystyle\int_C \frac{z(1 + z^6)\, dz}{z^4 - (p + p^{-1})\, z^2 + 1}$ ... (2) where C is the circle $|z| = 1$. The poles of $\dfrac{z(1 + z^6)}{z^4 - (p + p^{-1})\, z^2 + 1} = \dfrac{z(1 + z^6)}{(z^2 - p)\,(z^2 - p^{-1})}$ are obtained by solving $(z^2 - p)\,(z^2 - p^{-1}) = 0$

$\Rightarrow z^2 = p,\ p^{-1} \Rightarrow z = \pm\sqrt{p},\ \pm\sqrt{p^{-1}}$ and they are simple poles.

Of them $z = \pm\sqrt{p}$ lies inside C since $p < 1$. Now

$$\text{Res }(\sqrt{p}) = \lim_{z \to \sqrt{p}} \left\{(z - \sqrt{p})\, \frac{z(1 + z^6)}{(z^2 - p)\,(z^2 - p^{-1})}\right\}$$

$$= \frac{\sqrt{p}\,(1 + p^3)}{2\sqrt{p}\,(p - p^{-1})} = \frac{p(1 + p^3)}{2(p^2 - 1)} = \frac{-p(1 - p + p^2)}{2(1 - p)}$$

Similarly, $\text{Res }(-\sqrt{p}) = \dfrac{-p(1 - p + p^2)}{2(1 - p)}$.

Hence by the Cauchy's residue theorem, $(2) \Rightarrow$

$$I = \text{Real part of } \frac{-1}{2ip}\left[2\pi i\left\{\text{Res }(\sqrt{p}) + \text{Res }(-\sqrt{p})\right\}\right]$$

$$\Rightarrow I = \frac{-\pi}{p}\left[\frac{-p(1 - p + p^2)}{2(1 - p)} + \frac{-p(1 - p + p^2)}{2(1 - p)}\right] = \frac{\pi(1 - p + p^2)}{1 - p}$$

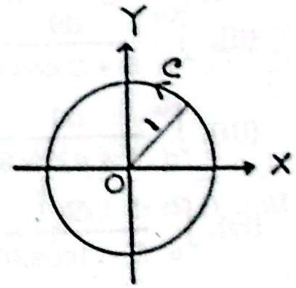
and the required result is obtained.

Example – 286 : Show that $\displaystyle\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{5}}(3 - \sqrt{5})^n.$

Solution : Suppose $I = \displaystyle\int_0^{2\pi} \frac{(1 + 2\cos\theta)^n \cos n\theta}{3 + 2\cos\theta}\, d\theta$

$$= \text{Real part of } \int_0^{2\pi} \frac{(1 + e^{i\theta} + e^{-i\theta})^n\, e^{in\theta}}{3 + e^{i\theta} + e^{-i\theta}}\, d\theta \quad \dots (1)$$

Let $z = e^{i\theta}$, then $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z^2 + 1}{2z}$ and $dz = ie^{i\theta}\, d\theta = iz\,d\theta$ or $d\theta = \dfrac{dz}{iz}$. Using these $(1) \Rightarrow$



$$I = \text{Real part of} \oint_C \frac{\left(1 + \dfrac{z^2 + 1}{z}\right)^n z^n}{3 + \dfrac{z^2 + 1}{z}}\, \frac{dz}{iz}$$

$$= \text{Real part of } \frac{1}{i}\oint_C \frac{(1 + z + z^2)^n}{1 + 3z + z^2}\, dz$$

... (2)

where C is the unit circle $|z| = 1$. The poles of $\frac{(1+z+z^2)^n}{1+3z+z^2}$ are obtained by solving $1 + 3z + z^2 = 0$

$\Rightarrow z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$ and they are simple.

Let $\alpha = \frac{-3+\sqrt{5}}{2}$ and $\beta = \frac{-3-\sqrt{5}}{2}$. Of them α lies inside C.

$$\text{Res}(\alpha) = \lim_{z\to\alpha} \left\{(z-\alpha)\frac{(1+z+z^2)^n}{1+3z+z^2}\right\}$$

$$= \lim_{z\to\alpha} \left\{(z-\alpha)\frac{(1+z+z^2)^n}{(z-\alpha)(z-\beta)}\right\} = \frac{(1+\alpha+\alpha^2)^n}{\alpha-\beta} \quad \ldots (3)$$

We have $\alpha - \beta = \sqrt{5}$ and $1 + \alpha + \alpha^2 = -2\alpha = 3 - \sqrt{5}$ since α is a root of $1 + 3z + z^2 = 0$. i. e. $1 + 3\alpha + \alpha^2 = 0$

Now using these (3) $\Rightarrow \text{Res}(\alpha) = \frac{(3-\sqrt{5})^n}{\sqrt{5}}$.

Hence by the Cauchy's residue theorem, (2) $\Rightarrow$

$I = \text{Real part of } \frac{1}{i}\left\{2\pi i.\ \text{Res}(\alpha)\right\}$

$\Rightarrow I = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$ and the required result is obtained.

Example – 287 : Show that :

(i). $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$;

(ii). $\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$;

(iii). $\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ if $a^2 < 1$;

(iv). $\int_0^{2\pi} \frac{d\theta}{5-4\cos\theta} = \frac{2\pi}{3}$.

Solution : Try yourself.

235. (Form 2). Integrals of the form : $\int_{-\infty}^{\infty} f(x)\,dx$, where $f(x)$ is a rational function of x : Consider $\oint_C f(z)\,dz$, where

C is the contour consisting of :

(i). the x- axis from $-R$ to R, where R is large;

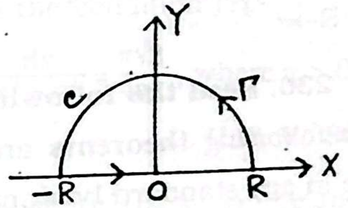
(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Then let $R\to\infty$ and if $f(x)$ is an even function this can be used to evalute the integral $\int_0^{\infty} f(x)\,dx$.

Theorem – 216 : If $|f(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 1$ and M is a constant, then show that $\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$,

where Γ is the upper semi-circle of radius R.

Proof : We have $\frac{M}{R^k}$ is the upper bound of $|f(z)|$ and πR is the upper semi circular arc length of Γ, then



$$\left|\int_\Gamma f(z)\,dz\right| \le \left(\frac{M}{R^k}\right)(\pi R)$$

$= \frac{\pi M}{R^{k-1}} \ldots (1)$.

Now taking $R\to\infty$ in both sides of (1)

$\Rightarrow \lim_{R\to\infty}\left|\int_\Gamma f(z)\,dz\right| = 0 \Rightarrow \lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ and the theorem is proved.

N. B. The above theorem is valid if the integration along the upper-semi circle Γ is replaced by the integration along the lower semi-circle Γ.

Theorem - 217 : Let $f(z) = \frac{P(z)}{Q(z)}$ be the quotient of two polynomials. If degree $Q(z) \geq 2 +$ degree $P(z)$, then

$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0$, where Γ is the upper semi-circle of radius R.

Proof : We have degree $Q(z) \geq 2 +$ degree $P(z)$ ... (1). For large $|z|$, (1) $\Rightarrow$ $|f(z)| \leq \frac{K}{|z|^2}$ where K is some constant.

Then $\left|\int_\Gamma f(z)\, dz\right| \leq \frac{K}{R^2} \cdot \pi R = \frac{K\pi}{R}$ ... (2) where $|z| = R$ and length of $\Gamma = \pi R$.

Now letting $R\to\infty$ in (2) we have $\lim_{R\to\infty} \left|\int_\Gamma f(z)\, dz\right| \leq 0$.

$\Rightarrow \lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0$.

236. Read the following three theorems : The proofs of the following theorems are not given here. Try yourself or see in any standard book of complex variables and these are :

Theorem - 218 : Let $f(z)$ be analytic in the upper half of the z-plane or Argand plane except at a finite number of poles in it and having no poles on the real axis. If $zf(z) \to 0$ as $z \to \infty$, then $\int_{-\infty}^{\infty} f(x)\, dx = 2\pi i$ [sum of the residues at the poles in the upper plane].

Theorem -219 : If $\lim_{z\to\infty} (z - a) f(z) = K$ and if Γ is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z - a| = r$, then

$$\lim_{r\to 0} \int_\Gamma f(z)\, dz = iK(\theta_2 - \theta_1).$$

N. B. In the above theorem if $z = a$ is a simple pole of $f(z)$, then $K = \text{Res}\,[f(z), a]$ and we have

$$\lim_{r\to 0} \int_\Gamma f(z)\, dz = i(\theta_2 - \theta_1)\, \text{Res}\,[f(z), a].$$

Theorem -220 : If Γ is an arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and if $\lim_{R\to\infty} zf(z) = K$, then $\lim_{R\to\infty} \int_\Gamma f(z)\, dz = iK(\theta_2 - \theta_1)$.

N. B. In the above theorem if $\lim_{R\to\infty} zf(z) = 0$ ... (1), then

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \text{ ... (2)}.$$

In the following some examples we will use the condition (2) directly if it satisfies the condition (1).

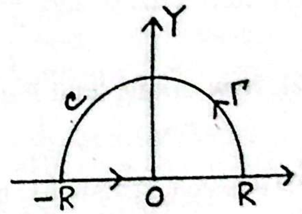
Example -288 : Show that $\int_0^\infty \frac{dx}{x^4 + a^4} = \frac{\pi\sqrt{2}}{4a^3}$, where $a > 0$.

D. U. H. T. '88.

Solation : Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{1}{z^4 + a^4}$ and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$ $= a^4 e^{(2n+1)\pi i} \Rightarrow z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3$.

$\Rightarrow z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$ and they are simple poles. Only the poles $z = \alpha = ae^{\pi i/4}$ and $z = \alpha = ae^{3\pi i/4}$ lie within C. Now using L. Hospital's rule, the residues at these poles $z = \alpha = \lim\limits_{z\to\alpha} (z-\alpha) f(z) = \lim\limits_{z\to\alpha} \dfrac{z-\alpha}{z^4+a^4}$

$$= \lim_{z\to\alpha} \frac{1}{4z^3} = \frac{1}{4\alpha^3} = \begin{cases} \dfrac{1}{4a^3} e^{-3\pi i/4} \text{ at } z = \alpha = a\,e^{\pi i/4} \\ \dfrac{1}{4a^3} e^{-9\pi i/4} \text{ at } z = \alpha = ae^{3\pi i/4} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$= 2\pi i$ (sum of residues)

$$= 2\pi i \left(\frac{1}{4a^3} e^{-3\pi i/4} + \frac{1}{4a^3} e^{-9\pi i/4} \right)$$

$$= \frac{\pi i}{2a^3} \left[e^{-\pi i} e^{\pi i/4} + e^{-2\pi i} e^{-\pi i/4} \right]$$

$$= \frac{\pi i}{2a^3} \left[-e^{\pi i/4} + e^{-\pi i/4} \right] = \frac{\pi i}{2a^3} \left(-2i \sin\frac{\pi}{4}\right) = \frac{\pi}{a^3\sqrt{2}} \quad \text{... (1).}$$

Here $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty} \dfrac{z}{z^4+a^4} = 0$, then $\lim\limits_{R\to\infty} \displaystyle\int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi}{a^3\sqrt{2}} \Rightarrow 2\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi}{a^3\sqrt{2}} \Rightarrow \int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^3}.$$

Example 289 : Show that $\displaystyle\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

D. U. '84; D. U. H. S. T. '85; D. U. M. SC. P. 88.

Solution : Try yourself or, see the above example.

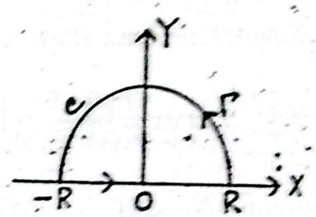
Example – 290 : Show that $\displaystyle\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$.

D. U. H. T. '86, '88.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{1}{z^6+1}$ and C is the contour consisting of :

(i). the x-axis from –R to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $z^6 + 1 = 0 \Rightarrow z^6 = -1 = e^{(2n+1)\pi i} \Rightarrow z = e^{(2n+1)\pi i/6}$, where $n = 0, 1, 2, 3, 4, 5$

$\Rightarrow z = e^{\pi i/6}, e^{\pi i/2}, e^{5\pi i/6}, e^{7\pi i/6}, e^{9\pi i/6}, e^{11\pi i/6}$ and they are simple. Of them only the poles $z = e^{\pi i/6}, e^{\pi i/2}$ and $e^{5\pi i/6}$ lie within C.

Now using L' Hospital's rule, the residue at the pole $z = \alpha = \lim\limits_{z\to\alpha} (z-\alpha) f(z) = \lim\limits_{z\to\alpha} \dfrac{z-\alpha}{z^6+1} = \lim\limits_{z\to\alpha} \dfrac{1}{6z^5}$

$$= \frac{1}{6\alpha^5} = \begin{cases} \dfrac{1}{6} e^{-5\pi i/6} = \dfrac{1}{6} e^{-\pi i} e^{\pi i/6} = -\dfrac{1}{6} e^{\pi i/6} \text{ at } z = e^{\pi i/6} \\ \dfrac{1}{6} e^{-5\pi i/2} = \dfrac{1}{6} e^{-2\pi i} e^{-\pi i/2} = -\dfrac{i}{6} \text{ at } z = e^{\pi i/2} \\ \dfrac{1}{6} e^{-25\pi i/6} = \dfrac{1}{6} e^{-4\pi i} e^{-\pi i/6} = \dfrac{1}{6} e^{-\pi i/6} \text{ at } z = e^{5\pi i/6} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$= 2\pi i$ (sum of residues)

$$= 2\pi i\left(-\frac{1}{6}e^{\pi i/6} - \frac{1}{6} + \frac{1}{6}e^{-\pi i/6}\right) = \frac{-\pi i}{3}\left(e^{\pi i/6} - e^{-\pi i/6} + i\right)$$

$= \left(-\frac{\pi i}{3}\right)\left(2i \sin\frac{\pi}{6} + i\right) = \frac{\pi}{3}(1 + 1) = \frac{2\pi}{3}$ ... (1) . Here

$\lim\limits_{z\to\infty} z\, f(z) = \lim\limits_{z\to\infty} \frac{z}{z^6 + 1} = 0$, then $\lim\limits_{R\to\infty} \int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2)

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{2\pi}{3} \Rightarrow 2\int_0^\infty \frac{dx}{x^6+1} = \frac{2\pi}{3} \Rightarrow \int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$$ and

the result is proved.

Example-29 : Show that $\int_0^\infty \frac{x^6}{(a^4 + x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16a}$, $(a > 0)$.

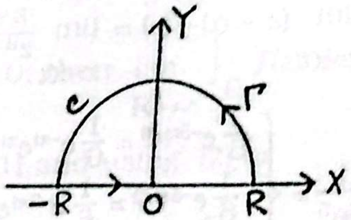
D. U. M. SC. P. '80; D. U. H. T. '77, '85.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{z^6}{(a^4 + z^4)^2}$ and

C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by

$(z^4 + a^4)^2 = 0 \Rightarrow z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = a^4 e^{(2n+1)\pi i} \Rightarrow$

$z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3$. Of them only $z = ae^{\pi i/4}$ and $z = ae^{3\pi i/4}$ lie inside C and each of order 2. Now we will find the residue at $z = ae^{\pi i/4} = \alpha$ (say). Now putting

$z = ae^{\pi i/4} + t = \alpha + t$, where t being very small, then we get

$$f(t + \alpha) = \frac{(\alpha + t)^6}{\{a^4 + (\alpha + t)^4\}^2} = \frac{\alpha^6 + 6\alpha^5 t + \dots\dots\dots}{(a^4 + \alpha^4 + 4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \dots\dots}{(a^4 - a^4 + 4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2} \left[\because \alpha^4 = (a\, e^{\pi i/4})^4 = a^4 e^{\pi i} = -a^4\right.$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \dots\dots}{(4\alpha^3 t + 6\alpha^2 t^2 + 4\alpha t^3 + t^4)^2} = \frac{\alpha^6 + 6\alpha^5 t + \dots\dots}{(4\alpha^3 t + 6\alpha^2 t^2 + \dots)^2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \dots\dots}{16\alpha^6 t^2 (1 + \frac{6t}{4\alpha} + \dots)^2} = \frac{\alpha^6 + 6\alpha^5 t + \dots}{16\alpha^6 t^2} \times \left(1 + \frac{6t}{4\alpha} + \dots\dots\right)^{-2}$$

$$= \frac{\alpha^6 + 6\alpha^5 t + \dots\dots}{16\alpha^6 t^2} \times \left(1 - 2.\frac{6t}{4\alpha} + \dots\dots\right)$$

Now the residue at $z = \alpha$ is the coefficient of

$$\frac{1}{t} \text{ in } f(t + \alpha) = \frac{6\alpha^5}{16\alpha^6} - \frac{12\alpha^6}{16.4\,\alpha^7} = \frac{3}{8\alpha} - \frac{3}{16\alpha}$$

$$= \frac{6-3}{16\alpha} = \frac{3}{16\alpha} = \frac{3}{16a} e^{-\pi i 4}.$$

Similarly, the residue at $z = \beta = ae^{3\pi i/4}$

$$= \frac{3}{16\beta} = \frac{3}{16a} e^{-3\pi i/4} = -\frac{3}{16a} e^{\pi i/4}.$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\left[\text{Res}(ae^{\pi i/4}) + \text{Res}\,(ae^{3\pi i/3})\right]$$

$$= 2\pi i\left[\frac{3}{16a}e^{-\pi i/4} - \frac{3}{16a}\,e^{\pi i/4}\right]$$

$$= -\frac{3\pi i}{8a}\left(e^{\pi i/4} - e^{-\pi i/4}\right) = \left(-\frac{3\pi i}{8a}\right)\left(2i\sin\frac{\pi}{4}\right)$$

$= \dfrac{3\pi}{4\sqrt{2}a}$ ... (1). Here $\lim\limits_{z\to\infty} zf(z) = \lim\limits_{z\to\infty}\dfrac{z^7}{(a^4+z^4)^2}$

$= 0$, then $\lim\limits_{R\to\infty}\displaystyle\int_\Gamma f(z)\,dz = 0$ ... (2).

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{3\pi}{4\sqrt{2}\,a} \quad\Rightarrow$$

$$2\int_0^{\infty} f(x)\,dx = \frac{3\pi}{4\sqrt{2}\,a} \quad\Rightarrow\int_0^{\infty} f(x)\,dx = \frac{3\pi}{8\sqrt{2}\,a}$$

$\Rightarrow \displaystyle\int_0^{\infty}\frac{x^6}{(a^4+x^4)^2}\,dx = \frac{3\pi\sqrt{2}}{16\,a}$ where $a > 0$ and the result is proved.

Example -292 : Show that $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{(x^2+b^2)(x^2+c^2)^2} = \frac{(b+2c)\pi}{2bc^3(b+c)^2}$ where $b > 0$, $c > 0$.

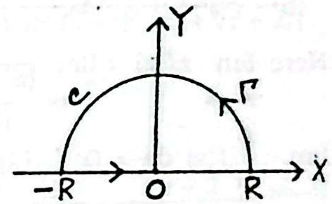
D. U. H. '80, '82, '88; R. U. H. '79; D. U. M. SC. P. '84.

Solution : Consider $\oint_C f(z)\,dz$

where $f(z) = \dfrac{1}{(z^2+b^2)(z^2+c^2)^2}$ and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $(z^2+b^2)(z^2+c^2)^2 = 0$

$\Rightarrow z = \pm bi$ are simple poles and $z = \pm ci$ are poles of order two. Here only $z = bi$ and $z = ci$ lie within C.

Now $\text{Res}\,(bi) = \lim\limits_{z\to bi}\left\{(z-bi)\dfrac{1}{(z^2+b^2)(z^2+c^2)^2}\right\}$

$$= \lim_{z\to bi}\left\{(z-bi)\frac{1}{(z-bi)(z+bi)(z^2+c^2)^2}\right\} = \frac{-i}{2b(b^2-c^2)^2}$$

Again $\text{Res}\,(ci) = \lim\limits_{z\to ci}\dfrac{d}{dz}\left\{(z-ci)^2\dfrac{1}{(z^2+b^2)(z^2+c^2)^2}\right\}$

$$= \lim_{z\to ci}\frac{d}{dz}\left\{(z-ci)^2\frac{1}{(z^2+b^2)(z-ci)^2(z+ci)^2}\right\}$$

$$= \lim_{z\to ci}\frac{d}{dz}\left\{\frac{1}{(z^2+b^2)(z+ci)^2}\right\}$$

$$= \lim_{z\to ci}\frac{-1}{(z^2+b^2)(z+ci)^2}\left[\frac{2z}{z^2+b^2} + \frac{2}{z+ci}\right]$$

$$= \frac{1}{4(b^2-c^2)c^2}\left[\frac{2ci}{b^2-c^2}-\frac{i}{c}\right] = \frac{i(3c^2-b^2)}{4c^3(b^2-c^2)^2}$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,[\text{Res}\,(bi) + \text{Res}\,(ci)]$$

$$= 2\pi i\left[\frac{-i}{2b(b^2-c^2)^2} + \frac{i(3c^2-b^2)}{4c^3(b^2-c^2)^2}\right]$$

$$= \frac{\pi}{(b^2-c^2)^2}\,\frac{b^3-3bc^2+2c^3}{2bc^3}$$

$$= \frac{\pi}{(b^2-c^2)^2}\cdot\frac{(b+2c)(b-c)^2}{2bc^3} = \frac{(b+2c)\pi}{2bc^3(b+c)^2} \quad \dots (1)$$

Here $\lim\limits_{z\to\infty} z f(z) = \lim\limits_{z\to\infty} \dfrac{z}{(z^2+b^2)(z^2+c^2)^2} = 0$, then $\lim\limits_{R\to\infty} \int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1)

and using (2) $\Rightarrow \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \frac{(b+3c)\pi}{2bc^3(b+c)^2}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{dz}{(x^2+b^2)(x^2+c^2)^2} = \frac{(b+2c)\pi}{2bc^3(b+c)^2}.$$ Hence the result.

Example -293 : $\displaystyle\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)^2(x^2+2x+2)} = \frac{7\pi}{50}.$

R. U. M. SC. P. '84.

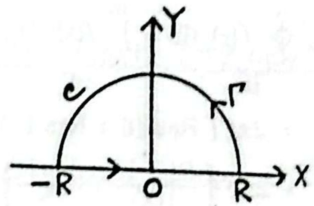
Solution : Consider $\oint_C f(z)\,dz$ where

$$f(z) = \frac{z^2}{(z^2+1)^2(z^2+2z+2)}$$

and C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.



Now the poles of $f(z)$ are given by $(z^2+1)^2(z^2+2z+2) = 0 \Rightarrow z = \pm i$ of order 2 and $z = -1 \pm i$ of order 1. Of them only $z = i$ and $z = -1+i$ lie inside C. Now

$$\text{Res}\,(i) = \lim_{z\to i} \frac{d}{dz}\left\{(z-i)^2\,\frac{z^2}{(z+i)^2(z-i)^2(z^2+2z+2)}\right\}$$

$$= \lim_{z\to i} \frac{d}{dz}\,\frac{z^2}{(z+i)^2(z^2+2z+2)}$$

$$= \lim_{z\to i} \frac{z^2}{(z+i)^2(z^2+2z+2)}\left[\frac{2}{z}-\frac{2}{z+i}-\frac{2z+2}{z^2+2z+2}\right]$$

$$= \frac{i^2}{(2i)^2(i^2+2i+2)}\left[\frac{2}{i}-\frac{2}{2i}-\frac{2i+2}{i^2+2i+2}\right]$$

$$= \frac{1}{4(1+2i)}\left(\frac{1}{i}-\frac{2i+2}{1+2i}\right).$$

$$= \frac{1+2i-2i^2-2i}{4(1+2i)i(1+2i)} = \frac{3}{4i(1+4i+4i^2)} = \frac{3}{4i(-3+4i)}$$

$$= \frac{-3}{4(4+3i)} = \frac{9i-12}{100}$$

Again Res $(-1+i)$

$$= \lim_{z\to -1+i}\left\{(z+1-i)\,\frac{z^2}{(z^2+1)^2(z+1-i)(z+1+i)}\right\}$$

$$= \frac{(-1+i)^2}{\{(-1+i)^2+1\}^2(2i)} = \frac{-2i}{(-2i+1)^2(2i)} = \frac{1}{3+4i} = \frac{3-4i}{25}.$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,[\,\text{Res}\,(i) + \text{Res}\,(-1+i)]$$

$$= 2\pi i\left(\frac{9i-12}{100}+\frac{3-4i}{25}\right) = 2\pi i\left(\frac{-7i}{100}\right) = \frac{7\pi}{50}$$ ... (1). Here

$\lim\limits_{z\to\infty} zf(z) = 0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{7\pi}{50} \Rightarrow$

$$\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+1)^2\,(x^2+2x+2)} = \frac{7\pi}{50}$$ and the result is proved.

Example 294 : Show that $\displaystyle\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \pi\log 2.$

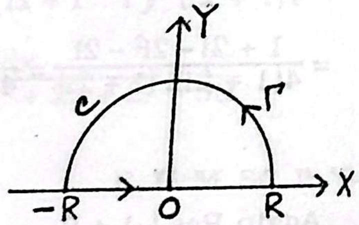
C. U. '81; D. U. H. 87; D. U. M. SC. P. T. '89; C. U. H. '89; R. U. H. '82.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{\log(z+i)}{z^2+1}$ and

C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ, of the circle $|z| = R$, which lies above the x-axis.



The poles of $f(z)$ are given by $z^2+1=0 \Rightarrow z=\pm i$ which are simple poles and of which only the pole $z = i$ lies inside C. Now

$$\text{Res}\,(i) = \lim_{z\to i}(z-i)\,f(z) = \lim_{z\to i}(z-i)\,\frac{\log(z+i)}{(z+i)(z-i)}$$

$$= \lim_{z\to i}\frac{\log(z+i)}{z+i} = \frac{\log(2i)}{2i} = \frac{\log 2+\log i}{2i} = \frac{\log 2+\pi i/2}{2i}$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,.\,\text{Res}\,(i) = 2\pi i\left(\frac{\log 2+\frac{\pi i}{2}}{2i}\right)$$

$$= \pi\left(\log 2+\frac{\pi i}{2}\right)$$ ... (1). Now $\lim\limits_{z\to\infty} zf(z)$

$$= \lim_{z\to\infty}\frac{z\log(z+i)}{(z+i)(z-i)} = \lim_{z\to\infty}\frac{z}{z-i}\,.\,\lim_{z\to\infty}\frac{\log(z+i)}{z+i}$$

$= 1\,.\,0 = 0$, then $\lim\limits_{R\to\infty}\int_\Gamma f(z)\,dz = 0$ ... (2)

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi\left(\log 2+\frac{\pi i}{2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\log(x+i)}{x^2+1}\,dx = \pi\left(\log 2+\frac{\pi i}{2}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\frac{1}{2}\log(x^2+1)+i\tan^{-1}1/x}{x^2+1}\,dx = \pi\left(\log 2+\frac{\pi i}{2}\right)$$

Now equating real part, we have

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2 \Rightarrow \frac{2}{2}\int_0^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx$$

$$= \pi \log 2 \Rightarrow \int_0^{\infty} \frac{\log(x^2+1)}{x^2+1}\,dx = \pi \log 2.$$ Hence the result.

Example : Show that $\int_0^{\pi/2} \log \sin x\, dx = \int_0^{\pi/2} \log \cos x\, dx = -\frac{\pi}{2}\log 2.$

Solution : Using $\int_0^{\infty} \frac{\log(1+x^2)}{1+x^2}\,dx = \pi \log 2$, try yurself.

Example – 295 : Show that :

(i). $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$;

(ii). $\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{\pi}{2}$;

(iii). $\int_0^{\infty} \frac{dx}{(1+x^2)^3} = \frac{3\pi}{16}$;

(iv). $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1} = \frac{\pi}{3}$;

(v). $\int_0^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx = \frac{5\pi}{24}$;

(vi). $\int_0^{\infty} \frac{dx}{(a+bx^2)^n} = \frac{1.\,3.\,5\ldots(2n-3)\pi}{1.\,2.\,3.\ldots(2n-1)\,2^n\,b^{1/2}\,a^{(2n-1)/2}}$

(vii). $\int_0^{\infty} \frac{dx}{(x^2+a^2)^{n+1}} = \frac{(2n)!\pi}{(n!)^2\,2^{2n+1}\,a^{2n+1}}.$

237. (Form 3). Integrals of the form :

$\int_{-\infty}^{\infty} f(x)\cos mx\,dx$ **or** $\int_{-\infty}^{\infty} f(x)\sin mx\,dx$ **where** $f(x)$ **is a rational function of x :** Consider $\oint_C e^{imz} f(z)\,dz$, $m > 0$ where C is the contour consisting of :

(i). the x-axis from $-R$ to R, where R is large;

(ii). the upper semi-circle Γ of the circle $|z| = R$, which lies above the x-axis.

Then let $R \to \infty$ and if $f(x)$ is an even function, this can be used to evalute the integral $\int_{-\infty}^{\infty} e^{imx} f(x)\,dx$, $m > 0$.

238. Jordan's inequality : The inequality $\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$ or $\frac{2\theta}{\pi} \le \sin\theta \le \theta$ is called the Jordan's inequality, where $0 \le \theta \le \frac{\pi}{2}$.

Jordan lemma (Theorem —221) : If $|f(z)| \le \frac{M}{R^k}$ for $z = Re^{i\theta}$, where $k > 0$ and M is a constant, then $\lim_{R\to\infty} \int_\Gamma e^{imz} f(z)\,dz = 0$, where Γ is the upper semi-circle of radius R and m is a positive constant.

Proof : If $z = Re^{i\theta}$, then $\int_\Gamma e^{imz} f(z)\,dz$

$$= \int_0^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\, d(Re^{i\theta}) = i\int_0^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\, Re^{i\theta}\, d\theta$$

$$\Rightarrow \left|\int_\Gamma e^{imz} f(z)\,dz\right| = \left|\int_0^{\pi} e^{imRe^{i\theta}} f(Re^{i\theta})\, Re^{i\theta}\, d\theta\right|$$

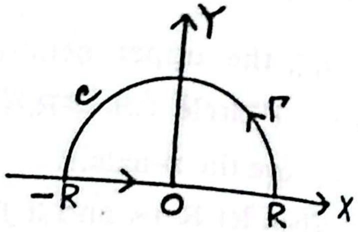
$$\leq \int_0^{\pi} \left| e^{imRe^{i\theta}} f(Re^{i\theta})\, Re^{i\theta}\, d\theta \right| = \int_0^{\pi} e^{-mR\sin\theta} \left| f(Re^{i\theta}) \right| Rd\theta$$

$$\leq \frac{M}{R^{k-1}} \int_0^{\pi} e^{-mR\sin\theta}\, d\theta$$

$$= \frac{2M}{R^{k-1}} \int_0^{2\pi} e^{-mR\sin\theta}\, d\theta \;\ldots (1).$$

But we have $\sin\theta \geq 2\theta/\pi$. (2)

for $0 \leq \theta \leq \pi/2$. Using (2),



$$(1) \Rightarrow \left| \int_{\Gamma} e^{imz} f(z) dz \right|$$

$$\leq \frac{2M}{R^{k-1}} \int_0^{\pi/2} e^{-2mR\theta/\pi}\, d\theta$$

$$= \frac{2M}{R^{k-1}} \left[\frac{\pi e^{-2mR\theta/\pi}}{-2MR} \right]_0^{\pi/2} = \frac{\pi M}{mR^k}(1 - e^{-mR}) \;\ldots (3)$$

Now taking $R \to \infty$ in both sides of (3) $\Rightarrow$

$$\lim_{R\to\infty} \left| \int_{\Gamma} e^{imz} f(z)\, dz \right| = 0 \Rightarrow \lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$$

and the theorem is proved.

N. B. If $m<0$, then also $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$ where Γ is the lower semi-circle of radius R and it can be proved by changing $z = -w$ in the above theorem.

Theorem -222 : Let $f(z)$ be analytic in the upper half plane of the z-plane or Argand plane except at a finite number of poles in it and having no poles on the real axis. If $f(z) \to 0$ as $z \to \infty$, then

$$\int_{-\infty}^{\infty} e^{imx} f(x)\, dx = 2\pi i \text{ [sum of the residues at the poles in}$$

the upper half plane] if $m>0$.

Proof : Try yourself.

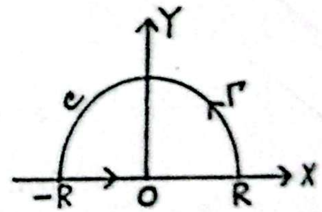
239. Other forms of Jordan lemma :

(i). If $f(z) = \dfrac{P(z)}{Q(z)}$ is the quotient of two polynomials $P(z)$ and $Q(z)$ such that degree $Q(z) \geq 1 +$ degree $P(z)$, then $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$, $m > 0$ where Γ is the upper half-circle of radius R.

(ii). If $f(z)$ is analytic at finite number of singularities and if $\lim_{z\to\infty} f(z) = 0 \ldots (1)$ uniformly, then $\lim_{R\to\infty} \int_{\Gamma} e^{imz} f(z)\, dz = 0$, $m > 0 \ldots (2)$. where Γ denotes the upper half-circle of radius R. In the following some examples we will use condition (2) if it satisfies the condition (1).

Example -236 : Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^2 + a^2}\, dx = \frac{\pi}{2a} e^{-ma}$, $m \geq 0$ and $a > 0$. **D. U. H. '78, '88; R. U. H. '76.**

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{e^{imz}}{z^2 + a^2}$ and C is the contour consisting of :



(i). the x-axis from $-R$ to R, where R is large.

(ii). the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.

The poles of $f(z) = \frac{e^{imz}}{z^2+a^2}$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$ and they are simple poles. Of which only $z = ai$ lies inside C.

$$\text{Res}\,[f(z), ai] = \lim_{z \to ai} \left\{ (z - ai) \frac{e^{imz}}{(z - ai)(z + ai)} \right\} = \frac{e^{-ma}}{2ai}.$$

Now by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dz + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,\text{Res}\,[f(z), ai] = 2\pi i \left(\frac{e^{-ma}}{2ai}\right) = \frac{\pi e^{-ma}}{a} \dots (1).$$

Here $\lim_{z \to \infty} \frac{1}{z^2+a^2} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \frac{1}{z^2+a^2}$ and then by the Jordan's lemma, $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0 \dots (2)$. Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi e^{-ma}}{a} \Rightarrow 2\int_0^\infty f(x)\,dx = \frac{\pi e^{-ma}}{a}$.

$\Rightarrow \int_0^\infty \frac{\cos mx}{x^2+a^2} = \frac{\pi e^{-ma}}{2a}$ and the required result is obtained.

Example – 297 : Show that (a) $\int_0^\infty \frac{\cos x}{1+x^2}\,dx = \frac{\pi}{2e}$;

(b) $\int_0^\infty \frac{\cos mx}{x^2+1} = \frac{\pi}{2}$; **R. U. M. SC. P. '84.**

(c) $\int_0^\infty \frac{\cos x}{x^2+a^2} = \frac{\pi e^{-a}}{2a}$, $a > 0$.

Solution : Use the above example.

Example – 298 : Using the above example, show that $\int_0^\infty \frac{x \sin mx}{x^2+a^2}\,dz = \frac{\pi}{2} e^{-ma}$, where $m \geq 0$ and $a > 0$.

Solution : By the above example, we have $\int_0^\infty \frac{\cos mx}{x^2+a^2}\,dx = \frac{\pi}{2a} e^{-ma} \dots (1)$ where $m \geq 0$ and $a > 0$.

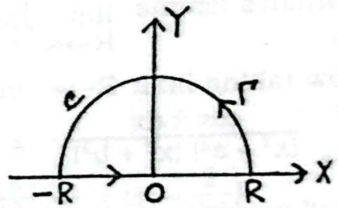
Now differentiating both sides of (1), with respect to m, we have $\int_0^\infty \frac{-x \sin mx}{x^2+a^2}\,dx = -\frac{a\pi}{2a} e^{-ma}$

$\Rightarrow \int_0^\infty \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}$ and the required result is obtained.

Example 299 : Show that $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$ where $a > b > 0$.

D. U. H. '89, '90; D. U. H. T. '77, 82, 87.

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$ are obtained by solving $(z^2 + a^2)(z^2 + b^2) = 0 \Rightarrow z = \pm ai, \pm bi$ which are simple poles. Only the poles $z = ai$ and $z = bi$ lie inside C.

Now Res (ai) $= \lim\limits_{z \to ai} \left\{ (z - ai) \dfrac{e^{iz}}{(z - ai)(z + ai)(z^2 + b^2)} \right\}$

$$= \lim_{z \to ai} \frac{e^{iz}}{(z + ai)(z^2 + b^2)} = \frac{e^{-a}}{2ai(b^2 - a^2)} = -\frac{e^{-a}}{2ai(a^2 - b^2)}.$$

Again Res (bi) $= \dfrac{e^{-b}}{2bi(a^2 - b^2)}$.

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz$$

$$= 2\pi i\,[\text{Res}(ai) + \text{Res}(bi)]$$

$$= 2\pi i\left[-\frac{e^{-a}}{2ai(a^2 - b^2)} + \frac{e^{-b}}{2bi(a^2 - b^2)}\right] = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$
... (1)

Here $\lim\limits_{z \to \infty} \dfrac{1}{(z^2 + a^2)(z^2 + b^2)} = 0$ and $f(z)$ is a function of the form $e^{imz}F(z)$, where $F(z) = \dfrac{1}{(z^2 + a^2)(z^2 + b^2)}$ and then by the Jordan's lemma $\lim\limits_{R \to \infty} \displaystyle\int_\Gamma f(z)\,dz = 0$ ... (2).

Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{\cos x\,dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$ and the required result is obtained.

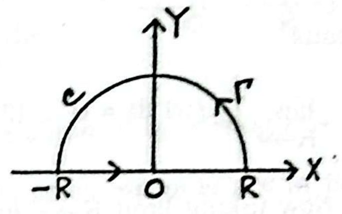
Example – 300 : Show that (a) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 8)(x^2 + 12)}\,dx = \frac{\pi}{8}\left(\frac{e^{-2\sqrt{2}}}{\sqrt{2}} - \frac{e^{-2\sqrt{3}}}{\sqrt{3}}\right)$; **D. U. H. '77.**

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + 1)(x^2 + 9)}\,dx = \frac{\pi}{8}\left(\frac{e^{-1}}{1} - \frac{e^{-3}}{3}\right)$. **R. U. H. '77.**

Solution : Try yourself.

Example -301 : Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{(x^2 + a^2)^2}\,dx = \frac{\pi(1 + ma)e^{-ma}}{4a^3}$, where $a > 0$ and $m \geq 0$. **D. U. H. T. '86; C. U. '81.**

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{imz}}{(z^2 + a^2)^2}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \dfrac{e^{imz}}{(z^2 + a^2)^2}$ are obtained by solving $(z^2 + a^2)^2 = 0 \Rightarrow z = \pm ai$ which are of order 2. Only the pole $z = ai$ lies inside C. Now Res (ai) $=$

$$\lim_{z \to ai} \frac{d}{dz}\left\{(z - ai)^2 \frac{e^{imz}}{(z - ai)^2 (z + ai)^2}\right\}$$

$$= \lim_{z \to ai} \frac{d}{dz}\left\{\frac{e^{imz}}{(z + ai)^2}\right\}$$

$$= \lim_{z \to ai}\left[\frac{e^{imz}}{(z + ai)^2}\left\{im - \frac{2}{z + ai}\right\}\right] = \frac{e^{-ma}}{(2ai)^2}\left\{im - \frac{2}{2ai}\right\}$$

$$= -\frac{e^{-ma}}{4a^2}\left(\frac{-am - 1}{ai}\right) = \frac{e^{-ma}(ma + 1)}{4a^3 i}.$$ Then by the Cauchy's residue theorem, we have $\oint_C f(z)\,dz$

$$= \int_{-R}^{R} f(x)\,dx + \int_\Gamma f(z)\,dz = 2\pi i\,\text{Res}(ai)$$

$$= 2\pi i \left\{ \frac{e^{-ma}(ma+1)}{4a^3 i} \right\} = \frac{\pi(1+ma)e^{-ma}}{2a^3} \ldots (1). \text{ Here}$$

$\lim_{z\to\infty} \frac{1}{(z^2+a^2)^2} = 0$ and $f(z)$ is a function of the form

$e^{imz} F(z)$, where $F(z) = \frac{1}{(z^2+a^2)^2}$ and then by the Jordan's

lemma

$$\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \ldots (2).$$

Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx$

$$= \frac{\pi(1+ma)e^{-ma}}{2a^3} \Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx}{(x^2+a^2)^2}\, dx = \frac{\pi(1+ma)e^{-ma}}{2a^3}$$

$$\Rightarrow 2\int_0^\infty \frac{\cos mx}{(x^2+a^2)^2}\, dx = \frac{\pi(1+ma)e^{-ma}}{2a^3} \Rightarrow \int_0^\infty \frac{\cos mx}{(x^2+a^2)^2}\, dx$$

$= \frac{\pi(1+ma)e^{-ma}}{4a^3}$ and the required result is proved.

Example –302 : Show that $\int_0^\infty \frac{\cos x}{(1+x^2)^2}\, dx = \frac{\pi}{2e}$.

R. U. '84; R. U. H. '72.

Solution : Try yourself.

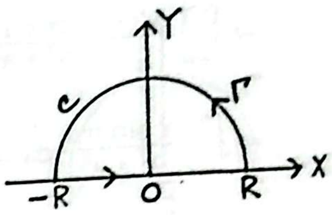
Example 303 : Show that $\int_0^\infty \frac{x \sin mx}{x^2+a^2}\, dx = \frac{\pi}{2} e^{-ma}$,

where $m \geq 0$ and $a > 0$.

D. U. H. T. '76, '78; R. U. '83; R. U. H. '73, '82; D. U. H. '87; D. U. M. SC. P. '78, '88.

Solution : Consider $\oint_C f(z)\, dz$

where $f(z) = \frac{z e^{imz}}{z^2+a^2}$ and C is the contour consisting of : (i) the x-axis from –R to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \frac{z e^{imz}}{z^2+a^2}$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$, which are simple poles but only $z = ai$ lies inside C. Now

$$\text{Res}(ai) = \lim_{z\to ai} \left\{ (z-ai) \frac{z e^{imz}}{(z-ai)(z+ai)} \right\}$$

$$= \lim_{z\to ai} \frac{z e^{imz}}{z+ai} = \frac{ai\, e^{-ma}}{2ai} = \frac{e^{-ma}}{2}.$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz$$

$$= 2\pi i \,\text{Res}(ai) = 2\pi i. \left(\frac{e^{-ma}}{2}\right) = \pi i e^{-ma} \ldots (1).$$

Here $\lim_{z\to\infty} \frac{1}{z^2+a^2} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \frac{1}{z^2+a^2}$ and then by the Jordan's lemma $\lim_{R\to\infty} \int_\Gamma f(z)\, dz = 0 \ldots (2)$. Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \pi i e^{-ma} \Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\,dx = \pi i e^{-ma} \Rightarrow \int_{-\infty}^{\infty} \frac{xe^{imx}}{x^2+a^2}\,dx = \pi i e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos mx + i \sin mx)}{x^2+a^2}\,dx = \pi i e^{-ma}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \pi e^{-ma} \qquad [\text{ equating imaginary parts}]$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \pi e^{-ma}$$

$$\Rightarrow \int_{0}^{\infty} \frac{x \sin mx}{x^2+a^2}\,dx = \frac{\pi}{2} e^{-ma}$$ and the required result is obtained.

Example -304 : Show that (a) $\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\,dx = \frac{\pi e^{-a}}{2}$;

C. U. H. '89.

(b) $\int_0^{\infty} \frac{x \sin x}{x^2+4}\,dx = \frac{\pi}{2} e^{-2}$. **R. U. '82; R. U. H. '75.**

Solution : Try yourself.

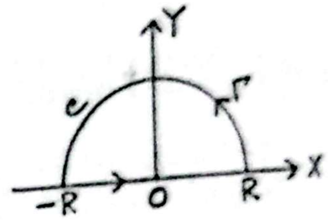
Example -305 : Show that (a) $\int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2+2x+5}\,dx = \frac{\pi}{2} e^{-2\pi}$;

(b) $\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}$. **R. U. M. SC. P. '84.**

Solution : Consider $\oint_C f(z)\,dz$ where $f(z) = \frac{ze^{i\pi z}}{z^2+2z+5}$ and C is the contour consisting of :

(i) the x-axis from $-R$ to R, where R is large;

(ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z) = \frac{ze^{i\pi z}}{z^2+2z+5}$ are obtained by solving

$z^2 + 2z + 5 = 0 \Rightarrow z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ and they are simple poles. But only $z = -1 + 2i$ lies inside C.

Now Res $(-1 + 2i)$

$$= \lim_{z \to -1+2i} \left\{ (z+1-2i) \cdot \frac{z e^{i\pi z}}{(z+1-2i)(z+1+2i)} \right\}$$

$= \frac{(-1+2i)e^{-\pi i - 2\pi}}{4i}$. Then by the Caucly's residue theorem. we have $\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz$

$$= 2\pi i \text{ Res } (-1+2i) = 2\pi i. \left\{ \frac{(-1+2i)\, e^{-\pi i - 2\pi}}{4i} \right\}$$

$= \frac{\pi}{2}(1-2i)e^{-2\pi}$ ... (1). Here $\lim_{z \to \infty} \frac{z}{z^2+2z+5} = 0$ and $f(z)$ is a function of the form e^{imz} F(z), where $F(z) = \frac{z}{z^2+2z+5}$ and then by the Jordan's lemma. $\lim_{R \to \infty} \int_{\Gamma} f(z)\,dz = 0$ ... (2).

Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx =$

$\frac{\pi}{2}(1-2i)e^{-2\pi}$.

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\, e^{i\pi x}}{x^2 + 2x + 5}\, dx = \frac{\pi}{2}(1 - 2i)\, e^{-2\pi} \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \cos \pi x + ix \sin \pi x}{x^2 + 2x + 5}\, dx = \frac{\pi}{2}(1 - 2i) e^{-2\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x \cos \pi x}{x^2 + 2x + 5}\, dx = \frac{\pi}{2} e^{-2\pi} \text{ and}$$

$\displaystyle\int_{-\infty}^{\infty} \frac{x \sin \pi x}{x^2 + 2x + 5}\, dx = -\pi e^{-2\pi}$ and the required result of (a) and (b) are obtained.

Example -306 : Show that $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)(x^2 + b^2)}\, dx$

$$= \frac{\pi}{2(a^2 - b^2)}(e^{-a} a^2 - e^{-b} b^2) \text{ where } a, b > 0.$$

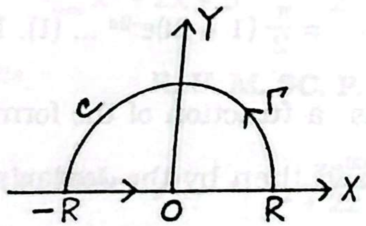
Solution : Try yourself.

Example -307 : Show that $\displaystyle\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)^2}\, dx = -\frac{\pi}{4}(a - 2)e^{-a}$

where $a > 2$.

Solution : Consider $\oint_C f(z)\, dz$

where $f(z) = \dfrac{z^3 e^{iz}}{(z^2 + a^2)^2}$ and C is the contour consisting of : **(i)** the x-axis from $-R$ to R, where R is large; **(ii)** the upper semi-circle Γ of the circle $|z| = R$



which lies above the x-axis. The poles of $f(z)$ inside C is $z = ai$, which is of order two.

Now $\displaystyle \text{Res}(ai) = \lim_{z \to ai} \frac{d}{dz}\left\{(z - ai)^2 \frac{z^3 e^{iz}}{(z - ai)^2 (z + ai)^2}\right\}$

$$= \lim_{z \to ai} \frac{d}{dz}\left\{\frac{z^3 e^{iz}}{(z + ai)^2}\right\} = \lim_{z \to ai}\left[\frac{z^3 e^{iz}}{(z + ai)^2}\left\{\frac{3}{z} + i - \frac{2}{z + ai}\right\}\right]$$

$= \dfrac{a^3 e^{-a}(2 - a)}{4a^3 i} = \dfrac{e^{-a}(2 - a)}{4i}$. Then by the Cauchy's residue theorem, we have $\displaystyle\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz$

$= 2\pi i \text{ Res}(ai)$

$= 2\pi i \left\{\dfrac{e^{-a}(2 - a)}{4i}\right\} = \dfrac{\pi e^{-a}(2 - a)}{2}$ ... (1). Here

$\displaystyle\lim_{z \to \infty} \frac{z^3}{(z^2 + a^2)^2} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$.. where $F(z) = \dfrac{z^3}{(z^2 + a^2)^2}$ and then by the Jordan's lemma $\displaystyle\lim_{R \to \infty} \int_{\Gamma} f(z)\, dz = 0$ ... (2)

Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow$

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi e^{-a}(2 - a)}{2} \Rightarrow \int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)^2}\, dx = \frac{\pi e^{-a}(2 - a)}{2}$$

$$\Rightarrow 2\int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)^2}\, dx = \frac{\pi e^{-a}(2 - a)}{2}$$

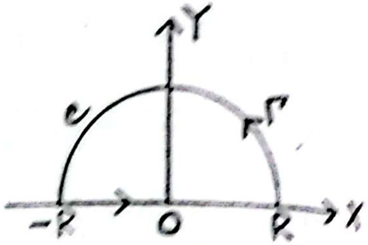
$\displaystyle\Rightarrow \int_0^{\infty} \frac{x^3 \sin x}{(x^2 + a^2)^2}\, dx = -\frac{\pi e^{-a}(a - 2)}{4}$ where $a > 2$ and the required result is obtained.

Example -308 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2 + a^2}\, dx = 2\pi e^{-a}$

where $a > 0$.

Solution : Consider $\oint_C f(z)\, dz$

where $f(z) = \dfrac{e^{iz}}{z-ai}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The pole of $f(z) = \dfrac{e^{iz}}{z-ai}$ is ai which is simple. Now

$$\text{Res}(ai) = \lim_{z\to ai} \left\{ (z-ai)\frac{e^{iz}}{z-ai} \right\}$$

$= \lim\limits_{z\to ai} e^{iz} = e^{-a}$. Then by the residue theorem, we have

$$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\Gamma} f(z)\, dz = 2\pi i\ \text{Res}(ai) = 2\pi i e^{-a}$$

... (1). Here $\lim\limits_{z\to\infty} \dfrac{1}{z-ai} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$ where $F(z) = \dfrac{1}{z-ai}$ and then by the Jordan's lemma $\lim\limits_{R\to\infty} \int_{\Gamma} f(z)\, dz = 0$ ... (2). Now taking limit $R\to\infty$ in (1) and using (2) $\Rightarrow \displaystyle\int_{-\infty}^{\infty} \frac{e^{ix}}{x-ai} dx = 2\pi i\, e^{-a} \Rightarrow \int_{-\infty}^{\infty} \frac{(x+ai)e^{ix}}{x^2+a^2} dx$

$$= 2\pi i e^{-a} \Rightarrow \int_{-\infty}^{\infty} \frac{(x+ai)(\cos x + i \sin x)}{x^2+a^2}\, dx = 2\pi i\, e^{-a} \Rightarrow$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(x \cos x - a \sin x) + i (a \cos x + x \sin x)}{x^2+a^2}\, dx = 2\pi i\, e^{-a}.$$

Equating imiginary parts, we have

$\displaystyle\int_{-\infty}^{\infty} \frac{a \cos x + x \sin x}{x^2+a^2} dx = 2\pi e^{-a}$ and the required result is obtained.

Example – 309 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{x \cos x - a \sin x}{x^2+a^2} dx = 0$

Solution : Try yourself.

Example–310 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{-a \cos x + x \sin x}{x^2+a^2} dx = 0$

where $a>0$.

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{e^{iz}}{z+ai}$ and C is the contour consisting of : (i) the x-axis from $-R$ to R, where R is large; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis. Then : try yourself or use the above example.

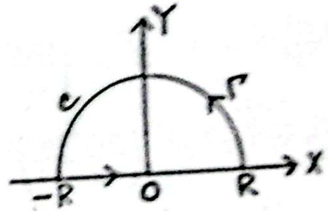
Example –311 : Show that $\displaystyle\int_0^{\infty} \frac{\cos mx}{x^4+a^4} dx$

$= \dfrac{\pi}{2a^3} e^{-ma/\sqrt{2}} \sin\left(\dfrac{ma}{\sqrt{2}} + \dfrac{\pi}{4}\right)$ where m, a > 0.

D. U. H. T. '75; D. U. H. '86.

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{e^{imz}}{z^4+a^4}$ and C is the contour consisting of :

(i) the x-axis from $-R$ to R; (ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis.



The poles of $f(z)$ are obtained by solving $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4$

$= e^{(2n+1)\pi i} a^4 \Rightarrow z = a e^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3 \Rightarrow$

$z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$. Only $z = \alpha = ae^{\pi i/4}$

$= a(1+i)/\sqrt{2}$ and $z = \alpha = ae^{3\pi i/4} = a(-1+i)/\sqrt{2}$ lie inside C

Now the residue at $z = \alpha = \lim\limits_{z \to \alpha} \left\{ (z-\alpha) \dfrac{e^{imz}}{z^4+a^4} \right\}$

$$= \lim_{z \to \alpha} \frac{e^{imz}}{4z^3} = \frac{e^{im\alpha}}{4\alpha^3} \qquad [\text{ by L' Hospital's rule }]$$

$$= \begin{cases} \dfrac{-1}{4a^3} e^{ima(1+i)/\sqrt{2} + \pi i/4} \text{ at } z = a(1+i)/\sqrt{2} \ ; \ [\alpha^3 = -a^3 e^{-\pi i/4}] \\ \dfrac{1}{4a^3} e^{ima(-1+i)/\sqrt{2} - \pi i/4} \text{ at } z = a(-1+i)/\sqrt{2} \ ; \ [\alpha^3 = a^3 e^{\pi i/4}] \end{cases}$$

$$= \begin{cases} -\dfrac{1}{4a^3} e^{-ma/\sqrt{2}} e^{i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} \quad \text{at } z = a(1+i)/\sqrt{2} \\ \dfrac{1}{4a^3} e^{-ma/\sqrt{2}} e^{-i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} \quad \text{at } z = a(-1+i)/\sqrt{2} \end{cases}$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz$$

$= 2\pi i$ (sum of the residues inside C)

$$= 2\pi i \left[-\frac{1}{4a^3} e^{-ma/\sqrt{2}} \left\{ e^{i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} - e^{-i\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)} \right\} \right]$$

$$= \frac{-\pi i}{2a^3} e^{-ma/\sqrt{2}} \cdot 2i \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$= \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right) \quad \ldots (1) \text{ . Here}$$

$\lim\limits_{z \to \infty} \dfrac{1}{z^4+a^4} = 0$ and $f(z)$ is a function of the form

$e^{imz} F(z)$ where $F(z) = \dfrac{1}{z^4+a^4}$ and then by the Jordan's lemma $\lim\limits_{R \to \infty} \displaystyle\int_\Gamma f(z)\, dz = 0$ ... (2). Now taking limit $R \to \infty$ in (1) and using (2) $\Rightarrow \displaystyle\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+a^4}\, dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos mx + i \sin mx}{x^4+a^4}\, dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

Now equating real parts, we have

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+a^4}\, dx = \frac{\pi}{a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{x^4+a^4}\, dx = \frac{\pi}{2a^3} e^{-ma/\sqrt{2}} \sin\left(\frac{ma}{\sqrt{2}} + \frac{\pi}{4}\right)$$

and the required result is obtained.

Example -312 : Show that $\displaystyle\int_0^{\infty} \frac{x \sin mx}{x^4+a^4}\, dx$

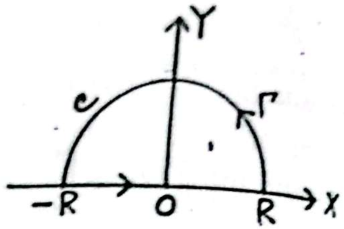
$= \dfrac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin \dfrac{ma}{\sqrt{2}}$ where $m, a > 0$. **D. U. H. '86.**

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \dfrac{z e^{imz}}{z^4+a^4}$ and C

is the contour consisting of :

(i) the x-axis from $-R$ to R.

(ii) the upper semi-circle Γ of the circle $|z| = R$ which lies above the x-axis. The poles of $f(z)$ are obtained by solving $z^4 + a^4 = 0 \Rightarrow z^4 = -a^4 = e^{(2n+1)\pi i} a^4$



$\Rightarrow z = ae^{(2n+1)\pi i/4}$ where $n = 0, 1, 2, 3 \Rightarrow z = ae^{\pi i/4}, ae^{3\pi i/4}, ae^{5\pi i/4}, ae^{7\pi i/4}$. Only $z = \alpha = a\, e^{\pi i/4} = a(\cos \pi/4 + i \sin \pi/4)$ $= \frac{a}{\sqrt{2}}(1 + i)$ and $z = \alpha = ae^{3\pi i/4} = a(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}) = \frac{a}{\sqrt{2}}(-1 + i)$ lie inside C. Now the residue at $z = \alpha =$

$$\lim_{z \to \alpha} \left\{(z - \alpha) \frac{ze^{imz}}{z^4 + a^4}\right\} = \lim_{z \to \alpha} \frac{e^{im\alpha}}{4z^3} = \lim_{z \to \alpha} \frac{e^{im\alpha}}{4z^2}$$

[by L' Hospital's rule]

$$= \frac{e^{im\alpha}}{4\alpha^2} = \begin{cases} \dfrac{e^{ima(1+i)/\sqrt{2}}}{2a^2(1+i)^2} & \text{at } \alpha = a(1 + i)/\sqrt{2} \\ \dfrac{e^{ima(-1+i)/\sqrt{2}}}{2a^2(-1+i)^2} & \text{at } a = a(-1 + i)/\sqrt{2} \end{cases}$$

$$= \begin{cases} \dfrac{e^{-ma/\sqrt{2}} \cdot e^{ima/\sqrt{2}}}{4a^2 i} & \text{at } \alpha = a(1 + i)/\sqrt{2} \\ \dfrac{e^{-ma/\sqrt{2}} \cdot e^{-im\,a/\sqrt{2}}}{-4a^2 i} & \text{at } \alpha = a(-1 + i)/\sqrt{2} \end{cases}$$

Now by the Cauchy's residue theorem, we have

$\oint_C f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_\Gamma f(z)\, dz = 2\pi i$ (sum of the residues within C) $= 2\pi i. \frac{e^{-ma/\sqrt{2}}}{4a^2 i}\left[e^{ima/\sqrt{2}} - e^{-ima/\sqrt{2}}\right]$

$$= \frac{\pi e^{-ma/\sqrt{2}}}{2a^2} \cdot 2i \sin \frac{ma}{\sqrt{2}} = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}} \quad \ldots (1)$$

Here $\lim_{z \to \infty} \frac{z}{z^4 + a^4} = 0$ and $f(z)$ is a function of the form $e^{imz}\, F(z)$ where $F(z) = \frac{z}{z^4 + a^4}$ and then by the Jordan's lemma, $\lim_{R \to \infty} \oint_\Gamma f(z)\, dz = 0 \ldots (2)$. Now taking limit $R \to \infty$ in (1) and using (2)

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x\, e^{imx}}{x^4 + a^4} dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{x(\cos mx + i \sin mx)}{x^4 + a^4} dx = \frac{\pi i}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

Now equating imaginary parts, we have

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^4 + a^4} dx = \frac{\pi}{a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin mx}{x^4 + a^4} dx = \frac{\pi}{2a^2} e^{-ma/\sqrt{2}} \sin \frac{ma}{\sqrt{2}}$$ and the required result is obtained.

Example – 313 : Show that $\int_0^{\infty} \frac{x^3 \sin mx}{x^4 + a^4} dx$

$= \frac{\pi}{2} e^{-ma/\sqrt{2}} \cos \frac{ma}{\sqrt{2}}$ where $m, a > 0$.

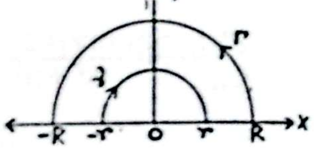
Solution : Try yourself.

239. (Form 4). Case of poles on the real axis : If $|f(x)| \to \infty$ as x tends to certain finite points on the x— axis, then we will use the **indented contours.**

Definition : A contour is said to be **indented** at the singularity when a part of a small semi—circle is described to avoid the singularity of the integrand keeping the singular point as centre. In the following examples we will use some **indented contour.**

Example – 314 : Show that $\int_0^\infty \frac{\sin x}{x}\, dx = \pi/2.$

D. U. H. S. 85; D. U. H. T. 85, 88; D. U. M. Sc. P. 80, 88; C. U. H. 90.



Solution : Consider the integral $\oint_C f(z)\, dz$, where $f(z) = \frac{e^{iz}}{z}$.

The function $f(z)$ has a singularity at the point z = 0 on the real axis where integration is not possible and also there is no any other singularity in the upper half plane. Now we consider the contour C consisting of a large semicircle Γ of radius R in the upper half plane and the real axis indented by a small semicircle γ of radius r at z = 0. Thus C have four parts : **(i)** x – axis from—r to —R; **(ii)** the small semi circle γ of radius r where $r \to 0$; **(iii)** x — axis from r to R and (iv) the great semi circle Γ of radius R where $R \to \infty$. Now by the Cauchy's integral theorem,

$$\oint_C f(z)\, dz = \int_{-R}^{-r} f(x)\, dx + \int_\gamma f(z)\, dz + \int_r^R f(x)\, dx + \int_\Gamma f(z)\, dz = 0 \quad \text{...... (1)}$$

If $r \to 0$ and $R \to \infty$, then $\int_{-R}^{-r} f(x)\, dx + \int_r^R f(x)\, dx$

$$\Rightarrow \int_{-\infty}^{0} f(x)\, dx + \int_0^\infty f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, dx$$

$= \int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx$ .... (2). Here $\lim_{z \to \infty} \frac{1}{z} = 0$ and $f(z)$ is a function of the form $e^{iz} F(z)$, where $F(z) = \frac{1}{z}$. Hence by the Jordan's lemma, $\lim_{R \to \infty} \int_\Gamma f(z)\, dz = 0$ ..... (3)

Again we have $\lim_{z \to 0} z\, f(z) = \lim_{z \to 0} z \frac{e^{iz}}{z} = e^0 = 1.$

Then $\lim_{r \to 0} \int_\gamma f(z)\, dz = -i \cdot 1 \cdot (\pi - 0) = -\pi i$ ..... (4),

where the negative sign being taken as the contour γ is in the clock wise direction. Now taking $r \to 0$ ansd $R \to \infty$ and using (2), (3) and (4), the equation (1)

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx - \pi i = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x + i \sin x}{x} dx = \pi i \Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x}\, dx = 0 \text{ and}$$

$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$. But $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi \Rightarrow 2\int_0^\infty \frac{\sin x}{x} dx = \pi$

$$\Rightarrow \int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Example 315 : show that (i) $\int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi$;

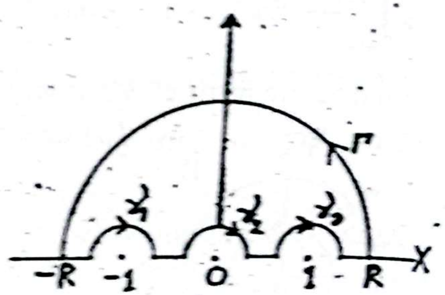
(ii) $\int_{-\infty}^{\infty} \cos x \, dx = 0$; (iii) $\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = \pi/2$.

Solution : Try yourself.

Example 316 : Show that. $\int_0^{\infty} \frac{\sin \pi x}{x(1-x^2)} dx = \pi$.

R. U. H. 73, 74, 81.

Solution : Consider the integral $\oint_C f(z)\, dz$ where $f(z) = \frac{e^{\pi z i}}{z(1-z^2)}$. The function $f(z)$ have the singularities at the points $z = -1, 0, 1$ on the x-axis where integrations are not possible and also there are no other singularities in the upper half plane. Now we consider the contour C consisting of a large semicircle Γ of radius R in the upper half plane and the real axis indented at $z = -1, 0, 1$ by small semicircles γ_1, γ_2 and γ_3 of radius r_1, r_2 and r_3 respectively. Here C have eight parts : on the x— axis from $-R$ to $-1-r_1$, $-1+r_1$ to $-r_2$, r_2 to $1-r_3$ and $1+r_3$ to R; the small semicircles γ_1, γ_2 and γ_3 of radi r_1, r_2 and r_3 respectively where $r_1, r_2, r_3 \to 0$ and the large semi circle Γ of radius R where $R \to \infty$. Since $f(z)$ is analytic within and on C, then by the Cauchy's integral theorem, we have



$$\oint_C f(z)\,dz = \int_{-R}^{-1-r_1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1+r_1}^{-r_2} f(x)\,dx$$

$$+ \int_{\gamma_2} f(z)\,dz + \int_{r_2}^{1-r_3} f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_{1+r_3}^{R} f(x)\,dx +$$

$\int_\Gamma f(z)\,dz = 0$ ..... (1). If $r_1, r_2, r_3 \to 0$ and $R \to \infty$, then (1) $\Rightarrow$

$$\int_{-\infty}^{-1} f(x)\,dx + \int_{\gamma_1} f(z)\,dz + \int_{-1}^{0} f(x)\,dx + \int_{\gamma_2} f(z)\,dz +$$

$$\int_0^1 f(x)\,dx + \int_{\gamma_3} f(z)\,dz + \int_1^{\infty} f(x)\,dx + \int_\Gamma f(z)\,dz = 0 \text{ .... (2)}$$

Now we have : $\lim_{z \to -1} (z+1) f(z) = -\frac{1}{2} e^{-\pi i} = \frac{1}{2}$.

$\lim_{z \to 0} z f(z) = 1$ and $\lim_{z \to 1} (z-1) f(z) = \frac{-1}{2} e^{\pi i} = \frac{1}{2}$.

Using these again we have $\lim_{r_1 \to 0} \int_{\gamma_1} f(z)\,dz = \frac{1}{2}(0 - \pi)$ $= -\frac{i\pi}{2}$, $\lim_{r_2 \to 0} \int_{\gamma_2} f(z)\,dz = i(0-\pi) = -\pi i$ and $\lim_{r_3 \to 0} \int_{\gamma_3} f(z)\,dz = \frac{i}{2}(0-\pi) = -\frac{\pi i}{2}$ where γ_1, γ_2 and γ_3 are described in the clockwise direction. Here $\lim_{z \to \infty} \frac{1}{z(1-z^2)} = 0$ and $f(z)$ is a function of the form $e^{imz} F(z)$, where $F(z) = \frac{1}{z(1-z^2)}$.

Hence by the Jordan's lemma, $\lim_{R \to \infty} \int_\Gamma f(z)\,dz = 0$.

Then (2) $\Rightarrow \int_{-\infty}^{-1} f(x)\,dx - \frac{\pi i}{2} + \int_{-1}^{0} f(x)\,dx - \pi i$

$$+ \int_0^1 f(x)\,dx - \frac{\pi i}{2} + \int_1^{\infty} f(x)\,dx + 0 \Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i$$

$$\Rightarrow 2\int_0^\infty \frac{e^{\pi x i}}{x(1-x^2)}\,dx = 2\pi i$$

$$\Rightarrow \int_0^\infty \frac{\cos \pi x + i \sin \pi x}{x(1-x^2)}\,dx = \pi i \Rightarrow \int_0^\infty \frac{\sin \pi x}{x(1-x^2)}\,dx = \pi \text{ and}$$

the required result is proved.

Example –317 : Show that :

(i) $\int_0^\infty \frac{\cos \pi x}{x(1-x^2)}\,dx = 0;$

(ii) $\int_{-0}^\infty \frac{\cos 2ax - \cos 2bx}{x^2}\,dx = \pi(b-a),\ a \ge b \ge 0;$

(iii) $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \pi/2;$

(iv) $\int_0^\infty \frac{\sin mx}{x(x^2+a^2)^2}\,dx = \frac{\pi}{2a^4} - \frac{\pi}{4a^3}\,e^{-ma}\left(m + \frac{2}{a}\right)$

$m > 0,\ a > 0;$

(vi) $\int_0^\infty \frac{\sin^2 mx}{x^2(x^2+a^2)} = \frac{\pi}{4a^3}(e^{-2ma} - 1 + 2ma),\ m > 0,\ a > 0;$

(vii) $\int_0^\infty \frac{\cos x}{a^2-x^2}\,dx = \frac{\pi}{2a}\sin a,\ a > 0;$

(ix) $\int_0^\infty \frac{x^4}{x^6-1}\,dx = \frac{\pi}{6}\sqrt{3};$

(x) $\int_0^\infty \frac{x-\sin x}{x^3(a^2+x^2)}\,dx = \frac{\pi}{2a^4}[a^2/2 - a + 1 - e^{-a}],\ a > 0;$

(xi) $\int_0^\infty \frac{(\log x)^2}{1+x^2}\,dx = \frac{\pi^3}{8};$

(xii) $\int_0^\infty \frac{\log x}{1+x^2}\,dx = 0;$ **D. U. H. T. 75.**

(xiii) $\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx = -\frac{\pi}{4};$ **R. U. H. 80; D. U. H. T. 78.**

(xiv) $\int_0^\infty \frac{x^a}{1+x^2}\,dx = \frac{\pi}{2}\sec\frac{\pi a}{2},\ -1 < a < 1;$

(xv) $\int_0^\infty \frac{\log(1+x^2)}{x^{1+a}}\,dx = 0,\ 0 < a < 1;$

(xvi) $\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx = \frac{\pi}{2}\operatorname{cosec} \pi a/2,\ 0 < a < 2;$

(xvii) $\int_0^\infty \frac{x^a}{x^2-x+1}\,dx = \frac{2\pi}{\sqrt{3}}\sin\left(\frac{2\pi a}{3}\right)\operatorname{cosec} a\pi, -1 < a < 1;$

(xviii) $\int_0^\infty \frac{x^a}{(1+x^2)^2}\,dx = \frac{\pi(1-a)}{4}\sec\frac{\pi a}{2},\ -1 < a < 3;$

(xix) $\int_0^\infty \frac{x^{a-1}}{x^2+x+1}\,dx = \frac{2\pi}{\sqrt{3}}\operatorname{cosec} \pi a \cos\frac{2\pi a+\pi}{6},\ 0 < a < 2.$

D. U. H. T. '82

(xxx) $\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi},\ 0 < a < 1;$

R. U. M. Sc. P. 86 ; D. U. H. T. 86, 89; R. U. H. 76, 88; D. U. M. Sc. P. 78.

(xxxi) $\int_0^\infty \frac{x^{a-1}}{1-x}\,dx = \pi \cot a\pi,\ 0 < a < 1.$ **R. U. M. Sc. P. 84.**

Solution : Try yourself.

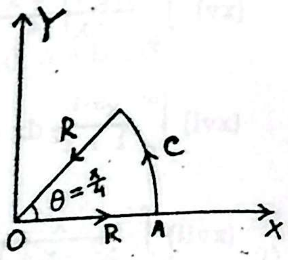
(Form 5) : Integral of the form $\int_0^\infty \sin x^2\,dx$ or $\int_0^\infty \cos x^2\,dx$

These two integals are called the **Fresnel integrals.** See the following example.

Example – 318 : Show that $\int_0^\infty \sin x^2\,dx = \int_0^\infty \cos x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

R. U. H. 81; D. U. M. Sc. P. 88.

Solution : consider $\oint_C f(z)\,dz$.



where $f(z) = e^{iz^2}$ and C or OABO is the contour shown in the Fig. consisting of : (i) OA which lies on the x—axis from $x = 0$ to $x = R$; **(ii)** AB the arc of the circle $|z| = R$ i. e. $z = R e^{i\theta}$ from $\theta = 0$ to $\theta = \pi/4$; **(iii)** BO, the line $z = r e^{\pi i/4}$ from $r = R$ to $r = 0$. There are no singularties inside C and hence by the Cauchy residue theorem, we have

$$\oint_C f(z)\,dz = \int_{OA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BO} f(z)\,dz = 0 \text{ .... (1)}$$

Now on OA : We have $z = x$ and limit from $x = 0$ to $x = R$,

then $\int_{OA} f(z)\,dz = \int_0^R f(x)\,dx = \int_0^R e^{ix^2}\,dx$

$$= \int_0^R (\cos x^2 + i \sin x^2)\,dx \Rightarrow \int_0^\infty (\cos x^2 + i \sin x^2)\,dx \text{ ..... (2)}$$

as $R \to \infty$.

On AB : we have $z = Re^{i\theta}$ and limit from $\theta = o$ to $\theta = \pi/4$.

then $\int_{AB} f(z)\,dz = \int_0^{\pi/4} e^{iR^2 e^{2i\theta}}\,d\,(Re^{i\theta})$

$$= \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\, e^{iR^2 \cos 2\theta}\, iRe^{i\theta}\,d\theta$$

$$\Rightarrow \left|\int_{AB} f(z)\,dz\right| \le \int_0^{\pi/4} e^{-R^2 \sin 2\theta} R\,d\theta = R/2 \int_0^{\pi/4} e^{-R^2 \sin\phi}\,d\phi$$

[putting $2\theta = \phi$

$$\le \frac{R}{2}\int_0^{\pi/2} e^{-2R^2\phi/\pi}\,d\phi = \frac{\pi}{4R}(1 - e^{-R^2}) \text{ .... (3)}$$

[Using Jordan's in equality $\sin\phi \ge 2\phi/\pi$, $0 \le \phi \le \pi/2$

If $R \to \infty$, then (3) $\Rightarrow \lim_{R\to\infty}\left|\int_{AB} f(z)dz\right| \le 0$

$$= \lim_{R\to\infty}\int_{AB} f(z)dz = 0 \text{ ... (4)}$$

On BO : we have $z = r\,e^{\pi i/4}$ and limit from $r = R$ to $r = 0$.

then $\int_{BO} f(z)\,dz = \int_R^0 e^{ir^2 e^{\pi i/2}}\,d\,(re^{\pi i/4})$

$= -e^{\pi i/4}\int_0^R e^{-r^2}\,dr$ .... (5). Now if $R \to \infty$, then (5)

$$\Rightarrow -e^{\pi i/4}\int_0^\infty e^{-r^2}\,dr = -e^{\pi i/4}\frac{\Gamma(1/2)}{2}$$

$$= -(\cos \pi/4 + i \sin \pi/4)\,\frac{\sqrt{\pi}}{2}$$

$= -\dfrac{\sqrt{\pi}}{2\sqrt{2}} - i\,\dfrac{\sqrt{\pi}}{2\sqrt{2}}$ .... (6). Now taking limit $R \to \infty$ in (1)

and using (2) (4) and (6)

$$\Rightarrow \int_0^\infty (\cos x^2 + i \sin x^2) = -\frac{1}{2}\sqrt{\frac{\pi}{2}} - i\frac{1}{2}\sqrt{\frac{\pi}{2}} = 0$$

$$\Rightarrow \int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and } \int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \text{ and}$$

our required result is obtained.

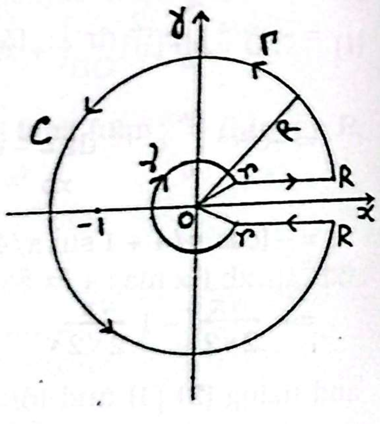
240. (Form 6) . Integrals involving many valued functions i. e. the integrals of the form $\int_0^\infty x^{p-1} f(x)\, dx$, where p is not an integer.

In the following examples we will integrate integrals involving many valued functions.

Example -319 : *If* $0 < p < 1$, then show that $\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$.

R. U. 84; D. U. H. 86; R. U. H. 72, 76, 81, 88; R. U. M. Sc. P. 86; D. U. M. Sc. P. T. 91; D. U. H. T. 86, 89; D. U. M. Sc. P. 78.

Solution : Consider $\oint_C f(z)\, dz$ where $f(z) = \frac{z^{p-1}}{1+z}$.



Since $z = 0$ is a branch point and we choose C as the contour consisting of :

(i) the x—axis from r to R;

(ii) the large circle Γ :

$|z| = R$ where Γ is in the positive direction and $R \to \infty$; (iii) the x—axis from R to r; (iv) the small circle γ : $|z| = r$ where γ is in the negative direction and $r \to 0$. Here C is a closed contour which excludes the the origin and within this contour $f(z)$ is single—valued. The integrand $f(z) = \frac{z^{p-1}}{1+z}$ has a simple pole at $z = -1$ inside C.

Now we consider amp$z = 0$ on (i). Then on (iii) we have amp$z = 2\pi$ since the amplitude of z is increased by 2π in going around the circle Γ.

Now at the point $z = -1 \Rightarrow z = e^{\pi i}$ since amp $z = \pi$ and the residue at this point is $\lim_{z \to -1} \left\{ (z+1) \frac{z^{p-1}}{1+z} \right\} = (e^{\pi i})^{p-1}$

$= e^{(p-1)\pi i}$

Hence we have $\oint_C f(z)\, dz = 2\pi i\, e^{(p-1)\pi i}$

$$\Rightarrow \int_r^R \frac{x^{p-1}}{1+x} dx + \int_\Gamma \frac{z^{p-1}}{1+z} dz + \int_R^r \frac{(x e^{2\pi i})^{p-1}}{1+x e^{2\pi i}} dx + \int_\gamma \frac{z^{p-1}}{1+z} dz$$

$= 2\pi i\, e^{(p-1)\pi i}$ ... (1)

Here $\int_r^R \frac{x^{p-1}}{1+x} dx + \int_R^r \frac{(x e^{2\pi i})^{p-1}}{1+x e^{2\pi i}} dx$

$= \{1 - e^{2(p-1)\pi i}\} \int_r^R \frac{x^{p-1}}{1+x} dx$ .... (2) $[\because e^{2\pi i} = 1]$

Now on Γ we have $\left| \frac{z^{p-1}}{1+z} \right| \le \frac{R^{p-1}}{R-1}$ so that

$\left| \int_\Gamma \frac{z^{p-1}}{1+z} dz \right| \le \frac{R^{p-1}}{R-1} . 2\pi R = \frac{2\pi R^p}{R-1} \to 0$ as $R \to \infty$ since $0 < p < 1$. Thus $\lim_{R \to \infty} \int_\Gamma \frac{z^{p-1}}{1+z} dz = 0$ .... (3)

Similarly $\left| \int_\gamma \frac{z^{p-1}}{1+z} dz \right| \le \frac{2\pi r^p}{1-r} \to 0$ as $r \to 0$ since $p > 0$.

Thus $\lim\limits_{r \to 0} \int_\gamma \frac{z^{p-1}}{1+z} dz = 0$ .... (4)

Now taking limit $r \to 0$ and $R \to \infty$ in (1) and (2) and using (3) and (4) , we have

$$\{1 - e^{2(p-1)\pi i}\} \int_0^\infty \frac{x^{p-1}}{1+x} dx = 2\pi i\, e^{(p-1)\pi i}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{2\pi i\, e^{(p-1)\pi i}}{1 - e^{2(p-1)\pi i}} = \frac{2\pi i}{e^{p\pi i} - e^{-p\pi i}}$$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$$ and the result is Proved.

N. B. The above example can also be solved, by indenting at the points $z = -1, 0$ on the x—axis. It can be solved in other ways also.

Example -320 : Using the above example,

show that $\int_{-\infty}^\infty \frac{e^{pt}}{1+e^t} dt = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

Solution : In the above example we have

$$\int_0^\infty \frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi} \text{ .... (1) where } 0 < p < 1.$$

Let $x = e^t$, then $dx = e^t\, dt$ and when $x = 0 \Rightarrow t = -\infty$ and when $x = \infty \Rightarrow t = \infty$ Now Using these (1) $\Rightarrow$

$$\int_{-\infty}^\infty \frac{(e^t)^{p-1}}{1+e^t} e^t\, dt = \frac{\pi}{\sin p\pi}$$

$$\Rightarrow \int_{-\infty}^\infty \frac{e^{pt}}{1+e^t} dt = \frac{\pi}{\sin p\pi}$$ where $0 < p < 1$ and the result is proved.

Example -321 : Show that $\int_0^\infty \frac{x^p}{1 + 2x\cos\theta + x^2} dx$

$= \frac{\pi \sin(p\theta)}{\sin(\pi p)\sin\theta}$ where $-1 < p < 1,\ p \ne 0,\ -\pi < \theta < \pi,\ \theta \ne 0$

Solution : Try yourself.

Example -322 : Show that

$$\int_0^\infty \frac{x^p}{(x+9)^2} dx = \frac{9^{p-1}\pi p}{\sin \pi p} \text{ where } -1 < p < 1,\ p \ne 0$$

Solution : Try yourself.

Example -323 : Show that $\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

D. U. H. T. 87, 88, 89; C. U. 86; D. U. S. 86; D. U. 73, 75, 86, 88, 89, 91.

Solution : We know if $0 < p < 1$, then $\int$ \s(\s\up12(∞),\s(0))

$\frac{x^{p-1}}{1+x} dx = \frac{\pi}{\sin p\pi}$ .... (1)

Again we know $\beta(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx$

.... (2)

Now let $p + q = 1$, then $q = 1 - P$ and using these (2) $\Rightarrow$

$\frac{\Gamma(p)\Gamma(1-p)}{\Gamma(1)} = \int_0^\infty \frac{x^{p-1}}{1+x} dx \Rightarrow \Gamma(p)\Gamma(1-p) = \int_0^\infty \frac{x^{p-1}}{1+x} dx$

.... (3) $[\because \Gamma(1) = 1]$

Now by (1) and (3), we have

$$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad \text{where } 0 < p < 1$$

and the result is proved.

241. (Form 7). Rectangular contours :

In the following examples we will use the rectangular contours.

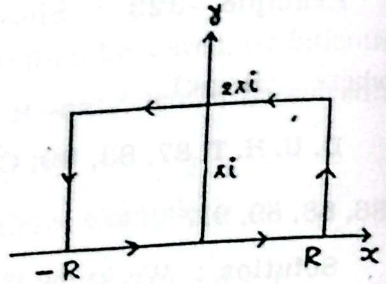
Example – 324 : Show that $\displaystyle\int_{-\infty}^{\infty} \frac{e^{px}}{1+e^{x}}\,dx = \frac{\pi}{\sin p\pi}$

where $0 < p < 1$.

R. U. 82; R. U. H. 75, 77, 87; R. U. M. Sc. P. 88; D. U. M. Sc. P. 78; D. U. H. T. 76, 78, 86, 87; D. U. H. 90.

Solution : Consider

$\displaystyle\oint_C f(z)\,dz$ where $f(z) = \dfrac{e^{pz}}{1+e^{z}}$



and C is a rectangle having sides consisting of the x—axis and the lines $x = \pm R$, $y = 2\pi$.

The poles of $f(z) = \dfrac{e^{pz}}{1+e^{z}}$ are obtained by solving $1 + e^{z} = 0$

$\Rightarrow e^{z} = -1$

$= e^{(2n+1)\pi i}$ where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$ The only pole enclosed by C is πi and it is simple. Now Res $[f(z), \pi i]$

$$\lim_{z\to\pi i} \left\{(z-\pi i)\,\frac{e^{pz}}{1+e^{z}}\right\}$$

$$= \lim_{z\to\pi i} e^{pz} \cdot \lim_{z\to\pi i} \frac{z-\pi i}{1+e^{z}} = e^{p\pi i} \lim_{z\to\pi i} \frac{1}{e^{z}} \text{ [by L' Hospital rule]}$$

$$= \frac{e^{p\pi i}}{e^{\pi i}} = -e^{p\pi i} \text{ since } e^{\pi i} = -1;$$

Then by the Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{0}^{2\pi} f(R+iy)\,i\,dy + \int_{R}^{-R} f(x+2\pi i)\,dx$$

$$+ \int_{2\pi}^{0} f(-R+iy)\,i\,dy = -2\pi i\, e^{p\pi i} \quad \ldots (1)$$

Here $\displaystyle\int_{-R}^{R} f(x)\,dx + \int_{R}^{-R} f(x+2\pi i)\,dx$

$$= \int_{-R}^{R} \frac{e^{px}}{1+e^{x}}\,dx - \int_{-R}^{R} \frac{e^{p(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx.$$

$$= (1-e^{2p\pi i}) \int_{-R}^{R} \frac{e^{px}}{1+e^{x}}\,dx \quad \ldots (2) \quad \left[\because \begin{matrix} e^{2p\pi} = 1 \\ e^{2\pi i} = 1 \end{matrix}\right]$$

Now we have $\displaystyle |f(R+yi)| = \left|\frac{e^{p(R+yi)}}{1+e^{R+yi}}\right| \le \frac{e^{pR}}{e^{R}-1}$

$$\therefore \left|\int_{0}^{2\pi} f(R+yi)\,i\,dy\right| \le \frac{e^{pR}\,2\pi}{e^{R}-1} \to 0 \text{ as } R\to\infty$$

Thus $\displaystyle\lim_{R\to\infty} \int_{0}^{2\pi} f(R+yi)\,i\,dy = 0 \quad \ldots (3)$

Similarly we can show that $\displaystyle\lim_{R\to\infty} \int_{2\pi}^{0} f(-R+yi)\,i\,dy = 0 \ldots (4)$

Now taking limit $R\to\infty$ in (1) an (2) and using (3) and (4) we have $\displaystyle(1-e^{2p\pi i}) \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^{x}}\,dx = -2\pi i\, e^{p\pi i}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^{x}}\,dx = \frac{-2\pi i e^{p\pi i}}{1-e^{2p\pi i}} = \frac{2\pi i}{e^{p\pi i}-e^{-p\pi i}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{px}}{1+e^{x}}\,dx = \frac{\pi}{\sin p\pi} \quad \text{where } 0<p<1 \text{ and the result is}$$

Proved.

Example -325 : Using the above example, show that $\int_{-\infty}^{\infty} \frac{t^{p-1}}{1+t}\,dt = \frac{\pi}{\sin p\pi}$ where $0 < p < 1$.

Solution : In the above example we have

$$\int_0^\infty \frac{e^{px}}{1+e^x}\,dx = \frac{\pi}{\sin px} \; ..... (1) \text{ where } 0 < p < 1.$$

Let $e^x = t$, then $dx = \frac{dt}{t}$ and when $x = \infty \Rightarrow t = \infty$ and when $x = \infty \Rightarrow t = 0$

Now using these (1) $\Rightarrow$

$$\int_{-\infty}^{\infty} \frac{t^p}{1+t}\,\frac{dt}{t} = \frac{\pi}{\sin p\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{t^{p-1}}{1+t}\,dt = \frac{\pi}{\sin p\pi} \quad \text{where } 0 < p < 1$$

and the require result is proved.

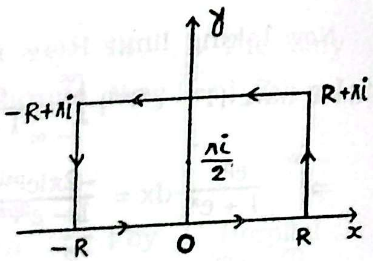
Example -326 : Show that $\int_0^\infty \frac{\cosh ax}{\cosh x}\,dx = \frac{\pi}{2\cos \pi a/2}$

D. U. H. T. 75; D. U. H. 89.

where $|a| < 1$.

Solution: Consider $\oint_C f(z)\,dz$

where $f(z) = \frac{e^{az}}{\cosh z}$ and C is a rectangle having vertices at $-R$, R, $R + \pi i$, $-R + \pi i$. The poles of $f(z) = \frac{e^{az}}{\cosh z}$ are obtained by solving $\cosh z = 0$



$\Rightarrow e^z + e^{-z} = 0 \Rightarrow e^{2z} + 1 \Rightarrow e^{2z} =$

$-1 = e^{(2n+1)\pi i} \Rightarrow z = (2n+1)\pi i/2$ where $n = 0, \pm 1, \pm 2, \pm 3, ...$ The only pole enclosed by C is $\pi i/2$ and it is simple.

Now $\text{Res}\left[f(z), \frac{\pi i}{2}\right] = \lim_{z \to \pi i/2} \left\{\left(z - \frac{\pi i}{2}\right)\frac{e^{az}}{\cosh z}\right\}$

$$= \lim_{z \to \pi i/2} e^{az} \cdot \lim_{z \to \pi i/2} \frac{z - \pi i/2}{\cosh z}$$

$$= e^{a\pi i/2} \lim_{z \to \pi i/2} \frac{1}{\sinh z} \quad [\text{by L' Hospital rule}]$$

$$= \frac{e^{a\pi i/2}}{\sinh \pi i/2} = \frac{e^{a\pi i/2}}{i \sin \pi/2} = -i\, e^{a\pi i/2}.$$

Then by the residue theorem, we have $\oint_C f(z)\,dz =$

$$\int_{-R}^{R} f(x)\,dx + \int_0^{\pi} f(R+iy)\, i\,dy + \int_R^{-R} f(x+\pi i)\,dx +$$

$$\int_{\pi}^{0} f(-R+iy)\, i\,dy = 2\pi i\,(-ie^{a\pi i/2}) \; ...... (1).$$ In (1) we have

$$\int_{-R}^{R} f(x)\,dx + \int_R^{-R} f(x+i\pi)\,dx$$

$$= \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx - \int_{-R}^{R} \frac{2e^{ax}\,e^{a\pi}}{e^{x+i\pi} + e^{-x-i\pi}}\,dx$$

$$= \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx + \int_{-R}^{R} \frac{2e^{ax}\,e^{a\pi i}}{e^x + e^{-x}}\,dx \quad [e^{\pi i} = e^{-\pi i} = -1]$$

$$= (1 + e^{a\pi i}) \int_{-R}^{R} \frac{e^{ax}}{\cosh x}\,dx \; ..... (2)$$

Now $f(R+iy) = \frac{e^{a(R+iy)}}{\cosh(R+iy)} = \frac{2e^{aR}e^{aiy}}{e^R e^{iy} - e^{-R}e^{-iy}}$

Then $|f((R+iy)| \le \dfrac{2|e^{aR}|.|e^{aiy}|}{|e^{R}||e^{iy}|-|e^{-R}||e^{-iy}|} = \dfrac{2e^{aR}}{e^{R}-e^{-R}}$

$\Rightarrow |f(R+idy| \le \dfrac{4e^{aR}}{e^{R}}$ since $e^{R}-e^{-R} \ge \frac{1}{2}e^{R}$

$\Rightarrow \left|\int_0^{\pi} f(R+idy)\, idy\right| \le \int_0^{\pi} \dfrac{4e^{aR}}{e^{R}}dy = 4\pi e^{(a-1)R} \to 0$ as $R\to\infty$

since $|a| < 1$.

$$\therefore \lim_{R\to\infty} \int_0^{\pi} f(R+iy)\, idy = 0 \quad .....\ (3)$$

Similarly we can show that

$$\lim_{R\to\infty} \int_{\pi}^{0} f(-R+iy)\, idy = 0 \quad ....\ (4)$$

Now taking limit $R\to\infty$ in (1) and (2) and using (3) and (4) we have

$$(1+e^{a\pi i})\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}dx = 2\pi e^{a\pi i/2}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x}dx = \frac{2\pi e^{a\pi i/2}}{1+e^{a\pi i}} = \frac{2\pi}{e^{a\pi i/2}+e^{-a\pi i/2}} = \frac{2\pi}{2\cos\frac{a\pi}{2}}$$

$$\Rightarrow \int_{-\infty}^{0} \frac{e^{ax}}{\cosh x}dx + \int_0^{\infty} \frac{e^{ax}}{\cosh x}dx = \frac{\pi}{\cos a\pi/2} \quad ....\ (5)$$

Now replacing x by $-x$ in the first integral of (5), then we get.

$$\int_0^{\infty} \frac{e^{-ax}}{\cosh x}dx + \int_0^{\infty} \frac{e^{ax}}{\cosh x}dx = \frac{\pi}{\cos a\pi/2}$$

$$\Rightarrow \int_0^{\infty} \frac{e^{ax}+e^{-ax}}{\cosh x}dx = \frac{\pi}{\cos a\pi/2}$$

$$\Rightarrow \int_0^{\infty} \frac{2\cosh ax}{\cosh x}dx = \frac{\pi}{\cos a\pi/2}$$

$\Rightarrow \displaystyle\int_0^{\infty} \frac{\cosh ax}{\cosh x}dx = \frac{\pi}{2\cos a\pi/2}$ and the required result is obtained.

Example -327 : Show that

(i) $\displaystyle\int_0^{\infty} \frac{\cosh ax}{\cosh \pi x}dx = \frac{1}{2}\sec a/2$ where $-\pi < a < \pi$;

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(ii) $\displaystyle\int_0^{\infty} \frac{\sinh ax}{\sinh \pi x}dx = \frac{1}{2}\tan\frac{a}{2}$ where $-\pi < a < \pi$;

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(iii) $\displaystyle\int_0^{\infty} \frac{x\sin x}{1+a^2-2a\cos x}dx = \frac{\pi}{a}\log(1+a)$ where $0 < a < 1$;

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(iv) $\displaystyle\int_0^{\infty} e^{-x^2}\cos 2ax\, dx = \frac{1}{2}\sqrt{\pi}\, e^{-a^2}$.

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Solution : Try yourself.