CONIC SECTION

BY

ADEPOJU (ARITHMETICIAN)

With over solved 50 Questions.

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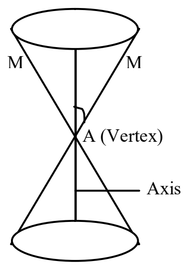
CONIC SECTIONS

Under this topic, we shall study about the following curves;

1. Parabola
2. Ellipse
3. Hyperbola

All these curves can be obtained as intersections of a plane with double-napped right circular cone.

However, a critical observation of cone reveals that there exist different shapes in a cone when it its dissected.



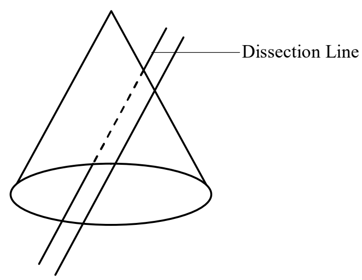
Another Definition

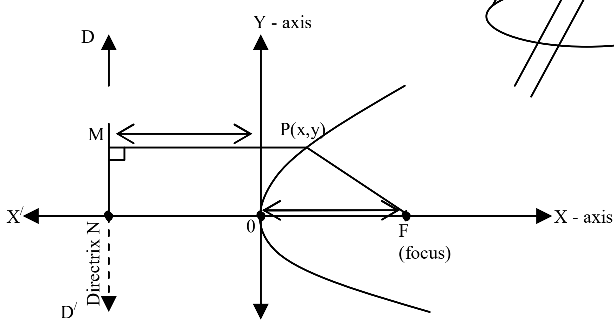
Conic Section/Conics:- When a right circular cone is intersected by a plane, the curves obtained are known as conic sections.

PARABOLA AND IT'S PROPERTIES

These shape/curves are obtained when a cone is sliced parallel to the slanting side. It is the pata traced by a point which moves in a plane in such a way that it distance from a fixed points it always equal to its distance from a fixed line, both lying in the same plane, whereas the given fixed point does not lie on the given line.

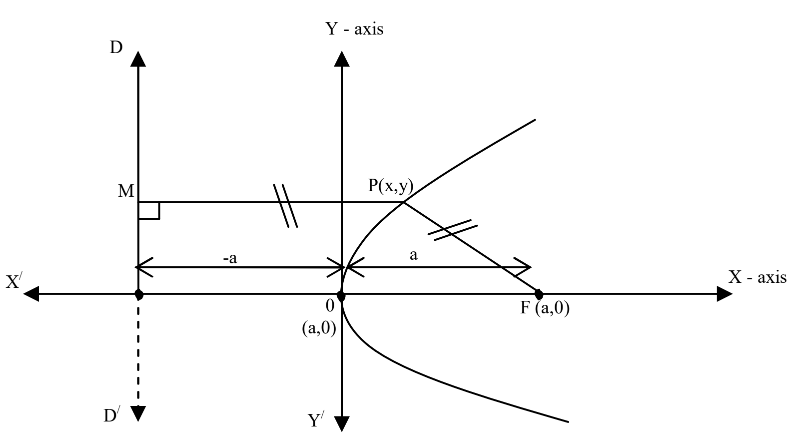
Thus, the fixed point is called *The Focus* of the parabola and the fixed line is called *The Directrix.*





1. A line through the focus and perpendicular to the directrix is called the axis of the parabola .i.e. MP
2. The point of intersection of the parabola with axis is called the vertex of the parabola.
3. In the Adjoining figure, C is a parabola with focus and the line DD$^{/}$ is the directrix while MP=PF.
4. x-axis is the axis of parabola at origin (0,0)
5. Origin (0,0) it is the vertex to the parabola.

RIGHT-HANDED PARABOLA



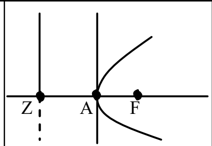
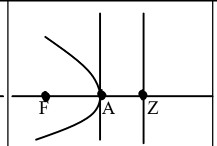
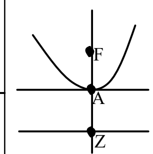
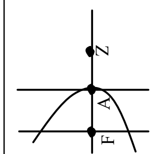
Let us consider a parabola whose focus is f (a,0) and the directrix is the $|DD^I|$ whose parabola, the directrix is the Negative of the Abisca to the focus. i.e from the diagram; focus is (a,0) then the directrix will be $-a$.
Equation of directrix is $x + a = 0$.
Let P(x,y) be arbitrary point on the parabola and PM perpendicular to DD^I, then by the definition of parabola, we have $PF = PM$
Now, $PF = PM$
Square both side
$|PF|^2 = |PM|^2$
Recall that, length of perpendicular from P(x,y) on the line directrix (x+y) is 0.
$(x+y) = (x-a)^2 + y^2 = (x+a)^2$

Thus, every point on the parabola satisfies the equation, $y^2=4ax$.

Definition Relation to Parabola

i. Vertex: The intersection point of parabola and axis.
ii. Centre: The point which bisects every chord of the conics passing through it.
iii. Focal chord: Any chord passing through the focus.
iv. Double Ordinate: A chord perpendicular to the axis of conics.
v. Latus Rectum: A double Ordinate passing through the focus of the parabola and perpendicular to the axis.
vi. Focus distance: The distance of a point P(x,y) from the focus is called the focal distance of the P.

Standard Forms of a Parabola and Related Terms Equation of Parabola

S/N	TERMS	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
1.	Graph				
2.	Vertex	A(0,0)	A(0,0)	A(0,0)	A(0,0)
3.	Focus	F(a,0)	F(-a,0)	F(0,a)	F(0,-a)
4.	Equation Of Directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
5.	Extremilies	$(a, \pm 2a)$	$(-a, \pm 2a)$	$(\pm 2a, a)$	$(\pm 2a, -a)$
6.	Eccentricity	$e = 1$	$e = 1$	$e = 1$	$e = 1$
7.	Latus Rectum	4a	4a	4a	4a
8.	Equation of L.R	$x = a$	$x + a = 0$	$y = a$	$y + a = 0$
9.	Parametric Equation	$x = at^2$ $y = 2at$	$x = -at^2$ $y = 2at$	$x = 2at$ $y = at^2$	$x = 2at$ $y = -at$

Notation

1. Eccentricity =
2. The Standard equation of a parabola which is symmetrical about the x-axis. The axis of the parabola is the x-axis while the vertex is at the origin, (0,0).

Examples

1. Given a parabola $y^2 = 12x$, find the focus and the directrix given the vertex is at the origin.

Solution

The given equation:
By comparison with the standard equation.
Standard equation
Hence, focus = f(a,0) i.e. the focus to the equation $y^2=4ax$.
Hence, focus = f(3,0) and directrix is $x = -a$ which is $x = -3$.

2. Find the coordinates of the focus and the vertex. The equation of directrix and the axis, and the length of lactus rectum of the parabola $x^2 = 6y$.

Solution

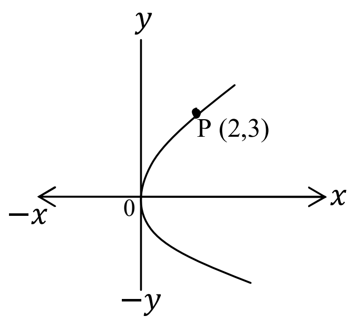
The given equation:
By comparing with the standard from of the equation
Standard equation

So, in this case, since a is positive and the equation depend on y-axis, then the parabola is upward and its focus is f (0,a) i.e. f (0,3) , its vertex is at origin i.e. A (0,0)
The equation of directrix is $y = -a$
Therefore, $y = \frac{-3}{2}$ implies that $2y + 3 = 0$.
Its axis $y - axis$, whose equation is $x = 0$
Length of latus rectum = $4a = 4 \times \frac{3}{2} = 6units.$

3. Find the equation of the parabola with vertex at origin, the axis along $x - axis$ and passing through the point $P(2,3)$.

<u>*Solution*</u>

It's given that the vertex is at origin i.e. $A(0,0)$ and it's axis lies along the $x - axis$.
The graph is shown below:



The equation is $y^2 = 4ax$. $P(2,3)$ lies on it, therefore
$3^2 = 4a \times 2 \Rightarrow a = \frac{9}{8}$.
Hence, the required equation is
$y^2 = 4\left(\frac{9}{8}\right)x \Rightarrow y^2 = \frac{9}{2}x.$

<u>*Explanation*</u>

Recall that your coordinate is always in form of (x, y) then $y = 3$ & $x = 2$. From your formula $y^2 = 4ax$; since we are solving for "a", then we substitute the value of x and y in our equation. After getting the value for "a" we substitute it in the equation without the value of y and x.

4. Find the equation of the parabola whose axis is the x-axis with vertex at origin and position of its focus is at (3.0). Find also the equation of directrix.

<u>*Solution*</u>

Since the Parabola is symmetric at x-axis, the standard equation is $y^2 = 4ax$ and vertex = $A(0,0)$. Focus is given as $f(3,0) \Rightarrow a = 3$.
$y^2 = 4ax \Rightarrow y^2 = 4 \times 3 \times x = 12x$
$\therefore$ The required equation is $y^2 = 12x$.
Equation of directrix would be $x = -a$
then $x = -3$

5. Find the equation of parabola with vertex at the origin, through the point (3,-4) and symmetric about the y-axis.

<u>*Solution*</u>

It is given the vertex of the parabola is the origin and it is symmetric about y-axis.

So, the equation is $x^2 = 4ay \; or \; x^2 = -4ay$
Since it passes through the point P(3,-4), it lies within the 4th quadrant, i.e. y is negative.
Then the equation will be $x^2 = -4ay$.
We have, $3^2 = -4 \times a \times -4$

$$9 = -16a \;\Rightarrow a = \frac{9}{16}$$

The required equation is $x^2 = -4ay$

$$x^2 = -4 \times \frac{9}{16} \times y$$

$\Rightarrow 4x^2 + ay = 0.$

6. If the lactus rectum of a parabola is $\frac{\sqrt{7}}{3}$, find it's focal length.

   a. $\frac{\sqrt{7}}{12}$ b. $\frac{12}{\sqrt{7}}$ c. $-\frac{\sqrt{7}}{12}$ d. $-\frac{12}{\sqrt{7}}$

Solution

Recall that, Lactus Rectum = L.R = 4a

$\Rightarrow 4a = \frac{\sqrt{7}}{3}$

$12a = \sqrt{7}$

$$a = \frac{\sqrt{7}}{12} \;\; (a)$$

7. The focus of the equation $y^2 - 3x = 0$, is?

Solution

$y^2 - 3x = 0$
$y^2 = 3x$; By comparing with $y^2 = 4ax$ then,
$y^2 = y^2 \Rightarrow 3x = 4ax \Rightarrow a = \frac{3}{4}$
the focus is f $(\frac{3}{4}, 0)$

At General Point (h, k)

To obtain the equation of parabola whose vertex is at general point (h,k), we need to reconstruct the coordinate from (x,y) to $(x - h, y - k)$ then the equation of parabola will be:

$(y - k)^2 = 4a(x - h)$ or $(x - h)^2 = 4a(y - k)$ and the focus will be

$f(a + h, k) \; or \; f(h, k + a)$ respectively.

The equation of directrix will be $(x - h) = -a \;\Rightarrow y - k = -a \; or \; y = k - a$ respectively.

Then the equation of the axis will become $y = k$.

General Equation of Parabola

A parabola is an equation represented either quadratic in x and linearly in y or quadratic in y and linear in x.

The general equation of a parabola is given as:

$ax^2 + bx + cy + d = 0$ or

$ay^2 + bx + cy + d = 0$

Let's choose any of the equation above and deduce it to semi-reduce form/standard form.

Taking $ax^2 + bx + cy + d = 0$, we are going to apply completing the square method to convert it to standard form.

$ax^2 + bx + cy + d = 0$

collecting like terms

$ax^2 + bx = -cy - d$

divide through by a

$$\frac{ax^2}{a} + \frac{bx}{a} = -\frac{c}{a}y - \frac{d}{a}$$

$$\Rightarrow x^2 + \frac{bx}{a} = -\frac{c}{a}y - \frac{d}{a}$$

Taking the coefficient of x by multiplying it by half and take the square of the result.

$$\left(\frac{b}{a} \times \frac{1}{2}\right)^2 = (\frac{b}{2a})^2$$

Add $\left(\frac{b}{2a}\right)^2$ to both sides.

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a}y - \frac{d}{a} + \left(\frac{b}{2a}\right)^2$$

$$= \left(x + \frac{b}{2a}\right)^2 = \frac{-c}{a}y - \frac{d}{a} + \frac{b^2}{4a^2}$$

$$\left(x+\frac{b}{2a}\right)^2=\frac{-c}{a}\left(y+\frac{4ad-b^2}{4ac}\right)$$

This is the standard equation of parabola

Examples:-

8. Given that $x^2-6x-8y+1=0$. Transform the equation to standard form and give full description of the parabola.

Solution

Given equation $x^2-6x-8y+1=0.\Rightarrow x^2-6xx=8y-1$

By using completing the square method. Take the half of co-efficient of x and square the result.

$\left(-6\times\frac{1}{2}\right)^2=(-3)^2$

Add $(-3)^2$ to both sides of the equation.

$X^2-6x+(-3)^2=8y-1+(-3)^2$

$(x-3)^2=8y-1+9$

$(x-3)^2=8y+8$

$(x-3)^2=8(y+1)$

The description goes thus:

By comparing $(x-3)^2=8(y+1)$ with $(x-h)^2=4\text{a}(y-k)$

Vividly, we can see that,

$(x-h)=(x-3)$, therefore, h = 3

And $\text{y}+1=\text{y}-\text{k}$ therefore, k = -1

The vertex at (h,k) is (-3, -1) and

$4\text{a}=12$, therefore, $a=\frac{8}{4}=2$

The parabola opens upwards since its symmetric to y-axis. The focus unit is 2. Since a = 2.

The focus is (h, k+a) giving (3, -1+2)

The focus therefore is (3, 1). The directrix will be y = -3 and the axis equals x = 3.

Lactus retum = $4\Leftrightarrow 4\times 2\Rightarrow 8$

9. If $y^2 - 2y - 8x - 7 = 0$ is a parabolic equation, find the focus.

Solution

Given the equation: $y^2 - 2y - 8x - 7 = 0$

$\Rightarrow y^2 - 2y = 8x + 7$

$\Rightarrow y^2 - 2y + (-1)^2 = 8x + 7 + (-1)^2$

$\Rightarrow (\text{y} - 1)^2 = 8x + 8$

$\Rightarrow (y - 1)^2 = 8(x + 1)$

By comparing with the original equation

$(x - h)^2 = 4\text{a}(y - k)$

$\Rightarrow 4a = 8, a = 2$

$x - h = -a$

$x = h - a$

$x = -1 - 2$

$x = -3$

Focus = $f(a + h, k)$

$f(2 + (-1), 1)$

$= f(1,1)$

The focus is (1,1)

10. Describe the graph of the equation.
$y^2 - 8 - 6y - 23 = 0$
Solution
Given equation $y^2 - 8 - 6y - 23 = 0$
By using completing the square method.
$y^2 - 6y = 8x + 23$
We complete the square on the y-terms by adding $(-3)^2$ to both sides
$y^2 - 6y + (-3)^2 = 8c + 23 + (-3)^2$
$\Rightarrow (y - 3)^2 = 8x + 23 + 9$
$\Rightarrow (y - 3)^2 = 8x + 32$
$\Rightarrow (y - 3)^2 = 8(x + 4)$
$\therefore$ the equation is in the form of

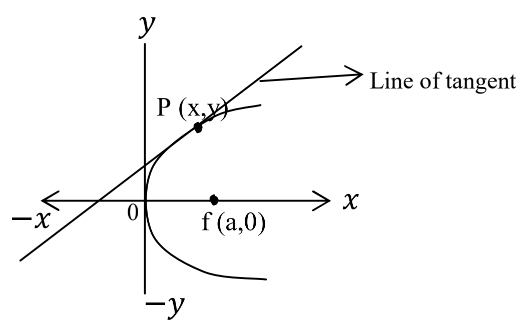
$(y - k)^2 = 4\text{a}(\text{x} - \text{h})$

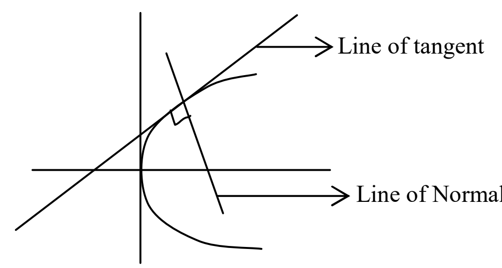
By comparism

$h = 4, k = 3 \; and \; a = 2$

the graph is parabola with vertex $(-4,3)$ opening to the right. Since a = 2 and the focus unit is 2 to the right vertex which places it at the point (-2, 3), then the directrix is 2unit to the left of the vertex which means that its equation is x = -6.

Tangents and Normals of a Parabola





From the standard equation of parabola: $y^2 = 4ax$ the slope of the parabola at any point p(x,y).

By differentiating $y^2 = 4ax$ with respect to x; where a is constant.

$\Rightarrow y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$

$\therefore \frac{dy}{dx} = \frac{2a}{y}$

At point p(x,y)

$\left.\frac{dy}{dx}\right|_{(x,y)} = \frac{2a}{y} = \frac{2a}{y'}$.

From the Cartesian coordinate; recall the equation o straight line at a given point

i.e. $y - y_1 = m(x - x_1)$.

Where m is the gradient/slope. Which is $\frac{dy}{dx}$.
Since $m = \frac{dy}{dx} = \frac{2a}{y'}$.

Therefore, $y - y_1 = \frac{2a}{y'}(x - x_1)$

By cross multiplying,

$y - y_1 = \frac{2a}{y'}(x - x_1)$

$y_1 y - y^2 = 2a(x - x_1)$ _________ i.)

The equation i.) is the equation of tangent at vertex (0,0)

EQUATION OF NORMAL

Recall that;

If the slope of tangent is m_1; then the slope of normal will be $m_2 = \frac{-1}{m_1}$. (Line of tangent and line of normal are perpendicular. Hence, the product of their slope is -1.)

Hence, $m_2 = \frac{-1}{\frac{2a}{y'}} = \frac{-y'}{2a}$

Our given equation of slope at one point

$y - y_1 = m(x - x_1)$

$\Rightarrow y - y_1 = m_2(x - x_1)$

But $m_2 = \frac{-y}{2a}$; by substituting,

$y - y_1 = \frac{y'}{2\text{a}}(x - x_1)$

$2a(y - y_1) = -y_1(\text{x} - x_1)$

$2ay - 2ay_1 = -\text{x}y_1 + x_1 y_1$

Hence, $2ay - 2ay_1 + \text{x}y_1 - x_1 y_1 = 0$ _____ii.

The equation ii. is the equation of normal at vertex (0,0).

NB: It's advisable to differentiate any given equation under the equation of tangent and normal and processed your working that to memorize the formula.

Equation Of Tangent And Normal At General Point

Consider the general equation

$ax^2 + bx + cy + d = 0$

By differentiating the general equation above with respect to x where a, b, c and d are constant at $p(x_1, y_1)$

$\Rightarrow ax^2 + bx + cy + d = 0$

$2ax + b + c\frac{dy}{dx} = 0$

$c\frac{dy}{dx} = -2ax - b$ (by dividing both side by c)

Hence, $\frac{dy}{dx} = \frac{-2ax - b}{c}$

at point (x,y)

$\left.\frac{dy}{dx}\right|_{(x,y)} = \frac{-2a(x_1) - \text{b}}{y} = \frac{-2ax_1 - \text{b}}{\text{c}}$

$\frac{dy}{dx} = \frac{-2ax_1 - \mathrm{b}}{\mathrm{c}}$ is the gradient/slope of equation of tangent.

By using $y - y_1 = m(x - x_1)$.

$m = \frac{-2ax_1 - \mathrm{b}}{\mathrm{c}}$

$\Rightarrow y - y_1 = \frac{-2ax_1 - \mathrm{b}}{\mathrm{c}}(x - x_1)$

By cross multiplying,

$c(y - y_1) = (-2ax_1 - \mathrm{b})(x - x_1)$

$cy - cy_1 = -2axx_1 + 2\mathrm{a}x_1{}^2 - \mathrm{b}x + \mathrm{b}x_1$

$cy - cy_1 + 2axx_1 - 2\mathrm{a}x_1{}^2 + \mathrm{b}x - \mathrm{b}x_1 = 0$ _______iii.

The equation iii. is the equation of tangent of parabola in the form of

$ax^2 + bx + cy + d = 0$ at a point p(x,y).

Equation Of Normal At Point P(X,Y)

From $m_1 m_2 = -1$; where m_1 is the gradient of tangent and m_2 is the gradient to normal

$\Rightarrow m_2 = \frac{-1}{m_1}$

By $m_1 = \frac{-2ax_1 - \mathrm{b}}{\mathrm{c}}$

$m_2 = \frac{-1}{\frac{-2ax_1 - \mathrm{b}}{\mathrm{c}}} = \frac{-\mathrm{c}}{-2ax_1 - \mathrm{b}} = \frac{\mathrm{c}}{2ax_1 + \mathrm{b}}$

Hence; the gradient of normal is $m_2 = \frac{\mathrm{c}}{2ax_1 + \mathrm{b}}$

By substituting m_2 with m_1 in equation $y - y_1 = m(x - x_1)$,

We have $y - y_1 = m_2(x - x_1)$

$\Rightarrow y - y_1 = \frac{\mathrm{c}}{2ax_1 + \mathrm{b}}(x - x_1)$

By cross multiplying,

$(2ax_1 + \mathrm{b})(y - y_1) = \mathrm{c}(x - x_1)$

$2ax_1\mathrm{y} - 2ax_1y_1 + \mathrm{by} - by_1 - \mathrm{c}x + cx_1 = 0$ _______iv.

Equation iv. is the equation normal to the parabola in the form of

$ax^2 + bx + cy + d = 0$ at point $p(x_1, y_1)$.

NB: It's advisable to be differentiating the given equation than to memorize the formula.

Example 11.

Find the equation of tangent and normal of $x^2 - 4x - 4y + 8 = 0$ at point (0,2).

Solution

$x^2 - 4x - 4y + 8 = 0$

$2x - 4 - 4y - 4\frac{dy}{dx} = 0$

$-4\frac{dy}{dx} = 4 - 2x$

$\therefore \frac{dy}{dx} = \frac{4-2x}{-4}$

at point (0,2)

$\frac{dy}{dx} = \frac{4-2(0)}{-4} = -1$; hence, $\frac{dy}{dx} = -1$

Recall that, $y - y_1 = m(x - x_1)$

where $\mathrm{m} = \frac{dy}{\mathrm{dx}} = -1$

$\Rightarrow y - 2 = -1(x - 0)$

$y - 2 = -x$

$y + x = 2$

Slope of normal $= \frac{-1}{slope\ of\ tangent}$

$m = \frac{-1}{m}$

$m = \frac{-1}{-1} = 1$

$\Rightarrow y - y_1 = m(x - x_1)$

$y - 2 = 1(x - 0)$

$y - 2 = x$

$\underline{\underline{y = x + 2}}$

Example 12.

Find the equation of tangent and normal to the parabola equation $3x^2 - 6x - 12y = 0$ at point p(0,-2).

Solution

To find equation of tangent, firstly differenciate the given equation

$3x^2 - 6x - 12y = 0$

$6x + 6 - 12\frac{dy}{dx} = 0$

$6x + 6 = 12\frac{dy}{dx}$

Divide through by 12

$\frac{dy}{dx} = \frac{6x+6}{12} = \frac{x+1}{2}$

$\therefore \frac{dy}{dx} = \frac{x+1}{2}$

at p(0,-2) i.e. $x = 0$ and $y = -2$

$\frac{dy}{dx} = \frac{0+1}{2} = \frac{1}{2}$

Slope at a point $m = \frac{y-y_1}{x-x_1}$

$\frac{dy}{dx} = \frac{y-y_1}{x-x_1}$

$\frac{1}{2} = \frac{y-y_1}{x-x_1}$

$\frac{1}{2}(x - 0) = (y - (-2))$

$\frac{x}{2} = y + 2$

$x = 2(y + 2)$

$\Rightarrow 2y - x + 4 = 0$

To find the Equation of Normal

Note: Line of Tangent and Normal are perpendicular to each other then, the product of their slope is equal to -1.

$\Rightarrow m_1 m_2 = -1$

where m_1 is the slope of tangent and m_2 is the slope of normal

$\Rightarrow m_2 = \frac{-1}{m_1}$

$\frac{dy}{dx} = \frac{1}{2}$ i.e. $m_1 = \frac{1}{2}$

$m_2 = \frac{-1}{m_1} = \frac{-1}{\frac{1}{2}} = -2$

$\Rightarrow$slope at a point

$m = \frac{y-y_1}{(x-x_1)}$

$-2 = \frac{y-(-2)}{(x-0)}$

$-2(x) = (y + 2)$

$-2x = y + 2$

$\Rightarrow y + 2x + 2 = 0$

Note: The eccentricity of Parabola is always equal to 1.

Example 13:

Find the equation of tangent and normal to the parabola

$y^2 = 4x$ at (1,1)

Solution

$y^2 = 4x$

by differentiating the given equation

$2y\frac{dy}{dx} = 4$

$\frac{dy}{dx} = \frac{4}{2y} = \frac{2}{y}$

at point (1,1),

$\frac{dy}{dx} = \frac{2}{1} = 2$

Slope at a point $m = \frac{y-y_1}{x-x_1}$

$2 = \frac{y-1}{x-1}$

$2(x-1) = (\mathrm{y}-1) \Rightarrow 2\mathrm{x}-2 = \mathrm{y}-1$

$y - 2x + 1 = 0$ __________(i)

Slope of normal = $\frac{-1}{m_1} \Rightarrow \frac{-1}{2}$

Slope at a point $m = \frac{y-y_1}{x-x_1}$

$-1 = \frac{y-1}{x-1}$

$-1(x-1) = 2(y-1)$

$-x + 1 = 2y - 2$

$2y + x - 3 = 0$ ____________(ii)

Hence equation (i) is the equation of tangent and equation (ii) is the equation of normal.

Note:- Line of tangent is the line that touches one point of a circumference of a curve.

Example 14:

Find the equation of tangent and normal to the equation $3y^2 + 6y - 12x = 0$ at point (1,-2).

Solution

To find the equation of tangent, first differentiate the given equation.

$3y^2 + 6y - 12x = 0$

$6y\frac{dy}{dx} + 6\frac{dy}{dx} - 12 = 0$

$\frac{dy}{dx}(6\mathrm{y} + 6) = 12$

$\frac{dy}{dx} = \frac{12}{6y+6}$

$\frac{dy}{dx} = \frac{2\times6}{6(y+1)} = \frac{2}{y+1}$

$\left.\frac{dy}{dx}\right|_{(1,-2)} = \frac{2}{y+1} = \frac{2}{-2+1} = \frac{2}{-1}$

$\frac{dy}{dx} = -2$

Slope at point $m = \frac{y-y_1}{x-x_1}$

$y - y_1 = m(x - x_1)$

$(y - -2) = -2(x-1)$

$y + 2 = -2x + 2$

$y + 2x = 0$ ________(*)

Equation (*) is the equation of tangent.

To find Equation of Normal

Recall that; $m_1 m_2 = -1$

$m_2 = \frac{-1}{m_1}$; where $m_1 \rightarrow slope\ of\ tangent\ \&$

$m_2 \rightarrow slope\ of\ Normal.$

$m_2 = \frac{-1}{-2}$

$m_2 = \frac{1}{2}$

⇒slope at a point

$y - y_1 = m(x - x_1)$

$(y - -2) = \frac{1}{2}(x - 1)$

$(y + 2) = \frac{1}{2}(x - 1)$

$2(y + 2) = x - 1$

$2y + 4 = x - 1$

$2y - x + 5 = 0$ ________(ii)

Equation (ii) is the equation of Normal.

Condition for a Straight Line to Touch a Parabola

Recall that; Equation of Parabola at general points at (h,k).

⇒$(y - k)^2 = 4a(x - h)$_________(i)

Recall the equation of straight line

$y = mx + c$ __________(ii)

Substitute y in equation (i)

$(mx + c - k)^2 = 4a(x - h)$

Expand the bracket.

$(mx)^2 + c^2 + k^2 - 2mxk + 2mxc - 2ck = 4ax - 4ah$

$(mx)^2 + c^2 + k^2 - 2mxk + 2mxc - 2ck - 4ax + 4ah = 0$

$(mx)^2 + 2(mc - mk - 2a)x + c^2 + k^2 - 2ck + 4ah = 0$

Let $A = (m)^2$

B= $2(mc - mk - 2a)$

C= $(c^2 + k^2 - 2ck + 4ah)$

Hence we have,

$Ax^2 + Bx + C = 0$

⇒for real root in quadratic equation, we have,

$b^2 - 4ac = 0$

Hence, $b^2 = 4ac$

$a = A, \;\; b = B \; and \;\; c = C$

⇒$B^2 = 4AC$

But $B = 2(mc - mk - 2a)$

$A = (m)^2$

$C = c^2 + k^2 - 2ck + 4ah$

By substituting in $B^2 = 4AC$

⇒$[2(mc - mk - 2a)]^2 = 4m^2[(c^2 + k^2 - 2ck + 4ah)]$

$4[(mc)^2 + (mk)^2 + 4a^2 - m^2ck - 4amc + 4amk] = 4m^2c^2 + 4m^2k^2 - 8m^2ck + 16m^2ah$

⇒$4\,(mc)^2 + 4(mk)^2 + 16a^2 - 8m^2ck - 16amc + 16amk = 4m^2c^2 + 4m^2k^2 - 8m^2ck + 16m^2ah$

⇒$16a^2 - 16amc + 16amk - 16m^2ah = 0$

Divide all through by $16a$

$\frac{16a^2 - 16amc + 16amk - \quad {}^2ah = 0}{16a} = 0$

⇒$a - cm + mk - hm^2 = 0$

Make "c" the subject of the formula

$-cm = hm^2 - mk - a$

Multiply through by -1

$cm = a - hm^2 + mk$

Divide through by m

$c = \frac{a}{m} - hm + k$ ______(i)

Equation (i) is the condition for which line $y = mx + c$ is a tangential to the parabola $(y-k)^2 = 4a(x-h)$.

Note:- If the vertex is at origin i.e. (0,0), the condition will be $c = \frac{a}{m}$;just substitute $h = 0$ and $k = 0$ in equation (i).

The equation of tangent will be $y = mx + \frac{a}{m}$.

Equation at General Point (p,q)

$y = mx + \frac{a}{m}$, but $x = p$ and $y = q$;

$q = mp + \frac{a}{m}$ $\Rightarrow q = \frac{m^2p+a}{m}$

$\Rightarrow mq = m^2p + a$ $\Rightarrow m^2p - qm + a = 0$

Example 15:

Show whether or not $y = mx + \frac{3}{4}m + \frac{1}{m}$ is a tangent to the parabola $y^2 = 4x + 3$.

Solution:

Equation given:

$y = mx + \frac{3}{4}m + \frac{1}{m}$

Equation of straight line:

$y = mx + c$

By comparison,

$c = \frac{3}{4}m + \frac{1}{m}$ _______(i)

Also, the equation of parabola given:

$y^2 = 4x + 3$

Writing in standard form

$y^2 = 4(x + \frac{3}{4})$

By comparing with $(y-k)^2 = 4a(x-h)$,

$k = 0, h = \frac{-3}{4}$

$4a = 4; a = 1$

Using the condition and substituting our values,

$c = \frac{a}{m} - hm + k$

$c = \frac{1}{m} - (\frac{-3}{4})m + 0$

$c = \frac{1}{m} + \frac{3}{4}m$ ______(ii)

Since equation (i) and (ii) are equal then the line is a tangent to the parabola.

Example 16:

Find the condition that the line $lx + my + n = 0$ may touch the parabola $y^2 = 4ax$.

Solution:

Suppose the line $lx + my + n = 0$ touches the parabola $y^2 = 4ax$ at point (x_1y_1).

Recall: The equation of tangent at (x_1y_1) is

$yy_1 = 2a(x + x_1)$

$\Rightarrow yy_1 = 2ax + 2ax_1$

$\Rightarrow 2ax - yy_1 + 2ax_1 = 0$

By comparing the given equation $lx + my + n = 0$ and equation of tangent $2ax - yy_1 + 2ax_1 = 0$,

$2ax = lx; my = yy_1 and\ 2ax_1 = n$

$l = 2a; m = y_1\ and\ x_1 = \frac{n}{2a}$

From $x_1 = \frac{n}{2a}$, since $c = 2a$

Hence, $x_1 = \frac{n}{c}$ and $y_1 = \frac{-2am}{c}$

Since (x_1, y_1) lies on the parabola $y^2 = 4ax$, it gives ${y_1}^2 = 4ax_1$

By substituting $x_1\ and\ y_1$

$\left(\frac{-2am}{c}\right)^2 = 4a\left(\frac{n}{c}\right)$

$\frac{4a^2m^2}{c^2} = \frac{4a}{c}$

$am^2 = nc$. [which is the required condition for $lx + my + n = 0$ to touch the parabola $y^2 = 4ax$] the point of contact is$\left(\frac{n}{c}, -2\sqrt{\frac{na}{c}}\right)$

Example 17:

Find the equation of tangents from the point (-1,-2) to the parabola $(y + 2)^2 = 12(x - 2)$.

Solution:

If you try to expand the given equation and find the derivative; after substituting point $(-1, -2)$ into $\frac{dy}{dx}$; then $\frac{dy}{dx} = \infty$; so let us do it this way.

Recall that the condition for a line $y = mx + c$ to be a tangential to a parabola $(y - k)^2 = 4a(x - h)$ is

$$c = \frac{a}{m} - hm + k$$

From the given equation: $(y + 2)^2 = 12(x - 2)$

Standard equation: $(y - k)^2 = 4a(x - h)$

By comparing,

$12 = 4a; a = 3$

$h = 2\ and\ k = -2$

Also, the equation of the tangent is

$y = mx + c$ [But $c = \frac{a}{m} - \mathrm{hm} + \mathrm{k}$]

$y = mx + \frac{a}{m} - \mathrm{hm} + \mathrm{k}$

Multiply through by m

$my = m^2x + a - hm^2 + km$

But $a = 3\ \ y = -2\ \ \&\ \ x = -1$. By substituting,

$m(-2) = m^2(-1) + (3) - (2)m^2 - 2m$

$\Rightarrow 3m^2 - 3 = 0$

$\Rightarrow m = \pm 1$

When $3m^2 - 3 = 0$

$c = 3 - 2 + (-2) \qquad \Rightarrow c = -1$

When $m = -1$,

$c = \frac{3}{-1} - 2(-1) + (-2) \qquad \Rightarrow c = -3$

Hence; $m_1 = 1; c_1 = 1\ and$

$m_2 = -1;\ c_2 = -3$

So the equation of the tangent will be

$y = m_1x + c_1$ and $y = m_2x + c_2$

$\Rightarrow y = (1)x + (-1)$ and $y = (-1)x + (-3)$

$\Rightarrow y = x - 1$ and $y = -x - 3$

Hence, $y - x + 1 = 0$ and $y + x + 3 = 0$

Therefore the above equations are the equation tangents to the parabola $(y + 2)^2 = 12(x - 2)$ at point P(-1,-2).

Example 18: Find the equation of the tangents and normals to the parabola $y^2 = 4ax$ at the ends of its latus rectum.

Solution:

Recall from page 5 of this material, the extremilies of the latus rectum are $(a, \pm 2)$ i.e $(a, 2a)$ and $(a, -2a)$.

Equation of tangent at $(a, 2a)$ is

$$y^2 = 4ax$$

$$2y\frac{dy}{dx} = 4a$$

$$\left.\frac{dy}{dx}\right|_{(a,2a)} = \frac{4a}{2y} = \frac{2a}{y} = \frac{2a}{2a} = 1$$

Hence, m=1

But slope at a point is

$$m = \frac{y - y_1}{x - x_1}$$

Then,

$$1 = \frac{(y - 2a)}{(x - a)}$$

$$(x - a) = (y - 2a)$$

$$y - 2a - x + a = 0$$

$$\boxed{y = x + a}$$

Equation of normal at (a, 2a)

$$m_1m_2 = -1$$

$$m_2 = \frac{-1}{m_1} \Rightarrow m_2 = -1$$

$$-1 = \frac{y - 2a}{x - a}$$

$$-1(x - a) = y - 2a$$

$$-x + a = y - 2a$$

$$y - 2a + x - a = 0$$

$$\boxed{y + x - 3a = 0}$$

Similarly, at point (a, -2a)

Tangent: $y = -(x + a)$

Normal: $y - x + 3a = 0$

Note:- The diameter $y = \frac{2a}{m}$; meets the parabola $y^2 = 4ax$ in at the point $(\frac{a}{m^2}, \frac{2a}{m})$. The tangent to the parabola at this point is:

$$y - \frac{2a}{m} = 2a\left(x + \frac{a}{m^2}\right) \text{ or}$$

$$y = mx + \frac{a}{m}$$

Where slope (m) is parallel to the given chord.

Note:- From page 21 of this material, the tangent at (p, q) is given at

$m^2p - ym + a = 0$; which is a quadratic in m given two values of m.

Corresponding to each value of m in equation of the tangent at origin $y = mx + \frac{a}{m}$ gives a tangent through (p, q).

$y = m_1x + c_1$ and $y = m_2x + c_2$

$\Rightarrow y = (1)x + (-1)$ and $y = (-1)x + (-3)$

$\Rightarrow y = x - 1$ and $y = -x - 3$

Hence, $y - x + 1 = 0$ and $y + x + 3 = 0$

Therefore the above equations are the equation tangents to the parabola $(y + 2)^2 = 12(x - 2)$ at point P(-1,-2).

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Solution:

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Equation of tangent at $(a, 2a)$ is

$y^2 = 4ax$

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx}\Big|_{(a,2a)} = \frac{4a}{2y} = \frac{2a}{y} = \frac{2a}{2a} = 1$

Hence, m=1

But slope at a point is

$m = \frac{y - y_1}{x - x_1}$

Then,

$1 = \frac{(y-2a)}{(x-a)}$

$(x - a) = (y - 2a)$

$y - 2a - x + a = 0$

$y = x + a$

Equation of normal at (a, 2a)

$m_1m_2 = -1$

$m_2 = \frac{-1}{m_1} \Rightarrow m_2 = -1$

$-1 = \frac{y-2a}{x-a}$

$-1(x - a) = y - 2a$

$-x + a = y - 2a$

$y - 2a + x - a = 0$

$y + x - 3a = 0$

Similarly, at point (a, -2a)

Tangent: $y = -(x + a)$

Normal: $y - x + 3a = 0$

Note:- The diameter $y = \frac{2a}{m}$; meets the parabola $y^2 = 4ax$ in at the point $(\frac{a}{m^2}, \frac{2a}{m})$. The tangent to the parabola at this point is:

$y - \frac{2a}{m} = 2a\left(x + \frac{a}{m^2}\right)$ or

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Where slope (m) is parallel to the given chord.

Note:- From page 21 of this material, the tangent at (p, q) is given at

$m^2p - ym + a = 0$; which is a quadratic in m given two values of m.

Corresponding to each value of m in equation of the tangent at origin $y = mx + \frac{a}{m}$ gives a tangent through (p, q).

Parametric Equation Representing Parabola

Recall the equation of a parabola: $y^2 = 4ax$

By square rooting both side

$\sqrt{y^2} = \sqrt{4ax}$

$y = 2\sqrt{ax}$ _________(i)

Let t be the parameter then $x = at^2$

But subtituting $x = at^2$ in equation (i)

Then; $y = 2\sqrt{a(at^2)}$.

$y = 2\sqrt{\mathrm{a}^2\mathrm{t}^2}$

$y = 2at$

Hence, the coordinate of the parabola will be

$(at^2, 2at)$.

To verify this:

The point $(at^2, 2at)$ lies on the parabola

$y^2 = 4ax$ for any value of t;

Substitute the value of $x = at^2$ and $y = 2at$. In the equation of parabola $y^2 = 4ax$, it show that

$(2at)^2 = 4a(at^2)$

$4a^2t^2 = 4a^2t^2$

Since, R.H.S=L.H.S, then the point lies on the parabola.

The Gradient/Slope to the Parabola at Point $(at^2, 2at)$

From the equation of Parabola: $y^2 = 4ax$ by differentiating the equation

$2y\frac{dy}{dx} = 4a$

$\frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}$

$\frac{dy}{dx}\Big|_{(a^2,2at)} = \frac{2\mathrm{a}}{2\mathrm{a}} = \frac{1}{\mathrm{t}}$

Then the slope of the tangent is $\frac{1}{\mathrm{t}}$ at point $(at^2, 2at)$.

The equation of tangent will be

$(y - y_1) = m(x - x_1)$

$y - 2at = \frac{1}{\mathrm{t}}(\mathrm{x} - at^2)$

$t(y - 2at) = (x - at^2)$

$yt - 2at^2 = x - at^2$

$ty - 2at^2 - x + at^2 = 0$

$\Rightarrow ty - x - at^2 = 0$ _________(ii)

The slope of the normal will be negative of the reciprocal of the tangent

Therefore, $m_2 = -t$ at $p(at^2, at)$

Equation of Normal;

$y - y_1 = m(x - x_1)$

$y - 2at = -t(x - at^2)$

$y - 2at = -tx + at^3$

$y + tx - 2at - at^3 = 0$ _________(iii)

Both equations (ii) and (iii) are the tangential equation and equation of normal respectively at point $(at^2, 2at)$.

Equation 19:

The conic represented by the equation $x = 4t^2$ and $y = 8t$ is?

Solution:

$x = 4t^2; y = 8t$

Recall that $y^2 = 4ax$.

Substitute the given parameters

$8t^2 = 4a(4t^2)$

$64t^2 = 16at^2$

$16a = 64$

$a = \frac{64}{4} = 4$

Then, substitute the value of a in the parabola

$y^2 = 4(4)x$

$y^2 = 16x$

Example 20:

The conic represented by $x = 6t^2$ and $y = 12t$ is?

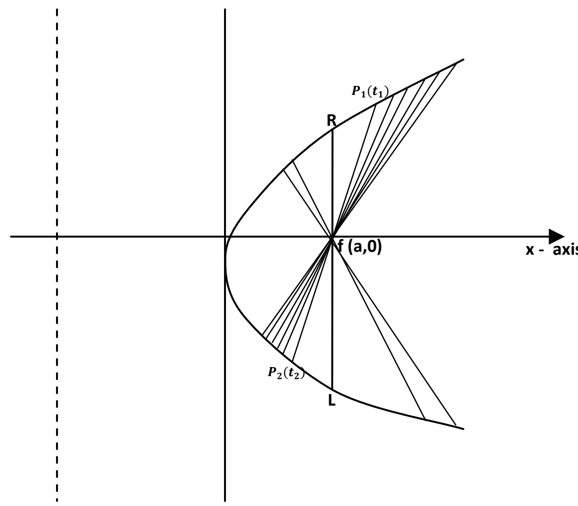
⇒Do this following the same procedure.

CHORD

Any Chord of a parabola passing the focus is called

Focal Chord

When a straight line bisects parabola at more than a point, such line is called *Chord of a Parabola.*



FOCAL CHORDS [Fig. (i)]

A focal chord passes through the point $(at^2, 2at)$.

Let P_1 and P_2 be a point that touches the parabola in opposites sides.

Equation of tangent at point $(at^2, 2at)$ is

$ty = x + at^2$;

Now, the slope of the focal chord will be at point $(a, 0)$ and $(at^2, 2at)$ is

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$\Rightarrow m = \frac{2at - 0}{at^2 - a}$

$m = \frac{2at}{a(t^2-1)} = \frac{2t}{t^2-1}$

The equation of focal chord will now be;

$y - y_1 = m(x - x_1)$

$y - 0 = \frac{2t}{t^2-1}(x - a)$

$y = \frac{2t}{t^2-1}(x - a)$

$y(t^2 - 1) = 2t(x - a)$

$t^2y - y = 2tx - 2at.$

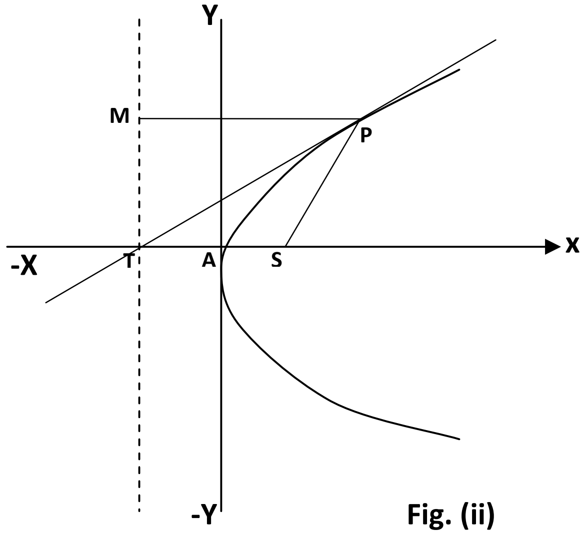


Fig. (ii)

Recall that, the slope of the focal chord is

$m_f = \frac{2t}{t^2-1}$

Also, slope of the tangent at point P$(at^2, 2at)$ is

$M_{PT} = \frac{1}{t}$

From fig. (ii)

Thus $tan\theta = tanSPT$

Recall; the formula to calculate the angle between two lines from Mat103

$tanSPT = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

m_1= slope of focal chord

m_2= slope of point $(at^2, 2at)$

$tanSPT = \left|\frac{\frac{2t}{t^2-1}-\frac{1}{t}}{1+\left(\frac{2t}{t^2-1}\times\frac{1}{t}\right)}\right|$

$= \frac{2t^2-t^2+1}{t(t^2-1)} \times \frac{(t^2-1)}{t^2-1+2}$

$\frac{t^2+1}{t} \times \frac{1}{t^2+1} = \frac{1}{t}$

$tanSPT = \frac{1}{t}$ _________(i)

Hence, the equation above gives,

$< SPT =< TPM.$

Note:

i. The normal at any point of a parabola bisects the angle between the focal chord and the diameter through that point.
ii. The tangents at the extremities of a focal chord of a parabola intersect at right angle on the directrix.

Proof:

Let $P(at_1{}^2, 2at_1)$ and $Q(at_2{}^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$, the equations of tangents at P and Q are;

For P; the equation of tangent is

$t_1y = x + at_1{}^2$

For Q; the equation of tangent is

$t_2y = x + at_2{}^2$ {from the equation of tangent at point $(at^2, 2at)$}

Solving these two equations, we set the point of intersection of the two tangents as

$(at_1t_2, a(t_1 + t_2))$. The equation of the chord PQ is

$y - 2at_1 = \frac{2at_2-2at_1}{at_2{}^2-at_1{}^2}(x - at_1{}^2)$

$\Rightarrow (t_1 + t_2)y = 2(x + at_1t_2).$

But focal chord PQ, passes through the focus f(a, 0) therefore we have, $x = a$ and $y = 0$.

$0 \times (t_1 + t_2) = 2(a + at_1t_2)$

$0 = 2a + 2at_1t_2$

$-2a = 2at_1t_2$

Hence, $t_1t_2 = -1$

Corollary: The extremities of a focal chord of the parabola $y^2 = 4ax$ have co-ordinates $(at^2, 2at)$ and $(\frac{a}{t^2}, \frac{-2a}{t})$ i.e that is they are the point t and $\frac{-1}{t}$.

Note:

The otho centre of the triangle formed by three tangents to a parabola lies on the directrix.

Example 21:

Find the locus of the points of intersection of two tangents of the parabola $y^2 = 4ax$, which are at right angles to each other.

Solution:

Let equation of the two tangents be

$y = m_1x + \frac{a}{m_1}$ and $y = m_2x + \frac{a}{m_2}$

⇒ Since the two are at right angles to one another. i.e they are perpendicular to each other.

$m_1m_2 = -1$

Which is $m_2 = \frac{-1}{m_1}$. Thus the two tangents will have equations:

$y = m_1x + \frac{a}{m_1}$ and

$y = \frac{-1}{m_1}x - am_1$

To see the locus of the point of intersection, we solve two equations.

$y = m_1x + \frac{a}{m_1}$

$y = \frac{-1}{m_1}x - am_1$

$0 = m_1x - \left(\frac{-1}{m_1}x\right) + \frac{a}{m_1} - -am_1$

$\Rightarrow 0 = \left(m_1 + \frac{1}{m_1}\right)x + \frac{a}{m_1} + am_1$

$\Rightarrow x\left(m_1 + \frac{1}{m_1}\right) + a\left(m_1 + \frac{1}{m_1}\right) = 0$

Where $m_1 + \frac{1}{m_1} = 1$

Hence, we have $x + a = 0$, which is the required locus.

Note that the locus is the equation of the directrix.

OVERVIEW QUESTION

Example 22:

Find the co-ordinate of the vertex, focus and the equations of the directrix and the axis of the parabola $x^2 + 20y + 4x = 56$.

Solution:

The equation given

$x^2 + 20y + 4x = 56$

$x^2 + 4x = 56 - 20y$

By completing the square method.

$(x+2)^2 = 4 + 56 - 20y$

$(x+2)^2 = 60 - 20y$

$(x+2)^2 = -20(y-3)$

By comparing with the original equation

$(x-h)^2 = 4a(y-k)$

$h = -2$ and $k = 3$

hence, the vertex is (-2, 3)

the equation of the axis is

$x = h \Rightarrow x = -2$

$\Rightarrow x + 2 = 0$

By comparing; $4a = -20$, $a = -5$

The equation of directrix is

$y - k = a \Rightarrow y - 3 = 5 \Rightarrow y = 8$

Now, y co-ordinate of focus should be (-a, 0)

Hence, focus is -5.

Original axes are;

$-a + k \Rightarrow -5 + 3 = -2$

Hence, the focus has the co-ordinates (-2, -2) i.e. (h, k-a)

Example 23: Try this

Find the coordinate of the vertex, focus and the equation of the directrix and the axis of the parabola

$y^2 + 2y + 4x + 5 = 0.$

Firstly reduce it to semi-reduced form/standard form.

Example 24:

Find the equation of a parabola whose focus is the point (-1, 1) and the directrix is the line $x + y = -1.$

<u>*Solution:*</u>

Let $f(-1,1)$ be the focus

Let's take $P(x, y)$ and point on the parabola.

If line PM is the perpendicular to the directrix:

By definition

$PM = PF$

$\Rightarrow x + y = -1 \Rightarrow x + y + 1 = 0$

Recall that to find the equation of parabola when the coordinate of a focus was given and the equation directrix to be (a, b) and $ax + by + c = 0$ respectively at point (x_1, y_1) then we have

$\frac{ax+by}{\sqrt{a^2+b^2}} = D$

From coordinate geometry

$D = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

Hence,

$\frac{ax+by}{\sqrt{a^2+b^2}} = \sqrt{(x-x_1)^2 + (y-y_1)^2}$

Apply this to the question

$\Rightarrow \frac{x+y+1}{\sqrt{(1)^2+1^2}} = \sqrt{(x-(-1))^2 + (y-1)^2}$

$\frac{x+y+1}{\sqrt{1+1}} = \sqrt{(x+1)^2 + (y-1)^2}$

Square both sides

$\Rightarrow \frac{(x+y+1)^2}{2} = (x+1)^2 + (y-1)^2$

$\Rightarrow (x+y+1)^2 = 2[(x+1)^2 + (y-1)^2]$

By expanding and simply

$\Rightarrow (x-y)^2 + 2x - 6y + 3 = 0$

This is the required equation of the parabola.

Example 25: Try this

Find the equation of the parabola with it vertex at (-2, 3) and the focus at (-7, 3).

NOTE: In the formula apply in the question above; the "a" and "b" in the square root in the denominator of L.H.S is the co-efficient of x and y respectively from the equation directrix.

THE ELLIPSE AND IT'S PROPERTIES

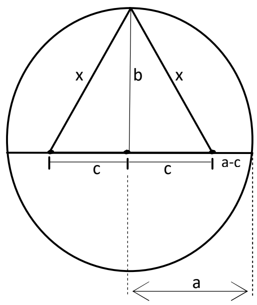
An ellipse is the locus of a point which moves in such a way that it's distance for a fixed point (focus) bear a constant (e) i.e eccentricity (0<e<1) to it's distance from a fixed like the directrix not passing through the focus.

Note:

1. The two fixed points are called foci of the ellipse
2. The midpoint of line segment joining the foci is called the centre
3. The end points of the major axis are called vertices
4. The length of major axis is 2a (when a and b are major and minor)
5. The length of minor axis is 2b (when a and b are major and minor)
6. The distance between the foci is 2c
7. The semi-major is a (when a & b are the major & minor respectively)
8. The semi-minor is b (when a & b are the major & minor respectively)
9. The eccentriticity is always less than 1 (i.e. 0<e<1)

EQUATION OF ELLIPSE

Since "a" and "b" are numbers, not geometric axes, there is a basic relationship between a, b and c that can be obtained by examining the sum of the distance to the foci from point P at the major axis from and Q at the end of the minor axis i.e.



By using Pythagoras's theorem

$x^2 = b^2 + c^2$

$x = \sqrt{b^2 + c^2}$

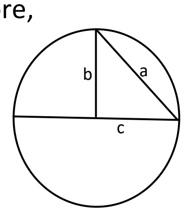
Since a is also a radius and $x \leq a$, by considering the equality sign

$a = \sqrt{b^2 + c^2}$ _______(i)

From equation (i)

$c = \sqrt{a^2 - b^2}$ _______(ii)

Therefore,

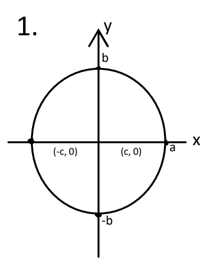


From equation (i) $a \geq b$. The equality holds only when $c \geq 0$.

The equation of an ellipse is simplest if the center of origin and the foci are on the $x - axis$ or $y - axis$.

The two possible outcomes are showed below:

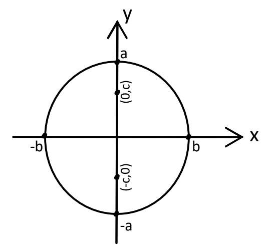
1.



$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$

When "a" is major axis

2.

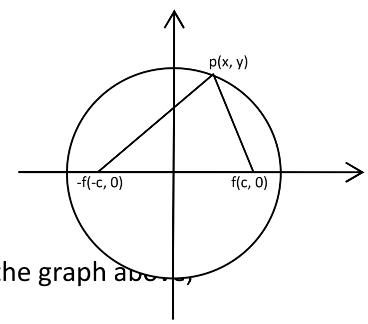


$\frac{x^2}{y^2} + \frac{y^2}{b^2} = 1$

When "b" is the major axis.

Proof of the Equation Of An Ellipse

Consider the following graph.



From the graph above,

$pf' + pf = 2a$

The distance between $pf' = \sqrt{(\mathrm{x} + \mathrm{c})^2 + \mathrm{y}^2}$

The distance between $pf = \sqrt{(\mathrm{x} - \mathrm{c})^2 + \mathrm{y}^2}$ {using $0 = \sqrt{(\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2}$ from coordinate geometry}

$\Rightarrow$
$pf' + pf =$
$\sqrt{(\mathrm{x} + \mathrm{c})^2 + \mathrm{y}^2} + \sqrt{(\mathrm{x} - \mathrm{c})^2 + \mathrm{y}^2} = 2a$

Transposing the second radical to the right sides of the equations

$\Rightarrow \sqrt{(\mathrm{x} + \mathrm{c})^2 + \mathrm{y}^2} = 2\mathrm{a}\sqrt{(\mathrm{x} - \mathrm{c})^2 + \mathrm{y}^2}$

Square both sides

$(\mathrm{x} + \mathrm{c})^2 + \mathrm{y}^2 = 4\mathrm{a}^2 - 4\mathrm{a}\sqrt{(\mathrm{x} - \mathrm{c})^2 + \mathrm{y}^2} + (\mathrm{x} - \mathrm{c})^2 + \mathrm{y}^2$

Collecting the like terms

$4a^2 - 4a\sqrt{(x-c)^2 + y^2} = (x+c)^2 + y^2 - (x-c)^2 - y^2$

$4a^2 - 4a\sqrt{(x-c)^2 + y^2} = x^2 + 2cx + c^2 - [x^2 - 2cx + c^2]$

$4a^2 - 4a\sqrt{(x-c)^2 + y^2} = 4cx$

$-4a\sqrt{(x-c)^2 + y^2} = 4cx - 4a^2$

$\Rightarrow 4a\sqrt{(x-c)^2 + y^2} = 4a^2 - 4cx$

Divide through by $4a$

$\sqrt{(x-c)^2 + y^2} = a - \frac{c}{a}x$

Square both sides

$(x-c)^2 + y^2 = \left(a - \frac{c}{a}x\right)^2 \Rightarrow x^2 - 2cx + c^2 + y^2 = a^2 - 2cx + \frac{c^2}{a^2}x^2$

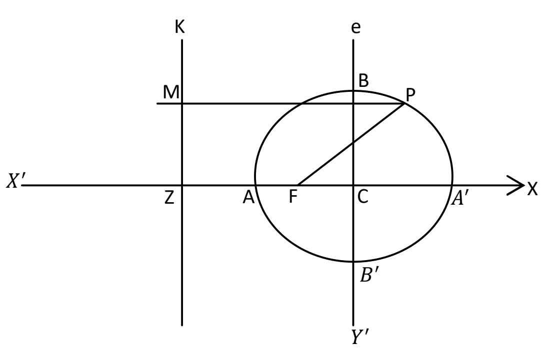
But $b^2 = a^2 - c^2$

Hence, we have

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ______*Proved*

There are many ways of proving it.

Second way of proving Equation of an Ellipse



From the diagram above, let F be the focus, ZK be the directrix and e be the eccentricity of the ellipse. Since $0 < e < 1$; we can divide ZF both internally and externally in the same ratio and the two points will be on the same side with ZK. Let the points of division be A and A' respectively.

$AF = e \times ZA$ ________(i)

$FA' = e \times ZA'$ ________(ii)

By definition, the points A and A' lie on ellipse. Let the C be the middle point of AA' and let $AC = CA' = a$.

Equation (i) and (ii) can be formed as

$AC - FC = e(CZ - CA) \Rightarrow a - FC = e(CZ - a)$

$FC - CA' = e(CZ + CA') \Rightarrow FC + a = e(CZ + a)$

By addition, $2a = 2e \times CZ$ or $CZ = \frac{a}{e}$.

From $a - FC = e(CZ - a)$

$FC + a = e(CZ + a)$

$-2FC = -2ea$

$\Rightarrow FC = ae$

The co-ordinate of the focus S are $(-ae, 0)$ and the equation directrix ZK is $x = \frac{-a}{e}$ [where "a" is the major axis]

Take P(x, y) any point on the ellipse.

$PF = ePM$

Square both sides

$|PF|^2 = e^2|PM|^2$

The distance $|PF|^2 = (x + (-ae))^2 + (y - 0)$ at $(-ae, 0)$, (x_1, y_1)

$|PF|^2 = (x + ae)^2 + y^2$

$\Rightarrow |PF|^2 = e^2|PM|^2$

$(x + ae)^2 + y^2 = \left(e^2 \times \frac{a}{e}\right)^2$

$x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$\Rightarrow \frac{x^2(1-e^2)}{a^2(1-e^2)} + \frac{y^2}{a^2(1-e^2)} = \frac{a^2(1-e^2)}{a^2(1-e^2)}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Note: $[1 - e^2 > 0 \; as \; e < 1]$

If we put $x = 0$ in the equation of ellipse, we get $y = \pm b$ which gives the intercepts CB and CB' on the $y - axis$.

Example 26:

An ellipse has a major axis of 22 an minor axis of 12. A possible equation for the ellipse is?

Solution:

Major axis $\Rightarrow 2a = 22; a = 11$

Minor axis $\Rightarrow 2b = 12; b = 6$

Recall the equation ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

By substituting $a = 11$ and $b = 6$

$\frac{x^2}{(11)^2} + \frac{y^2}{6^2} = 1$

$\frac{x^2}{121} + \frac{y^2}{36} = 1$

Example 27:

An ellipse axis 20 and 12. Find the equation of the ellipse.

Solution:

The major axis is the longest sides which 20

$2a = 20; a = 10$

The minor axis is the long side which is 12.

$2b = 12; b = 6$

By substituting $a = 10$ and $b = 6$ in equation of the ellipse.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \;\Rightarrow\; \frac{x^2}{(10)^2} + \frac{y^2}{6^2} = 1$

$\frac{x^2}{100} + \frac{y^2}{36} = 1$.

Example 28:

Find the four vertices and foci of the ellipse.

$\frac{x^2}{9} + \frac{y^2}{25} = 1$

Solution:

By comparing $\frac{x^2}{9} + \frac{y^2}{25} = 1$ with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$9 = a^2; a = \pm 3$

$25 = b^2; b = \pm 5$

"b" is the major axis. Hence the four vertices are $V_1(0,5), V_2(0,-5), V_3(3,0) \; and \; V_4(-3,0)$

Follow the patterns of $V_{1,2}(0, \pm a) and \; V_{3,4}(\pm b, 0)$

Since $c^2 = b^2 - a^2$ [where a is major]

but when "b" is major, we have;

$c^2 = a^2 - a^2$

$c^2 = (\pm 5)^2 - (\pm 3)^2$

$c^2 = 25 - 9$

$c^2 = 16$

$c = \pm 4$

Hence, the foci are $F_1(0,4) and\ F_2(0,-4)$

Example 29:

Find

(i) The co-ordinate of the foci

(ii) The equation of directrix to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Solution:

Given equation: $\frac{x^2}{25} + \frac{y^2}{16} = 1$

Standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

By compare the standard and the given equation.

Hence; $a = 5, b = 4$

(i) To find the eccentricity from $b^2 = a^2(1 - e^2)$

By substituting a and b to the above equation

$4^2 = 5^2(1 - e^2)$

$16 = 25(1 - e^2)$

$16 - 25 = -25e^2$

$-9 = -25e^2$

$e^2 = \frac{9}{25}$

$e = \frac{3}{5}$

(ii) The co-ordinates of the foci are $(\pm ae, 0)$ that is

$\left(\pm 5\left(\frac{3}{5}\right), 0\right) \Rightarrow (\pm 3,0)$

(iii) The equations of the directrix are $x = \pm \frac{a}{e} \Rightarrow x = \pm \frac{5}{\frac{3}{5}} \Rightarrow x = \pm \frac{25}{3}$

Note: The length of the latus rectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{2b^2}{a}$; where "b" is minor and "a" is the major axis.

Example 30:

If the foci of and ellipse are (0, 2), (0,-2) and the point P(3, 2) lies on the ellipse. Find the equation of the ellipse.

Solution:

F(0, 2) and $F'(0, -2)$ the distance between F and F' is

$(0,2) \& (0, -2)$

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$\sqrt{(0 - 0)^2 + (-2 - 2)^2}$

Distance between the two foci $|FF'| = 4$ i.e. $2ae = 4$

Divide through by 2

$ae = 2$

Recall that the sum of distance from moving point to the foci is

$FP + FP' = 2\text{a}$

From $P(3,2)$

$\Rightarrow 3 + \sqrt{3^2 + 4^2} = 2a$

$3 + \sqrt{9 + 16} = 2a$

$3 + \sqrt{25} = 2a$

$8 = 2a$

$a = 4$

From $ae = 2$

$4e = 2$

$e = {}^{2}/_{4} = {}^{1}/_{2}$

From the equation of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Where $b^2 = a^2(1 - e^2)$

$b^2 = 4^2\left(1 - \left(\frac{1}{2}\right)^2\right)$

$b^2 = 16\left(1 - \frac{1}{4}\right)$

$b^2 = 16\left(\frac{3}{4}\right) = 12$

$b = \sqrt{12}$

Hence,

$\frac{x^2}{4^2} + \frac{y^2}{\left(\sqrt{12}\right)^2} = 1$

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

Equation of an Ellipse at General Point $(\boldsymbol{h}, \boldsymbol{k})$

When an ellipse has vertex (h, k), the equation will be

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

General Equation of an Ellipse

Note: An ellipse represented by an equation in quadratic both x and y i.e. the equations has the term x^2 and y^2 and both x^2 and y^2 have different co-efficient.

The skeleton of the general equation of an ellipse is given as:

$Ax^2 + By^2 + Cx + Dy + E = 0$

Let's deduce the above equation into the standard form.

By collecting like terms,

$Ax^2 + Cx + By^2 + Dy = -E$

By completing the square method,

$A\left(x^2 + \frac{Cx}{A}\right) + B\left(y^2 + \frac{Dy}{B}\right) = -E$

$A\left[\left(x + \frac{C}{2A}\right)^2 - \frac{C^2}{4A^2}\right] + B\left[\left(y + \frac{D}{2B}\right)^2 - \frac{D^2}{4B^2}\right] = -E$

$A\left(x + \frac{C}{2A}\right)^2 - \frac{AC^2}{2A^2} + B\left(y + \frac{D}{2B}\right)^2 - \frac{BD^2}{4B^2} = -E$

$= A\left(x + \frac{C}{2A}\right)^2 + B\left(y + \frac{D}{2B}\right)^2 = -E + \frac{C^2}{4A} + \frac{D^2}{4B}$

$\Rightarrow\ A\left(x + \frac{C}{2A}\right)^2 + B\left(y + \frac{D}{2B}\right)^2 = \frac{C^2}{4A} + \frac{D^2}{4B} - E$

Let $Q = A\left(x + \frac{C}{2A}\right)^2 + B\left(y + \frac{D}{2B}\right)^2$ ______(i)

Hence, we have

$Q = \frac{C^2}{4a} + \frac{D^2}{4B} - E$

From equation (i), let $a^2 = \frac{Q}{A}\ and\ b^2 = \frac{Q}{B}$

Hence, $A = \frac{Q}{a^2}\ and\ B = \frac{Q}{b^2}$

$\Rightarrow Q = \frac{Q}{a^2}\left(x + \frac{C}{2A}\right)^2 + \frac{Q}{b^2}\left(y + \frac{D}{2B}\right)^2$

Divide through by Q

$\frac{Q}{Q} = \left[\frac{Q}{a^2}\left(x + \frac{C}{2A}\right)^2 \times \frac{1}{Q}\right] + \left[\frac{Q}{b^2}\left(y + \frac{D}{2B}\right)^2 \times \frac{1}{Q}\right]$

$1 = \frac{\left(x+\frac{C}{2A}\right)^2}{a^2} + \frac{\left(y+\frac{D}{2B}\right)^2}{b^2}$

Hence,

$$\boxed{\frac{\left(x+\frac{C}{2A}\right)^2}{a^2} + \frac{\left(y+\frac{D}{2B}\right)^2}{b^2} = 1}$$

is the General Equation of an ellipse.

Example 31:

Transform the equation $x^2 + 4y^2 - 2x + 4y - 2 = 0$ to a semi-reduced form (standard from) and give a full description of the ellipse.

<u>Solution:</u>

$x^2 + 4y^2 - 2x + 4y - 2 = 0$

By completing the square method,

$x^2 - 2x + 4y^2 + 4y = 2$

Take the half of the coefficient of x and y and square the result; then add final result to both sides.

$\Rightarrow$ the coefficient of x is -2; the half is -1

$\Rightarrow$ the coefficient of y is 1 and the half is $\frac{1}{2}$

Hence;

$x^2 - 2x + (-1)^2 + 4\left[y^2 + y + \left(\frac{1}{2}\right)^2\right]$

$2 + (-1)^2 + 4\left(\frac{1}{2}\right)^2$

Notice:- Since the terms y has a coefficient of 4 multiplying the whole term $\left[y^2 + y + \left(\frac{1}{2}\right)^2\right]$; then

$\Rightarrow x^2 - 2x + 1 + 4\left[y^2 + y + \left(\frac{1}{2}\right)^2\right] = 2 + 1 + 1$

$\Rightarrow\ x^2 - 2x + 1 + 4\left[y^2 + y + \left(\frac{1}{2}\right)^2\right] = 4$

$(x-1)^2 + 4\left(y + 6\frac{1}{2}\right)^2 = 4$

Divide through by 4

$\frac{(x-1)^2}{4} + \left(y + 6\frac{1}{2}\right)^2 = 1$

By comparing the above equation with

$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$a^2 = 4; a = 2$

$b^2 = 1; b = 1$

From $b^2 = a^2(1 - e^2)$

$1 = 4(1 - e^2)$

$\frac{1}{4} = 1 - e^2$

$\frac{1}{4} - 1 = -e^2$

$-\frac{3}{4} = -e^2$

$e^2 = \frac{3}{4} \quad e = \frac{\sqrt{3}}{2}$

$ae = 2 \times \frac{\sqrt{3}}{2}$ and $\frac{a}{e} = \frac{2}{\frac{\sqrt{3}}{2}} = \frac{4}{\sqrt{3}}$

$\frac{a}{e} = \frac{4\sqrt{3}}{3}$

Therefore, the foci of the ellipse are $\left(1+\sqrt{3}, -\frac{1}{2}\right)$ and $\left(1-\sqrt{3}, -\frac{1}{2}\right)$follow the $(b \pm ae, k)$

The equation of the directrix are

$x = b \pm ae$

$x = 1 \pm \frac{a}{e}$

$x = 1 \pm \frac{4}{3}$

While the vertices of the ellipse are $(h \pm a, k)$

$\Rightarrow \left(1 \pm 2, -\frac{1}{2}\right)$

$\Rightarrow \left(1+2, -\frac{1}{2}\right)$ and $\left(1-2, -\frac{1}{2}\right)$

$\Rightarrow \left(3, -\frac{1}{2}\right)$ and $\left(-1, -\frac{1}{2}\right)$

Example 32:

Find the eccentricities and co-ordinate of foci with the equation directrix of the following ellipse.

(i) $2x^2 + y^2 = 2$

(ii) $\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$

Solution:

(i) $2x^2 + y^2 = 2$

Divide through by 2

$x^2 + \frac{y^2}{2} = 1$

By comparing with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$a^2 = 1 \Rightarrow a = 1$

$b^2 = 2 \Rightarrow b = \sqrt{2}$

From the value of "a" and "b"; we can see that "b" is the major axis and "a" is the minor axis.

Hence, $a^2 = b^2(1-e^2)$

$$1 = 2(1-e^2)$$

$$\frac{1}{2} = 1 - e^2$$

$$\frac{1}{2} - 1 = -e^2$$

$$\frac{-1}{2} = -e^2$$

$$\frac{1}{2} = e^2$$

$$e = \frac{\sqrt{2}}{2}$$

Hence, the eccentricity is

$e = \frac{\sqrt{2}}{2}$

The coordinate of the foci is $(\pm ae, 0)$ i.e. $(\pm be, 0)$, since "b" is the major axis.

$\Rightarrow \left(\pm\sqrt{2} \times \frac{\sqrt{2}}{2}, 0\right)$; $(\pm 1, 0)$

The equation of the directrix is $x = \pm\frac{b}{e}$ since "b" is the major axis.

$\Rightarrow x = \frac{\pm\sqrt{2}}{\frac{\sqrt{2}}{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

(ii) $\frac{(x-1)^2}{25} + \frac{(y+2)^2}{16} = 1$

By comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$\Rightarrow a^2 = 25; a = 5$

$\Rightarrow b^2 = 16; b = 4$

$b^2 = a^2(1 - e^2)$

$4^2 = 5^2(1 - e^2)$

$16 = 25(1 - e^2)$

$\frac{16}{25} = 1 - e^2$

$\frac{16}{25} - 1 = -e^2$

$\frac{-9}{25} = -e^2 ; \frac{9}{25} = e^2$

$e = \frac{3}{5}$

Hence, the eccentricity is $e = \frac{3}{5}$

$ae = 5\left(\frac{3}{5}\right) = 3$

$\frac{a}{e} = \frac{5}{\frac{3}{5}} = \frac{25}{3}$

The co-ordinate of the foci $(b \pm ae, k)$

$(4 \pm 3, -2)$

$(7, -2)$ or $(1, -2)$

The vertices of the ellipse are $(h \pm a, k)$

$(1 \pm 5, -2)$

$\Rightarrow (6, -2)$ and $(-4, -2)$

The equation of directrix $x = b \pm \frac{a}{e}$

$x = 4 \pm \frac{25}{3}$

$x = \frac{37}{3}$ or $x = \frac{-13}{3}$

Example 33:

Find the length of the latus rectum of the ellipse $\frac{x^2}{169} + \frac{y^2}{144} = 1$. Hence, find the co-ordinate for the four points in which the latera recta meet the ellipse.

Solution:

Given the equation: $\frac{x^2}{169} + \frac{y^2}{144} = 1$

Standard equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

By comparing both equation both equations

$a^2 = 169; a = 13$

$b^2 = 144; b = 12$

From the length of lactus rectum of an ellipse $\frac{2b^2}{a}$

By substituting “b” and “a” in $\frac{2b^2}{a}$

Hence,

$LR = \frac{2(12)^2}{13} = \frac{2\times144}{13}$

$LR = \frac{288}{13}$

To find the co-ordinate;

From $b^2 = a^2(1 - e^2)$

$144 = 169(1 - e^2)$

$\frac{144}{169} = 1 - e^2$

$\frac{144}{169} - 1 = -e^2$

$\frac{-25}{169} = -e^2 \qquad \Rightarrow \frac{25}{169} = e^2$

$e = \frac{5}{13}$ OR

$e = \frac{c}{major\ axis}$

Recall that; $c = \sqrt{a^2 - b^2}$

$c = \sqrt{169 - 144} = \sqrt{25} = 5$

Therefore, $e = \frac{5}{13}$

Since the ellipse is at origin, $h = 1$ and $k = 1$

$ae = 13\left(\frac{5}{13}\right) = 5$

$\frac{a}{e} = \frac{5}{\frac{5}{13}} = 13$

Hence the four vertices are

$V_{1,2}(0, \pm 13)$ and $V_{3,4}(\pm 12{,}0)$

Example 34:

An ellipse has eccentricity $e = \frac{4}{5}$. It's foci are the point $(0, \pm 4)$. Find the length of it's semi-minor axes and hence write down the equation of the ellipses.

Solution:

Given $e = \frac{4}{5}$

From $e = \frac{c}{major\ axis}$, let "a" be the major axis

$e = \frac{c}{a} = \frac{4}{5}$

$\Rightarrow a = 5; c = 4$

By Pythagoras triple, $b = 3$

Semi-major axes is 5

Semi-minor axis is 3

To write the equation of ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1 \qquad \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$

Example 35:

Describe the graph of the equation

$16x^2 + ay^2 - 64x - 54y + 1 = 0$

Solution:

The equation involves quadratic terms in y and x. So we will collect like terms.

$(16x^2 - 64x) + (ay^2 - 54y) = -1$

Next, factor the co-efficient of x^2 and y^2 and complete the square.

$16(x^2 - 4x + 4) + 9(y^2 - 6y + 9) = -1 + 64 + 81$

$16(x - 2)^2 + 9(y - 3)^2 = 144$

Divide through by 144

$\frac{16(x-2)^2}{144} + \frac{9(y-3)^2}{144} = 1$

$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{16} = 1$

By comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$a = 3; b = 4; h = 2; k = 3$

The graph is an ellipse with center (2,3) and major axis parallel to the y-axis. It co-ordinates are (2,7) and 2, −1); (−1,3) and (5,3).

Since $c = \sqrt{a^2 - b^2}$; $c = \sqrt{b^2 - a^2}$

Since b is the major axis, $c = \sqrt{b^2 - a^2}$; $c = \sqrt{16 - 9} = \sqrt{7}$

The foci lie $\sqrt{7}$ units above and below the center.

Hence, the foci is $(h, k \pm ae)$

$\Rightarrow (2, 3 + \sqrt{7})$ and $\left(2, 3 - \sqrt{7}\right)$

Tanjents And Normals To An Ellipse

Tangent at point (x_1, y_1). The standard equation of an ellipse; $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, multiply through by a^2b^2

$$a^2b^2\frac{x^2}{a^2} + a^2b^2\frac{y^2}{b^2} = 1 \times a^2b^2$$

$$b^2x^2 + a^2y^2 = a^2b^2 \text{ \_\_\_\_\_\_\_\_\_\_(i)}$$

Differentiating the equation (i) using implicating differentiation with respect to x.

Hence, $2b^2x + 2a^2y\frac{dy}{dx} = 0$

$$2a^2y\frac{dy}{dx} = -2b^2x$$

Divide through by $2a^2y$

$$\frac{2a^2y\frac{dy}{dx}}{2a^2y} = \frac{-2b^2x}{2a^2y}; \qquad \left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{-b^2x_1}{a^2y_1}$$

The slope of tangent is $\frac{-b^2x_1}{a^2y_1}$. Therefore the equation of tangent will be;

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = \frac{-b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2y_1(y - y_1) = -b^2x_1(x - x_1)$$

$$a^2yy_1 - a^2{y_1}^2 = -b^2xx_1 + b^2{x_1}^2$$

$$\Rightarrow a^2yy_1 + b^2xx_1 = b^2{x_1}^2 + a^2{y_1}^2$$

The above equation is the equation of tangent.

<u>Further Simplification</u>

From the equation an ellipse

$b^2x^2 + a^2y^2 = a^2b^2$ at point (x_1, y_1). It became $b^2{x_1}^2 + a^2{y_1}^2 = a^2b^2$.

By comparing the equation $a^2yy_1 + b^2xx_1 = b^2{x_1}^2 + a^2{y_1}^2$ and equation of an ellipse at point (x_1, y_1) i.e. $b^2{x_1}^2 + a^2{y_1}^2 = a^2b^2$.

Hence, if $a{y_1}^2 + b^2{x_1}^2 = a{y_1}^2 + b^2{x_1}^2$

We have;

$$b^2xx_1 + a^2yy_1 = a^2b^2$$

Divide through by a^2b^2

$$\frac{b^2xx_1}{a^2b^2} + \frac{a^2yy_1}{a^2b^2} = \frac{a^2b^2}{a^2b^2}$$

Hence; the equation of an ellipse is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

However, the equation of tangent to the ellipse is the above equation. If the center of the ellipse at the General point (h, k). Hence, the equation becomes;

$$\frac{(x-h)(x_1-h)}{a^2} + \frac{(y-k)(y_1-k)}{b^2} = 1$$

Normal

Recall that the slope of normal and slope of tangent are perpendicular. Hence, $m_1 m_2 = 1$. Since the slope of tangent is $m_1 = \frac{-b^2 x_1}{a^2 y_1}$; slope of normal $m_2 = \frac{-1}{m} = \frac{-1}{\frac{-b^2 x_1}{a^2 y_1}} = \frac{a^2 y_1}{-b^2 x_1}$

Hence, the slope of normal is, $\frac{a^2 y_1}{-b^2 x_1}$.

The equation of normal at point (x_1, y_1)

$$y - y_1 = \frac{a^2 y_1}{-b^2 x_1}(x - x_1)$$

$$-b^2 x_1 (y - y_1) = a^2 y_1 (x - x_1)$$

Divide through by $x_1 y_1$

$$\frac{-b^2 x_1 (y-y_1)}{x_1 y_1} = \frac{a^2 y_1 (x-x_1)}{x_1 y_1}$$

Hence, $\frac{-b^2 (y-y_1)}{y_1} = \frac{a^2 (x-x_1)}{x_1}$

The above equation is the equation of normal at origin. However, equation of normal at general point (h, k) is stated as:

$$\frac{b^2[(y-k)-(y_1-k)]}{(y_1-k)} = \frac{a^2[(x-h)-(x_1-h)]}{(x_1-h)}$$

Example 37:

Obtain the equation of tangent and normal to the ellipse $4x^2 + 25y^2 = 100$ at the point $\left(-3, \frac{8}{5}\right)$.

<u>Solution:</u>

$$4x^2 + 25y^2 = 100$$

Divide through by 100

$$\frac{4x^2}{100} + \frac{25y^2}{100} = \frac{100}{100}$$

$$\frac{x^2}{25} + \frac{y^2}{4} = 1$$

By comparing with standard equation
$a = 5; b = 2$

Using the equation tangent at point (x_1, y_1)

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \left[x_1 = -3; y_1 = \frac{8}{5}\right]$$

$$\frac{x(-3)}{25} + \frac{y\left(\frac{8}{5}\right)}{4} = 1$$

$$\frac{-3x}{25} + \frac{8y}{20} = 1$$

Multiply through by 100

$$\Rightarrow 100\left(\frac{-3x}{25}\right) + 100\left(\frac{8y}{20}\right) = 1 \times 100$$

$$= 12x + 40y = 100$$

Divide through by 4

$$-3x + 10y = 25;$$

$$10y - 3x - 25 = 0$$

However, the equation above is the equation of tangent to the ellipse.

$4x^2 + 25y^2 = 100$ at point $\left(-3, \frac{8}{5}\right)$

Equation of Normal

By using $\frac{-b^2(y-y_1)}{y_1} = \frac{a^2(x-x_1)}{x_1}$

$\Rightarrow$ by substituting the values

$$\frac{2^2\left(y-\frac{8}{5}\right)}{\frac{8}{5}} = \frac{5^2(x-(-3))}{-3}$$

$$\frac{4\left(y-\frac{8}{5}\right)}{\frac{8}{5}} = \frac{-2 \ \ (x+3)}{3}$$

$$\Rightarrow 3(20y - 32) = 8(-25x - 75)$$

$$60y - 96 = -120x - 600$$

Divide through by 6

$10y - 16 = -20x - 100$

$\Rightarrow 10y + 20x + 84 = 0$

The above equation is the equation of Normal.

Example 38:

Obtain the equation of tangent and Normal to the equation $\frac{(x+3)^2}{3} + \frac{(y-4)^2}{2} = 1$ at point $\left(-3, 4 + \sqrt{2}\right)$.