2022 Recall:

finit: A sequence is a function whose domain is a rational number and the range a set of real numbers. i.e A real sequence is a function

$x_n : N \longrightarrow \mathbb{R}$.

A sequence $x_n : N \to \mathbb{R}$ is denot by $\{x_n\}_{n \in N}$ or $\{x_n\}$ or (x_n) or $(x_1, x_2, x_3 \cdots)$, or $\{x_1, x_2, x_3 \cdots\}$.

The terms $x_1, x_2, x_3 \cdots$ are called th first, second, third terms of the sequenc

Observe also that terms of the seque are infinite. However the range may be finite.

ple: The following are sequences

$\{x_n\} = \{1/n\}$, $\{x_n\} = \{1 + 1/n\}$,

$\{x_n\} = \{(-1)^n\}$ etc

ion: A sequence x_n is said to be **boun** **above** if there exist a positive real integer $\alpha \ni x_n \leq \alpha \; \forall n$. Also, x is said to be **bounded below** if $\exists$ real number $\beta \ni x_n \geq \beta \; \forall n$.

x_n is said to be bounded above and below, if (x_n) is bounded above by α and bounded below by β, the α and β are called the upper and lowe bounds of x_n respectively.

DEFINITION: A sequence x_n is said to **converge** to a real number P. If for every $\epsilon > 0$ $\exists$ a Posi integer $m \ni |x_n - P| < \epsilon$ $\forall n \geq m$.

POINT WISE CONVERGENCE

DEFINITION: **A sequence of functions** $\{f_n\}$ is a seq uence whose terms are real valued func defined on some interval say $I = (a,b)$. Hence, for each $P \in I$, $\{f_n\}$ corresponds to $f_1(P), f_2(P), f_3(P) \cdots$

EFINITION: A sequence of function $\{f_n\}$ is said be **Pointwise convergent** on $[a,b]$ if for each $\epsilon > 0$ and if for each $x \in [$ $\exists$ a positive integer $m = m(\epsilon, x) \ni$ $|f_n(x) - f(x)| < \epsilon$ $\forall n \geq m \cdots$ ①

The function f in ① is called the **Pointwise limit** of $f_n(x)$ from ①, we also write $\lim_{n \to \infty} f_n(x) = f(x)$ or

$f_n(x) \longrightarrow f(x)$ or

$f_n(x) \longrightarrow f(x)$ or

$f_n(x) \longrightarrow f(x)$ as $n \longrightarrow \infty$.

The notation $m(\epsilon, x)$ means that pointwise convergence of (f_n) to the limit function depends on both ε and

FINITION: A sequence of functions $\{f_n\}$ is said to converge uniformly on $[a,b]$ to a fun f defined on $[a,b]$ if for each $\epsilon >$ and $\forall x \in [a,b]$ $\exists$ a positive intege $m(\epsilon)$ (depending on ϵ) $\ni \forall x \in [a,b]$

$|f_n(x) - f(x)| < \epsilon \ \forall n \geq m(\varepsilon)$ .. ②

The function ② is called the uniform limit of f_n.

OTE => In a case of Point-wise convergen for each $\epsilon > 0$ and for each $x \in [a,b)$ $\exists$ Positive integer $m(\varepsilon, x)$ depending on bo ε and $x \ni$ equation ① holds i.e $|f_n(x) - f(x)| < \epsilon \ \forall n \geq m$. Wherease for uniform convergence, for each $\varepsilon > 0$, it is possible to find a positive integer $m(\varepsilon$ dependent only on ε which will suffice if $x \in [a,b]$.

It follows that every uniform convergence is Pointwise convergence. However, not Pointwise convergence implies not uniform convergence. This also means that Pointwise limit is the same as the uniform limit.

EFINITION: A sequence of functions $\sum_{n=1}^{\infty} f_n(x)$ is said to ~~be~~ convergent Pointwise to a function f defined on $[a,b]$ if its sequence of nth partial sum $\{S_n\}$ converges Pointwise to f.

$|S_n - f| < \epsilon$.

$\Rightarrow \left|\sum_{i=1}^{n} f_i(x) - f\right| < \epsilon$

Similarly, we define uniform convergence of series of functions.

10/1/2022.

FINITION: A sequence $\{x_n\}$ of numbers is said to Cauchy if for every $\epsilon > 0$, $\exists\, \alpha \in \mathbb{N} \ni$
$|x_n - x_m| < \epsilon,\ \forall\, m, n \geq \alpha$.

THEOREM: **CAUCHY CONVERGENCE CRITERION**

A sequence $\{x_n\}$ is convergent iff it
a Cauchy sequence.

CAUCHY CONVERGENCE CRITERION FO
SEQUENCE OF FUNCTIONS

HEOREM: A sequence of functions $\{f_n\}$ is converg
uniformly on $[a,b]$, iff for any $\epsilon>0$
for all $x \in [a,b]$, there exists a natur
number $\lambda(\epsilon)$ such that

$$|f_n(x) - f_m(x)| < \epsilon, \ \forall \ n,m \geq \lambda(\epsilon) \ldots$$

PROOF: **Necessary** $\Rightarrow$ Suppose $\{f_n\}$ converge
uniformly on $[a,b]$ to the limit functi
This means for every $\epsilon>0$ and for al
$x \in [a,b]$, $\exists$ a positive integers $\eta_1(\epsilon), \eta_2$

$$|f_n(x) - f(x)| < \epsilon/2 \quad \forall \ n \geq \eta_1(\epsilon) \ldots$$

$$|f_m(x) - f(x)| < \epsilon/2 \quad \forall \ m \geq \eta_2(\epsilon) \ldots$$

Let $\lambda(\epsilon) = \max(\eta_1(\epsilon), \eta_2(\epsilon))$

Then for the given $\epsilon>0$ and for all x
we have

$$|f_n(x) - f_m(x)| = |f_n(x) - f(x) + f(x) - f_m$$
$$\leq |f_n(x) - f(x)| + |f_m(x) - f$$
$$< \epsilon/2 + \epsilon/2 = \epsilon \ \forall \ n,m$$

sufficiency $\Leftarrow$ Assume that $\circledast$ holds then by cauchy convergence criterion, $\{f_n\}$ converges to f. (say) Pointwise To see that the convergence is uniform, from $\circledast$ let n be fixed and $m \to \infty$ Then for the given $\epsilon > 0$, and for all $x \in [a,b]$, we have

$$|f_n(x) - f(x)| < \epsilon \quad \forall\, n \geq \lambda(\epsilon)$$

$$\Leftrightarrow f_n \xrightarrow[n \to \infty]{\text{conv. uniformly}} f \qquad \square$$

Example 1: Test for uniform convergence of the seq. $\{f_n\}$, where $f_n(x) = \frac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$.

Solution.

First, we show that the sequence converges Pointwise. for this we see that $\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{nx}{1+n^2x^2} = 0 = f(x)$. That is, $f_n(x) \xrightarrow[n \to \infty]{} f(x) = 0$ on $\mathbb{R}$.

For uniform convergence, let $\epsilon > 0$ be given then for all $x \in [a,b]$ we need to show that $\exists$ a +ve number $\Omega(\epsilon)$ such that

$$|f_n(x) - f(x)| < \epsilon, \ \forall\, n \geq \Omega(\epsilon) \text{ - - - ①}$$

Consider the LHS of - - - ①

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+n^2x^2} - 0 \right| = \frac{nx}{1+n^2x^2} < \frac{nx}{n^2x^2}$$

$< \epsilon$, Provided $n > \frac{1}{\epsilon x} = \Omega(\epsilon, x)$.

Obviously, for $x = 0 \in \mathbb{R}$, $\Omega(\epsilon, 0) = \frac{1}{0} \rightarrow$

This means that we cannot find an $\Omega(\epsilon)$ that depends only on $\epsilon \ni$ ① is vali

Hence, the sequence does not converge un formly for all $x \in \mathbb{R}$.

NOTE ⇒ Every uniform convergence implies poi wise convergence but not conversely. However, not pointwise convergence impli not uniform convergence.

But not all [illegible]

2. Show that the sequence $\{f_n\}$ where $f_n(x) = 1/x+n$ is uniformly convergent i the interval $[0, b]$, $b > 0$

Solution.

First, for pointwise limit,

$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1}{x+n} = 0 = f(x)$

$\Rightarrow f_n(x) \xrightarrow[n \to \infty]{\text{Pointwise}} f(x) = 0$ on $[a, b]$

For uniform convergence, given any ε and $\forall x \in [a, b]$, we need to find a num $\omega(\epsilon) > 0 \ni$

$|f_n(x) - f(x)| < \epsilon, \forall n \geq \omega(\epsilon)$ ... ①

From the L.H.S of ①

$|f_n(x) - f(x)| = \left|\frac{1}{x+n} - 0\right| = \frac{1}{x+n} < \frac{1}{n} < \epsilon,$

$n > 1/\epsilon = \omega(\epsilon)$

$\therefore f_n(x) \longrightarrow f(x) = 0$ uniformly as $n \to \infty$ on $[a,b]$.

3. Show that the seq. $\{f_n\}$ where $f_n(x) = x^n$ is uniformly convergent on $[0,c]$, $c < 1$ and only pointwise on $[0,1]$.

Solution.

For Pointwise limit, we see that

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

We see that $f_n(x)$ converges pointwise to a discontinous function $f(x)$.

For uniform continuity, we consider for $0 < x \le c < 1$, we have

$|f_n(x) - f(x)| = |x^n - 0| = x^n < \epsilon$

if $\left(\frac{1}{x}\right)^n > 1/\epsilon$

if $n \log\left[\frac{1}{x}\right] > \log(1/\epsilon)$

if $n > \dfrac{\log(\frac{1}{\epsilon})}{\log(\frac{1}{x})}$

Notice that the maximum value of $\dfrac{\log(1/\epsilon)}{\log(1/x)}$ is $\dfrac{\log(1/\epsilon)}{\log(1/c)} = \epsilon = m(\epsilon)$

$\Rightarrow$ for the given $\epsilon > 0$ and $\forall\ x \in (0,1)$

$\exists\ m(\epsilon) \ni |f_n(x) - f(x)| < \epsilon,\ \forall\ n \geq \epsilon$

$$\Rightarrow \frac{n(1-nx^2)}{(1+n^2x^2)^2} = 0$$

$\Rightarrow n(1-n^2x^2) = 0 \Rightarrow x = 1/n$

$\therefore \text{Max} f_n(x)\big|_{x=1/n} = \dfrac{n(1/2)}{1+n^2(1/2)^2} = 1/2$

$M_n = \sup |f_n(x) - f(x)|$

$= \sup\limits_{x\in[a,b]} \left| \dfrac{nx}{1+n^2x^2} - 0 \right| = \sup\limits_{x\in[a,b]} \left| \dfrac{nx}{1+n^2} \right.$

$= \dfrac{1}{2} \begin{pmatrix} n \longrightarrow \infty \\ \not\rightarrow 0 \end{pmatrix}.$

Hence, $\{f_n\}$ does not converge unifo mly on any interval $[a,b]$ containing ze

2 Prove that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{x}{1+nx^2}$; $x \in \mathbb{R}$ converges uniform on a closed interval D of $\mathbb{R}$.

Solution

The pointwise limit function $f(x) = \lim\limits_{n\to\infty} f(x)$ $= \lim\limits_{n\to\infty} \dfrac{x}{1+nx^2} = 0$

It is easy to see that the functio attains it's maximum value $\dfrac{1}{2\sqrt{n}}$ at $x = \dfrac{1}{\sqrt{n}}$. Therefore,

$M_n = \sup\limits_{x\in D} |f_n(x) - f(x)| = \sup\limits_{x\in D} \left| \dfrac{x}{1+nx^2} \right| =$

$\therefore \lim_{n\to\infty} M_n = 0 \Rightarrow f_n(x) \xrightarrow[n\to\infty]{} f(x)$ on D by M_n

EXERCISE.

1. Show that the following sequences are not
formly convergent on the indicated interval
i. $\{u^n\}$ on $[0,1]$ ii) $\{e^{-nx}\}$ on $[0,k]$, $k>0$.

2. Test the following sequence for uniform co
i. $\left\{\frac{\sin(nx)}{\sqrt{n}}\right\}$, $0 \leq x \leq 2\pi$ ii. $\left\{\frac{x}{n+x}\right\}$, $0 \leq x$ $k>$
iii. $\left\{\frac{k}{n+x}\right\}$, $0 \leq x \leq \infty$, iv. $\left\{\frac{n^2 x}{1+n^3x^2}\right\}$, $0 \leq x \leq 1$

TESTS FOR UNIFORM CONVERGENCE OF SERIES OF FUNCTIONS

REM: (Weierstrass M-test): A series of func
$\sum f_n$ will converge uniformly on $[a,b]$ if there
exists a convergent series of +ve numb
$\sum M_n$ if $\forall x \in [a,b]$,
$|f_n(x)| \leq M_n$, $\forall n$.

of: let $\epsilon > 0$ be given. Since $\sum M_n$ converge
$\Rightarrow \lim_{n\to\infty} M_n = 0$. Hence, for the given $\epsilon > 0$
then we can find a natural number λ
$\ni |M_{n+1} + M_{n+2} + \cdots + M_{n+p}| < \epsilon$, $\forall n >$
$p \geq 1$. - - ①

Now, for the given $\epsilon > 0$ and $\forall\ x \in [a$

$|f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)| \leq$

$|f_{n+1}(x)| + |f_{n+2}(x)| + \cdots + |f_{n+p}(x)| \leq$

$M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \epsilon, \forall n \geq \lambda(\epsilon)$.

$\Rightarrow \sum f_n$ converges uniformly on $[a, b]$.

REMARK 1. The **converse** of the above theorem is not always true, i.e non-convergence of $\sum M$ does not imply non-uniform convergence of

2. Series that satisfy Weierstrass M-tes are sometimes called **normally converge** series to emphasize the fact that such series are both uniformly and absolut convergent.

Example: Test for uniform convergence, the series

i. $\sum r^n \cos(n\theta)$, ii. $\sum r^n \sin(n\theta)$

iii. $\sum r^n \sin(a^n\theta)$, $0 < r < 1$.

Solution:

i. Let $\sum f_n = \sum r^n \cos(n\theta)$

Now, $|f_n(x)| = |r^n \cos(n\theta)| \leq r^n = M_n$.

i.e $\sum M_n = \sum r^n$

We see that $\lim_{n\to\infty} r^n = 0 \ [\because 0 < r < 1]$

$r \in (0,1) \Rightarrow r = \frac{1}{k}, k > 1 \Rightarrow r^n = \left(\frac{1}{k}\right)^n = \frac{1^n}{k}$

$\Rightarrow r^n = 1/k^n$

$\Rightarrow \lim_{n\to\infty} r^n = \lim_{n\to\infty} \frac{1}{k^n} = \frac{1}{\lim_{n\to\infty} k^n} = \frac{1}{\infty} = 0.$

It is easy to see that by cauchy root test, $\sum r^n$ is convergent. Hence, $\sum r^n \cos(n\theta)$ converges uniformly by Weierstrass M-te

for ii, iii, the whole idea is the same.

Example: Test for uniform convergence the series

i. $\sum \frac{\sin(x^2+n^2x)}{n(n+1)}$, $x \in \mathbb{R}$

ii. $\sum \frac{(n\theta)}{n^p}$, $p>1$.

iii. $\sum \frac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$, $p>1$.

Solution

i. Let $\sum f_n(x) = \sum \frac{\sin(x^2+n^2x)}{n(n+1)}$

Now, $|f_n(x)| = \left|\frac{\sin(x^2+n^2x)}{n(n+1)}\right| \leq \frac{1}{n(n+1)} < \frac{1}{n^2}$

$= M_n$

i.e $\sum M_n = \sum 1/n^2$

clearly, $M_n \xrightarrow{n\to\infty} 0$ and $\sum 1/n^2$ conver by ratio test. Therefore, $\sum f_n$ converg uniformly for all $x \in \mathbb{R}$.

For ii, iii, the whole idea is the sam

Example: Show that the series $\sum \frac{1}{n^p+x^2n^q}$ conve uniformly over any finite interval $[a,b]$ f $p>1$, $q \geq 0$

3/7

Solution.

Let $\sum f_n(x) = \sum \frac{x}{n^2+x^2n^2}$

Now, $|f_n(x)| = \left|\frac{x}{n^2+x^2n^2}\right| \le \left|\frac{x}{n^2}\right| = \frac{|x|}{n^2} \le$

Obviously, $\lim_{n\to\infty} \frac{b}{n^2} = 0$; and the series $b\sum\frac{1}{n^2}$ is convergent, being a p-series with $p>1$. Therefore, $\sum\frac{x}{n^2+x^2n^2}$ is uniform convergent on $[a,b]$ by Weierstrass M-

02|2022

EFINITION: A sequence $\{x_n\}$ of numbers is said to be **monotonic increasing** if $x_{n+1} \geq x_n$, ∀ And is said to be monotonic ~~in~~decreasing if $x_{n+1} \le x_n, \forall n$. (x_n) is said to be monoto if it is either monotonic increasing or decreasing.

EFINITION: A sequence (f_n) of functions is said to **uniformly bounded** on $[a,b]$ if $\exists$ a numbe $\eta \in \mathbb{N} \ni \forall x \in [a,b]$ and $n \in \mathbb{N}$,

$|f_n(x)| \le \eta$.

EFINITION: A function $f_n(x)$ is said to be **positive** $f_n(x) \ge 0, \forall x$.

LEMMA: (ABEL'S LEMMA): If $\{b_n\}$ is a positiv monotone decreasing sequence and l, g denote respectively the least and the atest values of the sums $\sum_{r=m}^{p} u_r$, where $p = m, m+1, m+2, \cdots n$, then

$b_m \cdot l \leq \sum_{r=m}^{p} u_r \leq \sum_{r=m}^{p} u_r \leq L_m g]$.

THEOREM: [ABELS THEOREM]: If $b_n(x)$ is a Positiv monotone decreasing function of n and $x \in [a,b]$, and if $b_n(x)$ is uniformly bou on $[a,b]$ and if the series $\sum U_n(x)$ converg uniformly on $[a,b]$, then the series $\sum b_n(x) U_n(x)$ converges uniformly on $[a,b]$

PROOF: Since the function $b_n(x)$ is bounded fo all n and $x \in [a,b]$, this means there exist a number $g > 0$ ∋

$0 \leq B_n(x) \leq g$, $\forall n$ and $\forall x \in [a,b]$

Again, since $\sum u_n(x)$ is uniformly conve gent on $[a,b]$, then it's nth Partial sum $\sum_{r=n+1}^{n+p} U_r(x)$ is convengent. This means giv any $\epsilon > 0$, ∃ a number α ∋

$$\left| \sum_{r=n+1}^{n+p} U_r(x) \right| < \epsilon, \ \forall n \geq \alpha, \ p \geq 1 \cdots ①$$

Now, by Abel's Lemma,

$$\left| \sum_{r=n+1}^{n+p} b_r(x) U_r(x) \right| \leq b_{n+1}(x)\, g \longrightarrow \left| \sum_{r=n+1}^{n+q} U_r \right.$$

$$= b_{n+1}(x) \max_{q=1,2,\ldots p}$$

$$\leq g \frac{\epsilon}{g} = \epsilon$$

=> $\sum b_n(x)U_n(x)$ converges uniformly o
[a, b]
$\square$

Example: Determine whether the series $\sum \frac{(-1)^n}{n}|x|^n$ i
uniformly convergent on [-1,1] by using Abe
theorem.

Solution

Since $b_n(x) = |x|^n$ is positive monotonic n
decreasing and uniformly bounded on [-1,1]
and the series $\sum U_n = \sum \frac{(-1)^n}{n}$ converges
uniformly on [-1,1], it follows from Abel
theorem that $\sum \frac{(-1)^n}{n}|x|^n$ converges unifor
ly on [-1,1]

Example: Show that $\sum \frac{b_n}{n^x}$ converges uniformly
[0,1] if $\sum b_n$ is convergent.

Solution.

Since $b_n(x) = \frac{1}{n^x}$ is positive monoton
decreasing and uniformly bounded for
n and $x \in [0,1]$ and since $\sum b_n$ conver
it is uniformly convergent. It follows th
fore from Abels theorem that $\sum \frac{b_n}{n^x}$ co
rges on [0,1].

EXERCISE

If $\sum a_n$ is convergent, then use Abel theorem to show that each of the followin series is uniformly convergent on $[0,1]$:

i. $\sum t_n x^n$ ii. $\frac{t_n x^n}{1+x^n}$ iii. $\sum \frac{t_n x^n}{1+x^{2n}}$

iv. $\sum \frac{nx^n(1-x)t^n}{1+t^n}$ v. $\sum \frac{2nt_n x^n(1-x)}{1+x^{2n}}$.

10/06/22

THEOREM: (DIRICHLET'S TEST): If $b_n(x)$ is a monotone function and tends uniformly to zero on $[a,b]$ and if $\exists$ a number $k>0$ independent of n and x such that

$$\left|\sum_{r=1}^{n} U_r(x)\right| \leq k, \forall n,$$

then the series $\sum b_n(x)U_n(x)$ converges uniformly on $[a,b]$.

PROOF: (the whole idea can be followed from Abel's theorem).

Example: Prove that the series $\sum\frac{(-1)^n x^2+n}{n^2}$ ~~$\sum b_n(x)U_n(x)$~~ converges uniformly in ~~$[a,b]$~~ every bounded interval but does not converge absolutely, using Dirichlet's test.

Solution

Let $U_n = (-1)^n, n \in \mathbb{N}$. clearly, $\exists$ $k=1$ $\ni$

$$\left|\sum_{r=1}^{n} U_r(x)\right| = 1$$

Taking $b_n(x) = \frac{x^2+n}{n^2}$ and let D be bounde interval. This implies that $\exists\ K > 0$ such that $|x| < K,\ \forall\ x \in D$. Then, we can see that

$$b_n(x) = \frac{x^2+n}{n^2} < \frac{K^2+n}{n^2}$$

is a monotic decreasing function and tends uniformly to 0. Hence by Dirichlet's test, the series $\sum b_n(x) U_n(x)$ converges uniformly on D.

For absolute convergence, we see that

$$\left|\sum (-1)^n \frac{x^2+n}{n^2}\right| = \sum \frac{x^2+n}{n^2} \sim \sum \frac{1}{n}$$

which diverges. Hence the series

$\sum (-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely on any bounded interval.

Some Properties of Uniformly convergent series and functions

We shall learn that the sufficient condition for a limit function or series to enjoy/ inherit all the fundamental properties of a sequence of functions or series is that the convergence is uniform.

THEOREM: If a sequence $\{f_n\}$ converges uniformly on $[a, b]$ and x_0 is a point in $[a, b]$ such that

$\lim_{x \to x_0} f(x) = y_n$, $n \in N$, then

i. $\{y_n\}$ converges

ii. $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} y_n$.

PROOF :: i. let $\{f_n\}$ converges uniformly on $[a, b]$. Th means $\{f_n\}$ is a cauchy sequence. Hence, given any $\epsilon > 0$, we can find a rational nu ~~of~~ $\eta(\epsilon) \ni$

$$|f_n(x) - f_m(x)| < \frac{\epsilon}{2}, \ \forall \ n, m \geq \eta(\epsilon) \ldots$$

Since $\lim_{x \to x_0} f_n(x) = y_n$, $\forall n$, then letting $x \to$ in ①, we get

$|y_n - y_m| < \epsilon/2 < \epsilon$, $\forall \ n, m \geq \eta(\epsilon)$.

$\Rightarrow \{y_n\}$ is a cauchy sequences, and hence converges.

ii. Assume that $\{f_n\}$ converges uniformly on [This means for any $\epsilon > 0$, we can find an inte $\beta_1(\epsilon) \in N \ni$

$|f_n(x) - f(x)| < \epsilon/3$, $\forall \ n \geq \beta_1(\epsilon)$ . . . . ②

Since $\{y_n\}$ converges to P, then for a giv $\epsilon > 0$, $\exists$ a number $\beta_2(\epsilon) > 0 \ni$

$|y_n - P| < \epsilon/3$, $\forall \ n \geq \beta_2(\epsilon)$ . . . ③

Let $\alpha = \max(\beta_1(\epsilon), \beta_2(\epsilon))$.

By ~~hypho~~ hypothesis, $\lim_{x \to x_0} f_n(x) = y_n, \forall n$. Therefore,

$\lim_{x \to x_0} f_\alpha(x) = y_\alpha$. Hence, for any $\epsilon > 0, \exists\ \delta > $

$\ni$

$|x - x_0| < \delta \Rightarrow |f_\alpha(x) - y_\alpha| < \epsilon/3 \quad \ldots \quad (ii)$

Therefore, for the given $\epsilon > 0$, and $|x - x_0| <$
we see that $|f(x) - P| = |f(x) - f_\alpha(x) + f_\alpha(x) -$

$+ y_\alpha - P|$

$\leq |f(x) - f_\alpha(x)| + |f_\alpha(x) -$

$y_\alpha| + |y_\alpha - P|$

$< \epsilon/3 + \epsilon/3 + \epsilon/3 =$

[illegible]/2022

THEOREM: If a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a su
function f on $[a,b]$ and x_0 is a point in
$\ni \lim_{x \to x_0} f_n = y_n,\ n \in \mathbb{N}$, then

i. $\sum_{n=1}^{\infty} y_n$ converges

ii. $\lim_{x \to x_0} f_x = \sum_{n=1}^{\infty} y_n$

PROOF: (same idea with previous result).

REMARK: The consequence of the above theorem
that $\lim_{x \to x_0} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n$.

UNIFORM CONVERGENCE AND CONTINUI

Recall, A function f is said to be continuous a
$x_0 \in D(f)$ if given any $\epsilon > 0$, $\exists$ a $\delta_\epsilon > 0$

$|x_1 - x_0| < \delta_\epsilon$

$\Rightarrow |f(x) - f(x_0)| < \epsilon; \ \forall \ x \in D(f)$.

THEOREM: If $\{f_n\}$ is a sequence of continuous func ions on an interval $[a, b]$ and if $f_n \to f$ uniformly on $[a, b]$, then the sum functi f is also continuous on $[a, b]$.

PROOF: EXERCISE.

THEOREM: If a series $\sum f_n$ converges uniform to a sum function f on an interval $[a,b]$ and if f_n is continuous at $x_0 \in [a, b]$ for each n, then the sum fun ction f is also continuous at x_0.

PROOF: Since $\sum f_n$ converges uniformly to f o $[a, b]$, then given any $\epsilon > 0$, we can Produc an $7 \in \mathbb{N} \ni \forall x \in [a, b]$ ~~Produce an $7 \in \mathbb{N}$~~

$|\sum_{i=1}^{n} f_i - f(x)| < \epsilon/3 \ \forall \ n \geq 7 \cdots$ ①

In particular, for $x = x_0$ and $n = 7$, we get $|\sum_{i=1}^{n} f_i(x_0) - f(x_0)| < \epsilon/3 \cdots$ ②

Again, since $\sum f_n$ is continuous at x_0, the sum function $\sum_{i=1}^{n} f_i$ is continuou because finite sum of continuous func always continuous. Therefore, given any $\epsilon >$ there is a $\delta_0 > 0 \ni |x - x_0| < \delta_0$ gives

$$\left|\sum_{i=1}^{n} f(x) - \sum_{i=1}^{n} f_i(x_0)\right| < \epsilon/3 \quad \cdots ③$$

Now for the given $\epsilon > 0$, with $\delta_\epsilon > 0$, we see that $|x - x_0| < \delta_\epsilon$ yields,

$$|f(x) - f(x_0)| = \left|f(x) - \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} f(x_0) + \sum_{i=1}^{n} f(x_0) - f(x_0)\right|$$

$$\leq \left|f(x) - \sum_{i=1}^{n} f_i(x)\right| + \left|\sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} f_i(x_0)\right| + \left|\sum_{i=1}^{n} f_i (x_0) - f(x_0)\right|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.$$

i.e $|x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \epsilon \ \forall \ x \in [a, b]$.

This proves that f is continuous at $x_0 \in [$

Remark: The converse of the above theorem is n always true, i.e $\exists$ series or sequence continuous term which have a contin sum or limit but which are not unifo continuous.

However, if the sum or limit function not continuous, then convergence cann be uniform on the given interval.

Example: Show that the ff do not converge u rmly on the indicated interval converg uniformly on the indicated interval.

1. $\left\{\dfrac{nx}{1 + n^2x^2}\right\}$ on $[0, 1]$

ii. $\sum_{n=1}^{\infty}(1-x)x^n$ on $[0,1]$

Solution.

i. Clearly, $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

Now, $|f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2} - 0\right| = \frac{nx}{1+n^2x^2} \leq$

$= 1/nx$.

$= 1/nx < \epsilon$, provided $n > 1/\epsilon x = m($

Hence, we conclude that the sequence

not converge uniformly on $[0,1]$.

ii. Obviously, $f(x) = \begin{cases} 1, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

We see that $f(x)$ is not continuou

Hence the series cannot converge uni

ormly on $[0,1]$.

NOTE. There is a special class of sequence series of functions for which uniform convergence is equivalent to the continuit of the limit of sum function. In this co ection. We have a theorem due to an Ita mathematician called Dini.

HEOREM [DINI'S THEOREM ON UNIFORM CONV.

If a sequence of continuous functi $\{f_n\}$ defined on $[a,b]$ is monotonic incre sing and converges pointwise to a contin

function f, then the convergence is unif
rm.

THEOREM [DINI'S THEOREM ON UNIFORM CONVERG
OF A SERIES OF FUNCTIONS]

If the sum function f of a series Σf_n with non-negative continuous terms defin on an interval $[a,b]$ is continuous, the the series is uniformly convergence.

REMARK. If the pointwise limit or sum functi is not continuous then the convergen cannot be uniform.

Example. Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x}$
$+ \frac{x^4}{(1+x^4)^3} + \cdots$

$$\Sigma f_n = \sum_{n=0}^{\infty} \frac{x^4}{(1+x^4)^n}$$

$a = x^4$, $r = \frac{1}{1+x^4} < 1$ for $x \neq 0$

$$f(x) = S_\infty = \frac{a}{1-r} \quad \cancel{\frac{x^4}{1-\frac{1}{1+x^4}}} = \frac{x^4}{1-\frac{1}{1+x^4}} = 1$$

for $x = 0$, $\Sigma f_n = 0 = f(x)$

$$\therefore f(x) = \begin{cases} 1+x^4 & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

We see that the sum function is not continuous on $[0,1]$. Hence, the s can not converge uniformly on $[0,1]$.

Example. Show that $\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$ is uniform convergent on any interval $[a,b]$ $0<a<b$ but only Pointwise on $[0,b]$.

~~Solution~~ Solution.

Let $f_n(x) = \frac{x}{(nx+1)\{(n-1)x+1\}}$

$= \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$ (the said part fraction)

The nth partial sum of $\sum f_n$ is give

$$S_k(x) = \sum_{n=1}^{k} f_n(x) = \sum_{n=1}^{k}\left[\frac{1}{(n-1)x+1} - \frac{1}{nx+1}\right].$$

$$= \left(1 - \frac{1}{x+1}\right) + \left(\frac{1}{x+1} - \frac{1}{2x+1}\right) + \left(\frac{1}{2x+1} - \frac{1}{3x+1}\right) + \cdots + \left(\frac{1}{(k-1)(x+1)} - \frac{1}{kx+1}\right).$$

$$= 1 - \frac{1}{kx+1}$$

$$\therefore \lim_{k\to\infty} S_k(x) = \lim_{k\to\infty}\left[1 - \frac{1}{kx+1}\right] = 1 = f(x)$$

$$f(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

$\therefore$ The sum function is discontinuous on $[0, b]$ and therefore, convergence can not be uniform on $[0,b]$. i.e $\sum f_n$ is only pointwise on $[a,b]$. Now for $x \neq 0$ i.e $x \in [a,b]$ with $0 < a < b$, let $|S_n(x) - f(x)| = \left|\left(1 - \frac{1}{nx+1}\right) - 1\right| = \frac{1}{nx+1} < \frac{1}{nx} < \epsilon$ provided $n > 1/\epsilon x$.

Observe that $\frac{1}{\epsilon x}$ decreases with x. Let it's minimum be $1/a\epsilon = \alpha$ in $[a,b]$. Therefore, for all $x \in [a,b]$ $\exists$ an $\alpha \in \mathbb{N}$ such that $|S_n(x) - f(x)| < \epsilon$, $\forall$ $n \geq \alpha$.

Hence, the series converges uniformly on $[a,b]$, $0 < a < b$.

1/01/2023

efinition Let $[a,b]$ be a given closed interval. A Partition of $[a,b]$ is a finite set P of number $x_0, x_1, x_2, \cdots x_n$, where $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$.

The intervals $[x_0, x_1], [x_1, x_2], \cdots [x_{n-1}, x_n]$ are the subintervals of $[a,b]$. Note that the lengths of the interval $[a,b]$ is given by $L([a,b]) = b - a$

We denote the lengths of the subintervals of $[a,b]$ by Δx_i ($i = 1, 2, \cdots n$) that is,

$\Delta x_i = x_i - x_{i-1} \; (i = 1, 2, 3, \dots n)$.

Let f be a bounded function on [a,b] and Mi, mi denote respectively the sup mum and the infimum of f on [a, b]. Then con der the sums

$U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ ... ①

$L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$ ... ②

The sums in ① and ② are respectively called the upper and the lower daboux su of f with respect to the Partition P on [a

finition: For any Partition P of [a,b], the length the maximum sub-interval is called the m or the norm of the Partition P, and is denoted by $\mu(P)$.

That is,

$$\mu(P) = \max_{1 \le i \le n} \Delta x_i$$

$$= \max\{x_i - x_{i-1}, 1 \le i \le n\}$$

e.g. $\max\{1, 3, 8, 10, 0, 12\} = 12$.

orem (Necessary and sufficient conditions for integ ility).

A bounded function f on [a,b] is integra on [a,b] iff $\forall\, \epsilon > 0$, $\exists$ a Portion P

$[a,b] \ni$

$U(P,f) - L(P,f) < \epsilon$ ...... ④

By ②, $\lim (U(P,f) - L(P,f)) = 0$

Theorem: If a sequence $\{f_n\}$ of functions converge uniformly on $[a,b]$ to a limit function f, an each f_n is integrable on $[a,b]$ then the limi function is integrable on $[a,b]$ and the sequence $\{\int_a^x f_n\,dt\}$ converges uniformly to $\{\int_a^x f\,dt\}$.

That is,

$$\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt, \forall x \in [a,b]$$

Proof: Suppose that $\{f_n\}$ converges uniformly to f on $[a,b]$. Then, by defination, for every $\epsilon >$ and for all $x \in [a,b]$, we can find a positiv integer $2 \ni$

$|f_n(x) - f(x)| < \epsilon/3(b-a), \forall n \geq 2$ ...... ①

In particular, for $n = 2$, we get

$|f_2(x) - f(x)| < \epsilon/3(b-a)$ ...... ②

For this fixed 2, and since f_n is integrab we can choose a Partition P of $[a,b] \ni$

$U(P,f_2) - L(P,f_2) < \epsilon/3$ ...... ③

from ② we can write

$$-\frac{\epsilon}{3(b-a)} < f_2(x) - f(x) < \frac{\epsilon}{3(b-a)} \cdots$$

Now, from LHS of ④, we have

$$-\frac{\epsilon}{3(b-a)} < f_2(x) - f(x) \Rightarrow f(x) - \frac{\epsilon}{3(b-a)} < f_2$$

$$\Rightarrow f(x) < f_2(x) + \frac{\epsilon}{3(b-a)} \quad \cdots ⑤$$

$$\Rightarrow U(P,f) < U(P,f_2) + \epsilon/3 \quad \cdots ⑥$$

Again, from RHS of ④, we get

$$f_2(x) - f(x) < \frac{\epsilon}{3(b-a)}$$

$$\Rightarrow f_2(x) - \frac{\epsilon}{3(b-a)} < f(x).$$

$$\Rightarrow f(x) > f_2(x) - \frac{\epsilon}{3(b-a)}.$$

$$\Rightarrow L(P,f) > L(P,f_2) - \epsilon/3 \quad \cdots ⑦.$$

from ⑥ and ⑦, we have

$$U(P,f) - L(P,f) < U(P,f_2) - L(P,f_2) +$$

$$\Rightarrow U(P,f) - L(P,f) < \epsilon/3 + 2\epsilon/3 = \epsilon$$

$\therefore$ f is integrable on $[a, b]$.

Now, to show that $\int_a^x f\,dt = \lim_{n\to\infty} \int_a^x f_n\,dt$

Let $\{f_n\}$ converges uniformly to f on $[a,b$

$\Rightarrow$ For every $\epsilon > 0$, we can find a positive integer $2 \ni \forall x \in [a,b]$,

$|f_n(x) - f(x)| < \epsilon/3(b-a)$, $\forall n \geq 2$. ⑧

Then, for all $x \in [a,b]$ and for $n > 2$ we get

$$\left|\int_a^x f\,dt - \int_a^x f_n\,dt\right| = \left|\int_a^x (f\,dt - f_n\,dt)\right|$$

$$\leq \int_a^x |f - f_n|\,dt .$$

$$\leq |f - f_n|\,(x-a)$$

$$\leq |f - f_n| \sup_{x \in [a,b]} (x-a)$$

$$\leq |f - f_n)\,(b-a)$$

$$\leq \frac{\epsilon}{3(b-a)}\,(b-a) = \frac{\epsilon}{3}$$

That is, $\left|\int_a^x f\,dt - \int_a^x f_n\,dt\right. < \epsilon$, $\forall x \in$

$\Rightarrow \lim_{n\to\infty} \int_a^x f_n\,dt = \int_a^x f\,dt$.

□

REMARK: The converse of the above theorem need not be true.

That is, if the limit function f is integra the sequence of function ~~f~~ $\{f(x)\}$ is not necessarily uniformly convergent.

Note, however that if the limit functio f is not integrable then the convergence of $\{f_n\}$ cannot be uniform.

2|14|2022

THEOREM: If a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to a li function f on $[a,b]$ and each f_n is integrable, each n, then f is also integrable on $[a,b]$, a $\sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right)$ converges uniformly to $\int_a^x f\,dt$. Th is. $\int_a^x f\,dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right)$

$$= \int_a^x f_1\,dt + \int_a^x f_2\,dt + \int_a^x f_3\,dt + \cdots$$

In this case, we say that the series $\sum_{n=1}^{\infty} f_n\,dt$ is term-by-term integrable.

ROOF: (the whole idea is the same when the series replaced with $\{f_n\}_{n\in\mathbb{N}}$.

EMARK: The converse of the above theorem is not always true. That is, $\sum f_n$ may converge to integrable limit f, but the convergence of Σ may not be uniform. But, if the pointwise li f is not integrable or integrable the integr not equal to the sum of the series, then term integral is not possible and hence the co gence is not uniform.

Example: Show that for the series $1-x+x^2-x^3+x^4+\cdots$
$=\frac{1}{1+x}, 0\leq x\leq 1, \cdots$ ①

term-by-term integration is possible bu
does not converge uniformly.

Now, integrating the R.H.S of ① over [0
to have

$\int_0^1 \frac{1}{1+x}dx = \ln(1+x)\Big|_{x=0}^{1} = \ln x$.

Again, integrate the LHS of ① over [0,
$\int_0^1[1-x+x^2-x^3+\cdots]dx = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

We know that $\ln 2 = 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$

$\therefore$ LHS = RHS $\Rightarrow$ term-by-term integ
tion is possible. However, it is clear tha
the series does not converge uniformly on

Example: Show that the sequence $\{f_n\}$ where
$f_n(x) = nxe^{-nx^2}, n = 1, 2, 3\cdots$ cannot conve
uniformly on $[0,1]$

Solution: It is enough to show that
$\int_0^1 f(x)dx \neq \lim_{n\to\infty}\int_0^1 f_n(x)dx$.

Now, $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{-nx} = \lim_{n\to\infty}\frac{nx}{e}$
i.e $f(x) = 0$. We see that
$\int_0^1 f(x)dx = \int_0^1 0dx = 0$

$\int_0^1 f_n(x)dx = \int_0^1 nxe^{-nx^2}dx = \frac{1}{2}(1-e^{-n})$

$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)dx$.

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \frac{1}{2}(1-e^{-n}) = 1/2.$

$\therefore \int_0^1 f(x)\,dx = 0 \neq 1/2 = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx.$

Hence, the sequence does not converge ni
mly.

16/04/2028. UNIFORM CONVERGENCE AND DIFFERENTIAB

Recall (Lagrange's mean value Theorem | First Mea
value theorem):

If a function f defined on $[a,b]$ is:

(i.) continuous on $[a,b]$

(i₂) differentiable on (a,b),

then $\exists$ a real number $c \in (a,b) \ni$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Recall: A function f is said to be differentia
at c if $f'(c) = \lim_{x\to c} \frac{f(x) - f(c)}{x - c}$.

THEOREM: Let $\{f_n\}$ be a sequence of differentiable fun
on $[a,b] \ni$ it converges at least at one poi
x_0 in $[a,b]$. If the sequence of differentiable
$\{f_n'\}$ converges uniformly to a limit function
on $[a,b]$, then the given sequence $\{f_n\}$ con
uniformly to a function f on $[a,b]$ and
$f'(x) = \varphi(x), \forall x \in [a,b]$.

PROOF: Let $\epsilon > 0$ be given. By the convergence of ~~$\epsilon>0$.~~ $\{f_n(x_0)\}$ and $\{f_n'\}$, then for a given $\epsilon>0$, we can find a natural number $N \ni \forall x \in [a,b]$, we have

$|f_n(x_0) - f_m(x_0)| < \epsilon/2, \; \forall n,m \geq N$ ... ①

and $|f_n'(x) - f_m'(x)| < \frac{\epsilon}{2(b-a)}, \; \forall n,m \geq N$ ... ②

Since $(f_n - f_m)$ is differentiable and hence continuous on $[a,b]$, then by the Lagrange's MVT, for any two points $x,t \in [a,b]$, $\exists$ a real number $c \in (x,t) \ni$

$$\frac{|(f_n(x) - f_m(x)) - (f_n(t) - f_m(t)|}{|x-t|} = |f_n'(c) - f_m'(c)|$$

$$\Rightarrow |f_n(x) - f_m(x) - f_n(t)' + f_m'(t)| = |x-t||f_n'(c) - f_m'(c)|$$

$$\leq \sup_{x,t\in[a,b]} |x-t| |f_n'(c) - f_m'(c)|$$

$$= |b-a||f_n'(c) - f_m'(c)|$$

Hence, using ① and ②, we get, $< |b-a| \frac{\epsilon}{2(b-a)} = \epsilon/2$ ... ③

$|f_n(x) - f_m(x)| = |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0) + f_n(x_0) - f_m(x_0)|$

$\leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)| < \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow$ The sequence $\{f_n\}$ converges uniformly on $[a,b]$ to a function f^n (say).

Now, for $x \in [a,b]$, consider the auxili
functions defined as follows:

$\phi_n(t) = \dfrac{f_n(t) - f_n(x)}{t-x}, x \neq t$ ... ④

$\phi(t) = \dfrac{f(b) - f(x)}{t-x}, x \neq t$ ... ⑤

Since f_n is differentiable for each
then from ④, we see that

$\lim_{t \to x^n} \phi t = \lim_{t \to x} \dfrac{f_n(t) - f_n(x)}{t-x} = f_n'(x)$ ..

$\therefore |\phi_n(t) - \phi_n(t)| = \dfrac{|f_n(t) - f_n(x) - f_m(t) + f_m(}{|t-x)}$

$< \epsilon/2(b-a) < \epsilon/2 < \epsilon, \forall n,m$

$\Rightarrow \{\phi_n'(t)\}$ converges uniformly on $[a,b]$.

Since $\{f_n\}$ also converges uniformly on [
to f, it follows from ④ that

$\lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \dfrac{f_n(t) - f_n(x)}{t-x} = \dfrac{f(t) - f(x)}{t-x}$

$= \phi(t)$.

Therefore, $\{\phi_n(t)\}$ converges to $\phi(t)$

Now, recall that if a sequence $\{f_n\}$ con
uniformly to f [a,b] and $x_0 \in [a,b]$ ⇒
$\lim_{x \to x_0} f_n(x) = \lim_{n \to \infty} y_n$.

Applying this result to the uniformly converg sequence $\{\phi_n(t)\}$ and using ⑥, we get

$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x) = P(x)$ ... ⑦

$\Rightarrow \lim_{t \to x} \phi(t)$ exists.

Hence, from ⑤, we obtain

$\lim_{t \to x} \phi(t) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$ ... ⑧

That is, f is differentiable.

Thus, by uniqueness of limit, it follows f

② and ⑧ that $f'(x) = P(x)$. $\square$

THEOREM: If a series $\sum f_n$ of differentiable fun

converges pointwise to f on $[a,b]$ and a

f_n' is continuous and the series $\sum f_n'$ con

uniformly to P on $[a,b]$, then the given s

converges uniformly to P on $[a,b]$. Theref

the sum function P is also continuous

$[a,b]$. Consequently,

$\int_a^x P(t)\,dt$ is differentiable and

$\frac{d}{dx}\int_a^x P(t)\,dt = P(x) \ \forall x \in [a,b]$ ... ①

For every $x \in [a,b]$, let $f(x) = \sum f(x)$ ..

Since each ~~uniform~~ function f_n', being

tinous is integrable on $[a,b]$, then by

fundamental theorem of calculus,

$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a), \ \forall x \in [a,b]$

$\therefore \sum_{n=1}^{\infty} f_n'(t)\,dt = f(x) - f(a), \ \forall x \in [a,b] \cdots ③$

Again, since the series $\sum f_n'$ of integrab functions converges uniformly ~~on~~ to p on $[a,b]$. therefore, term by term integration is valid.

i.e $\int_a p(t)\,dt = \sum_{n=1}^{\infty} \int_a^x f_n'(t)\,dt, \ \forall x \in [a,b]$.

$= f(x) - f(a) \cdots ④$

$\frac{d}{dx}\int_a^x p(t)\,dt = \frac{d}{dx}\left[f(x) - f(a)\right]$

$\Rightarrow p(x) = f'(x) - 0 = f'(x), \ \forall x \in [a,b]$.

or equivalently $\frac{d}{dx}\sum_{n=1} f_n(x) = \sum \frac{d}{dx} f_n(x)$

i.e term-by-term differentiation is valid. □

xample: Show that the sequence $\{f_n\}$ were

$$f_n(x) = \begin{cases} n^2x & 0 \le x \le 1/n \\ -n^2x + 2n, & 1/n \le x \le 2/n \\ 0 & 2/n \le x \le 1. \end{cases}$$

is not uniformly convergent on $[0,1]$.

Solution.

For all $x \in [0,1]$, we have

$f(x) = \lim_{n\to\infty} f_n(x) = 0$.

Observe that each function f_n and f are continuous on $[0,1]$. Also,

$\int_0^1 f_n(x) = \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n}(-n^2x + 2n)\,dx + \int_{2/n}^1 0\,dx = 1$

But, $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$.

$\therefore \lim_{n\to\infty}\int_0 f_n(x)\,dx \neq \int_0^1 f(x)\,dx$.

Hence, the sequence $\{f_n\}$ cannot converge un

$\therefore \sum_{n=1}^{\infty} f_n'(t)dt = f(x) - f(a), \ \forall x \in [a,b] \cdots$ ③

Again, since the series $\sum f_n'$ of integrabl functions converges uniformly ~~on~~ to p on $[a,b]$. therefore, term by term integration is valid.

i.e $\int_a p(t)dt = \sum_{n=1}^{\infty} \int_a^x f_n'(t)dt, \ \forall x \in [a,b].$

$= f(x) - f(a) \cdots$ ④

$\frac{d}{dx}\int_a^x p(t)dt = \frac{d}{dx}[f(x) - f(a)]$

$\Rightarrow p(x) = f'(x) - 0 = f'(x), \ \forall x \in [a,b].$

or equivalently $\frac{d}{dx}\sum_{n=1} f_n(x) = \sum \frac{d}{dx} f_n(x)$

i.e term-by-term differentiation is valid. □

xample: Show that the sequence $\{f_n\}$ were

$f_n(x) = \begin{cases} n^2x & 0 \le x \le 1/n \\ -n^2x + 2n, & 1/n \le x \le 2/n \\ 0 & 2/n \le x \le 1. \end{cases}$

is not uniformly convergent on $[0,1]$.

Solution.

For all $x \in [0,1]$, we have

$f(x) = \lim_{n\to\infty} f_n(x) = 0.$

Observe that each function f_n and f are continuous on $[0,1]$. Also,

$\int_0^1 f_n(x) = \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n}(-n^2x + 2n)dx + \int_{2/n}^1 0\,dx = 1$

But, $\int_0^1 f(x)dx = \int_0^1 0\,dx = 0.$

$\therefore \lim_{n\to\infty}\int_0 f_n(x)dx \ne \int_0^1 f(x)dx.$

Hence, the sequence $\{f_n\}$ cannot converge un

on $[a,b]$ if its total variation on $[a,b]$ is
That is, $V(f,a,b) < \infty$.

Recall, a function f on $[a,b]$ is calle
monotone if it is either increasing or de
asing. That is, f is monotonic increasing o
$[a,b]$ if $f(a) \leq f(b)$ whenever $a \leq b$. On the
flip side, f is monotonic decreasing if $a \leq b$
implies $f(a) \geq f(b)$.

mple 1. A bounded monotonic function is of bou
variation.

of: Let f be a monotonic increasing function o
$[a,b]$ and $P = \{a = x_0 \leq x_1 \leq \dots \leq x_n = b\}$ be
a Partition of $[a,b]$

Note that f being monotonic increasing on
$[a,b]$ implies that $f(a) \leq f(b)$. Now,

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = f(b) - f(a)$$

ie $$\sup_P \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} [f(b) - f(a)]$$
$$= f(b) - f(a) < \infty$$

ie The total variation w.r.t f,
$V(f,a,b) < \infty$. Hence, f is of boun
variation on $[a,b]$.

2. If the derivative f' of f exists and is bo
on $[a,b]$, then the function f is of bounde