App. A: Sequences and difference equations

Hans Petter Langtangen[1,2]

[1]Simula Research Laboratory
[2]University of Oslo, Dept. of Informatics

Sep 23, 2014

Sequences

Sequences is a central topic in mathematics:

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_n, \ldots,$$

Example: all odd numbers

$$1, 3, 5, 7, \ldots, 2n+1, \ldots$$

For this sequence we have a formula for the n-th term:

$$x_n = 2n+1$$

and we can write the sequence more compactly as

$$(x_n)_{n=0}^{\infty},\quad x_n = 2n+1$$

Other examples of sequences

$$1,\ 4,\ 9,\ 16,\ 25,\ \ldots\quad (x_n)_{n=0}^{\infty},\ x_n = n^2$$

$$1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \ldots\quad (x_n)_{n=0}^{\infty},\ x_n = \frac{1}{n+1}$$

$$1,\ 1,\ 2,\ 6,\ 24,\ \ldots\quad (x_n)_{n=0}^{\infty},\ x_n = n!$$

$$1,\ 1+x,\ 1+x+\frac{1}{2}x^2,\ 1+x+\frac{1}{2}x^2+\frac{1}{6}x^3,\ \ldots\quad (x_n)_{n=0}^{\infty},\ x_n = \sum_{j=0}^{n}\frac{x^j}{j!}$$

Finite and infinite sequences

- Infinite sequences have an infinite number of terms ($n \to \infty$)
- In mathematics, infinite sequences are widely used
- In real-life applications, sequences are usually finite: $(x_n)_{n=0}^{N}$
- Example: number of approved exercises every week in INF1100
  $x_0, x_1, x_2, \ldots, x_{15}$
- Example: the annual value of a loan
  $x_0, x_1, \ldots, x_{20}$

Difference equations

- For sequences occuring in modeling of real-world phenomena, there is seldom a formula for the n-th term
- However, we can often set up one or more equations governing the sequence
- Such equations are called difference equations
- With a computer it is then very easy to generate the sequence by solving the difference equations
- Difference equations have lots of applications and are very easy to solve on a computer, but often complicated or impossible to solve for x_n (as a formula) by pen and paper!
- The programs require only loops and arrays

Modeling interest rates

Problem: Put x_0 money in a bank at year 0. What is the value after N years if the interest rate is p percent per year?

Solution: The fundamental information relates the value at year n, x_n, to the value of the previous year, x_{n-1}:

$$x_n = x_{n-1} + \frac{p}{100} x_{n-1}$$

How to solve for x_n? Start with x_0, compute x_1, x_2, ...

Simulating the difference equation for interest rates

What does it mean to simulate? Solve math equations by repeating a simple procedure (relation) many times (boring, but well suited for a computer!)

Program for $x_n = x_{n-1} + (p/100)x_{n-1}$:

```
from scitools.std import *
x0 = 100                      # initial amount
p = 5                         # interest rate
N = 4                         # number of years
index_set = range(N+1)
x = zeros(len(index_set))

# Solution:
x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (p/100.0)*x[n-1]
print x
plot(index_set, x, 'ro', xlabel='years', ylabel='amount')
```

We do not need to store the entire sequence, but it is convenient for programming and later plotting

- Previous program stores all the x_n values in a NumPy array
- To compute x_n, we only need one previous value, x_{n-1}

Thus, we could only store the two last values in memory:

```
x_old = x0
for n in index_set[1:]:
    x_new = x_old + (p/100.)*x_old
    x_old = x_new  # x_new becomes x_old at next step
```

However, programming with an array `x[n]` is simpler, safer, and enables plotting the sequence, so we will continue to use arrays in the examples

Daily interest rate

- A more relevant model is to add the interest every day
- The interest rate per day is $r = p/D$ if p is the annual interest rate and D is the number of days in a year
- A common model in business applies $D = 360$, but n counts exact (all) days

Just a minor change in the model:

$$x_n = x_{n-1} + \frac{r}{100} x_{n-1}$$

How can we find the number of days between two dates?

```
>> import datetime
>> date1 = datetime.date(2007, 8, 3)  # Aug 3, 2007
>> date2 = datetime.date(2008, 8, 4)  # Aug 4, 2008
>> diff = date2 - date1
>> print diff.days
367
```

Program for daily interest rate

```
from scitools.std import *
x0 = 100                              # initial amount
p = 5                                 # annual interest rate
r = p/360.0                           # daily interest rate
import datetime
date1 = datetime.date(2007, 8, 3)
date2 = datetime.date(2011, 8, 3)
diff = date2 - date1
N = diff.days
index_set = range(N+1)
x = zeros(len(index_set))

# Solution:
x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (r/100.0)*x[n-1]
print x
plot(index_set, x, 'ro', xlabel='days', ylabel='amount')
```

But the annual interest rate may change quite often...

Varying p means p_n:

- Could not be handled in school (cannot apply $x_n = x_0(1 + \frac{p}{100})^n$)
- A varying p causes no problems in the program: just fill an array `p` with correct interest rate for day `n`

Modified program:

```
p = zeros(len(index_set))
# fill p[n] for n in index_set (might be non-trivial...)

r = p/360.0                         # daily interest rate
x = zeros(len(index_set))

x[0] = x0
for n in index_set[1:]:
    x[n] = x[n-1] + (r[n-1]/100.0)*x[n-1]
```

Payback of a loan

- A loan L is paid back with a fixed amount L/N every month over N months + the interest rate of the loan
- p: annual interest rate, $p/12$: monthly rate
- Let x_n be the value of the loan at the end of month n

The fundamental relation from one month to the text:

$$x_n = x_{n-1} + \frac{p}{12 \cdot 100} x_{n-1} - (\frac{p}{12 \cdot 100} x_{n-1} + \frac{L}{N})$$

which simplifies to

$$x_n = x_{n-1} - \frac{L}{N}$$

(L/N makes the equation *nonhomogeneous*)

How to make a living from a fortune with constant consumption

- We have a fortune F invested with an annual interest rate of p percent
- Every year we plan to consume an amount c_n (n counts years)
- Let x_n be our fortune at year n

A fundamental relation from one year to the other is

$$x_n = x_{n-1} + \frac{p}{100} x_{n-1} - c_n$$

Simplest possibility: keep c_n constant, but inflation demands c_n to increase...

How to make a living from a fortune with inflation-adjusted consumption

- Assume I percent inflation per year
- Start with c_0 as q percent of the interest the first year
- c_n then develops as money with interest rate I

x_n develops with rate p but with a loss c_n every year:

$$x_n = x_{n-1} + \frac{p}{100}x_{n-1} - c_{n-1}, \quad x_0 = F,\ c_0 = \frac{pq}{10^4}F$$
$$c_n = c_{n-1} + \frac{I}{100}c_{n-1}$$

This is a coupled system of *two* difference equations, but the programming is still simple: we update two arrays, first `x[n]`, then `c[n]`, inside the loop (good exercise!)

The mathematics of Fibonacci numbers

No programming or math course is complete without an example on Fibonacci numbers:

$$x_n = x_{n-1} + x_{n-2}, \quad x_0 = 1,\ x_1 = 1$$

Mathematical classification. This is a *homogeneous difference equation of second order* (second order means three levels: n, $n-1$, $n-2$). This classification is important for mathematical solution technique, but not for simulation in a program.

Fibonacci derived the sequence by modeling rat populations, but the sequence of numbers has a range of peculiar mathematical properties and has therefore attracted much attention from mathematicians.

Program for generating Fibonacci numbers

```
N = int(sys.argv[1])
from numpy import zeros
x = zeros(N+1, int)
x[0] = 1
x[1] = 1
for n in range(2, N+1):
    x[n] = x[n-1] + x[n-2]
    print n, x[n]
```

Fibonacci numbers can cause overflow in NumPy arrays

Run the program with $N = 50$:

```
2 2
3 3
4 5
5 8
6 13
...
45 1836311903
```

```
Warning: overflow encountered in long_scalars
46 -1323752223
```

Note:

- Changing `int` to `long` or `int64` for array elements allows $N \leq 91$
- Can use `float96` (though x_n is integer): $N \leq 23600$

No overflow when using Python int types

- Best: use Python scalars of type `int` - these automatically changes to `long` when overflow in `int`
- The `long` type in Python has arbitrarily many digits (as many as required in a computation!)
- Note: `long` for arrays is 64-bit integer (`int64`), while scalar `long` in Python is an integer with as "infinitely" many digits

Program with Python's int type for integers

The program now avoids arrays and makes use of three `int` objects (which automatically changes to `long` when needed):

```
import sys
N = int(sys.argv[1])
xnm1 = 1                              # "x_n minus 1"
xnm2 = 1                              # "x_n minus 2"
n = 2
while n <= N:
    xn = xnm1 + xnm2
    print 'x_%d = %d' % (n, xn)
    xnm2 = xnm1
    xnm1 = xn
    n += 1
```

Run with $N = 200$:

```
x_2 = 2
x_3 = 3
...
x_198 = 173402521172797813159685037284371942044301
x_199 = 280571172992510140037611932413038677189525
x_200 = 453973694165307953197296969697410619233826
```

Limition: your computer's memory

New problem setting: exponential growth with limited environmental resources

The model for growth of money in a bank has a solution of the type

$$x_n = x_0 C^n \quad (= x_0 e^{n \ln C})$$

Note:

- This is exponential growth in time (n)
- Populations of humans, animals, and cells also exhibit the same type of growth as long as there are unlimited resources (space and food)
- Most environments can only support a maximum number M of individuals
- How can we model this limitation?

Modeling growth in an environment with limited resources

Initially, when there are enough resources, the growth is exponential:

$$x_n = x_{n-1} + \frac{r}{100} x_{n-1}$$

The growth rate r must decay to zero as x_n approaches M. The simplest variation of $r(n)$ is a linear:

$$r(n) = \varrho \left(1 - \frac{x_n}{M}\right)$$

Observe: $r(n) \approx \varrho$ for small n when $x_n \ll M$, and $r(n) \to 0$ as $x_n \to M$ and n is big

Logistic growth model:

$$x_n = x_{n-1} + \frac{\varrho}{100} x_{n-1} \left(1 - \frac{x_{n-1}}{M}\right)$$

(This is a *nonlinear* difference equation)

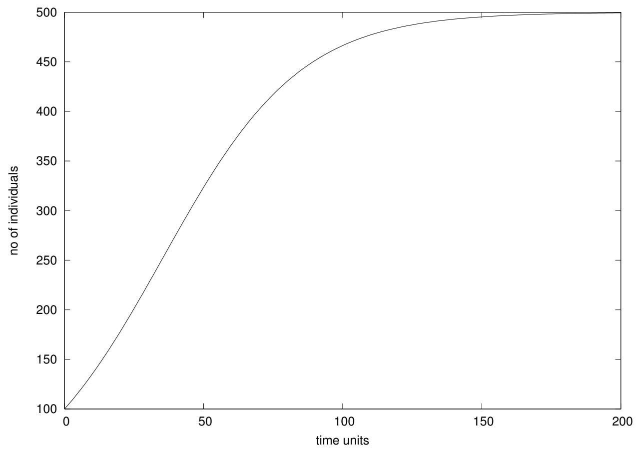
The evolution of logistic growth

In a program it is easy to introduce logistic instead of exponential growth, just replace

```
x[n] = x[n-1] + p/100.0)*x[n-1]
```

by

```
x[n] = x[n-1] + (rho/100.0)*x[n-1]*(1 - x[n-1]/float(M))
```



The factorial as a difference equation

The factorial $n!$ is defined as

$$n(n-1)(n-2)\cdots 1, \quad 0! = 1$$

The following difference equation has $x_n = n!$ as solution and can be used to compute the factorial:

$$x_n = n x_{n-1}, \quad x_0 = 1$$

Difference equations must have an initial condition

- In mathematics, it is much stressed that a difference equation for x_n must have an initial condition x_0
- The initial condition is obvious when programming: otherwise we cannot start the program (x_0 is needed to compute x_n)
- However: if you forget `x[0] = x0` in the program, you get $x_0 = 0$ (because `x = zeroes(N+1)`), which (usually) gives unintended results!

How you ever though about how $\sin x$ is really calculated?

- How can you *calculate* $\sin x$, $\ln x$, e^x without a calculator or program?
- These functions were originally defined to have some desired mathematical properties, but without an algorithm for how to evaluate function values

- Idea: approximate $\sin x$, etc. by polynomials, since they are easy to calculate (sum, multiplication), but how??

Would you expect these fantastic mathematical results?

Amazing result by Gregory, 1667:

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

Even more amazing result by Taylor, 1715:

$$f(x) = \sum_{k=0}^{\infty} (\frac{d^k}{dx^k} f(0)) x^k$$

For "any" $f(x)$, if we can differentiate, add, and multiply x^k, we can evaluate f at any x (!!!)

Taylor polynomials

Practical applications works with a truncated sum:

$$f(x) \approx \sum_{k=0}^{N} (\frac{d^k}{dx^k} f(0)) x^k$$

$N = 1$ is *very* popular and has been essential in developing physics and technology

Example:

$$\begin{aligned} e^x &= \sum_{k=0}^{\infty} \frac{x^k}{k!} \\ &\approx 1 + x + x^2 + x^3 \\ &\approx 1 + x \end{aligned}$$

Taylor polynomials around an arbitrary point

The previous Taylor polynomials are most accurate around $x = 0$. Can make the polynomials accurate around any point $x = a$:

$$f(x) \approx \sum_{k=0}^{N} (\frac{d^k}{dx^k} f(a))(x-a)^k$$

Taylor polynomials as one difference equation

The Taylor series for e^x around $x = 0$ reads

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Define

$$e_n = \sum_{k=0}^{n-1} \frac{x^k}{k!} = \sum_{k=0}^{n-2} \frac{x^k}{k!} + \frac{x^{n-1}}{(n-1)!}$$

We can formulate the sum in e_n as the following difference equation:

$$e_n = e_{n-1} + \frac{x^{n-1}}{(n-1)!}, \quad e_0 = 0$$

More efficient computation: the Taylor polynomial as two difference equations

Observe:

$$\frac{x^n}{n!} = \frac{x^{n-1}}{(n-1)!} \cdot \frac{x}{n}$$

Let $a_n = x^n/n!$. Then we can efficiently compute a_n via

$$a_n = a_{n-1}\frac{x}{n}, \quad a_0 = 1$$

Now we can update each term via the a_n equation and sum the terms via the e_n equation:

$$\begin{aligned} e_n &= e_{n-1} + a_{n-1}, \quad e_0 = 0,\ a_0 = 1 \\ a_n &= \frac{x}{n}a_{n-1} \end{aligned}$$

See the book for more details

Nonlinear algebraic equations

Generic form of any (algebraic) equation in x:

$$f(x) = 0$$

Examples that can be solved by hand:

$$ax + b = 0$$

$$ax^2 + bx + c = 0$$

$$\sin x + \cos x = 1$$

- Simple numerical algorithms can solve "any" equation $f(x) = 0$
- Safest: Bisection
- Fastest: Newton's method
- Don't like $f'(x)$ in Newton's method? Use the Secant method
- Secant and Newton are difference equations!

Newton's method for finding zeros

Newton's method. Simpson (1740) came up with the following general method for solving $f(x) = 0$ (based on ideas by Newton):

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad x_0 \text{ given}$$

Note:

- This is a (nonlinear!) difference equation
- As $n \to \infty$, we hope that $x_n \to x_s$, where x_s solves $f(x_s) = 0$
- How to choose N when what we want is x_N close to x_s?
- Need a slightly different program: simulate until $f(x) \leq \epsilon$, where ϵ is a small tolerance
- Caution: Newton's method may (easily) diverge, so $f(x) \leq \epsilon$ may never occur!

A program for Newton's method

Quick implementation:

```
def Newton(f, x, dfdx, epsilon=1.0E-7, max_n=100):
    n = 0
    while abs(f(x)) > epsilon and n <= max_n:
        x = x - f(x)/dfdx(x)
        n += 1
    return x, n, f(x)
```

Note:

- $f(x)$ is evaluated twice in each pass of the loop - only one evaluation is strictly necessary (can store the value in a variable and reuse it)
- `f(x)/dfdx(x)` can give integer division
- It could be handy to store the `x` and `f(x)` values in each iteration (for plotting or printing a convergence table)

An improved function for Newton's method

Only one $f(x)$ call in each iteration, optional storage of $(x, f(x))$ values during the iterations, and ensured float division:

```
def Newton(f, x, dfdx, epsilon=1.0E-7, max_n=100,
           store=False):
    f_value = f(x)
    n = 0
    if store: info = [(x, f_value)]
    while abs(f_value) > epsilon and n <= max_n:
        x = x - float(f_value)/dfdx(x)
        n += 1
        f_value = f(x)
        if store: info.append((x, f_value))
    if store:
        return x, info
    else:
        return x, n, f_value
```

Application of Newton's method

$$e^{-0.1x^2} \sin(\frac{\pi}{2}x) = 0$$

Solutions: $x = 0, \pm 2, \pm 4, \pm 6, \ldots$

Main program:

```
from math import sin, cos, exp, pi
import sys

def g(x):
    return exp(-0.1*x**2)*sin(pi/2*x)

def dg(x):
    return -2*0.1*x*exp(-0.1*x**2)*sin(pi/2*x) + \
           pi/2*exp(-0.1*x**2)*cos(pi/2*x)

x0 = float(sys.argv[1])
x, info = Newton(g, x0, dg, store=True)
print 'Computed zero:', x

# Print the evolution of the difference equation
# (i.e., the search for the root)
for i in range(len(info)):
    print 'Iteration %3d: f(%g)=%g' % (i, info[i][0], info[i][1])
```

Results from this test problem

$x_0 = 1.7$ gives quick convergence towards the closest root $x = 0$:

```
zero: 1.999999999768449
Iteration  0: f(1.7)=0.340044
Iteration  1: f(1.99215)=0.00828786
```

```
Iteration   2: f(1.99998)=2.53347e-05
Iteration   3: f(2)=2.43808e-10
```

Start value $x_0 = 3$ (closest root $x = 2$ or $x = 4$):

```
zero: 42.49723316011362
Iteration   0: f(3)=-0.40657
Iteration   1: f(4.66667)=0.0981146
Iteration   2: f(42.4972)=-2.59037e-79
```

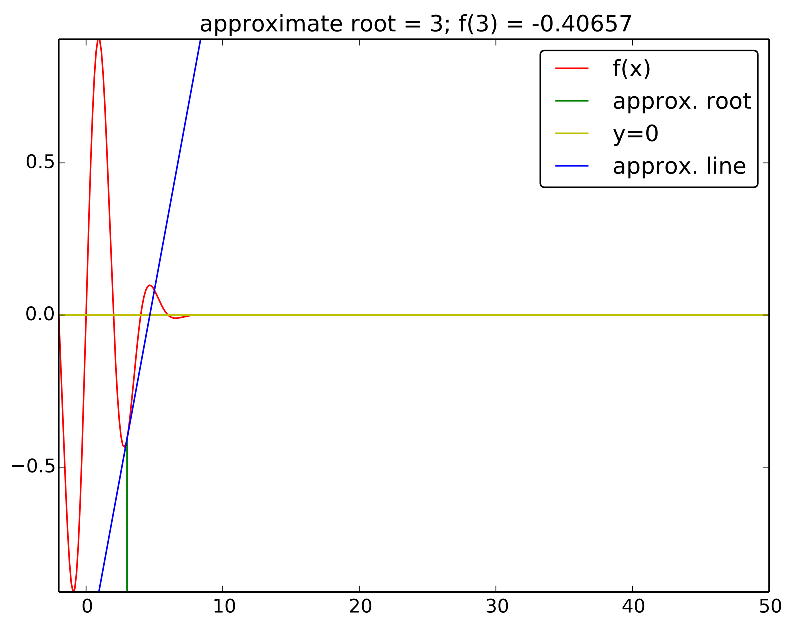
What happened here??

Try the demo program `src/diffeq/Newton_movie.py` with $x_0 = 3$, $x \in [-2, 50]$ for plotting and numerical approximation of $f'(x)$:

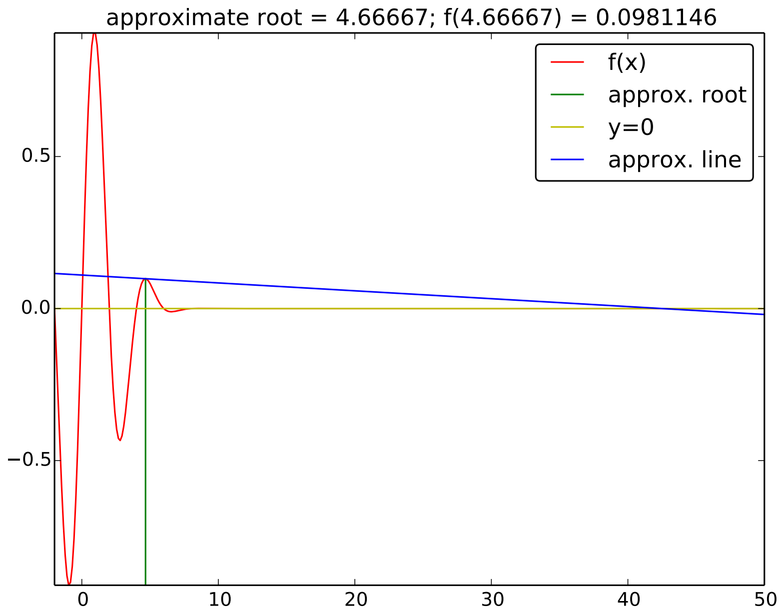
```
Terminal> python Newton_movie.py "exp(-0.1*x**2)*sin(pi/2*x)" \
          numeric 3 -2 50
```

Lesson learned: Newton's method may work fine or give wrong results! You need to understand the method to interpret the results!

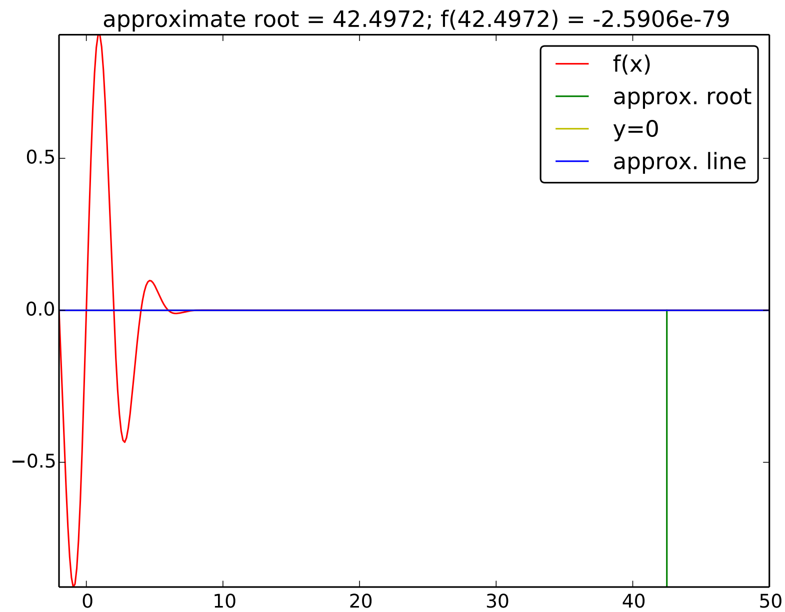
First step: we're moving to the right ($x = 4$?)



Second step: oops, too much to the right...



Third step: disaster since we're "done" ($f(x) \approx 0$)



Programming with sound

Tones are sine waves: A tone **A** (440 Hz) is a sine wave with frequency 440 Hz:

$$s(t) = A \sin (2\pi f t), \quad f = 440$$

On a computer we represent $s(t)$ by a discrete set of points on the function curve (exactly as we do when we plot $s(t)$). CD quality needs 44100 samples per second.

Making a sound file with single tone (part 1)

- r: sampling rate (samples per second, default 44100)
- f: frequency of the tone
- m: duration of the tone (seconds)

Sampled sine function for this tone:

$$s_n = A \sin \left(2\pi f \frac{n}{r}\right), \quad n = 0, 1, \ldots, m \cdot r$$

Code (we use descriptive names: `frequency` f, `length` m, `amplitude` A, `sample_rate` r):

```
import numpy
def note(frequency, length, amplitude=1,
         sample_rate=44100):
    time_points = numpy.linspace(0, length,
                                 length*sample_rate)
    data = numpy.sin(2*numpy.pi*frequency*time_points)
    data = amplitude*data
    return data
```

Making a sound file with single tone (part 2)

- We have `data` as an array with `float` and unit amplitude
- Sound data in a file should have 2-byte integers (`int16`) as data elements and amplitudes up to $2^{15} - 1$ (max value for `int16` data)

```
data = note(440, 2)
data = data.astype(numpy.int16)
max_amplitude = 2**15 - 1
data = max_amplitude*data
import scitools.sound
scitools.sound.write(data, 'Atone.wav')
scitools.sound.play('Atone.wav')
```

Reading sound from file

- Let us read a sound file and add echo
- Sound = array `s[n]`
- Echo means to add a delay of the sound

```
# echo: e[n] = beta*s[n] + (1-beta)*s[n-b]

def add_echo(data, beta=0.8, delay=0.002,
             sample_rate=44100):
    newdata = data.copy()
    shift = int(delay*sample_rate)  # b (math symbol)
    for i in xrange(shift, len(data)):
        newdata[i] = beta*data[i] + (1-beta)*data[i-shift]
    return newdata
```

Load data, add echo and play:

```
data = scitools.sound.read(filename)
data = data.astype(float)
data = add_echo(data, beta=0.6)
data = data.astype(int16)
scitools.sound.play(data)
```

Playing many notes

- Each note is an array of samples from a sine with a frequency corresponding to the note
- Assume we have several note arrays `data1`, `data2`, ...:

```
# put data1, data2, ... after each other in a new array:
data = numpy.concatenate((data1, data2, data3, ...))
```

The start of "Nothing Else Matters" (Metallica):

```
E1 = note(164.81, .5)
G = note(392, .5)
B = note(493.88, .5)
E2 = note(659.26, .5)
intro = numpy.concatenate((E1, G, B, E2, B, G))
...
song = numpy.concatenate((intro, intro, ...))
scitools.sound.play(song)
scitools.sound.write(song, 'tmp.wav')
```

Summary of difference equations

- Sequence: x_0, x_1, x_2, $\ldots$, x_n, $\ldots$, x_N
- Difference equation: relation between x_n, x_{n-1} and maybe x_{n-2} (or more terms in the "past") + known start value x_0 (and more values x_1, ... if more levels enter the equation)

Solution of difference equations by simulation:

```
index_set = <array of n-values: 0, 1, ..., N>
x = zeros(N+1)
x[0] = x0
for n in index_set[1:]:
    x[n] = <formula involving x[n-1]>
```

Can have (simple) systems of difference equations:

```
for n in index_set[1:]:
    x[n] = <formula involving x[n-1]>
    y[n] = <formula involving y[n-1] and x[n]>
```

Taylor series and numerical methods such as Newton's method can be formulated as difference equations, often resulting in a good way of programming the formulas

Summarizing example: music of sequences

- Given a x_0, x_1, x_2, ..., x_n, ..., x_N
- Can we listen to this sequence as "music"?
- Yes, we just transform the x_n values to suitable frequencies and use the functions in `scitools.sound` to generate tones

We will study two sequences:

$$x_n = e^{-4n/N} \sin(8\pi n/N)$$

and

$$x_n = x_{n-1} + q x_{n-1} \left(1 - x_{n-1}\right), \quad x = x_0$$

The first has values in $[-1, 1]$, the other from $x_0 = 0.01$ up to around 1

Transformation from "unit" x_n to frequencies:

$$y_n = 440 + 200 x_n$$

(first sequence then gives tones between 240 Hz and 640 Hz)

Module file: soundeq.py

- Three functions: two for generating sequences, one for the sound
- Look at `files/soundeq.py` for complete code

Try it out in these examples:

```
Terminal> python soundseq.py oscillations 40
Terminal> python soundseq.py logistic 100
```

Try to change the frequency range from 200 to 400.