1

FUNDAMENTAL CONCEPTS RELATED TO NUMERICAL ERRORS

Introduction, Type of Errors, Rate of Convergence & Propagation of Errors

INTRODUCTION :

Numeric means related to numbers. While dealing with complex numbers for computational and other similar purposes, we come across a number of errors which are, in most cases, unavoidable. For example, while dealing with a numerical problem through computer, there will occur an unavoidable error due to approximation if the numerical problem (concerned) contains quite a large number of digits as computer has a limit with regards to the storage of digits. Such errors need to be studied income detail and this chapter is purposefully aimed at achieving the same.

With regards to numbers. It is worthwhile to make a note of the term ***'slignificant figures'.*** The figures or digits which are used to express a number are called **signigicant figures or significant digits.** For e.g. the numbers 7.2896, 0.54547 and 8.0189 contain five significant digits each while the number 0.00013 contains, however only two significant digits. Viz., 1 and 3 since the zeros serve only to fix the position of the decimal point.

Now, let us march towards our main aim of studying numerical errors through this chapter as follows :-

VARIOUS TRYPES OF ERRORS :

Rounding Errors :

Rounding errors are unavoidable in most of the calculations since some of the quantities in the calculations are often non-terminating decimals and for practical reasons, only certain number of figures need to be carried in a calculation. Due to this, number have to be rounded-off, causing what are called **rounding errors.**

For example, consider an integer in base 10 namely 975423. When it is rounded to two figures, we obtain 98, 101. This is called a **rounded number** and it contains a certain proportion of rounding error.

Truncation errors :

Truncation errors are caused by the use of a closed form, such as the first few terms of an infinite series to express a quantity defined by limiting process. Such errors often appear when a definite integral is computed by Simpson's rule or when a caused but is mostly due to the method used in case of calculations.

Absolute, Relative and Percentage Errors :

The **absolute error** of number, measurement or calculation is the numerical difference between the true value of the concerned quantity and its approximate value as given or obtained by measurement or calculation.

The **relative error** is the absolute error divided by the true value of the quantity while the **percentage error** is 100 times the relative error.

Fog example, if a be the true value of a quantity and $a^{\cdot}$ its approximate value. then

$$\text{Absolute error} = a - a^{\cdot}$$

$$\text{Absolute error} = \frac{a - a^{\cdot}}{a}$$

$$\text{Percentage error} = 100 \times \frac{a - a^{\cdot}}{a}$$

If the first significant figure of a number is k_1 and the number is correct to n significant figure, then the relative error in this number is less than $\frac{1}{(k \times 10^{n-1})}$

Now, if the relative error in an approximate number is less than $\frac{1}{[(k + 1) \times 10^{n-1}]}$ then the number is correct to n significant figures.

Also, if the relative error of any number is not greater than $\frac{1}{(2 \times 10^{n})}$ the number is said to be correct to n significant figures.

Inherent Error – is that quantity which is already present in the statement of the problem before its solu.

A Special Mention of Rate of Convergence :

Numerical approximation are often arrived at by computing a sequence of approximations that get closer and closer to the desired answer. The order or rate of convergence of a sequence is analogous to the order of an approximation. It is therefore not surprising that many numerical methods for finding an answer, say of a given problem merely produce (the first few terms of) a sequence a_1, a_2, a_3, …. Which is shown to converge to the desired answer.

Basic concepts such as derivative, integral and continuity are defined in terms of convergent sequences. While elementary functions such as In χ or sin χ are defined by convergent series. At the same time, it is worthwhile to note that numerical answers to engineering and scientific problems are never needed exactly. Rather, an approximation to the answer is enquired which is accurate is a certain number of decimal places. Or accurate to within a given tolerance.

Now, the general definition for the rate of convergence can be given as :

(A sequence α_1, α_2, …. of (real or complex) numbers converges to α if and only it, for all $\alpha > 0$, there exists an integer $n_0(\Sigma)$ such that for all n $\geq n_0$, $|\alpha-\alpha_n| < \Sigma$.)

Hence, if we have a numerical method which produces a sequence α_1, α_2, …. converging to the desired answer α, then we can calculate α to any desired accuracy merely by calculating α_n for “ large enough” n.

A Special Mention of Propagation of Errors :

Once an error is committed, it contaminates subsequent results. Such a tendency of an error to contaminate or to spread onto subsequent results or steps is termed as ***Propagation of Error’.***

Now, let us investigate how error might be propagated in successive computations. Consider the addition of two numbers p and q (the true values) with the approximate values $\bar{p}$ and $\bar{q}$, which contain errors Σ_n and Σ_q respectively. Starting with p = $\bar{p} \pm \Sigma_p$ and q = $\bar{q} + \Sigma_p$, the sum is

$$p + q = (\bar{p} \pm \Sigma_p) + (\bar{q} \pm \Sigma_q)$$

$$= (\bar{p} + \bar{q}) + (\Sigma_p + \Sigma_q)$$

Hence, for addition, the error in the sum is the sum of the errors of the addends, the product

$$p + q = (\bar{p} \pm \Sigma_p) + (\bar{q} \pm \Sigma_q)$$

$$= \bar{p} \, \bar{q} + \bar{p}\Sigma_q + q\Sigma_p + \Sigma_p \Sigma_q$$

Hence if $\bar{p}$ and $\bar{q}$ are larger than 1 in absolute value, the terms $\bar{p}\Sigma_q$ and $\bar{q}\Sigma_p$ show that there is a possibility of magnification of the original errors Σ_q and Σ_p Insights are gained if we look at the relative error.

Let us again consider the above equation.

$$\bar{p} + \bar{q} = \bar{p} \, \bar{q} + p\Sigma_q + q\Sigma_p + \Sigma_p \Sigma_q$$

$$pq - \bar{p} \, \bar{q} = \bar{p}\Sigma_q + q\Sigma_p + \Sigma_p \Sigma_q$$

Suppose that p = 0 and q = 0, then we can divided (3) by pq and obtain

$$\frac{pq - \bar{p} \, \bar{q}}{pq} = \frac{p\Sigma_q + q\Sigma_p + \Sigma_p \Sigma_q}{pq}$$

$$= \frac{p\Sigma_q}{pq} + \frac{q\Sigma_p}{pq} + \frac{\Sigma_p \Sigma_q}{pq}$$

Furthermore suppose that $\frac{\bar{p}}{P} = 1$, $\frac{\bar{q}}{q} = 1$, and $\frac{\Sigma_p}{p}$ $\frac{\Sigma_q}{q} = R_p$ $R_q = 0$

Thus, making these substitutions yields that simplified relationship

$$\frac{pq - \bar{p} \, \bar{q}}{pq} = \frac{\Sigma_q}{q} + \frac{\Sigma_p}{p}$$

$$= R_p \, R_q = 0$$

This shows that the relative error in the product pq is approximately the sum of the relative errors in the approximation $\bar{q}$ and $\bar{p}$.

Like this way, based on above, we can see that often an initial error will be propagated in a sequence of calculations. A quality which is desirable for any numerical process is that a small error in the initial conditions will produce small changes in the **unstable.** Whenever possible we shall choose methods that are relatively relatively stable.

Derivation of a General Error Formula

A GENERAL ERROR FORMULA

In this unit, we will derive a general formula for the error committed in using a certain formula or a functional relation.

Now, let $\omega = f(n_u, n_2 \ldots. n_m)$

be a function of independent numbers $n_1, n_2 \ldots. n_m$ and let $\Delta\omega$ be the error in ω corresponding to the errors $\Delta n_1, \Delta n_2 \ldots. \Delta n_m$ in these independent numbers respectively.

$$\therefore \quad \omega + \Delta\omega = f(n_1 + \Delta n_1, n_1 + \Delta n_2 \ldots. \; n_1 + \Delta n_m)$$

In order to obtain an expression for $\Delta\omega$, we write Taylor's expansion of the right side. Then

$$\therefore \quad \omega + \Delta\omega = f(n_1, n_2 \ldots. n_m) \left(+ \; \Delta n_1 \frac{\partial f}{\partial n_1} + \Delta n_2 \frac{\partial f}{\partial n_2} + \ldots. + \Delta n_m \frac{\partial f}{\partial n_m} \right)$$

$$+ \frac{1}{21} \left((\Delta n_1)^2 \frac{\partial^2 f}{\partial n^2_1} + \ldots. + (\Delta n_m)^2 \frac{\partial^2 f}{\partial n^2_m} + \; 2\Delta n_2, \Delta n_2 \frac{\partial^2 f}{\partial n_1 \partial n_1} + \right) \ldots.$$

$+ ..$

Since errors $\Delta n_2, \Delta n_2, \ldots. \Delta n_m$ are assumed to be so small that their squares and products and highest powers can be neglected, we have

$$\therefore \quad \omega + \Delta\omega = f(n_1, n_2 \ldots. n_m) \left(+ \; \Delta n_1 \frac{\partial f}{\partial n_1} + \Delta n_2 \frac{\partial f}{\partial n_2} + \ldots. + \Delta n_m \frac{\partial f}{\partial n_m} \right)$$

$$\left(\partial f \qquad \partial f \qquad \partial^2 f \right)$$

$$\therefore \quad \omega + \Delta\omega = \omega + \Delta n_1 \frac{}{\partial n_1} + \Delta n_2 \frac{}{\partial n_n} + \ldots + \Delta n_m \frac{}{\partial n_m} \quad [\therefore f(n_1, n_2 \ldots n_m) = \omega]$$

$$\therefore \quad \Delta\omega = \Delta n_1 \frac{\partial\omega}{\partial n_1} + \Delta n_2 \frac{\partial\omega}{\partial n_n} + \ldots + \Delta n_m \frac{\partial\omega}{\partial n_m} \quad [\text{Putting } \omega \text{ for } f]$$

This is the general form which enables us to compute the error of a function of m independent numbers.

$$\text{Relative error} = \frac{\Delta\omega}{\omega}$$

$$= \frac{\Delta n_1}{\omega}, \frac{\partial\omega}{\partial n_1}, \frac{\Delta n_2}{\omega}, \frac{\partial\omega}{\partial n_2} + \ldots + \frac{\Delta n_m}{\omega}, \frac{\partial\omega}{\partial n_m}$$

Example Based on General Error Formula :

Ex. 1 Find the maximum relative error in using the following formula :

$$u = \frac{5\,xy^2}{z^3}$$

at x = y = z = 1 when the error in each one of them is 0.001.

Solution :

It is given that $u = \dfrac{5\,xy^2}{z^3}$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{5\,y^2}{z^3}, \quad \frac{\partial u}{\partial y} = \frac{10y}{z^3}, \quad \frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

$$\text{Now,} \quad \Delta u \frac{\partial u}{} \Delta x + \frac{\partial u}{} \Delta x + \frac{\partial u}{} \Delta z$$

$$\partial x \quad z^3 \quad \partial y \qquad \partial z$$

$$\therefore \quad \Delta u \; \frac{5y^2}{z^3}\,\Delta x + \frac{10y^2}{z^3}\,\Delta y + \left(- \frac{-15xy^2}{z^4}\,\Delta z \right)$$

It is given that x = y = z = 1

and $\Delta x = \Delta y = \Delta z = 0.001$

The errors Δx, Δy and Δz may be positive or negative and hence we take the absolute values of the term on R.H.S.

$$\therefore \quad \Delta u_{max} \left(\frac{5y^2}{z^3}\,\Delta x \right) + \left(\frac{10y^2}{z^3}\,\Delta y \right) + \left(- \frac{-15xy^2}{z^4}\,\Delta z \right)$$

$$\therefore \quad \Delta u_{max} = [5 \times 0.001] + [10 \times 0.001] - [15 \times 0.0001]$$

$$\therefore \quad \Delta u_{max} = 0.005 + 0.01 - 0.015$$

$$\therefore \quad \Delta u_{max} = 0.03$$

$$\therefore \quad \text{Maximum Relative Error, } (E_R)_{max} = \frac{(\Delta u)_{max}}{\mu}$$

$$= \frac{0.03}{5}$$

$$= 0.006$$

Examples based on Error Analysis

Examples of Rounding the numbers :

Procedure to Solve Examples of Rounding the Numbers :

Following rule will be applied for rounding off a number correct to n significant digits :

To round – off a number to n significant digits, discard all digits to the right of the n^{th} digit, and if this discarded number is

(i) less than half a unit in the n^{th} place, leave the n^{th} digit unchanged.

(ii) greater than half a unit in then n^{th} place, increase the n^{th} digit by unity.

(iii) exactly half a unit in the n^{th} place then leave the n^{th} digit by unity, an even number but increase n^{th} digit by 1 if it is odd.

Ex. 2 Round the following numbers to two decimal places :

(i) 48.2141 (ii) 2.375 (iii) 81.255

Solution :

(i) 48.21 (ii) 2.38 (iii) 81.26

Ex. 3 Round the following numbers to two decimal places :

(i) 24.5431 (ii) 7.4679 (iii) 102.6554

Solution :

(i) 24.54 (ii) 7.47 (iii) 102.66

Ex. 4 Round of the following number to four significant figures :

(i) 38.46235 (ii) 0.70029 (iii) 102.66

Solution :

(i) 38.46
(ii) 0.7003
(iii) 0.002222

Examples of Finding Error (i.e. Absolute Error), Relative Error,

Percentage Error and Significant Digits in the approximation :

If x is the true value and $\bar{x}$ is the measured value, then

(i) Error, $E_x = x - \bar{x}$

(ii) Relative Error, $E_R = \dfrac{x - \bar{x}}{x}$

(iii) Percentage Error = $E_R \times 100$

(iv) The condition of Significant Digits in the Approximation is

$$\frac{x - \bar{x}}{x} < \frac{10^{-d}}{2}$$

In the above condition, d is the number of Significant Digits.

Ex. 5 Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 and 9 cm respectively. If the true values are 10,000 and 10 cm respectively, compute :

(i) the error, and
(ii) the percentage relative error in each case.

Solution :

Step I : Calculation of error & percentage relative error in measurement of bridge :

Given : True length of bridge = 10,000 cm

Measured length of bridge = 9999 cm

$\therefore$ **Error in the measurement of bridge,**

ΔB_A = 10000 cm – 9999 cm = 1 cm

$\therefore$ Relative error in the measurement of bridge.

$$\Delta B_R = \frac{\Delta B_A}{\text{True length of bridge}}$$

$$= \frac{1}{10000}$$

$\therefore$ Percentage relative error = $\Delta B_R \times 100$

$$= \frac{1}{10000} \times 100$$

= 0.01 %

Step II : Calculation of error & percentage relative error in measurement of river :

Given : True length of rivet = 10 cm

Measured length of bridge = 9 cm

$\therefore$ **Error in the measurement of rivet,**

ΔR_A = 10 cm – 9 cm = 1 cm

$\therefore$ Relative error in the measurement of rivet.

$$\Delta R_R = \frac{\Delta R_A}{\text{True length of rivet}}$$

$$= \frac{1}{10}$$

$\therefore$ Percentage relative error = $\Delta R_R \times 100$

$$= \frac{1}{10} \times 100$$

$$\mathbf{= 10\ \%}$$

Ex. 6 Find the error E, and the relative error R_x and also determine the number of significant digits in the approximation for the following :

(i) Let x = 2.71828182 and $\bar{x}$ = 2.7182

(ii) Let y = 98.350 and $\bar{y}$ = 98,000

(iii) Let z = 0.000068 and $\bar{z}$ = 0.00006

Solution :

Case (i) :

Error, $E_x = x - \bar{x}$

= 2.71828182 – 2.7182

$= \mathbf{0.00008182}$

Relative error, $R_x = \dfrac{E_x}{x} = \dfrac{0.00008182}{2.71828182} = \mathbf{0.000030099}$

From above, we see that, $\dfrac{x - \bar{x}}{x} < \dfrac{10^{-4}}{2}$ $\left(\dfrac{E_x}{x} = \dfrac{x - \bar{x}}{x} = 0.000030099 \right)$ and $0.000030099 < \dfrac{10^{-4}}{2}$

$\bar{x}$ approximate x to 4 significant digits.

Case (ii) :

Error, $E_y = y - \bar{y}$

$= 98350 – 98000$

$= \mathbf{350}$

Relative error, $R_y = \dfrac{E_y}{y} = \dfrac{350}{98350} = \mathbf{0.003558}$

From above, we see that, $\dfrac{y - \bar{y}}{y} < \dfrac{10^{-2}}{2}$ $\left(\dfrac{E_y}{y} = \dfrac{y - \bar{y}}{y} = 0.003558 \right)$ and $0.003558 < \dfrac{10^{-2}}{2}$

$\bar{y}$ approximate y to 2 significant digits.

Case (ii) :

Error, $E_z = z - \bar{z}$

$= 0.000068 - 0.00006$

$\mathbf{= 0.000008}$

Relative error, $R_z = \dfrac{E_z}{z} = \dfrac{0.000008}{0.000068} = \mathbf{0.1176}$

From above, we see that, $\dfrac{z - \bar{z}}{z} < \dfrac{10^{-0}}{2}$ $\left(\dfrac{E_z}{z} = \dfrac{z - \bar{z}}{z} = 0.1176 \right.$ and $\left. 0.1176 < \dfrac{10^{-0}}{2} \right)$

$\bar{z}$ approximate z to no significant digits.

Miscellaneous Examples :

Ex. 7 Find the number of terms of the exponential series such that their sum gives the value of e^x correct to five decimal places for all values of x in the range $0 \le x \le 1$.

Solution :

For 5 decimal accuracy at x = 1, we must have.

$$\frac{x^n}{n!} > \frac{1}{2} \times 10^{-5}$$

$$\frac{x^n}{n!} > \frac{1}{2} \times 10^{-5}$$

$$n! > \frac{1}{}$$

0.5×10^{-5}

Now, $9! > 2 \times 10^{-5}$

$n = 9$

We need to take 9 terms of the exponential series in order that its sum is correct to five decimal places.

Ex. 8 Current flows through n 10 ohm resistance that is accurate with in 10%. The current is measured as 2.0 A with $\pm$ 0.1 Amps. What are absolute and relative errors in the computed voltage? Neglect round of errors.

Solution :

Step I : Finding absolute error in the measurement of resistance and current :

It is given that error in measurement of resistance is 10%

$\therefore$ Error in resistance measurement $= \pm 10 \times \frac{10}{100} = \pm 1\ \Omega$

It is also given that error in current measurement = $\pm$ 0.1 A

$\therefore$ Absolute error in measurement of resistance $\Delta R = |\pm 1| = 1\ \Omega$

Absolute error in measurement of current $\Delta I = |\pm 1| = 1$ A

Step II : Finding absolute error in the measurement of voltage

We have. $V = IR$

Absolute error in the measurement of voltage,

$$\Delta V_A = (R + \Delta R)(I + \Delta I) - IR$$

$$= IR + I\Delta R + R\Delta I + \Delta I\Delta R - IR$$

$$= I\Delta R + R\Delta I + \Delta I\Delta R$$

$$= 2 \times 1 + 10 \times 0.1 + 0.1 \times 1$$

$$= 2 + 1 + 0.1$$

$= 3.1$

Step III : Finding relative error in the measurement of voltage :

Relative error in the measurement of voltage is given by :

$$V_R = \frac{\Delta V_A}{V}$$

$$= \frac{\Delta V_A}{IR}$$

$$= \frac{3.1}{2 \times 10}$$

$$= \frac{3.1}{20}$$

$$\mathbf{= 0.155}$$

Ex. 9 Use zero 1st , 2nd , 3rd and 4th order Taylor's series expansion to approximate the function :

$f(x) = -0.1x^4 – 0.15x^3 – 0.5x^2 – 0.25 x + 1.2$

from $x_1 = 0$ with $h = \pm 1$, i.e., predict the function values at $x_{i+1} = 1$
State the truncation error in each case.

Solution :

Step I :

We are given that

$f(x) = -0.1x^4 – 0.15x^3 – 0.5x^2 – 0.25 x + 1.2$ $\therefore$
$f(0) = 1.2$

$f '(x) = -0.4x^3 – 0.45x^2 – 1.0 x + 1.2$ $\therefore$
$f '(0) = -0.25$

$f ''(x) = –1.2x^2 – 0.90 x + 1.0$ $\therefore$ $f ''(0)$
$= -1.0$

$f'''(x) = -2.4x + 0.90 \qquad \therefore \qquad f'''(0) = -0.90$

$f''(x) = -2.4 \qquad \therefore \qquad f''(0) = -2.4$

According to Taylor's series. we have

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) + \frac{h^3}{4!} f''(x_0)$$

Putting $x_0 = 0$ and $h = 1$, we get

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{6} f'''(0) + \frac{1}{24} f''(0)$$

Step II : Approximating the function using zero order Taylor's series :

Zero order Taylor's series from equation (I) is given by :

$f(1) = f(0)$

Putting values of f(0) in above equation, we get

$f(1) = 1.2$

1.2 is the function value at x = 1 using zero order Taylor's series.

$$\text{Truncation error} = f'(0) + \frac{1}{2} f''(0) + \frac{1}{6} f'''(0) + \frac{1}{24} f''(0)$$

$$= (-0.25) + \frac{1}{2}(-1.0) + \frac{1}{6}(-0.90) + \frac{1}{24}(-2.4)$$

$$= -1$$

Step III : Approximating the function using 1st order Taylor's series :

1^{st} order Taylor's series from equation (I) is given by :

f(1) = f(0) + f '(0)

Putting values of f(0) and f(0) in above equation, we get

f(1) = 1.2 – 0.25

f(1) = 0.95

0.95 is the function values at x = 1 using 1^{st} order Taylor's series.

$$\text{Truncation error} = \frac{1}{2} f''(0) + \frac{1}{6} f'''(0) + \frac{1}{24} f''(0)$$

$$= \frac{1}{2}(-1.0) + \frac{1}{6}(-0.90) + \frac{1}{24}(-2.4)$$

$$= -0.75$$

Step IV : Approximating the function using 2^{nd} order Taylor's series.

2^{nd} order Taylor's series from equation (I) is given by

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0)$$

Putting value of f(0), f '(0) and f ''(0) in above equation we get

$$f(1) = 1.2 - 0.25 + \frac{1}{2}(-1.0)$$

f(1) = 0.45

0.45 is the function value at x = 1 using 2^{nd} order Taylor's series

$$\text{Truncation error} = \frac{1}{} f'''(0) + \frac{1}{} f''(0)$$

$$\quad 6 \qquad 24$$

$$= \frac{1}{6}(-0.90) + \frac{1}{24}(-2.4)$$

$$= -0.25$$

Step V : Approximating the function using 3rd order Taylor's series :

3rd order Taylor's series from equation (I) is given by

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) \ \ \frac{1}{6} f'''(0)$$

Putting value of f(0), f '(0), f ''(0) and f '''(0) in above equation we get

$$f(1) = 1.2 - 0.25 + \frac{1}{2}(-1.0) + \frac{1}{6}(-0.90)$$

$$f(1) = 0.30$$

0.30 is the function value at x = 1 using 3rd order Taylor's series

$$\text{Truncation error} = \frac{1}{24} f''(0)$$

$$= \frac{1}{24}(-2.4)$$

$$= -0.1$$

Step VI : Approximating the function using 4th order Taylor's series :

4th order Taylor's series from equation (I) is given by

$$f(1) = f(0) + f'(0) + \frac{1}{2} f''(0) + \frac{1}{6} f'''(0) + \frac{1}{24} f''(0)$$

Putting value of f(0), f '(0), f "(0), f "'(0) and f "(0) in above equation we get

$$f(1) = 1.2 - 0.25 + \frac{1}{2}(-1.0) + \frac{1}{6}(-0.90) + \frac{1}{24}(-2.4)$$

$$f(1) = 0.20$$

0.20 is the function value at x = 1 using 4^{th} order Taylor's series

$$\text{Truncation error} = \frac{1}{120} f''(0)$$

$$= \frac{1}{120}(0)$$

$$= 0$$

Ex. 10 Consider the expression for $x^2 - y^2$ in a computer program. Assume that the expression is computed for values of x = a and y = b with relative errors e_a and e_b in a and b respectively. Evaluation of the expression can be made using the two forms :

(i) $f_1 = a*a - b*b$

(ii) $f_2 = (a + b)*(a - b)$

Express the errors of f_1 and f_2 in terms of e_a and e_b

Solution :

Step I :

It is given that relative error in a and b e_a and e_b respectively

$$\text{Now, Relative Error} = \frac{\text{Error in measurement of a}}{\text{True value}}$$

$$e_a = \frac{\text{Error in measurement of a}}{a}$$

Similarly ,

Error in measurement of a = e_a

Error in measurement of b = e_b

Step II : Finding error in f_1 :

We are given that $f_1 = a*a – b*b$

$f_1 = (ae_a + ae_a) - (be_b + be_b)$

$f_1 = 2ae_a + 2be_b$

Step III : Finding error in f_2 :

We are given that $f_2 = (a + b)*(a – b)$

$f_2 = (a + b)(ae_a + ae_a) + (a – b)(be_b + be_b)$

$f_2 = ae_a + ae_b + be_a + be_b + ae_a + ae_b - be_a – be_b$

$f_2 = 2ae_a + 2ae_b$

Introduction

In scientific engineering and applied mathematics, the problem of finding the roots of an equation of the type f(x) = 0 is very much important.

The equation f(x) = 0 may be an **algebraic equation** or **a transcendental equation.**

The **algebraic equation** of degree n is of the form :

$$a_n x^n + a_{n-1} x^{n-1} - . . . . + a_1 x + a_0 = 0$$

The above form of algebraic equation has n roots, some of which may be real and distinct, real and unequal or complex conjugates.

Any non – algebraic equation is called a **transcendental equation.** The following are some of the examples of transcendental equations.

$$\sin x - \frac{x^2}{2} + 2 = 0$$

$$e^{3x} - \frac{x}{} = 0$$

$$\log x - \frac{\pi^{2}}{2} = 0 \text{ etc.}$$

A transcendental equation may have a finite or infinite number of real roots or may have no real roots at all.

The equation of the form f(x) = 0 can be solved by the following two type of methods :

(i) Direct Method, and

(ii) Indirect Methods (or Iterative Methods).

Direct methods are generally applied to solve algebraic equations. These are the simplest methods for the evaluation of the roots of a algebraic equation. These methods give the exact value of the root in a finite number of steps. These methods determine all the roots at the same time.

Indirect methods are generally applied to solve transcendental equations. These methods are based on the idea of successive approximations. Staring with one or more initial approximations to the root. We obtain a sequence of iterates (x) which in the limit converges to the root. These methods determine only one root at a line Bisection Method, Secant Method, Regular Falsi Method, Newton – Raphson Method are the examples of indirect methods.

Bisection Method

Bisection method is the simplest of all iterative method and is based on repeated application of the **intermediate value theorem.**

[According to this theorem, if a function f(x) is continuous between a and b, and f(a) and f(b) are of opposite signs, then there exists at least one root between a and b.]

For example, let the given equation be of the form f(x) = 0 and let a and b are the initial approximation New, if f(a) is negative and f(b) is positive then the root lies between a and b and the approximate value of the root is given by $x_o = \frac{a - b}{2}$] known

$1(x_o) = 0$ then we conclude that x_o is the root of the equation $f(x) = 0$, otherwise the root lies between x_o and b or between x_o and a depending whether $f(x_o)$ is negative or positive. Then, as before, we bisect the interval and repeat the process until the root is known to the desired accuracy.

Number of Iterations required in Bisection Method :

If we have to calculate the root of the given equation with a permissible error then approximate number of iteration required can be obtained by using relation.

$$\frac{b - a}{2^n} \leq \in$$

$$b - a \leq \in 2^n$$

$$\log (b - a) \leq \log \in n\log 2$$

$$n \geq \frac{\log (b - a) - \log \in}{\log 2}$$

The shows that bisection method requires large number of iterations to achieve a reasonable degree of accuracy for the root.

Procedure to Solve Problems based on Bisection Method.

Step I : Find initial approximations if they are not given in the data of the problem

Step II : Find the number of iterations required.

Step III : Find the root of the given equation by successive iterations.

The above procedure will get clear to you as you will go through the following solved example.

Ex. 1 Obtain an initial approximation to a root of the equation :

$$f(x) = \cos x - xe^x = 0$$

Solution :

We have given that

$$f(x) = \cos x - xe^x$$

Now

$$f(0) = \cos 0 - 0 = 1$$

$$f(1) = \cos 1 - e^1 = 1.7184$$

$f(1) < 0$ and $f(0) > 0$

Initial approximations are $a = 0$ and $b = 1$

Ex. 2 Find the roots correct to two decimals using bisection method for equation $x^3 - x - 4 = 0$. How many iterations are required if permissible error is 0.02.

Solution :

Step I : Finding initial approximation :

We are given that

$$f(x) = x^3 - x - 4$$

Now

$$f(0) = -4$$

$$f(1) = -4$$

$$f(2) = 2$$

$f(1) < 0$ and $f(2) > 0$

Root lies between (1.2)

Initial approximations are $a = 1$ and $b = 2$

Step II : Finding the number of iterations required :

Number of iterations required is given by :

$$n \geq \frac{\log (b - a) - \log \in}{\log 2}$$

$$n \geq \frac{\log (2 - 1) - \log 0.02}{\log 2} \geq 5.6438$$

n = 6 iterations

Step III : **1^{st} Iteration :**

Here, a = 1 and b = 2

$$c = \frac{1 + 2}{2} = 1.5$$

Now $f(1.5) = (1.5)^3 – 1.5 – 4 = -2.125$

$f(1.5) < 0$ and $f(2) > 0$

Root lies between (1.5, 2)

New approximations are a = 2 and b =2

Step IV : **2^{nd} Iteration :**

Here, a = 1.5 and b = 2

$$a = \frac{1.5 + 2}{2} = 1.75$$

Now $f(1.75) = (1.75)^3 – 1.75 – 4 = -0.3906$

$f(1.75) < 0$ and $f(2) > 0$

Root lies between (1.75, 2)

New approximations are a = 1.75 and b =2

Step V : **3^{rd} Iteration :**

Here, a = 1.75 and b = 2

$$c = \frac{1.75 + 2}{2} = 1.875$$

Now $f(1.875) = (1.875)^3 – 1.875 – 4 = 0.7167$

$f(1.75) < 0$ and $f(1.875) > 0$

Root lies between (1.75, 1.875)

New approximations are a = 1.75 and b =1.875

Step VI : 4^{th} Iteration :

Here, a = 1.75 and b = 1.875

$$c = \frac{1.75 + 1.875}{2} = 1.8125$$

Now $f(1.8125) = (1.8125)^3 - 1.8125 - 4 = 0.1418$

$f(1.75) < 0$ and $f(1.8125) > 0$

Root lies between (1.75, 1.8125)

New approximations are a = 1.75 and b =1.8125

Step VII : 5^{th} Iteration :

Here, a = 1.75 and b = 1.8125

$$c = \frac{1.75 + 1.8125}{2} = 1.78125$$

Now $f(1.78125) = (1.78125)^3 - 1.78125 - 4 = -0.1296$

$f(1.75) < 0$ and $f(1.78125) > 0$

Root lies between (1.78125, 1.8125)

New approximations are a = 1.78125 and b =1.8125

Step VIII : 6^{th} Iteration :

Here, a = 1.78125 and b = 1.8125

$$c = \frac{1.78125 + 1.8125}{2} = 1.796875$$

Now $f(1.\ 796875) = (1.\ 796875)^3 - 1.\ 796875 - 4 = 0.0048$

$f(1.78125) < 0$ and $f(1.\ 796875) > 0$

Root lies between (1.78125, 1. 796875)

New approximations are a = 1.78125 and b =1. 796875

Step IX : Finding the root:

Here, a = 1.78125 and b = 1.796875

$$c = \frac{1.78125 + 1.796875}{2} = 1.7890625$$

Since, we have to calculate upto 6^{th} iteration for the desired accuracy.

Root of the given equation is 1.7890625

Ex. 3 Use bisection method to determine the root of $f(x) = e^{-x} - x$.

Solution :

Step I : Finding initial approximations :

We are given that

$f(x) = e^{-x} - x$

Now

$f(0) = -0$

$f(1) = e^{-1} - 1 = 0.6321$

$f(2) = 2$

$f(1) < 0$ and $f(0) > 0$

Root lies between (0.1)

Initial approximations are a = 0 and b = 1

Step II : Finding the number of iterations required :

Since the permissible error is not given, so we will calculate until the value of f(0) is same upto 3 decimal places.

Step III : 1^{st} Iteration :

Here, a = 1 and b = 2

$$c = \frac{0 + 1}{2} = 0.5$$

Now $f(0.5) = e^{-0.5} – 0.5 = 0.1065$

$f(1) < 0$ and $f(0.5) > 0$

Root lies between (0.5, 1)

New approximations are a = 0.5 and b =1

Step IV : 2^{nd} Iteration :

Here, a = 0.5 and b = 1

$$c = \frac{0.5 + 1}{2} = 0.75$$

Now $f(0.75) = e^{-0.75} – 0.75 = 0.2776$

$f(0.75) < 0$ and $f(0.5) > 0$

Root lies between (0.5, 0.75)

New approximations are a = 0.5 and b =0.75

Step V : 3^{rd} Iteration :

Here, a = 0.5 and b = 0.75

$$\therefore \quad c = \frac{0.5 + 0.75}{2} = 0.625$$

Now $f(0.625) = e^{-0.625} – 0.625 = 0.0.897$

$\therefore$ $f(0.625) < 0$ and $f(0.5) > 0$

$\therefore$ Root lies between (0.5, 0.625)

$\therefore$ New approximations are a = 0.5 and b =0.625

Step VI : 4^{th} Iteration :

Here, a = 0.5 and b = 0.625

$$\therefore \quad c = \frac{0.5 + 0.625}{2} = 0.5625$$

Now $f(0.5625) = e^{-0.5625} - 0.5625 = 0.00728$

$\therefore$ $f(0.625) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.625)

$\therefore$ New approximations are a = 0.5625 and b =0.625

Step VII : 5^{th} Iteration :

Here, a = 0.5625 and b = 0.625

$$\therefore \quad c = \frac{0.5625 + 0.625}{2} = 0.59375$$

Now $f(0.59375) = e^{-0.59375} - 0.59375 = 0.0414$

$\therefore$ $f(0.59375) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.59375)

$\therefore$ New approximations are a = 0.5625 and b =0.59375

Step VIII : 6^{th} Iteration :

Here, a = 0.5625 and b = 0.59375

$$\therefore \quad c = \frac{0.5625 + 0.59375}{2} = 0.578125$$

Now $f(0.579625) = e^{-0.579625} – 0.579625 = 0.0195$

$\therefore$ $f(0.579625) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.579625)

$\therefore$ New approximations are a = 0.5625 and b =0.579625

Step IX : 7^{th} Iteration :

Here, a = 0.5625 and b = 0.579625

$$\therefore \quad c = \frac{0.5625 + 0.579625}{2} = 0.5710625$$

Now $f(0.5710625) = e^{-0.5710625} – 0.5710625 = -0.0061$

$\therefore$ $f(0.5710625) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.5710625)

$\therefore$ New approximations are a = 0.5625 and b =0.5710625

Step X : 8^{th} Iteration :

Here, a = 0.5625 and b = 0.5710625

$$\therefore \quad c = \frac{0.5625 + 0.5710625}{2} = 0.5667812$$

Now $f(0.5667812) = e^{-0.5667812} – 0.5667812 = -0.0005$

Since, the value of c is consistent upto 3^{rd} decimal place.

Root of the given equation is 0.5667812

Ex. 4 Locate the root of $f(x) = x^{10} – 1$ between x = 0 and 1.3 using bisection method.

Solution :

Step I : Finding initial approximations :

We are given that we have to locate root between x = 0 and 1.3

Initial approximations are a = 0 and b = 1.3

Also $f(0) = (0)^{10} – 1 = – 1$

$f(1.3) = (1.3)^{10} – 1 = 12.7858$

Step III : 1^{st} Finding the number of iterations required :

Here, a = 0 and b = 1.3

$$\therefore \quad c = \frac{0 + 1.3}{2} = 0.65$$

Now $f(0.65) = (0.65)^{10} – 1 = 0.9865$

$\therefore$ f(0.65) < 0 and f(1.3) > 0

$\therefore$ Root lies between (0.65, 1.3)

$\therefore$ New approximations are a = 0.65 and b =1.3

Step IV : 2^{nd} Iteration :

Here, a = 6.65 and b = 1.3

$$\therefore \quad c = \frac{0.65 + 1.3}{2} = 0.975$$

Now $f(0.975) = (0.975)^{10} – 1 = 0.2236$

$\therefore$ f(0.975) < 0 and f(1.3) > 0

$\therefore$ Root lies between (0.975, 1.3)

$\therefore$ New approximations are a = 0.975 and b =1.3

Step V : 3^{rd} Iteration :

Here, $a = 0.975$ and $b = 1.3$

$$\therefore \quad c = \frac{0.975 + 1.3}{2} = 1.1375$$

Now $f(1.1375) = (1.1375)^{10} - 1 = 2.6267$

$\therefore$ $f(0.975) < 0$ and $f(1.1375) > 0$

$\therefore$ Root lies between (0.975, 1.1375)

$\therefore$ New approximations are $a = 0.975$ and $b = 1.1375$

Step VI : 4th Iteration :

Here, $a = 0.975$ and $b = 1.1375$

$$\therefore \quad c = \frac{0.975 + 1.1375}{2} = 1.05625$$

Now $f(1.05625) = (1.05625)^{10} - 1 = 0.7284$

$\therefore$ $f(0.975) < 0$ and $f(1.05625) > 0$

$\therefore$ Root lies between (0.975, 1.05625)

$\therefore$ New approximations are $a = 0.975$ and $b = 1.05625$

Step VII : 5th Iteration :

Here, $a = 0.975$ and $b = 1.05625$

$$\therefore \quad c = \frac{0.975 + 1.05625}{2} = 1.015625$$

Now $f(1.015625) = (1.015625)^{10} - 1 = 0.1677$

$\therefore$ $f(0.975) < 0$ and $f(1.015625) > 0$

$\therefore$ Root lies between (0.975, 1.015625)

$\therefore$ New approximations are a = 0.975 and b =1.015625

Step VII : 6^{th} Iteration :

Here, a = 0.975 and b = 1.015625

$$\therefore \quad c = \frac{0.975 + 1.015625}{2} = 0.9953125$$

Now $f(0.9953125) = (0.9953125)^{10} - 1 = 0.0458$

$\therefore$ $f(0.9953125) < 0$ and $f(1.015625) > 0$

$\therefore$ Root lies between (0.9953125, 1.015625)

$\therefore$ New approximations are a = 0.9953125 and b =1.015625

Step IX : 7^{th} Iteration :

Here, a = 0.9953125 and b = 1.015625

$$\therefore \quad c = \frac{0.9953125 + 1.015625}{2} = 1.0054688$$

Now $f(0.0054688) = (1.0054688)^{10} - 1 = 0.0560$

$\therefore$ $f(0.9953125) < 0$ and $f(1.0054688) > 0$

$\therefore$ Root lies between (0.9953125, 1.0054688)

$\therefore$ New approximations are a = 0.9953125 and b =1.0054688

Step X : 8^{th} Iteration :

Here, a = 0.9953125 and b = 1.0054688

$$\therefore \quad c = \frac{0.9953125 + 1.0054688}{2} = 1.0003907$$

Now $f(1.0003907) = (1.0003907)^{10} - 1 = 0.0039$

$\therefore$ f(0.9953125) < 0 and f(1.0003907) > 0

$\therefore$ Root lies between (0.9953125, 1.0003907)

$\therefore$ New approximations are a = 0.9953125 and b =1.0003907

Step XI : 9^{th} Iteration :

Here, a = 0.9953125 and b = 1.0003907

$$\therefore \quad c = \frac{0.9953125 + 1.0003907}{2} = 0.9978516$$

Now f(0.9978516) = $(0.9978516)^{10} - 1$ = 0.0212

$\therefore$ f(0.9978516) < 0 and f(1.0003907) > 0

$\therefore$ Root lies between (0.9978516, 1.0003907)

$\therefore$ New approximations are a = 0.9978516 and b =1.0003907

Step XII : 10^{th} Iteration :

Here, a = 0.9978516 and b = 1.0003907

$$\therefore \quad c = \frac{0.9978516 + 1.0003907}{2} = 0.9991211$$

Now f(0.9991211) = $(0.9991211)^{10} - 1$ = -0.0087

$\therefore$ f(0.9991211) < 0 and f(1.0003907) > 0

$\therefore$ Root lies between (0.9991211, 1.0003907)

$\therefore$ New approximations are a = 0.9991211 and b =1.0003907

Step XIII : 11^{th} Iteration :

Here, a = 0.9991211 and b = 1.0003907

$$\therefore \quad c = \frac{0.9991211 + 1.0003907}{2} = 0.9997559$$

$\therefore$ Root of the given equation is 0.9997559 since valued of c is same up to 3rd decimal place

Ex. 5 Evaluate the root for the equation $f(x) = \cos x - xc^x$ using bisection method.

Solution

Step I : Finding initial approximation :

We are given that

$$f(x) = \cos x - xc^x$$

Now

$$f(0) = \cos 0 – 0 = 1$$

$$f(1) = \cos 1 - e^{-1} = 1.7184$$

$$f(2) = 2$$

$f(1) < 0$ and $f(0) > 0$

Root lies between (0.1)

Initial approximations are $a = 0$ and $b = 1$

Step II : Finding the number of iterations required :

Since the permissible error is not given, so we will calculate until the value of f(0) is same upto 3 decimal places.

Step III : 1st Iteration :

Here, $a = 0$ and $b = 1$

$$\therefore \quad c = \frac{0 + 1}{2} = 0.5$$

Now $f(0.5) = \cos 0.5 - 0.05\,(e^{0.5}) = 0.1756$

$\therefore$ $f(1) < 0$ and $f(0.5) > 0$

$\therefore$ Root lies between (0.5, 1)

$\therefore$ New approximations are a = 0.5 and b =1

Step IV : 2^{nd} Iteration :

Here, a = 0.5 and b = 1

$$\therefore \quad c = \frac{0.5 + 1}{2} = 0.75$$

Now $f(0.75) = \cos 0.75 - 0.75\,(e^{0.75}) = 0.5878$

$\therefore$ $f(0.75) < 0$ and $f(0.5) > 0$

$\therefore$ Root lies between (0.5, 0.75)

$\therefore$ New approximations are a = 0.5 and b =0.75

Step V : 3^{rd} Iteration :

Here, a = 0.5 and b = 0.75

$$\therefore \quad c = \frac{0.5 + 0.75}{2} = 0.625$$

Now $f(0.625) = \cos 0.625 - 0.625\,(e^{0.625}) = 0.1677$

$\therefore$ $f(0.625) < 0$ and $f(0.5) > 0$

$\therefore$ Root lies between (0.5, 0.625)

$\therefore$ New approximations are a = 0.5 and b =0.625

Step VI : 4^{th} Iteration :

Here, a = 0.5 and b = 0.625

$$\therefore \quad c = \frac{0.5 + 0.625}{2} = 0.5625$$

Now $f(0.5625) = \cos 0.5625 – 0.5625\ (e^{0.5625}) = 0.0127$

$\therefore$ $f(0.625) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.625)

$\therefore$ New approximations are a = 0.5625 and b =0.625

Step VII : 5th Iteration :

Here, a = 0.5625 and b = 0.625

$$\therefore \quad c = \frac{0.5625 + 0.625}{2} = 0.59375$$

Now $f(0.59375) = \cos 0.59375 – 0.59375\ (e^{0.59375}) = 0.075$

$\therefore$ $f(0.\ 59375) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.59375)

$\therefore$ New approximations are a = 0.5625 and b =0.59375

Step VIII : 6th Iteration :

Here, a = 0.5625 and b = 0.59375

$$\therefore \quad c = \frac{0.5625 + 0.59375}{2} = 0.578125$$

Now $f(0.57812) = \cos 0.578125 – 0.\ 578125\ (e^{0.59375}) = 0.030$

$\therefore$ $f(0.\ 578125) < 0$ and $f(0.5625) > 0$

$\therefore$ Root lies between (0.5625, 0.578125)

$\therefore$ New approximations are a = 0.5625 and b =0.578125

Step IX : 7^{th} Iteration :

Here, a = 0.5625 and b = 0.578125

$$\therefore \quad c = \frac{0.5625 + 0.578125}{2} = 0.5703125$$

Now f(0.5703125) = cos 0.5703125 – 0.5703125 ($e^{0.5703125}$) = -0.008

$\therefore$ f(0.5703125) < 0 and f(0.5625) > 0

$\therefore$ Root lies between (0.5625, 0.5703125)

$\therefore$ New approximations are a = 0.5625 and b =0.5703125

Step X : 8^{th} Iteration :

Here, a = 0.5625 and b = 0.5703125

$$\therefore \quad c = \frac{0.5625 + 0.5703125}{2} = 0.5664062$$

Now f(0.5664062) = cos 0.5664062 – 0.5664062 ($e^{0.5664062}$) = 0.0019

$\therefore$ f(0.5703125) < 0 and f(0.5664062) > 0

$\therefore$ Root lies between (0.5664062, 0.5703125)

$\therefore$ New approximations are a = 0.5664062 and b =0.5703125

Step XI : 9^{th} Iteration :

Here, a = 0.5664062 and b = 0.5703125

$$\therefore \quad c = \frac{0.5664062 + 0.5703125}{2} = 0.5683593$$

Now $f(0.5683593) = \cos 0.5683593 - 0.5683593 \, (e^{0.5683593}) = 0.0034$

$\therefore$ $f(0.5683593) < 0$ and $f(0.5664062) > 0$

$\therefore$ Root lies between (0.5664062, 0.5683593)

$\therefore$ New approximations are a = 0.5664062 and b =0.5683593

Step XII : 10th Iteration :

Here, a = 0.5664062 and b = 5683593

$$\therefore \quad c = \frac{0.5664062 + 0.5683593}{2} = 0.5673827$$

Now $f(0.5673827) = \cos 0.56732827 - 0.5673827 \, (e^{0.56732827}) = -0.0007$

Since the values of c is consistent upto 3rd decimal place.

Root of the given equation is 0.5673827

Unit 4

Regula – Falsi Method

(Method of False Position)

Regula Falsi is the oldest method for finding the real root of an equation, and closely resembles the secant method. Fig 1 gives the graphical representation of this method.

Fig. 1 Method of false position (regula Falsi Method)

In this method, we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of apposite signs so that the root lies in between these two points. Now, the equation of the chord joining the two point $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ is :

$$\frac{y - f(x_0)}{x - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

The **Regula Flasi method (or the method of false position)** consists in replacing the part of the curve between the point $[x_0, f(x_0)]$ and $[x_1, f(x_1)]$ by means of the chord joining these points, and taking the point of intersection of the chord with the x axis as an approximation to the root. The point of intersection in the present case it given by putting y = 0 in equation (1). Thus, we obtain:

$$x = x_0 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_0)$$

Hence the second approximation to the root of f(x) = 0 is given by

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

If now $f(x_2)$ and $f(x_0)$ are of opposite signs – then the root lies between x_0 and x_2 and we replace x_1 by x_2 in equation (II), and obtain the next approximation. Otherwise, we replace x_0 by x_2 and gerlerate the next approximation.

The above procedure of getting next approximation is repeated till the root is obtained to the desired accuracy. **The rate of convergence of Regula – Falsi method is 1.6.**

Procedure to Solve Problems based on Regula Falsi Method :

Step I : Find initial approximation if they are not given in the data of the problem.

Step II : Find the number of iterations required.

Step III : Find the root of the given equation by successive iterations using the Regula – Flasi formula given by.

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

Step IV : Calculate $f(x_2)$

Step V : If $f(x_0)$ and $f(x_2)$ are of the opposite sign, then replace x_1 by x_2 and repeat the above procedure from Step III, otherwise

If $f(x_0)$ and $f(x_2)$ are of the same sign, then replace x_0 by x_2 and Repeat the above procedure from Step III.

The above procedure will get clear to you as you will go through the following solved example

Ex. 8 Obtain the smallest positive root for the equation $x^3 - x - 4 = 0$ using Regula Falsi method.

Solution :

Step I: **Finding initial approximation :**

We are given that

$f(x) = x^3 - x - x - 4$

Now

$f(0) = (0)^3 - 0 - 4 = -4$

$f(1) = (1)^3 - 1 - 4 = -4$

$f(2) = (2)^3 - 2 - 4 = 2$

$\therefore$ $f(1) < 0$ and $f(2) > 0$

$\therefore$ Root lies between (1, 2)

Initial approximation are $x_0 = 1$ and $x_1 = 2$

Step II : **Finding the number of iterations required :**

Since, it is given that value of the root should be correct to 4 significant figures so we will calculate until the value of root becomes same upto 4 decimal places in two consecutive iterations.

Step III : 1^{st} Iteration :

Here, $x_0 = 0.5625$ and $x_1 = 0.5703125$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 - 1)}{2 - (-4)} (2)$$

$$\therefore \quad x_2 = 1.6666667$$

Also, $f(x_2) = (1.6666667)^3 – 1.6666667 -4 = -1.0370371$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 1.6666667$ and $x_1 = 2$

Step IV : 2^{nd} Iteration :

Here, $x_0 = 1.6666667$ and $x_1 = 2$

Also $f(x_0) = f\,(1.6666667) = -1.0370371$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 – 1.6666667)}{2 - (-1.0370371)} (2)$$

$$\therefore \quad x_2 = 1.7804878$$

Also, $f(x_2) = (1.7804878)^3 – 1.7804878 - 4 = -0.1360975$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 1.7804878$ and $x_1 = 2$

Step V : 3^{rd} Iteration :

Here, $x_0 = 1.7804878$ and $x_1 = 2$

Also $f(x_0) = f\,(1.7804878) = -0.1360975$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 - 1.7804878)}{2 - (-0.1360975)}\,(2)$$

$$\therefore \quad x_2 = 1.7944736$$

Also, $f(x_2) = (1.7944736)^3 - 1.7944736 - 4 = -0.0160253$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 1.7944736$ and $x_1 = 2$

Step VI : 4^{th} Iteration :

Here, $x_0 = 1.7944736$ and $x_1 = 2$

Also $f(x_0) = f\,(1.7944736) = -0.0160253$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 - 1.7944736)}{2 - (-0.0160253)}\,(2)$$

$\therefore \qquad x_2 = 1.7961073$

Also, $f(x_2) = (1.7961073)^3 – 1.7961073 - 4 = -0.0018623$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 1.7961073$ and $x_1 = 2$

Step VII : 5^{th} Iteration :

Here, $x_0 = 1.7961073$ and $x_1 = 2$

Also $f(x_0) = f\ (1.7961073) = -0.0018623$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \qquad x_2 = 2 - \frac{(2 – 1.7961073)}{2 - (-0.0018623)} (2)$$

$\therefore \qquad x_2 = 1.796297$

Also, $f(x_2) = (1.796297)^3 – 1.796297 - 4 = -0.0002163$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 1.796297$ and $x_1 = 2$

Step VIII : 6^{th} Iteration :

Here, $x_0 = 1.796297$ and $x_1 = 2$

Also $f(x_0) = f\ (1.796297) = -0.0002163$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 - 1.796297)}{2 - (-0.002163)} (2)$$

$$\therefore \quad x_2 = 1.796319$$

Also, $f(x_2) = (1.796319)^3 - 1.796319 - 4 = -0.0000249$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$$x_0 = 1.796319 \quad \text{and} \quad x_1 = 2$$

Step IX : 7^{th} Iteration :

Here, $x_0 = 1.796319$ and $x_1 = 2$

Also $f(x_0) = f(1.796319) = -0.0000249$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 2 - \frac{(2 - 1.796319)}{2 - (-0.0000249)} (2)$$

$$\therefore \quad x_2 = 1.7963215$$

Since, the value of x_2 in 6^{th} and 7^{th} iterations are same upto 4 decimal places.

$\therefore$ Smallest positive root of the given equation is **1.7963215**

Ex. 9 Evaluate the root for the equation $f(x) = \cos x - xe^x = 0$ using false position method.

Solution :

Step I : Finding initial approximations :

We are given that

$$f(x) = \cos x - xe^x$$

Now

$$f(0) = \cos 0 – 0 = 1$$

$$f(1) = \cos 1 - e^1 = 1.781341$$

$\therefore$ $f(1) < 0$ and $f(2) > 0$

$\therefore$ Root lies between (0, 1)

Initial approximation are $x_0 = 0$ and $x_1 = 1$

Step II : Finding the number of iterations required :

Since, the permissible error is not given, so we will calculated until the value of root becomes same upto 3 decimal places in two consecutive iterations.

Step III : 1^{st} Iteration :

Here, $x_0 = 0$ and $x_1 = 1$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0)}{(-1.7184341) - (1)} (-1.7184341)$$

$$\therefore \quad x_2 = 0.3678588$$

Also, $f(x_2) = \cos (0.3678588) – (0.3678588)e^{(0.3678588)} = 0.4685564$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.3678588$ and $x_1 = 1$

Step IV : 2^{nd} Iteration :

Here, $x_0 = 0$ and $x_1 = 1$

and $f(x_0) = f (0.3678588) = 0.4685564$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.3678588)}{(-1.7184341) - (0.4685564)} (-1.7184341)$$

$$\therefore \quad x_2 = 0.5032932$$

Also, $f(x_2) = \cos(0.5032932) - (0.5032932)e^{(0.5032932)} = 0.1674339$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$$x_0 = 0.5032932 \quad \text{and} \quad x_1 = 1$$

Step V : 3^{rd} Iteration :

Here, $x_0 = 0.5032932$ and $x_1 = 1$

and $f(x_0) = f(0.5032932) = 0.1674339$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.5032932)}{(-1.7184341) - (0.1674339)} (-1.7184341)$$

$$\therefore \quad x_2 = 0.5473925$$

Also, $f(x_2) = \cos(0.5473925) - (0.5473925)e^{(0.5473925)} = 0.0536551$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$$x_0 = 0.5473925 \quad \text{and} \quad x_1 = 1$$

Step VI : 4^{th} Iteration :

Here, $x_0 = 0.5473925$ and $x_1 = 1$

and $f(x_0) = f\ (0.5473925) = 0.0536551$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.5473925)}{(-1.7184341) - (0.0536551)} (-1.7184341)$$

$$\therefore \quad x_2 = 0.5610965$$

Also, $f(x_2) = \cos(0.5610965) - (0.5610965)e^{(0.5610965)} = 0.0165781$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.5610965$ and $x_1 = 1$

Step VII : 5^{th} Iteration :

Here, $x_0 = 0.5610965$ and $x_1 = 1$

and $f(x_0) = f\ (0.5610965) = 0.0165781$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.5610965)}{(-1.7184341) - (0.0165781)} (-1.7184341)$$

$$\therefore \quad x_2 = 0.5652902$$

Also, $f(x_2) = \cos(0.5652902) - (0.5652902)e^{(0.5652902)} = 0.0050639$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.5652902$ and $x_1 = 1$

Step VIII : 6^{th} Iteration :

Here, $x_0 = 0.5652902$ and $x_1 = 1$

and $f(x_0) = f\,(0.5652902) = 0.0050639$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.5652902)}{(-1.7184341) - (0.0050639)}\,(-1.7184341)$$

$$\therefore \quad x_2 = 0.5665674$$

Also, $f(x_2) = \cos\,(0.5665674) - (0.5665674)e^{(0.5665674)} = 0.0015415$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.5665674$ and $x_1 = 1$

Step IX : 7^{th} Iteration :

Here, $x_0 = 0.5665674$ and $x_1 = 1$

and $f(x_0) = f\,(0.5665674) = 0.0015415$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\, f(x_1)$$

$$\therefore \quad x_2 = 1 - \frac{(1 - 0.5665674)}{(-1.7184341) - (0.0015415)}\,(-1.7184341)$$

$$\therefore \quad x_2 = 0.5669558$$

Since, the value of x_2 in 6^{th} and 7^{th} iterations are same upto 3 decimal places.

$\therefore$ **Root of the given equation is 0.5669558**

Ex. 10 Evaluate the root for the equation $f(x) = x^{10} - 1$ between $x = 0$ and 1.3 using

Solution :

Step I : Finding initial approximations :

We are given that we have to locate root between x = 0 and 1.3

Initial approximation are $x_0 = 0$ and $x_1 = 1.3$

Also, $f(x_0) = (0)^{10} - 1 = -1$

Now, $f(x_1) = (1.3)^{10} - 1 = 12.7858$

Step II : Finding the number of iterations required :

Since the permissible error is not given, so we will calculate until the value of root becomes same upto 4 decimal places in two consecutive iterations.

Here, $x_0 = 0$ and $x_1 = 1.3$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1.3 - \frac{(1.3 - 0)}{12.7858 - (-1)} (12.7858)$$

$\therefore$ $x_2 = 0.0942999$

Also $f(x_2) = f\,(0.0942999)^{10} - 1 = -1$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.0942999$ and $x_1 = 1.3$

Step IV : 2^{nd} Iteration :

Here, $x_0 = 0.0942999$ and $x_1 = 1.3$

and $f(x_0) = f\ (0.0942999) = -1$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\ f(x_1)$$

$$\therefore \quad x_2 = 1.3 - \frac{(1.3 - 0.0942999)}{12.7858 - (-1)}\ (12.7858)$$

$$\therefore \quad x_2 = 0.1817594$$

Also $f(x_2) = f\ (0.1817594)^{10} - 1 = -0.9999999$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.1817594$ and $x_1 = 1.3$

Step V : 3^{rd} Iteration :

Here, $x_0 = 0.1817594$ and $x_1 = 1.3$

and $f(x_0) = f\ (0.1817594) = -0.9999999$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)}\ f(x_1)$$

$$\therefore \quad x_2 = 1.3 - \frac{(1.3 - 0.1817594)}{12.7858 - (-0.9999999)}\ (12.7858)$$

$$\therefore \quad x_2 = 0.2628747$$

Also $f(x_2) = f\ (0.2628747)^{10} - 1 = 0.9999984$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.2628747$ and $x_1 = 1.3$

Step VI : 4^{th} Iteration :

Here, $x_0 = 0.2628747$ and $x_1 = 1.3$

and $f(x_0) = f(0.2628747) = -0.9999984$

Now,

$$x_2 = x_1 - \frac{(x_1 - x_0)}{f(x_1) - f(x_0)} f(x_1)$$

$$\therefore \quad x_2 = 1.3 - \frac{(1.3 - 0.2628747)}{12.7858 - (-0.9999984)} (12.7858)$$

$$\therefore \quad x_2 = 0.338106$$

Also $f(x_2) = f(0.338106)^{10} - 1 = -0.9999804$

Now, $f(x_0)$ and $f(x_2)$ are of the same sign, so replacing x_0 by x_2 we get the new approximation as

$x_0 = 0.338106$ and $x_1 = 1.3$

Step VII :

Similarly, we get

$x_2 = 0.407879$ after 5^{th} Iteration

$x_2 = 0.4725843$ after 6^{th} Iteration

$x_2 = 0.5325727$ after 7^{th} Iteration

$x_2 = 0.5881458$ after 8^{th} Iteration

$x_2 = 0.6395452$ after 9^{th} Iteration

$x_2 = 0.6869444$ after 10^{th} Iteration

$x_2 = 0.7304477$	after 11^{th} Iteration
$x_2 = 0.7701$	after 12^{th} Iteration
$x_2 = 0.8059087$	after 13^{th} Iteration
$x_2 = 0.837875$	after 14^{th} Iteration
$x_2 = 0.8660287$	after 15^{th} Iteration
$x_2 = 0.8904582$	after 16^{th} Iteration
$x_2 = 0.9113291$	after 17^{th} Iteration
$x_2 = 0.9288854$	after 18^{th} Iteration
$x_2 = 0.9434367$	after 19^{th} Iteration
$x_2 = 0.9553345$	after 20^{th} Iteration
$x_2 = 0.964946$	after 21^{st} Iteration
$x_2 = 0.97263$	after 22^{nd} Iteration
$x_2 = 0.9787193$	after 23^{rd} Iteration
$x_2 = 0.98351$	after 24^{th} Iteration
$x_2 = 0.9872569$	after 25^{th} Iteration
$x_2 = 0.9901736$	after 26^{th} Iteration
$x_2 = 0.9924355$	after 27^{th} Iteration
$x_2 = 0.9941844$	after 28^{th} Iteration
$x_2 = 0.9955335$	after 29^{th} Iteration
$x_2 = 0.9965724$	after 30^{th} Iteration
$x_2 = 0.9973712$	after 31^{st} Iteration
$x_2 = 0.9979848$	after 32^{nd} Iteration
$x_2 = 0.9984557$	after 33^{rd} Iteration

$x_2 = 0.9988169$ after 34^{th} Iteration

$x_2 = 0.9990938$ after 35^{th} Iteration

$x_2 = 0.999306$ after 36^{th} Iteration

$x_2 = 0.9994686$ after 37^{th} Iteration

$x_2 = 0.9995931$ after 38^{th} Iteration

$x_2 = 0.9996884$ after 39^{th} Iteration

$x_2 = 0.9997614$ after 40^{th} Iteration

$x_2 = 0.9998173$ after 41^{st} Iteration

$x_2 = 0.9998601$ after 42^{nd} Iteration

Since, the value of x2 in 41^{st} iterations are same upto 4 decimal places.

Smallest positive root of the given equation is 0.9998601

Step VIII :

For the given equation, the Bisection Method gives the root after 12^{th} iterations while Regula Falsi Method gives root after 42^{nd} iterations. So, for the given equation, Bisection method is preferable to Regula Falsi Method.

Unit 5

Newton – Rephson Method

Newton Raphson Method is much better and quicker method to find the root of a transcendental equation that Bisection or Regula Falsi method. It fact, it is has been mathematically proved that. Newton Raphson method is superior to other method.

Derivation of Newton – Rephson Method from Taylor's Series :

Let x_0 bean approximation root of the equation f(x) = 0 and let $x_1 = x_0 + h$ be the exact root so that $f(x_1) = 0$

Expanding f $(x_0 + h)$ by Taylor's series, we get

h^2

$$f(x_0 + h) = f(x_0) + h\, f(x_0) - \frac{\quad}{2!} f''(x_0)$$

Neglecting the second and higher order derivatives, we get

$$f(x_0 + h) = f(x_0) + h\, f'(x_0)$$

$$f(x_0) + h\, f(x_0) = 0 \qquad [\because f(x_1) = f(x_0 - h) = 0]$$

$$h = - \frac{f(x_0)}{f'(x_0)}$$

But $\quad x_1 = x_0 + h$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Successive approximation are given by $x_2, x_3, x_4, \ldots\ldots x_{n-1}$

Where

$$x_{n-1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Which is a general formula for Newton – Rephson's method.

GEOMETRICAL REPRESENTATION :

Geometrically the method consists in **replacing the part of the curse between [x_0, f (x_0)] the point and the x-asix by means of the tangent to the cure at the point and is described graphically as shown in Fig. 3**

In Newton – Raplison's method, we begin with just one point and consider a tangent to the curve at that point, for example in Fig 3. we start with a point p [x_0, f (x_0)] and draw a tangent to the curve at that point. The point at which this langent meets the x = asix is the next esfimated root and is denoted by x_1 Similarly x_2, x_3, …etc can be found. In this way, root of the given function is determined with the help of Newton – Raphson's method.

Fig. Newton – Raphson Method.

Questions asked from the above Topic in University Paper :

Ex. Prove that Newton – Raphson process has a quadrate convergence.

Procedure to Solve Problems based on Newton – Raphson's Method.

Step I : Find initial approximations if they are not given in the data of the problem.

Step II : Find the number of iterations required.

Step III : Find the root of the given equation by successive iterations using the Newton – Raphon's formula given by :

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The above procedure will get clear to you will go through the following solved example.

Ex. 11 Apply Newton Raphson's method to find a root of the equation $x^4 - x - 10 = 0$ correct to four decimal places.

Solution :

Step I : Finding initial approximations :

We are given that

$$f(x) = x^4 – x - 10$$

Now

$$f(x) = x^4 – 1$$

$$f(0) = (0)^4 – 0 - 10 = - 10$$

$$f(1) = (1)^4 – 1 – 10 = - 10$$

$$f(2) = (2)^4 – 2 – 10 = 4$$

$\therefore$ $f(1) < 0$ and $f(2) > 0$

$\therefore$ Root lies between (1, 2)

We will start with initial approximation of $x_0 = 2$

Step II : Finding the number of iterations required :

Since, it is given that value of the root should be correct to 4 decimal places, so we will calculate until the value of root becomes same upto 4 decimal places in two consecutive iterations.

Step III : 1st Iteration :

Here, $x_0 = 0$

$\therefore$ $f(x_0) = f(2) = (2)^4 – 2 – 10 = 4$

and $f(x_0) = f(2) = 4 \times (2)^4 – 1 = 31$

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \quad x_1 = 2 - \frac{4}{31}$$

$$\therefore \quad x_1 = 1.8709677$$

Step IV : 2nd Iteration :

Here, $x_1 = 1.8709677$

$\therefore$ $f(x_1) = (1.8709677)^4 – (1.8709677) – 10 = 0.3826746$

and $f(x_1) = 4 \times (1.8709677)^3 – 1 = 25.19744$

Now,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\therefore \quad x_2 = 1.8709677 - \frac{0.3826746}{25.19744}$$

$\therefore$ $x_2 = 1.8557807$

Step V : 3rd Iteration :

Here, $x_2 = 1.8557807$

$\therefore$ $f\,'(x_2) = f\,'(1.8557807)^4 – (1.8557807) – 10 = 0.0048169$

and $f\,'(x_2) = 4 \times (1.8557807)^3 – 1 = 24.564656$

Now,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\therefore \quad x_3 = 1.8557807 - \frac{0.0048169}{24.564656}$$

$\therefore$ $x_3 = 1.8555846$

Step VI : 4th Iteration :

Here, $x_3 = 1.8555846$

$\therefore$ $f\,'(x_3) = f\,'(1.8555846)^4 – (1.8555846) – 10 = 0.0000019$

and $f\,'(x_3) = 4 \times (1.8555846)^3 – 1 = 24.556553$

Now,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\therefore \quad x_4 = 1.8555846 - \frac{0.0000019}{24.556553}$$

$$\therefore \quad x_4 = 1.8555845$$

Since the value of root in 3rd and 4th iterations are same upto 4 decimal places.

$\therefore$ **Root of the given equation is 1.8555845**

Ex. 12 Use Newton – Raphson method to obtain a root to three decimal places of the following equation :

x sin x + cos x = 0

Solution :

Step I : Finding initial approximations :

We are given that

$$f(x) = x \sin x + \cos x$$

$$f'(x) = x \cos x - \sin x$$

Now,

$$f(0) = (0) \sin 0 - \cos 0 = 0$$

$$f\left(\frac{\pi}{2}\right) = \left(\frac{\pi}{2}\right) \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = \frac{\pi}{2}$$

2

$$f(\pi) = (\pi) \sin \pi + \text{con } \pi = -1$$

$$\therefore \quad f(\pi) < 0 \quad \text{and} \quad f\left(\frac{\pi}{2}\right) > 0$$

$$\therefore \quad \text{Root lies between} \left(\frac{\pi}{2}\ \pi\right)$$

$\therefore$ We will start with initial approximation of $x_0 = \pi$

Step II : Finding the number of iterations required :

Since, it is given that value of the root should be correct to 3 decimal places so we will calculate until the value of root becomes same upto 3 decimal places in two consecutive iterations.

Step III : 1st Iteration :

Here, $x_0 = \pi = 3.1415927$

$\therefore$ $f(x_0) = f(\pi) = (\pi) \sin \pi + \cos \pi = -1$

and $f'(x_0) = f'(\pi) = (\pi) \cos \pi - \sin \pi = -\pi = -3.1415927$

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \quad x_1 = 3.1415927 - \frac{(-1)}{(-3.1415927)}$$

$$\therefore \quad x_1 = 2.8232828$$

Step IV : 2nd Iteration :

Here, $x_1 = 2.8232828$

$$\therefore \quad f(x_1) = (2.8232828) \sin \left(2.8232828 \times \frac{180}{\pi}\right) + \cos \left(2.8232828 \times \frac{180}{\pi}\right) = -0.0661861$$

and

$$\therefore \quad f'(x_1) = (2.8232828) \cos \left(2.8232828 \times \frac{180}{\pi}\right) + \cos \left(2.8232828 \times \frac{180}{\pi}\right) = 2.994419$$

Now,

$f(x_1)$

$$x_2 = x_1 - \frac{\quad}{f'(x_1)}$$

$$\therefore \quad x_2 = 2.8232828 - \frac{(-0.0661861)}{(-2.994419)}$$

$$\therefore \quad x_2 = 2.8011796$$

Step V : 3rd Iteration :

Here, $x_2 = 2.8011796$

$$\therefore \quad f(x_2) = (2.8011796)\sin\left(2.8011796 \times \frac{180}{\pi}\right) + \cos\left(2.8011796 \times \frac{180}{\pi}\right) = -0.0073690$$

and

$$\therefore \quad f'(x_2) = (2.8011796)\cos\left(2.8011796 \times \frac{180}{\pi}\right) + \cos\left(2.8011796) \times \frac{180}{\pi}\right) = 2.9719914$$

Now,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\therefore \quad x_3 = 2.8011796 - \frac{(-0.0073690)}{(-2.9743155)}$$

$$\therefore \quad x_3 = 2.7987021$$

Step VI : 4th Iteration :

Here, $x_3 = 2.7987021$

$$\therefore \quad f(x_3) = (2.7987021)\sin\left(2.7987021 \times \frac{180}{\pi}\right) + \cos\left(2.7987021 \times \frac{180}{\pi}\right) = -0008328$$

and

$\therefore$ $f'(x_3)=(2.7987021)\cos\left(2.7987021 \times \frac{180}{\pi}\right) + \cos\left(2.7987021) \times \frac{180}{\pi}\right)$
= 2.9719914

Now,

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$\therefore \quad x_4 = 2.7987021 - \frac{(-0.0008328)}{(-2.9719914)}$$

$$\therefore \quad x_4 = 2.7984219$$

Since, the values of root in 3rd and 4th iterations are same upto 3 decimal places.

$\therefore$ **Root of the given equation is 2.7984219**

Ex. 13 Use Newton Raphson method to obtain the root up to 4 decimal places for the
following equation:

sin (x) = 1 – x

Solution :

Step I : Finding initial approximations :

We are given that

$\text{Sin } x = 1 - x$

$f(x) = x + \sin x - 1$

$f'(x) = 1 + \cos x$

Now,

$f(0) = 0 + \sin - 1 = -1$

$f(\pi) = \pi + \sin \pi - 1 = 2.1415927$

$\therefore$ $f(0) < 0$ and $f(\pi) > 0$

$\therefore$ Root lies between $(0, \pi)$

$\therefore$ We will start with initial approximation of $x_0 = 0$

Step II : Finding the number of iterations required :

Since, it is given that value of the root should be correct to 4 decimal places so we will calculate until the value of root becomes same upto 4 decimal places in two consecutive iterations.

Step III : 1st Iteration :

Here, $x_0 = 0$

$\therefore$ $f(x_0) = 0 + \sin 0 + 1 = -1$

and $f'(x_0) = 1 + \cos 0 = 2$

Now,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$\therefore \quad x_1 = x_0 - \frac{(-1)}{(2)}$$

$\therefore$ $x_1 = 0.5$

Step IV : 2nd Iteration :

Here, $x_1 = 0.5$

$$\therefore \quad f(x_1) = (0.5) + \sin\left(0.5 \times \frac{180}{\pi}\right) - 1 = 0.0205744$$

and

$$\therefore \quad f'(x_1) = 1 + \cos\left(0.5 \times \frac{180}{\pi}\right) = 1.8775826$$

[Not : $\frac{180}{\pi}$ multiplication factor is used to convert the value in degrees]

Now,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$\therefore \quad x_2 = 0.5 - \frac{(-0.0205744)}{(1.8775826)}$$

$$\therefore \quad x_2 = 0.5109579$$

Step V : 3^{rd} Iteration :

Here, $x_1 = 0.5109579$

$$\therefore \quad f(x_2) = (0.5109579) + \sin\left(0.5109579 \times \frac{180}{\pi}\right) - 1 = 0.000029$$

and

$$\therefore \quad f'(x_2) = 1 + \cos\left(0.5109579 \times \frac{180}{\pi}\right) = 1.8722765$$

Now,

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$\therefore \quad x_3 = 0.5109579 - \frac{(-0.000029)}{(1.8722765)}$$

$$\therefore \quad x_3 = 0.5109733$$

Since the value of root in 2^{nd} and 3^{rd} iterations are same upto 4 decimal places.

∴

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3

INTERPOLATION

Introduction

Interpolation means the art of reading between the lines of table. It also means filling up intermediate terms of a set of given values.

Consider a function $\gamma = (\chi)$, $a \leq \chi \leq b$. Since the expression for $f(\chi)$ is known, for and value of χ between a and b, we can always find the corresponding values of f.

However in practice, many times we known a set of values of χ between a and b and the corresponding values of f are also known, but the explicit expression for $f(\chi)$ is not known.

The process of finding such a expression (function) using a set of values is known as ***interpolation.***

Finite Difference Operators

Various finite difference operators along with their definition and symbols are as follows :

(1) Shift Operator (E) :

Shift operator is the operator which when operated on a function $f(\chi)$ gives the next values of that function. Shift operator is denoted by E.

i.e., $Ef(\chi) = f(\chi + h)$

(2) Forward Difference Operator (Δ) :

Forward Difference operator is the operator which when operated on a function $f(\chi)$ given the difference between the present and next value of that function. Forward Difference Operator is denoted by Δ.

i.e., $\Delta f(\chi) - f(\chi + h) - f(\chi)$

(3) Backward Difference Operator (∇) :

Backward difference operator is the operator which when operated on a function $f(\chi)$ gives the difference between the present and previous values of that function. Backward Difference Operator is denoted by ∇.

i.e., $\nabla f(\chi) - f(\chi\ ) - f(\chi - h)$

(4) Central Difference Operator (δ) :

Central Difference Operator is denoted by δ. Central Difference Operator is the operator which when operated on a function $f(\chi)$ gives the central difference between two values of that function.

i.e., $$\delta f(\chi) = f\left(\chi + \frac{h}{2}\right) - f\left(\chi - \frac{h}{2}\right)$$

(5) Averaging Operator (μ) :

Averaging Operator is the operator which when operated on a function $f(\chi)$ gives the average of two values of the given function. Averaging Operator is denoted by μ.

i.e., $$\mu f(\chi) = \frac{1}{2}\left(f\left(\chi + \frac{h}{2}\right) - f\left(\chi - \frac{h}{2}\right)\right)$$

RELATIONS BETWEEN FINITE DIFFERENCE OPERATORS

(a) Relation between E and Δ:

As per definition of E we have

$$Ef(\chi) = f(\chi + h)$$

And as per definition of Δ we have

$$\Delta f(\chi) = f(\chi + h) - f(\chi)$$

$$\Delta f(\chi) = Ef(\chi) - f(\chi) \qquad [□\ f(\chi) = f(\chi + h)$$

$$Ef(\chi) = f(\chi) + \Delta f(\chi)$$

$$Ef(\chi) = (1 + \Delta)\, f(\chi)$$

$$E = 1 + \Delta$$

(b) Relation between E and ∇:

As per definition of E we have

$$Ef(\chi) = f(\chi + h)$$

$$E^{-1}f(\chi) = f(\chi - h) \qquad \text{(Negative value)}$$

And as per definition of Δ we have

$$\nabla f(\chi) = f(\chi) - f(\chi - h)$$

$$\nabla f(\chi) = f(\chi) - E^{-1}f(\chi) \qquad [□E^{-1}f(\chi) = f(\chi - h)]$$

$$\nabla f(\chi) = (1-E^{-1})\, f(\chi)$$

$$\nabla = 1 + E^{-1}$$

(c) Relation between E and δ:

As per definition of E we have

$$Ef(\chi) = f(\chi + h)$$

$$E^{1/2}f(\chi) = f\left(\chi + \frac{h}{2}\right) \quad \text{and} \quad E^{-1/2}f(\chi) = f\left(\chi - \frac{h}{2}\right)$$

and as per definition of δ, we have

$$\delta f(\chi) = f\left(\chi + \frac{h}{}\right) - f\left(\chi - \frac{h}{}\right)$$

2 2

$$\delta f(\chi) = E^{1/2} f(\chi) - E^{-1/2} f(\chi)$$

$$\delta f(\chi) = (E^{1/2} - E^{-1/2}) f(\chi)$$

$$\delta = E^{1/2} - E^{-1/2}$$

(d) Relation between E and μ.

As per definition of E we have

$$E f(\chi) = f(\chi + h)$$

$$E^{1/2} f(\chi) = f\left(\chi + \frac{h}{2}\right) \quad \text{and} \quad E^{-1/2} f(\chi) = f\left(\chi - \frac{h}{\mathbf{2}}\right)$$

and as per definition of μ, we have

$$\mu f(\chi) = \frac{1}{2}\left(f\left(\chi + \frac{h}{2}\right) - f\left(\chi - \frac{h}{2}\right)\right)$$

$$\mu f(\chi) = \frac{1}{2} [E^{1/2} f(\chi) - E^{-1/2} f(\chi)]$$

$$\mu f(\chi) = \frac{1}{2} [E^{1/2} - E^{-1/2}] f(\chi)$$

$$\mu = \frac{1}{2} [E^{1/2} - E^{-1/2}]$$

Ex. 1 Show that : $\Delta = \nabla (1- \nabla)^{-1}$

Solution :

R. H. S. $= \nabla (1- \nabla)^{-1}$

$= [(1- E^{-1}) 1 - (1- E^{-1})]^{-1}$ [$\square \nabla$ = 1 E^{-1}]

$= (1- E^{-1}) [E^{-1}]^{-1}$

$$= (1 - E^{-1})\, E$$

$$= \left(1 - \frac{1}{E}\right) E$$

$$= \left(\frac{E-1}{E}\right) E$$

$$= E - 1$$

$$= \Delta = \text{L.H.S.}$$

$$= \Delta = \nabla\,(1 - \nabla)^{-1} \qquad [\square\Delta = E - 1]$$

Ex. 2 Show that : $1 + \Delta = (E - 1)\,\nabla^{-1}$

Solution :

R. H. S. $= (E - 1)\nabla^{-1}$

$$= \frac{E-1}{\nabla} \qquad [\square\nabla = 1\ E^{-1}]$$

$$= \frac{E-1}{1 - E^{-1}}$$

$$= \frac{E-1}{1 - 1/E}$$

$$= \frac{E-1}{(E-1)/E}$$

$$= E$$

$$= 1 + \Delta$$

$$= \text{L.H.S.} \qquad [\square E = 1 - \Delta]$$

$$1 + \Delta = (E - 1)\nabla^{-1}$$

Ex. 3 Show that $\mu = \dfrac{2 + \Delta}{2\sqrt{(1 + \Delta)}} = \dfrac{2 - \nabla}{2\sqrt{(1 - \nabla)}}$

Solution :

From the relation between μ and E, we have

$$\mu = \frac{1}{2} \left(E^{1/2} + E^{-1/2} \right)$$

Step I :

Substituting the relation E = 1 + Δ in equation (I), we have

$$\mu = \frac{1}{2}\left((1 + \Delta)^{1/2} + (1 + \Delta)^{-1/2} \right)$$

$$\mu = \frac{1}{2}\left((1 + \Delta)^{1/2} + \frac{1}{(1 + \Delta)^{-1/2}} \right)$$

$$\mu = \frac{1}{2}\left(\frac{(1 + \Delta) + 1}{(1 + \Delta)^{1/2}} \right)$$

… (II)

$$\mu = \frac{2 + \Delta}{2 \sqrt{(1 + \Delta)}}$$

Step II :

Substituting the relation E = 1 + ∇ in equation (I), we have

$$\mu = \frac{1}{2}\left((1 - \nabla)^{1/2} + (1 - \nabla)^{1/2} \right)$$

$$\mu = \frac{1}{}\left(\frac{1}{} + (1 - \nabla)^{1/2} \right)$$

$$2 \qquad (1 - \nabla)^{1/2}$$

$$\mu = \frac{1}{2}\left(\frac{1 + (1 - \nabla)}{(1 - \nabla)^{1/2}}\right)$$

… (III)

$$\mu = \frac{2 - \nabla}{2\sqrt{(1 - \nabla)}}$$

Step III :

From equations (II) and (III), we have

$$\mu = \frac{2 + \Delta}{2\sqrt{(1 + \Delta)}} = \frac{2 - \nabla}{2\sqrt{(1 - \nabla)}}$$

Ex. 4 Show that $\mu\,\delta = \frac{1}{2}(\Delta + \nabla)$

Solution :

From the relation between $\mu\,\delta$ and E, we have

$$\mu = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) \qquad \text{…(I)}$$

and from the relation between δ and E we have

$$\delta = E^{1/2} - E^{-1/2} \qquad \text{…(I)}$$

Multiplying equation (I) and (II), we have

$$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) \times \left(E^{1/2} - E^{-1/2}\right)$$

$$\mu\delta = \frac{1}{2}\left(E - E^{-1}\right)$$

$$\mu\delta = \frac{1}{2} (1 + \Delta) - (1 - \nabla) \qquad [\because E = 1 + \Delta \text{ and } \nabla = 1 - E^{-1}]$$

$$\mu\delta = \frac{1}{2} (\Delta + \nabla)$$

Ex. 5 Show that $\mu^2 = 1 + \dfrac{\delta^2}{4}$

Solution :

Step I :

From the relation between μ and E, we have

$$\mu = \frac{1}{2} (E^{1/2} + E^{-1/2})$$

Squaring above equation, we have

$$\mu^2 = \frac{1}{4} (E + 2 + E^{-1}) \qquad \ldots(I)$$

Step II :

From the relation between δ and E, we have

$$\delta = E^{1/2} - E^{-1/2}$$

Squaring above equation, we have

$$\delta^2 = E - 2 + E^{-1} \qquad \ldots(II)$$

Step III :

Equation (II) can be written as

$$\delta^2 = (E + 2 + E^{-1}) - 4$$

$$\delta^2 = 4\mu^2 - 4 \qquad [\because \mu^2 = \frac{1}{4} (E + 2 + E^{-1}) \text{ from equations (I)}$$

$$\frac{\delta^2}{4} = \mu^2 - 1$$

$$\mu^2 = 1 + \frac{\delta^2}{4}$$

Unit 3

Difference Tables

(i) Forward Difference Table :

If y_0, y_1, y_2 …. y_n denote a set of values of y, then ($y_1 - y_0$), ($y_2 - y_1$) …. ($y_n - y_{n-1}$) are called the first forward differences of y and are denoted by Δy_0 , Δy_1 …. Δy_2 respectively. Thus, we have

$$\Delta y_0 = y_1 - y_0,$$

$$\Delta y_1 = y_2 - y_1,$$

$$\Delta y_{n-1} = y_n - y_{n-1},$$

The difference of the first forward difference are called second forward difference and are denoted by $\Delta^2 y_0$, $\Delta^2 y_1$ …. Similarly, we can define third forward differences, fourth forward differences, etc.

The following table shows how the forward differences of all orders can be formed.

χ	Y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
χ_0	y_0						
		Δy_0					
χ_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
χ_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		

		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
χ_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
χ_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
χ_5	y_5		$\Delta^2 y_4$				
		Δy_5					
χ_6	y_6						

It should be noted that differences $\Delta^k y_6$ i.e., on straight line stopping downward to the right

(II) Backward Difference Table :

If y_0, y_1, y_2 …. y_n denote a set of values of y, then ($y_1 – y_0$), ($y_2 – y_1$) …. ($y_n – y_{n-1}$) are called the **first backward differences of y** and are denoted by ∇y_0 , ∇y_1 …. ∇y_2 respectively. Thus, we have

$$\nabla y_0 = y_1 - y_0,$$

$$\nabla y_1 = y_2 – y_1,$$

$$\nabla y_{n-1} = y_n – y_{n-1},$$

The difference of the first forward difference are called second backward difference and are denoted by $\nabla^2 y_0$, $\nabla^2 y_1$ …. Similarly, we can define third backward differences, fourth backward differences, etc.

The following table shows how the backward differences of all orders can be formed.

χ	Y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
χ_0	y_0						
		∇y_0					
χ_1	y_1		$\nabla^2 y_0$				
		∇y_1		$\nabla^3 y_0$			
χ_2	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$		
		vy_2		$\nabla^3 y_1$		$\nabla^5 y_0$	
χ_3	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$		$\nabla^6 y_0$
		∇y_3		$\nabla^3 y_2$		$\nabla^5 y_1$	
χ_4	y_4		$\nabla^2 y_3$		$V^4 y_2$		
		∇y_4		$\nabla^3 y_3$			
χ_5	y_5		$\nabla^2 y_4$				

		∇y_5					
χ_6	y_6						

It should be noted that the differences $\nabla^k y_6$ lie on a straight line sloping upward to the right.

Unit 5

Newton's Formulae for Interpolation

(i) Newton's Forward Difference Interpolation formula :

Given a set of (n + 1) values, viz., $(\chi_0, y_0), (\chi_1, y_1), (\chi_2, y_2), \ldots (\chi_n, y_n)$, of χ and y, it is required to find $y_n(\chi)$ a polynomial of the n^{th} degree such that y and $y_n(\chi)$ agree at the tabulated points.

Let the values of χ be equidistant.

i.e., Let $\chi_i = \chi_0 + ih.$ i = 0,1,2…..n.

Step I :

Since $y_n(\chi)$ is a polynomial of the n^{th} degree, it may be written as

$$y_n(\chi) = a_0 + a_1 (\chi - \chi_0) + a_2 (\chi - \chi_0)(\chi - \chi_1) + a_3 (\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_2) + a_n (\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_2) \ldots (\chi - \chi_{n-1}) \quad \ldots(I)$$

Step III :

Setting $\chi = \chi_0 + ph$ and substituting values of $a_0, a_1, \ldots.. a_n$ in equation (I) we get

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{p (p-1)}{2!} \Delta^2 y_0 + \frac{p (p-1)(p-2)}{3!} \Delta^3 y_0 + \ldots + \frac{p (p-1)(p-2) \ldots (p + n-1)}{n!} \Delta^n y_0$$

where $p = \dfrac{\chi - \chi_0}{h}$

The above formula is known as **Newton's forward difference interpolation formula.**

Newton's forward difference interpolation formula is useful for interpolation near beginning of a set of tabular values or for extrapolation for the values slightly less than the tabulated values.

Procedure to solve problems using Newton's Forward Difference formula :

Step I : Find the interpolation formula to be used

Step II : Find χ_0, y_0, h and p from the given data.

Step III : Make forward difference table.

Step IV : Find the required value using Newton's forward difference formula

The above procedure will get clear to you as you will go through the following solved example.

Ex. 7 The population of a town in the decennial census were as given below.
Estimate the population for the year 1895.

Year, χ	1891	1901	1911	1921	1931
Population, y (in thousands)	46	66	81	93	101

Solution :

Step I : Finding the interpolation formula to be used :

In this example, interpolation is desired near the beginning of the table and so we will use Newton's forward difference interpolation formula.

Step II : Find χ_0, y_0, *h* and p from the given data.

From the table, we see that

$\chi_0 = 1891$, $y_0 = 46$ and $h = 10$

$$p = \frac{\chi - \chi_0}{h} = \frac{1895 - 1891}{10} = 0.4$$

h 10

Step III : Make forward difference table.

From the data given in the problem, we will form the forward difference table as follows.

χ	y	Δ	Δ^2	Δ^3	Δ^4
1891	46				
		20			
1901	66		-5		
		15		2	
1911	81		-3		-3
		12		-1	
1921	93		-4		
		8			
1931	101				

Step IV : Finding y at χ = 1895 :

Newton's forward difference formula is given by :

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{p\ (p-1)}{2!}\Delta^2 y_0 + \frac{p\ (p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots$$

$$+ \frac{p\ (p-1)(p-2)\ (p+3)}{4!}\ \Delta^4 y_0$$

$$\therefore \qquad y(1895 = 46 + (0.4)(20) + \frac{(0.4)(0.4 - 1)}{2}(5)$$

$$+ \frac{(0.4)(0.4 - 1)(0.4 - 2)}{}(2) + \frac{(0.4)(0.4 - 1)(0.4 - 2)(0.4 - 3)}{}(-3)$$

6 24

$\therefore$ y(1895 = 46 + (0.6) + (0.4533) + (0.1248)

$\therefore$ y(1895 = 55.1781 thousand

Population for the year 1895 was 55,178

(ii) Newton's Backward Difference Interpolation formula :

Given a set of (n + 1) values, viz., (χ_0, y_0), (χ_1, y_1), (χ_2, y_2), … (χ_n, y_n), of χ and y, it is required to find $y_n(\chi)$ a polynomial of the n^{th} degree such that y and $y_n(\chi)$ agree at the tabulated points.

Let the values of χ be equidistant.

i.e., Let $\chi_i = \chi_0 + ih.$ i = 0,1,2…..n.

Step I :

Since $y_n(\chi)$ is a polynomial of the n^{th} degree, it may be written as

$$y_n(\chi) = a_0 + a_1 (\chi - \chi_n) + a_2 (\chi - \chi_n)(\chi - \chi_{n-1}) + a_3 (\chi - \chi_n)(\chi - \chi_{n-1})(\chi - \chi_{n-2})$$
$$+\ldots+ a_n (\chi - \chi_n)(\chi - \chi_{n-1})(\chi - \chi_{n-2}) \ldots (\chi - \chi_1) \qquad \ldots(I)$$

Step III :

Imposing now the condition that y and $y_n(\chi)$ should agree at the set of tabulated point, we obtain

$$a_0 = y_n$$

$$a_1 = \frac{y_n - y_{n-1}}{\chi_1 - \chi_0} = \frac{\nabla y_n}{h}$$

$$a_2 = \frac{\nabla^2 y_n}{h^2\, 2!}$$

$$a_3 = \frac{\nabla^3 y_n}{h^3\, 3!}$$

……………….

$$a_n = \frac{\nabla^n y_n}{h^n\, n!}$$

Step III :

Setting $\chi = \chi_n + ph$ and substituting values of a_0, a_1, …, a_n in equation (I), we get

$$y_n(\chi) = y_n + p\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \ldots + \frac{p\,(p+1)\ldots(p+n-1)}{n!}\nabla^n y_n$$

$$\text{where } p = \frac{\chi - \chi_n}{h}$$

The above formula is known as Newton's backward difference interpolation formula.

Newton's backward difference interpolation formula is useful for interpolation near the end of the tabular values or for extrapolation for the values slightly greater than the tabulated values.

Procedure to solve problems using Newton's Backward Difference formula :

Step I : Find the interpolation formula to be used

Step II : Find χ_n, y_n, h and p from the given data.

Step III : Make backward difference table.

Step IV : Find the required value using Newton's backward difference formula

The above procedure will get clear to you as you will go through the following solved example.

Ex. 8 From the data find y at 4.5.

χ	1	2	3	4	5
y	2.38	3.65	5.85	9.95	14.85

Solution :

Step I : Finding the interpolation formula to be used :

In this example, interpolation is desired at the end of the table and so we will use Newton's backward difference interpolation formula.

Step II : Find χ_n, y_n, h and p from the given data.

From the table, we see that

$\chi_n = 5$, $y_n = 14.85$ and $h = 1$

$$p = \frac{\chi - \chi_n}{h} = \frac{4.5 - 5}{1} = 0.5$$

Step III : Forming backward difference table :

From the data given in the problem, we will form the backward difference table as follows.

χ	y	∇	∇^2	∇^3	∇^4
1	2.38				
		1.27			
2	3.65		0.93		
		2.20		1.90	
3	5.85		1.90		-2.07
		4.10		-1.10	
4	9.95		0.80		
		4.90			
5	14.85				

Step IV : Finding y = (4.5) :

Newton's backward difference formula is given by :

$$y_n(\chi) = y_n + p\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots + \frac{p\,(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$\therefore \quad y(4.5 = 14.85 + (-0.5)(4.90) + \frac{(-0.5)(-0.5+1)}{} (0.80)$$

$$+ \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(-1.10) + \frac{(-0.5)(-0.5+1)(-0.5+2)(-0.5+3)}{24}(-2.07)$$

2

$\therefore$ y(4.5) = 14.85 – 2.45 + (-0.125)(0.80) + (0.0625)(-1.10) + (-0.039)(-2.07)

$\therefore$ y(4.5) = 14.85 – 2.45 – 0.1 – 0.06875 + 0.08073

$\therefore$ y(4.5) = 12.44948

$\therefore$ **y(4.5) = 12.45**

Examples Based on application of both Newton's Forward Difference and Newton's Backward Difference Formula :

[Note : In examples based on application of both forward difference and backward difference formula, we have to prepare the difference table only once. From the same difference table, forward difference are taken diagonally down from top to bottom and backward difference are taken diagonally up from bottom to top. This will get clear to you as you will go through the following solved examples.

Ex. 9 Given the table

χ	0.15	0.17	0.19	0.21	0.23
Sin χ	0.14944	0.16918	0.18886	0.20846	0.27798

Solution :

Case (I) : Evaluation of sin(0.157)

Step I : Finding the interpolation formula to be used :

In this case interpolation is desired near the beginning of the table and so will use Newton's toward difference interpolation formula.

Step II : Finding χ_0, y_0, *h* and p :

From the table, we see that

$\chi_0 = 0.15$, $y_0 = 0.14944$ and $h = 0.02$

$$p = \frac{\chi - \chi_0}{h} = \frac{0.157 - 0.15}{0.02} = 0.35$$

Step III : Forming forward difference table :

From the data given in the problem, we will form the backward difference table as follows.

χ	y	Δ	Δ^2	Δ^3	Δ^4
0.15	0.14944				
		0.01974			
0.17	0.16918		-0.00006		
		0.01968		-0.00002	
0.19	0.18886		-0.00008		0.00002
		0.01960		0	
0.21	0.20846		-0.00008		
		0.01952			
0.23	0.22798				

Step IV : Finding sin = (0.157) :

Newton's forward difference formula is given by :

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{p\,(p+1)}{2!}\Delta^2 y_0 + \frac{p\,(p+1)(p+2)}{3!}\Delta^3 y_0 + \frac{p\,(p+1)(p+2)(p+3)}{4!}\Delta^4 y_0$$

$$\therefore \quad \sin(0.157) = 0.14944 + 0.35(0.01974) + \frac{(0.35)(0.35-1)}{2}(-0.00006)$$

$$+ \frac{(0.35)(0.35-1)(0.35-2)}{6}(-0.00002) + \frac{(0.35)(0.35-1)(0.35-2)(0.35-3)}{24}(0.00002)$$

$$\therefore \quad \sin(0.157) = 0.14944 + 0.006909 + (-0.11375)(-0.00006) + (0.6256)(-0.00002) + (0.04144)(0.00002)$$

$\therefore \quad \sin(0.157) = 0.14944 + 0.006909 + 0.0000068 - 0.0000012 - 0.0000008$

$\therefore \quad \sin(0.157) = 0.1563538$

$\therefore \quad \mathbf{\sin(0.157) = 0.15635}$

Case (II) : Evaluation of sin(0.235)

Step I : Finding the interpolation formula to be used :

In this case, extrapolation is desired for the value slightly greater than the tabulated values and so we will use. Newton's backward difference interpolation formula.

Step II : Finding χ_n, y_n, *h* and p :

From the table, we see that

$\chi_n = 0.23, \qquad y_n = 0.2279 \qquad \text{and} \qquad h = 0.02$

$$p = \frac{\chi - \chi_n}{h} = \frac{0.235 - 0.23}{0.02} = 0.25$$

Step IV : Finding sin = (0.235) :

Newton's backward difference formula is given by :

$$y_n(\chi) = y_n + p\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p\,(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

Backward difference are taken from the same forward difference table by considering the value 0.01952 for ∇y_n and moving diagonally upward from bottom to top.

$$\therefore \quad \sin(0.235) = 0.22798 + 0.25(0.01952) + \frac{(0.25)(0.25-1)}{2}(-0.00008)$$

$$+ \frac{(0.25)(0.25+1)(0.25+2)}{}(0) + \frac{(0.25)(0.25+1)(0.25+2)(0.25+3)}{}(0.00002)$$

6 24

$\therefore$ sin(0.235) = 0.22798 + 0.00488 +(0.15625)(-0.00008)+(0.11718)(0) + (0.09521)(0.00002)

$\therefore$ sin(0.235) = 0.22798 + 0.00488 + 0.0000125 + 0 + 0.00000019

$\therefore$ sin(0.235) = 0.2328744

$\therefore$ **sin(0.235) = 0.2328744**

Ex. 10 An experiment carried out on a circuit gave the following readings :

V_{in} (Volts)	0	0.01	0.02	0.03	0.04	0.05
V_{out} (Volts)	0	1.52	2.65	3.84	4.65	6.25

Using the above table, evaluate –

(i) V_{out} (Volts) = 0.025 volts

(ii) V_{in} (Volts) = 5.00 volts

Solution :

Case (I) : Evaluation of V_{out} at V_{in} = 0.025 volts

Step I : Finding the interpolation formula to be used :

In this case interpolation is desired near the beginning of the table and so will use Newton's forward difference interpolation formula.

Step II : Finding χ_0, y_0, h and p :

From the table, we see that

$\chi_0 = V_{in}(0) = 0.01$, $y_0 = V_{out}(0)$ 1.52 and $h = 0.01$

$$p = \frac{\chi - \chi_0}{h} = \frac{0.025 - 0.01}{0.02} = 1.5$$

Step III : Forming forward difference table :

From the data given in the problem, we will form the backward difference table as follows.

V_{in} (χ)	V_{out} (y)	Δ	Δ^2	Δ^3	Δ^4
0.01	1.52				
		1.13			
0.02	2.65		0.06		
		1.19		-0.44	
0.03	3.84		-0.38		1.61
		0.81		1.17	
0.04	4.65		0.79		
		1.6			
0.05	6.25				

Step IV : Finding V_{out} at V_{in} = 0.025 volts:

Newton's Forward difference formula is given by :

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{p\,(p+1)}{2!}\Delta^2 y_0 + \frac{p\,(p+1)(p+2)}{3!}\Delta^3 y_0 + \frac{p\,(p+1)(p+2)(p+3)}{4!}\Delta^4 y_0$$

$$\therefore \quad V_{out}(0.025) = 1.52 + 1.5(1.13) + \frac{(1.5)(1.5-1)}{2}(0.06)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)}{6}(-0.44) + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{24}(1.61)$$

$$\therefore \quad V_{out}(0.025) = 1.52 + 1.695 + (0.375)(0.06) + (-0.625)(-0.44) + (0.0234375)(1.61)$$

$$\therefore \quad V_{out}(0.025) = 1.52 + 1.695 + 0.0225 + 0.0275 + 0.0377343$$

$$\therefore \quad V_{out}(0.025) = 3.3027343$$

$\therefore$ **$V_{out}(0.025) = 3.30$**

$V_{out} = 3.30$ V at $V_{in} = 0.025$ V

Case (II) : Evaluation V_{in} at V_{out} = 5.00 volts:

[In this case, we have to find out p V_{out} = 5.00 and from p, we will find out the values of x i.e. V_{in}]

Step I : Finding the interpolation formula to be used :

In this case interpolation is desired near the end of the table and so will use Newton's backward difference interpolation formula.

Step II : Finding χ_n, y_n, *h* and p :

From the table, we see that

$\chi_n = 0.05$, $y_n = 6.25$ and $h = 0.01$

$$p = \frac{\chi - \chi_n}{h} = \frac{\chi - 0.05}{0.01}$$

Step III : Finding the value p :

Newton's backward difference formula is given by :

$$y_n(\chi) = y_n + p\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n + \frac{p\,(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p\,(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

Neglecting $\nabla^3 y_0$ and higher order difference, we have

$$y_0(\chi) = y_n + p\nabla y_n + \frac{p\,(p+1)}{2!}\nabla^2 y_n$$

Backward difference are taken from the same forward difference table by considering the value 1.6 for ∇y_n and moving diagonally upward from bottom to top.

$$\therefore \quad 5 = 6.25 + p\,(1.6) + \frac{p\,(p+1)}{2!}\,(0.79)$$

$$\therefore \quad 5 = 6.25 + p\,(1.6) + 0.395\,(p^2 + p)$$

$$\therefore \quad 0.395\,p^2 + 1.995p + 1.25 = 0$$

On solving above equation, we get

$$p = 0.7329251$$

Step IV : Finding V_{in} at V_{out} = 5.00 volts:

But, we have

$$p = \frac{\chi - 0.05}{0.01}$$

$$-0.7329251 = \frac{\chi - 0.05}{0.01}$$

$$-0.007329251 = \chi - 0.05$$

$$\boldsymbol{\chi = -0.0426707}$$

V_{in} = 0.4267 V at V_{out} = 5.00 V

Ex. 11 From the following table, find the number of students who

(i) Obtained less than 45 marks

(ii) Obtained more than 65 marks

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	30	41	52	36	31

Solution :

Step I : Making cumulative frequency table :

Since we want students with marks less than 45 and students with marks greater than 65, we will first mark cumulative frequency table

Marks (χ)	**No. of Students (y)**
Below 40	30
Below 50	30 + 41 = 71
Below 60	71 + 52 = 123
Below 70	123 + 36 = 159
Below 80	159 + 31 = 190

Step II : Forming difference table.

From the cumulative frequmcy table we will form the difference table as follows

χ	Y	Δ	Δ^2	Δ^3	Δ^4
Below 40	30				
		41			
Below 50	71		11		
		52		-27	
Below 60	123		-16		38
		36		11	
Below 70	159		-5		
		31			
Below 80	190				

Case (I) : Finding number of Students who obtained less than 45 marks :

Step I : Finding χ_0, y_0, h and p :

From the table, we see that

$\chi_0 = 40$, $y_0 = 30$ and $h = 10$

$$p = \frac{\chi - \chi_0}{h} = \frac{45 - 40}{0.02} = 0.5$$

Step II : Finding number of students getting less than 45 marks :

Newton's forward difference formula is given by :

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{p\,(p+1)}{2!}\Delta^2 y_0 + \frac{p\,(p+1)(p+2)}{3!}\Delta^3 y_0 + \frac{p\,(p+1)(p+2)(p+3)}{4!}\Delta^4 y_0$$

$$y(45) = 30 + (0.5)(41) + \frac{0.5(0.5-1)}{2}(11)$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{6}(-27) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(38)$$

y(45) = 30 + 20.5 + (-0.125(11) + (0.0625) + 0.0625)(-27) + (-0.0390625)(38)

y(45) = 30 + 20.5 – 1.375 – 1.6875 – 1.484375

y(45) = 45.953125

y(45) = 46

Students scoring marks below 45 is 46

Case (II) : Finding number of Students who obtained more than 65 marks :

Step I : Finding χ_0, y_0, *h* and p :

From the table, we see that

$\chi_0 = 40$, $y_0 = 30$ and $h = 10$

$$p = \frac{\chi - \chi_0}{h} = \frac{65 - 40}{10} = 2.5$$

Step II : Finding number of students getting less than 65 marks :

Newton's forward difference formula is given by :

p (p -1) p (p-1)(p-2) p (p-1)(p-2)(p-3)

$$y_n(\chi) = y_0 + p\Delta y_0 + \frac{\qquad}{2!}\Delta^2 y_0 + \frac{\qquad}{3!}\Delta^3 y_0 + \frac{\qquad}{4!}\Delta^4 y_0$$

$$\therefore \quad y(65) = 30 + (2.5)(41) + \frac{2.5(2.5 - 1)}{2}(11)$$

$$+ \frac{2.5(2.5-1)(2.5-2)}{6}(-27) + \frac{2.5(2.5-1)(2.5-2)(2.5-3)}{24}(38)$$

$\therefore$ $y(65) = 30 + 102.5 + (1.875)(11) + (0.3125)(-27) + (-0.0390625)(38)$

$\therefore$ $y(65) = 30 + 102.5 + 20.625 – 8.4375 – 1.484375$

$\therefore$ $y(65) = 143.20313$

$\therefore$ $y(65) = 143$

$\therefore$ Students scoring marks below 65 is 143

$\therefore$ **Students scoring marks above 65 is (190 – 143) = 47**

Lagrange's Interpolation Formula

Newton's forward difference and backward difference interpolation formulae derived in the previous unit possess the disadvantage that they required the values of the independent variable to be equally spaced. But in certain cases, the independent veritable is given with unequally spaced values of the argument. In such cases, another interpolation formula called **Lagrange's interpolation formula** is used.

Derivation of Lagrange's Interpolation Formula :

Step I :

Let $f(\chi_0)$, $f(\chi_1)$, $f(\chi_2)$, ... $f(\chi_n)$ be the values of the function $y = f(\chi)$ corresponding to the arguments χ_0, χ_1, χ_2, ... χ_n, which may not necessarily be equally spaced.

Suppose that $P_n(\chi)$, is a polynomial in χ of degree n. Then

$$P_n(\chi) = A_0(\chi - \chi_1)(\chi - \chi_2) \dots (\chi - \chi_n) + A_1(\chi - \chi_0)(\chi - \chi_2) \dots (\chi - \chi_n)$$

$$+ \dots A_n(\chi - \chi_0)(\chi - \chi_1) \dots (\chi - \chi_n) \quad \dots(I)$$

Where A's are constants.

The constants $A_0, A_1, A_2 \ldots . A_n$ are determined such that $P_n(\chi_0) = f(\chi_0)$, $P_n(\chi_1) = f(\chi_1)$, ... $P_n(\chi_n) = f(\chi_n)$.

Step II :

Putting $\chi = \chi_0$ and $P_n(\chi_0) = f(\chi_0)$ in equation (I), we get

$f(\chi_0), = A_0(\chi - \chi_1)(\chi - \chi_2) \ldots (\chi - \chi_n)$

$$A_0 = \frac{f(\chi_0)}{(\chi - \chi_1)(\chi - \chi_2) \ldots (\chi - \chi_n)}$$

Putting $\chi = \chi_1$ and $P_n(\chi_1) = f(\chi_1)$ in equation (I), we get

$f(\chi_1), = A_1(\chi_1 - \chi_0)(\chi_1 - \chi_2) \ldots (\chi_1 - \chi_n)$

$$A_1 = \frac{f(\chi_1)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2) \ldots (\chi_1 - \chi_n)}$$

Similarly $$A_1 = \frac{f(\chi_n)}{(\chi_n - \chi_0)(\chi_n - \chi_1) \ldots (\chi_n - \chi_{n-1})}$$

Substituting the values of A's in equation (I), we get

$$P_n(\chi) = f(\chi) = \frac{(\chi - \chi_1)(\chi - \chi_2) \ldots (\chi - \chi_n)}{(\chi_0 - \chi_1)(\chi_0 - \chi_2) \ldots (\chi_0 - \chi_n)} f(\chi_0) + \frac{(\chi - \chi_0)(\chi - \chi_2) \ldots (\chi - \chi_n)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2) \ldots (\chi_1 - \chi_n)}$$

$$f(\chi_1) + \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_3) \ldots (\chi - \chi_n)}{(\chi_2 - \chi_0)(\chi_2 - \chi_1)(\chi_2 - \chi_3) \ldots (\chi_2 - \chi_n)} f(\chi_1) + \ldots$$

$$+ \frac{(\chi - \chi_0)(\chi - \chi_1) \ldots (\chi - \chi_{n-1})}{} f(\chi_n)$$

$$(\chi_n - \chi_1)(\chi_n - \chi_1) \ldots (\chi_n - \chi_{n-1})$$

...(II)

Equation (II) represents **Lagrange's interpolation formula.**

Ex. 12 Given the following table :

χ	1	2	5	9
$y = f(\chi)$	1	3	6	10

Interpolate for y at χ = 6 using Lagrange's interpolation.

Solution :

Step I :

From the table, we have

$\chi = 1.$ $\chi_1 = 2.$ $\chi_2 = 5.$ $\chi_3 = 9.$

$f(\chi_0) = 1$ $f(\chi_1) = 3.$ $f(\chi_2) = 6$ $f(\chi_3) = 10$

Also, we are given that $\chi = 6$

Step II :

According to Lagrange's interpolation formula, we have

$$f(\chi) = \frac{(\chi - \chi_1)(\chi - \chi_2) \ldots (\chi - \chi_n)}{(\chi_0 - \chi_1)(\chi_0 - \chi_2) \ldots (\chi_0 - \chi_n)} f(\chi_0) + \frac{(\chi - \chi_0)(\chi - \chi_2) \ldots (\chi - \chi_n)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2) \ldots (\chi_1 - \chi_n)}$$

$$f(\chi_1) + \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_3) \ldots (\chi - \chi_n)}{(\chi_2 - \chi_0)(\chi_2 - \chi_1)(\chi_2 - \chi_3) \ldots (\chi_2 - \chi_n)} f(\chi_2) -$$

$$\frac{(\chi - \chi_0)(\chi - \chi_1) \ldots (\chi - \chi_2)}{(\chi_3 - \chi_0)(\chi_3 - \chi_1) \ldots (\chi_3 - \chi_2)} f(\chi_3)$$

...(II)

Putting values from Step I in above equation, we get

$$f(6) = \frac{(6-2)(6-5)(6-9)}{(1-2)(1-5)(1-9)} \times 1 + \frac{(6-1)(6-5)(6-9)}{(2-1)(2-5)(2-9)} \times 3$$

$$+ \frac{(6-1)(6-2)(6-9)}{(5-1)(5-2)(5-9)} \times 6 + \frac{(6-1)(6-2)(6-5)}{(9-1)(9-2)(9-5)} \times 10$$

$$f(6) = \frac{(-1)(1)(-3)}{(-1)(-4)(-8)} \times 1 + \frac{(5)(1)(-3)}{(1)(-3)(-7)} \times 3 + \frac{(5)(4)(-3)}{(4)(3)(-4)} \times 6 + \frac{(5)(4)(1)}{(8)(7)(4)} \times 10$$

$$f(6) = \frac{(-12)}{(-32)} \times 1 + \frac{(-15)}{(21)} \times 3 + \frac{(-60)}{(-48)} \times 6 + \frac{(20)}{(224)} \times 10$$

$f(6) = 0.375 + (-2.1428574) + 7.5 + 0.8928571$

y = $f(6)$ = 6.625

Ex. 13 Using Lagrange's formula, find a unique polynomial p(x) if degree 2 or less such that : p(1) = 1, p(3) = 27, p(4) = 64 and hence evaluate p(1.5)

[Using Lagrange's formula]

Solution :

Step I :

From the table, we have

$\chi_0 = 1.$ $\chi_1 = 3.$ $\chi_2 = 4.$

$P(\chi_0) = 1$ $P(\chi_1) = 27.$ $P(\chi_2) = 64$

Step II : Finding polynomial P(χ) :

According, to Lagrange's interpolation formula, we have

$$P(\chi) = \frac{(\chi - \chi_1)(\chi - \chi_2)}{(\chi_0 - \chi_1)(\chi_0 - \chi_2)} P(\chi_0) + \frac{(\chi - \chi_0)(\chi - \chi_2)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2)} P(\chi_1) + \frac{(\chi - \chi_0)(\chi - \chi_1)}{(\chi_2 - \chi_1)(\chi_2 - \chi_1)} P(\chi_2)$$

Putting values from Step I in above equation, we get

$$P(\chi) = \frac{(\chi - 3)(\chi - 4)}{(1 - 3)(1 - 4)} \times 1 + \frac{(\chi - 1)(\chi - 4)}{(1 - 3)(1 - 4)} \times 27 + \frac{(\chi - 1)(\chi - 3)}{(4 - 1)(4 - 3)} \times 64$$

$$P(\chi) = \frac{(\chi^2 - 7\chi + 12)}{(6)} \times 1 + \frac{(\chi^2 - 5\chi + 4)}{(-2)} \times 27 + \frac{(\chi^2 - 4\chi + 3)}{(3)} \times 64$$

$$P(\chi) = \frac{1}{6}(\chi^2 - 7\chi + 12) - \frac{27}{2}(\chi^2 - 5\chi + 4) + \frac{64}{3} - (\chi^2 - 4\chi + 3)$$

$$P(\chi) = \chi^2 - 19\chi + 12$$

Step III : Finding P(1.5) :

Putting $\chi = 1.5$ in the above polynomial, we have

$P(1.5) = 8(1.5)^2 - 19(1.5) + 12$

$P(1.5) = 18 – 28.5 + 12$

P(1.5) = 1.5

Ex. 14 Find the interpolating polynomial for data :

χ	0	1	2	5
$f(\chi)$	2	3	12	147

Solution :

Step I :

From the table, we have

$\chi_0 = 0.$ $\quad \chi_1 = 1.$ $\quad \chi_2 = 2.$ $\quad \chi_3 = 5.$

$f(\chi_0) = 2$ $\quad f(\chi_1) = 3.$ $\quad f(\chi_2) = 12$ $\quad f(\chi_3) = 147$

Step II :

According to Lagrange's interpolation formula, we have

$$f(\chi) = \frac{(\chi - \chi_1)(\chi - \chi_2)(\chi - \chi_3)}{(\chi_0 - \chi_1)(\chi_0 - \chi_2)(\chi_0 - \chi_3)} f(\chi_0) + \frac{(\chi - \chi_0)(\chi - \chi_2)(\chi - \chi_3)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2)(\chi_1 - \chi_3)} f(\chi_1)$$

$$+ \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_3)}{(\chi_2 - \chi_0)(\chi_2 - \chi_1)(\chi_2 - \chi_3)} f(\chi_2) + \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_2)}{(\chi_3 - \chi_0)(\chi_3 - \chi_1)(\chi_3 - \chi_2)} f(\chi_3)$$

Putting values from Step I in above equation, we get

$$f(\chi) = \frac{(\chi - 1)(\chi - 2)(\chi - 5)}{(0 - 1)(0 - 2)(0 - 5)} \times 2 + \frac{(\chi - 0)(\chi - 2)(\chi - 5)}{(1 - 0)(1 - 2)(1 - 5)} \times 3$$

$$+ = \frac{(\chi - 0)(\chi - 1)(\chi - 5)}{(2 - 0)(2 - 1)(2 - 5)} \times 12 + \frac{(\chi - 0)(\chi - 1)(\chi - 2)}{(5 - 0)(5 - 1)(5 - 2)} \times 147$$

$$f(\chi) = \frac{(\chi^2 - 3\chi + 2)(\chi - 5)}{(-10)} \times 2 + \frac{(\chi^2 - 7\chi + 10)}{4} \times 3$$

$$+ = \frac{\chi(\chi^2 - 6\chi + 5)}{(-6)} \times 12 + \frac{\chi(\chi^2 - 3\chi + 2)}{60} \times 147$$

$\therefore \quad f(\chi) = (-0.2)(\chi^3 - 8\chi^2 + 17\chi - 10) + (0.75)(\chi^3 - 7\chi^2 + 10\chi)$

$\quad + (-2)(\chi^3 - 6\chi^2 + 5\chi) + (2.45)(\chi^3 - 3\chi^2 + 2\chi)$

$\therefore \quad f(\chi) = \chi^3 - \chi^2\ \chi - 2$

Ex. 15 If y(1) = -3, y(3) = 9, y(4) = 30 and y(6) = 132. Find the four point Lagrange's interpolation polynomial that takes, the same values as the function y at given points.

Solution :

Step I :

From the given data, we have

$\chi_0 = 1.$ $\chi_1 = 3.$ $\chi_2 = 4.$ $\chi_3 = 6.$

$f(\chi_0) = 3$ $f(\chi_1) = 9.$ $f(\chi_2) = 30$ $f(\chi_3) = 132$

Step II :

According to Lagrange's interpolation formula, we have

$$f(\chi) = \frac{(\chi - \chi_1)(\chi - \chi_2)(\chi - \chi_3)}{(\chi_0 - \chi_1)(\chi_0 - \chi_2)(\chi_0 - \chi_3)} f(\chi_0) + \frac{(\chi - \chi_0)(\chi - \chi_2)(\chi - \chi_3)}{(\chi_1 - \chi_0)(\chi_1 - \chi_2)(\chi_1 - \chi_3)} f(\chi_1)$$

$$+ \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_3)}{(\chi_2 - \chi_0)(\chi_2 - \chi_1)(\chi_2 - \chi_3)} f(\chi_2) + \frac{(\chi - \chi_0)(\chi - \chi_1)(\chi - \chi_2)}{(\chi_3 - \chi_0)(\chi_3 - \chi_1)(\chi_3 - \chi_2)} f(\chi_3)$$

Putting values from Step I in above equation, we get

$$f(\chi) = \frac{(\chi - 3)(\chi - 4)(\chi - 6)}{(1 - 3)(1 - 4)(1 - 6)} \times (-3) + \frac{(\chi - 1)(\chi - 4)(\chi - 6)}{(3 - 1)(4 - 3)(4 - 6)} \times 9$$

$$+ = \frac{(\chi - 1)(\chi - 3)(\chi - 6)}{(4 - 1)(4 - 3)(4 - 6)} \times 30 + \frac{(\chi - 1)(\chi - 3)(\chi - 4)}{(6 - 1)(6 - 3)(6 - 4)} \times 132$$

$$f(\chi) = \frac{(\chi - 3)(\chi^2 - 10\chi + 24)}{(-30)} \times 2 + \frac{(\chi - 1)(\chi^2 - 10\chi + 24)}{(6)} \times 3$$

$$+ = \frac{(\chi - 1)(\chi^2 - 9\chi + 18)}{(-6)} \times 30 + \frac{(\chi - 1)(\chi^2 - 7\chi + 12)}{(30)} \times 132$$

$$f(\chi) = \frac{(\chi^3 - 13\chi^2 + 54\chi + 72)}{} \times (-3) + \frac{(\chi^3 - 11\chi^2 + 34\chi - 24)}{} \times 9$$

$$+ = \frac{(\chi^3 - 10\chi^2 + 27\chi - 18)}{(-30)(-6)} \times 30 + \frac{(\chi^3 - 8\chi^2 + 19\chi - 12)}{(6)(30)} \times 132$$

$$\therefore \quad f(\chi) = 0.1\,(\chi^3 - 13\chi^2 + 54\chi + 72) + 1.5\,(\chi^3 - 11\chi^2 + 34\chi - 24)$$

$$+ (-5)\,(\chi^3 - 10\chi^2 + 27\chi - 18) + 4.4\,(\chi^3 - 8\chi^2 + 19\chi - 12)$$

$$\therefore \quad f(\chi) = \chi^3 - 3\chi^2 + 5\chi - 6$$

Ex. 16 Using Lagrange's interpolation formula, express the function $\frac{3\chi^2 + \chi + 1}{(\chi - 1)(\chi - 2)(\chi - 3)}$ as the sum of partial fractions.

Solution :

Step I :

Consider the numerator $(\chi) = 3\chi^2 + \chi + 1$ and tabulate its values for $\chi = 1.2.3$

χ	1	2	3
$f(\chi) = 3\chi^2 + \chi + 1$	5	15	31

Step II :

According to Lagrange's interpolation formula, we have

$$f(\chi) = \frac{(\chi - 2)(\chi - 3)}{(1 - 2)(1 - 3)} \times (5) + \frac{(\chi - 1)(\chi - 3)}{(2 - 1)(2 - 3)} \times (15) + \frac{(\chi - 1)(\chi - 2)}{(3 - 1)(3 - 2)} \times (31)$$

$$f(\chi) = \frac{5}{2} \times (\chi - 2)(\chi - 3) - 15\,(\chi - 1)(\chi - 3) + \frac{31}{2}\,(\chi - 1)(\chi - 2)$$

Step II :

Dividing both sides of above equation by $(\chi - 1)(\chi - 2)(\chi - 3)$, we have

$$\frac{f(\chi)}{(\chi - 1)(\chi - 2)(\chi - 3)} = \frac{5}{2} - \frac{(\chi - 2)(\chi - 3)}{2(\chi - 1)(\chi - 2)(\chi - 3)} - 15\frac{(\chi - 1)(\chi - 3)}{(\chi - 1)(\chi - 2)(\chi - 3)} + \frac{31}{2} - \frac{(\chi - 1)(\chi - 2)}{(\chi - 1)(\chi - 2)(\chi - 3)}$$

$$\frac{3\chi^3 + \chi + 1}{(\chi - 1)(\chi - 2)(\chi - 3)} = \frac{5}{2(\chi - 1)} - \frac{15}{(\chi - 2)} + \frac{31}{2(\chi - 3)}$$

Ex. 17 Using Lagrange's interpolation formula, express the function $\frac{\chi^2 + \chi - 3}{\chi^3 - 1\chi^2 - \chi + 3}$ as the sum of partial fractions.

Solution :

Step I :

We have

$$\frac{\chi^2 + \chi - 3}{\chi^3 - 2\chi^2 - \chi + 3} = \frac{\chi^2 + \chi - 3}{(\chi - 1)(\chi - 1)(\chi - 2)}$$

Consider the numerator $f(\chi) = \chi^2 + \chi - 3$ and tabulate its values for $\chi = 1,1,2$

χ	-1	1	2
$f(\chi) = \chi^2 - \chi - 3$	-3	-1	3

Step II :

According to Lagrange's interpolation formula, we have

$$f(\chi) = \frac{(\chi - 1)(\chi - 2)}{(-3 + 1)(-3 - 3)} \times (-3) + \frac{(\chi + 1)(\chi - 2)}{(-1 + 3)(-1 - 3)} \times (-1) + \frac{(\chi + 1)(\chi - 1)}{(3 + 3)(3 + 1)} \times (3)$$

$$f(\chi) = -\left(\frac{1}{4}\right)(\chi - 1)(\chi - 2) + \left(\frac{1}{8}\right)(\chi + 1)(\chi - 2) + \left(\frac{1}{8}\right)(\chi + 1)(\chi - 1)$$

Step II :

Dividing both sides of above equation by $(\chi + 1)(\chi - 2)$, we have

$$\frac{f(\chi)}{(\chi + 1)(\chi - 1)(\chi - 2)} = -\left(\frac{1}{4}\right)\frac{(\chi - 1)(\chi - 2)}{(\chi + 1)(\chi - 1)(\chi - 2)} + \left(\frac{1}{8}\right)\frac{(\chi + 1)(\chi - 2)}{(\chi + 1)(\chi - 1)(\chi - 2)}$$

$$+ \frac{31}{2} - \frac{(\chi - 1)(\chi - 2)}{(\chi - 1)(\chi - 2)(\chi - 3)}$$

$$\frac{\chi^2 + \chi - 3}{(\chi + 1)(\chi - 1)(\chi - 2)} = \frac{-1}{4(\chi - 2)} - \frac{1}{8(\chi - 1)} + \frac{1}{8(\chi - 2)} \quad 1$$

Chapter – 4

Least Square Curve Fitting

The technique used to minimize the sum of squares of residual errors is known as least square approximation. Least square approximations are most commonly used approximations for approximating a function $f(\chi)$ which may be given in tabular from or known explicitly over a given interval. Least square approximation is performed by different least square approximation methods.

The least square approximation methods are used to find a polynomial $P(\chi)$ which can approximate all the given data point. The polynomial $P(\chi)$ which is an approximation to the function $f(\chi)$ is selected in such a way that it minimizes the squares of the error.

Some of the least square approximation methods are as follows :

(i) Fitting of a Straight Line :

For fitting a straight line, we will consider the equation of a straight line given by

$$y = a + b\chi$$

This straight line equation can be approximated for the given set of data points. The values of a and b in the straight line equation can be found by forming two normal equations given by :

$$\sum_{i=l}^{n} y_i = na - b\sum_{i=i}^{n} \chi_i$$

and by multiplying equation (I) by χ_i we get

$$\sum_{i=i}^{n} \chi_i y_i = a \sum_{i=i}^{n} \chi_i + b\sum_{i=i}^{n} \chi^2_i$$

The values of a and b can be found by solving equations (I) and (II) simultaneously :

(ii) Fitting of Second Degree Parabola :

This is the second method of least square approximation and it involves the use polynomials of higher degrees. Since the error involved in straight line approximations is much higher polynomials of higher degrees are used to achieve the smoothness and better approximation.

For fitting a parabola, we will consider the equation of a second degree parabola given by :

$$y = a + b\chi + c\chi^2$$

This equation can be approximated for the given set of data points. The values a, b and c in the given equation can be found by forming three normal equations given by :

$$\sum_{i=i}^{n} y_i = na + b \sum_{i=i}^{n} \chi_i + c \sum_{i=i}^{n} \chi^2_i$$

…(III)

Multiplying equation (III) by χ_i,we get

$$\sum_{i=i}^{n} \chi_i y_i = a \sum_{i=i}^{n} \chi_i + b \sum_{i=i}^{n} \chi^2_i + c \sum_{i=i}^{n} \chi^3_i$$

…(IV)

Again multiplying equation (IV) by χ_i,we get

$$\sum_{i=i}^{n} \chi^2_i y_i = a \sum_{i=i}^{n} \chi^2_i + b \sum_{i=i}^{n} \chi^3_i + c \sum_{i=i}^{n} \chi^4_i$$

…(V)

The values of a, b and c can be found by solving equations (III), (IV) and (V)

Procedure to Solve Problems of Fitting a Straight Line :

Step I : Write the equation of the straight line as : $y = a + b\chi$

Step II : Write the normal equation for the above straight line as

$$\sum y = na + b \sum \chi$$

…(I)

and

$$\sum \chi y = a \sum \chi + b \sum \chi^2$$

…(II)

Step III : From the given set of values prepare table for χ, y, χy and hence find $\sum\chi$, $\sum y$, $\sum\chi y$ and $\sum\chi^2$

Step IV : Put the values obtained in Step III in equation (I) and (II). Then solve these equation to get the values of a and b.

Setp V: Put the values of a and b in the equation $y = a + b\chi$ to get the required equation of straight line.

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 20 Fit a straight line to the χ and y values in the two columns

χ_1	1	2	3	4	5	6	7
y_1	0.5	2.5	2.0	4.0	3.5	6.0	5.5

Solution :

Here n = 7 (number of data set points)

Step I : **Writing equation of the straight line :**

Let the equation of straight line be $y = a + b\chi$

Step II : **Writing normal equations :**

$$\sum y = na + b \sum \chi \quad \ldots(I)$$

and

$$\sum \chi y = a \sum \chi + b \sum \chi^2 \quad \ldots(II)$$

Step III : **Preparing the table :**

χ	y	χy	χ^2
1	0.5	0.5	1
2	2.5	5.0	4
3	2.0	6.0	9
4	4.0	16.0	16
5	3.5	17.5	25
6	6.0	36.0	36
7	5.5	38.5	49
$\sum\chi = 28$	**$\sum y = 24$**	**$\sum\chi y = 119.5$**	**$\sum\chi^2 = 140$**

Step IV : **Finding values of a and b :**

Putting the values from Step III in equation of Step II, we get

$$24 = 7a + 28b \quad \text{...(III)}$$

and

$$119.5 = 28a + 140b \quad \text{...(IV)}$$

Multiplying equation (III) by 4 and subtracting it from equation (IV), we get

$$119.5 - 96 = 140b - 112b$$

$$23.5 = 28b$$

$$b = 0.8392$$

Putting the value of b in equation (III), we get

$$24 = 7a + 28(0.8392)$$

$$24 = 7a + 23.5$$

$$7a = 0.5$$

$$a = 0.07142$$

Step V : Writing equation of the straight line :

Putting the value of a and b in the equation of straight line, we get the requires equation as :

$$y = 0.07142 + 0.8392\chi$$

Ex. 21 Fit a straight line to the χ and y values in the two columns

χ_1	1	2	3	4	5	6	7
y_1	0.5	2.3	2.1	4.2	3.6	5.8	5.5

Solution :

Here n = 7 (number of data set points)

Step I : Writing equation of the straight line :

Let the equation of straight line be $y = a + b\chi$

Step II : Writing normal equations :

$$\sum y = na + b \sum \chi \quad \ldots(I)$$

and

$$\sum \chi y = a \sum \chi + b \sum \chi^2 \quad \ldots(II)$$

Step III : Preparing the table :

χ	y	χy	χ^2
1	0.5	0.5	1
2	2.3	4.6	4
3	2.1	6.3	9
4	4.2	16.8	16
5	3.6	18.0	25
6	5.8	34.8	36
7	5.5	38.5	49
$\sum\chi$ = 28	**$\sum$y = 24**	**$\sum\chi$y = 119.5**	**$\sum\chi^2$ = 140**

Step IV : Finding values of a and b:

Putting the values from Step in equations of Step II, we get

$$24 = 7a + 28b \quad \ldots(III)$$

and

$$119.5 = 28a + 140b \quad \ldots(IV)$$

Multiplying equation (III) by 4 and subtracting it from equation (IV), we get

$$119.5 - 96 = 140b - 112b$$

$$23.5 = 28b$$

$$b = 0.8392$$

Putting the value of b in equation (III), we get

$$24 = 7a + 28(0.8392)$$

$$24 = 7a + 23.5$$

$$7a = 0.5$$

$$a = 0.07142$$

Step V : Writing equation of the straight line :

Putting the value of a and b in the equation of straight line, we get the requires equation as :

$$y = 0.07142 + 0.8392\chi$$

Step VI : Finding the values of y at $\chi = 4.5$

Putting $\chi = 4.5$, in the above equation, we get

$$y = 0.07142 + 0.8392 \times 4.5$$

$$\mathbf{y = 3.8478}$$

Procedure to Solve Problems of Fitting a Parabola of Second Degree :

Step I : Write the equation of the parabola : $y = a + b\chi + c\chi^2$

Step II : Write the normal equation for the above straight line as

$$\sum y = na + b \sum \chi + c \sum \chi^2 \quad \text{…(I)}$$

$$\sum \chi y = a \sum \chi + b \sum \chi^2 + c \sum \chi^3 \quad \text{…(II)}$$

and

$$\sum \chi^2 y = a \sum \chi^2 + b \sum \chi^3 + c \sum \chi^4 \quad \text{…(III)}$$

Step III : From the given set of values prepare table for χ, y, χ^2, χ^3, χ^4, χy and $\chi^2 y$ and hence find $\sum\chi$, $\sum y$, $\sum\chi^2$, $\sum \chi^3$, $\sum\chi^4$, $\sum\chi y$ and $\sum\chi^2 y$.

Step IV : Put the values obtained in Step III in equation (I) and (II). Then solve these equation to get the values of a, b and c.

Setp V: Put the values of a, b and c in the equation $y = a + b\chi + c\chi^2$ to get the required equation of parabola.

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 22 Fit a parabola of second degree to the following data :

χ	**0**	**1**	**2**	**3**	**4**
y	**1**	**1.8**	**1.3**	**2.5**	**6.3**

Solution :

Here n = 5 (number of data set points)

Step I : **Writing equation of the straight line :**

Let the equation of straight line be $y = a + b\chi + c\chi^2$

Step II : **Writing normal equations :**

The normal equations for the above parabola is given by :

$$\sum y = na + b \sum \chi + c \sum \chi^2 \quad \ldots(I)$$

$$\sum \chi y = a \sum \chi + b \sum \chi^2 + c \sum \chi^3 \quad \ldots(II)$$

and

$$\sum \chi^2 y = a \sum \chi^2 + b \sum \chi^3 + c \sum \chi^4 \quad \ldots(III)$$

Step III : **Preparing the table :**

χ	y	χ^2	χ^3	χ^4	χy	$\chi^2 y$
0	1	0	0	0	0	0
1	1.8	1	1	1	1.8	1.8
2	1.3	4	8	16	2.6	5.2
3	2.5	9	27	81	7.5	22.5
4	6.3	16	64	256	25.2	100.8
$\sum\chi$ =	$\sum y$ =	$\sum\chi^2$ =	$\sum\chi^3$ =	$\sum\chi^4$ =	$\sum\chi y$ =	$\sum\chi^2 y$ =

10	**12.9**	**30**	**100**	**354**	**37.1**	**130.3**

Step IV : Finding values of a, b and c :

$$12.9 = 5a + 10b + 30c \quad \ldots(IV)$$

$$37.1 = 10a + 30b + 100c \quad \ldots(V)$$

and

$$130.3 = 30a + 100b + 354c \quad \ldots(VI)$$

Solving these three equation, we have

$a = 1.42 \qquad b = -1.07 \qquad c = 0.55$

Step V : Writing equation of the parabola :

Putting the value of a, b and c in the equation of parabola, we get the required equation as :

$$y = 1.42 - 1.07\chi + 0.55\chi^2$$

❖❖❖

5

SOLUTION OF SIMULTANEOUS ALGEBRAIC EQUATIONS (LINEAR)

INTRODUCTION

The solution of a linear system of equations can be accomplished by a numerical method which falls in one of two categories : ***direct methods and iterative methods.***

DIRECT METHODS :

These methods are based on elimination of variables to transform set of equations to a triangular form. These methods give exact values of unknowns. *Methods of Determinants (Cramer's Rule),* ***Matrix Inversion Method and Gaussian Elimination Method are examples of Direct Methods.***

ITERATIVE METHODS :

These methods are based on Successive Approximation. In these methods, cycle of computation is repeated till the required accuracy is obtained. ***Jacobi and Gauss – Seidet methods are examples of Iterative Methods.***

UNIT 2

Cramer's Rule

One of the classical methods used to solve the system f linear equations $A\chi = P$. where A is a *n* x *n* matrix, χ is an *n* + 1 unknown matrix and P is an *n* + 1 known vector, is based on the computation of the determinant of A. Cramer's Rule can be explained by considering a set of three equations and can be generalized to a set of *n* equations.

Consider a system of equations

$$a_1\chi + b_1\gamma + c_1 z = P_1$$

$$a_2\chi + b_2\gamma + c_2 z = P_2$$
$$a_3\chi + b_3\gamma + c_3 z = P_3$$

then by Cramer's Rule,

$$\chi = \frac{\Delta\chi}{\Delta} \qquad \gamma = \frac{\Delta\gamma}{\Delta} \qquad \text{and} \qquad z = \frac{\Delta Z}{\Delta}$$

Where

$$\Delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

$$\Delta X = \begin{pmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ & & \end{pmatrix} \qquad \Delta\gamma = \begin{pmatrix} a_1 & p_1 & c_1 \\ a_1 & p_1 & c_1 \\ & & \end{pmatrix}$$

$$\begin{matrix} p_3 & b_3 & c_3 \end{matrix} \qquad \begin{matrix} a_1 & p_1 & c_1 \end{matrix}$$

and

$$\Delta z = \begin{pmatrix} a_1 & b_1 & p_1 \\ a_2 & b_2 & p_2 \\ a_3 & b_3 & p_3 \end{pmatrix}$$

Procedure to Solve Problems of Cramer's Rule :

Step I : Find the determinant (Δ) of the given system of equation.

Step II : Find $\Delta\chi$, $\Delta\gamma$ and Δz

Step III : Find the solution of the given equation by the formula.

$$\chi = \frac{\Delta\chi}{\Delta} \qquad \gamma = \frac{\Delta\gamma}{\Delta} \quad \text{and} \quad z = \frac{\Delta Z}{\Delta}$$

The above procedure will get clear to y as you will go through the following solved examples

Ex. 4. Use Cramer's Rule to solve :

$$\chi - 3\gamma + z = 2$$

$$3\chi - \gamma + z = 6$$

$$5\chi - \gamma + 3z = 3$$

Solution :

The given system of equation can be written as.

$$\begin{pmatrix} 1 & -3 & 1 \\ 3 & 1 & 1 \\ 5 & 1 & 3 \end{pmatrix} \begin{bmatrix} \chi \\ \gamma \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$$

Step I : **Finding Δ :**

We have

$$\Delta \begin{pmatrix} 1 & -3 & 1 \\ 3 & 1 & 1 \\ 5 & 1 & 3 \end{pmatrix}$$

$\Delta = 1\ (3 - 1) + 3(9 - 5) + 1(3 - 5)$

$\Delta = 1\ (2) + 3(4) + 1(-2)$

$\Delta = 2 + 12 - 2$

$\Delta = 12$

Step II : **Finding $\Delta\chi$:**

We have

$$\Delta\chi = \begin{pmatrix} 2 & -3 & 1 \\ 6 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix}$$

$\Delta\chi = 2\ (3 - 1) + 3(18 - 5) + 1(6 - 3)$

$\Delta\chi = 2\ (2) + 3\ (5) + 1(3)$

$\Delta\chi = 4 + 45 + 3$

$\Delta\chi = 52$

Step III : **Finding $\Delta\gamma$:**

We have

$$\Delta\gamma = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 3 & 3 \end{pmatrix}$$

$\Delta\gamma = 1\ (18 - 3) — 2(9 - 5) + 1(9 - 30)$

$\Delta\gamma = 1\ (15) — 2(4) + 1(-21)$

$\Delta\gamma = 15 – 8 – 21$

$\Delta\gamma = -14$

Step IV : Finding Δz :

We have

$$\Delta z = \begin{pmatrix} 1 & -3 & 2 \\ 3 & 1 & 6 \\ 5 & 1 & 3 \end{pmatrix}$$

$\Delta z = 1\ (3 – 6) + 3\ (9 – 30) + 2\ (3 – 5)$

$\Delta z = 1\ (-3) + 3(-21) + 2(-2)$

$\Delta z = -3 – 63 - 4$

$\Delta z = -70$

Step V : Finding solution of the given equations :

We have

$$\chi = \frac{\Delta\chi}{\Delta} = \frac{52}{12} = \mathbf{\frac{13}{3}}$$

$$\gamma = \frac{\Delta\gamma}{\Delta} = \frac{-14}{12} = \mathbf{\frac{7}{6}}$$

$$z = \frac{\Delta z}{\Delta} = \frac{-70}{12} = \mathbf{\frac{35}{6}}$$

Ex. 2. Use Cramer's Rule to solve :

$$0.3\chi_1 + 0.52\chi_2 + \chi_3 = -0.01$$

$$0.5\chi_1 + \chi_2 + 1.9\chi_3 = 0.67$$

$$0.1\chi_1 + 0.3\chi_2 + 0.5\chi_3 = -0.44$$

Solution :

The given system of equation can be written as.

$$\begin{pmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{pmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \begin{bmatrix} -0.1 \\ 0.67 \\ -0.44 \end{bmatrix}$$

Step I : Finding Δ :

We have

$$\Delta = \begin{vmatrix} 0.3 & 0.52 & 1 \\ 0.5 & 1 & 1.9 \\ 0.1 & 0.3 & 0.5 \end{vmatrix}$$

$$\Delta = 0.3\,(0.5 - 0.57) - 0.52(0.25 - 0.19) + 1(0.15 - 0.1)$$

$$\Delta = 0.3\,(0.07) - 0.52(0.06) + 1(0.05)$$

$$\Delta = 0.021 - 0.0312 + 0.05$$

$$\Delta = 0.0022$$

Step II : Finding $\Delta\chi$:

We have

$$\Delta\chi_1 = \begin{vmatrix} -0.01 & 0.52 & 1 \\ 0.67 & 1 & 1.9 \\ -0.44 & 0.3 & 0.5 \end{vmatrix}$$

$$\Delta\chi_1 = 0.01\,(0.5 - 0.57) - 0.52(0.335 + 0.836) + 1(0.201 + 0.44)$$

$$\Delta\chi_1 = 0.01\,(-0.07) - 0.52\,(1.171) + 1\,(0.641)$$

$\Delta\chi_1 = 0.0007 – 0.60892 + 0.641$

$\Delta\chi_1 = 0.03278$

Step III : **Finding $\Delta\chi_2$:**

We have

$$\Delta\chi_2 = \begin{pmatrix} 0.3 & -0.01 & 1 \\ 0.5 & 0.67 & 1.9 \\ 0.1 & -0.44 & 0.5 \end{pmatrix}$$

$\Delta\chi_2 = 0.3\ (0.335 + 0.836) + 0.01(0.25 – 0.19) + 1(-0.22 - -0.067)$

$\Delta\chi_2 = 0.3\ (1.171) + 0.01\ (0.06) + 1\ (-0.287\ )$

$\Delta\chi_2 = 0.3513 + 0.0006 – 0.287$

$\Delta\chi_2 = 0.0649$

Step IV : **Finding $\Delta\chi_3$:**

We have

$$\Delta\chi_3 = \begin{pmatrix} 0.3 & 0.52 & -0.01 \\ 0.5 & 1 & 0.67 \\ 0.1 & 0.3 & -0.44 \end{pmatrix}$$

$\Delta\chi_3 = 0.3\ (-0.44 – 0.201) - 0.52(-0.22 – 0.067) – 0.01(0.15 – 0.1)$

$\Delta\chi_3 = 0.3\ (-0.641) - 0.52\ (-0.287) – 0.01\ (\ 0.05\ )$

$\Delta\chi_3 = 0.1923 + 0.14924 – 0.0005$

$\Delta\chi_3 = 0.04356$

Step V : Finding solution of the given equations :

We have

$$\chi_1 = \frac{\Delta\chi_1}{\Delta} = \frac{0.03278}{-0.0022} = \mathbf{-14.9}$$

$$\chi_2 = \frac{\Delta\chi_2}{\Delta} = \frac{0.0649}{-0.0022} = \mathbf{-29.5}$$

$$\chi_3 = \frac{\Delta\chi_3}{\Delta} = \frac{-0.04356}{-0.0022} = \mathbf{19.8}$$

Unit - 3

MATRIX INVERSION METHOD

Matrix inversion methods is another method of solving a system of linear equations. This method can be explained by considering a system of *n* linear equations in *n* unknowns :

$$\left.\begin{array}{l} a_1\chi_1 + a_1\chi_2 + \ldots + a_{1n}\chi_n = b_1 \\ a_2\chi_1 + a_2\chi_2 + \ldots + a_{2n}\chi_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}\chi_1 + a_{n2}\chi_2 + \ldots + a_{nn}\chi_n = b_n \end{array}\right\} \ldots (I)$$

The above system of equations can be written in the matrix form as

$$AX = B$$

Where $A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{12} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$ $\chi = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \ldots \\ \chi_n \end{bmatrix}$ $B = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$

Let A be non-singular so that A^{-1} exists. Then premultiplying both sides of equation (II) by A^{-1} we get

$$A^{-1} AX = A^{-1} B$$

$$\therefore X = A^{-1} B \qquad [\therefore A^{-1} A = 1 \text{ and } 1X = X]$$

If A^{-1} is known. Then the solution vector X can be found out form the above matrix relation. The inverse of matrix A (i.e., A^{-1}) can be found as follows :

Method of Finding Inverse of Matrix :

Step I : Check whether determinant of the given matrix is zero or not

If the determinant of the given matrix is zero then the inverse of that matrix does not exist.

Step II : Find adjoint of the given matrix.

Step III : Find inverse of the fiven matrix using the theorem

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

Ex. 3 Evaluate the inverse of the following matrix.

$$|A| = \begin{pmatrix} 0.7 & -5.4 & 1.0 \\ 3.5 & 2.2 & 0.8 \\ 1.0 & -1.5 & 4.3 \end{pmatrix}$$

$= 0.7 (9.46 + 1.2) + 5.4(15.05 – 0.8) + 1.0(-5.25 – 2.2)$

$= 0.7 (10.66) + 5.4 (14.25) + 1.0 (-7.45)$

$= 76.962$

$|A| \neq 0$

$\therefore$ A^{-1} exists.

Step II : **Finding co-factors of A:**

We have

$$|A| = \begin{pmatrix} 0.7 & -5.4 & 1.0 \\ 3.5 & 2.2 & 0.8 \\ 1.0 & -1.5 & 4.3 \end{pmatrix}$$

The co-factors of the elements of the first row of determinant $|A|$ are

$$\begin{bmatrix} 2.2 & 0.08 \\ -1.5 & 4.3 \end{bmatrix}, -\begin{bmatrix} 3.5 & 0.8 \\ 1.0 & 4.3 \end{bmatrix}, \begin{bmatrix} 3.5 & 2.2 \\ 1.0 & -1.5 \end{bmatrix}$$

, i.e., are 1066, -14.25, -7.45 respectively.

The co-factors of the elements of the Second row of determinant $|A|$ are

$$\begin{bmatrix} -5.4 & 1.0 \\ -1.5 & 4.3 \end{bmatrix}, -\begin{bmatrix} 0.7 & 1.0 \\ 1.0 & 4.3 \end{bmatrix}, \begin{bmatrix} 0.7 & -5.4 \\ 1.0 & -1.5 \end{bmatrix}$$

, i.e., are 21.72, 2.01, -4.33 respectively.

The co-factors of the elements of the third row of determinant $|A|$ are

$$\begin{pmatrix} -5.4 & 1.0 \\ 2.2 & 0.8 \end{pmatrix}, -\begin{pmatrix} 0.7 & 1.0 \\ 3.5 & 0.8 \end{pmatrix}, \begin{pmatrix} 0.7 & -5.4 \\ 3.5 & 2.2 \end{pmatrix}$$

, i.e., are –6.52, 2.94, 20.44 respectively.

$$C = \begin{pmatrix} 10.66 & -14.25 & -7.45 \\ 21.72 & 2.01 & -4.35 \\ -6.52 & 2.94 & 20.44 \end{pmatrix}$$

Step III : **Finding adjoint of A :**

Now, adj. **A = Transpose of matrix .C** = $\begin{pmatrix} 10.66 & 21.72 & -6.52 \\ -14.25 & 2.01 & 2.94 \\ -7.45 & -4.35 & 20.44 \end{pmatrix}$

Step IV : Finding adjoint of A :

We have

$$\left(A^{-1} = \frac{1}{|A|} \text{ adj}(A) \right)$$

$$|A| = \frac{1}{76.962} \begin{pmatrix} 10.66 & 21.72 & -6.52 \\ -14.25 & 2.01 & 2.94 \\ -7.45 & -4.35 & 20.44 \end{pmatrix}$$

$$|A| = \begin{pmatrix} 0.1385 & 0.2822 & -0.0847 \\ -0.1851 & 0.0261 & 0.0382 \\ -0.0968 & -0.0565 & 0.2655 \end{pmatrix}$$

Ex. 4 Compute the inverse of the matrix $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix}$ and use the result of solve

the system of equation :

$$3\chi + 2\gamma + 4z = 7$$

$$2\chi + \gamma + z = 7$$

$$\chi + 3\gamma + 5z = 2$$

Solution :

Step I : Writing the given system of equations in matrix form :

The given system of equation can be written in matrix form as follows :

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \\ 2 \end{bmatrix}$$

i.e. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = B \quad \left\{ \text{Where } A = \begin{pmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{pmatrix} \ldots \right\}$ (I)

Step II :Checking whether determinant of matrix A is zero or not :

$$|A| = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

$= 3\,(5 - 3) - 2(10 - 1) - 4(6 - 1\,)$

$= 8$

$\therefore \quad |A| \neq 0$

$\therefore \quad A^{-1}$ exists.

Step III : Finding co-factors of A:

We have

$$|A| = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$$

The co-factors of the elements of the first row of determinant $|A|$ are

$$\begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}, \ -\begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$$

, i.e., are 2, -9, 5 respectively.

The co-factors of the elements of the Second row of determinant $|A|$ are

$$\begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}, \ -\begin{pmatrix} 3 & 4 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 3 \end{pmatrix}$$

, i.e., are 2, 11, -7 respectively.

The co-factors of the elements of the third row of determinant $|A|$ are

$$\begin{pmatrix} 2 & 4 \\ 1 & 1 \end{pmatrix}, \ -\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

, i.e., are –2, 5, -1 respectively.

$$C = \begin{pmatrix} 2 & -9 & 5 \\ 2 & 11 & -7 \\ -2 & 5 & -1 \end{pmatrix}$$

Step IV : Finding adjoint of A :

Now, adj. **A = Transpose of matrix .C** = $\begin{pmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{pmatrix}$

Step IV : Finding adjoint of A :

we have $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} B$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 2 & 2 & -2 \\ -9 & 11 & 5 \\ 5 & -7 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ 7 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 14 - 14 - 4 \\ -63 + 77 + 10 \\ 35 - 49 - 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 24 \\ 24 \\ -16 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$$

$x = 3 \qquad y = 3 \qquad z = -2$

GAUSS ELIMINATION METHOD

This is an elementary elimination method and it reduces the system of equations to an equivalent upper triangular system which can be solved by back substitution. Although this method is quite general we shall describe this method by considering a system of three equation for the sake of clarity and simplicity.

Let the system be

$$\left.\begin{aligned} a_{11}\chi_1 + a_{12}\chi_2 + a_{13}\chi_3 &= b_1 \\ a_{21}\chi_1 + a_{22}\chi_2 + a_{23}\chi_3 &= b_2 \\ a_{31}\chi_1 + a_{32}\chi_2 + a_{33}\chi_3 &= b_3 \end{aligned}\right\} \quad \ldots \text{(I)}$$

Step I : First stage of Elimination :

In the first stage of elimination multiplying the first row of equation (I) by $\frac{a_{21}}{a_{11}}$ and $\frac{a_{21}}{a_{11}}$ respectively and subtracting it from second and third rows, we get

$$\left.\begin{aligned} a_{22}^{(2)}\chi_2 + a_{23}^{(2)}\chi_3 &= b_2^{(2)} \\ a_{32}^{(2)}\chi_2 + a_{33}^{(2)}\chi_3 &= b_3^{(2)} \end{aligned}\right\} \quad \ldots \text{(II)}$$

where

$$a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}} a_{12} \qquad a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}} a_{12}$$

$$a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}} a_{12} \qquad a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}} a_{13}$$

$$b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}} a_{12} \qquad b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}} b_1$$

Step II : Second stage of Elimination :

In the second stage of elimination, multiplying the first row of equation (II) by $\frac{a_{32}^{(2)}}{a_{22}}$ and subtracting it form the second row we get

$$a_{33}^{(3)}\chi_3 = b_3^{(3)}$$

where
…(III)

$$a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} a_{23}^{(2)} \quad b_3^{(3)} = b_3^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} a_2^{(2)}$$

Step III :

Collecting the first equation from each stage .i.e first equation from (I), (II) and (III) we get

$$a_{11}^{(1)}\chi_1 + a_{12}^{(1)}\chi_2 + a_{13}^{(1)}\chi_3 = b_1^{(1)}$$

$$a_{22}^{(2)}\chi_2 + a_{23}^{(2)}\chi_3 = b_2^{(2)}$$

$$a_{33}^{(3)}\chi_3 = b_3^{(3)}$$

The above system is an upper triangular system and can be solved using the <u>back substitution method.</u> The elements $a_{11}^{(1)}$, $a_{22}^{(2)}$ and $a_{33}^{(3)}$ are called pivot elements and are assumed to be non – zero.

The elimination procedure described above to determine the unknown is called Gauss Elimination method.

<u>PIVOTING METHODS :</u>

In the Gauss elimination method if any of the pivot element $a_{11}^{(1)}$, $a_{22}^{(2)}$ ….. $a_{nn}^{(n)}$ vanished or becomes very small compared to other elements in that row than we attempt to rearrange the remaining rows so as to obtain a non – vanishing pivot or to avoid the multiplication by a large number. This strategy is called **pivoting.**

The pivoting is of following two types :

(i) Partial Pivoting and

(ii) Complete Pivoting

(i) Partial Pivoting :

In this type of pivoting during the first stage of elimination the first column in searched for the largest element in magnitude and brought as the first pivot by interchanging the first equation with the equation having the largest element in magnitude. In second elimination stage, the second column is searched for the largest element in magnitude among n-1 elements leaving the first element, and this element is brought as the second pivot by an interchange of the second equation with the equation having the largest element in magnitude. This procedure is continued until we get the upper triangular system. This method of pivoting is called partial pivoting.

(ii) Complete Pivoting :

In case of complete pivoting the matrix is searched for the largest element in magnitude and it is brought as the first pivot. This requires not only an interchange of equations but also an interchange of the position of the variables. This method of pivoting is called complete pivoting.

It should also be noted that if the matrix is diagonally dominant or real symmetric and positive definite then no pivoting in necessary.

Ex. 5 Solve the equations :

$$10\chi_1 - \chi_2 + 2\chi_3 = 4$$

$$\chi_1 + 10\chi_2 - \chi_3 = 3$$

$$2\chi_1 + 3\chi_2 + 20\chi_3 = 7$$

using the Gauss elimination method.

Solution :

The given system of equations can be written as

$$10\chi_1 - \chi_2 + 2\chi_3 = 4 \qquad \ldots(I)$$

$$\chi_1 + 10\chi_2 - \chi_3 = 3 \qquad \ldots(II)$$

$$2\chi_1 + 3\chi_2 + 20\chi_3 = 7 \qquad \ldots(III)$$

The system is diagonally dominant and hence no pivoting is necessary.

Step I : First stage of Elimination :

Multiplying equation (I) by $\frac{1}{10}$ and $\frac{2}{10}$ respectively and subtracting it from equation (II) and (III) we get

$$10\chi_1 - \chi_2 + 2\chi_3 = 4 \qquad \ldots(IV)$$

$$\frac{101}{10}\chi_2 - \frac{12}{10}\chi_3 = \frac{26}{10} \qquad \ldots(V)$$

$$\frac{32}{10}\chi_2 - \frac{196}{10}\chi_3 = \frac{62}{10} \qquad \ldots(VI)$$

Step II : Second stage of Elimination :

Multiplying equation (V) by $\frac{32}{101}$ and subtracting it from equation (VI) we get

$$10\chi_1 - \chi_2 + 2\chi_3 = 4 \qquad \ldots(VII)$$

$$\frac{101}{10}\chi_2 - \frac{12}{10}\chi_3 = \frac{26}{10} \qquad \ldots(VIII)$$

$$\frac{20180}{1010}\chi_3 = \frac{5430}{1010} \qquad \ldots(XI)$$

Step III : Finding solution using back substitution :

From equation (IX), we get

$$\chi_3 = \frac{5430}{1010} \times \frac{1010}{20180} = 0.269$$

Putting the value of χ_3 in equation (VIII), we get

$$\frac{101}{10}\chi_2 - \frac{12}{10}(0.269) = \frac{26}{10}$$

$\therefore$ $\chi_2 = 0.289$

Putting values of χ_2 and χ_3 in equation (VII) we get

$$10\chi_1 - (0.269) + 2(0.289) = 4$$

$\therefore$ $\chi_1 = 0.375$

Ex. 6 Solve the equation :

$$\chi_1 + \chi_2 + \chi_3 = 6$$

$$3\chi_1 + 3\chi_2 + 4\chi_3 = 20$$

$$2\chi_1 + \chi_2 + 3\chi_3 = 13$$

using the Gauss elimination method.

Solution :

The given system of equations can be written as

$$\chi_1 + \chi_2 + \chi_3 = 6$$

$$3\chi_1 + 3\chi_2 + 4\chi_3 = 20$$

$$2\chi_1 + \chi_2 + 3\chi_3 = 13$$

Step I : First stage of Elimination :

Multiplying equation (I) by 3 and 2 respectively and subtracting it form equation (II) and (III) we get

$$\chi_1 + \chi_2 + \chi_3 = 6$$

$$\chi_3 = 2$$

$$-\chi_2 + \chi_3 = 1$$

Step II :

Here, the pivot in the second equation is zero and so we cannot proceed as usual. So interchanging equation (V) and (VI) we have

$$\chi_1 + \chi_2 + \chi_3 = 6$$

$$- \chi_2 + \chi_3 = 1$$

$$\chi_3 = 2$$

Step III : Finding solution using back substitution :

From equation (IX) we get

$$\chi_3 = 2$$

Putting the value of χ_3 in equation (VIII) we get

$$-\chi_1 + 2 = 1$$

$$\chi_2 = 1$$

Putting value of χ_2 and χ_3 in equation (VIII) we get

$$\chi_1 + 1 + 2 = 6$$

$$\chi_1 = 3$$

Ex. 7 Currents in a circuit are given by the following equations :

$$28 I_1 - 28 I_2 = 10$$

$$-3 I_1 + 38 I_2 - 10 I_3 - 5 I_5 = 0$$

$$-10 I_2 + 25 I_3 - 15 I_4 = 0$$

$$-15 I_3 + 45 I_4 = 0$$

$$-5 I_2 + 30 I_5 = 0$$

Estimate the current using Gauss Elimination Method.

Solution :

The given system of equations can be rewritten as

$$28 I_1 - 3 I_2 = 10 \qquad \ldots(I)$$

$$-3\ I_1 + 38\ I_2 - 10\ I_3 - 5\ I_5 = 0 \qquad \text{…(II)}$$

$$-2\ I_2 + 5\ I_3 - 3\ I_4 = 0 \qquad \text{…(III)}$$

$$-\ I_3 + 3\ I_4 = 0 \qquad \text{…(IV)}$$

$$-\ I_2 + 6\ I_5 = 0 \qquad \text{…(V)}$$

Step I : **First stage of Elimination :**

Multiplying equation (I) by $\frac{3}{28}$ and adding to equation (II) we get

$$28\ I_1 - 28\ I_2 = 10$$

$$37.6785\ I_2 - 10\ I_3 - 5\ I_5 = 1.0714286$$

$$-2\ I_2 + 5\ I_3 - 3\ I_4 = 0$$

$$-\ I_3 + 3\ I_4 = 0$$

$$-\ I_2 + 6\ I_5 = 0$$

Rearranging the above equations we get

$$28\ I_1 - 3\ I_2 = 10 \qquad \text{…(Ia)}$$

$$-\ I_2 + 6\ I_5 = 0 \qquad \text{…(IIa)}$$

$$-2\ I_2 + 5\ I_3 - 3\ I_4 = 0 \qquad \text{…(IIIa)}$$

$$+\ 37.6785 I_2 - 10\ I_3 - 5\ I_5 = 1.0714286 \qquad \text{…(IVa)}$$

$$-\ I_3 + 3\ I_4 = 0 \qquad \text{…(Va)}$$

Step II : **Second stage of Elimination :**

Multiplying equation (IIa) by 2 and subtracting from equation (IIIa) and multiplying equation (IIa) by 37.6785 and adding to equation (IVa) we get

$$28\ I_1 - 3\ I_2 = 10$$

$$-\ I_2 + 6\ I_5 = 0$$

$$+5\ I_3 - 3\ I_4 - 12\ I_5 = 0$$

$$-\ 10\ I_3 + 221.071\ I_5 = 1.0714286$$

$$-\ I_3 + 3\ I_4 = 0$$

Rearranging the above equations we get

$$28\ I_1 - 3\ I_2 = 10 \qquad \dots\text{(Ib)}$$

$$-\ I_2 + 6\ I_5 = 0 \qquad \dots\text{(IIb)}$$

$$-\ I_3 + 3\ I_4 - 3\ I_4 = 0 \qquad \dots\text{(IIIb)}$$

$$5\ I_2 - 3\ I_4 - 12\ I_5 = 0 \qquad \dots\text{(IVb)}$$

$$-\ 10\ I_3 + 221.071\ I_5 = 1.0714286 \qquad \dots\text{(Vb)}$$

Step III : Third stage of Elimination :

Multiplying equation (IIIb) by 5 and adding to equation (IVb) and multiplying equation (IIIb) by 10 and adding to equation (Vb) we get

$$28\ I_1 - 3\ I_2 = 10$$

$$-\ I_2 + 6\ I_5 = 0$$

$$-\ I_3 + 3\ I_4 = 0$$

$$12\ I_4 - 12\ I_5 = 0$$

$$-\ 30\ I_4 + 221.071\ I_5 = 1.0714286$$

Rearranging the above equations we get

$$28\ I_1 - 3\ I_2 = 10 \qquad \dots\text{(Ic)}$$

$$-\ I_2 + 6\ I_5 = 0 \qquad \dots\text{(IIc)}$$

$$-\ I_3 + 3\ I_4 = 0 \qquad \dots\text{(IIIc)}$$

$$12\ I_4 - 12\ I_5 = 0 \qquad \dots\text{(IVc)}$$

$- 30\, I_4 + 221.071\, I_5 = 1.0714286$ …(Vc)

Step IV : Fourth stage of Elimination :

Multiplying equation (IVc) by 30 and adding to equation (Vc) we get

$28\, I_1 - 3\, I_2 = 10$ …(Id)

$- I_2 + 6\, I_5 = 0$ …(IId)

$- I_3 + 3\, I_4 = 0$ …(IIId)

$I_4 - I_5 = 0$ …(IVd)

$191.071\, I_5 = 1.0714286$ …(Vd)

Step V : Finding solution using back substitution :

From equation (Vd). We have

$$I_5 = \frac{1.0714286}{191.071} = \mathbf{0.0056}$$

From equation (IVd). We have

$$I_4 = I_5 = \mathbf{0.0056}$$

From equation (IIId). We have

$$I_3 = 3\, I_4 = \mathbf{0.0.168}$$

From equation (IId). We have

$$I_2 = 6\, I_5 = \mathbf{0.0336}$$

From equation (Id). We have

$$I_1 = \frac{1}{28} [\, 10 + 3\, I_2 \,] = \mathbf{0.3607}$$

Unit 5

Gauss Jordan Method

Gauss Jordan method is a modified form of Gauss elimination method

Consider the system of equations

$$\left.\begin{aligned} a_{11}\chi_1 + a_{12}\chi_2 + a_{13}\chi_3 &= b_1 \\ a_{21}\chi_1 + a_{22}\chi_2 + a_{23}\chi_3 &= b_2 \\ a_{31}\chi_1 + a_{32}\chi_2 + a_{33}\chi_3 &= b_3 \end{aligned}\right\}$$

The above equations can be written as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

i.e., $AX = B$

Now in case of Gauss Jordan method, the coefficient matrix (A) is reduced to a diagonal matrix rather than a triangular matrix as done in case of Gauss Elimination method.

So the above system becomes

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Using back substitution, the solution of the given system of equations can be found as follows :

$$\chi_1 = b_1$$

$$\chi_2 = b_2$$

$$\chi_3 = b_3$$

Gauss Jordan method is generally not used for solving a system of equations because it is involves more computation than Gauss elimination method.

However Gauss Jordan method is very simple method of finding inverse of a matrix and is therefore used for finding solution of linear equation using inverse of a matrix.

Procedure for Finding Inverse of a Matrix using Gauss Jordan Method :

Step I : Write the given matrix A as Augmented matrix [A / I]

Step II : By applying now transformation, reduce the augmented matrix to the form [I / A^{-1}]

Step III : Write the inverse of the matrix A^{-1}

The above procedure will get clear to you as you will go through the following solved example.

Ex. 8 Use Gauss Jordan method to compute the inverse of matrix

$$[A] = \begin{pmatrix} 3 & -0.1 & -0.2 \\ 0.1 & 7 & -0.3 \\ 0.3 & -0.2 & 10 \end{pmatrix}$$

Solution :

Step I : **Writing Augmented matrix :**

$$[A \quad I] \sim \left(\begin{array}{ccc|ccc} 3 & -0.1 & -0.2 & 1 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right)$$

Step II : **Finding A^{-1} using transformations :**

$R_1 \to \dfrac{1}{3} R_1$ gives

$$\sim \left(\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0666 & 0.3333 & 0 & 0 \\ 0.1 & 7 & -0.3 & 0 & 1 & 0 \\ 0.3 & -0.2 & 10 & 0 & 0 & 1 \end{array}\right)$$

$R_2 \to R_2 - 0.1\, R_1$ and $R_3 \to R_3 - 0.3\, R_1$ gives

$$\sim \left(\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0666 & 0.3333 & 0 & 0 \\ 0 & 7.0033 & -0.2933 & -0.0333 & 1 & 0 \\ 0 & -0.19001 & 10.0199 & -0.0999 & 0 & 1 \end{array}\right)$$

$R_2 \to \dfrac{1}{7.0033} R_2$ gives

$$\sim \left(\begin{array}{ccc|ccc} 1 & -0.0333 & -0.0666 & 0.3333 & 0 & 0 \\ 0 & 1 & -0.0418 & -0.0047 & 0.14270 & \\ 0 & -0.19001 & 10.0199 & -0.0999 & 0 & 1 \end{array}\right)$$

$R_1 \to R_1 - 0.0333\, R_2$ and $R_3 \to R_3 + 0.19001\, R_2$ gives

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -0.0679 & 0.3334 & 0.0047 & 0 \\ 0 & 1 & -0.0418 & -0.0047 & 0.1427 & 0 \\ 0 & 0 & 10.0199 & -0.1007 & 0.0271 & 1 \end{array}\right)$$

$R_3 \to \dfrac{1}{10.0119} R_3$ gives

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -0.0679 & 0.3334 & 0.0047 & 0 \end{array}\right)$$

$$\sim \begin{matrix} 0 & 1 & -0.0418 & -0.0047 & 0.1427 & 0 \\ 0 & 0 & 1 & -0.01 & 0.027 & 0.0998 \end{matrix}$$

$R_1 \rightarrow R_1 - 0.0679\, R_3$ and $R_2 \rightarrow R_2 + 0.0418\, R_3$ gives

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.3327 & 0.0048 & 0.0067 \\ 0 & 1 & 0 & -0.0051 & 0.1428 & 0.0041 \\ 0 & 0 & 1 & -0.01 & 0.0027 & 0.0998 \end{array}\right)$$

$[\,A \;\; I\,] \sim [\,I \,/\, A^{-1}\,]$

Step III : **Writing A^{-1}**

From Step II, we get

$$\begin{pmatrix} 0.3327 & 0.0048 & 0.0067 \\ A^{-1} = 0.0051 & 0.1428 & 0.0041 \\ -0.01 & 0.0027 & 0.0998 \end{pmatrix}$$

Unit 5

Gauss – Seidel Method

Gauss Seidel Method is an Iterative or Indirect method to find solution of a system of equations.

$$\left.\begin{array}{l} a_{11}\chi_1 + a_{12}\chi_2 + a_{13}\chi_3 + \ldots + a_{1n}\chi_n = b_1 \\ a_{21}\chi_1 + a_{22}\chi_2 + a_{23}\chi_3 + \ldots + a_{2n}\chi_n = b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\chi_1 + a_{n2}\chi_2 + a_{33}\chi_3 + \ldots + a_{nn}\chi_n = b_n \end{array}\right\} \quad \ldots(I)$$

In which the diagonal elements a_{ii} do not vanish. If this is not the case. then equations should be rearranged so that this condition is satisfied. Now rewriting the system (I) as :

$$\chi_1 = \frac{1}{a_{11}} [b_1 - a_{12}\chi_2 - a_{13}\chi_3 + \ldots + a_{1n}\chi_n$$

$$\left.\begin{aligned} \chi_2 &= \frac{1}{a_{22}} [b_1 - a_{21}\chi_1 - a_{23}\chi_3 + \ldots + a_{2n}\chi_n] \\ \chi_n &= \frac{1}{a_{nn}} [b_n - a_n\chi_1 - a_{n2}\chi_2 + \ldots + a_{n(n-1)}\chi_{n-1}] \end{aligned}\right\} \ldots \text{(II)}$$

Now in the first equation of (II), we substitute the first approximation ($\chi_2^{(0)}$, $\chi_3^{(0)}$ … $\chi_n^{(0)}$) in to the right hand side and denote the result as $\chi_1^{(1)}$, . In the second equation, we substitute ($\chi_1^{(0)}$, $\chi_3^{(0)}$ … $\chi_n^{(0)}$) and denote the result as $\chi_2^{(1)}$, . Similarly, we get $\chi_3^{(1)}$, $\chi_4^{(1)}$ … $\chi_n^{(1)}$. This completes the first stage of iteration.

Generalizing the above procedure, we get the Gauss Seidel equation as follows

$$\left.\begin{aligned} \chi_1^{(k+1)} &= \frac{1}{a_{11}} [b_1 - a_{12}\chi_2^{(k)} - a_{13}\chi_3^{(k)} + \ldots + a_{1n}\chi^{(k)}{}_n \\ \chi_2^{(k+1)} &= \frac{1}{a_{22}} [b_1 - a_{21}\chi_1^{(k)} - a_{23}\chi_3^{(k)} + \ldots + a_{2n}\chi^{(k)}{}_n] \\ \chi_n^{(k+1)} &= \frac{1}{a_{nn}} [b_n - a_n\chi_1^{(k+1)} - a_{n2}\chi_2^{(k+1)} + \ldots + a_{n(n-1)}\chi^{(k+1)}{}_{n-1}] \end{aligned}\right\} \ldots \text{(II)}$$

Now second stage of iterations are performed in the same manner as the first stage of iterations using the values of first sage of iterations and $k = 1$.

More stages of iterations are performed in the same manner till the values of $\chi_1, \chi_2 \ldots \chi_3$ are obtained to the desired accuracy.

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 10 Use the Gauss – Seidel method to obtain the solution of the system :

$$3\chi_1 - 0.1\chi_2 - 0.2\chi_3 = 7.85$$
$$0.1\chi_1 + 7\chi_2 - 0.3\chi_3 = 19.3$$
$$0.3\chi_1 - 0.2\chi_2 - 10\chi_3 = 71.4$$

Solution :

Step I : Writing Gauss – Seidel iterative equations :

Since the diagonal elements are higher than any other elements of their respective columns, so the system of equations can be used without rearranging.

$$\chi_1^{(k+1)} = \frac{1}{3} [\, 7.85 + 0.1\chi_2^{(k)} + 0.2\chi_3^{(k)} \,] \qquad ..\text{I}$$

$$\chi_2^{(k+1)} = \frac{1}{7} [\, -19.3 - 0.1\chi_1^{(k+1)} + 0.3\chi_3^{(k)} \,] \qquad ..\text{II}$$

$$\chi_3^{(k+1)} = \frac{1}{10} [\, 71.4 - 0.3\chi_1^{(k+1)} + 0.2\chi_2^{(k+1)} \,] \qquad \ldots\text{III}$$

Step II : 1st Iteration :

For 1st iteration let k = 0 and let the initial approximation be

$$\chi_1^{(0)} = \chi_2^{(0)} = \chi_3^{(0)} = 0$$

Putting values of k, $\chi_2^{(0)}$ and $\chi_3^{(0)}$ in equation (I), we get

$$\chi_1^{(1)} = \frac{1}{3} [\, 7.85 + 0.1 \times (0) + 0.2 \times (0) \,]$$

$\therefore \quad \chi_1^{(1)} = 2.6166$

Putting values of k, $\chi_1^{(1)}$ and $d\chi_3^{(0)}$ in equation (II), we get

$$\chi_2^{(1)} = \frac{1}{7} [-19.3 - 0.1 \times (2.6156) + 0.3 \times (0)]$$

$\therefore \quad \chi_2^{(1)} = 2.7945$

Putting values of k, $\chi_1^{(1)}$ and $\chi_3^{(II)}$ in equation (III), we get

$$\chi_3^{(1)} = \frac{1}{10} [71.4 - 0.3 \times (2.6166) + 0.2 \times (-2.7945)]$$

$\therefore \quad \chi_3^{(1)} = 7.0056$

After 1st iteration, we have

$\chi_1^{(1)} = 2.6166, \qquad \chi_2^{(1)} = -2.7945, \qquad \chi_3^{(1)} = 7.0056$

Step III : 2nd Iteration :

For 2nd iteration, we have $k = 1$

Putting values of k, $\chi_1^{(I)}$ and $\chi_3^{(II)}$ in equation (I), we get

$$\chi_1^{(2)} = \frac{1}{3} [7.85 + 0.1 \times (-2.7945) + 0.2 \times (7.0056)]$$

$\therefore \quad \chi_1^{(2)} = 2.99055$

Putting values of k, $\chi_1^{(2)}$ and $\chi_3^{(I)}$ in equation (II), we get

$$\chi_2^{(2)} = \frac{1}{7} [-19.3 - 0.1 \times (2.99055) + 0.3 \times (7.0056)]$$

$\therefore \quad \chi_2^{(2)} = 2.49962$

Putting values of k, $\chi_1^{(2)}$ and $\chi_2^{(2)}$ in equation (III), we get

1

$$\chi_3^{(2)} = \frac{1}{10} [\, 71.4 - 0.3 \times (2.99055) + 0.2 \times (-2.49962) \,]$$

$$\therefore \quad \chi_3^{(2)} = 7.0029$$

After 2nd iteration, we have

$$\chi_1^{(2)} = 2.99055, \qquad \chi_2^{(2)} = -2.49962, \qquad \chi_3^{(2)} = 7.00029$$

Step IV : 3rd Iteration :

For 3rd iteration, we have $k = 2$

Putting values of k, $\chi_2^{(2)}$ and $\chi_3^{(2)}$ in equation (I), we get

$$\chi_1^{(3)} = \frac{1}{3} [\, 7.85 + 0.1 \times (-2.49962) + 0.2 \times (7.00029)]$$

$$\therefore \quad \chi_1^{(3)} = 3.000032$$

Putting values of k, $\chi_1^{(2)}$ and $\chi_3^{(I)}$ in equation (II), we get

$$\chi_2^{(3)} = \frac{1}{7} [\, -19.3 - 0.1 \times (3.000032) + 0.3 \times (7.00029) \,]$$

$$\therefore \quad \chi_2^{(3)} = 2.499988$$

Putting values of k, $\chi_1^{(3)}$ and $\chi_2^{(3)}$ in equation (III), we get

$$\chi_3^{(3)} = \frac{1}{10} [\, 71.4 - 0.3 \times (3.000032) + 0.2 \times (-2.4999888) \,]$$

$$\therefore \quad \chi_3^{(3)} = 6.999999$$

After 3rd iteration, we have

$$\chi_1^{(3)} = 3.000032, \qquad \chi_2^{(3)} = -2.499988, \qquad \chi_3^{(3)} = 6.999999$$

From the above iterations, it is clear that the variables values are tending to 3. , 2.5 and 7 respectively.

So, the solution of the given equations is

$\chi_1 = 3 \qquad \chi_2 = -2.5 \qquad \chi_3 = 7$

Ex. 11 Solve the following equations by Gauss – Seidel procedure. The answer should be correct to three significant digits.

$$\chi_1 - 10\chi_2 + 4\chi_3 = 6$$

$$2\chi_1 - 4\chi_2 + 10\chi_3 = -15$$

$$9\chi_1 + 2\chi_2 + 4\chi_3 = 20$$

Solution :

[Note : In this example, it is not give that how much iterations we have to compute. So, we will stop the iterations when the values of χ_1, χ_2, and χ_3 are same upto three decimal places]

Step I : Writing Gauss – Seidel iterative equations :

Since the diagonal elements are not higher than other elements of their respective columns, so the system of equations needs to be rearranged.

$$9\chi_1 + 2\chi_2 + 4\chi_3 = 20$$

$$\chi_1 + 10\chi_2 + 4\chi_3 = 6$$

$$2\chi_1 - 4\chi_2 + 10\chi_3 = -15$$

So, the Gauss – Seidel iterative equations are written as :

$$\chi_1^{(k+1)} = \frac{1}{9} [20 + 2\chi_2^{(k)} - 4\chi_3^{(k)}]$$

$$\chi_2^{(k+1)} = \frac{1}{10} [6 - 1\chi_1^{(k+1)} - 4\chi_3^{(k)}]$$

$$\chi_3^{(k+1)} = \frac{1}{10} [-15 - 2\chi_1^{(k+1)} - 4\chi_2^{(k+1)}]$$

Step II : 1st Iteration :

For 1st iteration let k = 0 and let the initial approximation be

$$\chi_1^{(0)} = \chi_2^{(0)} = \chi_3^{(0)} = 0$$

Putting values of k, $\chi_2^{(0)}$ and $\chi_3^{(0)}$ in equation (I), we get

$$\chi_1^{(1)} = \frac{1}{9} [\, 20 - 2 \text{ x } (2.2222) - 4 \text{ x } (0) \,]$$

$$\therefore \quad \chi_1^{(1)} = 2.222$$

Putting values of k, $\chi_1^{(1)}$ and $d\chi_3^{(0)}$ in equation (II), we get

$$\chi_2^{(1)} = \frac{1}{10} [\, 6 - 1 \text{ x } (2.2222) - 4 \text{ x } (0) \,]$$

$$\therefore \quad \chi_2^{(1)} = 0.3777$$

Putting values of k, $\chi_1^{(1)}$ and $\chi_2^{(II)}$ in equation (III), we get

$$\chi_3^{(1)} = \frac{1}{10} [\, -15 - 2 \text{ x } (2.2222) + 4 \text{ x } (0.3777) \,]$$

$$\therefore \quad \chi_3^{(1)} = -1.7933$$

After 1st iteration, we have

$\chi_1^{(1)} = 2.2222$, $\chi_2^{(1)} = 0.3777$, $\chi_3^{(1)} = -1.7933$

Step III : 2nd Iteration :

For 2nd iteration, we have $k = 1$

Putting values of k, $\chi_1^{(I)}$ and $\chi_3^{(II)}$ in equation (I), we get

$$\chi_1^{(2)} = \frac{1}{9} [\, 20 - 2 \text{ x } (0.3777) - 4 \text{ x } (-1.7933)]$$

$$\therefore \quad \chi_1^{(2)} = 2.9353$$

Putting values of k, $\chi_1^{(2)}$ and $\chi_3^{(I)}$ in equation (II), we get

$$\chi_2^{(2)} = \frac{1}{10} [6 - 1 \times (2.9353) - 4 \times (-1.7933)]$$

$\therefore$ $\chi_2^{(2)} = 1.0237$

Putting values of k, $\chi_1^{(2)}$ and $\chi_2^{(2)}$ in equation (III), we get

$$\chi_3^{(2)} = \frac{1}{10} [-15 - 2 \times (2.9353) - 4 \times (1.0237)]$$

$\therefore$ $\chi_3^{(2)} = -1.6775$

After 2nd iteration, we have

$\chi_1^{(2)} = 2.9353$, $\chi_2^{(2)} = 1.0237$, $\chi_3^{(2)} = -1.6775$

Step IV : 3rd Iteration :

For 3rd iteration, we have $k = 2$

Putting values of k, $\chi_2^{(2)}$ and $\chi_3^{(2)}$ in equation (I), we get

$$\chi_1^{(3)} = \frac{1}{9} [20 - 2 \times (1.0237) - 4 \times (-1.6775)]$$

$\therefore$ $\chi_1^{(3)} = 2.7403$

Putting values of k, $\chi_1^{(2)}$ and $\chi_3^{(I)}$ in equation (II), we get

$$\chi_2^{(3)} = \frac{1}{10} [6 - 1 \times (2.7403) - 4 \times (-1.6775)]$$

$\therefore$ $\chi_2^{(3)} = 0.9969$

Putting values of k, $\chi_1^{(3)}$ and $\chi_2^{(3)}$ in equation (III), we get

$$\chi_3^{(3)} = \frac{1}{10} [-15 - 2 \times (2.7403) + 4 \times (0.9969)]$$

$\therefore \quad \chi_3^{(3)} = -1.6492$

After 3rd iteration, we have

$\chi_1^{(3)} = 2.7403, \quad \chi_2^{(3)} = 0.9969, \quad \chi_3^{(3)} = -1.6492$

Step V : 4th Iteration :

For 4th iteration, we have *k* = 3

Putting values of k, $\chi_2^{(3)}$ and $\chi_3^{(3)}$ in equation (I), we get

$$\chi_1^{(4)} = \frac{1}{9} [\, 20 - 2 \times (0.9969) - 4 \times (-1.6492)]$$

$\therefore \quad \chi_1^{(4)} = 2.7336$

Putting values of k, $\chi_1^{(4)}$ and $\chi_3^{(4)}$ in equation (II), we get

$$\chi_2^{(4)} = \frac{1}{10} [\, 6 - 1 \times (2.7336) - 4 \times (-1.6492)\,]$$

$\therefore \quad \chi_2^{(4)} = 0.9863$

Putting values of k, $\chi_1^{(4)}$ and $\chi_2^{(4)}$ in equation (III), we get

$$\chi_3^{(4)} = \frac{1}{10} [\, -15 - 2 \times (2.7336) + 4 \times (0.9863)\,]$$

$\therefore \quad \chi_3^{(4)} = -1.6521$

After 4th iteration, we have

$\chi_1^{(4)} = 2.7336, \quad \chi_2^{(4)} = 0.9863, \quad \chi_3^{(4)} = -1.6521$

Step VI : 5th Iteration :

For 5th iteration, we have *k* = 4

Putting values of k, $\chi_2^{(4)}$ and $\chi_3^{(4)}$ in equation (I), we get

$$\chi_1^{(5)} = \frac{1}{9} [20 - 2 \times (0.9863) - 4 \times (-1.6521)]$$

$$\therefore \quad \chi_1^{(5)} = 2.7373$$

Putting values of k, $\chi_1^{(5)}$ and $\chi_3^{(4)}$ in equation (II), we get

$$\chi_2^{(5)} = \frac{1}{10} [6 - 1 \times (2.7373) - 4 \times (-1.6521)]$$

$$\therefore \quad \chi_2^{(5)} = 0.9871$$

Putting values of k, $\chi_1^{(5)}$ and $\chi_2^{(5)}$ in equation (III), we get

$$\chi_3^{(5)} = \frac{1}{10} [-15 - 2 \times (2.7373) + 4 \times (0.9871)]$$

$$\therefore \quad \chi_3^{(5)} = -1.6526$$

After 5th iteration, we have

$$\chi_1^{(5)} = 2.7373, \qquad \chi_2^{(5)} = 0.9871, \qquad \chi_3^{(5)} = -1.6526$$

Step VII : 6th Iteration :

For 6th iteration, we have *k* = 5

Putting values of k, $\chi_2^{(5)}$ and $\chi_3^{(5)}$ in equation (I), we get

$$\chi_1^{(6)} = \frac{1}{9} [20 - 2 \times (0.9871) - 4 \times (-1.6526)]$$

$$\therefore \quad \chi_1^{(6)} = 2.7373$$

Putting values of k, $\chi_1^{(6)}$ and $\chi_3^{(5)}$ in equation (II), we get

$$\chi_2^{(6)} = \frac{1}{[\ldots]} [6 - 1 \times (2.7373) - 4 \times (-1.6526)]$$

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$\therefore \quad \chi_2^{(6)} = 0.9873$

Putting values of k, $\chi_1^{(6)}$ and $\chi_2^{(6)}$ in equation (III), we get

$$\chi_3^{(6)} = \frac{1}{10} [-15 - 2 \times (2.7373) + 4 \times (0.9873)]$$

$\therefore \quad \chi_3^{(6)} = -1.6525$

After 6th iteration, we have

$\chi_1^{(6)} = 2.7373, \quad \chi_2^{(6)} = 0.9873, \quad \chi_3^{(6)} = -1.6525$

The values of χ_1, χ_2 and χ_3 are same upto three decimal places in 5th and iteration So, the solution of given system of equations is.

$\chi_1^{(6)} = 2.7373, \quad \chi_2^{(6)} = 0.9873, \quad \chi_3^{(6)} = -1.6525$

Ex. 12 Given the linear equations :

$$4\chi + \gamma + z = 5$$

$$\chi + 6\gamma + 2z = 19$$

$$-\chi - 2\gamma + 5z = 10$$

Obtain the values of χ, γ and z for three successive iterations using Gauss Seidel method with an initial guess of $\chi = \gamma = z = 0$

Solution :

[Note : In this example, it is given that we have to compute till three iterations]

Step I : Writing Gauss – Seidel iterative equations :

Since the diagonal elements are higher than any other elements of then respective columns, so the system of equations are written as:

$$\chi^{(k+1)} = \frac{1}{4} [5 - 1\gamma^{(k)} - 1z^{(k)}] \qquad \dots(I)$$

$$\gamma^{(k+1)} = \frac{1}{6} [\, 19 - 1\chi_1^{(k+1)} - 2z^{(k)} \,]$$

$$z_3^{(k+1)} = \frac{1}{10} [\, -15 - 2\chi_1^{(k+1)} - 2\gamma^{(k+1)} \,]$$

Step II : **1st Iteration :**

For 1st iteration let k = 0

It is given that Initial approximation

$$\chi^{(0)} = \gamma^{(0)} = z^{(0)} = 0$$

Putting values of k, $\gamma^{(0)}$ and $z^{(0)}$ in equation (I), we get

$$\chi^{(1)} = \frac{1}{4} [5 - 1 \times (0) - 1 \times (0) \,]$$

$\therefore$ $\chi^{(1)} = 1.25$

Putting values of k, $\chi_1^{(1)}$ and $\gamma^{(1)}$ in equation (II), we get

$$\gamma^{(1)} = \frac{1}{6} [19 - 1 \times (1.25) - 2 \times (0) \,]$$

$\therefore$ $\gamma^{(1)} = 2.9583$

Putting values of k, $\chi_1^{(1)}$ and $\gamma_2^{(l)}$ in equation (III), we get

$$z^{(1)} = \frac{1}{10} [\, 10 + 1 \times (1.25) + 2 \times (2.9583) \,]$$

$\therefore$ $z^{(1)} = 3.4333$

After 1st iteration, we have

$$\chi^{(1)} = 1.25, \qquad \gamma^{(1)} = 2.9583, \qquad z^{(1)} = 3.4333$$

Step III : **2^{nd} Iteration :**

For 2^{nd} iteration, we have $k = 1$

Putting values of k, $\gamma^{(1)}$ and $z^{(1)}$ in equation (I), we get

$$\chi^{(2)} = \frac{1}{4} [5 - 1 \times (2.9583) - 1 \times (3.4333)]$$

$$\therefore \quad \chi^{(2)} = -0.3479$$

Putting values of k, $\chi_1^{(2)}$ and $\gamma^{(2)}$ in equation (II), we get

$$\gamma^{(2)} = \frac{1}{6} [19 - 1 \times (-0.3479) - 2 \times (3.4333)]$$

$$\therefore \quad \gamma^{(2)} = 2.0802$$

Putting values of k, $\chi^{(2)}$ and $\gamma^{(2)}$ in equation (III), we get

$$z^{(2)} = \frac{1}{10} [10 + 1 \times (0.3479) + 2 \times (2.0802)]$$

$$\therefore \quad z^{(2)} = 2.7625$$

After 2^{nd} iteration, we have

$$\chi^{(2)} = 0.3479, \qquad \gamma^{(2)} = 2.0802, \qquad z^{(2)} = 2.7625$$

Step III : **3^{rd} Iteration :**

For 3^{rd} iteration, we have $k = 2$

Putting values of k, $\gamma^{(2)}$ and $z^{(2)}$ in equation (I), we get

$$\chi^{(3)} = \frac{1}{4} [5 - 1 \times (2.0802) - 1 \times (2.7625)]$$

$$\therefore \quad \chi^{(3)} = 0.3932$$

Putting values of k, $\chi_1^{(3)}$ and $\gamma^{(3)}$ in equation (II), we get

$$\gamma^{(3)} = \frac{1}{6} [19 - 1 \times (0.03932) - 2 \times (2.7625)]$$

$$\therefore \quad \gamma^{(3)} = 2.2392$$

Putting values of k, $\chi^{(3)}$ and $\gamma^{(3)}$ in equation (III), we get

$$z^{(3)} = \frac{1}{10} [10 + 1 \times (0.03932) + 2 \times (2.2392)]$$

$$\therefore \quad z^{(3)} = 2.9035$$

After 3rd iteration, we have

$$\boldsymbol{\chi^{(3)} = 0.3932, \qquad \gamma^{(3)} = 2.2392, \qquad z^{(3)} = 2.9035}$$

Ex. 13 Develop the Gauss – Seidel iterative scheme for the solution of the system.

$$10\chi_1 - 5\chi_2 - 2\chi_3 = 3$$

$$-4\chi_1 + 10\chi_2 - 3\chi_3 = 3$$

$$-\chi_1 - 6\chi_2 + 10\chi_3 = 3$$

Iterate upto a maximum of 10 times or upto an accuracy of 0.0001 starting with the initial solution vector $\chi^{(0)} = 0$

Solution :

[Note : In this example, it is given that we have to compute till ten iterations]

Step I : Writing Gauss – Seidel iterative equations :

Since the diagonal elements are higher than any other elements of the respective columns, so the system of equations can be used without rearranging

So, the Gauss – Seidel iterative equations are written as :

$$\chi_1^{(k+1)} = \frac{1}{10} [3 + 5\chi_2^{(k)} + 2\chi_3^{(k)}] \qquad \ldots(I)$$

$$\chi_2^{(k+1)} = \frac{1}{10} [3 + 4\chi_1^{(k+1)} + 3\chi_3^{(k)}] \qquad \ldots(II)$$

$$\chi_3^{(k+1)} = \frac{1}{10} [3 + 1\chi_1^{(k+1)} + 6\chi_2^{(k+1)}] \qquad \ldots(III)$$

Step II : 1st Iteration :

For 1st iteration let k = 0 and let the initial approximation be

$$\chi_1^{(0)} = \chi_2^{(0)} = \chi_3^{(0)} = 0$$

Putting values of k, $\chi_2^{(0)}$ and $\chi_3^{(0)}$ in equation (I), we get

$$\chi_1^{(1)} = \frac{1}{10} [3 + 5 \text{ x } (0) + 2 \text{ x } (0)]$$

$\therefore \qquad \chi_1^{(1)} = 0.3$

Putting values of k, $\chi_1^{(1)}$ and $d\chi_3^{(0)}$ in equation (II), we get

$$\chi_2^{(1)} = \frac{1}{10} [3 + 4 \text{ x } (0.3) + 3 \text{ x } (0)]$$

$\therefore \qquad \chi_2^{(1)} = 0.42$

Putting values of k, $\chi_1^{(1)}$ and $\chi_3^{(II)}$ in equation (III), we get

$$\chi_3^{(1)} = \frac{1}{10} [3 + 1 \text{ x } (0.3) + 6 \text{ x } (0.42)]$$

$\therefore \qquad \chi_3^{(1)} = 0.582$

After 1st iteration, we have

$\chi_1^{(1)} = 0.3, \qquad \chi_2^{(1)} = 0.42, \qquad \chi_3^{(1)} = 0.582$

Step III : 2nd Iteration :

For 2nd iteration, we have $k = 1$

Putting values of k, $\chi_1^{(l)}$ and $\chi_2^{(l)}$ in equation (I), we get

$$\chi_1^{(2)} = \frac{1}{10} [3 + 5 \text{ x } (0.42) + 2 \text{ x } (0.582)]$$

$$\therefore \qquad \chi_1^{(2)} = 0.72516$$

Putting values of k, $\chi_1^{(2)}$ and $\chi_3^{(l)}$ in equation (II), we get

$$\chi_2^{(2)} = \frac{1}{10} [3 + 4 \text{ x } (0.6264) + 3 \text{ x } (0.582)]$$

$$\therefore \qquad \chi_2^{(2)} = 0.72516$$

Putting values of k, $\chi_1^{(2)}$ and $\chi_2^{(2)}$ in equation (III), we get

$$\chi_3^{(2)} = \frac{1}{10} [3 + 1 \text{ x } (0.6264) + 6 \text{ x } (0.72516)]$$

$$\therefore \qquad \chi_3^{(2)} = 0.79773$$

After 2nd iteration, we have

$\chi_1^{(2)} = 0.6264, \qquad \chi_2^{(2)} = 0.72516, \qquad \chi_3^{(2)} = 0.79773$

Step IV : 3rd Iteration :

For 3rd iteration, we have $k = 2$

Putting values of k, $\chi_2^{(2)}$ and $\chi_3^{(2)}$ in equation (I), we get

$$\chi_1^{(3)} = \frac{1}{10} [3 + 5 \text{ x } (0.72516) + 2 \text{ x } (0.79773)]$$

$\therefore \quad \chi_1^{(3)} = 0.82212$

Putting values of k, $\chi_1^{(3)}$ and $\chi_3^{(2)}$ in equation (II), we get

$$\chi_2^{(3)} = \frac{1}{10} [\, 3 + 4 \text{ x } (0.82212) + 3 \text{ x } (0.79773) \,]$$

$\therefore \quad \chi_2^{(3)} = 0.86816$

Putting values of k, $\chi_1^{(3)}$ and $\chi_2^{(3)}$ in equation (III), we get

$$\chi_3^{(3)} = \frac{1}{10} [\, 3 + 1 \text{ x } (0.82212) + + \text{ x } (0.86816) \,]$$

$\therefore \quad \chi_3^{(3)} = 0.90310$

After 3rd iteration, we have

$\chi_1^{(3)} = 0.82212, \qquad \chi_2^{(3)} = 0.86816, \qquad \chi_3^{(3)} = 0.90310$

Step V : 4th Iteration :

For 4th iteration, we have $k = 3$

Putting values of k, $\chi_2^{(3)}$ and $\chi_3^{(3)}$ in equation (I), we get

$$\chi_1^{(4)} = \frac{1}{10} [\, 3 + 5 \text{ x } (0.86816) + 2 \text{ x } (090310)]$$

$\therefore \quad \chi_1^{(4)} = 0.9147$

Putting values of k, $\chi_1^{(4)}$ and $\chi_3^{(3)}$ in equation (II), we get

$$\chi_2^{(4)} = \frac{1}{10} [\, 3 + 4 \text{ x } (0.9147) + 3 \text{ x } (0.90310) \,]$$

$\therefore \quad \chi_2^{(4)} = 0.93681$

Putting values of k, $\chi_1^{(4)}$ and $\chi_2^{(4)}$ in equation (III), we get

1

$$\chi_3^{(4)} = \frac{}{10} [\, 3 + 1 \times (0.9147) + 6 \times (0.93681) \,]$$

$\therefore \qquad \chi_3^{(4)} = 0.95355$

After 4^{th} iteration, we have

$\chi_1^{(4)} = 0.9147, \qquad \chi_2^{(4)} = 0.93681, \qquad \chi_3^{(4)} = 0.95355$

Step VI : 5^{th} Iteration :

For 5^{th} iteration, we have $k = 4$

Putting values of k, $\chi_2^{(4)}$ and $\chi_3^{(4)}$ in equation (I), we get

$$\chi_1^{(5)} = \frac{1}{10} [\, 3 + 5 \times (0.93681) + 2 \times (0.95355)]$$

$\therefore \qquad \chi_1^{(5)} = 0.95911$

Putting values of k, $\chi_1^{(5)}$ and $\chi_3^{(4)}$ in equation (II), we get

$$\chi_2^{(5)} = \frac{1}{10} [\, 3 + 4 \times (0.9591) + 3 \times (0.95355) \,]$$

$\therefore \qquad \chi_2^{(5)} = 0.96971$

Putting values of k, $\chi_1^{(5)}$ and $\chi_2^{(5)}$ in equation (III), we get

$$\chi_3^{(5)} = \frac{1}{10} [\, 3 + 1 \times (0.95911) + 6 \times (0.96971) \,]$$

$\therefore \qquad \chi_3^{(5)} = 0.97773$

After 5^{th} iteration, we have

$\chi_1^{(5)} = 0.95911, \qquad \chi_2^{(5)} = 0.96971, \qquad \chi_3^{(5)} = 0.97773$

Step VII : 6^{th} Iteration :

For 6th iteration, we have $k = 5$

Putting values of k, $\chi_2^{(5)}$ and $\chi_3^{(5)}$ in equation (I), we get

$$\chi_1^{(6)} = \frac{1}{10} [\ 3 + 5 \times (0.96971) + 2 \times (0.97773)]$$

$\therefore$ $\chi_1^{(6)} = 0.98040$

Putting values of k, $\chi_1^{(6)}$ and $\chi_3^{(5)}$ in equation (II), we get

$$\chi_2^{(6)} = \frac{1}{10} [\ 3 + 4 \times (0.98040) + 3 \times (0.97773)\ ]$$

$\therefore$ $\chi_2^{(6)} = 0.98547$

Putting values of k, $\chi_1^{(6)}$ and $\chi_2^{(6)}$ in equation (III), we get

$$\chi_3^{(6)} = \frac{1}{10} [\ 3 + 1 \times (0.98040) + 6 \times (0.98547)\ ]$$

$\therefore$ $\chi_3^{(6)} = 0.98932$

After 6th iteration, we have

$\chi_1^{(6)} = 0.98040$, $\chi_2^{(6)} = 0.98547$, $\chi_3^{(6)} = 0.98932$

Step VIII : 7th Iteration :

For 7th iteration, we have $k = 6$

Putting values of k, $\chi_2^{(6)}$ and $\chi_3^{(6)}$ in equation (I), we get

$$\chi_1^{(7)} = \frac{1}{10} [\ 3 + 5 \times (0.98547) + 2 \times (0.98932)]$$

$\therefore$ $\chi_1^{(7)} = 0.99060$

Putting values of k, $\chi_1^{(7)}$ and $\chi_3^{(6)}$ in equation (II), we get

$$\chi_2^{(7)} = \frac{1}{10} [\, 3 + 4 \times (0.99060) + 3 \times (0.98932) \,]$$

$\therefore \quad \chi_2^{(7)} = 0.99303$

Putting values of k, $\chi_1^{(7)}$ and $\chi_2^{(7)}$ in equation (III), we get

$$\chi_3^{(7)} = \frac{1}{10} [\, 3 + 1 \times (0.99060) + 6 \times (0.99303) \,]$$

$\therefore \quad \chi_3^{(7)} = 0.99488$

After 7th iteration, we have

$\chi_1^{(7)} = 0.99060, \quad \chi_2^{(7)} = 0.99303, \quad \chi_3^{(7)} = 0.99488$

Step IX : 8th Iteration :

For 8th iteration, we have $k = 7$

Putting values of k, $\chi_2^{(7)}$ and $\chi_3^{(7)}$ in equation (I), we get

$$\chi_1^{(8)} = \frac{1}{10} [\, 3 + 5 \times (0.99303) + 2 \times (0.99488)]$$

$\therefore \quad \chi_1^{(8)} = 0.99549$

Putting values of k, $\chi_1^{(8)}$ and $\chi_3^{(7)}$ in equation (II), we get

$$\chi_2^{(8)} = \frac{1}{10} [\, 3 + 4 \times (0.99549) + 3 \times (0.99488) \,]$$

$\therefore \quad \chi_2^{(8)} = 0.99666$

Putting values of k, $\chi_1^{(8)}$ and $\chi_2^{(8)}$ in equation (III), we get

$$\chi_3^{(8)} = \frac{1}{} [\, 3 + 1 \times (0.99549) + 6 \times (0.99666) \,]$$

10

$\therefore \quad \chi_3^{(8)} = 0.99754$

After 8th iteration, we have

$$\chi_1^{(8)} = 0.99549, \quad \chi_2^{(8)} = 0.99666, \quad \chi_3^{(8)} = 0.99754$$

Step X : 9th Iteration :

For 9th iteration, we have $k = 8$

Putting values of k, $\chi_2^{(8)}$ and $\chi_3^{(8)}$ in equation (I), we get

$$\chi_1^{(9)} = \frac{1}{10} [\, 3 + 5 \text{ x } (0.99666) + 2 \text{ x } (0.99754)]$$

$\therefore \quad \chi_1^{(9)} = 0.99783$

Putting values of k, $\chi_1^{(9)}$ and $\chi_3^{(8)}$ in equation (II), we get

$$\chi_2^{(9)} = \frac{1}{10} [\, 3 + 4 \text{ x } (0.99783) + 3 \text{ x } (0.99754)\,]$$

$\therefore \quad \chi_2^{(9)} = 0.99839$

Putting values of k, $\chi_1^{(9)}$ and $\chi_2^{(9)}$ in equation (III), we get

$$\chi_3^{(9)} = \frac{1}{10} [\, 3 + 1 \text{ x } (0.99783) + 6 \text{ x } (0.99839)\,]$$

$\therefore \quad \chi_3^{(9)} = 0.99882$

After 9th iteration, we have

$$\chi_1^{(9)} = 0.99783, \quad \chi_2^{(9)} = 0.99839, \quad \chi_3^{(9)} = 0.99882$$

Step XI : 10th Iteration :

For 10th iteration, we have $k = 9$

Putting values of k, $\chi_2^{(9)}$ and $\chi_3^{(9)}$ in equation (I), we get

$$\chi_1^{(10)} = \frac{1}{10} [\, 3 + 5 \times (0.99839) + 2 \times (0.99882)]$$

$$\therefore \quad \chi_1^{(10)} = 0.99895$$

Putting values of k, $\chi_1^{(10)}$ and $\chi_3^{(9)}$ in equation (II), we get

$$\chi_2^{(10)} = \frac{1}{10} [\, 3 + 4 \times (0.99895) + 3 \times (0.99882) \,]$$

$$\therefore \quad \chi_2^{(10)} = 0.99922$$

Putting values of k, $\chi_1^{(10)}$ and $\chi_2^{(10)}$ in equation (III), we get

$$\chi_3^{(10)} = \frac{1}{10} [\, 3 + 1 \times (0.99895) + 6 \times (0.99922) \,]$$

$$\therefore \quad \chi_3^{(10)} = 0.99943$$

After 10th iteration, we have

$$\chi_1^{(10)} = 0.99895, \qquad \chi_2^{(10)} = 0.99922, \qquad \chi_3^{(10)} = 0.99943$$

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6

INTRODUCTION

Numerical methods for solving differential equations are classified in following two categories.

(I) SINGLE STEP METHODS :

These methods are also called **Methods of Starting the Solution.** These are **direct methods.** In these methods, we use the information about the curve at one point and we do not iterate the solution. These methods involves more evaluation of the function. The main drawback of Single Step Methods is that it is very difficult to estimate the error in these methods. There are two classes of Single Step Methods.

Taylor Series Solution and Picard's Method of Successive Approximations are the examples of First Class of Single Step Methods.

Euler's Method and Runge-Kutta Methods are the examples of Second Class of Single Step Methods.

(II) MULTI – STEP METHODS :

These methods are also called **Methods for Continuing the Solution.** These are **indirect methods.** In these methods, next point on the curve is evaluated by performing iterations till sufficient accuracy is achieved. These methods involves fewer evaluation of the function. Estimation of error is also possible in Multi – Step Methods.

Predictor – Corrector Methods are the examples of Multi – Step Methods.

TAYLOR SERIES METHODS

The Taylor's series method theoretically provides a solution to any differential equation.

Consider the differential equation

$$y^1 = f(x, y) \qquad \ldots(i)$$

With the initial condition

$$y = (x_0) = y_0$$

If $y(x)$ is the exact solution of equation (1), then the Taylor's series for $y(x)$ around $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)\, y_0 + \frac{(x - x_0)^2}{2!}\, y^{11} + \qquad \ldots(II)$$

If values of y_0^I, y_0^{II}, . . . are known, then equation (II) gives a power series for y.

$$\left(\frac{dy}{} \right)$$

Ex. 1 Obtain the solution of $y^1 = x - y^2$ or $\dfrac{----}{dx} = x - y^2$ using. Taylor series.

Given : $y(0) = 1$.

Also find $y\,(0.1)$ correct to four decimal places.

Solution :

The Taylor series for $y(x)$ is given by

$$y\,(x) = y_0 + (x - x)\,y_0^{i} + \frac{(x - x_0)^2}{2}\,y_0^{ii} + \frac{(x - x_0)^3}{6}\,y_0^{iii} + \frac{(x - x_0)^4}{24}\,y_0^{iv} + \frac{(x - x_0)^5}{120}\,y_0^{v}$$

Step 1 :

Here, initial condition given is $y\,(o) = 1$.
So, $y_0 = 1$ when $x_0 = 0$

Step II : Finding the derivatives :

The derivatives $y_{0,}\ y_0$ …. etc. are obtained thus :

$y^{I}(x) = x - y^2$ $\qquad$ $y_0^{I} = 0 - 1 = -1$

$y^{II}(x) = 1 - 2\,yy^{I}$ $\qquad$ $y_0^{II} = 0 - 2(1)(-1) = 3$

$y^{III}(x) = -2\,yy^{II} - 2\,y^{I2}$ $\qquad$ $y_0^{III} = -2(1)(3) - 2(-1)^2 = -8$

$y^{Iv}(x) = -2\,yy^{III} - 6y^{I}y^{II}$ $\qquad$ $y_0^{Iv} = -2(1)(-8)-6(-1)(3) = 34$

$y^{Iv}(x) = -2\,yy^{Iv} - 8\,y^{I}y^{II}\ -6y^{II2}$ $\qquad$ $y_o^{V} = -186$

$y^{v}(x) = -2\,yy^{Iv} - 2\,y^{I}y^{III} - 6\,y^{I}y^{III}\ -6y^{II2} = 2^{yyIv}\ -8\,y^{I}y^{III}\ -6y^{II2}$

Step III :

Putting these values in the Taylor series, we get

$$y\,(x) = 1 + (x - 0)(-1) + \frac{(x - 0)^2}{2}(3) + \frac{(x - 0)^3}{6}(-8) + \frac{(x - 0)^4}{24}(34) + \frac{(x - 0)^5}{120}(-186) + \ldots$$

$$y\,(x) = 1^{-}\,x + \frac{3}{-----}\,x^2 - \frac{4}{-------}\,x3 + \frac{17}{-------}\,x^4 - \frac{31}{--------}\,x^5 + \ldots$$

2 3 12 20

Step IV : Finding *y*(0.1) correct to four decimal places :

To Obtain the value of *y*(0.1) correct to four decimal places, it is found that the terms upto x^4 should be considered So. Putting x = 0.1 in the above equation and considering terms till x^4, we have

$$y(0.1) = 1-(0.1) + \frac{3}{2}(0.1)^2 - \frac{4}{3}(0.1)^3 + \frac{17}{12}(0.1)^4$$

$$y(0.1) = 1 - 0.1 + 0.015 – 0.001333 + 0.000141$$

y(0.1) = 0.9139

Ex. 2 Solve the following equation by Taylor's series method.

$$xy^1 = x – y, \; y(2) = 2$$

Also find *y* at *x = 2.1*

Solution :

The Taylor series for *y(x)* is given by

$$y(x) = y_0 + (x – x_0)\, y_0^{i} + \frac{(x - x_0)^2}{2} y_0^{ii} + \frac{(x - x_0)^3}{6} y_0^{iii} + \frac{(x - x_0)^4}{24} y_0^{iv} + \frac{(x - x_0)^5}{120} y_0^{v} +$$

Step I :

Here, initial condition given is *y(2) = 2*

So, $y_0 = 2$ *when* $x_0 = 2$

Step II : finding the derivatives :-

It is given that $xy^1 = x – y$

$$y^1 = \frac{x - y}{x}$$

The derivatives y_0^{i}, y_0^{ii}, *… etc. are obtained thus :*

$$y^{i}(x) = 1 - \frac{y}{x} \qquad\qquad y_o^{i} = 1 - \frac{2}{2} = 0$$

$$y^{ii}(x) = \frac{y}{x^2} - \frac{y^{i}}{x} \qquad\qquad y_o^{ii} = \frac{2}{2} - \frac{0}{2} = \frac{1}{2}$$

$$y^{ii}(x) = \frac{-2y}{x^3} + \frac{2\,y^{i}}{x^2} - \frac{y^{ii}}{x} \qquad\qquad y_o^{ii} = \frac{-2(2)}{(2)^3} - \frac{2(0)}{(2)^3} = \frac{(1/2)}{2} = -\frac{3}{4}$$

Step III :

Putting these values in the Taylor series. we get

$$y(x) = 2 + (x-2)(0) + \frac{(x-2)^2}{2}\left(\frac{1}{2}\right) - \frac{(x-2)^3}{6} - \left(\frac{3}{4}\right) - \ldots$$

$$y\,(x) = 2 + \frac{(x-2)^2}{4} - \frac{(x-2)^3}{8} + \;.\;.\;.$$

Step IV : Finding *y* at *x* = 2.1 :

Putting x = 2.1 in the above equation. We have

$$y\,(2.1) = 2 + \frac{(2.1-2)^2}{4} - \frac{(2.1-3)^3}{8} + \;.\;.\;.$$

$$y\,(2.1) = 2 + \frac{(0.1)^2}{4} - \frac{(0.1)^3}{8} + \;.\;.\;.$$

$$y\,(2.1) = 2 + 0.0025 - 0.000125 + \;.\;.\;.$$

y (2.1) = 2.002375

PICARD'S METHODS OF SUCCESSIVE APPROXIMATIONS

Consider the differential equation

$$y^{j} = f(x,y)$$

with the initial condition $y(x_0) = y_0$

Integrating differential equation (1) we get

$$y = y_0 + \int_{x_0}^{x} f(x, y).\, dx$$

Equation (II) in which the unknown function y appears under the integral sign. Is called an **integral equation.** Such an equation can be solved by the method of successive approximations in which the first approximation to y is obtained by putting y^0 for y on the right side of equation (II). Thus, we get

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0).\, dx$$

The integral on the right can now be solved and the resulting $y^{(1)}$ is substituted for y in the integrand of equation (II) to obtain the second approximation $y^{(2)}$. Thus.

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(ii)}).\, dx$$

Proceeding in this way, we obtain $y^{(3)}, y^{(4)} \dots y^{(n-1)}$ and $y^{(n)}$ where

$$y^{(n)} = y_0 - \int_{x_0}^{x} f(x, y^{(n-1)}).\, dx$$

With

$$y^{(0)} = y_0$$

Hence this method yields a sequence of approximations $y^{(1)}, y^{(2)} \dots y^{(n)}$ to get the solution of the given equation.

Ex. 3 Solve $\dfrac{dy}{dx} = x + y$ by Picard's methods with initial conditions $x_0 = 0, y_0$.

Solution :

Here. $f(x, y) = x + y$
$x_0 = 0$ and $y_0 = 1$

Step 1 : First Approximation :-

We have

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0).\, dx$$

$$y^{(1)} = 1 + \int_{0}^{x} (x + 1).\, dx$$

$$y^{(1)} = 1 + x + \frac{x^2}{2}$$

Step II : Second Approximation :

We have

$$y^{(Z)} = 1 + \int_{0}^{x} (x, y^{(1)}).\, dx$$

$$y^{(2)} = 1 + \int_{0}^{x} \left(1 + x + \frac{x^2}{2}\right).\, dx$$

$$y^{(2)} = 1 + x + x2 + \frac{x^3}{6}$$

Step III : Third Approximation :

We have

$$y^{(3)} = 1 + \int_{0}^{x} (x + y^{(2)}).\, dx$$

$$\int^{x} \left(\qquad x^3 \qquad \right)$$

$$y^{(3)} = 1 + \int_{0} \left(1 + x + x^2 + \frac{\quad}{6}\right) . dx$$

$$y^{(3)} = 1 + x + x2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Solution to given equation is :

$$y = 1 + x + x2 + \frac{x^3}{3} + \frac{x^4}{24}$$

EULER'S METHOD

Suppose that we wish to solve the equation $y^{/} = f(x, y)$ for values of y at
$x = x_r = x_0 + rh\ (r = 1,2, \ldots)$.

Integrating $y^{/} = f(x, y)$, we get

$$y_1 = y_0 + \int_{x_0}^{x_1} (x, y).\, dx \qquad \ldots \text{(i)}$$

Assuming that $f(x, y) = f(x_0, y_0)$ in $x_0 \le x \le x_1$, we get Euler's formula as :

$$y_1 = y_0 + h.\, f(x_0 + y_0) \qquad \ldots \text{(ii)}$$

Similarly for the range $x_1 \le x \le x_2$, we have

$$y_2 = y_1 + \int_{x_1}^{x_2} (x, y).\, dx$$

Substituting $f(x_1, y_1)$ for $f(x, y)$ in $x_1 \le x \le x_2$, we get

$$y_2 = y_1 + h.\, f(x_1, y_1) \qquad \ldots \text{(iii)}$$

Proceeding in this way, we obtain the general Euler's formula as:

$$y_{n+1} = y_n + h.\, f(x_n, y_n) \quad n = 0,1,2, \ldots \ldots \quad \text{(iv)}$$

Procedure to Solve Examples using Simple Euler's method:

Step I : From the data given in the problem, find x_0, y_o and h

Step II : Find y_1 from the Euler's formula $y_{n+1} = y_n + h.\ f(x_n, y_n)$.
Step III : Find x_1 from the formula $x_{i+1} = x_i + h$

Step IV : Repeat Step II and Step III to find $(x_2, y_2), (x_3, y_3), \ldots$
and so on
till x reaches the final values given in the example.

Step V : Tabulate the result.

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 4 Use Euler's method to numerically integrate

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

from $x = 0$ to $x = 4$ with a step of 0.5.

The initial condition at $x = \mathbf{0}$ is $y = \mathbf{1}$.

Solution :

Here it is given that $x_0 = 0,\ \ y_0 = 1$ and $h = 0.5$

According to Euler's method, we have

$$y_{n+1} = y_n + h.\ f(x_n, y_n) \qquad \ldots (i)$$

Step I :

Putting $n = 0$ in equation (I), we get

$$y_1 = y_0 + h.\ f(x_0, y_0)$$
$$y_1 = 1 + 0.5 \ x\ [-2(0)^3 + 12(0)^2 - 20(0) + 8.5]$$
$$y_1 = 1 + 0.5 \ x\ 8.5$$
$$y_1 = 5.25$$

Step II :

Now $x_1 = x_0 + h = 0 + 0.5 = 0.5$

Putting $n = 1$ in equation (1) we get

$$y_2 = y_1 + h.\ f(x_1, y_1)$$

$$y_2 = 5.25 + 0.5 \ x\ [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]$$

$y_2 = 5.25 + 0.5 \times [-0.25 + 3 - 10 + 8.5]$

$y_2 = 5.25 + 0.5 \times 1.25$

$y_2 = 5.875$

Step III :

Now $x_2 = x_1 + h = 0.5 + 0.5 = 1$

Putting $n = 2$ in equation (i) we get

$y_3 = y_2 + h. f (x_2, y_2)$

$y_3 = 5.875 + 0.5 \times [-2(1)^3 + 12(1)^2 – 20(1) + 8.5]$

$y_3 = 5.875 + 0.5 \times [-2 + 12 - 20 + 8.5]$

$y_3 = 5.875 + 0.5 \times (-1.5)$

$y_3 = 5.125$

Step IV :

Now $x_3 = x_2 + h = 1 + 0.5 = 1.5$

Putting $n = 3$ in equation (I) we get

$y_4 = y_3 + h. f (x_3, y_3)$

$y_4 = 5.125 + 0.5 \times [-2(1.5)^3 + 12(1.5)^2 – 20(1.5) + 8.5]$

$y_4 = 5.125 + 0.5 \times [-6.75 + 27 - 30 + 8.5]$

$y_4 = 5.125 + 0.5 \times (-1.25)$

$y_4 = 4.5$

Step V :

Now $x_4 = x_3 + h = 1.5 + 0.5 = 2$

Putting $n = 4$ in equation (I) we get

$y_5 = y_4 + h. f (x_4, y_4)$

$y_5 = 4.5 + 0.5 \times [-2(2)^3 + 12(2)^2 – 20(2) + 8.5]$

$y_5 = 4.5 + 0.5 \times [-16 + 48 - 40 + 8.5]$

$y_5 = 4.5 + 0.5 \times (0.5)$

$y_5 = 4.75$

Step VI :

Now $x_5 = x_4 + h = 2 + 0.5 = 2.5$

Putting $n = 5$ in equation (I) we get

$y_6 = y_5 + h. f (x_5, y_5)$

$y_6 = 4.75 + 0.5 \times [-2(2.5)^3 + 12(2.5)^2 – 20(2.5) + 8.5]$

$y_6 = 4.75 + 0.5 \times [-31.25 + 75 - 50 + 8.5]$

$y_6 = 4.75 + 0.5 \times (2.25)$

$y_6 = 5.875$

Step VII :

Now $x_6 = x_5 + h = 2.5 + 0.5 = 3$

Putting $n = 6$ in equation (I) we get

$y_7 = y_6 + h. f (x_6, y_6)$

$y_7 = 5.875 + 0.5 \times [-2(3)^3 + 12(3)^2 – 20(3) + 8.5]$

$y_7 = 5.875 + 0.5 \times [-54 + 108 - 60 + 8.5]$

$y_7 = 5.875 + 0.5 \times (2.5)$

$y_7 = 7.125$

Step VIII :

Now $x_7 = x_6 + h = 3 + 0.5 = 3.5$

Putting $n = 7$ in equation (I) we get

$y_8 = y_7 + h. f (x_7, y_7)$

$y_8 = 7.125 + 0.5 \times [-2(3.5)^3 + 12(3.5)^2 – 20(3.5) + 8.5]$

y_8 = 7.125 + 0.5 x [-85.75 + 147 - 70 + 8.5]

y_8 = 7.125 + 0.5 x (-0.25)

$y_8 = 7$

Also $x_8 = x_7 + h$ = 3.5 + 0.5 = 4

Step IX : Tabulating the result :

From above steps, we can tabulate the result as follows :

x_n	y_n	y_{n+1}	x_{n+1}
$x_0 = 0$	$Y_0 = 1$	$Y_1 = 5.25$	$x_1 = 0.5$
$x_1 = 0.5$	$Y_1 = 5.25$	$Y_2 = 5.875$	$x_2 = 1.0$
$x_2 = 1.0$	$Y_2 = 5.875$	$Y_3 = 5.125$	$x_3 = 1.5$
$x_3 = 1.5$	$Y_3 = 5.125$	$Y_4 = 4.5$	$x_4 = 2.0$
$x_4 = 2.0$	$Y_4 = 4.5$	$Y_5 = 4.75$	$x_5 = 2.5$
$x_5 = 2.5$	$Y_5 = 4.75$	$Y_6 = 5.875$	$x_6 = 3.0$
$x_6 = 3.0$	$Y_6 = 5.875$	$y_7 = 7.125$	$X_7 = 3.5$
$X_7 = 3.5$	$y_7 = 7.125$	$y_8 = 7$	$x_8 = 4.0$

Ex. 5 Apply simple Euler's method to solve :

$$\frac{dy}{dx} = -xy^2, \quad y(0) = 2$$

computing upto $x = 1$ with $h = 0.1$

Solution :

Here, it is given that $x_0 = 0$ $y_0 = 2$ and $h = 0.1$

And $f(x, y) = -xy^2$

According to Euler's method, we have

$$Y_n+1 = y_n + h f(x_n, y_n) \qquad \ldots (I)$$

Step I :

Putting n = 0 in equation (I) we get

$$y_1 = y_0 + h.\, f(x_0, y_0)$$

$$y_1 = 2 + 0.1 \text{ x } [-0 \text{ x } (2)^2]$$

$$y_1 = 2 + 0$$

$$y_1 = 2$$

Step II :

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

Putting n = 1 in equation (I) we get

$$y_2 = y_1 + h.\, f(x_1, y_1)$$

$$y_2 = 2 + 0.1 \text{ x } [-(0.1) \text{ x } (2)^2]$$

$$y_2 = 2 + 0.1 \text{ x } (-0.4)$$

$$y_2 = 1.96$$

Step III :

Now, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Putting n = 2 in equation (I) we get

$$y_3 = y_2 + h.\, f(x_2, y_2)$$

$$y_3 = 1.96 + 0.1 \text{ x } [-(0.2) \text{ x } (1.96)^2]$$

$$y_3 = 1.96 + 0.1 \text{ x } (-0.76832)$$

$$y_3 = 1.883168$$

Step IV :

Now, $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

Putting n = 3 in equation (I) we get

$$y_4 = y_3 + h.\ f(x_3,\ y_3)$$

$$y_4 = 1.883168 + 0.1 \times [-(0.3) \times (1.883168)^2]$$

$$y_4 = 1.883168 + 0.1 \times (-1.063896)$$

$$y_4 = 1.7767783$$

Step V :

Now, $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

Putting n = 4 in equation (I) we get

$$y_5 = y_4 + h.\ f(x_4,\ y_4)$$

$$y_5 = 1.7767783 + 0.1 \times [-(0.4) \times (1.7767783)^2]$$

$$y_5 = 1.7767783 + 0.1 \times (-1.2627765)$$

$$y_5 = 1.6505006$$

Step VI :

Now, $x_5 = x_4 + h = 0.4 + 0.1 = 0.5$

Putting n = 5 in equation (I) we get

$$y_6 = y_5 + h.\ f(x_5,\ y_5)$$

$$y_6 = 1.6505006 + 0.1 \times [-(0.5) \times (1.6505006)^2]$$

$$y_6 = 1.6505006 + 0.1 \times (-1.3620762)$$

$$y_6 = 1.514293$$

Step VII :

Now, $x_6 = x_5 + h = 0.5 + 0.1 = 0.6$

Putting n = 6 in equation (I) we get

$$y_7 = y_6 + h.\ f(x_6,\ y_6)$$

$$y_7 = 1.514293 + 0.1 \times [-(0.6) \times (1.514293)^2]$$

$y_7 = 1.514293 + 0.1 \times (-1.37585)$

$y_7 = 1.376708$

Step VIII :

Now, $x_7 = x_6 + h = 0.6 + 0.1 = 0.7$

Putting n = 7 in equation (I) we get

$y_8 = y_7 + h. f(x_7, y_7)$

$y_8 = 1.376708 + 0.1 \times [-(0.7) \times (1.376708)^2]$

$y_8 = 1.376708 + 0.1 \times (-1.3267274)$

$y_8 = 1.2440353$

Step IX :

Now, $x_8 = x_7 + h = 0.7 + 0.1 = 0.8$

Putting n = 8 in equation (I) we get

$y_9 = y_8 + h. f(x_8, y_8)$

$y_9 = 1.2440353 + 0.1 \times [-(0.8) \times (1.2440353)^2]$

$y_9 = 1.2440353 + 0.1 \times (-1.238099)$

$y_9 = 1.1202254$

Step X :

Now, $x_9 = x_8 + h = 0.8 + 0.1 = 0.9$

Putting n = 9 in equation (I) we get

$y_{10} = y_8 + h. f(x_9, y_9)$

$y_{10} = 1.1202254 + 0.1 \times [-(0.9) \times (1.1202254)^2]$

y_{10} = 1.1202254 + 0.1 x (-1.1294145)

y_{10} = 1.007284

Also $x_{10} = x_9 + h = 0.9 + 0.1 = 1$

Step XI :

Tabulating the result :

x_n	y_n	y_{n+1}	x_{n+1}
$x_0 = 0$	$y_0 = 2$	$y_1 = 2$	$x_1 = 0.1$
$x_1 = 0.1$	$y_1 = 2$	$Y_2 = 1.96$	$x_2 = 0.2$
$x_2 = 0.2$	$y_2 = 1.96$	$y_3 = 1.883168$	$x_3 = 0.3$
$x_3 = 0.3$	$y_3 = 1.883168$	$y_4 = 1.7767783$	$x_4 = 0.4$
$x_4 = 0.4$	$y_4 = 1.7767783$	$y_5 = 1.6505006$	$x_5 = 0.5$
$x_5 = 0.5$	$y_5 = 1.6505006$	$y_6 = 1.514293$	$x_6 = 0.6$
$x_6 = 0.6$	$y_6 = 1.514293$	$y_7 = 1.376708$	$x_7 = 0.7$
$x_7 = 0.7$	$y_7 = 1.376708$	$y_8 = 1.2440353$	$x_8 = 0.8$
$x_8 = 0.8$	$y_8 = 1.2440353$	$y_9 = 1.1202254$	$x_9 = 0.9$
$x_9 = 0.9$	$y_9 = 1.1202254$	$y_{10} = 1.007281$	$x_{10} = 1.0$

Ex. 6 Given : $\frac{dy}{dx} = 10 + y^2$ and $y(0) = 0$

Solve numerically for the interval $0 < x < 1$ with $h = 0.1$ and $h = 0.2$

Solution :

Case (I) : When *h* = 0.1

Here, it is given that $x_0 = 0$, $y_0 = 0$ and $h = 0.1$

and $f(x, y) = 10 + y^2$

According to Euler's method, we have

$$y_{n-1} = y_n + h\, f\,(x_n,\ y_n) \qquad \text{….(I)}$$

Step I :

Putting n = 0 in equation (I), we get

$$y_1 = y_0 + h.\, f\,(x_0,\ y_0)$$

$$y_1 = 0 + 0.1 \text{ x } [\,10 + (0)^2\,]$$

$$y_1 = 0 + 0.1 \text{ x } (10)$$

$$y_1 = 1$$

Step II :

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

Putting n = 1 in equation (I), we get

$$y_2 = y_1 + h.\, f\,(x_1,\ y_1)$$

$$y_2 = 1 + 0.1 \text{ x } [\,10 + (1)^2\,]$$

$$y_2 = 1 + 0.1 \text{ x } (11)$$

$$y_2 = 2.1$$

Step III :

Now, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Putting n = 2 in equation (I), we get

$$y_3 = y_2 + h.\, f\,(x_2,\ y_2)$$

$$y_3 = 2.1 + 0.1 \text{ x } [\,10 + (2.1)^2\,]$$

$$y_3 = 2.1 + 0.1 \text{ x } (14.4.1)$$

$$y_3 = 3.541$$

Step IV :

Now, $x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

Putting n = 3 in equation (I), we get

$$y_4 = y_3 + h.\ f(x_3,\ y_3)$$

$$y_4 = 3.541 + 0.1 \times [10 + (3.541)^2]$$

$$y_4 = 3.541 + 0.1 \times (22.53868)$$

$$y_4 = 5.7948681$$

Step V :

Now, $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

Putting n = 4 in equation (I), we get

$$y_5 = y_4 + h.\ f(x_4,\ y_4)$$

$$y_5 = 5.7948681 + 0.1 \times [10 + (5.7948681)^2]$$

$$y_5 = 5.7948681 + 0.1 \times (43.580496)$$

$$y_5 = 10.152918$$

Step VI :

Now, $x_5 = x_4 + h = 0.4 + 0.1 = 0.5$

Putting n = 5 in equation (I), we get

$$y_6 = y_5 + h.\ f(x_5,\ y_5)$$

$$y_6 = 10.152918 + 0.1 \times [10 + (10.152918)^2]$$

$$y_6 = 10.152918 + 0.1 \times (113.08174)$$

$$y_6 = 21.461092$$

Step VII :

Now, $x_6 = x_5 + h = 0.5 + 0.1 = 0.6$

Putting n = 6 in equation (I), we get

$$y_7 = y_6 + h.\ f(x_6,\ y_6)$$

$$y_7 = 21.461092 + 0.1 \times [10 + (21.461092)^2]$$

$y_7 = 21.461092 + 0.1 \times (470.57846)$

$y_7 = 68.518939$

Step VIII :

Now, $x_7 = x_6 + h = 0.6 + 0.1 = 0.7$

Putting $n = 7$ in equation (I), we get

$y_8 = y_7 + h.\ f\,(x_7,\ y_6)$

$y_8 = 68.518939 + 0.1 \times [\,10 + (68.518939)^2\,]$

$y_8 = 68.518939 + 0.1 \times (4704.845)$

$y_8 = 539.00344$

Step IX :

Now, $x_8 = x_7 + h = 0.7 + 0.1 = 0.8$

Putting $n = 8$ in equation (I), we get

$y_9 = y_8 + h.\ f\,(x_8,\ y_8)$

$y_9 = 539.00344 + 0.1 \times [\,10 + (539.00344)^2\,]$

$y_9 = 539.00344 + 0.1 \times (290534.71)$

$y_9 = 29592.474$

Step X :

Now, $x_9 = x_8 + h = 0.8 + 0.1 = 0.9$

Putting $n = 9$ in equation (I), we get

$y_{10} = y_9 + h.\ f\,(x_9,\ y_9)$

$y_{10} = 29592.474 + 0.1 \times [\,10 + (29592.474)^2\,]$

$y_{10} = 29592.474 + 0.1 \times (18.7571453 \times 10^8)$

$y_{10} = 87601046$

Step XI : Tabulating the result :

x_n	y_n	y_{n+1}	x_{n+1}
$x_0 = 0$	$y_0 = 0$	$y_1 = 1$	$x_1 = 0.1$
$x_1 = 0.1$	$y_1 = 1$	$y_2 = 2.1$	$x_2 = 0.2$
$x_2 = 0.2$	$y_2 = 2.1$	$y_3 = 3.541$	$x_3 = 0.3$
$x_3 = 0.3$	$Y_3 = 3.541$	$y_4 = 5.7948681$	$x_4 = 0.4$
$x_4 = 0.4$	$y_4 = 5.7948681$	$y_5 = 10.152918$	$x_5 = 0.5$
$x_5 = 0.5$	$y_5 = 10.152918$	$y_6 = 21.461092$	$x_6 = 0.6$
$x_6 = 0.6$	$y_6 = 21.461092$	$y_7 = 68.518939$	$x_7 = 0.7$
$x_7 = 0.7$	$y_7 = 68.518939$	$y_8 = 539.00344$	$x_8 = 0.8$
$x_8 = 0.8$	$y_8 = 539.00344$	$y_9 = 29592.474$	$x_9 = 0.9$
$x_9 = 0.9$	$y_9 = 29592.474$	$y_{10} = 87601046$	$x_{10} = 1.0$

Case (II) : When $h = 0.2$

Here, it is given that $x_0 = 0,\ y_0 = 0$ and $h = 0.2$

and $f(x, y) = 10 + y^2$

According to Euler's method, we have

$$y_{n-1} = y_n + h\, f(x_n, y_n) \qquad \ldots(I)$$

Step I :

Putting $n = 0$ in equation (I), we get

$$y_1 = y_0 + h.\, f(x_0, y_0)$$

$$y_1 = 0 + 0.2 \text{ x } [\, 10 + (0)^2 \,]$$

$$y_1 = 0 + 0.2 \text{ x } (10)$$

$$y_1 = 2$$

Step II :

Now, $x_1 = x_0 + h = 0 + 0.2 = 0.2$

Putting $n = 1$ in equation (I), we get

$$y_2 = y_1 + h.\, f(x_1, y_1)$$

$$y_2 = 2 + 0.2 \text{ x } [\, 10 + (2)^2 \,]$$

$y_2 = 2 + 0.2 \text{ x } (14)$

$y_2 = 4.8$

Step III :

Now, $x_2 = x_1 + h = 0.2 + 0.2 = 0.4$

Putting $n = 2$ in equation (I), we get

$y_3 = y_2 + h. f(x_2, y_2)$

$y_3 = 4.8 + 0.2 \text{ x } [10 + (4.8)^2]$

$y_3 = 4.8 + 0.2 \text{ x } (33.04)$

$y_3 = 11.40$

Step IV :

Now, $x_3 = x_2 + h = 0.4 + 0.2 = 0.6$

Putting $n = 3$ in equation (I), we get

$y_4 = y_3 + h. f(x_3, y_3)$

$y_4 = 11.40 + 0.2 \text{ x } [10 + (11.40)^2]$

$y_4 = 11.40 + 0.2 \text{ x } (139.96)$

$y_4 = 39.392$

Step IV :

Now, $x_4 = x_3 + h = 0.6 + 0.2 = 0.8$

Putting $n = 4$ in equation (I), we get

$y_5 = y_4 + h. f(x_4, y_4)$

$y_5 = 39.392 + 0.2 \text{ x } [10 + (39.392)^2]$

$y_5 = 39.392 + 0.2 \times (1561.7297)$

$y_5 = 351.73793$

Step VI : Tabulating the result :

x_n	y_n	y_{n+1}	x_{n+1}
$x_0 = 0$	$y_0 = 0$	$y_1 = 2$	$x_1 = 0.2$
$x_1 = 0.2$	$y_1 = 2$	$Y_2 = 4.8$	$x_2 = 0.4$
$x_2 = 0.4$	$y_2 = 4.8$	$Y_3 = 11.40$	$x_3 = 0.6$
$x_3 = 0.6$	$Y_3 = 11.40$	$y_4 = 39.392$	$x_4 = 0.8$
$x_4 = 0.8$	$y_4 = 39.392$	$y_5 = 351.73793$	$x_5 = 1.0$

MODIFIED EULER'S METHOD

Drawback of Euler's method :

The Euler's method described in the last unit is very slow. We also need to take a smaller value for *h* to obtain reasonable accuracy with Euler's method. Because of this restriction on *h.* Euler's method is unsuitable for practical use and a modification of it. Known as the ***modified Euler method,*** which gives more accurate results is used for practical purposes.

Modified Euler's Method :

Suppose that we wish to solve the equation $y^{\prime} = f(x, y)$ for values of y at $x = x_1 = x_o + rh$ $(r = 1,2 \ldots .)$

Integrating $y^{\prime} = f(x, y)$, we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y).\, dx \qquad \ldots(\mathrm{I})$$

Approximating the above integral by means of the trapezoidal rule we get

$$y_1 = y_0 + \int_{x_0}^{x} [\, f(x_0, y_0) + f(x_1, y_1)] \qquad \ldots(\mathrm{II})$$

We thus obtain the iteration formula :

h

$$y_1^{(n+1)} = y_0 + \frac{}{2} [f(x_0, y_0) + f(x_1, y_1^{(n+1)})] \; n = 0,1,2, \ldots \text{ (III)}$$

Where $y_1^{(n)}$ is the *Nth* approximation to y_l. The iteration formula (III) can be started by choosing $y_1^{(10)}$ from Euler's formula : $y_l^{(0)} = y_0 + h. f(x_0, y_0)$

Hence the general formula of Euler's modified method is give by :

$$y_{i-1}^{(n+1)} = y_i + \frac{h}{2} [f(x_i, y_i) - f(x_i, y_{i-1}^{(n+1)})] \; n\text{-}i = 0,1,2, \ldots$$

Procedure the Solve Examples using Modified Euler's method :

Step I : From the data given in the problem find x_0, y_0 and h

Step II : Find y_1 from the Euler's formula. $y_{n+1} = h \,\&\, (x_n, y_n)$
Let this $y_1 = y_1^{(0)}$

Step III : Find x_1 from the formula $x_{i-1} = x_1 + h$

Step IV : Apply Euler's modified method to get more correct approximations.

According to Euler's modified method, we have

$$y_{i+1}^{(n+1)} = y_1 + \frac{h}{2} [f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{(n)})] \; n\text{-} i = 0,1,2, \ldots$$

Find $y_1^{(1)}$, $y_1^{(2)}$, $y_1^{(3)}$ … using above equation. The value of y_1 which is same in two iterations (i.e. in $y_1^{(2)}$ and , $y_1^{(3)}$) while be selected as the final values of

Step V : Repeat Step II to Step IV, to find $(x_2, y_2), (x_3, y_3)$ … and so on till x reaches the final value given in the example.

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 7 Find $y(0.2)$ using Modified Euler's Method.

Given : $\dfrac{dy}{dx} = x + y, \; y(0) = 1$, step size = 0.1

Solution

Here, it is given that $x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Step I :

According to Euler's method we have

$$y_{n-1} = y_n + hf(x_n, y_n)$$

Putting $n = 0$ in above equation we get

$$y_1 = y_0 + h f(x_0, y_0)$$

$$y_1 = 1 + 0.1 \text{ x } [0 + 1]$$

$$y_1 = 1 + 0.1 \text{ x } 1$$

$$y_1 = 1.1$$

Let $y_1^{(0)} = y_1 = 1.1$

Also $x_1 = x_0 + h = 0 + 0.1 = 0.1$

Now we apply Euler's modified method to get more correct approximations.

According to Euler's modified method we have

$$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

$$y_1^{(1)} = 1 + \frac{0.1}{2}[(0 + 1) + (0.1 + 1.1)]$$

$$y_1^{(1)} = 1.11$$

Now

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$y_1^{(2)} = 1 - \frac{0.1}{2}[(0 + 1) + (0.1 + 1.11)]$$

$$y_1^{(2)} = 1.1105$$

Now

$$y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$y_1^{(3)} = 1 + \frac{0.1}{2} [(0 + 1) + (0.1 + 1.1105)]$$

$$y_1^{(3)} = 1.1105$$

$y_1^{(2)}$ and $y_1^{(3)}$ are same upto four decimal places. Hence we will move to next step by taking

$$y_1 = 0.1 \qquad \text{and} \qquad y_1 = 1.1105$$

Step II :

According to Euler's method we have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

Putting $n = 1$ in above equation we get

$$y_2 = y_1 + h f(x_1, y_1)$$

$$y_2 = 1.1105 + 0.1 \text{ x } [0 + 1.1105]$$

$$y_2 = 1.1105 + 0.1 \text{ x } 1.2105$$

$$y_2 = 1.23155$$

Let $y_2^{(0)} = y_2 = 1.23155$

Also $x_2 = x_1 + h = 0 + 0.1 = 0.2$

Now we apply Euler's modified method to get more correct approximations.

According to Euler's modified method we have

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

$$y_2^{(1)} = 1.1105 + \frac{0.1}{2}[(0.1 + 1.1105) + (0.2 + 1.23155)]$$

$$y_2^{(1)} = 1.2426$$

Now

$$y_2^{(2)} = y_1 + \frac{h}{2}[\,f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

$$y_2^{(2)} = 1.1105 + \frac{0.1}{2}[(0.1 + 1.1105) + (0.2 + 1.2426)]$$

$$y_2^{(2)} = 1.24315$$

Now

$$y_2^{(3)} = y_1 + \frac{h}{2}[\,f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

$$y_2^{(3)} = 1.1105 + \frac{0.1}{2}[(0.1 + 1.1105) + (0.2 + 1.24315)]$$

$$y_2^{(3)} = 1.24318$$

$y_2^{(2)}$ and $y_2^{(3)}$ are same upto four decimal places. Hence, the result of given equation is

$$y_2 = 1.2431 \qquad \text{and} \qquad x_2 = 0.2$$

$$y\,(0.2) = 1.2431$$

Ex. 8

** Given : $\dfrac{dy}{dx} - \sqrt{x\,y} = 2 \qquad y\,(1) = 1,$

Find the value of $y(2)$ in steps of 0.1 using Euler's Modified Method.

Solution

Here, it is given that $x_0 = 1,\ y_0 = 1$ and $h = 0.1$

$$\sqrt{\quad}$$

and $f(x, y) = 2 + \quad xy$

Step I :

According to Euler's method we have

$$y_1 = y_0 + hf(x_0, y_0)$$

** $y_1 = 1 + 0.1 \ \text{x} \left[2 + \sqrt{(1)(1)} \right]$

$$y_1 = 1 + 0.3$$

$$y_1 = 1.3$$

Let $y_1^{(0)} = y_1 = 1.3$

Also $x_1 = x_0 + h = 1 + 0.1 = 0.1$

Now we apply Euler's modified method to get more correct approximations.

According to Euler's modified method we have

$$y_1^{(1)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(0)})]$$

** $y_1^{(1)} = 1 + \frac{0.1}{2} [(2 + \sqrt{(1)(1)}) + (2 + \sqrt{(1.1)(1.3)}]$

$$y_1^{(1)} = 1.3097913$$

Now

$$y_1^{(2)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

** $y_1^{(2)} = 1 - \frac{0.1}{2} [(2 + \quad (1)(1) + (2 + \quad (1.1)(1.3))]$

$$y_1^{(2)} = 1.310016$$

Now

$$y_1^{(3)} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

**

$$y_1^{(3)} = 1 + \frac{0.1}{2} [(2 + (1)(1)) + (2 + \sqrt{(1.1)(1.310016))}]$$

$$y_1^{(3)} = 1.3100212$$

$y_1^{(2)}$ and $y_1^{(3)}$ are same upto four decimal places. Hence we will move to next step by taking

$x_1 = 1.1$ and $y_1 = 1.3100242$

Step II :

According to Euler's method we have

$$Y_2 = y_1 + hf(x_1, y_1)$$

** $\quad y_2 = 1.3100212 + 0.1 \text{ x } [2 + (1.1)(1.3100212)]$

$$y_2 = 1.6300638$$

Let $\quad y_2^{(0)} = y_2 = 1.6300638$

Also $\quad x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

Now we apply Euler's modified method to get more correct approximations.

According to Euler's modified method we have

$$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})]$$

**

$$y_2^{(1)} = 1.3100212 + \frac{0.1}{2} [(2 + (1.1)(1.3100212)) + (2 + (1.2)(1.6300638)]$$

$$y_2^{(1)} = 1.6399724$$

Now

$$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$$

**

$$y_2^{(2)} = 1.3100212 + \frac{0.1}{2}[(2 + \sqrt{(1.1)(1.3100212)}) + (2 + \sqrt{(1.2)(1.6399724)})]$$

$$y_2^{(2)} = 1.6401846$$

Now

$$y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$$

**

$$y_2^{(3)} = 1.3100212 + \frac{0.1}{2}[(2 + (1.1)(1.3100212)) + (2 + (1.2)(1.6401846))]$$

$$y_2^{(3)} = 1.6401892$$

$y_1^{(2)}$ and $y_1^{(3)}$ are same upto four decimal places. Hence we will move to next step by taking

$x_2 = 1.2$ and $y_2 = 1.6401892$

Step III:

Similarly we have

When	$x_3 = 1.3$	then	$y_3 = 1.9907721$
When	$x_4 = 1.4$	then	$y_4 = 2.3621340$
When	$x_5 = 1.5$	then	$y_5 = 2.7546967$
When	$x_6 = 1.6$	then	$y_6 = 3.1689201$
When	$x_7 = 1.7$	then	$y_7 = 3.6052903$
When	$x_8 = 1.8$	then	$y_8 = 4.06431260$
When	$x_9 = 1.9$	then	$y_9 = 4.5465063$
When	$x_{10} = 2.0$	then	$y_{10} = 5.0524019$

$y(2) = 5.0524019$

RUNGE – KUTTA METHODS

Euler's methods is less efficient in practical problems since it requires step size *h.* to be small for obtaining reasonable accuracy. The Rung-Kutta mehods are designed to give greater accuracy and they possess the advantage of requires only the function values at some selected points on the subinterval.

Runge – Kutta Second Order Method :

Consider the general differential equation

$$y' = f(x, y)$$

Integrating above equation, we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y).\, dx \qquad \dots \text{(I)}$$

Approximating the above integral by means of the trapezoidal rule we get

$$y_1 = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1)] \qquad \dots \text{(II)}$$

Putting $x_1 = x_0 + h$ and $y_1 = y_0 + h f(x_0, y_0)$ in the above equation we have

$$y_1 = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_0, y_0) + h f(x_0, y_0)]$$

Putting $f(x_0, y_0) = f_0$ in the above equation we have

$$y_1 = y_0 + \frac{h}{2} [f_0 + f(x_0 + h, y_0 + h f_0)] \qquad \dots \text{(III)}$$

If we now set

$$k_1 = h f_0$$

and

$$k_1 = h f(x_0 + h, y_0 + k_l)$$

then equation (III) becomes

$$y_1 = y_0 + \frac{1}{2} [k_1 + k_2]$$

This formula is called second order Runge – Kutta formula.

Questions asked from the above Topic in University Papers:

Ex. 10 Given : $\dfrac{dy}{dx} = y - x$ where $y(0) = 2$

Find $y(0.1)$ and $y(0.2)$ correct upto four decimal places using Runge-Kutta second order method.

Solution :

Runge – Kutta Second order formula is given by :

$$y_1 = y_0 + \frac{1}{2}[k_1 + k_2]$$

where $k_1 = h f_0$

and $k_1 = h f(x_0 + h, y_0 + k_l)$

Step I :

Here it is given that $x_0 = 0$ and $y_0 = 2$

and $f(x, y) = y – x$

Let $h = 0.1$

Step II : Finding *y*(0.1) :

Now, $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$k_1 = h f_0$

$= h \times (y_0 - x_0)$

$= 0.1 \times (2 – 0)$

$= 0.2$

and, $k_2 = hf(x_0 + h, y_0 + k_1)$

$= h \times [(2 + 0.2) – (0 + 0.1)]$

$= 0.1 \times (2.2 – 0.1)$

$= 0.21$

According to Runge – Kutta Second order formula, we have

$$y_1 = y_0 + \frac{1}{2} [k_1 + k_2]$$

$$y(0.1) = 2 + \frac{1}{2} (0.2 + 0.21) \qquad [\because y_1 = y(0.1)]$$

$$\boldsymbol{y(0.1) = 2.2050}$$

Step II : Finding *y*(0.2) :

Now, $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$k_1 = h f_1$

$= h \text{ x } (y_1 - x_1)$

$= 0.1 \text{ x } (2.2050 - 0.1)$

$k_1 = 0.2105$

and, $k_2 = hf (x_1 + h , y_0 + k_1)$

$= h \text{ x } [(2.2050 + 0.2105) - (0.1 + 0.1)]$

$= 0.1 \text{ x } (2.4155 - 0.2)$

$k_2 = 0.22155$

According to Runge – Kutta Second order formula, we have

$$y_2 = y_1 + \frac{1}{2} [k_1 + k_2]$$

$$y(0.2) = 2.2050 + \frac{1}{2} (0.2105 + 0.22155) \qquad [\because y_2 = y(0.2)]$$

$$\boldsymbol{y(0.2) = 2.4210}$$

RUNGE – KUTTA FOURTH ORDER METHOD :

In order to increase the accuracy of the Rung – Kutta second order method a more accurate method called ***Runge – Kutta fourth order method*** is used in which the inform for movement is given to x_0 is further divided into two increments each of $\frac{h}{2}$

Consider the general differential equation

$$y_I = f(x, y)$$

Integrating above equation we get

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x, y).\, dx \qquad \ldots \text{(I)}$$

Approximating the above integral by means of Simpson's rule with step size of we get

$$y_1 = y_0 + \frac{h}{2}\,[f(x_0 + y_0) + 4(x_{1/2} + y_{1/2}) + f(x_1 + y_1)] \quad \ldots\text{(II)}$$

Putting $f(x_0 + y_0) = f_0$ in the above equation and then simplifying it gives

$$y_1 = y_0 + \frac{1}{6}\,[k_1 + 2\ k_2 + 2\,k_3 + k_4]$$

Where

$$k_1 = hf_0 \qquad \text{[where } f_0 = f(x_0, y_0)\text{]}$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2}\right)$$

and

$$k_4 = hf\ [x_0 + h,\ y_0 + k_3]$$

This formula is called ***fourth order Runge – Kutta formula.***

Ex. 11 Use the classical fourth order Runge – Kutta method to integrate

$$f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$$

with a step size of 0.05 and an initial condition of $y = 1$ at $x = 0.$

Solution :

Runge – Kutta Fourth order formula is given by :

$$y_1 = y_0 + \frac{1}{6}[k_1 + 2 k_2 + 2k_3 + k_4]$$

Where

$$k_1 = hf_0 \qquad [\text{where } f_0 = f(x_0, y_0)]$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_2}{2}\right)$$

and

$$k_4 = hf\ [x_0 + h,\ y_0 + k_3]$$

Step I :

Here it is given that $x_0 = 0$ and $y_0 = 1$, $h = 0.05$

and $f(x, y) = -2x^3 + 12x^2 - 20x + 8.5$

Step II : Finding *y* (0.05) :

Now, $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$k_1 = h f_0$

$= h \text{ x } [-2x_0^3 + 12x_0^2 - 20x_0 + 8.5]$

$= 0.05 \times [-2(0)^3 + 12(0)^2 - 20(0) + 8.5]$

$= 0.425$

also

$$k_2 = hf\left(x_0 + \frac{h}{2},\; y_0 + \frac{k_1}{2}\right)$$

$$= hf\left(0 + \frac{0.05}{2},\; 1 + \frac{0.425}{2}\right) = hf\,(0.025, 1.2125)$$

$= 0.05 \times [-2(0.025)^3 + 12(0.025)^2 - 20(0.025) + 8.5]$

$= 0.05 \times [-0.00003125 + 0.0075 - 0.05 + 8.5]$

$= 0.4$

also

$$k_3 = hf\left(x_0 + \frac{h}{2},\; y_0 - \frac{k_2}{2}\right)$$

$$= hf\left(0 + \frac{0.05}{2},\; 1 + \frac{0.4}{2}\right) = hf\,(0.025, 1.2)$$

$= 0.05 \times [-2(0.025)^3 + 12(0.025)^2 - 20(0.025) + 8.5]$

$= 0.05 \times [-0.00003125 + 0.0075 - 0.05 + 8.5]$

$= 0.4$

and

$k_4 = hf\,[x_0 + h,\; y_0 + k_3]$

$= hf\,[(0 + 0.05, 1 + 0.4) = hf\,(0.05, 1.4)]$

$= 0.05 \times [-2(0.05)^3 + 12(0.05)^2 - 20(0.05) + 8.5]$

$= 0.05 \times [-0.00025 + 0.03 - 1 + 8.5]$

$= 0.3764875$

According to Runge – Kutta Fourth order formula, we have

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$y(0.5) = 1 + \frac{1}{6} (0.425 + 2(0.4) + 0.3764875) \quad [\therefore y_1 = y(0.5)]$$

$$y(0.5) = 1.4002479$$

$$y = 1.4002479 \text{ at } x = 0.05$$

Ex. 12 Use the Runge – Kutta fourth order method to find the value of y when $x = 1$ given that $y = 1$ when $x = 0$ and that :

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

Solution :

Runge – Kutta Fourth order formula is given by :

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

Where

$$k_1 = hf_0 \qquad [\text{where } f_0 = f(x_0, y_0)]$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, \ y_0 - \frac{k_1}{2} \right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, \ y_0 + \frac{k_2}{2} \right)$$

and

$$k_4 = hf\ [x_0 + h,\ y_0 + k_3]$$

Step I :

Here it is given that $x_0 = 0$ and $y_0 = 1$,

and $$f(x, y) = \frac{y - x}{y + x}$$

Let $h = 0.5$

Step II : Finding y (0.5) :

Now, $x_1 = x_0 + h = 0 + 0.5 = 0.5$

$$k_1 = h f_0$$

$$= h \times \left(\frac{y_0 - x_0}{y_0 + x_0} \right)$$

$$= 0.5 \times \left(\frac{1 - 0}{1 + 0} \right)$$

$$= 0.5$$

also

$$k_2 = hf \left(x_0 + \frac{h}{2},\ y_0 - \frac{k_1}{2} \right)$$

$$= hf \quad 0 + \left(\frac{0.5}{2},\ 1 + \frac{0.5}{2} \right) = hf\ (0.25, 1.25)$$

$$= 0.5 \times \left(\frac{1.25 - 0.25}{1.25 + 0.25} \right)$$

$$= 0.33333$$

also

$$k_3 = hf \left(x_0 + \frac{h}{2},\ y_0 - \frac{k_2}{\ } \right)$$

2 2

$$= hf\left(0 + \frac{0.5}{3}, 1 + \frac{0.33333}{2}\right) = hf\,(0.25, 1.16666)$$

$$= 0.5 \times \left(\frac{1.16666 - 0.25}{1.16666 + 0.25}\right)$$

$$= 0.3235285$$

and

$$k_4 = hf\,[\,x_0 + h\,,\ y_0 + k_3\,]$$

$$= hf\,(\,0 + 0.5 + 0.3235285) = hf\,(\,0.5\,,\,1.3235285)$$

$$= 0.5 \times \left(\frac{0.8235285}{1.8235286}\right)$$

$$= 0.2258063$$

According to Runge – Kutta Fourth order formula. we have

$$y_1 = y_0 + \frac{1}{6}\,[\,k_1 + 2k_2 + 2k_3 + k_4\,]$$

$$y(0.5) = 1 + \frac{1}{6}\,[0.\,5 - 2(0.33333) + 2(0.3235285) + 0.2258063\,] \quad [\therefore y_1 = y(0.5)]$$

$$y(0.5) = 1.3399206$$

After Step II we have $x_1 = 0.5$, $y_1 = 1.3399206$

Step III : Finding *y* (1) :

Now, $x_2 = x_1 + h = 0.5 + 0.5 = 1$

$$k_1 = h\,f_1$$

$$k = h \times \left(\frac{y_1 - x_1}{y_1 + x_1} \right)$$

$$= 0.5 \times \left(\frac{1.3399206 - 0.5}{1.3399206 + 0.5} \right)$$

$$= 0.2282491$$

also

$$k_2 = hf \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2} \right)$$

$$= hf \left(0.5 + \frac{0.5}{2},\ 1.3399206 + \frac{0.2282491}{2} \right) = hf\,(0.75,\ 1.4540452)$$

$$= 0.5 \times \left(\frac{1.4540452 - 0.75}{1.4540452 + 0.75} \right)$$

$$= 0.1597165$$

also

$$k_3 = hf \left(x_0 + \frac{h}{2},\ y_0 - \frac{k_2}{2} \right)$$

$$= hf \left(0.5 + \frac{0.5}{2},\ 1.3399206 + \frac{0.1597165}{2} \right) = hf\,(0.75,\ 1.4197789)$$

$$= 0.5 \times \left(\frac{1.4197789 - 0.75}{1.4197789 + 0.75} \right)$$

$$= 0.1543426$$

According to Runge – Kutta Fourth order formula. we have

$$y_2 = y_1 + \frac{1}{6} \left[k_1 + 2k_2 + 2k_3 + k_4 \right]$$

$$y_2 = 1.3399206 + \frac{1}{6}[0.228249 + 2(0.228249) + 2(0.1597165) + 0.09908] \qquad [\because y_2 = y(1)]$$

$$\boldsymbol{y(1) = 1.4991618}$$

Ex. 13 Using Runge – Kutta's fourth order method, find the numerical solution $x = 0.8$ for

$$\frac{dy}{dx} = \sqrt{x + y}, \qquad y(0.4) = 0.41$$

Assume step length $h = 0.2$

Solution :

Runge – Kutta Fourth order formula given by:

$$y_I = y + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

Where

$$k_1 = hf_0 \qquad [\text{where } f_0 = f(x_0, y_0)]$$

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_2}{2}\right)$$

and

$$k_4 = hf\,[x_0 + h,\ y_0 + k_3]$$

Step I :

Here it is given that $x_0 = 0.4$ $y_0 = 0.41$, $h = 0.2$

*****

and $f(x, y) = \sqrt{x + y}$

Step II : Finding *y* (0.06) :

Now, $x_1 = x_0 + h = 0.4 + 0.2 = 0.6$

$k_1 = h\, f_0$

$= h \times [\ \sqrt{x_0 + y_0}\ ]$

$= 0.2 \times [\ \sqrt{0.4 + 0.41}\ ]$

$= 0.18$

also

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_0 - \frac{k_1}{2} \right)$$

$$= hf\left(0.4 + \frac{0.2}{2},\ 0.41 + \frac{0.18}{2} \right) = hf\,(0.5, 0.5)$$

$= 0.2 \times [\quad 0.5 + 0.5\ ]$

$= 0.2$

also

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2} \right)$$

$$= hf\left(0.4 + \frac{0.2}{2},\ 0.41 + \frac{0.18}{2} \right) = hf\,(0.5, 0.5)$$

**** $= 0.2 \times [\ \sqrt{0.5 + 0.5}\ ]$

$= 0.20099$

and

$k_4 = hf\ [\,x_0 + h,\ y_0 + k_3\,]$

$= hf\ (\,0.4 + 0.2,\ 0.41 + 0.20099) = hf\ (\,0.6\,,\ 0.61099)$

$= 0.2 \times [\quad 0.6 + 0.61099\ ]$

$= 0.22$

According to Runge – Kutta Fourth order formula. we have

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$y(0.6) = 1 + \frac{1}{6} [(0.18 + 2(0.2) + 2(0.20099) + (0.22)] \qquad [\therefore y_1 = y(0.6)]$$

$$y(0.6) = 0.610345$$

After Step II. We have $x_1 = 0.6$, $y_1 = 0.610345$

Step III : Finding *y* (0.08) :

Now, $x_2 = x_1 + h = 0.6 + 0.2 = 0.8$

$$k_1 = h f_0$$

$$= h \times [\sqrt{x_1 + y_1}\ ]$$

$$= 0.2 \times [\sqrt{0.6 + 0.610345}\ ]$$

$$= 0.22093$$

also

$$k_2 = hf\left(x_0 + \frac{h}{2},\ y_l + \frac{k_1}{2} \right)$$

$$= hf\left(0.6 + \frac{0.2}{2},\ 0.610345 + \frac{0.18}{2} \right) = hf\,(0.7, 0.72081)$$

$$= 0.2 \times [\sqrt{0.7 + 0.72081}\ ]$$

$$= 0.23830$$

also

$$k_3 = hf\left(x_0 + \frac{h}{2},\ y_l + \frac{k_2}{2} \right)$$

$$= hf\left(0.6 + \frac{0.2}{2},\ 0.610345 + \frac{0.23839}{2}\right) = hf\ (0.7,\ 0.72954)$$

$$= 0.2 \text{ x } [\sqrt{0.7 + 0.72954}\,]$$

$$= 0.23912$$

and

$$k_4 = hf\ [x_0 + h\ ,\ y_0 + k_3]$$

$$= hf\ (0.6 + 0.2,\ 0.610345 + 0.23912) = hf\ (0.8\ ,\ 0.849465)$$

$$= 0.2 \text{ x } [\sqrt{0.8 + 0.849465}\,]$$

$$= 0.256863$$

According to Runge – Kutta Fourth order formula. we have

$$y_2 = y_1 + \frac{1}{6}\ [k_1 + 2k_2 + 2k_3 + k_4]$$

$$y(0.8) = 0.610345 + \frac{1}{6}\ [(0.22093 + 2(0.23839) + 2(0.256863) + 0.256863]$$

$$y(0.8) = 0.8491471$$

$$y = 0.8491471 \text{ at } x = 0.8$$

❖❖❖

7

NUMERICAL INTERGRATION

INTRODUCTION

Numerical methods of a function can be carried out using any algebraic method. But sometimes. We come across integrals which cannot be easily of exactly

calculated, for example, $\int_{0}^{\chi} . \ \gamma\, d\chi$. Further. If the values of the integrand for a few

values of the variables are given then we cannot integrate. In such circumstances. numerical methods for integration are used.

There are a large number of numerical methods available for integration, but, in this chapter, we will consider only three numerical methods which are used for integration. The methods covered by this chapter are :

(i) Trapezoidal Rule.

(ii) Simpson's $\frac{1}{3}$ Rule,

(iii) Simpson's $\frac{3}{8}$ Rule,

TRAPEZOIDAL RULE

Derivation of Trapezoidal Rule :

Consider a definite integral $I \int_{0}^{\chi} . \ \gamma\, d\chi$.

Let the interval [a, b] be divided into *n* equal subintervals such that $a = \chi_0 < \chi_1 < \chi_2 < . . . < \chi_n = b.$

$$\chi_n = \chi_0 + nh$$

Hence the given definite integral becomes :

$$I \int_{0}^{\chi} y\, d\chi.$$

Approximating *y* by Newton's forward difference formula, we get

$$I = \int_{\chi_0}^{\chi_n} \left(\gamma_0 + p\Delta\gamma_0 + \frac{p\,(p-1)}{2}\,\Delta^2\gamma_0 + \frac{p\,(p-1)\,(p-2)}{6}\,\Delta^3\gamma_0 + \ldots \right) d\chi.$$

Since $\chi_n = \chi_0 + ph$ and $d\chi = h.dp$ the above integral become

$$I = h \int_{0}^{n} \left(\gamma_0 + p\Delta\gamma_0 + \frac{p\,(p-1)}{2}\,\Delta^2\gamma_0 + \frac{p\,(p-1)\,(p-2)}{6}\,\Delta^3\gamma_0 + \ldots \right) dp.$$

Which on simplification gives

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = nh \left(\gamma_0 + \frac{n}{2}\,\Delta\gamma_0 + \frac{n\,(2n-3)}{12}\,\Delta^2\gamma_0 + \frac{n\,(n-2)^2}{24}\,\Delta^3\gamma_0 + \ldots \right) \quad (I)$$

Putting $n = 1$ in the above equation, all differences higher than the first order difference will become zero and hence we get

$$\int_{\chi_0}^{\chi_1} \gamma.d\chi. = h \left(\gamma_0 + \frac{1}{2}\,\Delta\gamma_0 \right)$$

$$\int_{\chi_0}^{\chi_1} \gamma.d\chi. = h \left(\gamma_0 + \frac{1}{2}\,(\gamma_1 - \gamma_0) \right)$$

$$\int_{\chi_0}^{\chi_1} \gamma.d\chi. = \frac{h}{2}\,(\gamma_0 - \gamma_1)$$

Similar lv for the next interval (χ_1, χ_2) we get

$$\int_{\chi_0}^{\chi_1} \gamma.d\chi. = \frac{h}{2}\,(\gamma_1 - \gamma_0)$$

and so on.

For the last internal (χ_{n+1}, χ_n) we have

$$\int_{\chi_{n+1}}^{\chi_1} \gamma . d\chi . = \frac{h}{2} (\gamma_n - \gamma_n)$$

Summing up all these, we get

$$\int_{\chi_0}^{\chi_1} \gamma . d\chi . = \frac{h}{2} [\gamma_0 + 2 (\gamma_1 + \gamma_2 + \ldots \gamma_{n-1}) + \gamma_n]$$

The above formula is known the Trapezoidal Rule.

ERROR IN TRAPEZOIDAL RULE :

The error in trapezoidal formula is given by :

$$E = - \frac{1}{12} h^3 n\gamma'' (\bar{\chi})$$

$$E = - \frac{(b - a)}{12} h^2 \gamma'' (\bar{\chi}) \qquad [\therefore nh = b - a]$$

Procedure to Solve Problems using Trapezoidal Rule :

Step I : Find step size h by the formula $h = \frac{b - q}{N}$

Step II : Find different values of given function using different values of χ incremented by *h.*

Step III : Find value of integration by using Trapezoidal rule given by :

$$\int_{\chi_0}^{\chi_n} . \gamma . d\chi . = \frac{h}{2} [\gamma_0 + 2 (\gamma_1 + \gamma_2 + \ldots \gamma_{n-1}) + \gamma_n)]$$

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 1 Use the two segment Trapezoidal rule to estimate integration of

$f(\ ) = 0.2 + 25\chi + 200\chi^2 + 675\chi^3 - 900\chi^4 + 400\chi^5$

from a = 0 to b = 0.8.

Solution :

Step I : Finding *h* :

We are given that a = 0, b = 0.8

Since it is given to use two segment trapezoidal rule, we have

$n = 2,$

Now, $$h = \frac{b - a}{n}$$

$$\therefore \quad h = \frac{0.8 - 0}{2} = 0.4$$

Step II : Finding value of function for different values of χ :

$\therefore$ $\chi_0 = 0$ gives $\gamma_0 = f(\chi_0) = 0.2$

$\chi_1 = 0.4$ gives $\gamma_1 = f(\chi_1) = 66.456$

$\chi_2 = 0.8$ gives $\gamma_2 = f(\chi_2) = 256.232$

Step III : Integrating using Trapezoidal rule :

According to Trapezoidal rule. We have

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{2} [\gamma_0 + 2(\gamma_1 + \gamma_2 + \ldots \gamma_{n-1}) + \gamma_n)]$$

$$\therefore \int_{\chi_0}^{\chi_2} f(\chi_0).d\chi . = \frac{h}{2} [\gamma_0 + 2\gamma_1 + \gamma_2]$$

$$\therefore \quad 1 = \frac{0.4}{2} [0.2 + 2(66.456) + 256.232]$$

2

$\therefore$ ***I* = 77.8688**

Ex. 2 Find from the following table, the area bounded by the curve and the χ - axis from $\chi = 7.47$ to $\chi = 7.52$

χ	7.47	7.48	7.49	7.50	7.51	7.52
$f(\chi)$	1.93	1.95	1.98	2.01	2.03	2.06

Solution :

Step I :

From the given data, area bounded by the curve and the χ - axis is given by

$$\int_{\chi_0}^{\chi_2} f(\chi_0).d\chi.$$

Step II : Finding *h* :

We are given that a = 7.47, b = 7.52

Here the interval [7.47, 7.52] is divided into 5 equal parts

$n = 5$,

Now, $$h = \frac{b - a}{n}$$

$$\therefore \quad h = \frac{7.52 - 7.47}{5} = 0.01$$

Step III : Finding value of function for different values of χ :

From the table given in the problem, we get

$\chi_0 = 7.47$ gives $\gamma_0 = f(\chi_0) = 1.93$

$\chi_1 = 7.48$ gives $\gamma_1 = f(\chi_1) = 1.95$

$\chi_2 = 7.49$ gives $\gamma_2 = f(\chi_2) = 1.98$

$\chi_3 = 7.50$ gives $\gamma_3 = f(\chi_3) = 1.98$

$\chi_4 = 7.51$ gives $\gamma_4 = f(\chi_4) = 2.03$

$\chi_5 = 7.52$ gives $\gamma_5 = f(\chi_5) = 2.06$

Step IV : Finding area bounded by curve using Trapezoidal rule :

According to Trapezoidal rule we have

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{2} [\gamma_0 + 2(\gamma_1 + \gamma_2 + \ldots \gamma_{n-1}) + \gamma_n)]$$

$$\therefore \int_{7.47}^{7.52} f(\chi_0).d\chi. = \frac{h}{2} [\gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) + \gamma_5]$$

$$\therefore \quad 1 = \frac{0.01}{2} [1.93 + 2(1.95 + 1.98 + 2.01 + 2.03) + 2.06]$$

$\therefore$ $\boldsymbol{I = 0.09965}$

Ex. 3 Using Trapezoidal rule, estimate approximately the area of the cross section of a river 80 meters wide. The depth *d* (in meters) at a distance χ from one bank is given by the following table :

χ	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Solution :

Step I :

Area of the cross – section of a river 80 meters wide is given by :

$$. \int^{80} \gamma . d\chi .$$

0

Where y = d = depth at a distance χ from one bank.

Step II : Finding *h* :

We are given that a = 0, b = 80

Here the interval [0, 80] is divided into 8 equal parts.

$n = 8,$

Now, $h = \dfrac{b - a}{n}$

$\therefore$ $h = \dfrac{80 - 0}{8} = 10$

Step III : Finding value of function for different values of χ :

From the table given in the problem, we get

$\chi_0 = 0$	gives	$\gamma_0 = d_0 = 0$
$\chi_1 = 10$	gives	$\gamma_1 = d_1 = 4$
$\chi_2 = 20$	gives	$\gamma_2 = d_2 = 7$
$\chi_3 = 30$	gives	$\gamma_3 = d_3 = 9$
$\chi_4 = 40$	gives	$\gamma_4 = d_4 = 12$
$\chi_5 = 50$	gives	$\gamma_5 = d_5 = 15$
$\chi_6 = 60$	gives	$\gamma_6 = d_6 = 14$
$\chi_7 = 70$	gives	$\gamma_7 = d_7 = 8$
$\chi_8 = 80$	gives	$\gamma_8 = d_8 = 3$

Step IV : Finding area of cross – section of the river using Trapezoidal rule :

According to Trapezoidal rule we have

$$\int^{\chi_n} \gamma . d\chi . = \frac{h}{} \left[\gamma_0 + 2\left(\gamma_1 + \gamma_2 + \ldots \gamma_{n-1}\right) + \gamma_n\right]$$

$$\chi_0 \qquad 2$$

$$\therefore \int_0^{80} \gamma . d\chi = \frac{h}{2} [\gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7) + \gamma_8]$$

$$\therefore \quad 1 = \frac{10}{2} [0 + 2(4 + 7 + + 9 + 12 + 15 + 14 + 8) + 3]$$

$\therefore$ ***I* = 705 sq. meters.**

Simpson's $\frac{1^{rd}}{3}$ Rule

Derivation of Simpson's $\frac{1^{rd}}{3}$ Rule :

Consider a definite integral $I \int_a^b \gamma \, d\chi$.

Let the interval [a, b] be divided into *n* equal subintervals such that $a = \chi_0 < \chi_1 < \chi_2 < . . . < \chi_n = b$.

$$\chi_n = \chi_0 + nh$$

Hence the given definite integral becomes :

$$I \int_{\chi_0}^{\chi_n} y \, d\chi.$$

Approximating *y* by Newton's forward difference formula, we get

$$I = \int_{\chi_0}^{\chi_n} \left(\gamma_0 + p\Delta\gamma_0 + \frac{p(p-1)}{2} \Delta^2\gamma_0 + \frac{p(p-1)(p-2)}{6} \Delta^3\gamma_0 + \ldots \right) d\chi.$$

Since $\chi_n = \chi_0 + ph$ and $d\chi = h.dp$ the above integral become

$$n \left(\qquad p(p-1) \qquad p(p-1)(p-2) \qquad \right)$$

$$I = h \int_{0} \left[y_0 + p\Delta y_0 + \frac{}{2} \Delta^2 y_0 + \frac{}{6} \Delta^3 y_0 + \dots \right] dp.$$

which on simplification gives

$$\int_{x_0}^{x_n} y.dx. = nh \left(y_0 + \frac{n}{2} \Delta y_0 + \frac{n(2n-3)}{12} \Delta^2 y_0 + \frac{n(n-2)^2}{24} \Delta^3 y_0 + \dots \right) \quad \dots \text{(I)}$$

Putting $n = 2$ in the above equation, all differences higher than the Second order difference will become zero and hence we get

$$\int_{x_0}^{x_2} y.dx. = 2h \left(y_0 + \Delta y_0 \; \frac{1}{6} \Delta^2 y_0 \right)$$

$$\int_{x_0}^{x_2} y.dx. = \frac{h}{3} \left(y_0 + (y_1 - y_2) + \frac{1}{6} (y_2 - 2y_1 + y_0) \right)$$

$$\int_{x_0}^{x_2} y.dx. = \frac{h}{3} [y_0 + 4y_1 + y_2] \quad ,,,, \text{(II)}$$

Similarly

$$\int_{x_2}^{x_4} y.dx. = \frac{h}{3} [y_2 + 4y_3 + y_1] \quad \dots.\text{(III)}$$

and finally

$$\int_{x_{n-2}}^{x_n} y.dx. = \frac{h}{3} [y_2 + 4y_{n-1} + y_n] \quad \dots\text{(IV)}$$

Summing up all these we get

$$\int_{x_0}^{x_n} y.dx. = \frac{h}{3} \left(y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n \right)$$

1^{rd}

The above formula is known Simpson's $\frac{}{3}$ Rule or simply Simpson's rule.

[Note : It should be noted that this rate requires the division of the whole range into an event number of subintervals of width h]

Error in Simpson's $\frac{1^{rd}}{3}$ Rule :

The error in Simpson's rule is given by :

$$E = -\frac{(b-a)}{180} h^4 \gamma^{iv}(\bar{\chi})$$

where γ^{iv} is the largest value of the fourth derivatives.

Procedure to Solve Problems using Simpson's $\frac{1^{rd}}{3}$ Rule :

Step I : Find step size *h* by the formula $h \ \frac{b-a}{h}$

[Note : Simpson's $\frac{1^{rd}}{3}$ rule requires that *n* should be even otherwise the rule fails]

Step II : Find different values of given function using different values of χ incremented by *h.*

Step III : Find value of integration by using Simpson's $\frac{1^{rd}}{3}$ rule given by :

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5 + \ldots + \gamma_{n-1}) + 2(\gamma_2 + \gamma_4 + \gamma_6 + \ldots + \gamma_{n-2}) + \gamma_n \right)$$

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 5 Use the Simpson's 1.3rd rule to calculate the area under the curve $\gamma = e^{-x^2}$ when χ range from to 1. Divide the given range into 10 intervals.

Solution :

Step I : Finding *h* :

We are given that a = 0, b = 1

Since it is given that the range is divide into 10 intervals we have

$n = 10,$

Now, $$h = \frac{b - a}{n}$$

$$h = \frac{1 - 0}{10} = 0.1$$

Step II : Finding value of function for different values of χ :

Putting different values of χ we get different values of function as follows :

$\chi_0 = 0$	gives	$\gamma_0 = 1$
$\chi_1 = 0.1$	gives	$\gamma_1 = 0.99$
$\chi_2 = 0.2$	gives	$\gamma_2 = 0.9608$
$\chi_3 = 0.3$	gives	$\gamma_3 = 0.9189$
$\chi_4 = 0.4$	gives	$\gamma_4 = 0.8521$
$\chi_5 = 0.5$	gives	$\gamma_5 = 0.7788$
$\chi_6 = 0.6$	gives	$\gamma_6 = 0.6977$
$\chi_7 = 0.7$	gives	$\gamma_7 = 0.6126$
$\chi_8 = 0.8$	gives	$\gamma_8 = 0.5273$
$\chi_9 = 0.9$	gives	$\gamma_9 = 0.4449$

$\chi_{10} = 10$ gives $\gamma_{10} = 0.3679$

Step III : Calculating area under the curve using Simpson's $\frac{1^{rd}}{3}$ rule :

According to Simpson's $\frac{1^{rd}}{3}$ rule we have

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5 + \ldots + \gamma_{n-1}) + 2(\gamma_2 + \gamma_4 + \gamma_6 + \ldots + \gamma_{n-2}) + \gamma_n\right)$$

$$\therefore \int_0^1 e^{-x^2}. = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5 + \gamma_7 + \gamma_9) + 2(\gamma_2 + \gamma_4 + \gamma_6 + \gamma_8) + \gamma_{10}\right)$$

Sol

$$\therefore \quad 1 = \frac{0.1}{3}\ [1 + 4(3.7402) + 2(3.0379) + 0.3679]$$

Ex.

$\therefore$ ***I* = 07468**

Ex. 6 Calculate by Simpson's rule an approximate value of $\int_{-3}^{+3} \chi^4 \, d\chi$ by taking

seven equidistant ordinates. Compare it with exact value and the value obtained by using the Trapezoidal rule.

Solution :

Step I : Finding *h* :

We are given that a = -3, b = 3

It is given that we have to take seven ordinates, therefore. The given range is divided into 6 intervals.

$n = 6,$

b – a

Now, $h = \dfrac{\quad}{n}$

$$h = \frac{3 - (-3)}{6} = 1$$

Step II : Finding value of function for different values of χ :

Putting different values of χ we get different values of function as follows :

$\chi_0 = 3$	gives	$\gamma_0 = f(\chi_0) = 81$
$\chi_1 = -2$	gives	$\gamma_1 = f(\chi_1) = 16$
$\chi_2 = 1$	gives	$\gamma_2 = f(\chi_2) = 1$
$\chi_3 = 0$	gives	$\gamma_3 = f(\chi_3) = 0$
$\chi_4 = 1$	gives	$\gamma_4 = f(\chi_4) = 1$
$\chi_5 = 2$	gives	$\gamma_5 = f(\chi_5) = 16$
$\chi_6 = 3$	gives	$\gamma_6 = f(\chi_6) = 8$

Step III : Calculating approximate value of function using Simpson's $\dfrac{1rd}{3}$ rule :

According to Simpson's $\dfrac{1^{rd}}{3}$ rule we have

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5 + \ldots + \gamma_{n-1}) + 2(\gamma_2 + \gamma_4 + \gamma_6 + \ldots + \gamma_{n-2}) - \gamma_n\right)$$

$$\therefore \quad \int_{-3}^{+3} \chi^4 \, d\chi . = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5) + 2(\gamma_2 + \gamma_4) + \gamma_6\right)$$

$$\therefore \quad 1 = \frac{0.1}{3}\,[\,81 + 4\,(16 + 0 + 16) + 2(1 + 1) + 81\,]$$

$\therefore$ ***I* = 98**

Step IV : Calculating approximate value of function using Trapezoidal rule :

According to Trapezoidal rule we have

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{2} \left(\gamma_0 + 4 (\gamma_1 + \gamma_3 + \gamma_5) + 2 (\gamma_2 + \gamma_4) + \gamma_6 \right)$$

$$\therefore \quad \int_{-3}^{+3} \chi^4 \, d\chi . = \frac{h}{2} \left(\gamma_0 + 2 (\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5) + \gamma_6 \right)$$

$$\therefore \quad 1 = \frac{1}{2} \; [81 + 2 (16 + 1 + 0 + 1 + 16) + 81]$$

$\therefore$ ***I* =115**

Step V : Finding exact value of function :

We have.

$$I = \int_{-3}^{+3} \chi^4 \, d\chi .$$

$$I = \left(\frac{\chi^3}{5} \right)_{-3}^{+3}$$

$$I = \frac{243 - (243)}{5}$$

I = 97.2

Step VI : Comparing the results :

Exact integration value of given function is 97.2

Integration value obtained using Simpson's rule is 98

and Integration value obtained using Trapezoidal rule is 115

This shows that, in this case, Simpson's rule gives a better results than the trapezoidal rule.

Ex. 7 Approximate the integral given below using the composite trapezoidal rule and Simpson's rule using 8 equal subdivisions

$$\int_{-1}^{1} \frac{1}{1+\chi^2}\, d\chi.$$

Solution :

Step I : **Finding *h* :**

We are given that a = -1, b = 1

It is given that we have to take 8 equal sub-divisions.

$n = 8$

Now, $$h = \frac{b-a}{n}$$

$$h = \frac{1-(-1)}{8} = 0.25$$

Step II : **Finding value of function for different values of χ :**

Putting different values of χ we get different values of function as follows :

$\chi_0 = 1$ gives $\gamma_0 = f(\chi_0) = 0.5$

$\chi_1 = -0.75$ gives $\gamma_1 = f(\chi_1) = 0.64$

$\chi_2 = -0.50$ gives $\gamma_2 = f(\chi_2) = 0.8$

$\chi_3 = -0.25$ gives $\gamma_3 = f(\chi_3) = 0.9411764$

$\chi_4 = 0$ gives $\gamma_4 = f(\chi_4) = 1$

$\chi_5 = 0.25$ gives $\gamma_5 = f(\chi_5) = 0.94411764$

$\chi_6 = 0.50$ gives $\gamma_6 = f(\chi_6) = 0.8$

$\chi_7 = 0.75$ gives $\gamma_7 = f(\chi_7) = 0.64$

$\chi_8 = 1$ gives $\gamma_8 = f(\chi_8) = 0.5$

Step III : Calculating approximate value of function using Simpson's $\frac{1^{rd}}{3}$ rule :

According to Simpson's $\frac{1^{rd}}{3}$ rule we have

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5 + \dots + \gamma_{n-1}) + 2(\gamma_2 + \gamma_4 + \gamma_6 + \dots + \gamma_{n-2}) - \gamma_n\right)$$

$$\therefore \int_{-1}^{+1} \frac{1}{1+\chi^2} d\chi. = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5) + 2(\gamma_2 + \gamma_4 + \gamma_6) + \gamma_8\right)$$

$$\therefore 1 = \frac{0.25}{2}[0.5 + 4(0.64 + 0.9411764 + 0.9411764 + 0.64) + 2(0.8 + 1 + 0.8) + 0.5]$$

$\therefore$ ***I* = 1.5707843**

Step IV : Calculating approximate value of function using Trapezoidal rule :

According to Trapezoidal rule we have

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{h}{2}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \dots + \gamma_{n+1}) + \gamma_n\right)$$

$$\therefore \int_{-1}^{+1} \frac{1}{1+\chi^2} d\chi. = \frac{h}{2}\left(\gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 + \gamma_5 + \gamma_6 + \gamma_7) + \gamma_8\right)$$

0.25

$\therefore$ 1 = ------ [0.5 + 2(0.64 + 0.8 + 0.9411764 + 1 + 0.9411764 + 0.8 + 0.64) + 0.5]
2

$\therefore$ ***I* = 1.5655882**

Step V : Finding exact value of function :

We have.

$$I = \int_{-1}^{+1} \frac{1}{1+\chi^2}\, d\chi.$$

$$I = \left[\tan^{-1}\chi\right]_{-1}^{+1}$$

$$I = \frac{\Pi}{4} - \left(\frac{\Pi}{4}\right)$$

$$\mathbf{I = \frac{\Pi}{2} = 1.5707963}$$

Step VI : Comparing the results :

Exact integration value of given function is 1.5707963

Integration value obtained using Simpson's rule is 1.5707843

and Integration value obtained using Trapezoidal rule is 1.5655882

Error in calculation by Simpson's rule = 0.000012 in defection

and Error in calculation by Trapezoidal rule = 0.0052081 in defection

There is a greater accuracy in the calculation by Simpson's rule.

Unit 4

Simpson's $\frac{3^{rd}}{8}$ Rule

Derivation of Simpson's $\frac{3^{rd}}{8}$ Rule :

Consider a definite integral $I = \int_a^2 \gamma\, d\chi.$

Let the interval [a, b] be divided into n equal subintervals such that $a = \chi_0 < \chi_1 < \chi_2 < . . . < \chi_n = b.$

$$\chi_n = \chi_0 + nh$$

Hence the given definite integral becomes :

$$I = \int_{\chi_0}^{\chi_n} y\, d\chi.$$

Approximating y by Newton's forward difference formula, we get

$$I = \int_{\chi_0}^{\chi_n} \left(\gamma_0 + p\Delta\gamma_0 + \frac{p(p-1)}{2} \Delta^2\gamma_0 + \frac{p(p-1)(p-2)}{6} \Delta^3\gamma_0 + \ldots \right) d\chi.$$

Since $\chi_n = \chi_0 + ph$ and $d\chi = h.dp$ the above integral become

$$I = h \int_0^p \left(\gamma_0 + p\Delta\gamma_0 + \frac{p(p-1)}{2} \Delta^2\gamma_0 + \frac{p(p-1)(p-2)}{6} \Delta^3\gamma_0 + \ldots \right) dp.$$

which on simplification gives

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = nh \left(\gamma_0 + \frac{n}{2} \Delta\gamma_0 + \frac{n(2n-3)}{12} \Delta^2\gamma_0 + \frac{n(n-2)^2}{24} \Delta^3\gamma_0 + \right) \ldots \qquad \ldots \text{(I)}$$

Putting n = 3 in the above equation, all differences higher than the Second order difference will become zero and hence we get

$$\int_{\chi_0}^{\chi_3} \gamma.d\chi. = 3h \left(\gamma_0 + \Delta\gamma_0 \ \frac{3}{4} \Delta^2\gamma_0 \ \frac{1}{8} \ \Delta^3\gamma_0 \right)$$

$$\int_{\chi_0}^{\chi_3} \gamma.d\chi. = 3h \left(\gamma_0 + \Delta\gamma_0 \ \frac{3}{4} \ \Delta^2\gamma_0 \ \frac{1}{8} \ \Delta^3\gamma_0 \right)$$

$$\int_{\chi_0}^{\chi_3} \gamma.d\chi. = \frac{3h}{8} \left(\gamma_0 + 3\gamma_1 + 3\gamma_2 + 3\gamma_3 \right)$$

Similarly

$$\int_{\chi_3}^{\chi_6} \gamma.d\chi. = \frac{3h}{8} \ [\gamma_3 + + 3\gamma_4 + 3\gamma_5 + \gamma_6]$$

and so on.

Summing up all these, we get

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{3h}{8} \left((\gamma_0+3\gamma_1+3\gamma_2+3\gamma_3)+(\gamma_3+3\gamma_4+3\gamma_5+3\gamma_6)+\ldots+ \gamma_{n-3})+3\gamma_{n-2} +3\gamma_{n-1} + \gamma_n) \right)$$

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{3h}{8} \left(\gamma_0 + 3\gamma_1 + 3\gamma_2 + 2\gamma_3 + 3\gamma_4 + 3\gamma_5 + 2\gamma_6 + \ldots + 2\gamma_{n-3} + 3\gamma_{n-2} + 3\gamma_{n-1} + \gamma_n \right)$$

$$\int_{\chi_0}^{\chi_n} \gamma.d\chi. = \frac{3h}{8} \left(\gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + +) + 2(\gamma_3 + \gamma_6 + \gamma_9 + \ldots + \gamma_{n-3}) + \gamma_n \right)$$

The above formula is known Simpson's $\frac{3^{rd}}{8}$ Rule.

Error in Simpson's $\frac{3^{rd}}{8}$ Rule :

The error in Simpson's $\frac{3^{rd}}{8}$ rule is given by :

3

$$E = - \frac{}{80} h^5 \gamma^{ic} (\bar{\chi})$$

where $\gamma^{iv} (\bar{\chi})$ is the largest value of the fourth derivatives.

Procedure to Solve Problems using Simpson's $\frac{3^{rd}}{8}$ Rule :

Step I : Find step size h by the formula $h \; \frac{b - a}{h}$

Step II : Find different values of given function using different values of χ incremented by *h.*

Step III : Find value of integration by using Simpson's $\frac{3^{rd}}{8}$ rule given by :

$$\int_{\chi_0}^{\chi_n} . \; \gamma.d\chi. = \frac{3h}{8} \left(\gamma_0 + 3 (\gamma_1 + \gamma_2 + \gamma_4 - \gamma_5 - \gamma_7 + \;) + 2 (\gamma_3 + \gamma_6 + \gamma_9 + \dots + \gamma_{n-3}) + \gamma_n \right)$$

The above procedure will get clear to you as you will go through the following solved examples.

Ex. 9 Using Simpson's $\frac{3^{rd}}{8}$ rule to integrate

$$f (\chi) = 0.2 + 25\chi + 200\chi^2 + 675\chi^2 - 900\chi^4 + 400\chi^5$$

From a = 0 to b = 0.8 where b – a is the width of integration.

Solution :

Step I : Finding *h* :

We are given that a = 0, b = 0.8

Let the given interval of (0.08) be divided into 8 equal intervals.

$n = 8$,

Now, $h = \dfrac{b - a}{n}$

$$h = \frac{0.8 - 0}{8} = 0.1$$

Step II : Finding value of function for different values of χ :

Putting different values of χ we get different values of function as follows :

$\chi_0 = 0$	gives	$\gamma_0 = f(\chi_0) = 0.2$
$\chi_1 = 0.1$	gives	$\gamma_1 = f(\chi_1) = 1.289$
$\chi_2 = -0.2$	gives	$\gamma_2 = f(\chi_2) = 1.288$
$\chi_3 = -0.3$	gives	$\gamma_3 = f(\chi_3) = 1.607$
$\chi_4 = 0.4$	gives	$\gamma_4 = f(\chi_4) = 2.456$
$\chi_5 = 0.5$	gives	$\gamma_5 = f(\chi_5) = 3.325$
$\chi_6 = 0.6$	gives	$\gamma_6 = f(\chi_6) = 3.464$
$\chi_7 = 0.7$	gives	$\gamma_7 = f(\chi_7) = 2.363$
$\chi_8 = 0.8$	gives	$\gamma_8 = f(\chi_8) = 0.232$

Step III : Integrating using Simpson's $\dfrac{3^{rd}}{8}$ rule :

According to Simpson's $\dfrac{3^{rd}}{8}$ rule we have

$$\int_0^{0.8} f(\chi). = \frac{3h}{8}\left(\gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7) + 2(\gamma_3 + \gamma_6) + \gamma_8\right)$$

$$\therefore 1 = \frac{3 \times 0.1}{8} [0.2+3(1.289+1.288 + 2.456 + 3.325 + 2.363) + 2(1.607 + 3.464) + 0.232]$$

$$\therefore 1 = \frac{3 \times 0.1}{8} [0.2 + 3 (10.721) + 2 (5.071) + 0.232]$$

$\therefore$ ***I* = 1.6026375**

Ex. 10 Using Simpson's $\frac{3^{rd}}{8}$ rule to integrate the function

$$f (\chi) = 0.2 + 20\chi + 25\chi^2 + 60\chi^2$$

From a = 0.0 to b = 1.0

Solution :

Step I : Finding *h* :

We are given that a = 0.0, b = 1.0

Let the given interval of (0, 1.0) be divided into 10 equal intervals.

$n = 10$,

Now, $h = \frac{b - a}{n}$

$$h = \frac{1.0 - 0}{10} = 0.1$$

Step II : Finding value of function for different values of χ :

Putting different values of χ we get different values of function as follows :

$\chi_0 = 0$ gives $\gamma_0 = f (\chi_0) = 0.2$

$\chi_1 = 0.1$ gives $\gamma_1 = f (\chi_1) = 2.51$

$\chi_2 = -0.2$	gives	$\gamma_2 = f(\chi_2) = 5.68$
$\chi_3 = -0.3$	gives	$\gamma_3 = f(\chi_3) = 10.07$
$\chi_4 = 0.4$	gives	$\gamma_4 = f(\chi_4) = 16.04$
$\chi_5 = 0.5$	gives	$\gamma_5 = f(\chi_5) = 23.95$
$\chi_6 = 0.6$	gives	$\gamma_6 = f(\chi_6) = 34.16$
$\chi_7 = 0.7$	gives	$\gamma_7 = f(\chi_7) = 47.03$
$\chi_8 = 0.8$	gives	$\gamma_8 = f(\chi_8) = 62.92$
$\chi_9 = 0.9$	gives	$\gamma_9 = f(\chi_9) = 82.19$
$\chi_{10} = 1.0$	gives	$\gamma_{10} = f(\chi_{10}) = 105.20$

Step III : Integrating using Simpson's $\frac{3^{rd}}{8}$ rule :

According to Simpson's $\frac{3^{rd}}{8}$ rule we have

$$\int_0^1 f(\chi). = \frac{3h}{8}\left(\gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + \gamma_8) + 2(\gamma_3 + \gamma_6 + \gamma_9) + \gamma_{10}\right)$$

$$\therefore 1 = \frac{3 \times 0.1}{8}[0.2+3(2.51 + 5.68 + 16.04 + 23.95 + 47.03 + 62.92) + 2(10.07+34.16+82.19) + 105.20]$$

$$\therefore 1 = \frac{3 \times 0.1}{8}[0.2 + 3(158.13) + 2(126.42) + 105.20]$$

$\therefore$ ***I* = 31.223625**

Ex. 11 Evaluate the integral

$$I \int^{\chi} \frac{1}{\quad} d\chi.$$

$_0 \quad 1 + \chi$

with $h = \dfrac{1}{6}$ by using Simpson's $\dfrac{1^{rd}}{3}$ rule and $\dfrac{3^{th}}{8}$ rule and compare the results

Solution :

Step I : Finding *h* :

We are given that a = 0.0, b = 1 and $h = \dfrac{1}{6}$

Step II : Finding value of function for different values of χ :

Putting different values of χ we get different values of function as follows :

$\chi_0 = 0$ gives $\gamma_0 = f(\chi_0) = 1$

$\chi_1 = \dfrac{1}{6}$ gives $\gamma_1 = f(\chi_1) = 0.8571428$

$\chi_2 = \dfrac{2}{6}$ gives $\gamma_2 = f(\chi_2) = 0.75$

$\chi_3 = \dfrac{3}{6}$ gives $\gamma_3 = f(\chi_3) = 0.6666666$

$\chi_4 = \dfrac{4}{6}$ gives $\gamma_4 = f(\chi_4) = 0.6$

$\chi_5 = \dfrac{5}{6}$ gives $\gamma_5 = f(\chi_5) = 0.5454545$

$\chi_6 = \dfrac{6}{6} = 1$ gives $\gamma_6 = f(\chi_6) = 0.5$

1^{rd}

Step III : Calculating approximate value of function using Simpson's 1rd ----- rule : 3

According to Simpson's $\frac{1^{rd}}{3}$ rule we have

$$\int_0^1 \frac{1}{1+\chi}\, d\chi = \frac{h}{3}\left(\gamma_0 + 4(\gamma_1 + \gamma_3 + \gamma_5) + 2(\gamma_2 + \gamma_4) + \gamma_6 \right)$$

$$I = \frac{1/6}{3} [\, 1 + 4 (0.8571428 + 0.6666666 + 0.5454545) + 2(0.75 + 0.6 + 0.5\,]$$

$$I = \frac{1}{18} [\, 1 + 4 (2.0692639) + 2(1.35) + 0.5]$$

***I* = 0.6931697**

Step III : Calculating approximate value of function using Simpson's ----- rule :

According to Simpson's $\frac{3^{rd}}{8}$ rule we have

$$\int_0^1 \frac{1}{1+\chi}\, d\chi = \frac{3h}{8}\left(\gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5) + 2(\gamma_3) + \gamma_6 \right)$$

$$I = \frac{3 \times 1/6}{8} [\, 1 + 3 (0.8571428 + 0.75 + 0.6 + 0.5454545) + 2(0.6666666) + 0.5\,]$$

$$I = \frac{1}{16} [\, 1 + 3 (2.7525973) + 2(0.6666666) + 0.5]$$

***I* = 0.6931953**

Step V : Finding exact value of the function :

$$I = \int_0^1 \frac{1}{1+\chi}\, d\chi.$$

$$I = \left[\log_e (1+\chi)\right]_0^1$$

$I = \log_e 2 - \log_e 1$

I = 0.6931471 – 0

I = 0.6931471

Step VI : Comparing the results :

Exact integration value of given function is **0.6931471**

Integration value obtained using Simpson's $\frac{1^{rd}}{3}$ rule is **0.6931697**

and Integration value obtained using Simpson's $\frac{3^{th}}{8}$ rule is **0.6931953**

Error in calculation by Simpson's $\frac{1^{rd}}{3}$ rule = **0.000022** in excess

and Error in calculation by Simpson's $\frac{3^{th}}{8}$ rule = **0.000048** in excess

There is a greater accuracy in the calculation by Simpson's $\frac{1^{rd}}{3}$ rule.

Unit 5

Comparison between Trapezoidal,
Simpson's $\frac{1^{rd}}{}$ & Simpson's $\frac{3^{th}}{}$ Rule

3 **8**

Trapezoidal Rule	Simpson's 1^{rd} ------ Rule 3	Simpson's 3^{TH} ------ Rule 8
The trapezoidal rule approximates the function by **a straight line.**	The Simpson's 1/3 rule approximates the function by **a second degree polynomial.**	The Simpson's 3/8 rule approximates the function by a cubic polynomial
Formula for Trapezoidal Rule is given by equation (I) given below.	Formula for Simpson's 1/3 Rule is given by equation (II) given below.	Formula for Simpson's 3/8 Rule is given by equation (III) given below.
For use of this rule, we require (n + 1) **ordinates.**	For use of this rule. we require (2n + 1) **ordinates.**	For us of this rule, we require (3n + 1) **ordinates.**
The equation of the curve to be integrated by Trapezoidal Rule is a linear function of the form $\gamma = a\chi + b$	The equation of the curve to be integrated by Simpson's 1/3 Rule is a polynomial of second degree of the form $\gamma = a\chi^2 + b\chi + c$	The equation of the curve to be integrated by Simpson's 3/8 Rule is a polynomial of third degree of the form $\gamma = a\chi^2 + b\chi + c\chi + d$
This rule requires the division of the whole range into either **odd or even number** of subintervals of width *h.*	This rule requires the division of the whole range into either **an even number** of subintervals of width *h.*	This rule requires the division of the whole range into either **either odd or even number** of subintervals of width *h.*
Error in Formula for Trapezoidal Rule is given by $E = - \frac{(b - a)}{12} h^2 \gamma^{ic} (\bar{\chi})$	Error in Formula for Trapezoidal 1/3 Rule is given by $E = - \frac{(b - a)}{180} h^4 \gamma^{ic} (\bar{\chi})$	Error in Formula for Trapezoidal 3/8 Rule is given by $E = - \frac{(b - a)}{80} h^2 \gamma^{ic} (\bar{\chi})$

Formula for Trapezoidal Rule is given by :

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{2} \left((\gamma_0 + 2 (\gamma_0 + 2\gamma_1 + \ldots + \gamma_{n-1}) + \gamma_n \right)$$

1^{th}

Formula for Simpson's $\frac{\quad}{3}$ Rule is given by :

$$. \int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{h}{3} \left(\gamma_0 + 4\gamma_1 + \gamma_3 + \gamma_5 + \ldots + \gamma_{n-1}) + 2\gamma_2 + 3\gamma_4 + 3\gamma_6 + \ldots + \gamma_{n-2}) + \gamma_n \right)$$

Formula for Simpson's $\frac{3^{rd}}{8}$ Rule is given by :

$$\int_{\chi_0}^{\chi_n} \gamma . d\chi . = \frac{3h}{8} \left(\gamma_0 + 3(\gamma_1 + \gamma_2 + \gamma_4 + \gamma_5 + \gamma_7 + \ldots +) + 2(\gamma_3 + \gamma_6 + \gamma_9 + \ldots + \gamma_{n-3}) + \gamma_n \right)$$

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