

Ch.7: Introduction to classes

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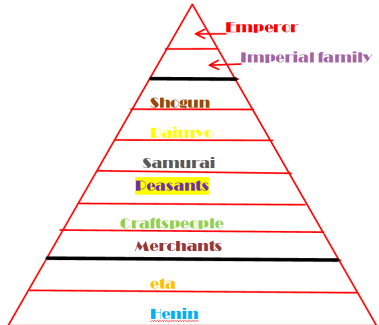
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1 Basics of classes

2 Special methods

Basics of classes



Class = functions + data (variables) in one unit

- A class packs together data (a collection of variables) and functions as *one single unit*
- As a programmer you can create a new class and thereby a new object type (like `float`, `list`, `file`, ...)
- A class is much like a module: a collection of “global” variables and functions that belong together
- There is only one instance of a module while a class can have many instances (copies)
- Modern programming applies classes to a large extent
- It will take some time to master the class concept
- Let's learn by doing!

Representing a function by a class; background

Consider a function of t with a parameter v_0 :

$$y(t; v_0) = v_0 t - \frac{1}{2} g t^2$$

We need both v_0 and t to evaluate y (and $g = 9.81$), but how should we implement this?

Having t and v_0 as arguments:

```
def y(t, v0):  
    g = 9.81  
    return v0*t - 0.5*g*t**2
```

Having t as argument and v_0 as global variable:

```
def y(t):  
    g = 9.81  
    return v0*t - 0.5*g*t**2
```

Motivation: $y(t)$ is a function of t only

Representing a function by a class; idea

- With a class, $y(t)$ can be a function of t only, but still have v_0 and g as parameters with given values.
- The class packs together a function $y(t)$ and data (v_0, g)

Representing a function by a class; technical overview

- We make a class `Y` for $y(t; v_0)$ with variables `v0` and `g` and a function `value(t)` for computing $y(t; v_0)$
- Any class should also have a function `__init__` for initialization of the variables

<code>Y</code>
<code>__init__</code> <code>value</code> <code>formula</code> <code>__call__</code> <code>__str__</code>
<code>g</code> <code>v0</code>

Representing a function by a class; the code

```
class Y:
    def __init__(self, v0):
        self.v0 = v0
        self.g = 9.81

    def value(self, t):
        return self.v0*t - 0.5*self.g*t**2
```

Usage:

```
y = Y(v0=3)           # create instance (object)
v = y.value(0.1)      # compute function value
```


Representing a function by a class; the constructor

When we write

```
y = Y(v0=3)
```

we create a new variable (instance) `y` of type `Y`. `Y(3)` is a call to the *constructor*:

```
def __init__(self, v0):  
    self.v0 = v0  
    self.g = 9.81
```

What is this `self` variable? Stay cool - it will be understood later as you get used to it

- Think of `self` as `y`, i.e., the new variable to be created.
`self.v0 = ...` means that we attach a variable `v0` to `self` (`y`).
- `Y(3)` means `Y.__init__(y, 3)`, i.e., set `self=y`, `v0=3`
- Remember: `self` is always first parameter in a function, but never inserted in the call!
- After `y = Y(3)`, `y` has two variables `v0` and `g`

```
print y.v0  
print y.g
```

In mathematics you don't understand things. You just get used to them. John von Neumann, mathematician, 1903-1957.

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print y.v0  
print y.g
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Representing a function by a class; the value method

- Functions in classes are called *methods*
- Variables in classes are called *attributes*

Here is the value method:

```
def value(self, t):  
    return self.v0*t - 0.5*self.g*t**2
```

Example on a call:

```
v = y.value(t=0.1)
```

`self` is left out in the call, but Python automatically inserts `y` as the `self` argument inside the `value` method. Think of the call as

```
Y.value(y, t=0.1)
```

Inside `value` things “appear” as

```
return y.v0*t - 0.5*y.g*t**2
```

`self` gives access to “global variables” in the class object.

Representing a function by a class; summary

- Class Y collects the attributes `v0` and `g` and the method `value` as one unit
- `value(t)` is function of `t` only, but has automatically access to the parameters `v0` and `g` as `self.v0` and `self.g`
- The great advantage: we can send `y.value` as an ordinary function of `t` to any other function that expects a function `f(t)` of one variable

```
def make_table(f, tstop, n):  
    for t in linspace(0, tstop, n):  
        print t, f(t)  
  
def g(t):  
    return sin(t)*exp(-t)  
  
table(g, 2*pi, 101)           # send ordinary function  
  
y = Y(6.5)  
table(y.value, 2*pi, 101)     # send class method
```

Representing a function by a class; the general case

Given a function with $n + 1$ parameters and one independent variable,

$$f(x; p_0, \dots, p_n)$$

it is wise to represent f by a class where p_0, \dots, p_n are attributes and where there is a method, say `value(self, x)`, for computing $f(x)$

```
class MyFunc:
    def __init__(self, p0, p1, p2, ..., pn):
        self.p0 = p0
        self.p1 = p1
        ...
        self.pn = pn

    def value(self, x):
        return ...
```

Class for a function with four parameters

$$v(r; \beta, \mu_0, n, R) = \left(\frac{\beta}{2\mu_0} \right)^{\frac{1}{n}} \frac{n}{n+1} \left(R^{1+\frac{1}{n}} - r^{1+\frac{1}{n}} \right)$$

```
class VelocityProfile:
    def __init__(self, beta, mu0, n, R):
        self.beta, self.mu0, self.n, self.R = \
            beta, mu0, n, R

    def value(self, r):
        beta, mu0, n, R = \
            self.beta, self.mu0, self.n, self.R
        n = float(n) # ensure float divisions
        v = (beta/(2.0*mu0))*(1/n)*(n/(n+1))*\
            (R**(1+1/n) - r**(1+1/n))
        return v

v = VelocityProfile(R=1, beta=0.06, mu0=0.02, n=0.1)
print v.value(r=0.1)
```


Rough sketch of a Python class

```
class MyClass:
    def __init__(self, p1, p2):
        self.attr1 = p1
        self.attr2 = p2

    def method1(self, arg):
        # can init new attribute outside constructor:
        self.attr3 = arg
        return self.attr1 + self.attr2 + self.attr3

    def method2(self):
        print 'Hello!'

m = MyClass(4, 10)
print m.method1(-2)
m.method2()
```

It is common to have a constructor where attributes are initialized, but this is not a requirement - attributes can be defined whenever desired

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You can learn about other versions and views of class Y in the course book

- The book features a section on a different version of class Y where there is no constructor (which is possible)
- The book also features a section on how to implement classes without using classes
- These sections may be clarifying - or confusing

Another class example: a bank account

- Attributes: name of owner, account number, balance
- Methods: deposit, withdraw, pretty print

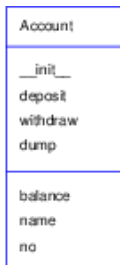
```
class Account:
    def __init__(self, name, account_number, initial_amount):
        self.name = name
        self.no = account_number
        self.balance = initial_amount

    def deposit(self, amount):
        self.balance += amount

    def withdraw(self, amount):
        self.balance -= amount

    def dump(self):
        s = '%s, %s, balance: %s' % \
            (self.name, self.no, self.balance)
        print s
```

UML diagram of class Account



Example on using class Account

```
>>> a1 = Account('John Olsson', '19371554951', 20000)
>>> a2 = Account('Liz Olsson', '19371564761', 20000)
>>> a1.deposit(1000)
>>> a1.withdraw(4000)
>>> a2.withdraw(10500)
>>> a1.withdraw(3500)
>>> print "a1's balance:", a1.balance
a1's balance: 13500
>>> a1.dump()
John Olsson, 19371554951, balance: 13500
>>> a2.dump()
Liz Olsson, 19371564761, balance: 9500
```

Use underscore in attribute names to avoid misuse

Possible, but not intended use:

```
>>> a1.name = 'Some other name'  
>>> a1.balance = 100000  
>>> a1.no = '19371564768'
```

The assumptions on correct usage:

- The attributes should *not* be changed!
- The balance attribute can be viewed
- Changing balance is done through `withdraw` or `deposit`

Remedy:

Attributes and methods not intended for use outside the class can be marked as *protected* by prefixing the name with an underscore (e.g., `_name`). This is just a convention - and no technical way of avoiding attributes and methods to be accessed.

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Improved class with attribute protection (underscore)

```
class AccountP:
    def __init__(self, name, account_number, initial_amount):
        self._name = name
        self._no = account_number
        self._balance = initial_amount

    def deposit(self, amount):
        self._balance += amount

    def withdraw(self, amount):
        self._balance -= amount

    def get_balance(self):      # NEW - read balance value
        return self._balance

    def dump(self):
        s = '%s, %s, balance: %s' % \
            (self._name, self._no, self._balance)
        print s
```

Usage of improved class AccountP

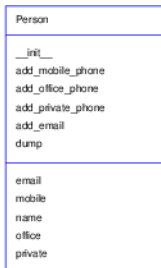
```
a1 = AccountP('John Olsson', '19371554951', 20000)
a1.withdraw(4000)

print a1._balance      # it works, but a convention is broken
print a1.get_balance() # correct way of viewing the balance
a1._no = '19371554955' # this is a "serious crime"!
```

Another example: a phone book

- A phone book is a list of data about persons
- Data about a person: name, mobile phone, office phone, private phone, email
- Let us create a class for data about a person!
- Methods:
 - Constructor for initializing name, plus one or more other data
 - Add new mobile number
 - Add new office number
 - Add new private number
 - Add new email
 - Write out person data

UML diagram of class Person



Basic code of class Person

```
class Person:
    def __init__(self, name,
                  mobile_phone=None, office_phone=None,
                  private_phone=None, email=None):
        self.name = name
        self.mobile = mobile_phone
        self.office = office_phone
        self.private = private_phone
        self.email = email

    def add_mobile_phone(self, number):
        self.mobile = number

    def add_office_phone(self, number):
        self.office = number

    def add_private_phone(self, number):
        self.private = number

    def add_email(self, address):
        self.email = address
```


Code of a dump method for printing all class contents

```
class Person:
    ...
    def dump(self):
        s = self.name + '\n'
        if self.mobile is not None:
            s += 'mobile phone:  %s\n' % self.mobile
        if self.office is not None:
            s += 'office phone:   %s\n' % self.office
        if self.private is not None:
            s += 'private phone:  %s\n' % self.private
        if self.email is not None:
            s += 'email address:  %s\n' % self.email
        print s
```

Usage:

```
p1 = Person('Hans Petter Langtangen', email='hpl@simula.no')
p1.add_office_phone('67828283'),
p2 = Person('Aslak Tveito', office_phone='67828282')
p2.add_email('aslak@simula.no')
phone_book = [p1, p2]                # list
phone_book = {'Langtangen': p1, 'Tveito': p2}  # better
for p in phone_book:
    p.dump()
```

Another example: a class for a circle

- A circle is defined by its center point x_0, y_0 and its radius R
- These data can be attributes in a class
- Possible methods in the class: area, circumference
- The constructor initializes x_0, y_0 and R

```
class Circle:
    def __init__(self, x0, y0, R):
        self.x0, self.y0, self.R = x0, y0, R

    def area(self):
        return pi*self.R**2

    def circumference(self):
        return 2*pi*self.R
```

```
>>> c = Circle(2, -1, 5)
>>> print 'A circle with radius %g at (%g, %g) has area %g' % \
...      (c.R, c.x0, c.y0, c.area())
A circle with radius 5 at (2, -1) has area 78.5398
```

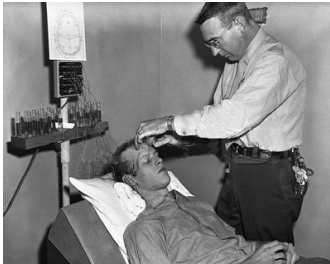
Test function for class Circle

```
def test_Circle():  
    R = 2.5  
    c = Circle(7.4, -8.1, R)  
  
    from math import pi  
    exact_area = pi*R**2  
    computed_area = c.area()  
    diff = abs(exact_area - computed_area)  
    tol = 1E-14  
    assert diff < tol, 'bug in Circle.area, diff=%s' % diff  
  
    exact_circumference = 2*pi*R  
    computed_circumference = c.circumference()  
    diff = abs(exact_circumference - computed_circumference)  
    assert diff < tol, 'bug in Circle.circumference, diff=%s' % dif
```

1 Basics of classes

2 Special methods

Special methods



```
class MyClass:  
    def __init__(self, a, b):  
        ...
```

```
p1 = MyClass(2, 5)  
p2 = MyClass(-1, 10)
```

```
p3 = p1 + p2  
p4 = p1 - p2  
p5 = p1*p2  
p6 = p1**7 + 4*p3
```

Special methods allow nice syntax and are recognized by double leading and trailing underscores

```
def __init__(self, ...)  
def __call__(self, ...)  
def __add__(self, other)  
  
# Python syntax  
y = Y(4)  
print y(2)  
z = Y(6)  
print y + z  
  
# What's actually going on  
Y.__init__(y, 4)  
print Y.__call__(y, 2)  
Y.__init__(z, 6)  
print Y.__add__(y, z)
```

We shall learn about many more such *special methods*

Example on a call special method

Replace the value method by a *call* special method:

```
class Y:
    def __init__(self, v0):
        self.v0 = v0
        self.g = 9.81

    def __call__(self, t):
        return self.v0*t - 0.5*self.g*t**2
```

Now we can write

```
y = Y(3)
v = y(0.1)  # same as v = y.__call__(0.1) or Y.__call__(y, 0.1)
```

Note:

- The instance `y` behaves and looks as a function!
- The `value(t)` method does the same, but `__call__` allows nicer syntax for computing function values

Representing a function by a class revisited

Given a function with $n + 1$ parameters and one independent variable,

$$f(x; p_0, \dots, p_n)$$

it is wise to represent f by a class where p_0, \dots, p_n are attributes and `__call__(x)` computes $f(x)$

```
class MyFunc:
    def __init__(self, p0, p1, p2, ..., pn):
        self.p0 = p0
        self.p1 = p1
        ...
        self.pn = pn

    def __call__(self, x):
        return ...
```


Can we automatically differentiate a function?

Given some mathematical function in Python, say

```
def f(x):  
    return x**3
```

can we make a class `Derivative` and write

```
dfdx = Derivative(f)
```

so that `dfdx` behaves as a function that computes the derivative of $f(x)$?

```
print dfdx(2)    # computes 3*x**2 for x=2
```

Automatic differentiation; solution

Method

We use numerical differentiation “behind the curtain”:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

for a small (yet moderate) h , say $h = 10^{-5}$

Implementation

```
class Derivative:
    def __init__(self, f, h=1E-5):
        self.f = f
        self.h = float(h)

    def __call__(self, x):
        f, h = self.f, self.h          # make short forms
        return (f(x+h) - f(x))/h
```

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        return (f(x+h) - f(x))/h
```

Automatic differentiation; demo

```
>>> from math import *
>>> df = Derivative(sin)
>>> x = pi
>>> df(x)
-1.0000000082740371
>>> cos(x)  # exact
-1.0
>>> def g(t):
...     return t**3
...
>>> dg = Derivative(g)
>>> t = 1
>>> dg(t)  # compare with 3 (exact)
3.000000248221113
```

Automatic differentiation; useful in Newton's method

Newton's method solves nonlinear equations $f(x) = 0$, but the method requires $f'(x)$

```
def Newton(f, xstart, dfdx, epsilon=1E-6):  
    ...  
    return x, no_of_iterations, f(x)
```

Suppose $f'(x)$ requires boring/lengthy derivation, then class Derivative is handy:

```
>>> def f(x):  
...     return 100000*(x - 0.9)**2 * (x - 1.1)**3  
...  
>>> df = Derivative(f)  
>>> xstart = 1.01  
>>> Newton(f, xstart, df, epsilon=1E-5)  
(1.0987610068093443, 8, -7.5139644257961411e-06)
```

Automatic differentiation; test function

- How can we test class Derivative?
- Method 1: compute $(f(x + h) - f(x))/h$ by hand for some f and h
- Method 2: utilize that linear functions are differentiated exactly by our numerical formula, regardless of h

Test function based on method 2:

```
def test_Derivative():  
    # The formula is exact for linear functions, regardless of h  
    f = lambda x: a*x + b  
    a = 3.5; b = 8  
    dfdx = Derivative(f, h=0.5)  
    diff = abs(dfdx(4.5) - a)  
    assert diff < 1E-14, 'bug in class Derivative, diff=%s' % diff
```

Automatic differentiation; explanation of the test function

Use of lambda functions:

```
f = lambda x: a*x + b
```

is equivalent to

```
def f(x):  
    return a*x + b
```

Lambda functions are convenient for producing quick, short code

Use of closure:

```
f = lambda x: a*x + b  
a = 3.5; b = 8  
dfdx = Derivative(f, h=0.5)  
dfdx(4.5)
```

Looks straightforward...but

- How can `Derivative.__call__` know `a` and `b` when it calls our `f(x)` function?
- Local functions inside functions remember (have access to) *all* local variables in the function they are defined (!)
- `f` can access `a` and `b` in `test_Derivative` even when called

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```
f = lambda x: a*x + b  
a = 3.5; b = 8  
dfdx = Derivative(f, h=0.5)  
dfdx(4.5)
```

Looks straightforward...but

- How can `Derivative.__call__` know `a` and `b` when it calls our `f(x)` function?
- Local functions inside functions remember (have access to) *all* local variables in the function they are defined (!)
- `f` can access `a` and `b` in `test.Derivative` even when called

Automatic differentiation detour; sympy solution (exact differentiation via symbolic expressions)

SymPy can perform exact, symbolic differentiation:

```
>>> from sympy import *
>>> def g(t):
...     return t**3
...
>>> t = Symbol('t')
>>> dgdt = diff(g(t), t)           # compute g'(t)
>>> dgdt
3*t**2

>>> # Turn sympy expression dgdt into Python function dg(t)
>>> dg = lambdify([t], dgdt)
>>> dg(1)
3
```

Automatic differentiation detour; class based on sympy

```
import sympy as sp

class Derivative_sympy:
    def __init__(self, f):
        # f: Python f(x)
        x = sp.Symbol('x')
        sympy_f = f(x)
        sympy_dfdx = sp.diff(sympy_f, x)
        self.__call__ = sp.lambdify([x], sympy_dfdx)
```

```
>>> def g(t):
...     return t**3

>>> def h(y):
...     return sp.sin(y)

>>> dg = Derivative_sympy(g)
>>> dh = Derivative_sympy(h)
>>> dg(1)    # 3*1**2 = 3
3
>>> from math import pi
>>> dh(pi)   # cos(pi) = -1
-1.0
```

Automatic integration; problem setting

Given a function $f(x)$, we want to compute

$$F(x; a) = \int_a^x f(t) dt$$

Technique: Trapezoidal rule

$$\int_a^x f(t) dt = h \left(\frac{1}{2} f(a) + \sum_{i=1}^{n-1} f(a + ih) + \frac{1}{2} f(x) \right)$$

Desired application code:

```
def f(x):  
    return exp(-x**2)*sin(10*x)  
  
a = 0; n = 200  
F = Integral(f, a, n)  
x = 1.2  
print F(x)
```

Automatic integration; implementation

```
def trapezoidal(f, a, x, n):  
    h = (x-a)/float(n)  
    I = 0.5*f(a)  
    for i in range(1, n):  
        I += f(a + i*h)  
    I += 0.5*f(x)  
    I *= h  
    return I
```

Class Integral holds f, a and n as attributes and has a call special method for computing the integral:

```
class Integral:  
    def __init__(self, f, a, n=100):  
        self.f, self.a, self.n = f, a, n  
  
    def __call__(self, x):  
        return trapezoidal(self.f, self.a, x, self.n)
```

Automatic differentiation; test function

- How can we test class Derivative?
- Method 1: compute $(f(x + h) - f(x))/h$ by hand for some f and h
- Method 2: utilize that linear functions are differentiated exactly by our numerical formula, regardless of h

Test function based on method 2:

```
def test_Derivative():  
    # The formula is exact for linear functions, regardless of h  
    f = lambda x: a*x + b  
    a = 3.5; b = 8  
    dfdx = Derivative(f, h=0.5)  
    diff = abs(dfdx(4.5) - a)  
    assert diff < 1E-14, 'bug in class Derivative, diff=%s' % diff
```

Automatic differentiation; explanation of the test function

Use of lambda functions:

```
f = lambda x: a*x + b
```

is equivalent to

```
def f(x):  
    return a*x + b
```

Lambda functions are convenient for producing quick, short code

Use of closure:

```
f = lambda x: a*x + b  
a = 3.5; b = 8  
dfdx = Derivative(f, h=0.5)  
dfdx(4.5)
```

Looks straightforward...but

- How can `Derivative.__call__` know `a` and `b` when it calls our `f(x)` function?
- Local functions inside functions remember (have access to) *all* local variables in the function they are defined (!)
- `f` can access `a` and `b` in `test_Derivative` even when called

Automatic differentiation; explanation of the test function

Use of lambda functions:

```
f = lambda x: a*x + b
```

is equivalent to

```
def f(x):  
    return a*x + b
```

Lambda functions are convenient for producing quick, short code

Use of closure:

```
f = lambda x: a*x + b  
a = 3.5; b = 8  
dfdx = Derivative(f, h=0.5)  
dfdx(4.5)
```

Looks straightforward...but

- How can `Derivative.__call__` know `a` and `b` when it calls our `f(x)` function?
- Local functions inside functions remember (have access to) *all* local variables in the function they are defined (!)
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Automatic differentiation; explanation of the test function

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Lambda functions are convenient for producing quick, short code

Use of closure:

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- How can `Derivative.__call__` know `a` and `b` when it calls our `f(x)` function?
- Local functions inside functions remember (have access to) *all* local variables in the function they are defined (!)
- `f` can access `a` and `b` in `test.Derivative` even when called

Automatic integration; test function

- Method 1: compute with the Trapezoidal rule by hand
- Method 2: utilize the fact that the Trapezoidal rule integrates linear functions exactly

Test function based on method 2:

```
def test_Integral():  
    # The Trapezoidal rule is exact for linear functions  
    f = lambda x: 2*x + 5  
    F = lambda x: x**2 + 5*x # integral of f  
    a = 2  
    I = Integral(f, a, n=4)  
    x = 6  
    diff = abs(I(x) - (F(x) - F(a)))  
    assert diff < 1E-15, 'bug in class Integral, diff=%s' % diff
```

Special method for printing

- In Python, we can usually print an object `a` by `print a`, works for built-in types (strings, lists, floats, ...)
- Python does not know how to print objects of a user-defined class, but if the class defines a method `__str__`, Python will use this method to convert an object to a string

Example:

```
class Y:
    ...
    def __call__(self, t):
        return self.v0*t - 0.5*self.g*t**2

    def __str__(self):
        return 'v0*t - 0.5*g*t**2; v0=%g' % self.v0
```

Demo:

```
>>> y = Y(1.5)
>>> y(0.2)
0.1038
>>> print y
v0*t - 0.5*g*t**2; v0=1.5
```

Class for polynomials; functionality

A polynomial can be specified by a list of its coefficients. For example, $1 - x^2 + 2x^3$ is

$$1 + 0 \cdot x - 1 \cdot x^2 + 2 \cdot x^3$$

and the coefficients can be stored as `[1, 0, -1, 2]`

Desired application code:

```
>>> p1 = Polynomial([1, -1])
>>> print p1
1 - x
>>> p2 = Polynomial([0, 1, 0, 0, -6, -1])
>>> p3 = p1 + p2
>>> print p3.coeff
[1, 0, 0, 0, -6, -1]
>>> print p3
1 - 6*x^4 - x^5
>>> p2.differentiate()
>>> print p2
1 - 24*x^3 - 5*x^4
```

How can we make class `Polynomial`?

Class Polynomial; basic code

```
class Polynomial:
    def __init__(self, coefficients):
        self.coeff = coefficients

    def __call__(self, x):
        s = 0
        for i in range(len(self.coeff)):
            s += self.coeff[i]*x**i
        return s
```

Class Polynomial; addition

```
class Polynomial:
    ...

    def __add__(self, other):
        # return self + other

        # start with the longest list and add in the other:
        if len(self.coeff) > len(other.coeff):
            coeffsum = self.coeff[:] # copy!
            for i in range(len(other.coeff)):
                coeffsum[i] += other.coeff[i]
        else:
            coeffsum = other.coeff[:] # copy!
            for i in range(len(self.coeff)):
                coeffsum[i] += self.coeff[i]
        return Polynomial(coeffsum)
```

Class Polynomial; multiplication

Mathematics:

Multiplication of two general polynomials:

$$\left(\sum_{i=0}^M c_i x^i \right) \left(\sum_{j=0}^N d_j x^j \right) = \sum_{i=0}^M \sum_{j=0}^N c_i d_j x^{i+j}$$

The coeff. corresponding to power $i+j$ is $c_i \cdot d_j$. The list `r` of coefficients of the result: `r[i+j] = c[i]*d[j]` (`i` and `j` running from 0 to M and N , resp.)

Implementation:

```
class Polynomial:
    ...
    def __mul__(self, other):
        M = len(self.coeff) - 1
        N = len(other.coeff) - 1
        coeff = [0]*(M+N+1) # or zeros(M+N+1)
        for i in range(0, M+1):
            for j in range(0, N+1):
                coeff[i+j] += self.coeff[i]*other.coeff[j]
        return Polynomial(coeff)
```


Class Polynomial; multiplication

Mathematics:

Multiplication of two general polynomials:

$$\left(\sum_{i=0}^M c_i x^i \right) \left(\sum_{j=0}^N d_j x^j \right) = \sum_{i=0}^M \sum_{j=0}^N c_i d_j x^{i+j}$$

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Implementation:

```
class Polynomial:
    ...
    def __mul__(self, other):
        M = len(self.coeff) - 1
        N = len(other.coeff) - 1
        coeff = [0]*(M+N+1) # or zeros(M+N+1)
        for i in range(0, M+1):
            for j in range(0, N+1):
                coeff[i+j] += self.coeff[i]*other.coeff[j]
        return Polynomial(coeff)
```

Class Polynomial; multiplication

Mathematics:

Multiplication of two general polynomials:

$$\left(\sum_{i=0}^M c_i x^i \right) \left(\sum_{j=0}^N d_j x^j \right) = \sum_{i=0}^M \sum_{j=0}^N c_i d_j x^{i+j}$$

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        M = len(self.coeff) - 1
        N = len(other.coeff) - 1
        coeff = [0]*(M+N+1) # or zeros(M+N+1)
        for i in range(0, M+1):
            for j in range(0, N+1):
                coeff[i+j] += self.coeff[i]*other.coeff[j]
        return Polynomial(coeff)
```

Class Polynomial; differentiation

Mathematics:

Rule for differentiating a general polynomial:

$$\frac{d}{dx} \sum_{i=0}^n c_i x^i = \sum_{i=1}^n i c_i x^{i-1}$$

If c is the list of coefficients, the derivative has a list of coefficients, dc , where $dc[i-1] = i*c[i]$ for i running from 1 to the largest index in c . Note that dc has one element less than c .

Implementation:

```
class Polynomial:
    ...
    def differentiate(self):      # change self
        for i in range(1, len(self.coeff)):
            self.coeff[i-1] = i*self.coeff[i]
        del self.coeff[-1]

    def derivative(self):        # return new polynomial
        dpdx = Polynomial(self.coeff[:]) # copy
        dpdx.differentiate()
        return dpdx
```

Class Polynomial; pretty print

```
class Polynomial:
    ...
    def __str__(self):
        s = ''
        for i in range(0, len(self.coeff)):
            if self.coeff[i] != 0:
                s += ' + %g*x^%d' % (self.coeff[i], i)
        # fix layout (lots of special cases):
        s = s.replace('+ -', '- ')
        s = s.replace(' 1*', ' ')
        s = s.replace('x^0', '1')
        s = s.replace('x^1 ', 'x ')
        s = s.replace('x^1', 'x')
        if s[0:3] == ' + ': # remove initial +
            s = s[3:]
        if s[0:3] == ' - ': # fix spaces for initial -
            s = '- ' + s[3:]
        return s
```

Class for polynomials; usage

Consider

$$p_1(x) = 1 - x, \quad p_2(x) = x - 6x^4 - x^5$$

and their sum

$$p_3(x) = p_1(x) + p_2(x) = 1 - 6x^4 - x^5$$

```
>>> p1 = Polynomial([1, -1])
>>> print p1
1 - x
>>> p2 = Polynomial([0, 1, 0, 0, -6, -1])
>>> p3 = p1 + p2
>>> print p3.coeff
[1, 0, 0, 0, -6, -1]
>>> p2.differentiate()
>>> print p2
1 - 24*x^3 - 5*x^4
```

The programmer is in charge of defining special methods!

How should, e.g., `__add__(self, other)` be defined? This is completely up to the programmer, depending on what is meaningful by `object1 + object2`.

An anthropologist was asking a primitive tribesman about arithmetic. When the anthropologist asked, *What does two and two make?* the tribesman replied, *Five*. Asked to explain, the tribesman said, *If I have a rope with two knots, and another rope with two knots, and I join the ropes together, then I have five knots.*

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How should, e.g., `__add__(self, other)` be defined? This is completely up to the programmer, depending on what is meaningful by `object1 + object2`.

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An anthropologist was asking a primitive tribesman about arithmetic. When the anthropologist asked, *What does two and two make?* the tribesman replied, *Five*. Asked to explain, the tribesman said, *If I have a rope with two knots, and another rope with two knots, and I join the ropes together, then I have five knots.*

Special methods for arithmetic operations

`c = a + b` `# c = a.__add__(b)`

`c = a - b` `# c = a.__sub__(b)`

`c = a*b` `# c = a.__mul__(b)`

`c = a/b` `# c = a.__div__(b)`

`c = a**e` `# c = a.__pow__(e)`

Special methods for comparisons

`a == b` `# a.__eq__(b)`

`a != b` `# a.__ne__(b)`

`a < b` `# a.__lt__(b)`

`a <= b` `# a.__le__(b)`

`a > b` `# a.__gt__(b)`

`a >= b` `# a.__ge__(b)`

Class for vectors in the plane

Mathematical operations for vectors in the plane:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b) - (c, d) = (a - c, b - d)$$

$$(a, b) \cdot (c, d) = ac + bd$$

$$(a, b) = (c, d) \text{ if } a = c \text{ and } b = d$$

Desired application code:

```
>>> u = Vec2D(0,1)
>>> v = Vec2D(1,0)
>>> print u + v
(1, 1)
>>> a = u + v
>>> w = Vec2D(1,1)
>>> a == w
True
>>> print u - v
(-1, 1)
>>> print u*v
0
```

Class for vectors; implementation

```
class Vec2D:
    def __init__(self, x, y):
        self.x = x; self.y = y

    def __add__(self, other):
        return Vec2D(self.x+other.x, self.y+other.y)

    def __sub__(self, other):
        return Vec2D(self.x-other.x, self.y-other.y)

    def __mul__(self, other):
        return self.x*other.x + self.y*other.y

    def __abs__(self):
        return math.sqrt(self.x**2 + self.y**2)

    def __eq__(self, other):
        return self.x == other.x and self.y == other.y

    def __str__(self):
        return '(%g, %g)' % (self.x, self.y)

    def __ne__(self, other):
        return not self.__eq__(other) # reuse __eq__
```

The repr special method: `eval(repr(p))` creates p

```
class MyClass:
    def __init__(self, a, b):
        self.a, self.b = a, b

    def __str__(self):
        """Return string with pretty print."""
        return 'a=%s, b=%s' % (self.a, self.b)

    def __repr__(self):
        """Return string such that eval(s) recreates self."""
        return 'MyClass(%s, %s)' % (self.a, self.b)
```

```
>>> m = MyClass(1, 5)
>>> print m          # calls m.__str__()
a=1, b=5
>>> str(m)           # calls m.__str__()
'a=1, b=5'
>>> s = repr(m)      # calls m.__repr__()
>>> s
'MyClass(1, 5)'
>>> m2 = eval(s)     # same as m2 = MyClass(1, 5)
>>> m2               # calls m.__repr__()
'MyClass(1, 5)'
```

Class Y revisited with repr print method

```
class Y:
    """Class for function  $y(t; v_0, g) = v_0*t - 0.5*g*t**2$ ."""

    def __init__(self, v0):
        """Store parameters."""
        self.v0 = v0
        self.g = 9.81

    def __call__(self, t):
        """Evaluate function."""
        return self.v0*t - 0.5*self.g*t**2

    def __str__(self):
        """Pretty print."""
        return 'v0*t - 0.5*g*t**2; v0=%g' % self.v0

    def __repr__(self):
        """Print code for regenerating this instance."""
        return 'Y(%s)' % self.v0
```

Class for complex numbers; functionality

Python already has a class `complex` for complex numbers, but implementing such a class is a good pedagogical example on class programming (especially with special methods).

Usage:

```
>>> u = Complex(2,-1)
>>> v = Complex(1)      # zero imaginary part
>>> w = u + v
>>> print w
(3, -1)
>>> w != u
True
>>> u*v
Complex(2, -1)
>>> u < v
illegal operation "<" for complex numbers
>>> print w + 4
(7, -1)
>>> print 4 - w
(1, 1)
```

Class for complex numbers; implementation (part 1)

```
class Complex:
    def __init__(self, real, imag=0.0):
        self.real = real
        self.imag = imag

    def __add__(self, other):
        return Complex(self.real + other.real,
                        self.imag + other.imag)

    def __sub__(self, other):
        return Complex(self.real - other.real,
                        self.imag - other.imag)

    def __mul__(self, other):
        return Complex(self.real*other.real - self.imag*other.imag,
                        self.imag*other.real + self.real*other.imag)

    def __div__(self, other):
        ar, ai, br, bi = self.real, self.imag, \
                           other.real, other.imag # short forms
        r = float(br**2 + bi**2)
        return Complex((ar*br+ai*bi)/r, (ai*br-ar*bi)/r)
```


Class for complex numbers; implementation (part 2)

```
def __abs__(self):
    return sqrt(self.real**2 + self.imag**2)

def __neg__(self):    # defines -c (c is Complex)
    return Complex(-self.real, -self.imag)

def __eq__(self, other):
    return self.real == other.real and \
           self.imag == other.imag

def __ne__(self, other):
    return not self.__eq__(other)

def __str__(self):
    return '(%g, %g)' % (self.real, self.imag)

def __repr__(self):
    return 'Complex' + str(self)

def __pow__(self, power):
    raise NotImplementedError(
        'self**power is not yet impl. for Complex')
```

Refining the special methods for arithmetics

Can we add a real number to a complex number?

```
>>> u = Complex(1, 2)
>>> w = u + 4.5
...
AttributeError: 'float' object has no attribute 'real'
```

Problem: we have assumed that other is Complex. Remedy:

```
class Complex:
    ...
    def __add__(self, other):
        if isinstance(other, (float,int)):
            other = Complex(other)
        return Complex(self.real + other.real,
                        self.imag + other.imag)

# or

    def __add__(self, other):
        if isinstance(other, (float,int)):
            return Complex(self.real + other, self.imag)
        else:
            return Complex(self.real + other.real,
                            self.imag + other.imag)
```

Special methods for “right” operands; addition

What if we try this:

```
>>> u = Complex(1, 2)
>>> w = 4.5 + u
...
TypeError: unsupported operand type(s) for +:
      'float' and 'instance'
```

Problem: Python's float objects cannot add a Complex.

Remedy: if a class has an `__radd__(self, other)` special method, Python applies this for `other + self`

```
class Complex:
    ...
    def __radd__(self, other):
        """Return other + self."""
        # other + self = self + other:
        return self.__add__(other)
```

Special methods for “right” operands; subtraction

Right operands for subtraction is a bit more complicated since $a - b \neq b - a$:

```
class Complex:
    ...
    def __sub__(self, other):
        if isinstance(other, (float,int)):
            other = Complex(other)
        return Complex(self.real - other.real,
                        self.imag - other.imag)

    def __rsub__(self, other):
        if isinstance(other, (float,int)):
            other = Complex(other)
        return other.__sub__(self)
```

What's in a class?

```
class A:
    """A class for demo purposes."""
    def __init__(self, value):
        self.v = value
```

Any instance holds its attributes in the `self.__dict__` dictionary
(Python automatically creates this dict)

```
>>> a = A([1,2])
>>> print a.__dict__    # all attributes
{'v': [1, 2]}
>>> dir(a)              # what's in object a?
['__doc__', '__init__', '__module__', 'dump', 'v']
>>> a.__doc__           # programmer's documentation of A
'A class for demo purposes.'
```

Ooops - we can add new attributes as we want!

```
>>> a.myvar = 10                                # add new attribute (!)
>>> a.__dict__
{'myvar': 10, 'v': [1, 2]}
>>> dir(a)
['__doc__', '__init__', '__module__', 'dump', 'myvar', 'v']

>>> b = A(-1)
>>> b.__dict__                                # b has no myvar attribute
{'v': -1}
>>> dir(b)
['__doc__', '__init__', '__module__', 'dump', 'v']
```

Summary of defining a class

Example on a defining a class with attributes and methods:

```
class Gravity:
    """Gravity force between two objects."""
    def __init__(self, m, M):
        self.m = m
        self.M = M
        self.G = 6.67428E-11 # gravity constant

    def force(self, r):
        G, m, M = self.G, self.m, self.M
        return G*m*M/r**2

    def visualize(self, r_start, r_stop, n=100):
        from scitools.std import plot, linspace
        r = linspace(r_start, r_stop, n)
        g = self.force(r)
        title='m=%g, M=%g' % (self.m, self.M)
        plot(r, g, title=title)
```

Summary of using a class

Example on using the class:

```
mass_moon = 7.35E+22
mass_earth = 5.97E+24

# make instance of class Gravity:
gravity = Gravity(mass_moon, mass_earth)

r = 3.85E+8 # earth-moon distance in meters
Fg = gravity.force(r) # call class method
```


Summary of special methods

- $c = a + b$ implies $c = a.__add__(b)$
- There are special methods for $a+b$, $a-b$, $a*b$, a/b , $a**b$, $-a$, `if a:`, `len(a)`, `str(a)` (pretty print), `repr(a)` (recreate a with `eval`), etc.
- With special methods we can create new mathematical objects like vectors, polynomials and complex numbers and write “mathematical code” (arithmetics)
- The call special method is particularly handy: $v = c(5)$ means $v = c.__call__(5)$
- Functions with parameters should be represented by a class with the parameters as attributes and with a call special method for evaluating the function

Summarizing example: interval arithmetics for uncertainty quantification in formulas

Uncertainty quantification:

Consider measuring gravity g by dropping a ball from $y = y_0$ to $y = 0$ in time T :

$$g = 2y_0 T^{-2}$$

What if y_0 and T are uncertain? Say $y_0 \in [0.99, 1.01]$ m and $T \in [0.43, 0.47]$ s. What is the uncertainty in g ?

The uncertainty can be computed by interval arithmetics

Interval arithmetics

Rules for computing with intervals, $p = [a, b]$ and $q = [c, d]$:

- $p + q = [a + c, b + d]$
- $p - q = [a - d, b - c]$
- $pq = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$
- $p/q = [\min(a/c, a/d, b/c, b/d), \max(a/c, a/d, b/c, b/d)]$
([c, d] cannot contain zero)

Obvious idea: make a class for interval arithmetics!

Class for interval arithmetics

```
class IntervalMath:
    def __init__(self, lower, upper):
        self.lo = float(lower)
        self.up = float(upper)

    def __add__(self, other):
        a, b, c, d = self.lo, self.up, other.lo, other.up
        return IntervalMath(a + c, b + d)

    def __sub__(self, other):
        a, b, c, d = self.lo, self.up, other.lo, other.up
        return IntervalMath(a - d, b - c)

    def __mul__(self, other):
        a, b, c, d = self.lo, self.up, other.lo, other.up
        return IntervalMath(min(a*c, a*d, b*c, b*d),
                             max(a*c, a*d, b*c, b*d))

    def __div__(self, other):
        a, b, c, d = self.lo, self.up, other.lo, other.up
        if c*d <= 0: return None
        return IntervalMath(min(a/c, a/d, b/c, b/d),
                             max(a/c, a/d, b/c, b/d))

    def __str__(self):
        return '[%g, %g]' % (self.lo, self.up)
```

Demo of the new class for interval arithmetics

Code:

```
I = IntervalMath    # abbreviate
a = I(-3,-2)
b = I(4,5)

expr = 'a+b', 'a-b', 'a*b', 'a/b'    # test expressions
for e in expr:
    print e, '=', eval(e)
```

Output:

```
a+b = [1, 3]
a-b = [-8, -6]
a*b = [-15, -8]
a/b = [-0.75, -0.4]
```

Shortcomings of the class

This code

```
a = I(4,5)
q = 2
b = a*q
```

leads to

```
File "IntervalMath.py", line 15, in __mul__
    a, b, c, d = self.lo, self.up, other.lo, other.up
AttributeError: 'float' object has no attribute 'lo'
```

Problem: IntervalMath times int is not defined.

Remedy: (cf. class Complex)

```
class IntervalArithmetics:
    ...
    def __mul__(self, other):
        if isinstance(other, (int, float)):          # NEW
            other = IntervalMath(other, other)        # NEW
            a, b, c, d = self.lo, self.up, other.lo, other.up
            return IntervalMath(min(a*c, a*d, b*c, b*d),
                                max(a*c, a*d, b*c, b*d))
```

(with similar adjustments of other special methods)

More shortcomings of the class

Try to compute $g = 2*y0*T**(-2)$: multiplication of `int` (2) and `IntervalMath` (`y0`), and power operation $T**(-2)$ are not defined

```
class IntervalArithmetics:
    ...
    def __rmul__(self, other):
        if isinstance(other, (int, float)):
            other = IntervalMath(other, other)
        return other*self

    def __pow__(self, exponent):
        if isinstance(exponent, int):
            p = 1
            if exponent > 0:
                for i in range(exponent):
                    p = p*self
            elif exponent < 0:
                for i in range(-exponent):
                    p = p*self
            p = 1/p
        else: # exponent == 0
            p = IntervalMath(1, 1)
        return p
    else:
        raise TypeError('exponent must int')
```

Adding more functionality to the class: rounding

“Rounding” to the midpoint value:

```
>>> a = IntervalMath(5,7)
>>> float(a)
6
```

is achieved by

```
class IntervalArithmetics:
    ...
    def __float__(self):
        return 0.5*(self.lo + self.up)
```


Adding more functionality to the class: repr and str methods

```
class IntervalArithmetics:
    ...
    def __str__(self):
        return '[%g, %g]' % (self.lo, self.up)

    def __repr__(self):
        return '%s(%g, %g)' % \
            (self.__class__.__name__, self.lo, self.up)
```

Demonstrating the class: $g = 2y_0 T^{-2}$

```
>>> g = 9.81
>>> y_0 = I(0.99, 1.01)
>>> Tm = 0.45          # mean T
>>> T = I(Tm*0.95, Tm*1.05) # 10% uncertainty
>>> print T
[0.4275, 0.4725]
>>> g = 2*y_0*T**(-2)
>>> g
IntervalMath(8.86873, 11.053)
>>> # computing with mean values:
>>> T = float(T)
>>> y = 1
>>> g = 2*y_0*T**(-2)
>>> print '%.2f' % g
9.88
```

Demonstrating the class: volume of a sphere

```
>>> R = I(6*0.9, 6*1.1)    # 20 % error
>>> V = (4./3)*pi*R**3
>>> V
IntervalMath(659.584, 1204.26)
>>> print V
[659.584, 1204.26]
>>> print float(V)
931.922044761
>>> # compute with mean values:
>>> R = float(R)
>>> V = (4./3)*pi*R**3
>>> print V
904.778684234
```

20% uncertainty in R gives almost 60% uncertainty in V