Chapter One: Topological Space

Definition:

Let X be a non empty set and let T be a subfamily of P(X), T is said to be a topology on X iff:

1) $X, \emptyset \in T$.
2) If $U, V \in T \Longrightarrow U \cap V \in T$.
3) If $U_\alpha \in T, \forall \alpha \in \Lambda \Longrightarrow \bigcup_{\alpha \in \Lambda} U_\alpha \in T$.

And (X,T) is called a topological space.

Example:

Let $X = \{1,2,3\}$, $T_1 = \{X, \{1\}, \{1,2\}\}$ is not a topology on X, since $\emptyset \notin T_1$.

$T_2 = \{\emptyset, \{1,2\}, \{1,3\}, \{1\}\}$ is not a topology on X, since $X \notin T_2$.

$T_3 = \{X, \emptyset, \{1,2\}, \{1,3\}\}$ is not a topology on X,

Since, $\{1,2\} \in T_3$ and $\{1,3\} \in T_3$, but $\{1,2\} \cap \{1,3\} = \{1\} \notin T_3$.

$T_4 = \{X, \emptyset, \{1\}, \{2\}\}$ is not a topology on X, Since, $\{1\} \in T_4$ and $\{2\} \in T_4$, but $\{1\} \cup \{2\} = \{1,2\} \notin T_4$.

$T_5 = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}$ is a topology on X, since

1. $X, \emptyset \in T_5$
2. $\{1\} \in T_5, \{2\} \in T_5 \Longrightarrow \{1\} \cap \{2\} = \emptyset \in T_5$
$\{1\} \in T_5, \{1,2\} \in T_5 \Longrightarrow \{1\} \cap \{1,2\} = \{1\} \in T_5$
$\{2\} \in T_5, \{1,2\} \in T_5 \Longrightarrow \{2\} \cap \{1,2\} = \{2\} \in T_5$
3. $\{1\} \in T_5, \{2\} \in T_5 \Longrightarrow \{1\} \cup \{2\} = \{1,2\} \in T_5$
$\{1\} \in T_5, \{1,2\} \in T_5 \Longrightarrow \{1\} \cup \{1,2\} = \{1,2\} \in T_5$
$\{2\} \in T_5, \{1,2\} \in T_5 \Longrightarrow \{2\} \cup \{1,2\} = \{1,2\} \in T_5$
$\{1\} \in T_5, \{2\} \in T_5, \{1,2\} \in T_5$
$\Longrightarrow \{1\} \cup \{2\} \cup \{1,2\} = \{1,2\} \in T_5$.

Homework: **On a set with (3) elements we can define (29) topologies.**

Definition:

Let (X,T) be a topological space. A subset U of X is said to be open set iff $U \in T$.

i.e. $U \subseteq X$ is open $\Leftrightarrow U \in T$.

A subset F of X is said to be closed set iff F^c is open set.

i.e. $F \subseteq X$ is closed $\Leftrightarrow$ F^c is open $\Leftrightarrow F^c \in T$.

The family of all closed subsets of X is denoted by $\mathcal{F}$.

i.e. T is the family of all open sets.

$\mathcal{F}$ is the family of all closed sets.

Example:

Let $X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,2\}\}$ T is a topology on X.

Now we can find the family of all closed sets in X as follows:

$T = \{X, \emptyset, \{1\}, \{1,2\}\}$

$\mathcal{F} = \{X^c, \emptyset^c, \{1\}^c, \{1,2\}^c\}$

$\Rightarrow \mathcal{F} = \{\emptyset, X, \{2,3\}, \{3\}\}$

The family of all closed subsets of X is $\mathcal{F} = \{X, \emptyset, \{2,3\}, \{3\}\}$.

Theorem:

Let (X,T) be a topological space and Let $\mathcal{F}$ be the family of all closed subsets of X, then:

1) $X, \emptyset \in T$.
2) If $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.
3) If $A_\alpha \in \mathcal{F}, \forall \alpha \in \Lambda \Rightarrow \bigcap_{\alpha \in \Lambda} A_\alpha \in \mathcal{F}$.

Proof:

1) $X^c = \emptyset \in T$, $\emptyset^c = X \in T$ by condition (1) of T.

$\Rightarrow X, \emptyset \in \mathcal{F}$ by definition of $\mathcal{F}$.

2) Let $A, B \in \mathcal{F}$, to prove $A \cup B \in \mathcal{F}$.

$A \in \mathcal{F} \Longrightarrow A^c \in T$, $B \in \mathcal{F} \Longrightarrow B^c \in T$ (By definition of $\mathcal{F}$).

$\Longrightarrow A^c \cap B^c \in T$ (By condition (2) of T).

$\Longrightarrow (A \cup B)^c \in T$ (By De-Morgan Laws).

$\Longrightarrow A \cup B \in \mathcal{F}$ (By definition of $\mathcal{F}$).

3) Let $A_\alpha \in \mathcal{F}, \forall \alpha \in \Lambda$, To prove $\bigcap_{\alpha \in \Lambda} A_\alpha \in \mathcal{F}$.

$(\bigcap_{\alpha \in \Lambda} A_\alpha)^c = \bigcup_{\alpha \in \Lambda} {A_\alpha}^c$ (By De-Morgan Laws)

But $A_\alpha^c \in T, \forall \alpha$ (By definition of $\mathcal{F}$).

$\bigcup_{\alpha \in \Lambda} A_\alpha^c \in T$ (By condition (3) of T)

$(\bigcap_{\alpha \in \Lambda} A_\alpha)^c \in T \Longrightarrow \bigcap_{\alpha \in \Lambda} A_\alpha \in \mathcal{F}$.

Theorem:

Let $X \neq \emptyset$ and let $\mathcal{F} \subseteq \mathbb{P}(X)$ such that:

1. **$X, \emptyset \in \mathcal{F}$.**
2. **If $A, B \in \mathcal{F} \Longrightarrow A \cup B \in \mathcal{F}$.**
3. **If $A_\alpha \in \mathcal{F}, \forall \alpha \in \Lambda \Longrightarrow \bigcap_{\alpha \in \Lambda} A_\alpha \in \mathcal{F}$, then**

$T = \{U \subseteq X \mid U^c \in \mathcal{F}\}$ is a topology on X.

Proof:

1) $X^c = \emptyset \in \mathcal{F}$, $\emptyset^c = X \in \mathcal{F}$ (By hypothesis (1)).

$\Longrightarrow X, \emptyset \in T$.

2) Let $U, V \in T$, to prove $U \cap V \in T$.

$U^c \in \mathcal{F}$, $V^c \in \mathcal{F}$ (By definition of T).

$\Longrightarrow U^c \cup V^c \in \mathcal{F}$ (By hypothesis (2)).

$\Longrightarrow (U \cap V)^c \in \mathcal{F}$ (By De-Morgan Laws).

$\Longrightarrow U \cap V \in T$.

3) Let $U_\alpha \in T, \forall\alpha \in \Lambda$, To prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

$U_\alpha{}^c \in \mathcal{F}$, $\forall\alpha \in \Lambda$ (By definition of T).

$\bigcap_{\alpha\in\Lambda} U_\alpha^c \in \mathcal{F}$ (By hypothesis (3))

. $(\bigcup_{\alpha\in\Lambda} U_\alpha)^c \in \mathcal{F}$ (By De-Morgan Laws).

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

T is a topology on X with family of closed sets $\mathcal{F}$.

Types Of Topological Spaces

Definition:(The Discrete Topology (D), D = $\mathbb{P}(\mathbf{X})$):

Let $X \neq \emptyset$ and $\mathbb{P}(\mathbf{X})$ is the power set of X, then $T = \mathbb{P}(\mathbf{X})$ is a topology on X which is called the discrete topology and denoted by (D).

Q: Prove that the discrete topology D is topological space.

Proof:

1) Since $X \subseteq X \Longrightarrow X \in \mathbb{P}(X)$ (By definition of $\mathbb{P}(X)$).

$\Longrightarrow X \in T$

And since $\emptyset \subseteq X \Longrightarrow \emptyset \in \mathbb{P}(X)$ (By definition of $\mathbb{P}(X)$).

$\emptyset \in T$.

2) Let $U, V \in T$, to prove $U \cap V \in T$.

$U, V \in T \Longrightarrow U, V \in \mathbb{P}(X)$

$\Longrightarrow U \subseteq X$ and $V \subseteq X$ (By definition of $\mathbb{P}(X)$).

$\Longrightarrow U \cap V \subseteq X$

$\Longrightarrow U \cap V \in \mathbb{P}(X)$

$\Longrightarrow U \cap V \in T$.

3) Let $U_\alpha \in T, \forall\alpha \in \Lambda$, To prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

$U_\alpha \in \mathbb{P}(X) \Longrightarrow U_\alpha \subseteq X$, $\forall\alpha \in \Lambda$.

$\bigcup_{\alpha\in\Lambda} U_\alpha \subseteq X \Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in \mathbb{P}(X)$.

$\bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

T is a topology on X which is the Largest topology we can defined on X.

Definition:(The Indiscrete Topology (I)):

Let $X \neq \emptyset$, then $T = \{X, \emptyset\}$ is a topology on X which is called the indiscrete topology and denoted by (I), this topology is the smallest topology we can defined on X.

Q: Prove that the indiscrete topology I is topological space.

1) $X, \emptyset \in T$ (By definition of T)
2) $X \cap \emptyset = \emptyset \in T$
3) $X \cup \emptyset = X \in T$

$T = I = \{X, \emptyset\}$ is a topology on X.

Definition:(The Fixed Point Topology):

Let $X \neq \emptyset$ and $p \in X$ then:-

i. $T = \{U \subseteq X |\ p \in X \text{ or } U = \emptyset\}$ is a topology on X.
ii. $T = \{U \subseteq X |\ p \notin X \text{ or } U = X\}$ is a topology on X.

Q: Prove that the fixed point topology is topological space.

Proof:

$T = \{U \subseteq X |\ p \in X \text{ or } U = \emptyset\}$.

1. $\emptyset \in T$ (By definition of T), since $p \in X \Longrightarrow X \in T$.
2. Let $U, V \in T$, to prove $U \cap V \in T$

$U \in T \Longrightarrow p \in U \text{ or } U = \emptyset$

$V \in T \Longrightarrow p \in V \text{ or } V = \emptyset$

$p \in U \cap V \Longrightarrow U \cap V \in T$

$U \cap V = \emptyset \Longrightarrow U \cap V \in T$

$U \cap V \in T$.

3. Let $U_\alpha \in T, \forall \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T$

$U_\alpha \in T \Longrightarrow p \in U_\alpha, \forall\alpha \Longrightarrow p \in \bigcup_{\alpha\in\Lambda} U_\alpha$.

$U_\alpha \in T \Longrightarrow U_\alpha = \emptyset, \forall\alpha \Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = \emptyset$.

$U_\alpha \in T \Longrightarrow p \in U_\alpha$, for some α and $U_\alpha = \emptyset$ for another some α $\Longrightarrow p \in \bigcup_{\alpha\in\Lambda} U_\alpha$.

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

$\mathbf{T = \{U \subseteq X \mid p \notin X \text{ or } U = X\}}$.

1. $X \in T$ (By definition of T), since $p \notin X \Longrightarrow \emptyset \in T$.

2. Let $U, V \in T$, to prove $U \cap V \in T$

$U \in T \Longrightarrow p \notin U$ or $U = X$

$V \in T \Longrightarrow p \notin V$ or $V = X$

$p \notin U \cap V \Longrightarrow U \cap V \in T$

$U \cap V = X \Longrightarrow U \cap V \in T$

$U \cap V \in T$.

3. Let $U_\alpha \in T, \forall \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T$

$U_\alpha \in T \Longrightarrow p \notin U_\alpha, \forall\alpha \Longrightarrow p \notin \bigcup_{\alpha\in\Lambda} U_\alpha$.

$U_\alpha \in T \Longrightarrow U_\alpha = X, \forall\alpha \Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X$.

$U_\alpha \in T \Longrightarrow p \notin U_\alpha$, for some α and $U_\alpha = X$ for another some α $\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha = X$.

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in T$.

T is a topology on X.

Definition:(The usual Topology):

Let $T_u = \{X, \emptyset, U; \forall x \in U \exists \text{open interval}(a, b); x \in (a, b) \subseteq U\}$ or $T_u = \{U \subseteq X; U = \text{union of family of open interval}\}$.

Q: Show that (X, T_u) is a topological space. (Exersices).

Definition:(The Cofinite Topology):

Let X be infinite set and $T_{cof} = \{U \subseteq X; U^c = \text{finite set}\} \cup \{\emptyset\}$.

Q: Show that (X, T_{cof}) is a topological space. (Exersices).

Q: The union of any family of closed sets is closed (prove or disprove).

For example: (disprove)

In $(\mathbb{N}, T_{cof})$, let $A_n = \{n+1\}, n \in \mathbb{N}$

i.e. $A_1 = \{2\}, A_2 = \{3\}, A_3 = \{4\}, \ldots$

Note that $\{A_n\}_{n\in\mathbb{N}}$ is a family of closed sets in $(\mathbb{N}, T_{cof})$.

But $\bigcup_{n\in\mathbb{N}} A_n = \{2,3,4,5,\ldots\} = \mathbb{N}/\{1\}$ is not closed set in $(\mathbb{N}, T_{cof})$.

Therefore, the union of any family of closed sets need not to be closed.

Theorem:

Let X be a nonempty set and let T_1, T_2 be two topologies on X, then $T_1 \cap T_2$ is atopology on X.

Proof:

1) $X, \emptyset \in T_1$ (By condition(1) of T_1).
$X, \emptyset \in T_2$ (By condition(1) of T_2).
$\Longrightarrow X, \emptyset \in T_1 \cap T_2$ (By definition of $\cap$).

2) Let $U, V \in T_1 \cap T_2$, to prove $U \cap V \in T_1 \cap T_2$

$U \in T_1 \wedge V \in T_1, U \cap V \in T_1$ (By condition(2) of T_1).

$U \in T_2 \wedge V \in T_2, U \cap V \in T_2$ (By condition(2) of T_2).

$\Longrightarrow U \cap V \in T_1 \cap T_2$ (By definition of $\cap$).

3) Let $U_\alpha \in T_1 \cap T_2, \forall \alpha \in \Lambda$, to prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T_1 \cap T_2$

$U_\alpha \in T_1, \forall \alpha \in \Lambda \wedge U_\alpha \in T_2, \forall \alpha \in \Lambda$

$\bigcup_{\alpha\in\Lambda} U_\alpha \in T_1$ (By condition(3) of T_1).

$\bigcup_{\alpha\in\Lambda} U_\alpha \in T_2$ (By condition(3) of T_2).

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in T_1 \cap T_2$.

$T_1 \cap T_2$ is a topology on X.

Q: $T_1 \cup T_2$ need not necessary to be a topology on X (prove or disprove).

For example:

Let $X = \{1,2,3\}, T_1 = \{X, \emptyset, \{1\}\}, T_2 = \{X, \emptyset, \{2\}\}$, T_1 and T_2 are two topologies on X.

$T_1 \cup T_2 = \{X, \emptyset, \{1\}, \{2\}\}$.

Note that $T_1 \cup T_2$ is not a topology on X, since $\{1\}, \{2\} \in T_1 \cup T_2$ but $\{1\} \cup \{2\} = \{1,2\} \notin T_1 \cup T_2$.

Theorem:

Let (X,T) be a topological space. $T = D \Leftrightarrow \{x\} \in T, \forall x \in X$.

Proof:

Let $T = D$, since $\{x\} \subseteq X, \forall x \in X \Longrightarrow \{x\} \in \mathbb{P}(X)\ \forall x \in X$

$\Longrightarrow \{x\} \in T, \forall x \in X\ (T = D = \mathbb{P}(X))$

$\Longleftarrow \{x\} \in T, \forall x \in X$, to prove $T = D$

$T \subseteq D$ ... (1) (By definition of T).

Let $A \in D \Longrightarrow A \subseteq X \Longrightarrow A = \cup_{x \in \Lambda}\{x\}$

But $\{x\} \in T \Longrightarrow \cup_{x \in \Lambda}\{x\} \in T$ (by condition (3) of T)

$\Longrightarrow A \in T \Longrightarrow D \subseteq T$ ... (2)

By (1) &(2)

$T = D$.

Neighborhood And Open Neighborhood

Definition:

Let (X,T) be a topological space and let $A \subseteq X$ and $x \in X$, then A is said to be a **Neighborhood** of x (nbh of x) iff there exists an open set U such that $x \in U \subseteq A$.

i.e. A is a neighborhood of x$\Leftrightarrow \exists U \in T \ni x \in U \subseteq A$.

If A is open set, then A is said to be an open neighborhood of x.

i.e. A is open neighborhood for x $\Leftrightarrow$ x $\in$ A $\in$ T.

Remarks: In any topological spaces (X,T):

1) **X is an open neighborhood for each x $\in$ X.**
2) **Every open set is an open neighborhood for each element in it.**
3) **If U and V are two neighborhoods of x, then U$\cap$V is a neighborhood for x.**

Example: In $(\mathbb{R}, \mathbf{T_u})$find:

1) **Two open neighborhood for 1.**
2) **three open neighborhood for 2.**
3) **Two open neighborhood for $\sqrt{2}$.**

Solution:

1)(-2,2) is open neighborhood for 1

(since $1 \in (-2,2) \in T_u$)

(0,3) is open neighborhood for 1

(since $1 \in (0,3)$ and $(0,3) \in T_u$).

2) $\mathbb{R}$ is open neighborhood for 2.

(since $2 \in \mathbb{R} \in T_u$)

(0,10) is open neighborhood for 2.

(since $2 \in (0,10) \in T_u$).

(-4,4) is open neighborhood for 2.

(since $2 \in (-4,4) \in T_u$).

3) (1,2) is open neighborhood for $\sqrt{2}$.

(since $\sqrt{2} \in (1,2) \in T_u$).

$(\frac{1}{2}, \frac{9}{2})$ is open neighborhood for $\sqrt{2}$.

(since $\sqrt{2} \in (\frac{1}{2}, \frac{9}{2}) \in \mathrm{T_u}$).

Example: In $(\mathbb{N}, \mathbf{T_{cof}})$, find:

1) Two open neighborhood for 3.
2) Two open neighborhood for 10.

Solution:

1) $\mathbb{N}\backslash\{1\}$ is open neighborhood for 3.

(since $\mathbb{N}\backslash\{1\} \in \mathrm{T_{cof}}$ and $3 \in \mathbb{N}\backslash\{1\}$).

$\mathbb{N}\backslash\{2,4,6\}$ is open neighborhood for 3.

(since $3 \in \mathbb{N}\backslash\{2,4,6\} \in \mathrm{T_{cof}}$).

2) $\mathbb{N}\backslash\{3\}$ is open neighborhood for 10.

(since $10 \in \mathbb{N}\backslash\{3\} \in \mathrm{T_{cof}}$).

$\mathbb{N}\backslash\{20,30,40\}$ is open neighborhood for 10.

(since $10 \in \mathbb{N}\backslash\{20,30,40\} \in \mathrm{T_{cof}}$).

Chapter Two Derived Sets

Definition(Interior of a set):

Let (X,T) be a topological space and let $A \subseteq X$ and $x \in A$, then x is called an interior point of A iff there exists an open set U such that $x \in U \subseteq A$.

The set of all interior points of A is said to be the interior set of A and denoted by A° or int(A)

i.e. $A^{\circ} = \{x \in A \mid \exists U \in T \ni x \in U \subseteq A\}$.

$x \in A^{\circ} \Leftrightarrow \exists U \in T \ni x \in U \subseteq A$

$x \notin A^{\circ} \Leftrightarrow \forall U \in T \ni x \in U \nsubseteq A$

Example:

Let $\mathbf{X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,2\}\}}$

1) If $\mathbf{A = \{1,3\}}$, find $\mathbf{A^{\circ}}$.

Solution:

$1 \in A^{\circ}$ since $\exists U = \{1\} \in T \ni 1 \in \{1\} \subseteq \{1,3\}$

$3 \notin A^{\circ}$ since $\exists U = X \in T \ni 3 \in X \nsubseteq \{1,3\}$

$\Longrightarrow A^{\circ} = \{1\}$.

Solution:

2) If $\mathbf{A = \{2,3\}}$, find $\mathbf{A^{\circ}}$.

$2 \notin A^{\circ}$ since $\forall U \in T \ni 2 \in \{1,2\} \nsubseteq \{1,3\}$

$3 \notin A^{\circ}$ since $\exists U \in T \ni 3 \in X \nsubseteq \{1,3\}$

$\Longrightarrow A^{\circ} = \emptyset$.

Theorem:

Let (X,T) be a topological space and let A,B be two subsets of X:

1) $A^{\circ} \subseteq A$.
2) if $A \subseteq B \Longrightarrow A^{\circ} \subseteq B^{\circ}$ but $A^{\circ} \subseteq B^{\circ} \nRightarrow A \subseteq B$.
3) $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$.
4) $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$, but $(A \cup B)^{\circ} \nsubseteq A^{\circ} \cup B^{\circ}$.
5) $A \in T \Leftrightarrow A^{\circ} = A$.
6) $A^{\circ} = \cup\{ U \in T \ni U \subseteq A\}$ =The Largest open set contained in A.
7) $X^{\circ} = X$, $\emptyset^{\circ} = \emptyset$.

Proof:

1) $A^{\circ} \subseteq A$

$x \in A^{\circ} \Longrightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of int(A))

$\Longrightarrow x \in A$

$\Longrightarrow A^{\circ} \subseteq A$.

2) Let $A \subseteq B$ to prove $A^{\circ} \subseteq B^{\circ}$

Let $x \in A^{\circ} \Longrightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of int(A))

$\Longrightarrow \exists U \in T \ni x \in U \subseteq B$ (since $A \subseteq B$)

$\Longrightarrow x \in B^{\circ}$(By definition of int(B))

$\Longrightarrow A^{\circ} \subseteq B^{\circ}$.

Note that if $A^{\circ} \subseteq B^{\circ} \nRightarrow A \subseteq B$ (disprove)

For example:

$X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,3\}\}$

$A = \{3\} \Longrightarrow A^{\circ} = \emptyset$

$B = \{1,2\} \Longrightarrow B^{\circ} = \{1\}$

$\Longrightarrow A^{\circ} \subseteq B^{\circ}$, but $A \subseteq B$.

3) To prove $(A\cap B)^{\circ} \subseteq A^{\circ}\cap B^{\circ}$ & $A^{\circ}\cap B^{\circ} \subseteq (A\cap B)^{\circ}$

Since $(A\cap B) \subseteq A$ & $(A\cap B) \subseteq B$

$\Longrightarrow (A\cap B)^{\circ} \subseteq A^{\circ}$ & $(A\cap B)^{\circ} \subseteq B^{\circ}$

$\Longrightarrow (A\cap B)^{\circ} \subseteq A^{\circ}\cap B^{\circ}$ ... (1)

Since $A^{\circ} \subseteq A$ & $B^{\circ} \subseteq B$ (By (1) in theorem)

$\Longrightarrow A^{\circ}\cap B^{\circ} \subseteq A\cap B$

$\Longrightarrow (A^{\circ}\cap B^{\circ})^{\circ} \subseteq A^{\circ}\cap B^{\circ}$ (By (2) in theorem)

But $A^{\circ}\cap B^{\circ} \in T$ (since A° & $B^{\circ}$are open)

$\Longrightarrow (A^{\circ}\cap B^{\circ})^{\circ} = A^{\circ}\cap B^{\circ}$ (By theorem. $A \in T \Leftrightarrow A^{\circ} = A$)

$\Longrightarrow A^{\circ}\cap B^{\circ} \subseteq (A\cap B)^{\circ}$ ... (2)

By (1) &(2) we get $(A\cap B)^{\circ} = A^{\circ}\cap B^{\circ}$.

4) To prove $A^{\circ}\cup B^{\circ} \subseteq (A\cup B)^{\circ}$

Since $A^{\circ} \subseteq A$ & $B^{\circ} \subseteq B$ (By (1) in theorem)

$\Longrightarrow A^{\circ}\cup B^{\circ} \subseteq A\cup B$

$\Longrightarrow (A^{\circ}\cup B^{\circ})^{\circ} \subseteq (A\cup B)^{\circ}$ (By (2) in theorem)

But $A^{\circ}\cup B^{\circ} \in T$ (since A° & B° are open sets)

$\Longrightarrow (A^{\circ}\cup B^{\circ})^{\circ} = A^{\circ}\cup B^{\circ}$ (By theorem. $A \in T \Leftrightarrow A = A^{\circ}$)

$\Longrightarrow A^{\circ}\cup B^{\circ} \subseteq (A\cup B)^{\circ}$

Note that: $(A\cup B)^{\circ} \nsubseteq A^{\circ}\cup B^{\circ}$ **(disprove)**

For example:

$X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,3\}\}$

$A = \{1\} \Longrightarrow A^{\circ} = \emptyset$

$B = \{3\} \Longrightarrow B^{\circ} = \emptyset$

$\Longrightarrow A^{\circ}\cup B^{\circ} = \emptyset$

$A \cup B = \{1,3\}$

$\Rightarrow (A \cup B)^{\circ} = \{1,3\}$

$A^{\circ} \cup B^{\circ} = \emptyset \nsubseteq \{1,3\} = (A \cup B)^{\circ}$

5) $\Rightarrow$ suppose that $A \in T$ to prove $A = A^{\circ}$

$A^{\circ} \subseteq A$ ... (1) (By (1) in theorem)

Let $x \in A \in T \Rightarrow \exists U \in T \ni x \in U \subseteq A$ (By theorem $A \in T \Leftrightarrow \forall x \in A\, \exists U \in T \ni x \in U \subseteq A$)

$\Rightarrow x \in A^{\circ}$ (By definition of int(A))

$A \subseteq A^{\circ}$ ... (2)

By (1) & (2) we get $A = A^{\circ}$

$\Leftarrow$ Let $A = A^{\circ}$ To prove $A \in T$

Let $x \in A = A^{\circ} \Rightarrow x \in A^{\circ}$

$\Rightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of A°)

$\Rightarrow \forall x \in A\, \exists U \in T \ni x \in U \subseteq A$

$\Rightarrow A \in T$(By theorem $A \in T \Leftrightarrow \forall x \in A\, \exists U \in T \ni x \in U \subseteq A$).

6) We have to show that

$A^{\circ} \subseteq \bigcup\{U \in T \ni U \subseteq A\}$&$\bigcup\{U \in T \ni U \subseteq A\} \subseteq A^{\circ}$

Let $x \in A^{\circ} \Rightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of int(A))

$\Rightarrow x \in \bigcup\{U \in T \ni U \subseteq A\}$ (By definition of union)

$\Rightarrow A^{\circ} \subseteq \bigcup\{U \in T \ni U \subseteq A\}$ ... (1)

Let $x \in \bigcup\{U \in T \ni U \subseteq A\}$

$\Rightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of union)

$\Rightarrow x \in A^{\circ}$ (By definition of int(A))

$\Rightarrow \bigcup\{U \in T \ni U \subseteq A\} \subseteq A^{\circ}$ ... (2)

By (1) & (2) we get

$A^{\circ} = \bigcup\{ U \in T \ni U \subseteq A\}$.

7) Since $X, \emptyset \in T$ (By condition (1) of T)

$\Longrightarrow X^{\circ} = X \wedge \emptyset^{\circ} = \emptyset$ (By Theorem $A \in T \Leftrightarrow A = A^{\circ}$).

Definition(Exterior of a set):

Let (X,T) be a topological space and let $A \subseteq X$ and $x \in A^c$, then x is called an Exterior point of A iff there exists an open set U such that $x \in U \subseteq A^c$.

The set of all Exterior points of A is said to be the Exterior set of A and denoted by A^x or Ext(A)

i.e. $A^x = \{x \in A^c \mid \exists U \in T \ni x \in U \subseteq A^c\}$.

$x \in A^x \Leftrightarrow \exists U \in T \ni x \in U \subseteq A^c$

$x \notin A^x \Leftrightarrow \forall U \in T \ni x \in U \nsubseteq A^c$

Example:

Let $X = \{1,2,3\}, T = D = \mathbb{P}(X),\ A = \{2,3\}$, find A^x.

Solution:

$T = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$A = \{2,3\} \Longrightarrow A^c = \{1\}$

$1 \in A^x$ since $\exists U = \{1\} \in T \ni 1 \in \{1\} \subseteq \{1\}$

$A^x = \{1\}$.

Theorem:

Let (X,T) be a topological space and let A,B be two subsets of X:

1) $A^x \subseteq A^c$.
2) $A^x = (A^c)^{\circ}$
3) if $A \subseteq B \Longrightarrow B^x \subseteq A^x$ but $B^x \subseteq A^x \nRightarrow A \subseteq B$.
4) $A^{\circ} \cap A^x = \emptyset$

5) $(\mathbf{A}\cup\mathbf{B})^{\mathbf{x}} = \mathbf{A}^{\mathbf{x}}\cup\mathbf{B}^{\mathbf{x}}$**.**

6) $\mathbf{A}^{\mathbf{c}} \in \mathbf{T} \Leftrightarrow \mathbf{A}^{\mathbf{x}} = \mathbf{A}^{\mathbf{c}}$**.**

7) $\emptyset^{\mathbf{x}} = \mathbf{X}\,, \mathbf{X}^{\mathbf{x}} = \emptyset$**.**

Proof:

1) Let $x \in A^x \Longrightarrow \exists U \in T \ni x \in U \subseteq A^c$ (By definition of Ext(A))

$\Longrightarrow x \in A^c$

$\Longrightarrow A^x \subseteq A^c$.

2) To prove $A^x \subseteq (A^c)^\circ$ & $(A^c)^\circ \subseteq A^x$

Let $x \in A^x \Leftrightarrow \exists U \in T \ni x \in U \subseteq A^c$

$\Leftrightarrow x \in (A^c)^\circ$ (By definition of ext(A) & int(A^c))

3) Let $A \subseteq B$, To prove $B^x \subseteq A^x$

Let $x \in B^x \Longrightarrow \exists U \in T \ni x \in U \subseteq B^c$

$\Longrightarrow \exists U \in T \ni x \in U \subseteq A^c$ (since $A \subseteq B \Longrightarrow B^c \subseteq A^c$)

$\Longrightarrow x \in A^x$

$B^x \subseteq A^x$

Note that: $\mathbf{B^x \subseteq A^x \not\Rightarrow A \subseteq B}$ (disprove)

For example:

$X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}\}$

$A = \{1,3\} \Longrightarrow A^c = \{2\} \Longrightarrow A^x = \{2\}$

$B = \{1,2\} \Longrightarrow B^c = \{3\} \Longrightarrow B^x = \emptyset$

$B^x \subseteq A^x \not\Rightarrow A \subseteq B$

$\emptyset \subseteq \{2\} \not\Rightarrow \{1,3\} \nsubseteq \{1,2\}$

4) Suppose that $A^\circ \cap A^x \neq \emptyset$

$\Longrightarrow x \in A^\circ \cap A^x$

$\Longrightarrow x \in A^\circ \wedge x \in A^x$

$\Longrightarrow \exists U \in T \ni x \in U$

$U \subseteq A$ (By definition of A°)

$U \subseteq A^c$ (By definition of A^x)

$\Longrightarrow x \in A \ \wedge x \in A^c$

$\Longrightarrow x \in A \cap A^c$ which is a contradiction

$A^{\circ} \cap A^x \neq \emptyset$.

5) $(A \cup B)^x = ((A \cup B)^c)^{\circ}$ (By (2) in theorem)

$= (A^c \cap B^c)^{\circ}$ (By De-Morgan Law)

$= (A^c)^{\circ} \cap (B^c)^{\circ}$ (By proposition of interior set)

$= A^x \cup B^x$ (By (2) in theorem)

6) $A^x = (A^c)^{\circ}$ (By (2) in theorem)

$A^c \in T \Leftrightarrow A^c = (A^c)^{\circ}$ (By proposition of interior set $A \in T \Leftrightarrow A = A^{\circ}$)

$A^c \in T \Leftrightarrow A^c = A^x$

7) Since $X^c = \emptyset$ and $\emptyset^c = X$ and since $\emptyset, X \in T$ (By condition (1) of T)

$\Longrightarrow X^c = \emptyset \ \& \ \emptyset^c = X$ (By theorem $A^c \in T \Leftrightarrow A^x = A^c$).

Definition(Boundary of a set):

Let (X,T) be a topological space and let $A \subseteq X$ and $x \in X$, then x is called an Boundary point of A iff there exists an open set contains x intersected with A and with A^c.

The set of all Boundary points of A is said to be the Boundary set of A and denoted by A^b or b(A) or $\partial(A)$.

i.e. $A^b = \{x \in X | \ \forall U \in T \ni x \in U, U \cap A \neq \emptyset \ \wedge U \cap A^c \neq \emptyset\}$.

i.e. $x \in A^b \Leftrightarrow \forall U \in T \ni x \in U, U \cap A \neq \emptyset \ \wedge U \cap A^c \neq \emptyset$

$x \notin A^b \Leftrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset \ \wedge U \cap A^c = \emptyset$

Example:

Let $\mathbf{X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,2\}\}}$

$\mathbf{A = \{1,3\}, B = \{1,2\}, C = \{2,3\}.}$ **Find** $\mathbf{A^b, B^b, C^b}$.

Solution:

$A = \{1,3\} \Longrightarrow A^c = \{2\}$

$1 \notin A^b$ since $\exists \{1\} \in T \ni 1 \in \{1\}, \{1\} \cap \{1,3\} = \{1\} \neq \emptyset \ \wedge \{1\} \cap \{2\} = \emptyset$

$2 \in A^b$ since $\forall \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\} \cap \{1,3\} = \{1\} \neq \emptyset \ \wedge \{1,2\} \cap \{2\} = \{2\} \neq \emptyset$

$3 \in A^b$ since $\forall X \in T \ni 3 \in X, X \cap \{1,3\} = \{1,3\} \neq \emptyset \ \wedge X \cap \{2\} = \{2\} \neq \emptyset$

$A^b = \{2,3\}$

$B = \{1,2\} \Longrightarrow B^c = \{3\}$

$1 \notin B^b$ since $\exists \{1\} \in T \ni 1 \in \{1\}, \{1\} \cap \{1,2\} = \{1\} \neq \emptyset \ \wedge \{1\} \cap \{3\} = \emptyset$

$2 \notin B^b$ since $\exists \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\} \cap \{1,2\} = \{1,2\} \neq \emptyset \ \wedge \{1,2\} \cap \{3\} = \emptyset$

$3 \in B^b$ since $\forall X \in T \ni 3 \in X, X \cap \{1,2\} = \{1,2\} \neq \emptyset \ \wedge X \cap \{3\} = \{3\} \neq \emptyset$

$\Longrightarrow B^b = \{3\}$

$C = \{2,3\} \Longrightarrow C^c = \{1\}$

$1 \notin C^b$ since $\exists \{1\} \in T \ni 1 \in \{1\}, \{1\} \cap \{2,3\} = \emptyset \ \wedge \{1\} \cap \{1\} = \{1\} \neq \emptyset$

$2 \in C^b$ since $\forall \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\} \cap \{2,3\} = \{2\} \neq \emptyset \ \wedge \ \{1,2\} \cap \{1\} = \{1\} \neq \emptyset$

$3 \in C^b$ since $\forall X \in T \ni 3 \in X, X \cap \{2,3\} = \{2,3\} \neq \emptyset \ \wedge X \cap \{1\} = \{1\} \neq \emptyset$

$\Longrightarrow C^b = \{2,3\}$

Remarks: In Any Topological Space:

1) $\mathbf{A^b}$ **may be a subset of A or a subset of** $\mathbf{A^c}$ **or** $\mathbf{A^b}$ **intersects A and** $\mathbf{A^c}$**.**

i.e. $\mathbf{A^b \subseteq A}$ **or** $\mathbf{A^b \subseteq A^c}$ **or** $\mathbf{A^b \cap A \neq \emptyset \ \wedge \ A^b \cap A^c \neq \emptyset}$

2) If $\{a\} \in T$, then $a \notin A^b$ for any $A \subseteq X$ since if $a \in A \Longrightarrow \{a\} \cap A^c = \emptyset$

And if $a \in A^c \Longrightarrow \{a\} \cap A = \emptyset \Longrightarrow a \notin A^b$

3) In (X,I), if $\emptyset \neq A \nsubseteq X$, then $A^b = X$ (since X is the only open neighborhood for each $x \in X$ and $X \cap A \neq \emptyset \wedge X \cap A^c \neq \emptyset$).

4) In (X,D), if $A \subseteq X$, then $A^b = \emptyset$ (since $\forall x \in X, \{x\} \in T \Longrightarrow x \notin A^b\ \forall x \in X$, $A^b = \emptyset$).

Example:

Define a topological space and find subset of it has six boundary points.

Solution:

Let $X = \{1,2,3,4,5,6,7\}$ and let $T = \{X, \emptyset, \{4\}\}$ and $A = \{1,3,5,7\} \Longrightarrow A^c = \{2,4,6\}$

$4 \notin A^b$ since $\exists \{4\} \in T \ni 4 \in \{4\}, \{4\} \cap \{1,3,5,7\} = \emptyset$ $\forall x \in X \ni x \neq 4, x \in A^b$

Since the only open neighborhood of x is X and $X \cap A \neq \emptyset \wedge X \cap A^c \neq \emptyset$.

$A^b = \{1,2,3,5,6,7\}$

Theorem:

Let (X,T) be a topological space and A,B be two subsets of X:

1) $A^b = (A^c)^b$
2) $A^b \cap A^\circ = \emptyset, A^b \cap A^x = \emptyset$
3) $(A \cup B)^b \subseteq A^b \cup B^b$
4) $A \in T \Leftrightarrow A^b \subseteq A^c$
5) $A^c \in T \Leftrightarrow A^b \subseteq A$
6) $A, A^c \in T \Leftrightarrow A^b = \emptyset$
7) $X^b = \emptyset$ & $\emptyset^b = X$
8) A^b is a closed set.

Proof:

1) $x \in A^b \Leftrightarrow \forall U \in T \ni x \in U, U \cap A \neq \emptyset \ \wedge U \cap A^c \neq \emptyset$ (By definition of b(A))

$\Leftrightarrow \forall U \in T \ni x \in U, U \cap (A^c)^c \neq \emptyset \ \wedge U \cap A^c \neq \emptyset$ $[\ (A^c)^c = A]$

$x \in (A^c)^b$ (By definition of $b(A^c)$)

$A^b = (A^c)^b$.

2) To prove $A^b \cap A^\circ = \emptyset$

Suppose that $A^b \cap A^\circ \neq \emptyset$

$\Longrightarrow x \in A^b \cap A^\circ \neq \emptyset$

$\Longrightarrow x \in A^b \wedge x \in A^\circ$ (By definition of $\cap$)

$x \in A^\circ \Longrightarrow \exists U \in T \ni x \in U \subseteq A$ (By definition of int(A))

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A^c = \emptyset$ ($since$ $A \cap A^c = \emptyset$

$\Longrightarrow x \notin A^b$ which is contradiction

$A^b \cap A^\circ = \emptyset$

To prove $A^b \cap A^x = \emptyset$

Suppose that $A^b \cap A^x \neq \emptyset$

$\Longrightarrow \exists x \in A^b \cap A^x \neq \emptyset$

$\Longrightarrow x \in A^b \wedge x \in A^x$ (By definition of $\cap$)

$x \in A^x \Longrightarrow \exists U \in T \ni x \in U \subseteq A^c$ (By definition of ext(A))

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset$ ($since$ $A \cap A^c = \emptyset$

$\Longrightarrow x \notin A^b$ which is contradiction

$A^b \cap A^x = \emptyset$.

3) To prove $(A \cup B)^b \subseteq A^b \cup B^b$

Let $x \in (A \cup B)^b$

$\Longrightarrow \forall U \in T \ni x \in U, U \cap (A \cup B) \neq \emptyset \ \wedge U \cap (A \cup B)^c \neq \emptyset$

(By definition of $b(A \cup B)$)

$\Longrightarrow (U \cap A) \cup (U \cap B) \neq \emptyset \ \wedge U \cap (A \cup B)^c \neq \emptyset$

(since $\cap$ distribution over $\cup$) (By De-Morgan Laws)

$\Longrightarrow (U \cap A) \cup (U \cap B) \neq \emptyset \ \wedge (U \cap A^c) \cup (U \cap B^c) \neq \emptyset$

(since $\cap$ distribution over $\cap$)

$\Longrightarrow [(U \cap A) \neq \emptyset \vee (U \cap B) \neq \emptyset] \ \wedge [(U \cap A^c) \neq \emptyset \wedge (U \cap B^c) \neq \emptyset]$

$\Longrightarrow [(U \cap A) \neq \emptyset \wedge (U \cap A^c) \neq \emptyset] \vee \ [(U \cap B) \neq \emptyset \wedge (U \cap B^c) \neq \emptyset]$

$\Longrightarrow x \in A^b \vee x \in B^b$

$\Longrightarrow x \in A^b \cup B^b$

$(A \cup B)^b \subseteq A^b \cup B^b$

Note that $A^b \cup B^b \nsubseteq (A \cup B)^b$ in general (disprove)

<u>For example:</u>

Let $X = \{1,2,3\}, T = \{X, \emptyset\} = I, A = \{1,2\}, B = \{3\}$

$A^b = X$ & $B^b = X$

$\Longrightarrow A^b \cup B^b = X \cup X = X$

$A \cup B = \{1,2\} \cup \{3\} = \{1,2,3\} = X$

$(A \cup B)^b = X^b = \emptyset$

$(A \cup B)^b \subseteq A^b \cup B^b$ but $A^b \cup B^b \nsubseteq (A \cup B)^b$.

4) $A \in T \Leftrightarrow A^b \subseteq A^c$

Suppose that $A \in T$ To prove $A^b \subseteq A^c$

Let $A^b \nsubseteq A^c$

$\Longrightarrow x \in A^b \wedge x \notin A^c$

$\Longrightarrow \forall U \in T \ni x \in U, U \cap A \neq \emptyset \ \wedge U \cap A^c \neq \emptyset$ (By definition of A^b)

But $A \in T$ (by hyperthesis) and $x \in A$(since $x \notin A^c$

$\Longrightarrow A \cap A^c \neq \emptyset$ which is contradiction. (since $A \cap A^c = \emptyset$)

$A^b \subseteq A^c$

$\Longleftarrow$ suppose that $A^b \subseteq A^c$ To prove $A \in T$

Let $x \in A \Longrightarrow x \notin A^c$ (since $A \cap A^c = \emptyset$)

$\Longrightarrow x \notin A^b$ (since $A^b \subseteq A^c$ by hyperthesis)

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset \ \vee U \cap A^c = \emptyset$ (By definition of A^b)

But $x \in A \wedge x \in U \Longrightarrow A \cap U \neq \emptyset$

$\Longrightarrow U \cap A^c = \emptyset$

$\Longrightarrow \exists U \in T \ni x \in U \subseteq A$

$A \in T$.

5) $A^c \in T \Leftrightarrow A^b \subseteq A$

Let $A^c \in T$ To prove $A^b \subseteq A$

Let $A^b \nsubseteq A$

$\Longrightarrow x \in A^b \wedge x \notin A$

$\Longrightarrow \forall U \in T \ni x \in U, U \cap A \neq \emptyset \ \wedge U \cap A^c \neq \emptyset$ (By definition of A^b)

But $A^c \in T$ (by hyperthesis) and $x \in A^c$(since $x \notin A$

$\Longrightarrow A^c \cap A \neq \emptyset$ which is contradiction. (since $A \cap A^c = \emptyset$)

$A^b \subseteq A$

$\Longleftarrow$ Suppose that $A^b \subseteq A$ To prove $A^c \in T$

Let $x \in A^c \Longrightarrow x \notin A$ (since $A^c \cap A \neq \emptyset$)

$\Longrightarrow x \notin A^b$ (since $A^b \subseteq A$ by hyperthesis)

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset \ \vee U \cap A^c = \emptyset$ (By definition of A^b)

$\Longrightarrow U \cap A = \emptyset$ (since $x \in A^c \cap U$)

$\Longrightarrow \exists U \in T \ni x \in U \subseteq A^c$

$A^c \in T$.

6) $A, A^c \in T \Leftrightarrow A^b = \emptyset$

$A, A^c \in T \Leftrightarrow A^b \subseteq A \wedge A^b \subseteq A^c$

$\Leftrightarrow A^b \subseteq A \cap A^c$

$\Leftrightarrow A^b \subseteq \emptyset$

$\Leftrightarrow A^b = \emptyset$ (since $\emptyset \subseteq A^b$ and $A^b \subseteq \emptyset$).

7) Since $X, \emptyset \in T$

$\Longrightarrow X, X^c \in T$ and $\emptyset, \emptyset^c \in T$

$\Longrightarrow X^b = \emptyset$ and $, \emptyset^b = \emptyset$ (By (6)).

8) A^b is closed set

$A^b = (A^\circ \cup A^x)^c, A^\circ \in T$ and $A^x \in T$

$A^\circ \cup A^x \in T$ (By condition (3) of T)

$\Longrightarrow (A^\circ \cup A^x)^c$ is closed set

$\Longrightarrow A^b$ is closed set.

<u>Remark:</u> In any topological space (X,T) and any subset A of X:

1) $\mathbf{A^\circ \cap A^b = \emptyset,\ A^\circ \cap A^x = \emptyset, A^x \cap A^b = \emptyset}$
2) $\mathbf{A^\circ \cup A^b \cup A^x = X}$**.**

The family $\{\mathbf{A^\circ, A^x, A^b}\}$ form a partition for X.

<u>Note that:</u>

$\mathbf{A^\circ = X \backslash A^x \cup A^b = \left(A^x \cup A^b\right)^c}$

$\mathbf{A^x = X \backslash A^\circ \cup A^b = \left(A^\circ \cup A^b\right)^c}$

$\mathbf{A^b = X \backslash A^\circ \cup A^x = (A^\circ \cup A^x)^c}$

Definition(Limit point or an acculumation point):

Let (X,T) be a topological space and let $A \subseteq X$ and $x \in X$, then x is called **Limit point of A** **or** **an acculumation point of A** iff every open neighborhood of x has another element y of A such that $y \neq x$.

The set of all Limit points of A is said to be the derived set of A and denoted by d(A) or A'.

i.e. $A' = \{x \in X | \forall U \in T \ni x \in U, U \backslash \{x\} \cap A \neq \emptyset\}$.

i.e. $x \in A' \Leftrightarrow \forall U \in T \ni x \in U, U \backslash \{x\} \cap A \neq \emptyset$

$x \notin A' \Leftrightarrow \exists U \in T \ni x \in U, U \backslash \{x\} \cap A = \emptyset$

Example:

Let $\mathbf{X} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \mathbf{T} = \{\mathbf{X}, \emptyset, \{\mathbf{1}\}, \{\mathbf{2}, \mathbf{3}\}\}$

$\mathbf{A} = \{\mathbf{1}, \mathbf{3}\}, \mathbf{B} = \{\mathbf{2}\}, \mathbf{C} = \{\mathbf{2}, \mathbf{3}\}$. **Find** $\mathbf{A}', \mathbf{B}', \mathbf{C}'$.

Solution:

$A = \{1,3\}$

$1 \notin A'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\} \backslash \{1\} \cap \{1,3\} = \emptyset$

$2 \in A'$ since $\forall U = \{2,3\} \in T \ni 2 \in \{2,3\}, \{2,3\} \backslash \{2\} \cap \{1,3\} = \{3\} \neq \emptyset$

$3 \notin A'$ since $\exists U = \{2,3\} \in T \ni 3 \in \{2,3\}, \{2,3\} \backslash \{3\} \cap \{1,3\} = \emptyset$

$A' = \{2\}$

$B = \{2\}$

$1 \notin B'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\} \backslash \{1\} \cap \{2\} = \emptyset$

$2 \notin B'$ since $\exists U = \{2,3\} \in T \ni 2 \in \{2,3\}, \{2,3\} \backslash \{2\} \cap \{2\} = \emptyset$

$3 \in B'$ since $\forall U = \{2,3\} \in T \ni 3 \in \{2,3\}, \{2,3\} \backslash \{3\} \cap \{2\} = \{2\} \neq \emptyset$

$B' = \{3\}$

$C = \{2,3\}$

$1 \notin C'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\} \backslash \{1\} \cap \{2,3\} = \emptyset$

$2 \in C'$ since $\forall U = \{2,3\} \in T \ni 2 \in \{2,3\}, \{2,3\} \setminus \{2\} \cap \{2,3\} = \{3\} \neq \emptyset$

$3 \in C'$ since $\forall U = \{2,3\} \in T \ni 3 \in \{2,3\}, \{2,3\} \setminus \{3\} \cap \{2,3\} = \{2\} \neq \emptyset$

$C' = \{2,3\}$.

Theorem:

Let (X,T) be a topological space and A,B be two subsets of X:

1) If **if** $\mathbf{A \subseteq B \Longrightarrow A' \subseteq B'}$ **but** $\mathbf{A' \subseteq B' \nRightarrow A \subseteq B}$
2) $(\mathbf{A} \cup \mathbf{B})' = \mathbf{A}' \cup \mathbf{B}'$
3) $(\mathbf{A} \cap \mathbf{B})' \subseteq \mathbf{A}' \cap \mathbf{B}'$ **but** $\mathbf{A}' \cap \mathbf{B}' \nsubseteq (\mathbf{A} \cap \mathbf{B})'$
4) $\mathbf{A^c} \in \mathbf{T} \Leftrightarrow \mathbf{A}' \subseteq \mathbf{A}.$

Proof:

1) Let $x \in A' \Longrightarrow \forall U \in T \ni x \in U, U \setminus \{x\} \cap A \neq \emptyset$ (By definition of d(A))

$\Longrightarrow \forall U \in T \ni x \in U, U \setminus \{x\} \cap B \neq \emptyset$ ($since$ $A \subseteq B$)

$\Longrightarrow x \in B'$ (By definition of d(B))

$A' \subseteq B'$

Note that, if $\mathbf{A' \subseteq B' \nRightarrow A \nsubseteq B}$ **(disprove)**

For example:

Let $X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{1,2\}\}$

$A = \{3\}, B = \{1,2\}$

$\Longrightarrow A' = \emptyset, B' = \{2,3\}$

$\emptyset \subseteq \{2,3\} \nRightarrow \{3\} \nsubseteq \{1,2\}$

2) To prove $A' \cup B' \subseteq (A \cup B)'$

$A \subseteq A \cup B \wedge B \subseteq A \cup B$ (By proposition of the union)

$A' \subseteq (A \cup B)' \wedge B' \subseteq (A \cup B)'$ (By $A \subseteq B \Longrightarrow A' \subseteq B'$)

$\Longrightarrow (A \cup B)' \subseteq A' \cup B' \ldots (1)$

To prove $(A \cup B)' \subseteq A' \cup B'$

Let $x \in (A \cup B)'$

$\Longrightarrow \forall U \in T \ni x \in U, U\backslash\{x\} \cap (A \cup B) \neq \emptyset$

$\Longrightarrow \forall U \in T \ni x \in U, (U\backslash\{x\} \cap A) \cup (U\backslash\{x\} \cap B) \neq \emptyset$

$\Longrightarrow \forall U \in T \ni x \in U, (U\backslash\{x\} \cap A) \neq \emptyset \vee (U\backslash\{x\} \cap B) \neq \emptyset$

$\Longrightarrow \forall U \in T \ni x \in U, (U\backslash\{x\} \cap A) \neq \emptyset \ \vee \ \forall U \in T \ni x \in U, (U\backslash\{x\} \cap B) \neq \emptyset$

$\Longrightarrow x \in A' \vee x \in B'$ (By definition of derived set)

$\Longrightarrow x \in A' \cup B'$ (By definition of the union)

$\Longrightarrow (A \cup B)' \subseteq A' \cup B' \dots (2)$

By (1)&(2) we get $(A \cup B)' = A' \cup B'$

3) To prove $(A \cap B)' \subseteq A' \cap B'$

Since $A \cap B \subseteq A \wedge A \cap B \subseteq B$ (By proposition of intersection)

$(A \cap B)' \subseteq A' \wedge (A \cap B)' \subseteq B'$ (By $A \subseteq B \Longrightarrow A' \subseteq B'$)

$\Longrightarrow (A \cap B)' \subseteq A' \cap B'$ (By definition of $\cap$)

Note that $A' \cap B' \nsubseteq (A \cap B)'$ in general (disprove)

<u>For example:</u>

Let $X = \{1,2,3\}, T = \{X, \emptyset, \{1\}\}$

$A = \{1\}, B = \{2\}$

$\Longrightarrow A' = \{2,3\}, B' = \{2,3\}$

$\Longrightarrow A \cap B = \{1\} \cap \{2\} = \emptyset$

$\Longrightarrow (A \cap B)' = \emptyset' = \emptyset \wedge A' \cap B' = \{2,3\}$

$A' \cap B' = \{2,3\} \cap \{2,3\} = \{2,3\} \nsubseteq \emptyset$

4) Let $A^c \in T$ to prove $A' \subseteq A$

$\Longrightarrow x \notin A \Longrightarrow x \in A^c$ which is open

$\Longrightarrow \exists U \in T \ni x \in U \subseteq A^c$ (By theorem $A \in T \Leftrightarrow \forall x \in A \exists U \in T \ni x \in U \subseteq A$)

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset$

$\Longrightarrow \exists U \in T \ni x \in U, U\backslash\{x\} \cap A = \emptyset$

$\Longrightarrow x \notin A'$ (By definition of d(A))

$A' \subseteq A$

$\Longleftarrow$ Let $A' \subseteq A$, to prove $A^c \in T$

$\Longrightarrow x \in A^c \Longrightarrow x \notin A \Longrightarrow x \notin A'$

$\Longrightarrow \exists U \in T \ni x \in U, U\backslash\{x\} \cap A = \emptyset$ (By definition of d(A))

$\Longrightarrow \exists U \in T \ni x \in U, U \cap A = \emptyset$ (since $x \notin A$)

$\Longrightarrow \exists U \in T \ni x \in U, U \subseteq A^c$

$\Longrightarrow A^c \in T$ (By theorem. $A \in T \Leftrightarrow \forall x \in A \exists U \in T \ni x \in U \subseteq A$).

Definition(Closure of a set):

Let (X,T) be a topological space and let $A \subseteq X$, the closure set of A is denoted by cl(A), $\bar{A}$ and defined by: $\bar{A} = A \cup A'$.

Example:

Let $\mathbf{X} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \mathbf{T} = \{\mathbf{X}, \emptyset, \{\mathbf{1}\}, \{\mathbf{1}, \mathbf{2}\}\}$

$\mathbf{A} = \{\mathbf{1}, \mathbf{3}\}, \mathbf{B} = \{\mathbf{2}, \mathbf{3}\}, \mathbf{C} = \{\mathbf{1}, \mathbf{2}\}$. Find $\bar{\boldsymbol{A}}, \bar{\boldsymbol{B}}, \bar{\boldsymbol{C}}$

Solution:

$\bar{A} = A \cup A'$. To find $\bar{A}$

$1 \notin A'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\}\backslash\{1\} \cap \{1,3\} = \emptyset$

$2 \in A'$ since $\forall U = \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\}\backslash\{2\} \cap \{1,3\} = \{1\} \neq \emptyset$

$3 \in A'$ since $\exists U = X \in T \ni 3 \in X, X\backslash\{3\} \cap \{1,2\} \neq \emptyset$

$A' = \{2,3\} \Longrightarrow \bar{A} = \{1,3\} \cup \{2,3\} = X$

$\bar{B} = B \cup B'$. To find $\bar{B}$

$1 \notin B'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\}\backslash\{1\} \cap \{2,3\} = \emptyset$

$2 \notin B'$ since $\exists U = \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\}\backslash\{2\} \cap \{2,3\} = \emptyset$

$3 \in B'$ since $\forall U = X \in T \ni 3 \in X, X\backslash\{3\} \cap \{2,3\} = \{2\} \neq \emptyset$

$B' = \{3\} \Longrightarrow \bar{B} = \{2,3\} \cup \{3\} = \{2,3\}$

$\bar{C} = C \cup C'$. To find $\bar{C}$

$1 \notin C'$ since $\exists U = \{1\} \in T \ni 1 \in \{1\}, \{1\}\backslash\{1\} \cap \{1,2\} = \emptyset$

$2 \in C'$ since $\forall U = \{1,2\} \in T \ni 2 \in \{1,2\}, \{1,2\}\backslash\{2\} \cap \{1,2\} = \{1\} \neq \emptyset$

$3 \in C'$ since $\forall U = X \in T \ni 3 \in X, X\backslash\{3\} \cap \{1,2\} = \{1,2\} \neq \emptyset$

$C' = \{2,3\} \Longrightarrow \bar{C} = \{1,2\} \cup \{2,3\} = X.$

Theorem:

Let (X,T) be a topological space and let $A \subseteq X$, then:

$\bar{\mathbf{A}} = \bigcap\{\mathbf{F} \subseteq \mathbf{X} | \mathbf{F}^c \in \mathbf{T} \wedge \mathbf{A} \subseteq \mathbf{F}\}$

i.e. $\bar{\mathbf{A}}$ = smallest closed set containing A

Proof:

We have to show that

$\bar{A} \subseteq \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$ & $\bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\} \subseteq \bar{A}$

To prove $\bar{A} \subseteq \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$

Let $x \in \bar{A} \Longrightarrow x \in A \cup A'$ (By definition of $\bar{A}$)

$\Longrightarrow x \in A \vee x \in A'$ (By definition of union)

If $x \in A \Longrightarrow x \in \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$ (since $A \subseteq F \ \forall F$)

If $x \in A'$ suppose that $x \notin \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$

$\Longrightarrow \exists F \subseteq X \ni F^c \in T \wedge A \subseteq F$ and $x \notin F$

$\Longrightarrow \exists U = F^c \in T \ni x \in U \wedge U \cap A = \emptyset$ ($x \in F^c$, $F^c \cap A = \emptyset$)

$\Longrightarrow \exists U \in T \ni x \in U \wedge U\backslash\{x\} \cap A = \emptyset$

$\Longrightarrow x \notin A'$ which is a contradiction

$x \in \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$

$\bar{A} \subseteq \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\} \ldots (1)$

To prove $\bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\} \subseteq \bar{A}$

Let $x \in \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$

To prove $x \in \bar{A}$

Suppose that $x \notin \bar{A}$

$\Longrightarrow x \notin A \cup A' \Longrightarrow x \notin A \wedge x \notin A'$ (By definition of $\bar{A}$)(By definition of union)

$x \notin A' \Longrightarrow \exists U \in T \ni x \in U \wedge U \backslash \{x\} \cap A = \emptyset$ (By definition of d(A))

$\Longrightarrow \exists U \in T \ni x \in U \wedge U \cap A = \emptyset$ (since $x \notin A$)

$\Longrightarrow U^c$ is closed set and $A \subseteq U^c$ and $x \notin U^c$

$x \notin \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$ which is a contradiction.

$\Longrightarrow x \in \bar{A}$

$\Longrightarrow \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\} \subseteq \bar{A}$ ... (2)

By (1) & (2) we get $\bar{A} = \bigcap\{F \subseteq X | F^c \in T \wedge A \subseteq F\}$

Theorem:

Let (X,T) be a topological space, and A,B be two subsets of X, then:

1) $\mathbf{A \subseteq \bar{A}}$
2) $\mathbf{A \subseteq B \Longrightarrow \bar{A} \subseteq \bar{B}}$ **but** $\mathbf{\bar{A} \subseteq \bar{B} \not\Rightarrow A \subseteq B}$
3) $\mathbf{\overline{(A \cap B)} \subseteq \bar{A} \cap \bar{B}}$ **but** $\mathbf{\bar{A} \cap \bar{B} \nsubseteq \overline{(A \cap B)}}$
4) $\mathbf{\overline{(A \cup B)} = \bar{A} \cup \bar{B}}$
5) $\mathbf{A^c \in T \Leftrightarrow A = \bar{A}}$
6) $\mathbf{\bar{\bar{A}} = \bar{A}, \bar{X} = X, \bar{\emptyset} = \emptyset}$

Proof:

1) Since $A \subseteq A \cup A'$

$\Longrightarrow A \subseteq \bar{A}$ (since $\bar{A} = A \cup A'$)

2) Let $A \subseteq B$ to prove $\bar{A} \subseteq \bar{B}$

$A \subseteq B \Longrightarrow A' \subseteq B'$ (By properties of A')

$\Longrightarrow A \cup A' \subseteq B \cup B'$ (By properties of $\cup$)

$\Longrightarrow \overline{A} \subseteq \overline{B}$ (By definition of cl(A)&cl(B))

Note that $\overline{A} \subseteq \overline{B} \nRightarrow A \subseteq B$ in general (disprove)

For example:

In $(\mathbb{R}, T_u)$, let $A = \{0\}, B = (0,1)$

$\overline{A} = \overline{\{0\}}, \overline{B} = \overline{(0,1)} = [0,1]$

Note that $\overline{A} = \overline{\{0\}} \subseteq [0,1] = \overline{B}$ but $A \nsubseteq B$.

3) To prove $\overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$

Since $A \cap B \subseteq A \ \wedge \ A \cap B \subseteq B$ (By properties of intersection)

$\Longrightarrow \overline{(A \cap B)} \subseteq \overline{A} \wedge \overline{(A \cap B)} \subseteq \overline{B}$ (By $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$)

$\Longrightarrow \overline{(A \cap B)} \subseteq \overline{A} \cap \overline{B}$

Note that $\overline{A} \cap \overline{B} \nsubseteq \overline{(A \cap B)}$ in general (disprove)

For example:

In $(\mathbb{R}, T_u)$, let $A = (2,3), B = [1,2]$

$\Longrightarrow A \cap B = \emptyset \Longrightarrow \overline{(A \cap B)} = \overline{\emptyset} = \emptyset$

$\overline{A} = [2,3], \overline{B} = [1,2]$

$\Longrightarrow \overline{A} \cap \overline{B} = \{2\} \nsubseteq \emptyset = \overline{(A \cap B)}$

4) We have to show that:

$\overline{(A \cup B)} \subseteq \overline{A} \cup \overline{B}$ & $\overline{A} \cup \overline{B} \subseteq \overline{(A \cup B)}$

Since $A \subseteq A \cup B \ \wedge \ B \subseteq A \cup B$

$\Longrightarrow \overline{A} \subseteq \overline{(A \cup B)} \wedge \overline{B} \subseteq \overline{(A \cup B)}$ (By $A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$)

$\Longrightarrow \overline{A} \cup \overline{B} \subseteq \overline{(A \cup B)}$ ... (1)

Since $A \subseteq \overline{A} \ \wedge B \subseteq \overline{B}$ (By (1) of theorem)

$\Longrightarrow A \cup B \subseteq \bar{A} \cup \bar{B}$

And since $\bar{A}, \bar{B}$ are two closed set (By $\bar{A}$ = smallest closed set $\ni A \subseteq \bar{A}$)

$\Longrightarrow \bar{A} \cup \bar{B}$ is closed set (the union of finite number of closed set is closed)

$\bar{A} \cup \bar{B}$ is closed set containing $A \cup B$

$\overline{(A \cup B)} \subseteq \bar{A} \cup \bar{B} \ldots (2)$

By (1)&(2) we get $\overline{(A \cup B)} = \bar{A} \cup \bar{B}$

5) Let $A^c \in T$,To prove $A = \bar{A}$

$A = \bar{A} \ldots (1)$ (By (1) of theorem)

To prove $\bar{A} \subseteq A$

Let $x \in \bar{A} \Longrightarrow \forall U \in T \ni x \in U, U \cap A \neq \emptyset$

If $x \notin A \Longrightarrow x \in A^c \in T$

$\Longrightarrow A^c \cap A \neq \emptyset$ which is contradiction

$x \in A \Longrightarrow \bar{A} \subseteq A \ldots (2)$

By (1)&(2) we get $A = \bar{A}$

Let $A = \bar{A}$, To prove $A^c \in T$

Since $\bar{A}$ is closed set

$\Longrightarrow A$ is closed set (since $A \subseteq \bar{A}$)

$\Longrightarrow A^c$ is open set

$\Longrightarrow A^c \in T$

6) Since $\bar{A}$ is closed set

$\Longrightarrow \bar{\bar{A}} = \bar{A}$ (By $A^c \in T \Leftrightarrow \bar{A} = A$)

Since X is closed set

$\Longrightarrow \bar{X} = X$ (By $A^c \in T \Leftrightarrow \bar{A} = A$)

Since $\emptyset$ is closed set

$\Longrightarrow \overline{\emptyset} = \emptyset$ (By $A^c \in T \Leftrightarrow \overline{A} = A$)

(Homework) Prove Or Disprove:

1) $(\boldsymbol{A} \cap \boldsymbol{B})^{\circ} = \boldsymbol{A}^{\circ} \cap \boldsymbol{B}^{\circ}$
2) $(\boldsymbol{A} \cup \boldsymbol{B})^{\circ} = \boldsymbol{A}^{\circ} \cup \boldsymbol{B}^{\circ}$
3) $(\boldsymbol{A} \cap \boldsymbol{B})^{\boldsymbol{x}} = \boldsymbol{A}^{\boldsymbol{x}} \cap \boldsymbol{B}^{\boldsymbol{x}}$
4) $(\boldsymbol{A} \cup \boldsymbol{B})^{\boldsymbol{x}} = \boldsymbol{A}^{\boldsymbol{x}} \cup \boldsymbol{B}^{\boldsymbol{x}}$
5) $(\boldsymbol{A} \cap \boldsymbol{B})^{\boldsymbol{b}} = \boldsymbol{A}^{\boldsymbol{b}} \cap \boldsymbol{B}^{\boldsymbol{b}}$
6) $(\boldsymbol{A} \cup \boldsymbol{B})^{\boldsymbol{b}} = \boldsymbol{A}^{\boldsymbol{b}} \cup \boldsymbol{B}^{\boldsymbol{b}}$
7) $\overline{(\boldsymbol{A} \cap \boldsymbol{B})} = \overline{\boldsymbol{A}} \cap \overline{\boldsymbol{B}}$
8) $\overline{(\boldsymbol{A} \cup \boldsymbol{B})} = \overline{\boldsymbol{A}} \cup \overline{\boldsymbol{B}}$

Chapter three:Metrizable Topological Spaces

Definition:

Let $X \neq \emptyset$ and $d: X \times X \longrightarrow \mathbb{R}$ is any map, then d is said to be **a metric map** iff:

1) $d(x, y) \geq 0, \forall x, y, z \in X$
2) $d(x, y) = d(y, x)$
3) $d(x, y) = 0 \Leftrightarrow x = y$
4) $d(x, z) \leq d(x, y) + d(y, z)$

Then (X,d) is called a metric space.

Definition:

Let (X,d) be a metric space and let $x \in X$ and $\epsilon > 0$, the ball with center x and radius ϵ is denoted by $B_{\epsilon}(x)$ and defined by:

$$B_{\epsilon}(x) = \{y \in X | \, d(x, y) < \epsilon\}$$

Definition:

Let (X,d) be a metric space and let U be a subset of X, then U is said to be **open set** iff for each $x \in U$, there exists $\epsilon > 0$ such that $B_{\epsilon}(x) \subseteq U$.

i.e. U is open set $\Leftrightarrow \forall x \in U, \exists \epsilon > 0$ such that $B_{\epsilon}(x) \subseteq U$.

Proposition:

Let (X,d) be a metric space and let U $\subseteq$ X, then

U is open set $\Leftrightarrow$ $\mathbf{U = \bigcup_{x\in U} B_\epsilon(x)}$

Now, if (X,d) is a metric space we can induced a topological space from it as follows:

$\mathbf{T_d = \{U \subseteq X | U \text{ is open set w.r.t } d\}}$

T_d is a topology on X since:

1) $X, \emptyset$ are open sets w.r.t d $\Longrightarrow X, \emptyset \in T_d$
2) If $U, V \in T_d$ to prove $U \cap V \in T_d$

U,V are two open sets w.r.t d

$\Longrightarrow U \cap V$ is open set w.r.t d

$\Longrightarrow U \cap V \in T_d$

3) Let $U_\alpha \in T_d \forall \alpha \in \Lambda$, To prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T_d$

$\Longrightarrow U_\alpha$ is open set w.r.t d , $\forall\alpha$

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha$ is open set w.r.t d

$\Longrightarrow \bigcup_{\alpha\in\Lambda} U_\alpha \in T_d$

So (X, T_d) is a topological space which is called a metrizable topological space from a metric space (X,d).

Example:

$(\mathbb{R}, T_u)$ is a metrizable topological space.

Since, in $(\mathbb{R}, d)$

Where $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \ni d(x,y) = |x-y| \forall x, y$ if $x \in \mathbb{R}$ and $\epsilon > 0$

$B_\epsilon(x) = \{d(x,y) < \epsilon; y \in \mathbb{R}\}$

$= \{y \in \mathbb{R};\ |y-x| < \epsilon\}$

$= \{y \in \mathbb{R} | -\epsilon < y - x < \epsilon\} = (x-\epsilon, x+\epsilon)$

That is every open ball in $(\mathbb{R}, d)$ is open interval and so every open set w.r.t d is the union of open intervals.

$\Longrightarrow T_d = \{U \subseteq \mathbb{R} | U = \text{union of open intervals}\} = T_u$

Remark:

We can get a topological space from any metric space, but we can not get a metric space from a topological space.

Example:

1) Let $X = \{1,2,3\}, T = \{X, \emptyset, \{1\}\}$

(X,T) is a topological space, we can not get any metric space from (X,T).

2) Let (X,d) be a metric space such that $d: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \ni d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$

We can defined T_d as follows:
$T_d = \{U \subseteq X | U \text{ is open set w.r.t } d\}$

In this metric space if U is open set then:

$U = \{x\}, x \in X \text{ or } U = X$

i.e. $\{x\} \in T_d \forall x \in X$

$T_d = D$

Therefore, we can say that (X,D) is a metrizable topological space for any $X \neq \emptyset$.

Base Or Basis

Definition:

Let (X,T) be a topological space and let $\mathfrak{B}$ be a subfamily of T, then $\mathfrak{B}$ is said to be **a base** for T iff every open set is a union of a members of $\mathfrak{B}$.

i.e. $\mathfrak{B}$ is a base for T$\Leftrightarrow \forall U \in T, U = \bigcup_{i \in \Lambda} B_i, B_i \in \mathfrak{B} \ \forall i$ and $\mathfrak{B} \subseteq T$

Remarks: In any topological space (X,T):

1) **T is a base for T, which is a trivial base.**
2) **T has more than one base.**
3) **Any base $\mathfrak{B}$ of T, must containing $\emptyset$. (i.e. $\emptyset \in \mathfrak{B}$, for any base $\mathfrak{B}$).**
4) **If $\mathfrak{B}$ is a base for T, then X need not be in $\mathfrak{B}$.**
5) **If $\{x\} \in T, x \in X$, then $\{x\} \in \mathfrak{B}$.**

Example:

Let $X = \{1, 2, 3\}, T = \{X, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$ define anon trivial base for T.

Solution:

Let $\mathfrak{B} = \{X, \emptyset, \{1\}, \{2\}\}$

Note that $\mathfrak{B} \subseteq T$ and $X = X \cup X$

$\emptyset = \emptyset \cup \emptyset$

$\{1\} = \{1\} \cup \{1\}$, $\{2\} = \{2\} \cup \{2\}$

$\{1,2\} = \{1\} \cup \{2\}$

$\mathfrak{B}$ is a base for T.

Example:

Let $X = \{1, 2, 3\}, T = D$ define two non trivial bases for T.

Solution:

$T = D = \mathbb{P}(X) = \{X, \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$

$\mathfrak{B}_1 = \{X, \emptyset, \{1\}, \{2\}, \{3\}\}$

$\mathfrak{B}_1$ is a base for T since:

$\emptyset = \emptyset \cup \emptyset$

$\{1\} = \{1\} \cup \{1\}$

$\{2\} = \{2\} \cup \{2\}$

$\{3\} = \{3\} \cup \{3\}$

$\{1,3\} = \{1\} \cup \{3\}$

$\{1,2\} = \{1\} \cup \{2\}$

$\{2,3\} = \{2\} \cup \{3\}$

$\mathfrak{B}_2 = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2,3\}\} \subseteq T$

$\mathfrak{B}_2$ is a base for T (check).

Q: Define anon trivial base for $(\mathbb{R}, \mathbf{T_u})$

Solution:

$T_u = \{U \subseteq \mathbb{R} | U = \text{union of open intervals}\}$

$\mathfrak{B}_u = \{(a,b): a, b \in \mathbb{R}\} \subseteq T_u$

Note that: $(0,1) \in T_u \wedge (0,1) \in \mathfrak{B}_u$

$(-\infty, 1) \in T_u$ but $(-\infty, 1) \notin \mathfrak{B}_u$

$(0, \infty) \in T_u$ but $(0, \infty) \notin \mathfrak{B}_u$

$\Longrightarrow \mathfrak{B}_u \not\subseteq T_u$

$(a,b) \in T_u \;\&\; (a,b) = (a,b) \cup (a,b)$

$\emptyset \in T_u \;\&\; \emptyset = (a,a) \in \mathfrak{B}_u \;\&\; \emptyset = \emptyset \cup \emptyset$

$(-\infty, b) \in T_u \;\&\; (-\infty, b) = \bigcup_{n \in \mathbb{N}} (a-n, b)$

$(a, \infty) \in T_u \;\&\; (a, \infty) = \bigcup_{n \in \mathbb{N}} (a, b+n)$

$(-\infty, \infty) = \mathbb{R} \in T_u \;\&\; (-\infty, \infty) = \bigcup_{n \in \mathbb{N}} (a-n, b+n)$

$\mathfrak{B}_u$ is anon trivial base for T_u.

Theorem:

Let (X,T) be a topological space and let $\mathfrak{B}$ be a base for T, then:

- $\boldsymbol{X = \bigcup_{i \in \Lambda} B_i \,,\, B_i \in \mathfrak{B} \;\forall i}$
- **If $\boldsymbol{B_1, B_2 \in \mathfrak{B}}$, *then* $\boldsymbol{B_1 \cap B_2 = \bigcup_{i \in \Lambda} B_i \,,\, B_i \in \mathfrak{B} \;\forall i}$**

Proof:

- Since $X \in T$ (By condition (1) of T) and $\mathfrak{B}$ is a base for T(by hyperthesis)

$\Longrightarrow X = \bigcup_{i \in \Lambda} B_i \,,\, B_i \in \mathfrak{B} \;\forall i$ (By definition of base)

- Let $B_1, B_2 \in \mathfrak{B} \subseteq T$

$\Longrightarrow B_1, B_2 \in T$

$\Longrightarrow B_1 \cap B_2 \in T$ (By condition (2) of T)

$\Longrightarrow B_1 \cap B_2 = \bigcup_{i \in \Lambda} B_i \,,\, B_i \in \mathfrak{B} \;\forall i$ (By definition of $\mathfrak{B}$)

Theorem:

Let $X \neq \emptyset$ and let $\mathfrak{B}$ be a subfamily of $\mathbb{P}(X)$ such that:

1) $X = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i$
2) If $B_1, B_2 \in \mathfrak{B} \Longrightarrow B_1 \cap B_2 = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i$
3) $\emptyset \in \mathfrak{B}$**. Then:** $T = \{U \subseteq X; U = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i\}$ **is a topology on X, which is the unique topology with base** $\mathfrak{B}$**.**

Proof:

1) Since $\emptyset = \emptyset \cup \emptyset$ and $\emptyset \in \mathfrak{B} \Longrightarrow \emptyset \in T$ (By (3))

Since $X = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i$ (By (1))

$\Longrightarrow X \in T$

2) Let $U, V \in T$ to prove $U \cap V \in T$

$U = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i$

$V = \bigcup_{j\in\Lambda} B_j , B_j \in \mathfrak{B}\ \forall j$ (By definition of T)

$\Longrightarrow U \cap V = \bigcup_{i\in\Lambda} B_i \cap \bigcup_{j\in\Lambda} B_j = \bigcup_{i,j\in\Lambda} (B_i \cap B_j)$

But $B_i \cap B_j = \bigcup_{k\in\Lambda} B_k , B_k \in \mathfrak{B}$ (By(2))

$\Longrightarrow U \cap V = \bigcup_{i,j\in\Lambda} (\bigcup_{k\in\Lambda} B_k) = \bigcup_{k\in\Lambda} B_k , B_k \in \mathfrak{B}$

$\Longrightarrow U \cap V \in T$

3) Let $U_\alpha \in T \forall \alpha \in \Lambda$, To prove $\bigcup_{\alpha\in\Lambda} U_\alpha \in T$

$\Longrightarrow U_\alpha = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i, \forall \alpha$

$\Longrightarrow \bigcup_{\forall\alpha\in\Lambda} U_\alpha = \bigcup_{\forall\alpha\in\Lambda} (\bigcup_{i\in\Lambda} B_i), B_i \in \mathfrak{B}\ \forall i$

$= \bigcup_{i\in\Lambda} B_i, B_i \in \mathfrak{B}\ \forall i$

$\bigcup_{\alpha\in\Lambda} U_\alpha \in T$

T is a topology on X, which is the unique topology with base $\mathfrak{B}$

Example:

Let $X = \{1,2,3\}, \mathfrak{B} \subseteq \mathbb{P}(X)$ such that $\mathfrak{B} = \{\emptyset, \{1\}, \{2,3\}\}$, find T which $\mathfrak{B}$ is a base for it.

Solution:

$T = \{U \subseteq X; U = \bigcup_{i\in\Lambda} B_i , B_i \in \mathfrak{B}\ \forall i\}$

$\emptyset = \emptyset \cup \{1\} = \{1\}$

$\emptyset \cup \emptyset = \emptyset$

$\emptyset = \emptyset \cup \{2,3\} = \{2,3\}$

$\{1\} \cup \{1\} = \{1\}$

$\{1\} \cup \{2,3\} = X$

$\{2,3\} \cup \{2,3\} = \{2,3\}$

$\Longrightarrow T = \{\emptyset, X, \{1\}, \{2,3\}\}$

Sub base

Definition:

Let (X,T) be a topological space and let $\mathfrak{B}$ be a base for T and let $\mathcal{S}$ be a subfamily of T, then $\mathcal{S}$ is said to be **a sub-base** for T, iff every element of $\mathfrak{B}$ is a finite intersection of members of $\mathcal{S}$.

i.e. $\mathcal{S}$ is a sub-base for T$\Leftrightarrow \mathcal{S} \subseteq T$ and $\forall B \in \mathfrak{B}, B = \bigcap_{j=1}^{n} S_j, S_j \in \mathcal{S} \forall j$

Remarks: In any topological space (X,T)

1) **T is a sub-base for T which is a trivial sub-base.**
2) **There are more than one sub-base for T.**
3) **If $\mathcal{S}$ is a sub-base for T, then $\emptyset$ need not be in $\mathcal{S}$.**
4) **If $\mathcal{S}$ is a sub-base for T, then X need not be in $\mathcal{S}$.**

Example:

Let $X = \{1,2,3\}, T = \{X, \emptyset, \{1\}, \{2\}, \{1,2\}, \{1,3\}\}$,

$\mathfrak{B} = \{\emptyset, \{1\}, \{1,2\}, \{1,3\}\}$ define a sub-base for T.

Solution:

$\mathcal{S} = \{\emptyset, \{1,2\}, \{1,3\}\} \subseteq \mathrm{T}$

$\mathcal{S}$ is a sub-base for T since:

$\emptyset = \emptyset \cap \emptyset$

$\{1\} = \{1,2\} \cap \{1,3\}$

$\{1,2\} = \{1,2\} \cap \{1,2\}$

$\{1,3\} = \{1,3\} \cap \{1,3\}$

Example:

Let $\mathbf{X} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}, \mathbf{T} = \mathbf{D} = \mathbb{P}(\mathbf{X}), \mathfrak{B} = \{\emptyset, \{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{3}\}\}$ **define a sub-base for T.**

Solution:

$\mathcal{S} = \{\{1,2\}, \{1,3\}, \{2,3\}, \{3\}\}$

Note that $\mathcal{S} \subseteq \mathrm{T}$ and

$\emptyset = \{1,2\} \cap \{3\}$

$\{1\} = \{1,2\} \cap \{1,3\}$

$\{2\} = \{1,2\} \cap \{2,3\}$

$\{3\} = \{1,3\} \cap \{2,3\}$

$\mathcal{S}$ is a sub-base for T.

Q: define anon-trivial sub-base for $(\mathbb{R}, \mathbf{T_u})$

Solution:

$\mathrm{T_u} = \{\mathrm{U} \subseteq \mathbb{R} | \mathrm{U} = \text{union of open intervals}\}$

$\mathfrak{B}_u = \{(a, b): a, b \in \mathbb{R}\}$

$\mathcal{S}_u = \{(a, b): a = -\infty \vee b = \infty\}$

Note that:

$(0,1) \in \mathrm{T_u} \wedge (0,1) \in \mathfrak{B}_u, (0,1) \notin \mathcal{S}_u$

$(0, \infty) \in \mathrm{T_u}, (0, \infty) \notin \mathfrak{B}_u, (0, \infty) \in \mathcal{S}_u$

$(-\infty, 1) \in \mathrm{T_u}, (-\infty, 1) \notin \mathfrak{B}_u, (-\infty, 1) \in \mathcal{S}_u$

$(-\infty, \infty) \in \mathrm{T_u}, (-\infty, \infty) \notin \mathfrak{B}_u, (-\infty, \infty) \in \mathcal{S}_u$

$\forall (a, b) \in \mathfrak{B}_u, (a, b) = (-\infty, b) \cap (a, \infty)$

$\mathcal{S}_u$ is anon trivial sub-base for $\mathrm{T_u}$.