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1st lecture (27/08/2023-Offline). (Midterm)

- **From slide:**
  - What is numerical method? How its used in engineering?
  - Why need to know numerical methods?
  - Real life application of numerical method.
  - **Suggested Book:** Numerical methods for engineers.
    By: Steven C. Chapra and Raymond P. Canale.
- **From book:**

❖ The Bisection methods: In general, if f(x) is real and continuous in the interval from x_l to x_u and $f(x_l)$ and $f(x_u)$ have opposite signs, that is, $f(x_l)f(x_u)<0$

then there is at least one real root between x_l and x_u.

➢ **Procedure:**

1. Choose 2 real numbers a & b such that f(a)*f(b)<0.
2. Define root, $c=\frac{a+b}{2}$.
3. Find f(c).
4. F(a) x f(c)<0

   then set b=c

   else set a=c

return the step-1 until finding the root matched twice.

➢ **Example-5.4:**

Problem Statement. Use the graphical approach to determine the drag coefficient c needed for a parachutist of mass $m = 68.1$ kg to have a velocity of 40 m/s after free-falling for time $t = 10$ s. *Note:* The acceleration due to gravity is 9.81 m/s^2.

Sol^n: Follow book for step-by-step solution.

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40 \qquad \text{(E5.1.1)}$$

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	12	16	14		5.413
2	14	16	15	6.667	1.344
3	14	15	14.5	3.448	2.035
4	14.5	15	14.75	1.695	0.345
5	14.75	15	14.875	0.840	0.499
6	14.75	14.875	14.8125	0.422	0.077

2^{nd} lecture (29/08/2023-Online). (Midterm)

- **Random example: Consider finding the root of f(x) = x^2 - 3. Let ε_{step} = 0.01, ε_{abs} = 0.01 and start with the interval [1, 2].**

Given: $x^2-3 = 0$
Let $f(x) = x^2-3$
Now, find the value of f(x) at a= 1 and b=2.
$f(x=1) = 1^2-3 = 1 – 3 = -2 < 0$
$f(x=2) = 2^2-3 = 4 – 3 = 1 > 0$
The given function is continuous, and the root lies in the interval [1, 2].
Let "c" be the midpoint of the interval.
I.e., c = (1+2)/2
c = 3 / 2
c = 1.5
Therefore, the value of the function at "c" is
$f(c) = f(1.5) = (1.5)^2-3 = 2.25 – 3 = -0.75 < 0$
If f(c)<0, assume a = c.
and
If f(c)>0, assume b = c.
f(c) is negative, so a is replaced with c = 1.5 for the next iterations.

Table 1. Bisection method applied to f(x) = x^2 - 3.

a	b	f(a)	f(b)	c = (a + b)/2	f(c)	Update	new b − a
1.0	2.0	-2.0	1.0	1.5	-0.75	a = c	0.5
1.5	2.0	-0.75	1.0	1.75	0.062	b = c	0.25
1.5	1.75	-0.75	0.0625	1.625	-0.359	a = c	0.125
1.625	1.75	-0.3594	0.0625	1.6875	-0.1523	a = c	0.0625
1.6875	1.75	-0.1523	0.0625	1.7188	-0.0457	a = c	0.0313
1.7188	1.75	-0.0457	0.0625	1.7344	0.0081	b = c	0.0156
1.71988/td>	1.7344	-0.0457	0.0081	1.7266	-0.0189	a = c	0.0078

Thus, with the seventh iteration, we note that the final interval, [1.7266, 1.7344], has a width less than 0.01 and |f(1.7344)| < 0.01, and therefore we chose b = 1.7344 to be our approximation of the root.

➢ **Related example: Consider finding the root of f(x) = e^{-x}(3.2 sin(x) - 0.5 cos(x)) on the interval [3, 4], this time with ε_{step} = 0.001, ε_{abs} = 0.001.**

Table 1. Bisection method applied to f(x) = e^{-x}(3.2 sin(x) - 0.5 cos(x)).

a	b	f(a)	f(b)	c = (a + b)/2	f(c)	Update	new b − a
3.0	4.0	0.047127	-0.038372	3.5	-0.019757	b = c	0.5
3.0	3.5	0.047127	-0.019757	3.25	0.0058479	a = c	0.25
3.25	3.5	0.0058479	-0.019757	3.375	-0.0086808	b = c	0.125
3.25	3.375	0.0058479	-0.0086808	3.3125	-0.0018773	b = c	0.0625
3.25	3.3125	0.0058479	-0.0018773	3.2812	0.0018739	a = c	0.0313
3.2812	3.3125	0.0018739	-0.0018773	3.2968	-0.000024791	b = c	0.0156
3.2812	3.2968	0.0018739	-0.000024791	3.289	0.00091736	a = c	0.0078
3.289	3.2968	0.00091736	-0.000024791	3.2929	0.00044352	a = c	0.0039
3.2929	3.2968	0.00044352	-0.000024791	3.2948	0.00021466	a = c	0.002
3.2948	3.2968	0.00021466	-0.000024791	3.2958	0.000094077	a = c	0.001
3.2958	3.2968	0.000094077	-0.000024791	3.2963	0.000034799	a = c	0.0005

Thus, after the 11th iteration, we note that the final interval, [3.2958, 3.2968] has a width less than 0.001 and |f(3.2968)| < 0.001 and therefore we chose b = 3.2968 to be our approximation of the root.

3rd lecture (31/08/2023-Online). (Midterm)

✪ Difference between determinants and matrices.

1. Matrix is the set of numbers which are covered by two brackets. Determinants is also set of numbers but it is covered by two bars.
2. It is not necessary that number of rows will be equal to the number of columns in matrix, But it is necessary that number of rows will be equal to the number of columns in determinant.
3. Matrix can be used for adding, subtracting and multiplying the coefficients. Determinants can be used for calculating the value of x, y and z with Cramer's Rule.

✪ Cramer's rule.

➢ **Example: Using Cramer's rule solve the following.**

$.3x_1+.52x_2+x_3=0.01$

$.5x_1+.3x_2+.5x_3=.67$

$.1x_1+.3x_2+.5x_3=-.44$

Sol^n: Let us write these equations in the form AX=B

$$\begin{bmatrix} .3 & .52 & 1 \\ .5 & .3 & .5 \\ .1 & .3 & .5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .01 \\ .67 \\ -.44 \end{bmatrix}$$

D=|A|=$\begin{vmatrix} .3 & .52 & 1 \\ .5 & .3 & .5 \\ .1 & .3 & .5 \end{vmatrix}$=.3(.5x.3-.5x.3)-.52(.5x.5-.5x.1)+1(.5x.3-.3x.1)=0.016

Dx_1=$\begin{vmatrix} .01 & .52 & 1 \\ ..67 & .3 & .5 \\ -.44 & .3 & .5 \end{vmatrix}$=0.0444

Dx_2=$\begin{vmatrix} .3 & .01 & 1 \\ .5 & .67 & .5 \\ .1 & -.44 & .5 \end{vmatrix}$= -0.1255

$Dx_3 = \begin{vmatrix} .3 & .52 & .01 \\ .5 & .3 & .67 \\ .1 & .3 & -.44 \end{vmatrix}$=0.05054

X_1= Dx_1/D=0.0444/0.016=2.775

X_2= Dx_2/D=-0.1255/0.016=-7.84375

X_3= Dx_3/D=0.05054/0.016=3.158

❖ When Cramer's rule become Impractical?

- For more than three equations, Cramer's rule becomes impractical because, as the number of equations increases, the determinants are time consuming to evaluate by hand or by computer.

✪ Gauss-elimination method.

➤ **In this method we basically follow two steps:**

- Forward Elimination.
- Backward Substitution.

➤ **Example-01: Solve the given set of equations by using Gauss elimination method:**

$3x_1+.1x_2-.2x_3 = 7.85$
$.1x_1+7x_2-.3x_3 = -19.3$
$.3x_1-2x_2+10x_3 = 71.4$

Sol^n: We'll create the augmented matrix and perform row operations:

$$\left[\begin{array}{ccc|c} 3 & 0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -2 & 10 & 71.4 \end{array}\right]$$

Step 1: Perform row operations to create zeros below the leading coefficient in the first column.

- ✓ R2 = R2 - (0.1/3) * R1
- ✓ R3 = R3 - (0.3/3) * R1

The augmented matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 0.1 & -0.2 & 7.85 \\ 0 & 6.99967 & -0.2933 & -19.56167 \\ 0 & -2.01 & 10.02 & 70.615 \end{array}\right]$$

Step 2: Create zeros below the leading coefficient in the second column.

- ✓ R3 = R3 + (2.01/6.97) * R2

The augmented matrix becomes:

$$\left[\begin{array}{ccc|c} 3 & 0.1 & -0.2 & 7.85 \\ 0 & 6.99967 & -0.2933 & -19.56167 \\ 0 & 0 & 9.935 & 64.99774 \end{array}\right]$$

Step 3: Solve for x_3 using the last row:

$9.935x_3 = 64.99774$

$x_3 \approx 64.99774/ 9.935$

$x_3 \approx 6.54$

Step 4: Substitute the value of x_3 into the second row to solve for x_2:

$6.99967x_2 - 0.2933x_3 = -19.56167$

$6.99967x_2 - -0.2933(6.54) = -19.56167$

$6.99967x_2 - 1.98 \approx -19.56167$

$6.99967x_2 \approx -19.56167 + 1.98$

$6.99967x_2 \approx -17.5817$

$x_2 \approx -17.5817/ 6.99967$

$x_2 \approx -2.51$

Step 5: Substitute the values of x_2 and x_3 into the first row to solve for x_1:

$3x_1 + 0.1x_2 - 0.2x_3 = 7.85$

$3x_1 + 0.1(-2.51) - 0.2(6.54) \approx 7.85$

$3x_1 - 0.251 – 1.308 \approx 7.85$

$3x_1 \approx 7.85 + 1.559$

$3x_1 \approx 9.409$

$x_1 \approx 9.409 / 3 \Rightarrow x_1 \approx 3.136$

4^{th} lecture (07/09/2023-Offline). (Midterm)

- **Gauss-Jordan Method.**
- **Triangularization /Factorization /Cholesky's Method.**

✪ Gauss-Jordan method.

➢ **Example-1: Solve the following system by the Gauss-Jordan method.**

$3x_1-0.1x_2-0.2x_3=7.85$
$0.1x_1+7x_2-0.3x_3=-19.3$
$0.3x_1-0.2x_2+10x_3=71.4$

- **Step-01:** First we have to express the coefficients and the right-hand side as an augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -0.1 & -0.2 & 7.85 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right]$$

- **Step-02:** Divide Row 1 by 3 to make the leading coefficient 1 in the first row:

$$\left[\begin{array}{ccc|c} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0.1 & 7 & -0.3 & -19.3 \\ 0.3 & -0.2 & 10 & 71.4 \end{array}\right] r_1'=r_1/3$$

- **Step-03:** Subtract 0.1 times Row 1 from Row 2 and 0.3 times Row 1 from Row 3 to make the entries below the leading 1 in Row 1 equal to 0:

$$\left[\begin{array}{ccc|c} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 7.00333 & -0.29333 & -19.878 \\ 0 & -0.19000 & 10.0200 & 70.6150 \end{array}\right] r_2'=r_2-r_1x0.1 \text{ and } r_3'=r_3-r_1x0.3$$

- **Step-04:** Divide Row 2 by 7.0033 to make the leading coefficient 1 in the second row:

$$\left[\begin{array}{ccc|c} 1 & -0.0333333 & -0.066667 & 2.61667 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & -0.19000 & 10.0200 & 70.6150 \end{array}\right] r_2'=r_2/7.00333$$

- **Step-05:** Reduction of x_2 terms from first and third equation we can use,

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 10.01200 & 70.0843 \end{array}\right] r_1'=r_1+r_2x0.0333 \text{ and } r_3'=r_3+r_2x0.1900$$

- **Step-06:** Divide Row 3 by 10.01200 to make the leading coefficient 1 in the third row:

$$\left[\begin{array}{ccc|c} 1 & 0 & -0.0680629 & 2.52356 \\ 0 & 1 & -0.0418848 & -2.79320 \\ 0 & 0 & 1 & 7 \end{array}\right] r_3'=r_3/10.01200$$

- **Step-07:** Finally, reducing x_3 terms from equation 1 and 2 we need,

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3.0 \\ 0 & 1 & 0 & -2.50 \\ 0 & 0 & 1 & 7.0 \end{array}\right]$$ $r_1'=r_1+r_3 x 0.0680629$ and $r_2'=r_2+r_3 x 0.0418848$

Now, we find the value of x_1, x_2 and x_3,

$x_1=3.0$

$x_2=-2.50$ and

$x_3=7.0$

❖ **Difference between Gauss-Jordan and Gauss-Elimination method.**

1. When an unknown is eliminated in the gauss-Jordan method, it is eliminated from all other equations but in Gauss-elimination, just the subsequent ones are eliminated.
2. In Gauss-Jordan method, the elimination steps result in an identity matrix whereas, in Gauss-elimination, the elimination steps result in a triangular matrix.
3. It is not necessary to employ back substitution to obtain the solution of inn Gauss-Jordan method, but in Gauss-elimination, the back substitution is necessary for the solution.

✪ Triangularization/Factorization/Cholesky's Method.

➢ **Example-1: Apply the Cholesky's process to locate the root of the following system:**

$x_1+x_2-x_3=2$

$2x_1+3x_2+5x_3=-3$

$3x_1+2x_2-3x_3=6$

○ **Sol^n:** Rules or technique→ AX=B => LUX=B => LY=B then UX=Y

$$A=\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix}, X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B=\begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

A matrix can be written as the product of a lower triangular matrix and an upper triangular matrix

$$A=\begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & 5 \\ 3 & 2 & -3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12}+u_{22} & l_{21}u_{13}+u_{23} \\ l_{31}u_{11} & l_{31}u_{12}+l_{32}u_{22} & l_{31}u_{13}+l_{32}u_{23}+u_{33} \end{bmatrix}$$

Comparing the values,

u_{11}=1, u_{12}=1, u_{13}=-1, $l_{21}u_{11}$=2, so that l_{21}=2

Similarly, we can find out another unknown variable,

AX=B => LUX=B

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & -1\\0 & 1 & 7\\0 & 0 & 7\end{bmatrix}X=\begin{bmatrix}2\\-3\\6\end{bmatrix}$$

Let, UX=Y => LY=B

$$\begin{bmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & -1 & 1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}=\begin{bmatrix}2\\-3\\6\end{bmatrix}$$

From above matrix we find,

- ✓ y_1=2
- ✓ $2y_1$+ y_2=-3
  => y_2=-7
- ✓ $3y_1$- y_2+y_3=6=> y_3=-7

Again, UX=Y

$$\begin{bmatrix}1 & 1 & -1\\0 & 1 & 7\\0 & 0 & 7\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\-7\\-7\end{bmatrix}$$

From above matrix we find,

- ✓ $7x_3$=-7
  x_3=-1
- ✓ x_2+ $7x_3$=-7
  => x_2=0
- ✓ x_1+ x_2-Y_3=2
  => x_1=1

5th lecture (13/09/2023-Online) For clear understanding, please complete lecture 6 from below.

- Polynomial regression**:** The least-squares procedure can be readily extended to fit the data to a higher-order polynomial. For example, suppose that we fit a second-order polynomial or quadratic:

  $y = a_0 + a_1x + a_2x^2 + e$

  For this case the sum of the squares of the residuals is [compare with Eq. (17.3)]

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1x_i)^2 \qquad (17.3)$$

$$S_r = \sum_{i=1}^{n} (y_i - a_0 - a_1x_i - a_2x_i^2)^2 \qquad (17.18)$$

Following the procedure of the previous section, we take the derivative of Eq. (17.18) with respect to each of the unknown coefficients of the polynomial, as in:

$$\frac{\partial S_r}{\partial a_0} = -2\sum \left(y_i - a_0 - a_1x_i - a_2x_i^2\right)$$

$$\frac{\partial S_r}{\partial a_1} = -2\sum x_i \left(y_i - a_0 - a_1x_i - a_2x_i^2\right)$$

$$\frac{\partial S_r}{\partial a_2} = -2\sum x_i^2 \left(y_i - a_0 - a_1x_i - a_2x_i^2\right)$$

- **Criteria for best fit:** These equations can be set equal to zero and rearranged to develop the following set of normal equations:

$$\begin{aligned}(n)a_0 + \left(\sum x_i\right)a_1 + \left(\sum x_i^2\right)a_2 &= \sum y_i \\ \left(\sum x_i\right)a_0 + \left(\sum x_i^2\right)a_1 + \left(\sum x_i^3\right)a_2 &= \sum x_iy_i \\ \left(\sum x_i^2\right)a_0 + \left(\sum x_i^3\right)a_1 + \left(\sum x_i^4\right)a_2 &= \sum x_i^2y_i\end{aligned} \qquad (17.19)$$

- **Standard error is formulated as:**

$$s_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}} \qquad (17.20)$$

❖ **Example-17.5:** Fit a second-order polynomial to the data in the first two columns of Table 17.4

TABLE 17.4 Computations for an error analysis of the quadratic least-squares fit.

x_i	y_i	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1x_i - a_2x_i^2)^2$
0	2.1	544.44	0.14332
1	7.7	314.47	1.00286
2	13.6	140.03	1.08158
3	27.2	3.12	0.80491
4	40.9	239.22	0.61951
5	61.1	1272.11	0.09439
Σ	152.6	2513.39	3.74657

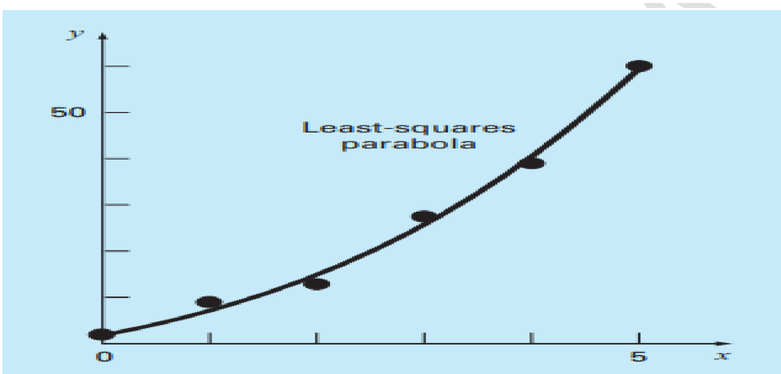
Solution. From the given data,

$m = 2 \quad \sum x_i = 15 \quad \sum x_i^4 = 979$

$n = 6 \quad \sum y_i = 152.6 \quad \sum x_i y_i = 585.6$

$\bar{x} = 2.5 \quad \sum x_i^2 = 55 \quad \sum x_i^2 y_i = 2488.8$

$\bar{y} = 25.433 \quad \sum x_i^3 = 225$



Therefore, the simultaneous linear equations are

$$\begin{bmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 152.6 \\ 585.6 \\ 2488.8 \end{Bmatrix}$$

Solving these equations through a technique such as Gauss elimination gives $a_0 = 2.47857$, $a_1 = 2.35929$, and $a_2 = 1.86071$. Therefore, the least-squares quadratic equation for this case is

$$y = 2.47857 + 2.35929x + 1.86071x^2$$

The standard error of the estimate based on the regression polynomial is [Eq. (17.20)]

$$s_{y/x} = \sqrt{\frac{3.74657}{6-3}} = 1.12$$

The coefficient of determination is

$$r^2 = \frac{2513.39 - 3.74657}{2513.39} = 0.99851$$

and the correlation coefficient is $r = 0.99925$.

These results indicate that 99.851 percent of the original uncertainty has been explained by the model. This result supports the conclusion that the quadratic equation represents an excellent fit, as is also evident from Fig. 17.11.

6th lecture (13/09/2023-Online)

- **Linear regression.**
- **Example-17.1.**
- **Re-cap polynomial regression.**

❖ Linear regression: The simplest example of a least-squares approximation is fitting a straight line to a set of paired observations: (x1, y1), (x2, y2), . . . , (xn, yn).

The mathematical expression for the straight line is:

$y = a_0 + a_1x + e$---17.1

where a_0 and a_1 are coefficients representing the intercept and the slope, respectively, and e is the error, or residual, between the model and the observations, which can be represented by rearranging Eq. (17.1) as

$e = y - a_0 - a_1x$

➢ **Criteria for a "best" fit:** One strategy for fitting a "best" line through the data would be to minimize the sum of the residual errors for all the available data, as in

$$\sum_{i=1}^{n} e_i = \sum_{i=1}^{n} (y_i - a_0 - a_1x_i) \qquad (17.2)$$

✓ where n = total number of points

Therefore, another logical criterion might be to minimize the sum of the absolute values of the discrepancies, as in

$$\sum_{i=1}^{n} |e_i| = \sum_{i=1}^{n} |y_i - a_0 - a_1x_i|$$

A strategy that overcomes the shortcomings of the aforementioned approaches is to minimize the sum of the squares of the residuals between the measured y and the y calculated with the linear model

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,\text{measured}} - y_{i,\text{model}})^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1x_i)^2 \qquad (17.3)$$

- **least-square fit of a straight line:** To determine values for a_0 and a_1, Eq. (17.3) is differentiated with respect to each coefficient:

$$\frac{\partial S_r}{\partial a_0} = -2 \sum (y_i - a_0 - a_1 x_i)$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum [(y_i - a_0 - a_1 x_i) x_i]$$

Note that we have simplified the summation symbols; unless otherwise indicated, all summations are from i = 1 to n. Setting these derivatives equal to zero will result in a minimum Sr. If this is done, the equations can be expressed as

$$0 = \sum y_i - \sum a_0 - \sum a_1 x_i$$

$$0 = \sum y_i x_i - \sum a_0 x_i - \sum a_1 x_i^2$$

Now, realizing that $\sum a_0 = na_0$, we can express the equations as a set of two simultaneous linear equations with two unknowns (a_0 and a_1):

$$na_0 + \left(\sum x_i\right) a_1 = \sum y_i \tag{17.4}$$

$$\left(\sum x_i\right) a_0 + \left(\sum x_i^2\right) a_1 = \sum x_i y_i \tag{17.5}$$

These are called the *normal equations*. They can be solved simultaneously

$$a_1 = \frac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2} \tag{17.6}$$

This result can then be used in conjunction with Eq. (17.4) to solve for

$$a_0 = \bar{y} - a_1 \bar{x} \tag{17.7}$$

where $\bar{y}$ and $\bar{x}$ are the means of y and x, respectively.

- **Example-17.1:** Fit a straight line to the x and y values in the first two columns of Table 17.1.

TABLE 17.1 Computations for an error analysis of the linear fit.

x_i	y_i	$(y_i - \bar{y})^2$	$(y_i - a_0 - a_1 x_i)^2$
1	0.5	8.5765	0.1687
2	2.5	0.8622	0.5625
3	2.0	2.0408	0.3473
4	4.0	0.3265	0.3265
5	3.5	0.0051	0.5896
6	6.0	6.6122	0.7972
7	5.5	4.2908	0.1993
Σ	24.0	22.7143	2.9911

Solution. The following quantities can be computed:

$$n = 7 \qquad \sum x_i y_i = 119.5 \qquad \sum x_i^2 = 140$$

$$\sum x_i = 28 \qquad \bar{x} = \frac{28}{7} = 4$$

$$\sum y_i = 24 \qquad \bar{y} = \frac{24}{7} = 3.428571$$

Using Eqs. (17.6) and (17.7),

$$a_1 = \frac{7(119.5) - 28(24)}{7(140) - (28)^2} = 0.8392857$$

$$a_0 = 3.428571 - 0.8392857(4) = 0.07142857$$

Therefore, the least-squares fit is

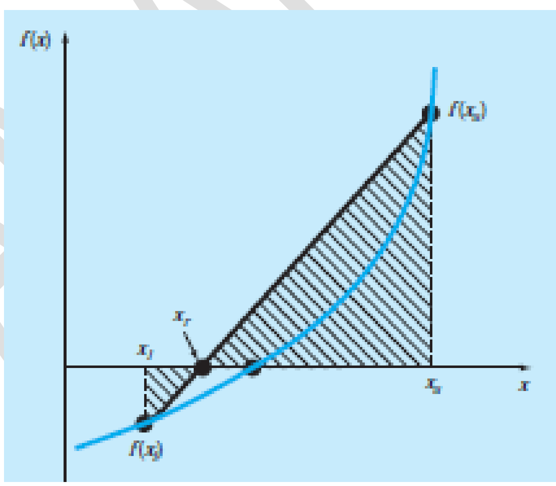
$$y = 0.07142857 + 0.8392857x$$

7th lecture (19/09/2023-Online). (Midterm)

✪ Regular-Falsi/False-position method:

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

❖ Proof False-position method:



Using similar triangles (Fig. 5.12), the intersection of the straight line with the x axis can be estimated as

$$\frac{f(x_l)}{x_r - x_l} = \frac{f(x_u)}{x_r - x_u} \tag{5.6}$$

Cross-multiply Eq. (5.6) to yield

$$f(x_l)(x_r - x_u) = f(x_u)(x_r - x_l)$$

Collect terms and rearrange:

$$x_r\,[\,f(x_l) - f(x_u)] = x_u\,f(x_l) - x_l\,f(x_u)$$

Divide by $f(x_l) - f(x_u)$:

$$x_r = \frac{x_u\,f(x_l) - x_l\,f(x_u)}{f(x_l) - f(x_u)} \tag{B5.1.1}$$

This is one form of the method of false position. Note that it allows the computation of the root x_r as a function of the lower and upper guesses x_l and x_u. It can be put in an alternative form by expanding it:

$$x_r = \frac{x_u\,f(x_l)}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

then adding and subtracting x_u on the right-hand side:

$$x_r = x_u + \frac{x_u\,f(x_l)}{f(x_l) - f(x_u)} - x_u - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

Collecting terms yields

$$x_r = x_u + \frac{x_u\,f(x_u)}{f(x_l) - f(x_u)} - \frac{x_l\,f(x_u)}{f(x_l) - f(x_u)}$$

or

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$

which is the same as Eq. (5.7). We use this form because it involves one less function evaluation and one less multiplication than Eq. (B5.1.1). In addition, it is directly comparable with the secant method which will be discussed in Chap. 6.

❖ **Example-5.5:** Use the false-position method to determine the root of the same equation investigated in Example 5.1 [Eq. (E5.1.1)].

Problem Statement. Use the graphical approach to determine the drag coefficient c needed for a parachutist of mass $m = 68.1$ kg to have a velocity of 40 m/s after free-falling for time $t = 10$ s. *Note:* The acceleration due to gravity is 9.8 m/s^2.

Solution. This problem can be solved by determining the root of Eq. (PT2.4) using the parameters $t = 10$, $g = 9.8$, $v = 40$, and $m = 68.1$:

$$f(c) = \frac{9.8(68.1)}{c}\left(1 - e^{-(c/68.1)10}\right) - 40$$

or

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40 \tag{E5.1.1}$$

Solution. As in Example 5.3, initiate the computation with guesses of $x_l = 12$ and $x_u = 16$.

First iteration:

$$x_l = 12 \qquad f(x_l) = 6.0699$$

$$x_u = 16 \qquad f(x_u) = -2.2688$$

$$x_r = 16 - \frac{-2.2688(12 - 16)}{6.0669 - (-2.2688)} = 14.9113$$

which has a true relative error of 0.89 percent.

Second iteration:

$$f(x_l)\,f(x_r) = -1.5426$$

Therefore, the root lies in the first subinterval, and x_r becomes the upper limit for the next iteration, $x_u = 14.9113$:

$$x_l = 12 \qquad f(x_l) = 6.0699$$

$$x_u = 14.9113 \qquad f(x_u) = -0.2543$$

$$x_r = 14.9113 - \frac{-0.2543(12 - 14.9113)}{6.0669 - (-0.2543)} = 14.7942$$

which has true and approximate relative errors of 0.09 and 0.79 percent. Additional iterations can be performed to refine the estimate of the roots.

✪ Where Bisection Is Preferable to False Position explain with appropriate example (5.6).

Problem Statement. Use bisection and false position to locate the root of

$$f(x) = x^{10} - 1$$

between $x = 0$ and 1.3.

Solution. Using bisection, the results can be summarized as

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

Thus, after five iterations, the true error is reduced to less than 2 percent. For false position, a very different outcome is obtained:

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.09430		90.6
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2

❖ Example-5.4: Solve again using False-position method and compare the result with the bisection method.

$$f(c) = \frac{667.38}{c}\left(1 - e^{-0.146843c}\right) - 40 \qquad \text{(E5.1.1)}$$

Iteration	x_l	x_u	$f(x_l)$	$f(x_u)$	x_r	$f(x_l).f(x_r)$	ε_a(%)	Update
1	12	16	6.0699	-2.2688	14.9113	-1.5426	100%	$x_u = x_r$
2	12	14.9113	6.0699	-0.2543	14.7942	-0.1638	0.78	$x_u = x_r$
3	12	14.794	6.0699	-0.027	14.7816	-0.0164	0.08	No need

- **Implement the following method using python:**
  1. **Gauss-elimination method and.**
  2. **Gauss-Jordan method.**

Already implemented, check python file.

8th lecture (28/09/2023-Online). (Midterm)

- **Trapezoidal rules error.**
- **How to minimize Trapezoidal rules error using Simpson's 1/3 formula.**

Check 15th lecture.

- Lagrange Interpolation. **(Midterm)**

  Let y = f(x) be a function such that f(x) takes the values y_0 , y_1 , y_2 ,......., y_n corresponding to x= x_0 , x_1, x_2 ..., x_n That is $y_i = f(x_i)$, i = 0,1,2,...,n .Now, there are (n + 1) paired values (x_i, y_i),i = 0, 1, 2, ..., n and hence f(x) can be represented by a polynomial function of degree n in x.

  Then the Lagrange's formula is:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)}y_1$$

$$+\dots+\frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})}y_n$$

- **Example 5.22**

Using Lagrange's interpolation formula find y(10) from the following table:

x	5	6	9	11
y	12	13	14	16

Solution:

Here the intervals are unequal. By Lagrange's interpolation formula we have

$$x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$$

$$y_0 = 12, y_1 = 13, y_2 = 14, y_3 = 16$$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times y_3$$

$$= \frac{(x-6)(x-9)(x-11)}{(5-6)(5-6)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-9)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16)$$

Put $x = 10$

$$y(10) = f(10) = \frac{4(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13) + \frac{5(4)(-1)}{4(3)(-2)}(14) + \frac{(5)(4)(1)}{6(5)(2)}(16)$$

$$= \frac{1}{6}(12) - \frac{13}{3} + \frac{5(14)}{3 \times 2} + \frac{4 \times 16}{12}$$

$$= 14.6663$$

9th lecture (03/10/2023-Online). (Midterm)

✪ Milne's Simpson Predictor-Corrector Method.

Formula

Milne's simpson predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

putting $n = 3$, we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right)$$

Milne's simpson corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

putting $n = 3$, we get

$$y_{4,c} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)$$

- **Example-01: Given $\frac{dy}{dx}=\frac{1}{2}(x+y)$, y(0)=2, y(0.5)=2.636, y(1.0)=3.595, y(1.5)=4.968 Now, find y(2) by Milne's method.**

Solution:

$$y' = \frac{x+y}{2}$$

Milne's simpson predictor formula is

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\left(2y'_{n-2} - y'_{n-1} + 2y'_n\right)$$

putting $n = 3$, we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right) \rightarrow (2)$$

We have given that
$x_0 = 0, x_1 = 0.5, x_2 = 1, x_3 = 1.5$

and using runge kutta 4 method, we get
$y_0 = 2, y_1 = 2.636, y_2 = 3.595, y_3 = 4.968$

$$y' = \frac{x+y}{2}$$

$$y'_1 = \frac{x+y}{2} = 1.568 \text{ (where } x = 0.5, y = 2.636)$$

$$y'_2 = \frac{x+y}{2} = 2.2975 \text{ (where } x = 1, y = 3.595)$$

$$y'_3 = \frac{x+y}{2} = 3.234 \text{ (where } x = 1.5, y = 4.968)$$

putting the values in (2), we get

$$y_{4,p} = y_0 + \frac{4h}{3}\left(2y'_1 - y'_2 + 2y'_3\right) \rightarrow (2)$$

$$y_{4,p} = 2 + \frac{4 \cdot 0.5}{3} \cdot (2 \cdot 1.568 - 2.2975 + 2 \cdot 3.234)$$

$$y_{4,p} = 6.871$$

So, the predicted value is 6.871

Now, we will correct it by corrector method to get the final value

$$y'_4 = \frac{x+y}{2} = 4.4355 \text{ (where } x = 2, y = 6.871)$$

Milne's simpson corrector formula is

$$y_{n+1,c} = y_{n-1} + \frac{h}{3}\left(y'_{n-1} + 4y'_n + y'_{n+1}\right)$$

putting $n = 3$, we get

$$y_{4,c} = y_2 + \frac{h}{3}\left(y'_2 + 4y'_3 + y'_4\right)$$

$$y_{4,c} = 3.595 + \frac{0.5}{3} \cdot (2.2975 + 4 \cdot 3.234 + 4.4355)$$

$$y_4 = 6.8732$$

$\therefore y(2) = 6.8732$ (Answer)

10th lecture (05/10/2023)

✪ Picard's Formula.

➢ **Example-01:** Given dy/dx=1+xy and y(0)=1. Calculate y(0.1), y(0.2) using Picard's method.

Sol^n: To calculate y(0.1) and y(0.2) using Picard's method, we'll use the iterative formula:

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} (1 + x y_n(y))\, dx$$

Given $y_0(x) = 1$ and $x_0 = 0$, let's perform three iterations for both $y(0.1)$ and $y(0.2)$.

Iteration 1:

$$y_1(x) = 1 + \int_0^x (1 + x \cdot 1)\, dx$$
$$y_1(x) = 1 + \int_0^x (1 + x)\, dx$$
$$y_1(x) = 1 + \left[x + \frac{x^2}{2}\right]_0^x$$
$$y_1(x) = 1 + \left(x + \frac{x^2}{2}\right)$$

Using $x = 0.1$:

$$y_1(0.1) = 1 + 0.1 + \frac{0.1^2}{2} = 1.105$$

Using $x = 0.2$:

$$y_1(0.2) = 1 + 0.2 + \frac{0.2^2}{2} = 1.21$$

Iteration 2:

$y_2(x)=1+\int_0^x (1+x.(1+x+\frac{x^2}{2}))\,dx$

$y_2(x)=1+\int_0^x (1+x+x^2+\frac{x^3}{2}))\,dx$

$y_2(x)=1+\left[x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right]_0^x dx$

Using x=0.1:
$y_2(0.1)=1.105$

Using x=0.2:
$y_2(0.2)=1.22$

iteration 3:

$y_3(x)= 1+\int_0^x (1+x.(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}))\,dx$

$y_3(x)= 1+\left[x+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}\right]_0^x\,dx$

using x=0.1:
$y_3(0.1)=1.1003$

using x=0.2:
$y_3(0.2)=1.202$

11^{th} lecture (10/10/2023)(Online)

✪ Simple fixed-point iteration:

- **Example: Find a real root of the equation $2x^2$-4x+1=0 using iteration process.**

Sol^n: given equation $2x^2-4x+1=0$
$f(x)=2x^2-4x+1$
$f(0)=1$
$f(1)=-1$
If $f(x)=0$ then we find $x=g(x)$
$=>x = \frac{1}{2}x^2 + \frac{1}{4}$
The root is between 0 and 1
So, $x_0=\frac{0+1}{2}=0.5$
We know that, $x_{i+1}=\phi(x_i)$
Putting i=0, we get
$X_1=\phi(x_0)= \frac{1}{2}x_0^2 + \frac{1}{4}=\frac{1}{2}(0.5)^2 + \frac{1}{4}=0.375$
$X_2=\phi(x_1)= \frac{1}{2}x_1^2 + \frac{1}{4}=\frac{1}{2}(0.375)^2 + \frac{1}{4}=0.320$
$X_3=\phi(x_2)= \frac{1}{2}x_2^2 + \frac{1}{4}=\frac{1}{2}(0.320)^2 + \frac{1}{4}=0.3012$
$X_4=\phi(x_3)= \frac{1}{2}x_3^2 + \frac{1}{4}=\frac{1}{2}(0.302)^2 + \frac{1}{4}=0.295$
$X_5=\phi(x_4)= \frac{1}{2}x_4^2 + \frac{1}{4}=\frac{1}{2}(0.295)^2 + \frac{1}{4}=.0.293$
$X_6=\phi(x_5)= 0.2929$
$X_7=\phi(x_6)= 0.292895205$
$X_8=\phi(x_7)= 0.292865792$
$X_9=\phi(x_8)= 0.2928852236$
$X_{10}=\phi(x_9)= 0.2928908$
$X_{11}=\phi(x_{10})= 0.292892553$
$X_{12}=\phi(x_{11})= 0.2928932018$
$X_{13}=\phi(x_{12})= 0.29289316$
$X_{14}=\phi(x_{13})= 0.2928932016$
$X_{15}=\phi(x_{14})= 0.2928932138$

$x = \frac{1}{2}x^2 + \frac{1}{4}$
$x=g(x)$

- **In convergence:**

1st step - 2nd step> 2nd step - 3rd step.

n	x_n
1	0.375
2	0.320
3	0.3012
===================================	==========================
===============================	==========================
13	0.29289316
14	0.2928932016
15	0.2928932138

- **In divergence:**

3rd step-2nd step > 2nd step-1st step

n	x_n
1	2.0
2	2.25
3	2.78125
4	4.117676
5	8.727627
6	38.335736

➢ Example-6.1: Use simple fixed-point iteration to locate the root of $f(x) = e^{-x} - x$.

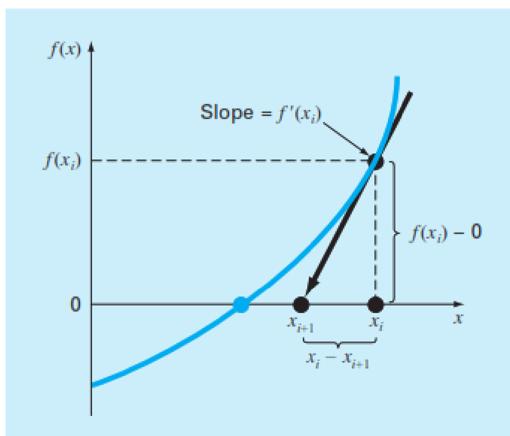
Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute

i	x_i	ε_a (%)	ε_t (%)
0	0		100.0
1	1.000000	100.0	76.3
2	0.367879	171.8	35.1
3	0.692201	46.9	22.1
4	0.500473	38.3	11.8
5	0.606244	17.4	6.89
6	0.545396	11.2	3.83
7	0.579612	5.90	2.20
8	0.560115	3.48	1.24
9	0.571143	1.93	0.705
10	0.564879	1.11	0.399

Thus, each iteration brings the estimate closer to the true value of the root: 0.56714329.

12th lecture (12/10/2023)

✪ Newton-Raphson (NR) method:



If the initial guess at the root is x_i, a tangent can be extended from the point [x_i, $f(x_i)$]. The point where this tangent crosses the x-axis usually represents an improved estimate of the root.

The Newton-Raphson method can be derived on the basis of this geometrical interpretation (an alternative method based on the Taylor series is described in Box 6.2). As in Fig. 6.5, the first derivative at x is equivalent to the slope:

$$f'(x_i) = \frac{f(x_i) - 0}{x_i - x_{i+1}} \tag{6.5}$$

which can be rearranged to yield

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \tag{6.6}$$

which is called the *Newton-Raphson formula.*

- **Example-6.3: Use the Newton-Raphson method to estimate the root of f $(x)=e^{-x}-x$, employing an initial guess of $x_0=0$.**

Solution. The first derivative of the function can be evaluated as

$$f'(x) = -e^{-x} - 1$$

which can be substituted along with the original function into Eq. (6.6) to give

$$x_{i+1} = x_i - \frac{e^{-x_i} - x_i}{-e^{-x_i} - 1}$$

Starting with an initial guess of $x_0 = 0$, this iterative equation can be applied to compute

i	x_i	ε_t (%)
0	0	100
1	0.500000000	11.8
2	0.566311003	0.147
3	0.567143165	0.0000220
4	0.567143290	$< 10^{-8}$

Thus, the approach rapidly converges on the true root. Notice that the true percent relative error at each iteration decreases much faster than it does in simple fixed-point iteration (compare with Example 6.1).

Where NR Method is used?

We use this method for root finding with less steps.

Advantages of NR method.

- One of the fastest convergences to the root.
- Converges on the root quadratically:
  - Near a root, the number of significant digits approximately doubles with each step.
  - This leads to the ability of the Newton-Raphson Method to "polish" a root from another convergence technique
- Easy to convert to multiple dimensions
- Can be used to "polish" a root found by other methods

Why NR method is used?

For finding the root of given equations. As it converges on the root faster and is less steps. Besides, it requires only one guess.

❖ **Pitfall of NR method:** Example of a Slowly Converging Function with Newton-Raphson.

Problem Statement. Determine the positive root of $f(x) = x^{10} - 1$ using the Newton-Raphson method and an initial guess of $x = 0.5$.

Solution. The Newton-Raphson formula for this case is

$$x_{i+1} = x_i - \frac{x_i^{10} - 1}{10x_i^9}$$

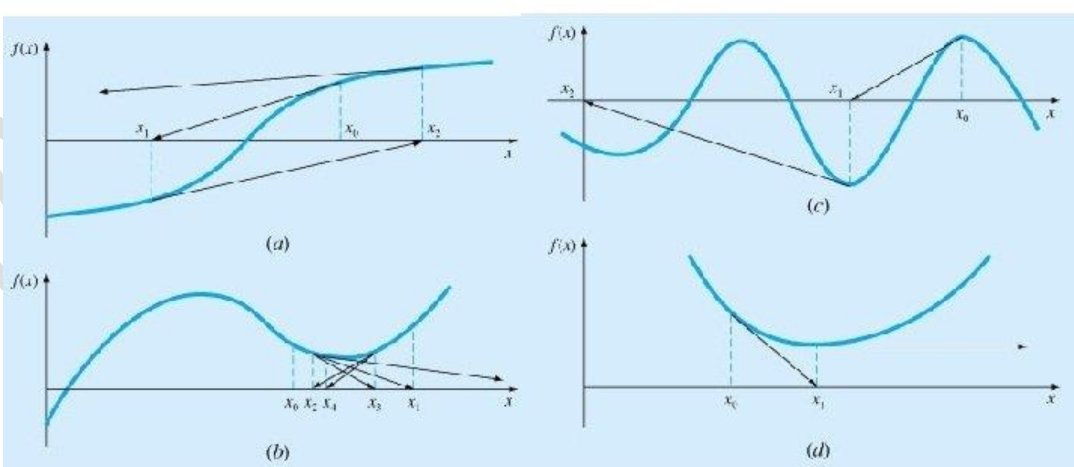
which can be used to compute

Iteration	x
0	0.5
1	51.65
2	46.485
3	41.8365
4	37.65285
5	33.887565
.	
.	
.	
∞	1.0000000

Thus, after the first poor prediction, the technique is converging on the true root of 1, but at a very slow rate.

❖ **Four cases where the NR method exhibits poor convergence.**

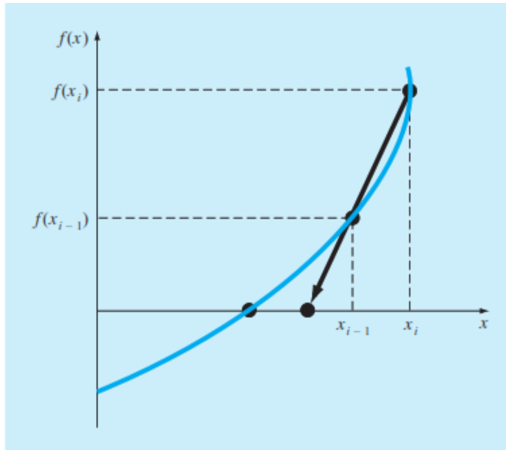
Newton-Raphson Method



Examples of poor convergence

13th lecture (18/10/2023)

✪ The secant method:



The derivative can be approximated by a backward finite divided difference, as in (above fig.)

$$f'(x_i) \cong \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$

- **Example-6.6:** Use the secant method to estimate the root of $f(x)=e^{-x}-x$. Start with initial estimates of $x_{-1}=0$ and $x_0=1.0$.

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \qquad (6.7)$$

Solution. Recall that the true root is 0.56714329. . . .

First iteration:

$$x_{-1} = 0 \qquad f(x_{-1}) = 1.00000$$
$$x_0 = 1 \qquad f(x_0) = -0.63212$$
$$x_1 = 1 - \frac{-0.63212(0-1)}{1-(-0.63212)} = 0.61270 \qquad \varepsilon_t = 8.0\%$$

Second iteration:

$$x_0 = 1 \qquad f(x_0) = -0.63212$$
$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$

(Note that both estimates are now on the same side of the root.)

$$x_2 = 0.61270 - \frac{-0.07081(1-0.61270)}{-0.63212-(-0.07081)} = 0.56384 \qquad \varepsilon_t = 0.58\%$$

Third iteration:

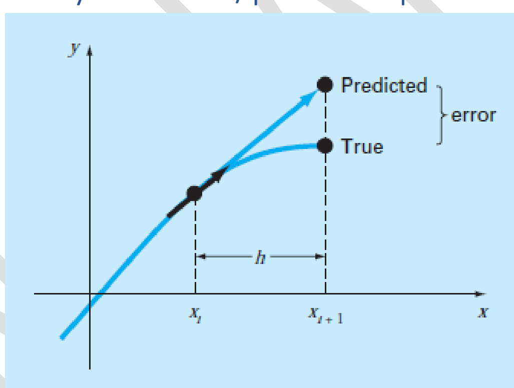
$$x_1 = 0.61270 \qquad f(x_1) = -0.07081$$
$$x_2 = 0.56384 \qquad f(x_2) = 0.00518$$
$$x_3 = 0.56384 - \frac{0.00518(0.61270-0.56384)}{-0.07081-(-0.00518)} = 0.56717 \qquad \varepsilon_t = 0.0048\%$$

❖ Difference between secant method and False-position method.

Aspect	Secant method	False-position method
Convergence rate	Faster	Slower
Root bracketing	Does not always guarantee the root stays bracketed	Guarantees the root stays bracketed.
Root stability	Can oscillate around the root.	Stable behavior, root remains bracketed.
Disadvantages	Can be sensitive to initial guesses, may not converge	Slower convergence, especially for functions with long intervals between roots

14th lecture (29/10/2023)

✪ Euler's method/Euler-Cauchy method/point slope method.



The first derivative provides a direct estimate of the slope at x_i (Above figure):

$\varphi = f(x_i , y_i)$

where $f(x_i, y_i)$ is the differential equation evaluated at x_i and y_i.

For the velocity of the falling parachutist. Recall that the method was of the general form

New value = old value + slope × step size

or, in mathematical terms,

$\boldsymbol{y_{i+1} = y_i + \varphi h}$

$\boldsymbol{y_{i+1} = y_i + f(x_i , y_i)h}$

This formula is referred to as *Euler's* (or the *Euler-Cauchy* or the *point-slope*) *method.*

❖ ***Example-25.1:*** Use Euler's method to numerically integrate the given equ^n.

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

From x=0 to x=4 with a step size of 0.5. The initial condition at x=0 in y=1. Recall that the exact equation is given by

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

Solution: equation $y_{i+1}=y_i+ f(x_i, y_i)h$ can be used to implement Euler's method

$$y(0.5) = y(0) + f(0, 1)0.5$$

where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0, 1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1.0 + 8.5(0.5) = 5.25$$

The true solution at $x = 0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1 = 3.21875$$

Thus, the error is

$$E_t = \text{true} - \text{approximate} = 3.21875 - 5.25 = -2.03125$$

or, expressed as percent relative error, $\varepsilon_t = -63.1\%$. For the second step,

$$\begin{aligned} y(1) &= y(0.5) + f(0.5, 5.25)0.5 \\ &= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]0.5 \\ &= 5.875 \end{aligned}$$

The true solution at $x = 1.0$ is 3.0, and therefore, the percent relative error is -95.8%. The computation is repeated, and the results are compiled in Table 25.1 and Fig. 25.3. Note that,

			Percent Relative Error	
x	y_{true}	y_{Euler}	Global	Local
0.0	1.00000	1.00000		
0.5	3.21875	5.25000	−63.1	−63.1
1.0	3.00000	5.87500	−95.8	−28.0
1.5	2.21875	5.12500	131.0	−1.41
2.0	2.00000	4.50000	−125.0	20.5
2.5	2.71875	4.75000	−74.7	17.3
3.0	4.00000	5.87500	46.9	4.0
3.5	4.71875	7.12500	−51.0	−11.3
4.0	3.00000	7.00000	−133.3	−53.0

Table 25.1: Comparison of true and approximate values of the integral.

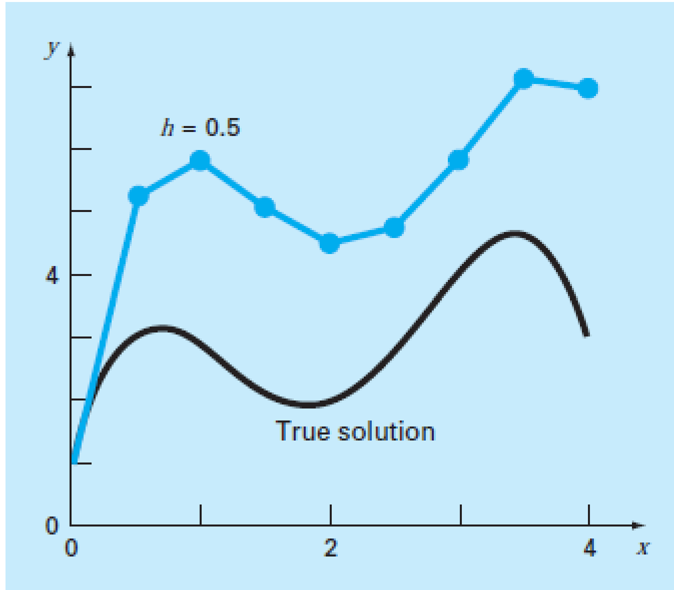


Fig: Comparison of the true solution with a numerical solution using Euler's method for the given integral.

15th lecture (02/11/2023)

❖ Why we use Trapezoidal rules?

- **Simplicity:** The Trapezoidal rule is straightforward and easy to understand, making it accessible for calculations and implementations.
- **Accuracy Improvement:** It offers better accuracy compared to simpler methods like the midpoint rule, especially for functions with varying slopes.
- **Wide Applicability:** The method is versatile and applicable to a wide range of functions, making it a useful tool for various fields such as physics, engineering, and computer science.
- **Ease of Implementation:** Its computational simplicity allows for quick implementation in numerical algorithms and computer programs.

❖ How to Minimize the error of Trapezoidal Rule by using Simpson's 1/3 rule?

To reduce the error of the Trapezoidal Rule, We can use Simpson's 1/3 Rule, which provides a more accurate approximation of the integral. Here's how we can do it:

1. **Divide the interval:** Divide the interval of integration into smaller subintervals. For Simpson's 1/3 Rule, we need an even number of intervals.
2. **Apply Simpson's 1/3 Rule:** Apply Simpson's 1/3 Rule to each pair of adjacent subintervals. For two adjacent subintervals [a, b] and [b, c] the approximation of the integral using Simpson's 1/3 Rule is:

   Integral approximation=$\frac{h}{3}[f(a)+4f(b)+f(c)]$

   Where h is the width of each subinterval (h=$\frac{b-a}{2}$)
3. **Sum up the results:** Sum up the results from step 2 for all the pairs of subintervals to get the final approximation of the interval.

✪ Simpson's 1/3 Rule:

Simpson's 1/3 rule results when a second-order interpolating polynomial is substituted into Eq. (21.1):

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_2(x)\,dx$$

If a and b are designated as x_0 and x_2 and $f_2(x)$ is represented by a second-order Lagrange polynomial [Eq. (18.23)], the integral becomes

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

After integration and algebraic manipulation, the following formula results:

$$I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \tag{21.14}$$

Where, for this case h=(b-a)/2

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}_{\text{Average height}} \tag{21.15}$$

❖ Example-21.4:

Single Application of Simpson's 1/3 Rule

Problem Statement. Use Eq. (21.15) to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Recall that the exact integral is 1.640533.

Solution.

$$f(0) = 0.2 \qquad f(0.4) = 2.456 \qquad f(0.8) = 0.232$$

Therefore, Eq. (21.15) can be used to compute

$$I \cong 0.8\frac{0.2 + 4(2.456) + 0.232}{6} = 1.367467$$

which represents an exact error of

$$E_t = 1.640533 - 1.367467 = 0.2730667 \qquad \varepsilon_t = 16.6\%$$

which is approximately 5 times more accurate than for a single application of the trapezoidal rule (Example 21.1).

The estimated error is [Eq . (21.16)]

$$E_a = -\frac{(0.8)^5}{2880}(-2400) = 0.2730667$$

where −2400 is the average fourth derivative for the interval as obtained using Eq. (PT6.4). As was the case in Example 21.1, the error is approximate (E_a) because the average fourth

✪ Simpson's 3/8 Rule:

In a similar manner to the derivation of the trapezoidal and Simpson's 1/3 rule, a third-order Lagrange polynomial can be fit to four points and integrated:

$$I = \int_a^b f(x)\,dx \cong \int_a^b f_3(x)\,dx$$

to yield

$$I \cong \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

Where, h=(b-a)/3. Hence,

$$I \cong \underbrace{(b-a)}_{\text{Width}} \underbrace{\frac{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)}{8}}_{\text{Average height}} \qquad (21.20)$$

Thus, the two interior points are given weights of three-eighths, whereas the end points are weighted with one-eighth. Simpson's 3/8 rule has an error of

$$E_t = -\frac{3}{80}h^5 f^{(4)}(\xi)$$

or, because $h = (b-a)/3$,

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi) \qquad (21.21)$$

❖ Example-21.6:

Problem Statement.

(a) Use Simpson's 3/8 rule to integrate

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$.

(b) Use it in conjunction with Simpson's 1/3 rule to integrate the same function for five segments.

Solution.

(a) A single application of Simpson's 3/8 rule requires four equally spaced points:

$$f(0) = 0.2 \qquad f(0.2667) = 1.432724$$
$$f(0.5333) = 3.487177 \qquad f(0.8) = 0.232$$

Using Eq. (21.20),

$$I \cong 0.8\,\frac{0.2 + 3(1.432724 + 3.487177) + 0.232}{8} = 1.519170$$

$$E_t = 1.640533 - 1.519170 = 0.1213630 \qquad \varepsilon_t = 7.4\%$$

$$E_a = -\frac{(0.8)^5}{6480}(-2400) = 0.1213630$$

(b) The data needed for a five-segment application ($h = 0.16$) is

$f(0) = 0.2$ $\quad$ $f(0.16) = 1.296919$

$f(0.32) = 1.743393$ $\quad$ $f(0.48) = 3.186015$

$f(0.64) = 3.181929$ $\quad$ $f(0.80) = 0.232$

The integral for the first two segments is obtained using Simpson's 1/3 rule:

$$I \cong 0.32\,\frac{0.2 + 4(1.296919) + 1.743393}{6} = 0.3803237$$

For the last three segments, the 3/8 rule can be used to obtain

$$I \cong 0.48\,\frac{1.743393 + 3(3.186015 + 3.181929) + 0.232}{8} = 1.264754$$

The total integral is computed by summing the two results:

$$I = 0.3803237 + 1.264753 = 1.645077$$

$$E_t = 1.640533 - 1.645077 = -0.00454383 \qquad \varepsilon_t = -0.28\%$$

16th lecture (07/11/2023) (Offline)

✪ **Discuss about syllabus and Previous Lecture.**

17th lecture (15/11/2023) (Online)

✪ Heun's Method.

➢ **Example-25.5:**

Problem Statement. Use Heun's method to integrate $y' = 4e^{0.8x} - 0.5y$ from $x = 0$ to $x = 4$ with a step size of 1. The initial condition at $x = 0$ is $y = 2$.

Mean=$\frac{\int_a^b f(x)dx}{b-a}$--(PT6.4)

$y_{i+1}^0 = y_i + f(x_i, y_i)h$--(25.15)

Solution. Before solving the problem numerically, we can use calculus to determine the following analytical solution:

$$y = \frac{4}{1.3}(e^{0.8x} - e^{-0.5x}) + 2e^{-0.5x} \qquad \text{(E25.5.1)}$$

This formula can be used to generate the true solution values in Table 25.2.

First, the slope at (x_0, y_0) is calculated as

$$y_0' = 4e^0 - 0.5(2) = 3$$

This result is quite different from the actual average slope for the interval from 0 to 1.0, which is equal to 4.1946, as calculated from the differential equation using Eq. (PT6.4).

The numerical solution is obtained by using the predictor [Eq. (25.15)] to obtain an estimate of y at 1.0:

$$y_1^0 = 2 + 3(1) = 5$$

		Iterations of Heun's Method			
		1		15	
x	y_{true}	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)	y_{Heun}	$\lvert\varepsilon_t\rvert$ (%)
0	2.0000000	2.0000000	0.00	2.0000000	0.00
1	6.1946314	6.7010819	8.18	6.3608655	2.68
2	14.8439219	16.3197819	9.94	15.3022367	3.09
3	33.6771718	37.1992489	10.46	34.7432761	3.17
4	75.3389626	83.3377674	10.62	77.7350962	3.18

Note that this is the result that would be obtained by the standard Euler method. The true value in Table 25.2 shows that it corresponds to a percent relative error of 19.3 percent.

Now, to improve the estimate for y_{i+1}, we use the value y_1^0 to predict the slope at the end of the interval

$$y_1' = f(x_1, y_1^0) = 4e^{0.8(1)} - 0.5(5) = 6.402164$$

which can be combined with the initial slope to yield an average slope over the interval from $x = 0$ to 1

$$y' = \frac{3 + 6.402164}{2} = 4.701082$$

which is closer to the true average slope of 4.1946. This result can then be substituted into the corrector [Eq. (25.16)] to give the prediction at $x = 1$

$$y_1 = 2 + 4.701082(1) = 6.701082$$

which represents a percent relative error of -8.18 percent. Thus, the Heun method without iteration of the corrector reduces the absolute value of the error by a factor of 2.4 as compared with Euler's method.

Now this estimate can be used to refine or correct the prediction of y_1 by substituting the new result back into the right-hand side of Eq. (25.16):

$$y_1 = 2 + \frac{[3 + 4e^{0.8(1)} - 0.5(6.701082)]}{2}1 = 6.275811$$

which represents an absolute percent relative error of 1.31 percent. This result, in turn, can be substituted back into Eq. (25.16) to further correct:

$$y_1 = 2 + \frac{[3 + 4e^{0.8(1)} - 0.5(6.275811)]}{2}1 = 6.382129$$

which represents an $|\varepsilon_t|$ of 3.03%. Notice how the errors sometimes grow as the iterations proceed. Such increases can occur, especially for large step sizes, and they prevent us from drawing the general conclusion that an additional iteration will always improve the result. However, for a sufficiently small step size, the iterations should eventually converge on a single value. For our case, 6.360865, which represents a relative error of 2.68 percent, is attained after 15 iterations. Table 25.2 shows results for the remainder of the computation using the method with 1 and 15 iterations per step.

18^{th} lecture (05/12/2023) (Offline)

- **Trapezoidal rules derivations:** Let f(x) be a continuous function on the interval [a, b]. Now divide the intervals [a, b] into n equal subintervals with each of width,
  $\Delta x = (b-a)/n$, Such that $a = x0 < x1 < x2 < x3 < \ldots < xn = b$
  Then the Trapezoidal Rule formula for area approximating the definite integral $\int_a^b f(x)dx$ is given by:
  $$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots .2f(x_{n-1}) + f(x_n)]$$
  Where, $xi = a+i\Delta x$
  If $n \to \infty$, R.H.S of the expression approaches the definite integral ∫ab f(x)dx.
- **Weddle's rule derivations:** putting n=6 in equation of Newton-cotes formula $I=\int_a^b f(x)dx \cong \int_a^b f_6(x)dx$ & neglecting the differences of order higher than six, we obtain Weddle's rule.
  $\int_{x_0}^{x_6} ydx = \frac{3h}{10}(y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6)$

Sir announced the official end of the course.

Solve minimum three years previous year questions.

5 set question from here.