① Let $G = \{(1), (12), (12)(34), (34)\}$ and consider the natural action of G on $X = \{1, 2, 3, 4\}$. Find the orbit of this action and the stabilizer of each point.

Solution

Given $G = \{(1), (12), (12)(34), (34)\}$, $X = \{1, 2, 3, 4\}$

Recall $\text{Orbit}(x) = \{gx \mid g \in G\}$ This explains the path an element follows

E.g

$\text{Orbit}(1) = \{1, 2\}$

$\text{Orbit}(2) = \{2, 1\}$

$\text{Orbit}(3) = \{3, 4\}$

$\text{orbit}(4) = \{4, 3\}$

So, The orbit of this action is $\{1, 2\}$, and $\{4, 3\}$.

Stabilizer of each point

Recall, $\text{Stab}_G(x) = \{g \in G \mid gx = x\}$ Set of elements in G that fixes x

For $\text{Stab}_G(1)$ we check how many elements in the G, fix (1) or left (1) constant. NOTE: $(1) = (2) = (3) = (4)$ all these are representations of identity element so

$\text{Stab}_G(1) = \{(1), (34)\}$

$\text{Stab}_G(2) = \{(1), (3,4)\}$

$\text{Stab}_G(3) = \{(1), (12)\}$

$\text{Stab}_G(4) = \{(1), (12)\}$

The Action of Conjugation is a group action

(15) Show that a group can act on itself by conjugation. Find the Orbit of this type of action and the stabilizer of each point. Deduce the class equation from this.

Solution

Let G be any group and X a non-empty set s.t $[X = G]$, Then, G is said to act on itself by conjugation if $gx = gxg^{-1}$. Two elements x and y are called conjugate if they are related by this action. i.e if $\exists\, g \in G$ such that

$gx = y$

$\Rightarrow gxg^{-1} = y$ for some $g \in G$

The orbits of this type of action are called conjugacy classes of G. And we write $C(x)$ for the conjugacy class of the element x

i.e $\quad C(x) = \{gxg^{-1} : g \in G\}$

The stabilizer of each point say x under this action is given by

$$\begin{aligned} Stab_G x &= \{g \in G \mid gx = x\} \\ &= \{g \in G \mid gxg^{-1} = x\} \\ &= \{g \in G \mid gx = xg\} \end{aligned}$$

which is called the centralizer of x in G, denoted $C_G(x)$.

Recall The class equation for G is $|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(x_i)]$ where $Z(G)$ is the Center of G and x_i denotes distinct representatives from each conjugacy class. Under this action we can write the class equation for G as $|G| = \sum_{i=1}^{r} [G : C_G(x_i)]$ since the center consist of self conjugating elements, and each x_i is a distinct representative of elements in each conjugacy class [either in the center or not].

1c. Let p be a prime and G a p-group ...
is non trivial

<u>Proof</u>

Let G be a p-group and consider the class equation for G

$|G| = |Z(G)| + \sum_{i=1}^{r} |G : C_G(x_i)|$ where $x_1, x_2, \ldots, x_r$ are distinct representatives of non central conjugacy classes

Since for each i x_i is not in $Z(G)$, then $|C_G(x_i)| < |G|$ and

So $p \mid [G : C_G(x_i)]$ for each i. Thus $p \mid \sum_{i=1}^{r} [G : C_G(x_i)]$

It follows that $p \mid |G|$ we must have

$p \mid (|G| - \sum_{i=1}^{r} [G : C_G(x_i)]) = |Z(G)|$ since p is a prime

$|Z(G)| > 1$ i.e. non-trivial as required □

2a. state the three Sylow's theorem

Let G be a finite group of order $|G| = p^r m$ where r, m are integers and p a prime such that $p \nmid m$. Then

1) G has a Sylow p-subgroup i.e. a subgroup of order p^r (1st Sylow's Thm)

) If P and Q are Sylow p-subgroups of G, Then $P = xQx^{-1}$ for some $x \in G$ i.e. any two Sylow p-subgroups are conjugate (2nd Sylow's theorem)

) The number n_p of Sylow p-subgroup is of the form $n_p = 1 + kp$ for some $k \in \mathbb{Z}$ $\Rightarrow n_p \equiv 1 \mod p$ (3rd Sylow's thm)

2b. Show that if H is a Sylow p-subgroup of a finite group G. Then H is the Unique Sylow p-subgroup of its normalizer $N_G(H)$

Proof

It is easy to see that H is a Sylow p-subgroup of every subgroup which contains H. In particular H is a Sylow p-subgroup of $N_G(H)$. Suppose K is any Sylow p-subgroup of $N_G(H)$. Then, by second part of Sylow's theorem, there is an element $x \in N_G(H)$ such that $K = xHx^{-1}$ but then since $x \in N_G(H)$, then $H = xHx^{-1}$. Hence, H is the Unique Sylow p-subgroup of $N_G(H)$. ■

2b. Provide the proof for the Sylow's third theorem
(Check your note)

(2c) Show that a group of order 225 is not simple

Solution

$|G| = 225 = 5^2 \cdot 3^2$

By Sylow's first theorem, G has a Sylow 5-subgroup of order 25. And by third Sylow's theorem, the number n_5 of Sylow 5-subgroup of G divides $3^2 = 9$ and $n_5 \equiv 1 \mod 5$. The factors of 3^2 are 1, 3 and 9 and which only $1 \equiv 1 \mod 5$. Thus $n_5 = 1$. i.e G has only one Sylow 5-subgroup say P. By second Sylow's theorem, $P = xPx^{-1}$ for all $x \in G$ and so, it is normal subgroup of G. Hence, G has a proper normal subgroup of order 25 and so it is not simple.

Similarly G has a Sylow 3-subgroup by first Sylow's theorem and by third Sylow's theorem the number n_3 divides 25 and $n_3 \equiv 1 \bmod 3$. The factors of 25 are 1, 5 and 25 and only $1 \equiv 1 \bmod 3$. So $n_3 = 1$ That is $\exists$ only 1 Sylow 3-subgroup of order 9 say Q Now by second Sylow's theorem $Q = xQx^{-1}$ for all $x \in G$ Thus $Q \trianglelefteq G$ of order 9 and so it is not simple

Thus a group G of order 225 is not simple

(3a) Show that a group of order p^2 is abelian

Proof
We prove this result by contradiction
Suppose $|G| = p^2$ and G is non-abelian Then $Z(G) \lneq G$ a proper subgroup of G By Lagrange's theorem $|Z(G)| = p$
Hence $\left| G/Z(G) \right| = p$, and so $G/Z(G)$ is cyclic
and it follows that G is abelian which is false
Thus, every group of order p^2 is abelian $\square$

The action of conjugation is a group action

i.e $\forall\, g, h \in G$, and $x \in G$

$$g(hx) = g(hxh^{-1})$$
$$= g(hxh^{-1})g^{-1}$$
$$= (gh)x(gh)^{-1}$$
$$= (gh)x$$

and also, $1x = 1x1^{-1} = x \quad \forall\, g, h, x \in G$.

(a)

$G=\{(1),(12),(12)(34),(34)\}$

$X=\{1\ 2\ 3\ 4\}$

find the orbits of this action and the stabilizer of each point

Sln

Orb(x) is defined as $orb(x)=\{gx \mid g\in G\}$ ~~where g~~ where $g\in G$ and $x\in X$

Then for gx, we have

Orb(1):

$(1)(1)=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}(1)=1$

Hint: Orb(1) means where does 1 maps to in the Symmetric group

$(12)(1)=\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}(1)=2$

$(12)(34)=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}(1)=2$

$(34)=\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}(1)=1$

$\therefore$ Orb(1) $=\{1\ 2\}$

Orb(2):

Where does 2 maps to in the Symmetric group

$(1)(2)=\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}(2)=2$

$(12)(2)=\begin{pmatrix}1&2&3&4\\2&1&3&4\end{pmatrix}(2)=1$

$(12)(34)=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}(2)=1$

$(34)(2)=\begin{pmatrix}1&2&3&4\\1&2&4&3\end{pmatrix}(2)=2$

$\therefore$ Orb(2) $=\{1\ 2\}$

Orb(3):

Where does 3 maps to in the Symmetric group

$(1)(3)=3$

$(12)(3)=3$

$(12)(34)(3)=4$

$(34)(3)=4$

$\therefore$ orb(3) $=\{3\ 4\}$

Orb(4)

$(1)(4)=4$

$(12)(4)=4$

$(12)(34)(4)=3$

$(34)(4)=3$

Orb(3) $=\{3\ 4\}$

$\Rightarrow$ The orbit of this action are: $\{(12)(34)\}$

Stabilizer is defined as ~~$Stab_G(x)=\{$~~

$Stab_G(x)=\{g\in G \mid gx=x\}$

Hint: Check the orbits, at each point x such that $gx=x$

$Stab_G(1)=\{(1)(34)\}$

$Stab_G(2)=\{(1)(34)\}$

$Stab_G(3)=\{(1)(12)\}$

$Stab_G(4)=\{(1)(12)\}$

$\Rightarrow$ The $Stab_G(x)$ at each point are

$Stab_G(1)=Stab_G(2)=\{(1)(34)\}$

$Stab_G(3)=Stab_G(4)=\{(1)(12)\}$

(1b) Show that a group can act on itself by conjugation. find the orbits of this type of action and the stabilizer of each point. Deduce the class equation from this.

Sln

Let G be any group and $X = G$.
The conjugation of G on itself is defined as $gx = gxg^{-1}$ $\forall g, x \in G$ since $X = G$.
Such that (i) $ex = exe^{-1} = x$
(ii) $g\cdot(h\cdot x) = g(hxh^{-1}) = g\cdot(hxh^{-1})g^{-1}$
$= (gh)x(gh)^{-1} = (gh)x$

which satisfied the group action axioms
Hence, the group act on itself by conjugation

The orbit of this action is given as
$Orb(x) = C(x) = \{gxg^{-1} \mid g \in G\}$
which is also known as conjugacy class

The stabilizer of this action is given as
$Stab_G(x) = \{g \in G \mid gxg^{-1} = x\}$
which is also called centralizer of x in G
denoted by $C_G(x)$

∴ Therefore we can deduce the class equation as
$|G| = |Z(G)| + \sum_{i=1}^{k} [G : C_G(x)]$

(1) (c) Let p be a prime number and G a p-group. Show that the center $Z(G)$ of G is non-trivial.

Sln

Let G be a p-group and consider the class equation for G. $|G| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]$ where $g_1, g_2 \cdots g_r$ are distinct representative of non-central conjugacy classes.

Since for each i, g_i is not in $Z(G)$, then the order of the centralizer $|C_G(g_i)| < |G|$, which means $p \mid [G : C_G(g_i)]$ for each i.

Thus $p \mid \sum_{i=1}^{r} [G : C_G(g_i)]$, therefore since $p \mid |G|$, we must have $p \mid \left(|G| - \sum_{i=1}^{r} [G : C_G(g_i)]\right) = Z(G)$. Since p is a prime $|Z(G)| > 1$.

Hence it is non-trivial.