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CHAPTER – 7

RESIDUES AND THE RESIDUE THEOREMS

227. Residue : D. U. H. '90; D. U. H. T. '83, '89.

If the function $f(z)$ has an isolated singularity at the point $z = a$, then the coefficient a_{-1} of $\frac{1}{z-a}$ in the Laurent's expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-a)^n}$$

$$= a_0 + a_1 (z-a) + a_2 (z-a)^2 + \dots \dots$$

$$+ \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-3}}{(z-a)^3} + \dots \dots$$

around $z = a$ is called the residue of $f(z)$ at $z = a$ and it is denoted by Res $[f(z), a]$ or Res (a) or a_{-1}.

N. B. In this book, by $[a_p]_{-1}$ we will mean the residue at the finite point $z = a_p$.

Theorem -200 : If $f(z)$ is analytic inside and on a simple closed curve C except at the point $z = a$ inside C, then

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz.$$

Proof : The Laurent series about $z = a$ is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \dots (1) \text{ where } a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz.$$

$n = 0, \pm 1, \pm 2, \dots (2)$

Now putting $n = -1$ in (2), we get

$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$ and the theorem is proved.

Theorem -201 : If $f(z)$ has a simple pole at $z = a$, then show that

$$a_{-1} = \lim_{z \to a} (z-a) f(z), \text{ where } a_{-1} \text{ is the residue of } f(z) \text{ at}$$

the point $z = a$.

Proof : If $f(z)$ has a simple pole at $z = a$, then the corresponding Laurent series of $f(z)$ is

$$f(z) = \frac{a_{-1}}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n \dots (1) \text{ where } a_{-1} \neq 0. \text{ Now}$$

multiplying (1) by $z - a$, then we have

$$(z-a) f(z) = a_{-1} + (z-a) \sum_{n=0}^{\infty} a_n (z-a)^n \dots (2).$$

Now on letting $z = a$ in (2), we have $a_{-1} = \lim_{z \to a} (z-a) f(z)$

and the theorem is proved.

Theorem -202 : The residue at a finite point $z = a$ is given by the equation Res $[f(z), a] = a_{-1}$ where $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n \dots (1)$ is a Laurent expansion of $f(z)$ at $z = a$.

Proof : We have Res $[f(z), a] = \frac{1}{2\pi i} \oint_C f(z)\, dz \dots (2)$

where a is finite. Let C be a circle $|z - a| = r$ in the region $\Re$ enclosing no singuarity other than a. Than the Laurent series converges uniformly on C' and (1) can be integrate term by term. But we know.

$$\oint_C \frac{1}{(z-a)^n}\, dz = \int_0^{2\pi} \frac{r\, i\, e^{i\theta}}{r^n e^{in\theta}}\, d\theta = \begin{cases} 2\pi i \text{ if } n = 1 \\ 0 \quad \text{if } n \neq 1 \end{cases} \quad \dots (3)$$

Now by (1), (2) and (3), we have Res $[f(z), a] = a_{-1}$ and the theorem is proved.

Theorem – 203 : If $f(z) = \dfrac{P(z)}{Q(z)}$, where P(z) and Q(z) are analytic at z = a but $P(a) \neq 0$, then $a_{-1} = \dfrac{P(a)}{Q'(a)}$ if Q(z) has a simple zero at z = a.

Proof : Since Q(z) has a simple zero at z = a, then $f(z) = \dfrac{P(z)}{Q(z)}$ has a simple pole at z = a where Q(a) = 0 and $P(a) \neq 0$. Now the residue of f(z) at z = a is $a_{-1} = \lim_{z \to a} (z-a) f(z)$

$$= \lim_{z \to a} (z-a) \frac{P(z)}{Q(z)}$$

$$= \lim_{z \to a} \frac{P(z)}{\dfrac{Q(z) - Q(a)}{z-a}} = \frac{P(a)}{Q'(a)} \text{ and the theorem is proved.}$$

Theorem – 204 : If f(z) has a double pole at z = a, then show that

$$a_{-1} = \lim_{z \to a} \frac{1}{1!} \frac{d}{dz} \{(z-a)^2 f(z)\}.$$

Proof : If f(z) has a double pole at z = a, then the corresponding Laurent series of f(z) is

$$f(z) = \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{z-a} + \sum_{n=0}^{\infty} a_n (z-a)^n \dots (1)$$

Now multiplying (1) by $(z-a)^2$, then we get

$$(z-a)^2 f(z) = a_{-2} + a_{-1}(z-a) + \sum_{n=0}^{\infty} a_n (z-a)^{n+2} \dots (2)$$

Now differentiating (2) with respect to z we get

$$\frac{d}{dz}\{(z-a)^2 f(z)\} = a_{-1} + \sum_{n=0}^{\infty} (n+2)\, a_n (z-a)^{n+1} \dots (3).$$

Now on letting $z \to a$ in (3) we get

$$a_{-1} = \lim_{z \to a} \frac{d}{dz} \{(z-a)^2 f(z)\} \text{ and the theorem is proved.}$$

Theorem – 205 : If $f(z) = \dfrac{P(z)}{Q(z)}$ where P(z) and Q(z) are analytic at z = a but $P(a) \neq 0$, then $a_{-1} = \dfrac{6P'(a)\, Q''(a) - 2P(a)\, Q'''(a)}{3\, [Q''(a)]^2}$.

where P(a), P'(a), Q''(a) and Q'''(a) are exist and Q(z) has a double zero at z = a.

Proof : Try yourself.

228. Residue at a multiple point :

Theorem – 206 : If f(z) is analytic inside and on a simple closed curve C except at pole z = a of order m inside C, then the residue of f(z) at z = a is

$$a_{-1} = \lim_{z \to a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\}.$$

D. U. H. T. '84, '86; D. U. M. SC. P. T. '88; D. U. '79; D. U. H. '88, '90; J. U. H.

Proof : (Method 1) : Since $f(z)$ has a pole of order m at z = a, then $f(a) = \frac{F(z)}{(z-a)^m}$, where F(z) is analytic inside and on C and also $F(a) \neq 0$. Then by the Cauchy's (m − 1)th differential integral formula, we have

$$\frac{1}{2\pi i}\oint_C f(z)\,dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{F(z)}{(z-a)^m}dz = \frac{1}{2\pi i}\frac{F^{(m-1)}(a)}{(m-1)!}$$

$$= \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} = a_{-1} \text{ and the}$$

answer line

theorem is proved.

(Method 2) : Since $f(z)$ has a pole at z = a of order m, then by the Laurent series we have

$$f(z) = \frac{a_{-m}}{(z-a)^m} + \frac{a_{-m+1}}{(z-a)^{m-1}} + \dots + \frac{a_{-1}}{z-a}$$

$$+ a_0 + a_1(z-a) + a_2(z-a)^2 + \dots\dots \quad (1)$$

Now multiplying (1) by $(z-a)^m \Rightarrow$

$$(z-a)^m f(z) = a_{-m} + a_{-m+1}(z-a) + \dots + a_{-1}(z-a)^{m-1}$$

$$+ a_0(z-a)^m + a_1(z-a)^{m+1} + \dots \quad (2)$$

Now differentiating both sides of (2) m − 1 times with respect to z, then (2) $\Rightarrow$

$$\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} = (m-1)!\,a_{-1} + \frac{m!}{1!}a_0(z-a) +$$

$$\frac{(m+1)!}{2!}a_1(z-a)^2 + \dots \quad (3)$$

Now letting $z \to a$ in (3)

$$\Rightarrow \lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\} = (m-1)!\,a_{-1} \Rightarrow$$

$$a_{-1} = \lim_{z\to a}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left\{(z-a)^m f(z)\right\} \text{ and the theorem}$$

is proved.

Theorem − 207 : If $f(z)$ is analytic at z = a, then the residue of $\frac{f(z)}{(z-a)^{n+1}}$ at z = a is $\frac{f^{(n)}(a)}{n!}$, where n is a positive integer.

Proof : We have $f(z) = \sum_{n=0}^{\infty}\frac{(z-a)^n}{n!}f^{(n)}(a)$

$$= f(a) + \frac{z-a}{1!}f'(a) + \dots + \frac{(z-a)^n}{n!}f^{(n)}(a) + \dots\dots$$

$$\Rightarrow \frac{f(z)}{(z-a)^{n+1}} = \frac{f(a)}{(z-a)^{n+1}} + \frac{f'(a)}{1!}\frac{1}{(z-a)^n} + \dots + \frac{f^{(n)}(a)}{n!}\frac{1}{z-a} + \dots$$

Here the coefficient of $\frac{1}{z-a}$ in $\frac{f(z)}{(z-a)^{n+1}}$ is $\frac{f^{(n)}(a)}{n!}$

$$= \text{Res}\left[\frac{f(z)}{(z-a)^{n+1}}, a\right]$$

and the theorem is proved.

Example − 249 : Evaluate the residues of $f(z) = \frac{z^2}{z^2+a^2}$ at the poles. → poles.

Solution : The poles of $f(z)$ are given by $z^2 + a^2 = 0$

$\Rightarrow z = \pm ai$ and they are simple poles.

Method 1 : Now Res (ai) $= \lim_{z\to ai}\left\{(z-ai)f(z)\right\}$

$$= \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{z^2+a^2}\right\} = \lim_{z\to ai}\left\{(z-ai)\frac{z^2}{(z-ai)(z+ai)}\right\}$$

$$= \lim_{z\to ai} \frac{z^2}{z+ai} = \frac{(ai)^2}{2ai} = \frac{1}{2}ai.$$

Similarly, Res $(-ai) = -\frac{1}{2}ai.$

Method 2 : Let $f(z) = \frac{P(z)}{Q(z)}$ where $P(z) = z^2$ and $Q(z) = z^2 + a^2$. Then $P(ai) = (ai)^2$, $Q'(z) = 2z$ and $Q'(ai) = 2ai$.

Now Res $(ai) = \frac{P(ai)}{Q'(ai)} = \frac{(ai)^2}{2ai} = \frac{1}{2}ai$. Similarly,

Res $(-ai) = \frac{P(-ai)}{Q'(-ai)} = \frac{(-ai)^2}{-2ai} = -\frac{1}{2}ai.$

Example 250 : Evalute the residues of $f(z) = \frac{z^4}{z^2+a^2}$ at the poles.

Solution : The poles of $f(z)$ are obtained by solving $z^2 + a^2 = 0 \Rightarrow z = \pm ai$ and they are simple poles. Now Res (ai)

$$= \lim_{z\to ai} \left\{(z-ai)\frac{z^4}{z^2+a^2}\right\} = \frac{(ai)^4}{2ai} = -i\frac{a^3}{2}$$

$\lim_{z\to ai} \left\{(z-ai)\frac{z^4}{(z-ai)(z+ai)}\right\} = \lim_{z\to ai} \frac{z^4}{z+ai}$

Similarly, Res $(-ai) = i\frac{a^3}{2}$

Method 2 : Let $f(z) = \frac{P(z)}{Q(z)}$. Then $P(z) = z^4$ and $Q(z) = z^2 + a^2$. Here $Q'(z) = 2z$. Now Res $(ai) = \frac{P(ai)}{Q'(ai)} = \frac{(ai)^4}{2(ai)} = -\frac{1}{2}ia^3$

and Res $(-ai) = \frac{P(-ai)}{Q'(-ai)} = \frac{(-ai)^4}{2(-ai)} = \frac{1}{2}ia^3.$

Example 251 : Find Res $[f(z), i]$ where $f(z) = \frac{e^{iz}}{(z^2+1)^4}$

Solution : Here $z = i$ is a pole of order 4. Now Res (i)

$$= \lim_{z\to i} \frac{1}{3!}\frac{d^3}{dz^3}\left[(z-i)^4 \frac{e^{iz}}{(z^2+1)^4}\right] = \lim_{z\to i} \frac{1}{6}\frac{d^3}{dz^3}\left[\frac{e^{iz}}{(z+i)^4}\right]$$

But $\frac{d^3}{dz^3}\left[\frac{e^{iz}}{(z+i)^4}\right] = i^3 e^{iz}(z+i)^{-4} + 3c_1\, i^2 e^{iz}(-4)(z+i)^{-5}$

$+ 3c_2\, i\, e^{iz}(-4)(-5)(z+i)^{-6} + e^{iz}(-4)(-5)(-6)(z+i)^{-7}$

Then $\lim_{z\to i} \frac{d^3}{dz^3}\left[\frac{e^{iz}}{(z+i)^4}\right] = \frac{e^{-1}}{(i+i)^4}(-i - 6i - 15i - 15i)$

$\Rightarrow$ Res $(i) = -\frac{37e^{-1}}{16}i.$

Example 252 : Find the residues of the function $f(z) = \frac{z^2-2z}{(z+1)^2(z^2+4)}$ D. U. H. T. '89; D. U. M. SC. P. '88.

Solution : The poles of $f(z)$ are obtained by solving $(z+1)^2(z^2+4) = 0 \Rightarrow f(z)$ has a double pole at $z = -1$ and simple poles at $z = \pm 2i$. Now the residue at $z = -1$ is

$$\lim_{z\to -1} \frac{1}{1!}\frac{d}{dz}\left\{(z+1)^2\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\} = \lim_{z\to -1}\frac{d}{dz}\left\{\frac{z^2-2z}{z^2+4}\right\}$$

$$= \lim_{z\to -1} \frac{z^2-2z}{z^2+4}\left[\frac{2z-2}{z^2-2z} - \frac{2z}{z^2+4}\right] = \frac{3}{5}\left(\frac{-4}{3} + \frac{2}{5}\right) = -\frac{14}{25}$$

Residue at $z = 2i$ is $\lim_{z\to 2i}\left\{(z-2i)\frac{z^2-2z}{(z+1)^2(z^2+4)}\right\}$

$$= \frac{(2i)^2 - 4i}{(2i+1)^2(4i)} = \frac{-4-4i}{(4i^2+4i+1)(4i)} = \frac{1+i}{4+3i} = \frac{7+i}{25}$$

Similarly, residue at $z = -2i$ is $\frac{7-i}{25}$.

Example 253 : Show that Res $\left[\frac{1}{z^2(z-1)}, 0\right] = -1.$

Solution : We have

$$\frac{1}{z^2(z-1)} = -\frac{1}{z^2}(1-z)^{-1} = -\frac{1}{z^2}(1 + z + z^2 + z^3 + \dots),$$

$0 < |z| < 1$

$$= -\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 - \dots\dots$$

Here the coefficient of $\frac{1}{z}$ is -1 and Res $\left[\frac{1}{z^2(z-1)}, 0\right] = -1$

Example – 254 : Show that Res $\left[\frac{1}{z^2(z^2-1)}, 0\right] = 0.$

Solution : Try yourself.

229. Another method of finding the residue at a multiple point : If $f(z)$ has a pole of order m, then the Laurent series of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \dots + \frac{a_{-m}}{(z-a)^m} \quad \dots (1)$$

Now putting $z = t + a$ in (1), where t small,

$$f(t+a) = \sum_{n=0}^{\infty} a_n t^n + \frac{a_{-1}}{t} + \frac{a_{-2}}{t^2} + \dots + \frac{a_{-m}}{t^m} \quad \dots (2)$$

In (1) a_{-1} is the coefficient of $\frac{1}{z-a}$ and in (2) it is the coefficient $\frac{1}{t}$ and it is called the residue of $f(z)$ at the point $z = a$.

Example – 255 : Find Res [$f(z)$, 1] where $f(z) = \frac{z^2}{(z-1)^3(z-2)} \dots (1).$

Solution : Here $z = 1$ is a pole of order 3 of the function $f(z)$. Now let $z = t + 1$ in (1), we get $f(t+1) = \frac{(t+1)^2}{t^3(t-1)}$

$$= -\frac{t^2+2t+1}{t^3}(1-t)^{-1}$$

$$= -\frac{(t^2+2t+1)}{t^3}(1+t+t^2+t^3+\dots)$$

$$= -\left(\frac{1}{t}+\frac{2}{t^2}+\frac{1}{t^3}\right)(1+t+t^2+t^3+\dots)$$

Here coefficient $\frac{1}{t} = -1-2-1 = -4 =$ Res $[f(z), 1]$.

Example – 256 : Find the residues at the poles of the following functions :

(i). $\frac{\sin z}{z^2}$; (ii). $\frac{\tan z}{z^2}$; (iii). $\frac{\cos z}{z}$; (iv). $\cot z$;

(v). cosec z; (vi). sech z; (vii). $\left(\frac{z+1}{z-1}\right)^2$.

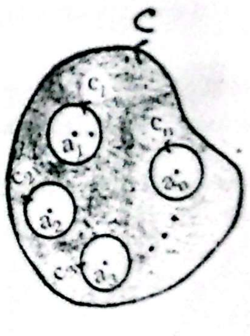
Solution : Try yourself.

Cauchy's residue theorem – 208 : If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n singular points $a_1, a_2, \dots, a_n$ inside C, then

$$\oint_C f(z)\,dz$$

$= 2\pi i$ [Res (a_1) + Res (a_2) + ... + Res (a_n)], i. e. $2\pi i$ times the sum of the residues at the singularities within C.

R. U. M. SC. P. '85; D. U. M. SC. P. '88; R. U. '76, '82; C. U. H. '81, '88; J. U. H. '87, '91; C. U. M. SC. P. '87; D. U. H. T. '82, '84; D. U. '72.



Proof : Let $C_1, C_2, \dots, C_n$ be n circles with centres at the points $a_1, a_2, \dots, a_n$. Let the radii of these circles are so small that they lie entirely inside C and do not overlap. Then $f(z)$ is analytic in the region between C and these circles and so by a corrolary of the Cauchy's theorem, we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots + \oint_{C_n} f(z)\,dz \ldots$$

(1)

But we have

$$\oint_{C_1} f(z)\,dz = 2\pi i \text{ Res } (a_1), \oint_{C_2} f(z)\,dz = 2\pi i \text{ Res } (a_2), \ldots .$$

$$\oint_{C_n} f(z)\,dz = 2\pi i \text{ Res } (a_n), \ldots (2)$$

Then, by (1) and (2) we have

$$\oint_C f(z)\,dz = 2\pi i \,[\text{ Res } (a_1) + \text{ Res } (a_2) + \ldots + \text{ Res } (a_n)\,]$$

$= 2\pi i$ [sum of the residues within C] and the theorem is proved.

Theorem -209 : If $f(z)$ is analytic inside and on a simple closed curve C except at a finite number of n poles $a_1, a_2, \ldots, a_n$ inside C, then

$$\oint_C f(z)\,dz = 2\pi i \,[\text{ Res } (a_1) + \text{ Res } (a_2) + \ldots + \text{ Res } (a_n)\,].$$

Proof : Use the above proof. In some books, this theorem is also known as the Cauchy's residue theorem.

BT **Example-257 :** Show that $I = \dfrac{1}{2\pi i} \oint_C \dfrac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \dfrac{t-1}{2} + \dfrac{1}{2} e^{-t} \cos t$ where C is the circle with equation $|z| = 3$. **C. U. H. '88; R. U. M. SC. P. '84.**

Solution : The poles of $\dfrac{e^{zt}}{z^2(z^2+2z+2)}$ are obtained by solving $z^2(z^2+2z+2) = 0 \Rightarrow$ the integrand has a double pole

at $z = 0$ and two simple poles at $z = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$ and also all these poles are inside C.

Now the residue at $z = 0$ is $\lim\limits_{z \to 0} \dfrac{1}{1!} \dfrac{d}{dz} \left\{ z^2 \dfrac{e^{zt}}{z^2(z^2+2z+2)} \right\}$

$$= \lim_{z \to 0} \frac{e^{zt}}{z^2+2z+2} \left[t - \frac{2z+2}{z^2+2z+2} \right] = \frac{t-1}{2}.$$

Residue at $z = -1 + i = \lim\limits_{z \to -1+i} \left\{ (z+1-i) \dfrac{e^{zt}}{z^2(z^2+2z+2)} \right\}$ $(z+1)^2+1$

$= \dfrac{e^{(-1+i)t}}{(-1+i)^2(-1+i+1+i)} = \dfrac{-e^{-t}}{4} e^{it}$. Similarly the residue at $z = -1 - i$ is $\dfrac{e^{-t}}{4} e^{-ti}$ $(z+1)^2 - i^2$ $(z+1+i)(z$

Thus by the Cauchy's residue theorem

$$I = 2\pi i \text{ (sum of the residues) } = 2\pi i \left[\frac{t-1}{2} + \frac{e^{-t}}{4}(e^{ti} + e^{-ti}) \right]$$

$$= 2\pi i \left[\frac{t-1}{2} + \frac{1}{2} e^{-t} \cos t \right]$$ $\left[\because \frac{1}{2}(e^{ti} + e^{-ti}) = \cos t\right]$

$$\Rightarrow \frac{1}{2\pi i} \oint_C \frac{e^{zt}}{z^2(z^2+2z+2)}\,dz = \frac{t-1}{2} + \frac{1}{2} e^{-t} \cos t.$$

BT **Example-258 :** Show that $\oint_C \dfrac{e^{tz}}{(z^2+1)^2}\,dz = \pi i (\sin t - t\cos t)$ where C is the circle $|z| = 3$ and $t > 0$. **R. U. H. '80, '82.**

Solution : Here the poles of $\dfrac{e^{tz}}{(z^2+1)^2} = \dfrac{e^{tz}}{(z-i)^2(z+i)^2}$ are at $z = \pm i$ inside C and both is of order two.

Now the residue at $z = i$ is $\lim\limits_{z \to i} \dfrac{1}{1!} \dfrac{d}{dz} \left\{ \dfrac{(z-i)^2 e^{tz}}{(z-i)^2 (z+i)^2} \right\}$

$$= \lim_{z\to i} \frac{d}{dz}\left\{\frac{e^{tz}}{(z+i)^2}\right\} = \lim_{z\to i} \frac{t(z+i)^2 e^{tz} - 2(z+i)\, e^{tz}}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{t(z+i)e^{tz} - 2e^{tz}}{(z+i)^3} = \frac{2(it-1)e^{it}}{(2i)^3} = \frac{-\cancel{2}(t+i)e^{it}}{4}$$

Similarly, the residue at $z = -i$ is $\frac{-\cancel{2}(t-i)e^{-it}}{4}$

Then by the Cauchy's residue theorem, $\oint_C \frac{e^{tz}}{(z^2+1)^2}\,dz$

$$= 2\pi i \text{ (sum of the residues)} = 2\pi i\left\{\frac{-\cancel{2}(t+i)e^{it}}{4} + \frac{-\cancel{2}(t-i)e^{-it}}{4}\right\}$$

$$= -\frac{\pi i}{2}\left\{t(e^{it}+e^{-it}) + i\,(e^{it}-e^{-it})\right\} = \boxed{\pi i(\sin t - t\cos t).}$$

Problem

Example 259 : Show that $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}$.

where C is the circle $|z| = 4$. C. U. M. SC. P. '87.

Solution : Here $f(z) = \frac{e^z}{(z^2+\pi^2)^2} = \frac{e^z}{(z-\pi i)^2\,(z+\pi i)^2}$ has double pole at $z = \pi i$ and also at $z = -\pi i$.

Now the residue at $z = \pi i$ is $\lim_{z\to\pi i}\frac{1}{1!}\frac{d}{dz}\left\{\frac{(z-\pi i)^2\, e^z}{(z-\pi i)^2\,(z+\pi i)^2}\right\}$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left\{\frac{e^z}{(z+\pi i)^2}\right\} = \lim_{z\to\pi i}\frac{(z+\pi i)^2 e^z - 2e^z(z+\pi i)}{(z+\pi i)^4}$$

$= \frac{(2\pi i - 2)e^{\pi i}}{(\pi i+\pi i)^3} = \frac{\pi+i}{4\pi^3}$ and the residue at $z = -\pi i$ is

$$\lim_{z\to-\pi i}\frac{d}{dz}\left\{\frac{e^z}{(z-\pi i)^2}\right\} = \lim_{z\to-\pi i}\frac{(z-\pi i)e^z - 2e^z}{(z-\pi i)^3} = \frac{\pi - i}{4\pi^3}$$

..., by the residue theorem

$\frac{\pi+i}{4\pi^3}$, $\frac{-2(t+i)e^{ti}}{4}$

$$\Rightarrow \oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\left(\frac{\pi+i}{4\pi^3}+\frac{\pi-i}{4\pi^3}\right) = \frac{i}{\pi}.$$

± 1

Example 260 : Show that

$\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i \cosh 1$, where C is the circle $|z| = 2$.

Solution : Since $f(z) = \frac{ze^z}{z^2-1}$ has poles at $z = \pm 1$ and they are simple, then by the Cauchy's residue theorem, we have $\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i\,[\,\text{Res}(1) + \text{Res}(-1)]$ ... (1) since both the poles lies inside C. Now

$$\text{Res}(1) = \lim_{z\to 1}\{(z-1)\,f(z)\} = \lim_{z\to 1}\frac{ze^z}{z+1} = \frac{e}{2} \;\ldots (2).$$

$$\text{Res}(-1) = \lim_{z\to -1}\{(z+1)\,f(z)\} = \lim_{z\to -1}\frac{ze^z}{z-1} = \frac{e^{-1}}{2} \;\ldots (3)$$

Then by (2) and (3), (1) $\Rightarrow$

$$\oint_C \frac{ze^z}{z^2-1}\,dz = 2\pi i\left(\frac{e}{2}+\frac{e^{-1}}{2}\right) = 2\pi i \cosh 1 \text{ (proved).}$$

$\left[\frac{e}{2} + \frac{e^{-1}}{2} = \cosh\right.$

Example 261 : Show that

$\oint_C \frac{e^z}{z(z-1)^2}\,dz = 2\pi i$, where C is the circle $|z| = 2$.

Solution : The poles of $f(z) = \frac{e^z}{z(z-1)^2}$ are obtained by solving $z(z-1)^2 = 0 \Rightarrow z = 0,\ 1$ and both the poles lies in side C.

Here $z = 0$ is a simple pole and $z = 1$ is a double pole

$$\text{Res}(0) = \lim_{z\to 0}\left\{z\,\frac{e^z}{z(z-1)^2}\right\} = \lim_{z\to 0}\frac{e^z}{(z-1)^2} = 1;$$

$$\text{Res } (1) = \lim_{z \to 1} \frac{d}{dz} \left\{ (z-1)^2 \frac{e^z}{z(z-1)^2} \right\} = \lim_{z \to 1} \frac{d}{dz} \left(\frac{e^z}{z} \right)$$

$$= \lim_{z \to 1} \frac{e^z(z-1)}{z^2} = 0.$$

Now by the residue theorem, we have $\oint_C \frac{e^z}{z(z-1)^2} dz$

$= 2\pi i$ [Res (0) + Res (1)] $= 2\pi i\ (1 + 0) = 2\pi i.$

Example -262 : Show that $\oint_C \frac{z}{z^4 - 1} dz = 0,$

where C is he circle $|z| = 2.$

Salution : ry yourself.

Ezample - 260 : Show that $\oint_C \frac{\tan z}{z} dz = 0,$

where C is th circle $|z| = 2.$

Solution : Tr, urself

230. .Laurent series at ∞.

A function $f(z)$ is said to be analytic in a deleted neighbourhood of ∞ if the function $f(z)$ is analytic for $|z| > r_2$ for some r_2. In this case the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,\ r_2 < |z-a| < r_1$$

$$\Rightarrow f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,\ |z| > r_2 \ldots (1)$$

if we take $r_1 = \infty$ and $a = 0$.

If there is no positive power of z in (1), the $f(z)$ is said to have a removal singularity at $z = \infty$ and we can make $f(z)$ analytic at $z = \infty$ by defining $f(\infty) = a_0$. In this case (1)

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n z^n,\ |z| > r_2 \Big\} \ldots (2)$$

$f(\infty) = a_0$

If $f(z)$ is analytic at $z = \infty$ as in (2) and $f(\infty) = a_0 = 0$, then $f(z)$ is said to have **zero** at the point $z = \infty$.

231. Zero at infinity : If $f(z)$ is analytic at $z = \infty$ and if $\frac{1}{f(z)}$ has a pole of order n at $z = \infty$, then $f(z)$ has a **zero of order n** at $z = \infty$ and conversely.

If n = 1, then $f(z)$ has a **first order zero or zero of order one** at $z = \infty$.

232. Residue at infinity : If $f(z)$ is analytic for $|z| > r$ for some r, then the residue of $f(z)$ at $z = \infty$ is defined by the following :

$$\text{Res } [\, f(z), \infty\,] = \frac{1}{2\pi i} \oint_C f(z)\, dz$$

where the integral is taken in the negative direction on a simple closed curve C, in the region of analyticity of $f(z)$ and outside of which $f(z)$ has no singularity other than infinity.

Theorem -210 : The residue of $f(z)$ at $z = \infty$ is given by the equation Res $[f(z), \infty] = -a_{-1}$, where $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \ldots$ (1) is a Laurent expansion of $f(z)$ at $z = \infty$ and a_{-1} is the coefficient of z^{-1} in (1).

Proof : Try yourself.

Example – 264 : The function $f(z) = e^{1/z}$

$= 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\dots$ is analytic at $z = \infty$. Here the coefficient of z^{-1} is $1 = a_{-1}$ and the residue at $z = \infty$ is -1 i. e.

$\text{Res}\,[f(z), \infty] = -a_{-1} = -1.$

Theorem – 211 : If $f(z)$ is analytic at $z = \infty$, then $\text{Res}\,[f(z), \infty] = \lim_{z\to\infty} \{-zf(z)\}$.

Proof : Try yourself.

Theorem – 212 : Show that

$$\text{Res}\left[f(z), \infty\right] = -\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right].$$

Proof : The Laurent expansion of $f(z)$ at $z = \infty$ is $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ ... (1) where $|z| > r$ for some r. Then $f\left(\frac{1}{z}\right)$

$$= \sum_{n=-\infty}^{\infty} \frac{a_n}{z^n} \text{ ... (2) where } 0 < |z| < \frac{1}{r}.$$

Now multiplying (2) by $\frac{1}{z^2}$, then we get

$$\frac{1}{z^2} f\left(\frac{1}{z^2}\right) = \dots + \frac{a_0}{z^2} + \frac{a_{-1}}{z} + a_{-2} + \dots \text{ (3).}$$

Now letting $z\to 0$ in (3), we have

$$\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right] = a_{-1} = -\text{Res}\,[f(z), \infty]$$

$$\Rightarrow \text{Res}\left[f(z), \infty\right] = -\text{Res}\left[\frac{1}{z^2}f\left(\frac{1}{z}\right), 0\right]$$ and the theorem is proved.

Theorem – 213 : If $f(z)$ has a zero of second or higher order at $z = \infty$, then $\text{Res}\,[f(z), \infty] = 0$.

Proof : Try yourself.

Theorem – 214 : If a single valued function $f(z)$ has only a finite number of singularities, then the sum of all the residues at these singularities of $f(z)$ including the residue at $z = \infty$ is zero.

Proof : Let C be a closed curve enclosing all the singularities of $f(z)$ except at ∞. Then the sum of the residues at these singularities is $\frac{1}{2\pi i}\oint_C f(z)\,dz$ ... (1). But the residue at ∞ is $-\frac{1}{2\pi i}\oint_C f(z)\,dz$ ... (2)

Adding (1) and (2), we get our required result. Thus the theorem is proved.

Example – 265 : Find $\text{Res}\left[f(z), 1\right]$ where

$$f(z) = \frac{z^3}{(z-1)^4(z-2)(z-3)}.$$

Solution : (Method 1). Here $z = 1$ is a pole of order 4.

$$\text{Now Res (1)} = \lim_{z\to 1} \frac{1}{3!}\frac{d^3}{dz^3}\left\{(z-1)^4 f(z)\right\}$$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[\frac{z^3}{(z-2)(z-3)}\right]$$

$$= \lim_{z\to 1} \frac{1}{6}\frac{d^3}{dz^3}\left[z \cdot A - \frac{8}{z-2} + \frac{27}{z-3}\right]$$ [by partial fractions where A is a constant]

$= \lim_{z\to 1} \frac{1}{6}\left[-8(-1)^3\, 3!\,(z-2)^{-4} + 27(-1)^3\, 3!\,(z-3)^{-4}\right]$

$= 8 - \frac{27}{16} = \frac{101}{16}.$

(Method 2). We have

Res (1) + Res (2) + Res (3) + R (∝) = 0 ... (1)

Here Res (2) = $\lim_{z\to 2}\{(z-2)\, f(z)\}$

$= \lim_{z\to 2} \frac{z^3}{(z-1)^4 (z-3)} = -8$ ... (2)

Res (3) = $\lim_{z\to 3}\{(z-3) f(z)\}$

$= \lim_{z\to 3} \frac{z^3}{(z-1)^4 (z-2)} = \frac{27}{16}$ .... (3)

Res (∝) = 0. ... (4) since at $z = \infty$, the function has a zero of order 3.

Now by (1), (2), (3) and (4) we have

Res (1) − 8 + $\frac{27}{16}$ + 0 = 0 ⇒ Res (1) = $\frac{101}{16}$.

Example – 266 : Find Res $[f(z), \infty]$, where $f(z) = \frac{z^3}{z^2-1}$.

Solution : The finite poles of $f(z)$ are $z = \pm 1$ and they are simple.

Here Res (1) = $\lim_{z\to 1}\left\{(z-1)\frac{z^3}{z^2-1}\right\} = \frac{1}{2}$ and

Res (−1) = $\lim_{z\to -1}\left\{(z+1)\frac{z^3}{z^2-1}\right\} = \frac{1}{2}$

Now Res (1) + Res (−1) + Res (∝) = 0

$\Rightarrow \frac{1}{2} + \frac{1}{2}$ + Res (∝) = 0 ⇒ Res (∝) = − 1.

Example –267 : Compute the residues at the singularities of $f(z) = \frac{z^3}{(z-1)(z-2)(z-3)}$ and also at ∝. Also show that their sum is zero including the residue at ∝.

Solution : The poles of $f(z)$ are $z = 1, 2, 3$ and they are simple poles. Now :

Res (1) = $\lim_{z\to 1} (z-1) f(z) = \lim_{z\to 1} \frac{z^3}{(z-2)(z-3)} = \frac{1}{2}$.

Res (2) = $\lim_{z\to 2} (z-2) f(z) = \lim_{z\to 2} \frac{z^3}{(z-1)(z-3)} = -8$.

Res (3) = $\lim_{z\to 3} (z-3) f(z) = \lim_{z\to 3} \frac{z^3}{(z-1)(z-2)} = \frac{27}{2}$.

Again we have $f(z) = 1 + \frac{1/2}{z-1} - \frac{8}{z-2} + \frac{27/2}{z-3}$.

$= 1 + \frac{1}{2z}\left(1-\frac{1}{z}\right)^{-1} - \frac{8}{z}\left(1-\frac{2}{z}\right)^{-1} + \frac{27}{2z}\left(1-\frac{3}{z}\right)^{-1}$ ... (1)

In (1), the coefficient of $\frac{1}{z}$ is $\frac{1}{2} - 8 + \frac{27}{2} = 6$ and the residue at ∝ is − 6.

Now the sum of the residues including the residue at ∝ =

$\frac{1}{2} - 8 + \frac{27}{2} - 6 = 0$.

233. Cauchy's residue theorem including the point infinity.

Theorem –215 : Let $f(z)$ be analytic in a region $\mathfrak{R}$ which includes a deleted neighbourhood of ∝. If C is a closed path in $\mathfrak{R}$ out side of which $f(z)$ is analytic except for isolated singularities at the points $a_1, a_2, \dots, a_n$, then $\oint_C f(z)\, dz =$ $2\pi i \left\{\sum_{p=1}^{n} \text{Res}\,[f(z), a_p] + \text{Res}\,[f(z), \infty]\right\}$ where the integral is

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taken on C in the negative direction and the residue at ∞ must be included on the right.

Proof : Try yourself.

Example – 268 : Show that $\int_C \frac{1}{(z-1)^3(z-7)}\,dz = -\frac{\pi i}{108}$

where C is the circle $|z| = 2$.

Solution : Using the Cauchy's residue theorem including the point at infinity we will prove the result. Outside the circle $|z| = 2$, the function has a first Order pole at $z = 7$ and a zero of order 4 at $z = \infty$. Then by the residue theorem including the point at infinity, we have

$$\oint_C \frac{1}{(z-1)^3(z-7)}\,dz = -2\pi i\,[\text{Res}(7) + \text{Res}(\infty)] \quad \dots (1)$$

Here $\text{Res}(7) = \lim_{z\to 7}\left\{(z-7)\frac{1}{(z-1)^3(z-7)}\right\} = \frac{1}{216} \dots (2)$

and $\text{Res}(\infty) = 0 \dots (3)$ since it is a zero of order 4.

Now by (2) and (3), (1) $\Rightarrow$

$$\oint_C \frac{1}{(z-1)^3(z-7)}\,dz = -2\pi i\left(\frac{1}{216}+0\right) = -\frac{\pi i}{108}$$ and the required result is obtained.

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Example – 269 : Show that $\oint_C \frac{z}{z^4-1}\,dz = 0$,

where C is the circle $|z| = 2$.

Solution : The function $f(z) = \frac{z}{z^4-1}$ has no singularity outside the circle $|z| = 2$ other than ∞. At the point $z = \infty$ the function $f(z)$ has a zero of order 3, then $\text{Res}(\infty) = 0$. Now by the Cauchy's residue theorem including the point at infinity, we have

$$\oint_C \frac{z}{z^4-1}\,dz = -2\pi i\,\text{Res}(\infty) = 0.$$

CHAPTER – 8

CONTOUR INTEGRATION

In this chapter we will evalute a varity types of real definite integrals with the help of the Cauchy's residue theorem using a suitable types of closed path or contour. For this reason, the process is called contour integration. Now we will discuss it dividing several forms.

234. (Form 1). Integrals of the form :

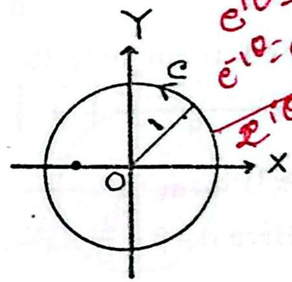
$\int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$, **where** $f(\cos\theta, \sin\theta)$ **is a rational function of** $\cos\theta$ **and** $\sin\theta$ **:**

Let $I = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta \dots (1)$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$. Let $z = e^{i\theta}$, then $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z^2+1}{2z}$

$e^{i\theta} = \cos\theta + i\sin\theta$
$e^{-i\theta} = \cos\theta - i\sin\theta$
$e^{i\theta} + e^{-i\theta} = 2\cos\theta$

$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z^2-1}{2iz}$ and $dz = ie^{i\theta}\,d\theta$

$= iz\,d\theta$ or $d\theta = \frac{dz}{iz}$. Now using these (1) $\Rightarrow I = \oint_C g(z)\,dz$,



where C is the unit circle $|z| = 1$, whose centre is at the origin and radius is equal to 1.

Example – 270 : Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ if $a > |b|$.

D. U. H. T. '75, 77, 87; D. U. H. '87.

$e^{-i\theta} = \frac{1}{z}$