Logic

Propositional Logic Concepts

- Logic is a study of principles used to
  - distinguish correct from incorrect reasoning.
- Formally it deals with
  - the notion of truth in an abstract sense and is concerned with the principles of valid inferencing.
- A proposition in logic is a declarative statements which are either **true or false** (but not both) in a given context. For example,
  - "Jack is a male"

Cont..

- Given some propositions to be true in a given context,
  - logic helps in inferencing new proposition, which is also true in the same context.
- Suppose we are given a set of propositions such as
  - “It is hot today" and
  - “If it is hot, it will rain", then
  - we can infer that
    - “It will rain today".

This Lecture

- There are various forms of logic, of which the simplest is probably propositional calculus (also known as sentence logic), and the most commonly used in AI is first order predicate calculus (also known as first order predicate logic).

A Story

- **You roommate comes home; he/she is completely wet**
- You know the following things:
  - Your roommate is wet
  - If your roommate is wet, it is because of rain, sprinklers, or both
  - If your roommate is wet because of sprinklers, the sprinklers must be on
  - If your roommate is wet because of rain, your roommate must not be carrying the umbrella
  - The umbrella is not in the umbrella holder
  - If the umbrella is not in the umbrella holder, either you must be carrying the umbrella, or your roommate must be carrying the umbrella
  - You are not carrying the umbrella
- **Can you conclude that the sprinklers are on?**
- **Can AI conclude that the sprinklers are on?**

Knowledge Base For The Story

- RoommateWet
- RoommateWet => (RoommateWetBecauseOfRain OR RoommateWetBecauseOfSprinklers)
- RoommateWetBecauseOfSprinklers => SprinklersOn
- RoommateWetBecauseOfRain => NOT(RoommateCarryingUmbrella)
- UmbrellaGone
- UmbrellaGone => (YouCarryingUmbrella OR RoommateCarryingUmbrella)
- NOT(YouCarryingUmbrella)

Syntax

- What do well-formed sentences in the knowledge base look like?
- A BNF grammar:
- *Symbol* → P, Q, R, …, RoommateWet, …
- *Sentence* → True | False | *Symbol* | NOT(*Sentence*) | (*Sentence* AND *Sentence*) | (*Sentence* OR *Sentence*) | (*Sentence* => *Sentence*)
- We will drop parentheses sometimes, but formally they really should always be there

Propositional Calculus

- Propositional calculus is built out of simple statements called propositions which are either **true or false**.
  - London is a city.
  - Ice is hot.

Syntax and Semantics of Logics

- Syntax
  - How we can construct legal sentences in the logic
  - Which symbols we can use (English: letters, punctuation)
  - How we are allowed to write down those symbols
- Semantics
  - How we interpret (read) sentences in the logic
  - i.e., what the meaning of a sentence is
- Example: “All lecturers are six foot tall”
  - Perfectly valid sentence (syntax)
  - And we can understand the meaning (semantics)
  - This sentence happens to be false (there is a counter-example)

Propositional Logic

- Syntax
  - Propositions such as P meaning “it is wet”
  - Connectives: and, or, not, implies, equivalent
  - Brackets, T (true) and F (false)

- Semantics
  - How to work out the truth of a sentence
    - Need to know how connectives affect truth
    - E.g., “P and Q” is true if and only if P is true and Q is true
    - “P implies Q” is true if P and Q are true or if P is false
  - Can draw up truth tables to work out the truth of statements

Well Formed Formulas (WFFs)

- Logical Sentences are also called Well Formed Formulas (WFFs).
- A WFF is defined as follows:
  - A symbol is a sentence
  - If **S** is a sentence, then $\neg$**S** is a sentence (negation)
  - If $\mathbf{S_1}$ and $\mathbf{S_2}$ are sentences, $(\mathbf{S_1} \wedge \mathbf{S_2})$ is a sentence (conjunction)
  - If $\mathbf{S_1}$ and $\mathbf{S_2}$ are sentences, $(\mathbf{S_1} \vee \mathbf{S_2})$ is a sentence (disjunction)
  - If $\mathbf{S_1}$ and $\mathbf{S_2}$ are sentences, $(\mathbf{S_1} \Rightarrow \mathbf{S_2})$ is a sentence (implication)
  - If $\mathbf{S_1}$ and $\mathbf{S_2}$ are sentences, $(\mathbf{S_1} \Leftrightarrow \mathbf{S_2})$ is a sentence (biconditional)
  - Precedence $\neg$, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$

Propositional Calculus

- These are joined together to form more complex statements by logical connectives, expressing simple ideas such as and, or, not, if…then…

Propositional Calculus

- There are standard symbols for these:

  $\wedge$ stands for "and",

  $\vee$ stands for "or",

  $\neg$ stands for "not",

  $\Rightarrow$ stands for "if … then …",

  $\Leftrightarrow$ stands for "if and only if".

Propositional Calculus

Example of a statement written in propositional calculus:

- **R** stands for "It is raining"
- **G** stands for "I have got a coat"
- **W** stands for "I will get wet".

$$\mathbf{R} \wedge \neg \mathbf{G} \Rightarrow \mathbf{W}$$

is a way of writing

"**If** it is raining **and** I have not got a coat, **then** I will get wet."

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
  - ❑ P=Q=true
  - ❑ P=Q=false
  - ❑ P=true, Q=false
  - ❑ P=false, Q=true

$P \rightarrow Q$

- When is $P \rightarrow Q$ true? Check all that apply
  - ☑ P=Q=true
  - ☑ P=Q=false
  - ☐ P=true, Q=false
  - ☑ P=false, Q=true
- We can get this from the truth table for $\rightarrow$

Semantics

- Given a model, It should be able to tell you whether a sentence is true or false

- Truth table defines semantics of operators:

A	B	NOT(A)	$A \wedge B$	$A \vee B$	A => B
false	false	true	false	false	true
false	true	true	false	true	true
true	false	false	false	true	false
true	true	false	true	true	true

Semantics

- With these symbols, *8* possible models, can be enumerated automatically.
- Rules for evaluating truth with respect to a model *m*:
  - $\neg S$ is true iff S is false
  - $S_1 \wedge S_2$ is true iff S_1 is true and S_2 is true
  - $S_1 \vee S_2$ is true iff S_1 is true or S_2 is true
  - $S_1 \Rightarrow S_2$ is true iff S_1 is false or S_2 is true
    - i.e., is false iff S_1 is true and S_2 is false
  - $S_1 \Leftrightarrow S_2$ is true iff $S_1 \Rightarrow S_2$ is true and $S_2 \Rightarrow S_1$ is true

Truth Tables For Connectives

- Truth tables are used to define logical connectives
- and to determine when a complex sentence is true given the values of the symbols in it

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
$false$	$false$	$true$	$false$	$false$	$true$	$true$
$false$	$true$	$true$	$false$	$true$	$true$	$false$
$true$	$false$	$false$	$false$	$true$	$false$	$false$
$true$	$true$	$false$	$true$	$true$	$true$	$true$

Example of A Truth Table Used for A Complex Sentence

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

More terms

- A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or what the semantics is.
  - Example: “It’s raining or it’s not raining”
- An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes.
  - “It’s raining and it’s not raining.”
- **P entails Q**, written P $\rightarrow$ Q, means that whenever P is True, so is Q. In other words, all models of P are also models of Q.

Validity

- A sentence is valid, if it is true in **all** models

$$True,$$

$$A \vee \neg A$$

$$A \Rightarrow A$$

$$(A \wedge (A \Rightarrow B)) \Rightarrow B$$

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A	B	NOT(A)	A ∧ B	A ∨ B	A => B
false	false	true	false	false	true
false	true	true	false	true	true
true	false	false	false	true	false
true	true	false	true	true	true

Example

- Show that " It is humid today **and if** it is humid **then** it will rain **so** it will rain today"   is a valid argument.
- **Solution:** Let us symbolize English sentences by propositional atoms as follows:

  A : It is humid

  B : It will rain

- Formula corresponding to a text:

  $\alpha : ((A \rightarrow B) \wedge A) \rightarrow B$

- Using truth table approach, one can see that α is true under all four interpretations and hence is valid argument.

Cont..

Truth Table for $((A \rightarrow B) \wedge A) \rightarrow B$				
A	B	$A \rightarrow B = X$	$X \wedge A = Y$	**$Y \rightarrow B$**
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Tautologies

- A sentence is a tautology if it is true for any setting of its propositional symbols

P	Q	P OR Q	NOT(P) AND NOT(Q)	(P OR Q) OR (NOT(P) AND NOT(Q))
false	false	false	true	true
false	true	true	false	true
true	false	true	false	true
true	true	true	false	true

- (P OR Q) OR (NOT(P) AND NOT(Q)) is a tautology

Is This A Tautology?

(P => Q) OR (Q => P)

Satisfiablity

- A sentence is satisable if it is true in **some** model

  $A \vee B$

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- A sentence is unsatisable if it is true in **no** models

  $A \wedge \neg A$

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A	B	NOT(A)	A ∧ B	A ∨ B	A => B
false	false	true	false	false	true
false	true	true	false	true	true
true	false	false	false	true	false
true	true	false	true	true	true

Logical Equivalences

- Two sentences α and β are equivalent, if they are true in the same set of models, which is written as α <=> β
  - they have the same truth value for every setting of their propositional variables

P	Q	P OR Q	NOT(NOT(P) AND NOT(Q))
false	false	false	false
false	true	true	true
true	false	true	true
true	true	true	true

- P OR Q and NOT(NOT(P) AND NOT(Q)) are logically equivalent
- Tautology = logically equivalent to True

Equivalence Laws

Commutation

1.	$P \wedge Q$	$\cong$	$Q \wedge P$
2.	$P \vee Q$	$\cong$	$Q \vee P$

Association

1.	$P \wedge (Q \wedge R)$	$\cong$	$(P \wedge Q) \wedge R$
2.	$P \vee (Q \vee R)$	$\cong$	$(P \vee Q) \vee R$

Double Negation

$\sim(\sim P) \cong P$

Distributive Laws

1.	$P \wedge (Q \vee R)$	$\cong$	$(P \wedge Q) \vee (P \wedge R)$
2.	$P \vee (Q \wedge R)$	$\cong$	$(P \vee Q) \wedge (P \vee R)$

De Morgan's Laws

1.	$\sim(P \wedge Q)$	$\cong$	$\sim P \vee \sim Q$
2.	$\sim(P \vee Q)$	$\cong$	$\sim P \wedge \sim Q$

Law of Excluded Middle

$P \vee \sim P \cong$ T (true)

Law of Contradiction

$P \wedge \sim P \cong$ F (false)