CHAPTER – 2

FUNCTIONS LIMITS AND CONTINUITY

81. Complex variable : D. U. H. S.-T. 85.

Let S be a set of complex numbers.

If z denotes any one of the numbers of S, then z is called a complex variable.

If x and y are real variables, then $z = x + iy$ is called a complex variable.

82. Function : Let S_1 and S_2 be two sets of complex numbers. Now if for each complex variable z of S_1 there corresponds one or more values of a complex variable w of S_2, then w is called a function of z and it is denoted by $w = f(z)$ or $w = F(z)$ or $w = g(z)$ or $w = G(z)$ etc.

Here the set S_1 is called a ***domain*** of definition of the function $w = f(z)$ and S_2 is called the ***range*** of the function w.

N. B. In this book all functions will be considered complex functions unless otherwise any other functions stated.

83. Independent and dependent variable of a function

Let $w = f(z)$ be a function, then the variable z is called an independent variable and the variable w is called a dependent variable.

84. Value of a function :

Let $w = f(z)$ be a function, then the value of this function at $z = a$ is written $f(a)$.

85. Single-valued function :

A function $w = f(z)$ is called a single-valued in a domain S if only one value of w corresponds to each value of z in S.

86. Multiple-valued function : R. U. H. 80.

A function $w = f(z)$ is called a multiple-valued in a domain S if more than one value of w corresponds to each value of z in S.

Any multiple valued function can be considered as a collection of single-valued functions where each single-valued member is called a ***branch*** of the function.

Example 64 : If $w = f(z) = z^2 + 2$, then w is called a single valued function of z since to each value of z there is only one value of w.

Example 65 : If $w = f(z) = z^{1/2}$, then w is called a multiple-valued function of z since to each value of z there are two values of w.

Example 66 : If $w = f(z) = z^2$, then $f(1+i) = (1+i)^2$
$= 1 + 2i + i^2 = 1 + 2i - 1 = 2i$.

87. Even function : A function $f(z)$ is called an even function if $f(-z) = f(z)$.

Example 67 : The function $f(z) = z^2$ is an even function since $f(-z) = (-z)^2 = z^2 = f(z)$. Similarly, $\cos z$, $z^4 + z^2 + c$, etc are even functions.

88. Odd function : A function $f(z)$ is called an odd function if $f(-z) = -f(z)$.

Example 68 : The function $f(z)=z^3$ is an odd function since $f(-z)=(-z)^3=-z^3=-f(z)$. Similarly, $\sin z$, $\tan z$, z^3+z, etc are odd functions.

Example 69 : The functions $\cos z+\sin z$, z^4+z^3+5, etc are neither even nor odd.

N. B. Next we will consider all functions are single-valued function unless otherwise stated.

89. *Inverse function*: Let $w=f(z)$ be a function, then we can consider z as a function of w and it is denoted by $z=g(w)=f^{-1}(w)$. Here the function f^{-1} is called the inverse function of f. The functions $w=f(z)$ and $w=f^{-1}(z)$ are inverse functions of each other.

90. *Real and imaginary parts of $w=f(z)$ corresponding to the complex variable $z=x+iy$.*

Let $w=f(z)=u+iv$ be a single-valued function of $z=x+iy$.

Now replacing $x+iy$ for z, we have $u+iv=f(x+iy)$.

Then equating real and imaginary parts we have $u=u(x, y)$ and $v=v(x, y)$.

Example 70 : If $w=e^z$, then $u+iv=e^{x+iy}$ $=e^x(\cos y+i\sin y) \Rightarrow u=e^x\cos y=u(x, y)$ and $v=e^x\sin y=v(x, y)$

91. *The polynomial function*: A function of the form $P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ is called a polynomial function of degree n where $a_0\neq 0, a_1, a_2\cdots, a_n$ are complex constants and n is a positive integer.

92. *The rational algebraic function*: A function of the form $w=\dfrac{P(z)}{Q(z)}$ is called a rational algebraic function where $P(z)$ and $Q(z)$ are polynomials.

93. *The exponential function*: *R. U. 76.*

A function of the form $w=e^z=e^{x+iy}=e^x e^{iy}$ $=e^x(\cos y+i\sin y)$ is called an exponential function where $e=2.71828\cdots$ is the natural base of logarithm.

It is clear that $e^{z_1}\, e^{z_2}=e^{z_1+z_2}$ and

$$\frac{e^{z_1}}{e^{z_2}}=e^{z_1-z_2}.$$

94. *Definition of a^z*: If a is real and positive, then a^z can be defined as follows :

$a^z=e^{z\ln a}=e^{z\log a}$ where $\ln a$ or $\log a$ is the ***natural logarithm*** of a.

N. B. In this book ln and log have the identical meaning but $\log_a$ is not a natural logarithm if $a\neq e$.

95. *Natural logarithm of z*:

If $z=e^w \Rightarrow w=\ln z$, which is called the ***natural logarithm*** of z. The natural logarithm function is the inverse of the exponential function and it can be defined by

$w=\ln z=\log r+i(2k\pi+\theta)$ where

$z=re^{i\theta}=re^{i(2k\pi+\theta)}$ and $k=0, \pm1, \pm2, \cdots\cdots$

It is clear that $\ln z$ is a multiple-valued function.

96. The principal value of log z :

The principal value or principal branch of ln z is defined as $\ln r+i\theta$ where $z=r\,e^{i\theta}$ and $\theta \in [0, 2\pi[$ or $\theta \in]-\pi, \pi]$ etc where the interval must be a length of 2π.

97. Definition** of a^w **if a is real :

If a is real, then if $z=a^w \Rightarrow w=\log_a z$ where $a>0$, $a\neq 0, 1$ and $a \neq e$. Also in this case we have $z=e^{w \ln a}$ and $w=\log_a z=\dfrac{\ln z}{\ln a}$.

98. Trigonometric or circular functions in terms of exponential functions :

The trigonometric or circular functions can be defined by the following :

(i) $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$ ***R. U. 76.*** *(ii)* $\cos z=\dfrac{e^{iz}+e^{-iz}}{2}$

(iii) $\sec z=\dfrac{1}{\cos z}=\dfrac{2}{e^{iz}+e^{-iz}}$, *(iv)* $\operatorname{cosec} z=\dfrac{1}{\sin z}$

$=\dfrac{2i}{e^{iz}-e^{-iz}}$, *(v)* $\tan z=\dfrac{\sin z}{\cos z}=\dfrac{e^{iz}-e^{-iz}}{i\,(e^{iz}+e^{-iz})}$

(vi) $\cot z=\dfrac{\cos z}{\sin z}=\dfrac{i\,(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}$

Example 71 : Show that :

(i) $\sin^2 z+\cos^2 z=1$, *(ii)* $\sec^2 z=1+\tan^2 z$, *(iii)* $\operatorname{cosec}^2 z=1+\cot^2 z$, *(iv)* $\sin(-z)=-\sin z$, *(v)* $\cos(-z)=\cos z$, *(vi)* $\tan(-z)=-\tan z$, *(vii)* $\cot(-z)=-\cot z$, *(viii)* $\sec(-z)=\sec z$, *(ix)* $\operatorname{cosec}(-z)=-\operatorname{cosec} z$,

(x) $\sin(z_1 \pm z_2)=\sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$, *(xi)* $\cos(z_1 \pm z_2)=\cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$, *(xii)* $\tan(z_1 \pm z_2)=\dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$, *(xiii)* $\cot(z_1 \pm z_2)=\dfrac{\cot z_2 \cot z_1 \mp 1}{\cot z_2 \pm \cot z_1}$

Solution : Try yourself.

99. Hyperbolic function : The hyperbolic functions are defined by the following :

(i) $\sinh z=\dfrac{e^{z}-e^{-z}}{2}$, ***R. U. 76 ;*** *(ii)* $\cosh z=\dfrac{e^{z}+e^{-z}}{2}$

(iii) $\operatorname{sech} z=\dfrac{1}{\cosh z}=\dfrac{2}{e^{z}+e^{-z}}$; *(iv)* $\operatorname{cosech} z=\dfrac{1}{\sinh z}$

$=\dfrac{2}{e^{z}-e^{-z}}$, *(v)* $\tanh z=\dfrac{\sinh z}{\cosh z}=\dfrac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$,

(vi) $\coth z=\dfrac{e^{z}+e^{-z}}{e^{z}-e^{-z}}$

Example 72 : Show that :

(i) $\cosh^2 z-\sinh^2 z=1$, *(ii)* $\operatorname{sech}^2 z=1-\tanh^2 z$, *(iii)* $\operatorname{cosech}^2 z=\coth^2 z-1$, *(iv)* $\sinh(-z)=-\sinh z$, *(v)* $\cosh(-z)=\cosh z$, *(vi)* $\tanh(-z)=-\tanh z$, *(vii)* $\operatorname{sech}(-z)=\operatorname{sech} z$, *(viii)* $\operatorname{cosech}(-z)=-\operatorname{cosech} z$, *(ix)* $\coth(-z)=-\coth z$,

(x) $\sinh(z_1 \pm z_2)=\sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$

(xi) $\cosh(z_1 \pm z_2)=\cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$,

(xii) $\tanh(z_1 \pm z_2)=\dfrac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$

(xiii) $\coth(z_1 \pm z_2)=\dfrac{\cosh z_2 \coth z_1 \pm 1}{\coth z_2 \pm \coth z_1}$,

Solution : Try yourself.

100. Relation between the trigonometric or circular functions and the hyperbolic functions:

(*i*) $\sin iz = i \sinh z \Rightarrow \sinh z = -i \sin iz$,

(*ii*) $\cos iz = \cosh z$, (*iii*) $\tan iz = i \tanh z$

$\Rightarrow \tanh z = -i \tan iz$, (*iv*) $\operatorname{cosec} iz = -i \operatorname{cosech} z \Rightarrow$

$\operatorname{cosech} z = i \operatorname{cosec} iz$, (*vi*) $\sec iz = \operatorname{sech} z$,

(*vii*) $\cot iz = -i \coth z \Rightarrow \coth z = i \cot iz$,

(*ix*) $\sinh iz = i \sin z \Rightarrow \sin z = -i \sinh iz$

(x) $\cosh iz = \cos z$, (*xi*) $\tanh iz = i \tan z \Rightarrow \tan z = -i \tanh iz$.

Example 73: Show that: (*i*) $\overline{\sin z} = \sin \bar{z}$;

(*ii*) $\overline{\cos z} = \cos \bar{z}$; (*iii*) $\overline{\tan z} = \tan \bar{z}$;

(*iv*) $\overline{\operatorname{cosec} z} = \operatorname{cosec} \bar{z}$. (v) $\overline{\sec z} = \sec \bar{z}$.

(*vi*) $\overline{\cot z} = \cot \bar{z}$.

Solution: (*i*) We have $\sin z = \sin (x + iy)$

$= \sin x \cosh y + i \cos x \sinh y \Rightarrow \overline{\sin z} = \sin (x - iy) = \sin \bar{z}$.

Others: Try yourself.

Example 74: Find all the roots of $\sinh z = i$.

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Solution: We have $\sinh z = i \Rightarrow \dfrac{e^z - e^{-z}}{2} = i$

$\Rightarrow e^{2z} - 2ie^z - 1 = 0 \Rightarrow e^z = \dfrac{2i \pm \sqrt{(4i^2 + 4)}}{2} = i = e^{\pi i/2}$

$= e^{\pi i/2}\, e^{2n\pi i} = e^{(2n+1/2)\pi i} \Rightarrow z = (2n + \frac{1}{2})\pi i$ where

$n = 0, \pm 1, \pm 2, \pm 3, \cdots \cdots$

Example 75: Show that $\ln z = 2n\pi i + \frac{1}{2} \ln (x^2 + y^2) + i \tan^{-1} y/x$ where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ and the principal value $= \frac{1}{2} \ln (x^2 + y^2) + i \tan^{-1} y/x$.

Solution 75: Try yourself.

101. Inverse trigonometric functions in terms of natural logarithms:

The inverse trigonometric functions are multiplevalued functions which can be expressed in terms of natural logarithms as follows;

(*i*) $\sin^{-1} z = \dfrac{1}{i} \ln (iz + \sqrt{1 - z^2}) + 2n\pi$;

(*ii*) $\cos^{-1} z = \dfrac{1}{i} \ln (z + \sqrt{(z^2 - 1)}) + 2n\pi$;

(*iii*) $\tan^{-1} z = \dfrac{1}{2i} \ln \left(\dfrac{1 + iz}{1 - iz} \right) + n\pi$;

(*iv*) $\operatorname{cosec}^{-1} z = \dfrac{1}{i} \ln \left(\dfrac{i + \sqrt{z^2 - 1}}{z} \right) + 2n\pi i$;

(*v*) $\sec^{-1} z = \dfrac{1}{i} \ln \left(\dfrac{1 + \sqrt{1 - z^2}}{z} \right) + 2n\pi$;

(*vi*) $\cot^{-1} z = \dfrac{1}{2i} \ln \left(\dfrac{z + i}{z - i} \right) + n\pi$,

where in each case $n = 0, \pm 1, \pm 2, \pm 3, \cdots \cdots$,

N. B. If $n = 0$, then the principal value can be obtained.

102. Inverse hyperbolic functions in terms of natural logarithms:

The inverse hyperbolic functions are multiple valued functions which can be expressed interms of natural logarithms as follows:

(*i*) $\sinh^{-1} z = \ln(z + \sqrt{z^2+1}) + 2n\pi i$;

(*ii*) $\cosh^{-1} z = \ln(z + \sqrt{z^2-1}) + 2n\pi i$;

(*iii*) $\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right) + n\pi i$;

(*iv*) $\operatorname{cosech}^{-1} z = \ln\left(\frac{1+\sqrt{z^2+1}}{z}\right) + 2n\pi i$;

(*v*) $\operatorname{sech}^{-1} z = \ln\left(\frac{1+\sqrt{1-z^2}}{z}\right) + 2n\pi i$;

(*vi*) $\coth^{-1} z = \frac{1}{2} \ln\left(\frac{z+1}{z-1}\right) + n\pi i$,

where in each case $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

N. B. If $n = 0$, then the principal value can be obtained.

103. The functions of the forms z^{α} **and** $f(z)^{g(z)}$ **:**

The function z^{α} is defined as $z^{\alpha} = e^{\alpha \ln z}$, where α may be complex. Again if $f(z)$ and $g(z)$ are two functions then the function $f(z)^{g(z)}$ is defined as $f(z)^{g(z)} = e^{g(z) \ln f(z)}$. Generally, the functions e^z and $f(z)^{g(z)}$ are multiple-valued functions.

104. Algebraic functions :

The function $w = f(z)$ is called an algebraic function of z if w is a solution of the polynomial equation $P_0(z) w^n + P_1(z) w^{n-1} + \cdots + P_{n-1}(z) w + P_n(z) = 0 \cdots$ (1) where $P_0(z) \neq 0$, $P_1(z), \cdots P_n(z)$ are polynomial in z and n is a positive integer.

Example 76 : The function $w = f(z) = z^{1/2}$ is an algebraic function since it is a solution of the polynomial equation $w^2 - z = 0$.

105. Transcendental functions :

The function which is not algebraic is called transcendental i. e. any function which can not be expressed as a solution of (1).

Example 77 : All trignometric, hyperbolic, logarithomic, inverse trigonometric, inverse hyperbolic etc functions are transcendental functions.

106. Limit at a finite point :

Let $f(z)$ be a single valued function which is defined in a neighbourhood of $z = z_0$ with the possible exception of $z = z_0$ itself. Then $f(z)$ is said to tend to the limit l as z tends to the value z_0 if corresponding to any positive number ϵ (however small) a positive number δ (which usually depends on ϵ) can be found such that $|f(z) - l| < \epsilon$ whenever $0 < |z - z_0| < \delta$ and it is denoted by $\lim\limits_{z \to z_0} f(z) = l$.

N. B. In above the limit l is independent of the path by which z tends to z_0.

Also, the limit l has not necessarily the same value as $f(z_0)$.

107. Theorem 38 : If $\lim\limits_{z \to z_0} f(z)$ exists, then it must be unique.

Proof : Try yourself.

108. Limit at infinity :

The single valued function $f(z)$ is said to tend to the limit l as z tends to infinity if corresponding to any positive number ϵ (however small) a positive number N can be found such that

$|f(z)-l|<\epsilon$ whenever $|z|>N$ and it is denoted by

$$\lim_{n\to\infty} f(z)=l.$$

109. Infinite limit :

The single valued function $f(z)$ is said to tend to the limit infinity as z tends to z_0 if corresponding to any positive number N (however large) a positive number δ can be found such that $|f(z)|>N$ whenever $|z-z_0|<\delta$ and it is denoted by

$$\lim_{z\to z_0} f(z)=\infty.$$

110. Four fundamental theorems on limit :

Theorems 39, 40, 41 and 42 :

If $\lim_{z\to z_0} f(z)$ and $\lim_{z\to z_0} g(z)$ are exist, then

39. $\lim_{z\to z_0}\{f(z)+g(z)\}=\lim_{z\to z_0} f(z)+\lim_{z\to z_0} g(z)$,

40. $\lim_{z\to z_0}\{f(z)-g(z)\}=\lim_{z\to z_0} f(z)-\lim_{z\to z_0} g(z)$,

41. $\lim_{z\to z_0}\{f(z)\,g(z)\}=\{\lim_{z\to z_0} f(z)\}\{\lim_{z\to z_0} g(z)\}$,

42. $\lim_{z\to z_0}\{f(z)/g(z)\}=\dfrac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$ where $\lim_{z\to z_0} g(z)\neq 0$.

Proofs : Try yourself.

Example 78 : Show that $\lim_{z\to 0}\dfrac{\bar{z}}{z}$ does not exist.

Solution : We have $z=x+iy$ and $\bar{z}=x-iy$. If $z\to 0$, then along the x-axis : $y=0$ and $x\to 0$, so the required limit is

$$\lim_{z\to 0}\frac{\bar{z}}{z}=\lim_{x\to 0}\frac{x}{x}=1.$$

Again if $z\to 0$, then along the y-axis : $x=0$ and $y\to 0$,

so the required limit is $=\lim_{z\to 0}\dfrac{\bar{z}}{z}=\lim_{y\to 0}\dfrac{-iy}{iy}=-1.$

Thus the two approaches are not equal and the limit does not exist.

111. Cotinuity : D. U. H. 93

The single valued function $f(z)$ is said to be continuous at the point $z=z_0$ if $f(z_0)$ has a definite value and if $\lim_{z\to z_0} f(z)=f(z_0)$.

Second definition : The single-valued function $f(z)$ is said to be continuous at the point $z=z_0$ if for any $\epsilon>0$ we can find $\delta>0$ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$.

112. Discontinuity : The function $f(z)$ is said to be discontinuous at the point $z=z_0$ if $f(z)$ is fails to be continuous at $z=z_0$.

113. Removal discontinuity : The function $f(z)$ is said to be removal discontinous at $z=z_0$ if $f(z)$ has a definite limit at $z=z_0$ but is not equal to $f(z_0)$.

114. Continuity at infinity : The continuity of $f(z)$ at $z=\infty$ can be examined by the continuity of $f(1/w)$ at $w=0$ by replacing $z=1/w$ in $f(z)$.

115. Continuity in a region :

A function $f(z)$ is said to be continuous in a region R if it is continuous at all points of the region R.

116. Four fundamental theorems on continuity :

Theorems 43, 44, 45 and 46 : If $f(z)$ and $g(z)$ are continuous at $z=z_0$, then the following functions are continuous at $z=z_0$.

(43) $f(z)+g(z)$; **(44)** $f(z)-g(z)$; **(45)** $f(z)\,g(z)$ and

(46) $f(z)/g(z)$ where $g(z_0)\neq 0$.

Proofs : Try yourself.

117. Theorem 47 : Every polynomial functions are continuous in a finite region.

Proof : Try yourself.

118. Theorem 48 : If $f(z)$ is continuous and has the value $f(z_1)$ at $z=z_1$. Again if $\phi(z)$ is continuous at $z=f(z_1)$, then $\phi(f(z))$ is continuous at $z=z_1$.

Proof : Try yourself.

119. Theorem 49 : If $w=f(z)$ is continuous at the point $z=z_0$ and $z=g(\xi)$ is continuous at $\xi=\xi_0$ and if $\xi_0=f(z_0)$ then the composite function or the function of function $w=g(f(z))$ is continuous at $z=z_0$.

Proof : Try yourself.

120. Theorem 50 : If $f(z)$ is continuous in a closed region R and if it is bounded in R. i. e. if there exists a real constanst M such that $|f(z)|<M$ for all points z in the region R.

Proof : Try yourself.

121. Theorem 51 : The real and imaginary parts of a continuous function $f(z)$ are continuous.

Proof : Try yourself

Example 79 : The functions e^z, $\sin z$ and $\cos z$ are continuous in every finite region

Solution : Try yourself:

122. Uniform continuity : A function $f(z)$ is said to be uniformly continuous in a region R if corresponding to any $\epsilon>0$ we can find $\delta>0$ (which is a function of ϵ only) such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$ for every point z_0 in the region R.

N. B. In continuity, δ depend on both ϵ and the particular point z_0. But in uniform continuity, δ depends only on ϵ.

Second definition : A function $f(z)$ is said to be uniformly continuous in a region R if for any $\epsilon>0$ we can find $\delta>0$ such that $|f(z_1)-f(z_2)|<\epsilon$ whenever $|z_1-z_2|<\delta$ for every points z_1 and z_2 in the region R.

123. Theorem 52 : If $f(z)$ is continuous in a closed region R, then it is uniformly continuous in R.

Proof : Try yourself.

Example 80 : If $f(z)=z^2$ then show that

(i) $\lim\limits_{z\to a} f(z)=a^2$; **D U. M. SC. P. 89.**

(ii) $f(z)$ is continuous at $z=a$;

(iii) $f(z)$ is uniformly continuous in the region $|z|<1$.

D U M SC. p, 89

Solution : **(i) :** We have to show that given any $\epsilon>0$, we can find $\delta>0$ such that $|z^2-a^2|<\epsilon$ whenever $0<|z-a|<\delta$. Now if $\delta\leq 1$, then $0<|z-a|<\delta\Rightarrow|z^2-a^2|=|z-a|\,|z+a|$ $=|z-a|\,|(z-a)+2a|<|z-a|\{|z-a|+|2a|\}$ $<\delta(1+2|a|)$. Now taking δ as 1 or $\epsilon/(1+2|a|)$ which ever is smaller. Thus $|z^2-a^2|<\epsilon$ whenever $0<|z-a|<\delta$ and we have $\lim\limits_{z\to a} f(z)=a^2$.

(ii) : By *(i)*, $\lim\limits_{z\to a} f(z)=a^2$. Again we have $f(a)=a^2$. Thus $\lim\limits_{z\to a} f(z)=f(a) \Rightarrow f(z)$ is continuous at $z=a$.

(iii) We have to show that given any $\epsilon>0$, we can find $\delta>0$ such that $|z^2-a^2|<\epsilon$ when $|z-a|<\delta$ where δ in a function of ϵ only.

Suppose z and a are any two point in $|z|<1$, then

$|z^2-a^2|=|z-a||z+a| \leqslant |z-a|(|z|+|a|)$

$<2|z-a|$ ... (1). Now if $|z-a|<\delta$, then (1)

$\Rightarrow |z^2-a^2|<2\delta \Rightarrow |z^2-a^2|<\epsilon$ choosing $\delta=\epsilon/2$.

Thus $|z^2-a^2|<\epsilon$ when $|z-a|<\delta$.

Hence the given function is uniformly continuous in the region $|z|<1$.

Example 81 : Show that $f(z)=1/z$ is not uniformly continuous in the region $|z|<1$.

Solution : We consider $f(z)$ is uniformly continuous in the region $|z|<1$.

Then for any $\epsilon>0$ it is possible to find δ which lies between 0 and 1 such that $|f(z)-f(a)|<\epsilon$ when $|z-a|<\delta$ for all z and a in the region $|z|<1$.

Let $z=\delta$ and $a=\dfrac{\delta}{1+\epsilon}$ then $|z-a|=\left|\delta-\dfrac{\delta}{1+\epsilon}\right|$

$=\left|\dfrac{\delta+\delta\epsilon-\delta}{1+\epsilon}\right|=\dfrac{\epsilon}{1+\epsilon}\delta<\delta.$

But $\left|\dfrac{1}{z}-\dfrac{1}{a}\right|=\left|\dfrac{1}{\delta}-\dfrac{1+\epsilon}{\delta}\right|=\left|\dfrac{-\epsilon}{\delta}\right|=\dfrac{\epsilon}{\delta}>\epsilon$ since we have considered $0<\delta<1$.

Thus we have a contradiction and the given function is not uniformly continuous in $|z|<1$.

124 Complex sequence :

A complex sequence $<f(n)>$ or $<u_n>$ is a function whose domain is the set of natural numbers ***N*** and range is the subset of the set of complex numbers **C** In this book, by a sequence we will mean the complex sequence.

The nth term of the sequence $<f(n)>$ or $<u_n>$ is $f(n)$ or u_n.

Example 82 : $<i^n> = <i, i^2, i^3, i^4, \cdots>$ is a sequence.

125. Limit of a sequence :

A number l is said to be the limit of the sequence $<u_n>$ if for any positive number ϵ we can determine a positive number N (depending on ϵ) such that $|u_n-l|<\epsilon$ for all $n>N$ and it is denoted by $\lim\limits_{n\to\infty} u_n=l$.

126. Convergent sequence :

If the limit of the sequence $<u_n>$ exists, then the sequence is called convergent.

127. Divergent sequence : If the limit of the sequence $<u_n>$ does not exist, then the sequence is called divergent.

128 Theorem 53 : If $\lim\limits_{n\to\infty} u_n=l$, where l is finite then it must be unique

Proof : Try yourself.

129 Four fundamental theorems on limits of sequences ;

Theorems 54, 55, 56 and 57 : If the sequences $< a_n >$ and $<b_n>$ both are convergent, then

54. $\lim\limits_{n\to\infty} (a_n+b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

55. $\lim\limits_{n\to\infty} (a_n-b_n) = \lim\limits_{n\to\infty} a_n - \lim\limits_{n\to\infty} b_n$

56. $\lim\limits_{n\to\infty} (a_n\, b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n)$

57 $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$ where $\lim\limits_{n\to\infty} b_n \neq 0$.

Proofs : Try yourself.

130. *Infinite series :* Let $<u_n>$ be a sequence, then

$\sum\limits_{n=1}^{\infty} u_n = u_1+u_2+u_3+$ is called an infinite series.

Example 83 : $1+z+z^2+z^3+ \ldots$ is an infinite series.

131. *nth partial sum*

Let $<u_n>$ be a sequence. Suppose $S_1 = u_1$, $S_2 = u_1+u_2$, $S_3 = u_1+u_2+u_3$, ..., $S_n = u_1+u_2+u_3+ \ldots +u_n$, where S_n is called the nth partial sum of the first n terms of the sequence $<u_n>$.

If $\lim\limits_{n\to\infty} S_n = S$ exists, then the series $\sum\limits_{n=1}^{\infty} u_n$ is called ***convergent*** and S is called its sum. If it is not convergent, then it is called ***divergent***

132. *Theorem 58 :* If the series $u_1+u_2+u_3+ \ldots \ldots$ is convergent, then $\lim\limits_{n\to\infty} u_n = 0$.

Proof : Try yourself.

Example 84 : Show that $1+z+z^2+z^3+ \ldots \ldots = \dfrac{1}{1-z}$ if $|z|<1$.

Solution : Try yourself.

133. *Theorem 59 :* If $\lim\limits_{n\to\infty} z_n = l$, then show that $\lim\limits_{n\to\infty} \operatorname{Re}\{z_n\} = \operatorname{Re}\{l\}$ and $\lim\limits_{n\to\infty} \operatorname{Im}\{z_n\} = \operatorname{Im}\{l\}$.

Proof : Try yourself.

Example 85 : Show that if $|a|<1$, then

(i) $\sum\limits_{n=0}^{\infty} a^n \cos n\theta = \dfrac{1-a\cos\theta}{1-2a\cos\theta+a^2}$;

(ii) $\sum\limits_{n=1}^{\infty} a^n \sin n\theta = \dfrac{a\sin\theta}{1-2a\cos\theta+a^2}$.

Solution : Try yourself.