Gauss Elimination method

🧮 Gauss Elimination Method

The **Gauss Elimination Method** is a **systematic technique for solving systems of linear equations.** It reduces a system to **row-echelon form** using **elementary row operations**, then solves it using **back-substitution.**

📌 Objective:

To solve a system of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

🔧 Steps of the Gauss Elimination Method

🔹 Step 1: Forward Elimination

- Convert the coefficient matrix into **upper triangular form** by eliminating variables below the pivot (leading 1s).

🔹 Step 2: Back Substitution

- Solve the last equation first and substitute back to find other unknowns.

🧮 Example:

Solve the system:

$$\begin{aligned} 2x + 3y + z &= 1 \\ 4x + 7y + 5z &= 2 \\ 6x + 18y + 6z &= 5 \end{aligned}$$

➤ Step 1: Augmented Matrix

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 4 & 7 & 5 & 2 \\ 6 & 18 & 6 & 5 \end{array}\right]$$

🔁 Step 1: Forward Elimination (with Full Detail)

We aim to **convert the system into upper triangular form**, where all elements **below the main diagonal are zero**.

🧮 Given System of Equations:

$$
\begin{aligned}
2x + 3y + z &= 1 \quad \text{(Eq1)} \\
4x + 7y + 5z &= 2 \quad \text{(Eq2)} \\
6x + 18y + 6z &= 5 \quad \text{(Eq3)}
\end{aligned}
$$

📌 Step 1.1: Write the Augmented Matrix

$$
\left[\begin{array}{ccc|c}
2 & 3 & 1 & 1 \\
4 & 7 & 5 & 2 \\
6 & 18 & 6 & 5
\end{array}\right]
$$

📌 Step 1.2: Make the First Pivot = 1 (optional for clarity)

You can divide **Row 1** by 2 to make the pivot element 1:

$$R_1 \leftarrow R_1 \div 2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 0.5 \\ 4 & 7 & 5 & 2 \\ 6 & 18 & 6 & 5 \end{array}\right]$$

📌 Step 1.3: Eliminate the first variable (x) from Rows 2 and 3

- **Eliminate Row 2:**
  $R_2 \leftarrow R_2 - 4 \cdot R_1$

$$R_2 = [4, 7, 5, 2] - 4 \cdot [1, 1.5, 0.5, 0.5] = [0, 1, 3, 0]$$

- **Eliminate Row 3:**
  $R_3 \leftarrow R_3 - 6 \cdot R_1$

$$R_3 = [6, 18, 6, 5] - 6 \cdot [1, 1.5, 0.5, 0.5] = [0, 9, 3, 2]$$

🆕 Updated Matrix after First Elimination:

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 0.5 \\ 0 & 1 & 3 & 0 \\ 0 & 9 & 3 & 2 \end{array}\right]$$

📌 Step 1.4: Eliminate y from Row 3 (2nd column below pivot)

We want to make the second pivot the only non-zero entry in its column.

$$R_3 \leftarrow R_3 - 9 \cdot R_2 \Rightarrow [0, 9, 3, 2] - 9 \cdot [0, 1, 3, 0] = [0, 0, -24, 2]$$

✅ Final Upper Triangular Matrix (after forward elimination):

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 0.5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -24 & 2 \end{array}\right]$$

This completes **Step 1: Forward Elimination.**

You can now proceed to **Step 2: Back Substitution** to solve for z, then y, then x.

🔁 Step 2: Back Substitution (Detailed)

We start with the **upper triangular matrix**:

$$\left[\begin{array}{ccc|c} 1 & 1.5 & 0.5 & 0.5 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -24 & 2 \end{array}\right]$$

This corresponds to the following system:

$$\begin{aligned} &(\text{Eq } 3) \quad -24z = 2 \\ &(\text{Eq } 2) \quad y + 3z = 0 \\ &(\text{Eq } 1) \quad x + 1.5y + 0.5z = 0.5 \end{aligned}$$

- **Step 2.1: Solve Eq 3 for z**

$$-24z = 2 \Rightarrow z = \frac{2}{-24} = -\frac{1}{12}$$

- **Step 2.2: Substitute z into Eq 2 to find y**

$$y + 3z = 0 \Rightarrow y + 3\left(-\frac{1}{12}\right) = 0 \Rightarrow y = \frac{1}{4}$$

- **Step 2.3: Substitute y and z into Eq 1 to find x**

$$x + 1.5y + 0.5z = 0.5$$

Substitute values:

$$x + 1.5\left(\frac{1}{4}\right) + 0.5\left(-\frac{1}{12}\right) = 0.5$$

$$x + \frac{3}{8} - \frac{1}{24} = 0.5$$

Common denominator: 24

$$x + \left(\frac{9}{24} - \frac{1}{24}\right) = \frac{12}{24} \Rightarrow x + \frac{8}{24} = \frac{12}{24}$$

$$x = \frac{12-8}{24} = \frac{4}{24} = \frac{1}{6}$$

✅ Final Answer:

$$x = \frac{1}{6}, \quad y = \frac{1}{4}, \quad z = -\frac{1}{12}$$

🧠 What We Did:

- Started with the last equation and substituted **upwards**.
- Solved each equation one-by-one for the unknowns.
- This is how **back substitution** works after Gauss Elimination.

🧾 Example: Solving for Three Unknowns (Shopping Example)

🛍️ Scenario:

You bought 3 different products — pens, notebooks, and erasers — on 3 separate days with different combinations. You want to find the price of each item.

💡 Given:

1. 2 pens + 1 notebook + 3 erasers = ৳20
2. 1 pen + 2 notebooks + 1 eraser = ৳18
3. 3 pens + 2 notebooks + 4 erasers = ৳32

Let:

- x = price of a pen
- y = price of a notebook
- z = price of an eraser

Step 1: Write as a 3×4 Augmented Matrix

$$\begin{bmatrix} 2 & 1 & 3 & | & 20 \\ 1 & 2 & 1 & | & 18 \\ 3 & 2 & 4 & | & 32 \end{bmatrix}$$

Step 2: Forward Elimination

Pivot 1: Use Row 1 as the pivot

- Eliminate column 1 in Row 2:
  $R2 \leftarrow R2 - 0.5 \times R1$

$$[1, 2, 1, |18] - 0.5 \times [2, 1, 3, |20] = [0, 1.5, -0.5, |8]$$

- Eliminate column 1 in Row 3:
  $R3 \leftarrow R3 - 1.5 \times R1$

$$[3, 2, 4, |32] - 1.5 \times [2, 1, 3, |20] = [0, 0.5, -0.5, |2]$$

✅ New Matrix:

$$\begin{bmatrix} 2 & 1 & 3 & | & 20 \\ 0 & 1.5 & -0.5 & | & 8 \\ 0 & 0.5 & -0.5 & | & 2 \end{bmatrix}$$

🔹 Pivot 2: Make leading coefficient of Row 2 = 1

$R2 \leftarrow R2 \div 1.5$

$$[0, 1, -\frac{1}{3}, | \frac{16}{3}]$$

Then eliminate column 2 from **Row 3**:

$R3 \leftarrow R3 - 0.5 \times R2$

$$[0, 0.5, -0.5, |2] - 0.5 \times [0, 1, -1/3, |16/3] = [0, 0, -0.333..., | -\frac{2}{3}]$$

✅ Final Matrix:

$$\begin{bmatrix} 2 & 1 & 3 & | & 20 \\ 0 & 1 & -\frac{1}{3} & | & \frac{16}{3} \\ 0 & 0 & -\frac{1}{3} & | & -\frac{2}{3} \end{bmatrix}$$

Step 3: Back Substitution

- From Row 3:
  $-\frac{1}{3}z = -\frac{2}{3} \Rightarrow z = 2$
- From Row 2:
  $y - \frac{1}{3}(2) = \frac{16}{3} \Rightarrow y = 6$
- From Row 1:
  $2x + y + 3z = 20 \Rightarrow 2x + 6 + 6 = 20 \Rightarrow x = 4$

Final Answer:

Item	Price (৳)
Pen (x)	4
Notebook (y)	6
Eraser (z)	2