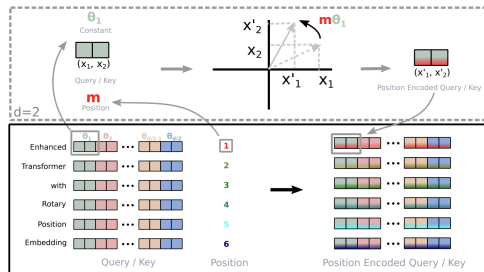


# Rotational Positional Encoding

## RoPE

A. Buynitsky

Jan 28, 2025



# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding
- ④ Derivation
- ⑤ Result

# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding
- ④ Derivation
- ⑤ Result

# What is a Complex Number

Define  $i = \sqrt{-1}$  so  $i^2 = -1$

Any complex number  $z = a + bi$  can be split into its real part:

$$\text{Re}(Z) = a$$

And imaginary part:

$$\text{Im}(z) = b$$

**Example:** Find Re and Im parts of  $(3 + 4i)(2 + 3i)$ :

$$\begin{aligned}(3 + 4i)(2 + 3i) &= 6 + 8i + 9i + 12(i^2) \\ &= 6 + 8i + 9i - 12 \\ &= -6 + 17i\end{aligned}$$

# Euler's Formula

Consider the power series expansion of  $e^z$ :

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$e^{iz} = 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \dots$$

$$= 1 + iz - \frac{(z)^2}{2!} - \frac{iz^3}{3!} + \frac{iz^4}{4!} + \dots$$

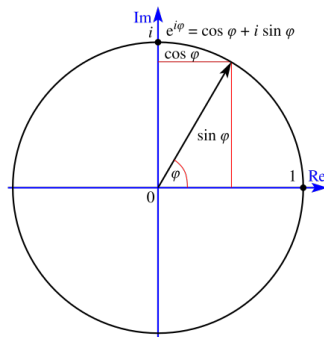
$$= \left(1 - \frac{z^2}{2!} + \frac{iz^4}{4!} + \dots\right) + \left(iz - \frac{iz^3}{3!} + \dots\right)$$

$$= \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\right) + \left(iz - \frac{iz^3}{3!} + \dots\right)$$

$$= \cos(z) + i\sin(z)$$

Therefore:

$$re^{i\theta} = r(\cos(\theta) + i\sin(\theta))$$



# Representation and Properties of Complex Numbers

For a complex number  $z$ :

$$z = re^{i\theta} = r(\cos\theta + \sin\theta) = a + bi$$

**Magnitude of Complex Number:**

$$R(z) = |z| = r = \sqrt{a^2 + b^2}$$

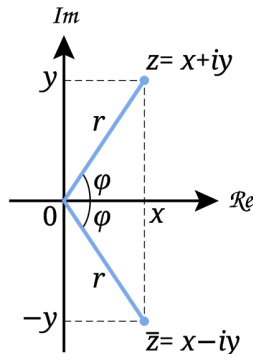
**Argument (angle) of Complex Number:**

$$\Theta(z) = \text{Arg}z = \theta = \arctan\left(\frac{b}{a}\right) \text{ for } -\pi \leq \theta < \pi$$

$$\arg z = \theta + 2\pi k = \arctan\left(\frac{b}{a}\right) + 2\pi k \text{ for } k \in \mathbb{Z}$$

**Conjugate of Complex Number:**

$$\bar{z} = re^{-i\theta} = r(\cos\theta - \sin\theta)$$



# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding
- ④ Derivation
- ⑤ Result

# Transformers (part 1)

Define a sequence of input tokens and corresponding word embeddings:

$$\mathbb{S}_N = \{w_i\}_{i=1}^N$$

$$\mathbb{E}_N = \{x_i\}_{i=1}^N$$

where  $x_i \in \mathbb{R}^d$  is embedding vector of token  $w_i$  **without positional info**:

**Self Attention:**

Calculates:  $q_m = f_q(x_m, m)$ ,  $k_n = f_k(x_n, n)$ ,  $v_n = f_v(x_n, n)$  with:

$$f_{t:t \in \{q,k,v\}} = \mathbf{W}_{t:t \in \{q,k,v\}}(x_i + p_i) \quad (1)$$

where  $p_i$  is:

$$\begin{cases} p_{2t} = \sin\left(\frac{k}{10000} \cdot \frac{2t}{d}\right), \\ p_{2t+1} = \cos\left(\frac{k}{10000} \cdot \frac{2t}{d}\right). \end{cases}$$



# Attention

Calculate attention score and normalize with softmax:

$$a_{m,n} = \frac{\exp\left(\frac{\mathbf{q}_m^\top \mathbf{k}_n}{\sqrt{d}}\right)}{\sum_{j=1}^N \exp\left(\frac{\mathbf{q}_m^\top \mathbf{k}_j}{\sqrt{d}}\right)}$$

Extract values

$$o_m = \sum_{n=1}^N a_{m,n} v_n$$

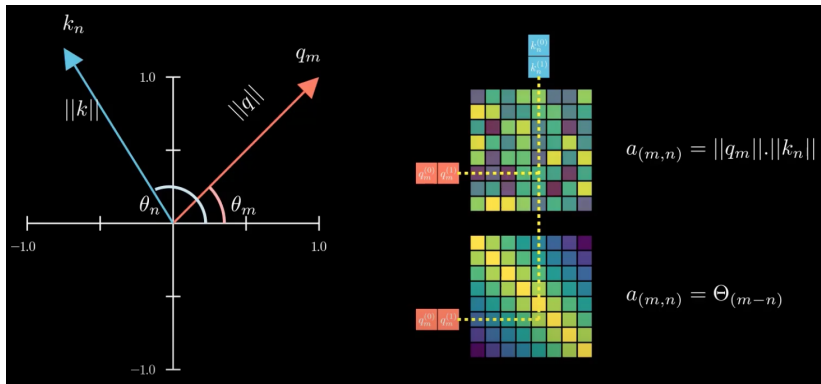
# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding**
- ④ Derivation
- ⑤ Result

# Issue with positional encodings

Let's consider a 2D case:

- Similar tokens should have higher score  $\|q_m\| \cdot \|k_n\|$  (magnitude)
- Closer tokens should have higher score  $\Theta_{(m-n)}$

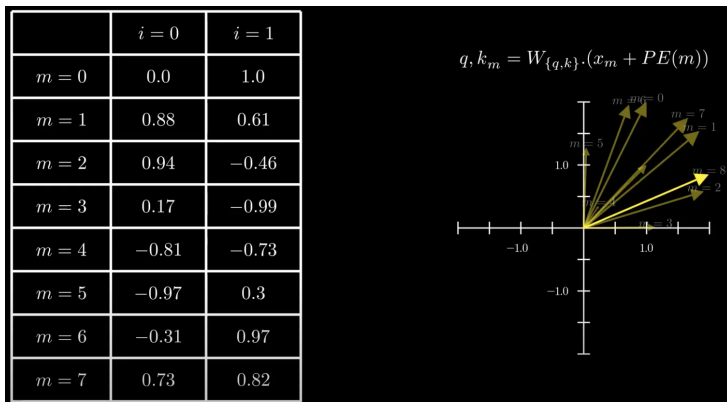


Shoutout to <https://www.youtube.com/watch?v=GQPOTyITy54> for visuals

# Issue with positional encodings (cont)

Let's consider a 2D case:

- Similar tokens should have higher score  $\|q_m\| \cdot \|k_n\|$  (magnitude)
- Closer tokens should have higher score  $\Theta_{(m-n)}$



Problem: Combined positional and token embeddings together

# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding
- ④ Derivation
- ⑤ Result

# Reformulation

For each inner product, we want:

$$q_m^T k_n = \langle f_q(x_m, m), f_k(x_n, n) \rangle = g(x_m, x_n, m - n)$$

In 2D case, convert vectors to complex numbers:

$$q_m = R_q(x_m, m)e^{i\Theta_q(x_m, m)} \quad (2)$$

$$k_n = R_k(x_n, n)e^{i\Theta_k(x_n, n)} \quad (3)$$

$$g(x_m, x_n, n - m) = R_g(x_m, x_n, n - m)e^{i\Theta_g(x_m, x_n, n - m)} \quad (4)$$

$$q_m^T = R_q(x_m, m)e^{-i\Theta_q(x_m, m)} \quad (5)$$

Multiply  $q_m^T \cdot k_n$  in complex numbers:

$$q_m^T \cdot k_n = R_q(x_m, m) \cdot R_k(x_n, n)e^{i(\Theta_k(x_n, n) - \Theta_q(x_m, m))} \quad (6)$$

# Determine Magintude

$$R_q(x_m, m)R_k(x_n, n) = R_g(x_m, x_n, n - m) \quad (7)$$

$$\Theta_k(x_n, n) - \Theta_q(x_m, m) = \Theta_g(x_m, x_n, n - m) \quad (8)$$

Now suppose that  $m = n$ . Then left with just 7

$$R_q(x_m, m)R_k(x_n, n) = R_g(x_m, x_n, 0)$$

Now set  $m = n = 0$

$$R_g(x_m, x_n, 0) = R_q(x_m, 0)R_k(x_n, 0) = ||q|| ||k||$$

Therefore:

$$R_q(x_m, m) = ||q|| \text{ and } R_k(x_n, n) = ||k||$$

# Determine Argument

By a similar trick setting  $m = n$ :

$$\Theta_q(x_m, m) - \Theta(x_n, n) = \Theta_g(x_m, x_n, 0) \quad (9)$$

$$= \Theta(x_m, 0) - \Theta(x_n, 0) = \theta_q - \theta_k \quad (10)$$

Rearranging, we get:

$$\Theta_q(x_m, m) - \theta_q = \Theta_k(x_n, m) - \theta_k$$

Observe that values only related to  $m$ , independent if  $x = x_n$  or  $x = x_m$

Let  $\phi(m)$  to be:

$$\phi(m) = \Theta_q(x_m, m) - \theta_q = \Theta_k(x_n, m) - \theta_k$$



# Determine Argument

Recall:

$$\Theta_q(x_m, m) - \Theta_k(x_n, n) = \Theta_g(x_m, x_n, n - m)$$

and let  $n = m + 1$  so:

$$\Theta_q(x_m, m) - \Theta_k(x_{m+1}, m + 1) = \Theta_g(x_m, x_{m+1}, 1) \quad (11)$$

Now:

$$\phi(m) = \Theta_q(x_m, m) - \theta_q \implies \Theta_q(x_m, m) = \phi(m) + \theta_q$$

$$\phi(m + 1) = \Theta_k(x_{m+1}, m + 1) - \theta_k \implies \Theta_k(x_{m+1}, m + 1) = \phi(m + 1) + \theta_k$$

Plugging into 11:

$$\phi(m) + \theta_q - (\phi(m + 1) + \theta_k) = \Theta_g(x_m, x_{m+1}, 1)$$

$$\phi(m) - \phi(m + 1) = \Theta_g(x_m, x_{m+1}, 1) - \theta_q + \theta_k$$

RHS is constant irrelevant to  $m$ !  $\phi(m)$  has constant difference between term regardless of  $m$  so its an arithmetic sequence!

$$\phi(m) = m\theta + \gamma$$

# Combining Everything

$$q_m = f_q(x_m, m) = R_q(x_m, m)e^{i\Theta_q(x_m, m)} \quad (12)$$

$$= \|q\| e^{i\Theta_q(x_m, m)} \quad (13)$$

$$= \|q\| e^{i(\phi(m) + \theta_q)} \quad (14)$$

$$= \|q\| e^{i(m\theta + \gamma + \theta_q)} \quad (15)$$

$$= \|q\| e^{i\theta_q} e^{i(m\theta + \gamma)} \quad (16)$$

$$= q e^{i(m\theta + \gamma)} \quad (17)$$

Recall that:

$$q_m = f_q(x_m, m) = W_q(x_m + p_m)$$

so if  $p_m = 0$ , then

$$q_m = W_q \cdot x_m$$

Therefore assume no position info when  $m = 0$  so let  $\gamma = 0$ .

**Final Result:**

$$f_q(x_m, m) = (W_q x_m) e^{im\theta}$$

$$f_k(x_n, n) = (W_k x_n) e^{in\theta}$$

# Matrix Equations

## Matrix form of equations in 2D

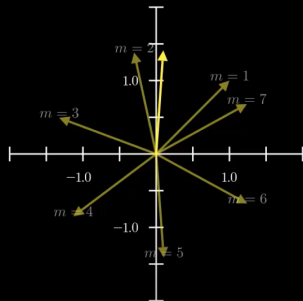
$$d = 2 \quad i = 0 \quad m = 7$$

$$\{q, k\}_m = R_{\Theta, m}^d \cdot \{q, k\}$$

$$\{q, k\} = W_{\{q, k\}} x_m$$

$$R_{\Theta, m}^d = \begin{pmatrix} \cos(m\theta_i) & -\sin(m\theta_i) \\ \sin(m\theta_i) & \cos(m\theta_i) \end{pmatrix}$$

$$\theta_i = 10000^{-2i/d}$$



# Extending beyond d=2

## Extending to multiple dimensions

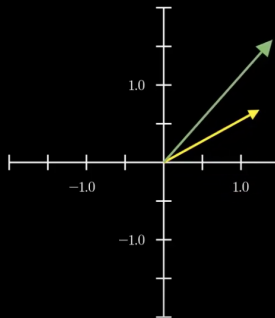
$$d = 4 \quad i = \{0, 1\} \quad m = 6$$

$$\{q, k\}_m = R_{\Theta, m}^d \cdot \{q, k\}$$

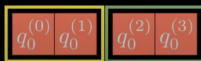
$$\{q, k\} = W_{\{q, k\}} x_m$$

$$R_{\Theta, m}^d = \begin{pmatrix} \boxed{\begin{matrix} \cos(m\theta_0) & -\sin(m\theta_0) \\ \sin(m\theta_0) & \cos(m\theta_0) \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \boxed{\begin{matrix} \cos(m\theta_1) & -\sin(m\theta_1) \\ \sin(m\theta_1) & \cos(m\theta_1) \end{matrix}} \end{pmatrix}$$

$$\theta_i = 10000^{-2i/d}$$



Block 0    Block 1

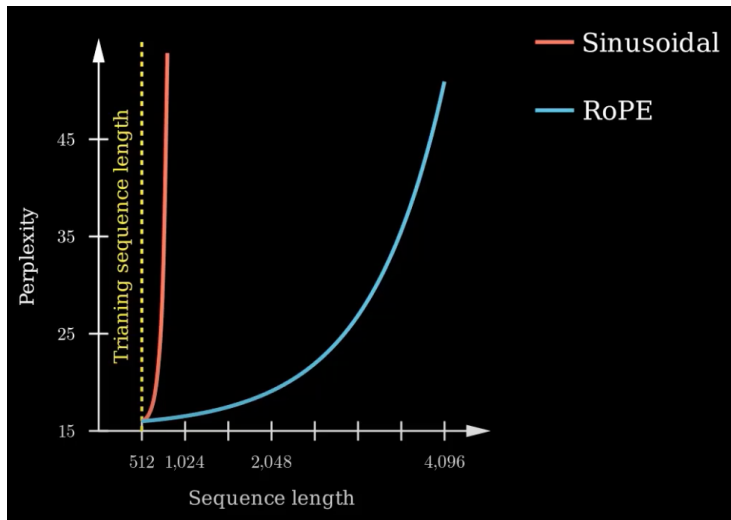


# Outline

- ① Complex Numbers
- ② Transformers
- ③ Issue with Positional Encoding
- ④ Derivation
- ⑤ Result

# Result

Increases Sequence prediction confidence (perplexity):



# Thank you!

Have a great rest of your Day!!!