University of Illinois at Urbana-Champaign

Department of Aerospace Engineering

AE - 410 Computational Aerodynamics

AE 410 Final Project 2024

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1). A finite volume solver on MATLAB was developed to solve the one-dimensional Euler equations. The Godunov method was implemented to construct the fluxes at the faces. The code was used to solve the one-dimensional shock tube problem.

The code of the solver is provided in the appendix section.

2).

To verify the solver, the results were compared against tests 1 and 2 of Toro. Figures 1 to 6 show the comparison between the numerical results and the test results. For both cases, the number of cells and timestep size were taken to be 500 and 0.0001s, respectively. A good match between the numerical and Toro's result was observed.

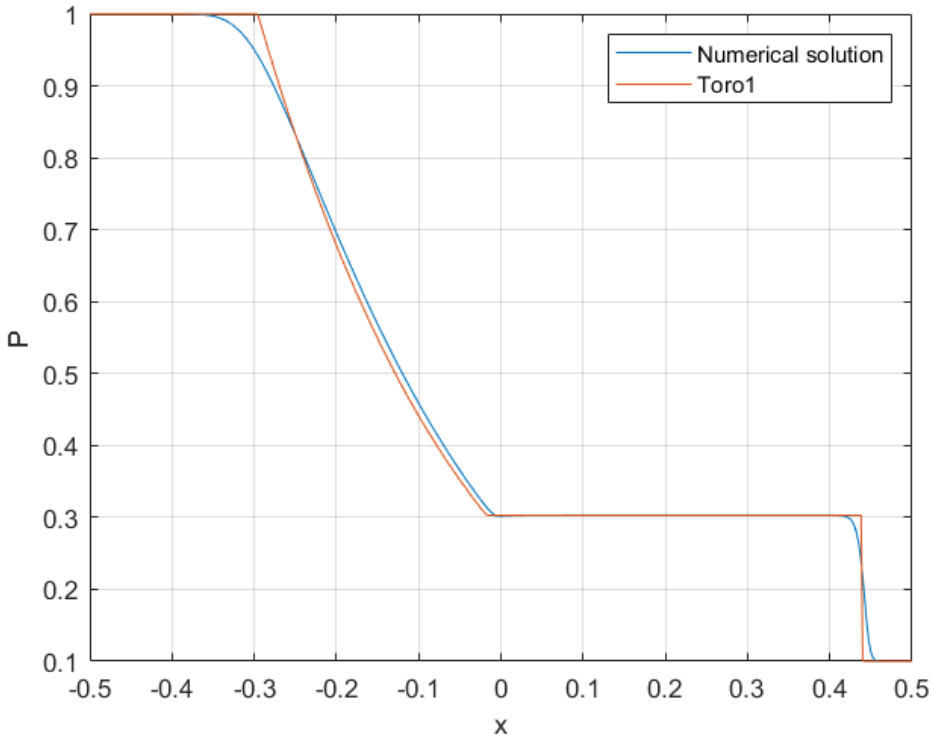


Figure 1: Comparison of pressure variation obtained by the numerical solutions against Toro's test1

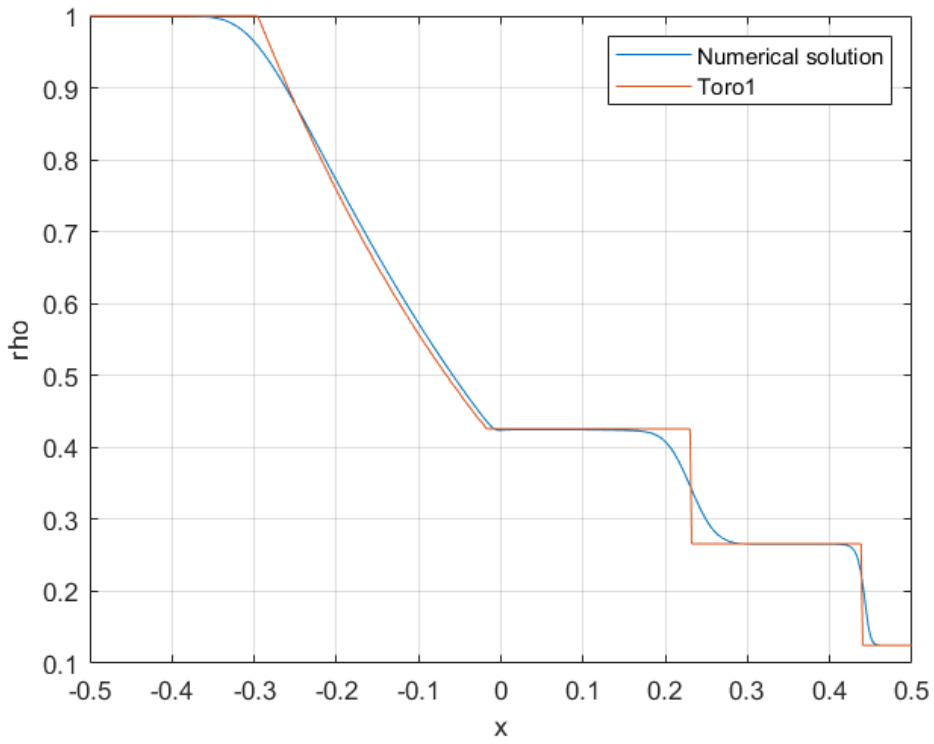


Figure 2: Comparison of density variation obtained by the numerical solutions against Toro's test1

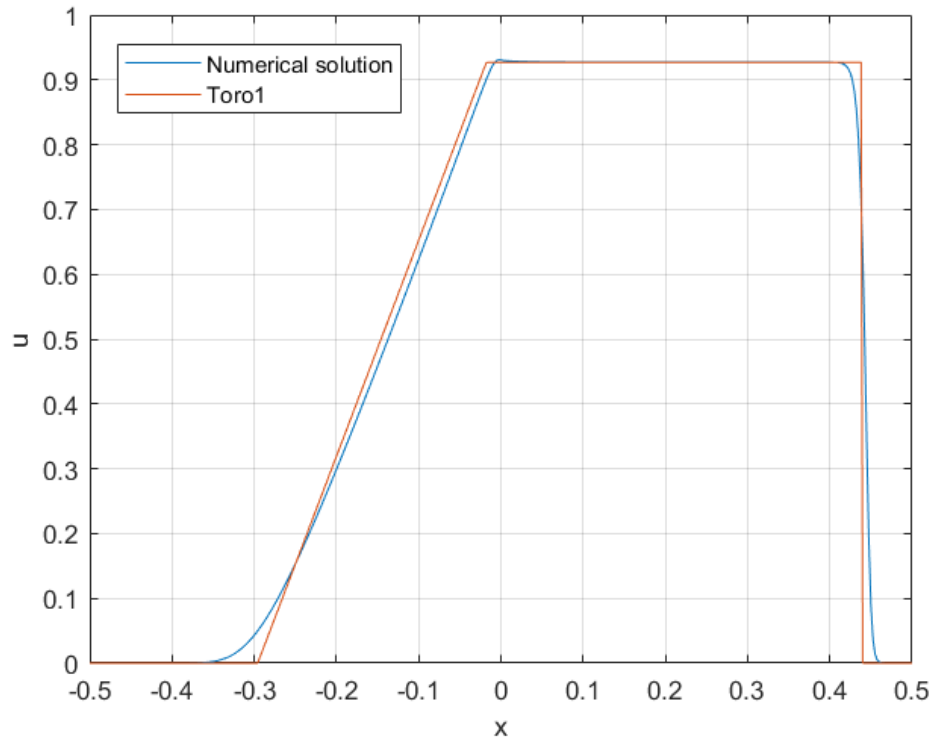


Figure 3: Comparison of velocity variation obtained by the numerical solutions against Toro's test1

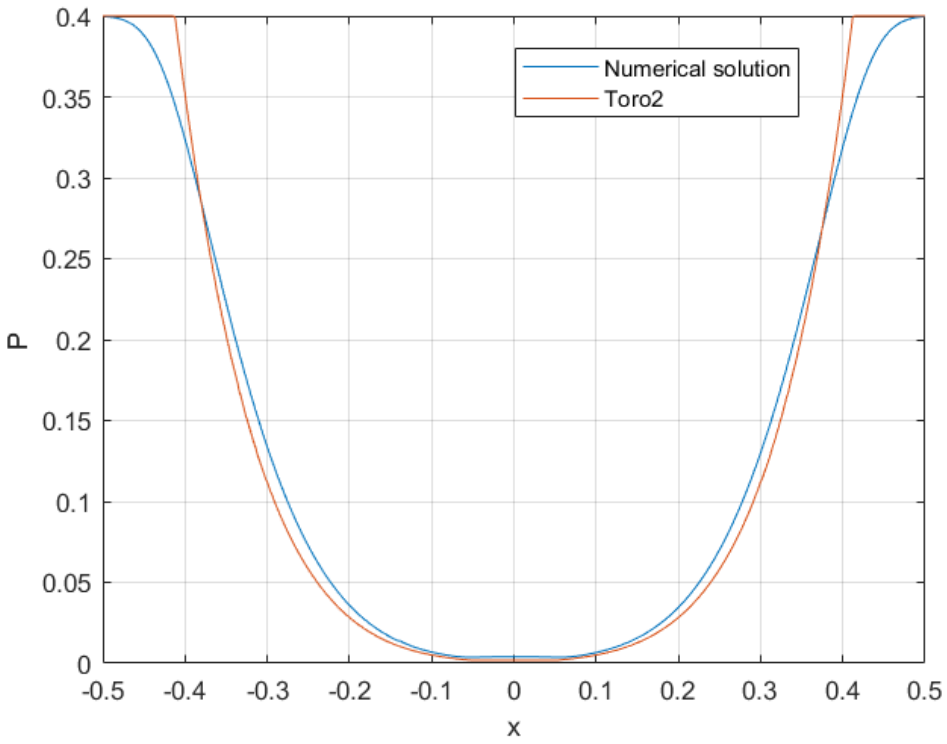


Figure 4: Comparison of pressure variation obtained by the numerical solutions against Toro's test2

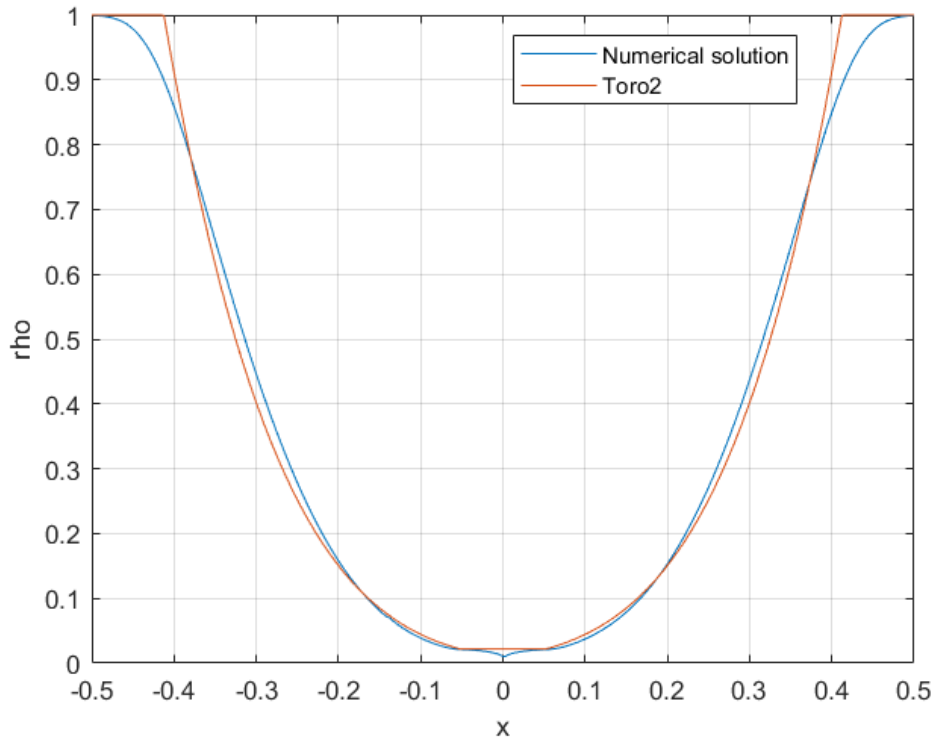


Figure 5: Comparison of density variation obtained by the numerical solutions against Toro's test2

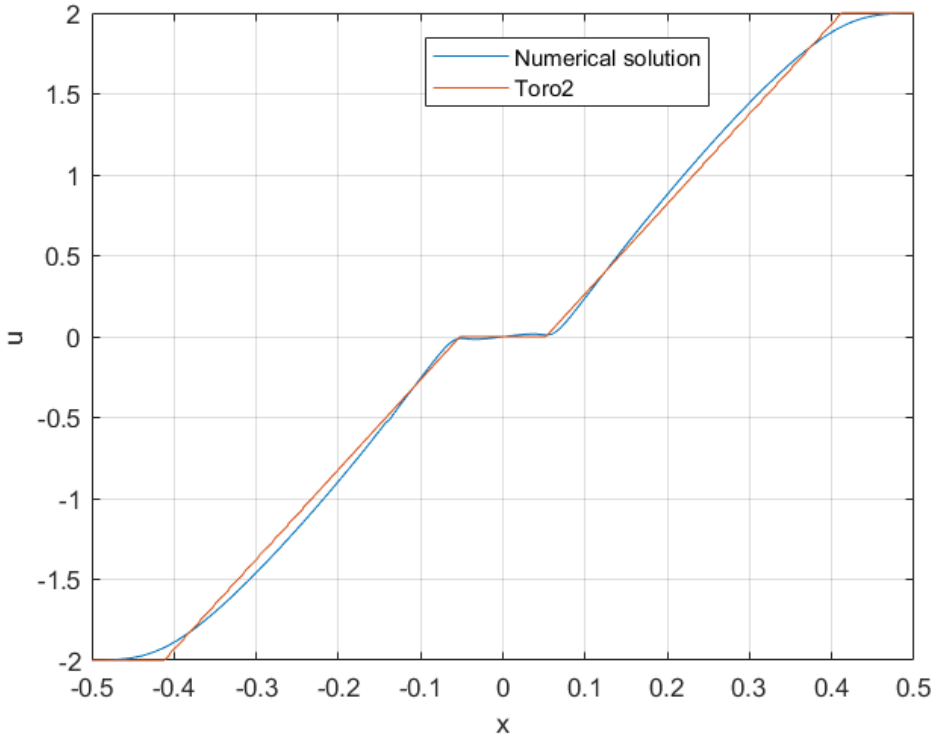


Figure 6: Comparison of velocity variation obtained by the numerical solutions against Toro's test2

3).

Figures 7 to 10 show the t vs x plot of Pressure, Velocity, density, and Temperature, respectively, along with the observed flow features. A grid spacing of 0.04m and time step size of 10^{-5}s was chosen for this case.

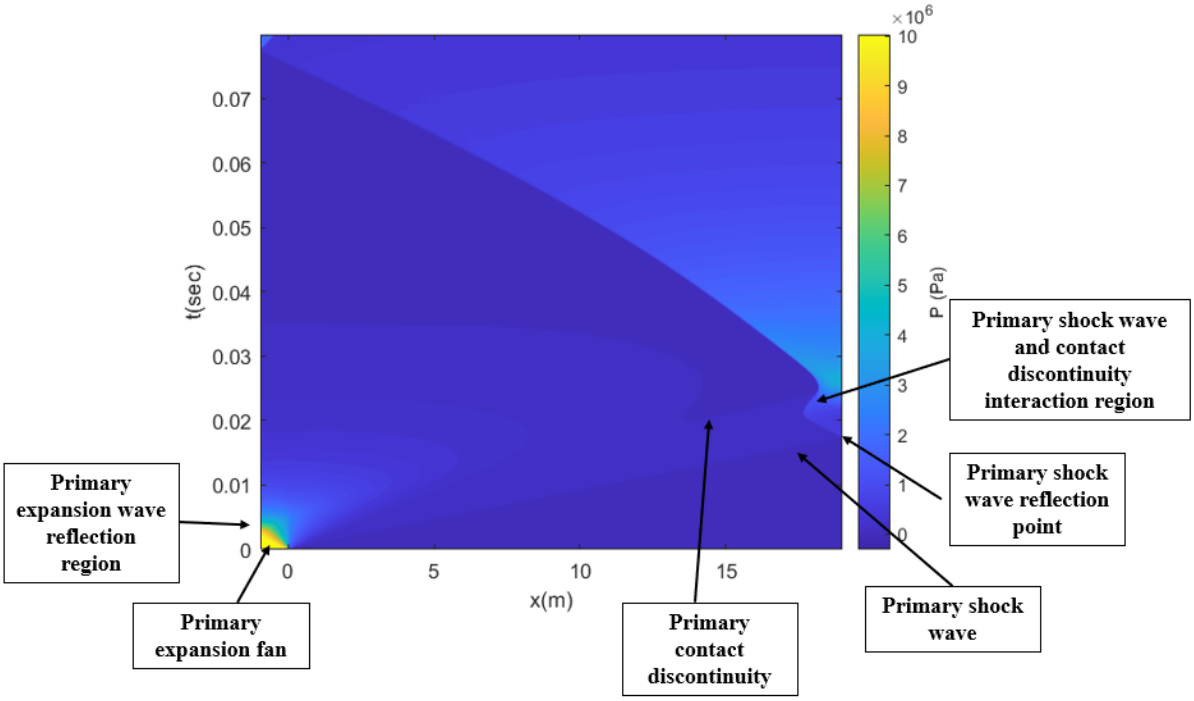


Figure 7: t vs x diagram showing variation of pressure with time

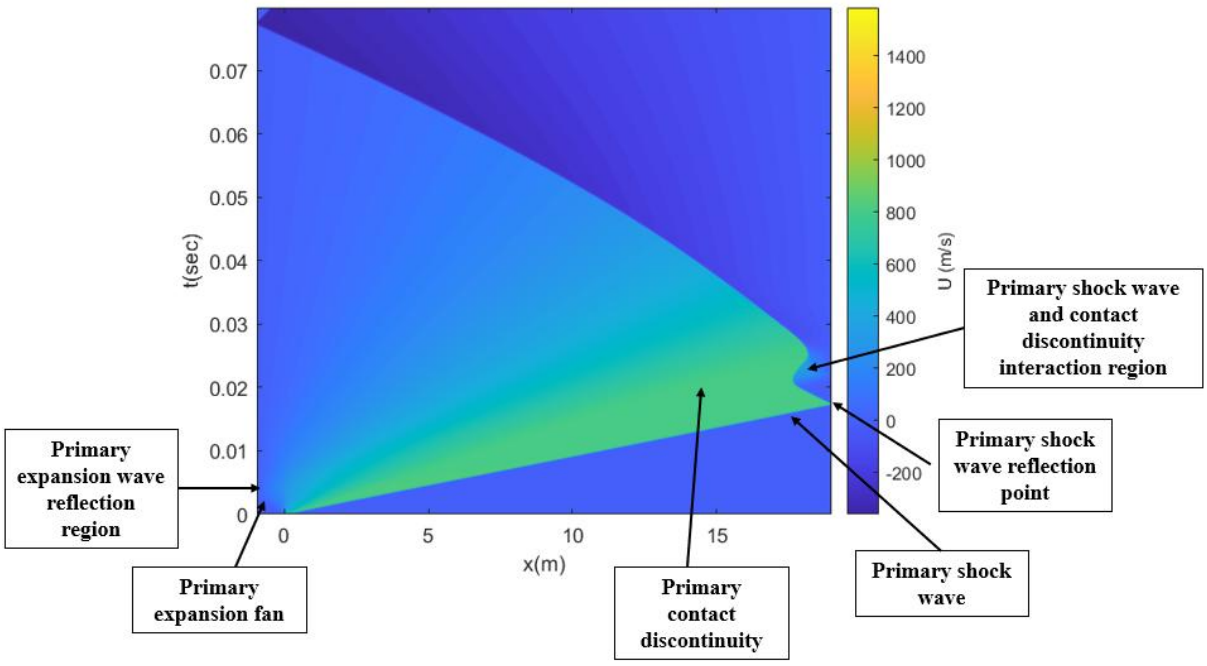


Figure 8: t vs x diagram showing variation of velocity with time

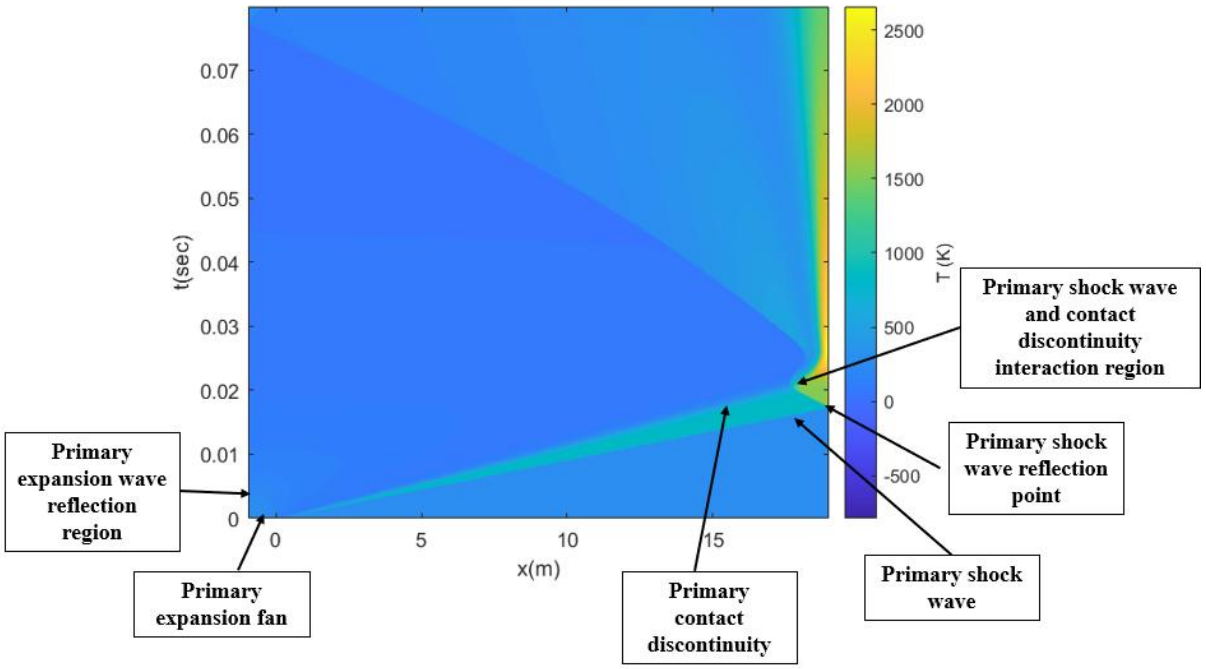


Figure 9: t vs x diagram showing variation of density with time

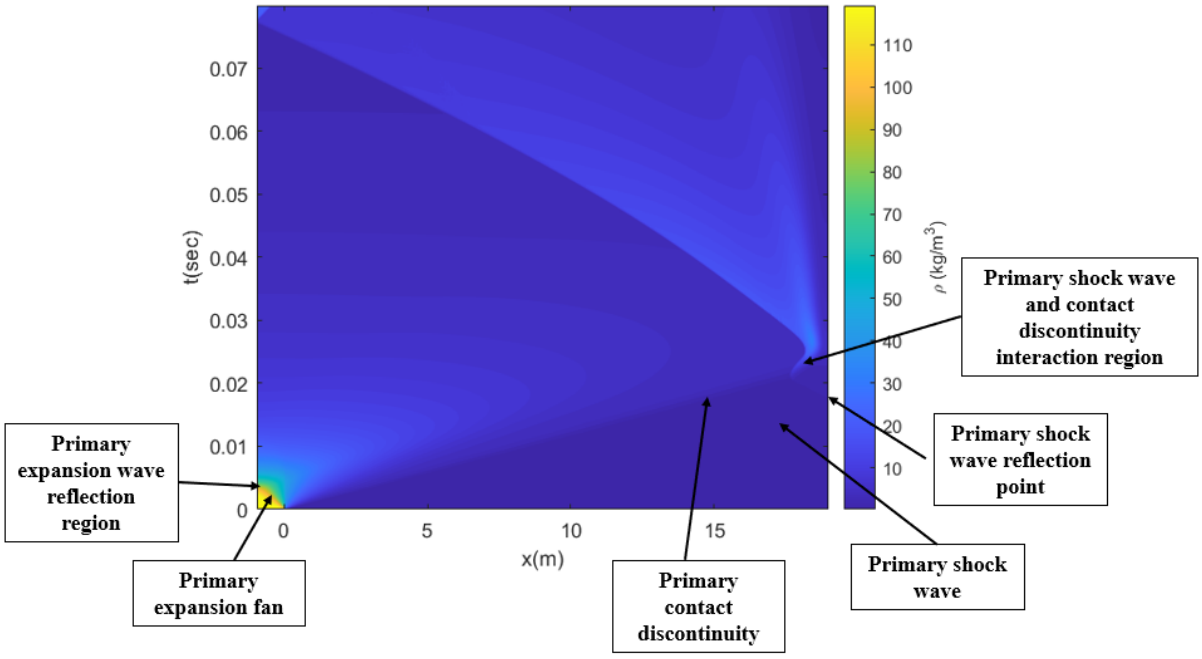


Figure 10: t vs x diagram showing the variation of density with time

Figure 11 shows the shock location at two different time instances. Using this, the shock speed can be calculated:

$$S = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x_2 - x_1}{y_2 - y_1} = \frac{16.04 - 10.36}{0.0145 - 0.0094}$$

$$S = 1113.72 \text{m/s}$$

From initial conditions, we can calculate the speed of sound in the driven section as:

$$a_1 = (\gamma R T_1)^{0.5}$$

$$a_1 = 343.1142 \text{m/s}$$

Hence, shock Mach number M_s will come out to be $M_s = 3.246$.

Now on solving the shock-tube equation (using fzero function on MATLAB), **the analytical value of the shock Mach number comes out to be 3.1505.**

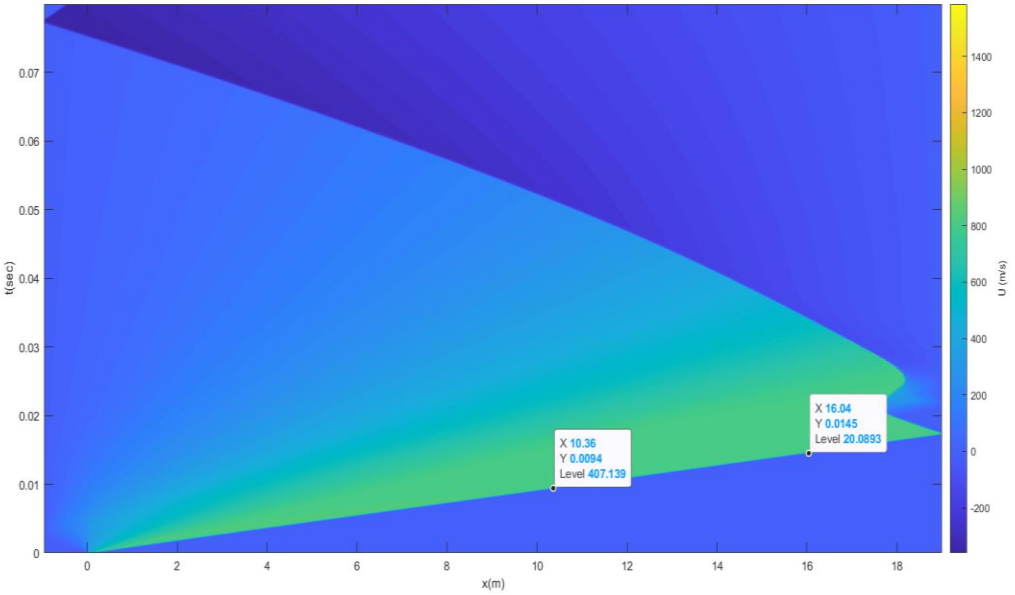
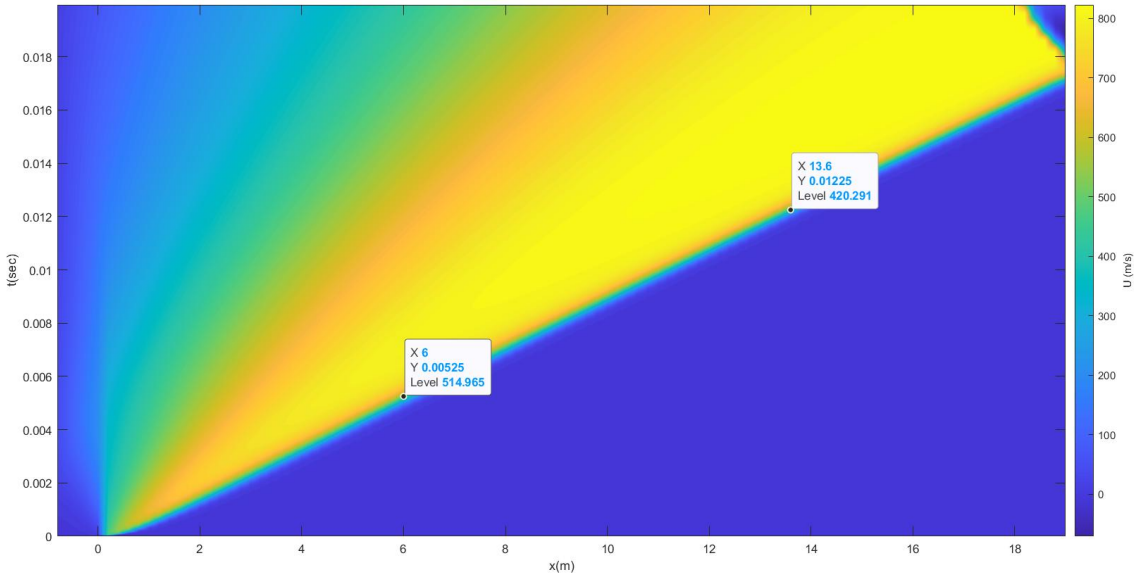
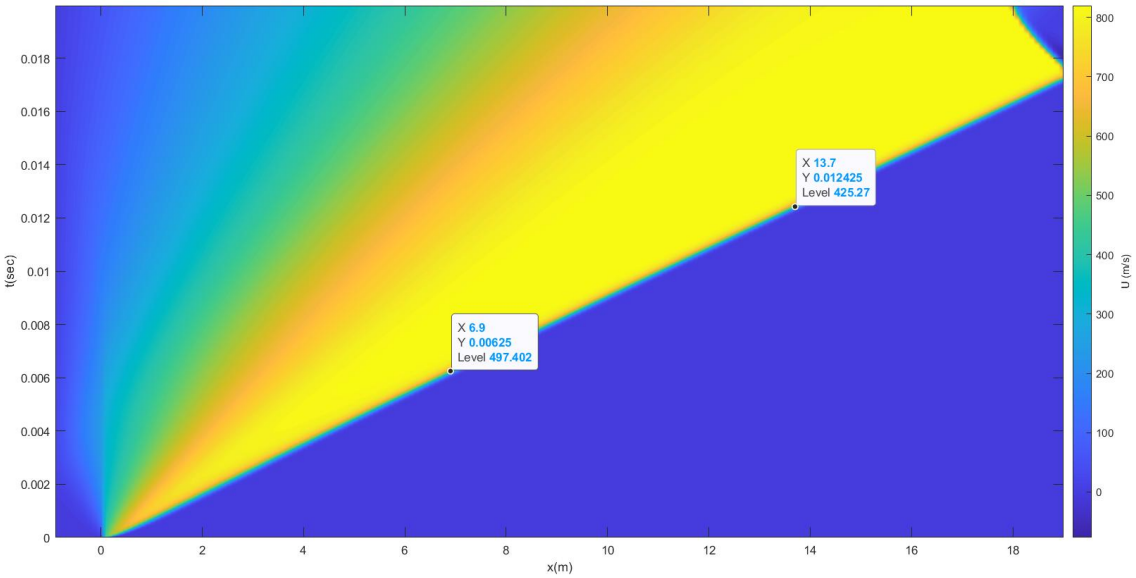


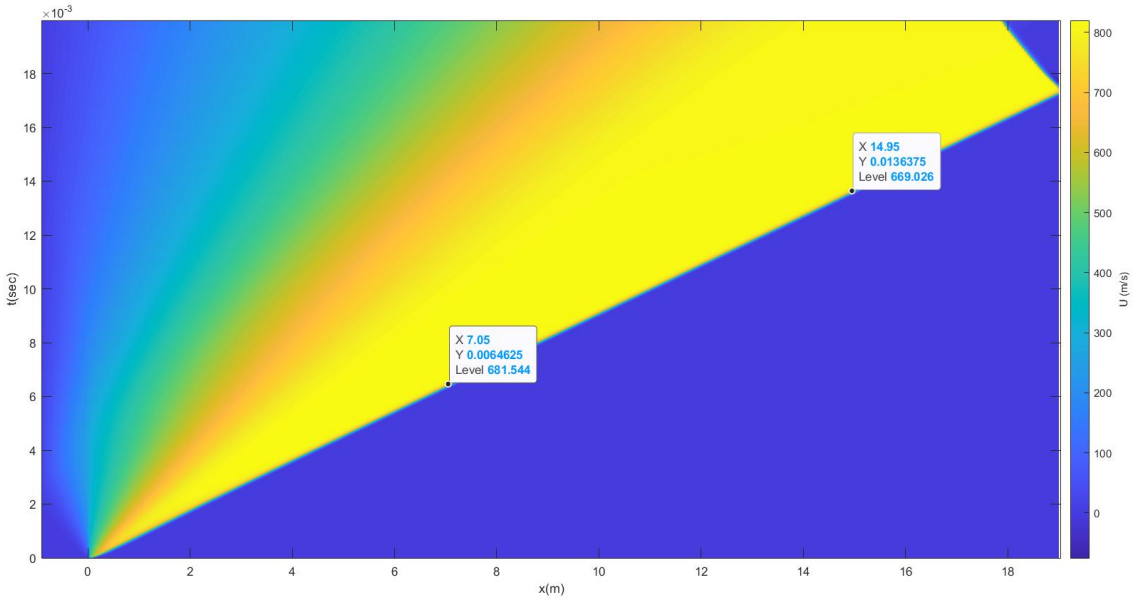
Figure 11: t vs x diagram showing shock location for two different time instances



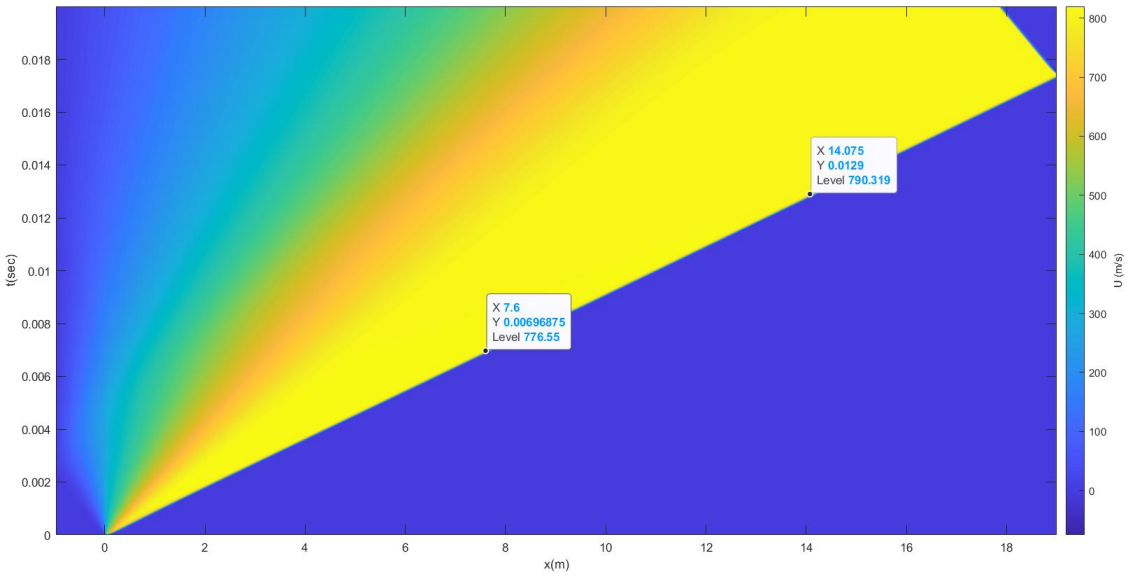
a)


X 13.7
Y 0.012425
Level 425.27
X 6.9
Y 0.00625
Level 497.402
t(sec)
x(m)
U (m/s)

b)


×10-3
X 14.95
Y 0.0136375
Level 669.026
X 7.05
Y 0.0064625
Level 681.544
t(sec)
x(m)
U (m/s)

c)



d)

Figure 12: t vs x diagram showing shock location for two different time instances for a) delx=0.2m delt=0.00005s, b) delx=0.1m delt=0.000025s, c) delx=0.05m delt=0.0000125s, and d) delx=0.01m delt=0.00000625s

From Figures 12 a to 12 d, the shock Mach numbers of cases with different grid spacing and time step sizes are calculated and shown in the table below

Case	Analytical	Δx=0.2m Δt=0.00005s	Δx=0.1m Δt=0.000025s	Δx=0.05m Δt=0.0000125s	Δx=0.01m Δt=0.00000625s
M_s	3.1505	3.1643	3.2094	3.2089	3.1816

Figure 13 shows the t vs diagram for Temperature, clearly indicating the primary shock and the contact discontinuity. A vertical line is plotted at x=10.5m, that is, at the model location. The line is plotted between the primary shock and the primary contact discontinuity. Using the endpoints of the line, the test time can be found, which comes to be 1.9×10^{-3}s or 1.9ms.

From figures 14 a) to 14 d), the post-shock conditions can be extracted, which come out to be:

Property	**Value**
Pressure	118592 Pa
Temperature	856.578 K
Density	0.4802 kg/m^3
Velocity	824.028 m/s
Mach number	1.404

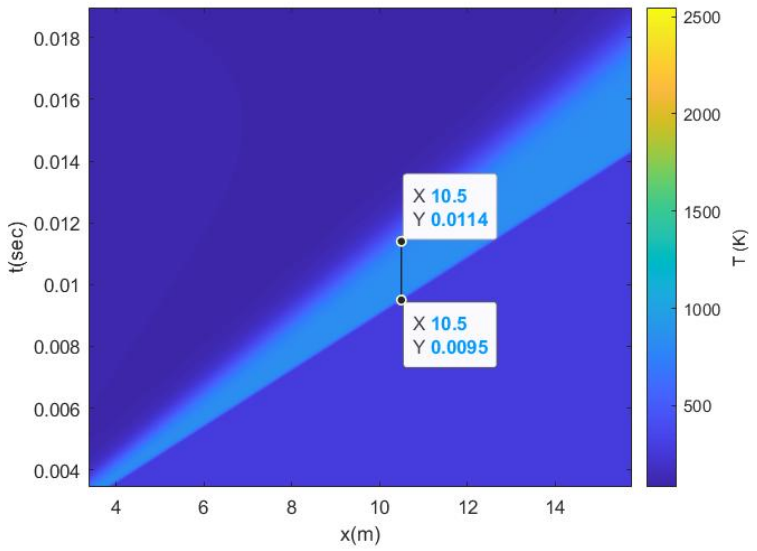
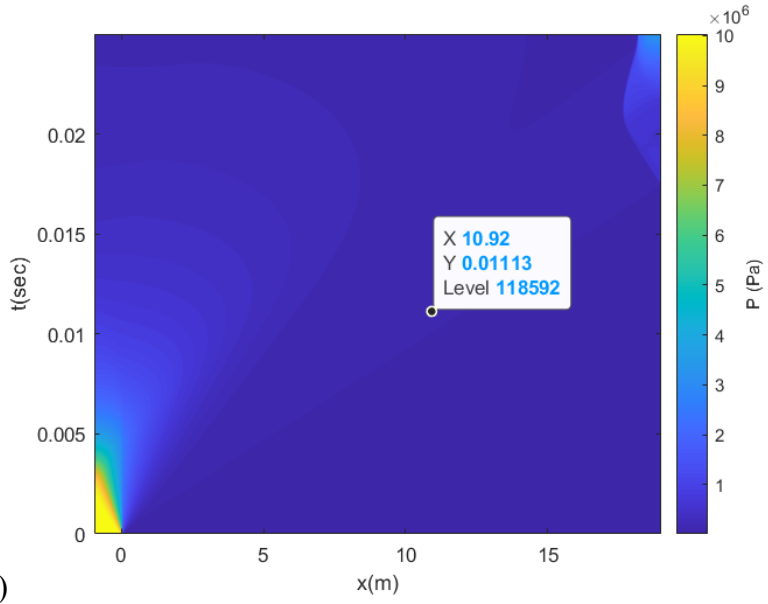
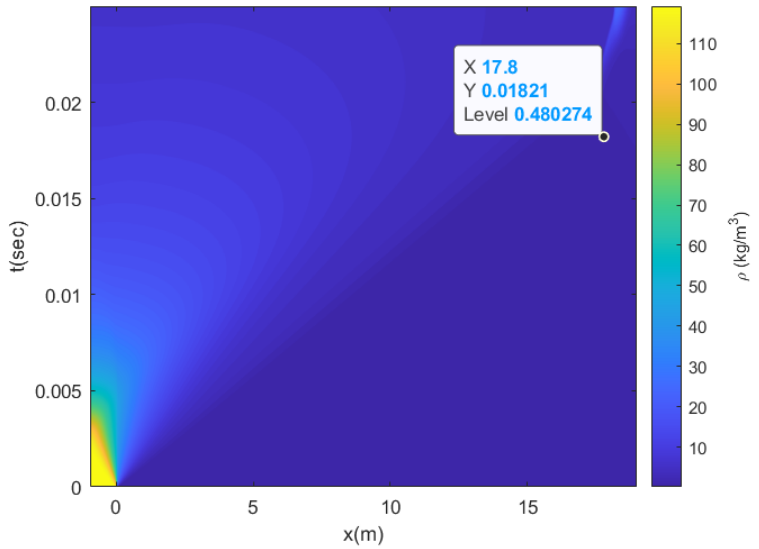


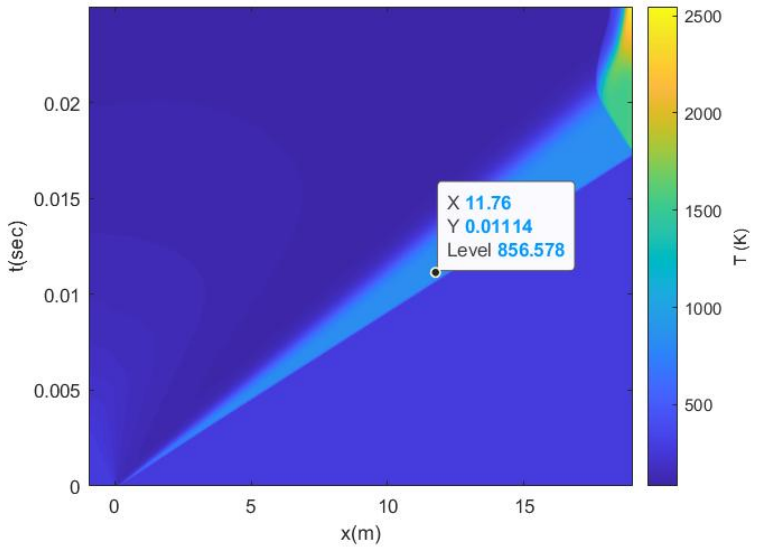
Figure 13: t vs x diagram for Temperature showing the test time region at the model location



a)


X 17.8
Y 0.01821
Level 0.480274
t(sec)
x(m)
ρ (kg/m3)

b)


X 11.76
Y 0.01114
Level 856.578
t(sec)
x(m)
T (K)

c)

d)

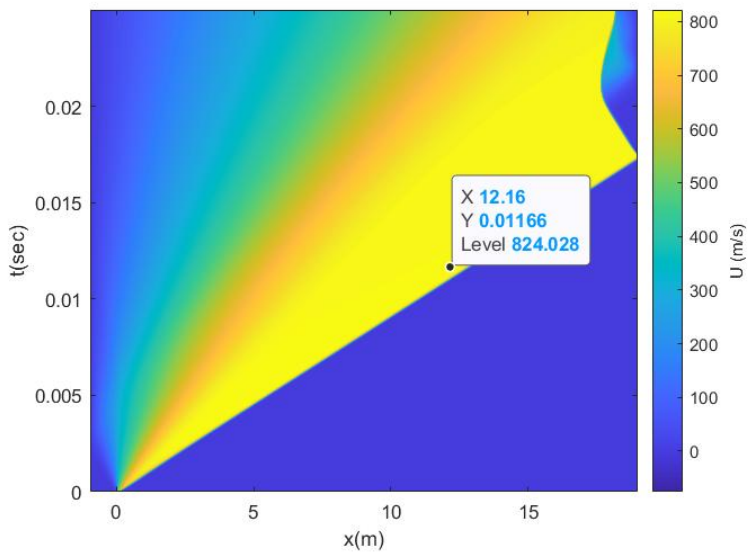


Figure 14: t vs x diagram for a) Pressure, b) Density, c) Temperature, and d) Velocity indicating the post-shock or the test conditions.

Appendix 1: Main

```
clc
clear
close all

% Author    : Abhyudaya Singh
% UIN       : 655691216
% NetID     : singh124@illinois.edu

%% Main

fprintf('\nEnter 1 for solving Toro case1 in Problem 2
\nEnter 2 for solving Toro case2 in Problem 2 \nEnter 3
for solving Problem 3 \nEnter 4 to enter custom
conditions')
ch=input('\nEnter choice');

switch ch
    case 1
        L1=0.5;
        L4=0.5;
        BC1="SW";
        BC4="SW";
        q1in=[1,1/287,0];
        q4in=[0.1,0.1/(287*0.125),0];
        Nx=500;
        T=0.25;
        delt=0.0001;

FUN_Finite_Volume_Solver(L1,BC1,q1in,L4,BC4,q4in,Nx,T,del
t,ch);
    case 2
        L1=0.5;
        L4=0.5;
        BC1="SW";
        BC4="SW";
        q1in=[0.4,0.4/287,-2];
        q4in=[0.4,0.4/287,2];
        Nx=500;
        T=0.15;
        delt=0.0001;

FUN_Finite_Volume_Solver(L1,BC1,q1in,L4,BC4,q4in,Nx,T,del
t,ch);
    case 3
```

```
        L1=19;
        L4=1;
        BC1="SW";
        BC4="SW";
        q1in=[100*101325,293,0];
        q4in=[0.1*101325,293,0];
        Nx=500;
        T=0.08;
        delt=0.00001;

FUN_Finite_Volume_Solver(L1,BC1,q1in,L4,BC4,q4in,Nx,T,del
t,ch);
    case 4
        L1=input('\nEnter length of the driven section');
%driven section length length
        L4=input('\nEnter length of the driver section');
%driver section length length

        q4in=input('\nEnter the driver section initial
conditions[p1,T1,u1]');
        q1in=input('\nEnter the driven section initial
conditions[p4,T4,u4]');

        fprintf('\nEnter boundary conditoin type: \nSW->
solid wall condition \nFF-> far-field condition')
        BC1=input('\ndriven section boundary type','s');
%driven section boundary type
        BC4=input('\ndrive4 section boundary type','s');
%driven section boundary type

        Nx=input('\n Enter number of grid points');
%number of cells
        T=input('\nEnter final simulation time');
%total time
        delt=input('\nEnter the time step size');
%time step size

FUN_Finite_Volume_Solver(L1,BC1,q1in,L4,BC4,q4in,Nx,T,del
t,ch);
    otherwise
        fprintf('\n wrong choice')
end
```

Appendix 2: Function FUN_Fininte_Volume_Solver

```
function
[]=FUN_Finite_Volume_Solver(L1,BC1,q1in,L4,BC4,q4in,Nx,T,
delt,ch)

% Author    : Abhyudaya Singh
% UIN       : 655691216
% NetID     : singh124@illinois.edu

% This code solves the 1D Euler's equation using Finite
Volume method using Godunov's method for flux
reconstruction

%% initialization

gama=1.4;
R=287;

L=L1+L4;
delx=L/(Nx);

N_dia=round(L4/delx);       % node at diaphram

q0=zeros(10000,Nx);
q1=zeros(10000,Nx);
q2=zeros(10000,Nx);

FL=zeros(3,Nx);
FR=zeros(3,Nx);

%% initalizing flow field
rho1in=q1in(1)/(R*q1in(2));
rho4in=q4in(1)/(R*q4in(2));
q0(1,1:N_dia) = ones(1,N_dia)*rho1in;
q1(1,1:N_dia) = q0(1,1:N_dia)*q1in(3);
q2(1,1:N_dia) = ones(1,N_dia)*(q1in(1)/(gama-1) +
0.5*rho1in*q1in(3)^2);

q0(1,N_dia+1:end) = ones(1,Nx-N_dia)*rho4in;
q1(1,N_dia+1:end) = q0(1,N_dia+1:end)*q4in(3);
q2(1,N_dia+1:end) = ones(1,Nx-N_dia)*(q4in(1)/(gama-1) +
0.5*rho4in*q4in(3)^2);


%% solution
fprintf('\nCalculating the solution!...')
```

```
fprintf('\n\titer \t Flow time')

t=zeros(10000,1);
ct=0;
while t(ct+1)<=T
    ct=ct+1;
    for n=1:Nx

        if n==1
            if BC1 == "SW"
                if q1(n)>0
                    qL=FUN_Godunov_flux_recon([q0(ct,n),
0, q2(ct,n)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
                else

qL=FUN_Godunov_flux_recon([2*q0(ct,n)-q0(ct,n+1), 0,
2*q2(ct,n)-
q2(ct,n+1)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
                end
            elseif BC1 == "FF"
                if q1(n)>0
                    qL=FUN_Godunov_flux_recon([q0(ct,n),
q1(ct,n), q2(ct,n)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
                else

qL=FUN_Godunov_flux_recon([2*q0(ct,n)-q0(ct,n+1),
2*q1(ct,n)-q1(ct,n+1), 2*q2(ct,n)-
q2(ct,n+1)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
                end
            end

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[q
0(ct,n+1),q1(ct,n+1),q2(ct,n+1)],gama,0);
            PL=(gama-1)*(qL(3)-(0.5*qL(2)^2 /qL(1)));
            FL(1:3,n)=[qL(2), qL(2)^2 /qL(1) + PL,
(qL(3)+PL)*qL(2)/qL(1)]';
            PR=(gama-1)*(qR(3)-(0.5*qR(2)^2 /qR(1)));
            FR(1:3,n)=[qR(2), qR(2)^2 /qR(1) + PR,
(qR(3)+PR)*qR(2)/qR(1)]';

        elseif n==Nx

            if BC4 == "SW"
                if q1(n)>0

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[2
*q0(ct,n)-q0(ct,n-1), 0, 2*q2(ct,n)-q2(ct,n-1)],gama,0);
```

```
                else

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[q
0(ct,n),0,q2(ct,n)],gama,0);
                end
            elseif BC4 == "FF"
                if q1(n)>0

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[2
*q0(ct,n)-q0(ct,n-1), 2*q1(ct,n)-q1(ct,n-1), 2*q2(ct,n)-
q2(ct,n-1)],gama,0);
                else

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[q
0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
                end
            end
            qL=FUN_Godunov_flux_recon([q0(ct,n-
1),q1(ct,n-1),q2(ct,n-
1)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);
            PL=(gama-1)*(qL(3)-(0.5*qL(2)^2 /qL(1)));
            FL(1:3,n)=[qL(2), qL(2)^2 /qL(1) + PL,
(qL(3)+PL)*qL(2)/qL(1)]';
            PR=(gama-1)*(qR(3)-(0.5*qR(2)^2 /qR(1)));
            FR(1:3,n)=[qR(2), qR(2)^2 /qR(1) + PR,
(qR(3)+PR)*qR(2)/qR(1)]';

        else

            qL=FUN_Godunov_flux_recon([q0(ct,n-
1),q1(ct,n-1),q2(ct,n-
1)],[q0(ct,n),q1(ct,n),q2(ct,n)],gama,0);

qR=FUN_Godunov_flux_recon([q0(ct,n),q1(ct,n),q2(ct,n)],[q
0(ct,n+1),q1(ct,n+1),q2(ct,n+1)],gama,0);
            PL=(gama-1)*(qL(3)-(0.5*qL(2)^2 /qL(1)));
            FL(1:3,n)=[qL(2), qL(2)^2 /qL(1) + PL,
(qL(3)+PL)*qL(2)/qL(1)]';
            PR=(gama-1)*(qR(3)-(0.5*qR(2)^2 /qR(1)));
            FR(1:3,n)=[qR(2), qR(2)^2 /qR(1) + PR,
(qR(3)+PR)*qR(2)/qR(1)]';
        end
    end

        q0(ct+1,:) = q0(ct,:) - (delt/delx)*(FR(1,:)-
FL(1,:));
```

```
        q1(ct+1,:) = q1(ct,:) - (delt/delx)*(FR(2,:)-
FL(2,:));
        q2(ct+1,:) = q2(ct,:) - (delt/delx)*(FR(3,:)-
FL(3,:));

        if ct>1
            fprintf(repmat('\b',1,lineLength))
        end
        lineLength = fprintf('\n\t%d \t %f',ct,t(ct));

        t(ct+1)=t(ct)+delt;

end

fprintf('\nCalculation complete!')

%% plotting
if ch == 1 || ch == 2
    if ch==1
        toro= table2array(readtable('Test1_Toro1.txt'));
    else
        toro= table2array(readtable('Test2_Toro1.txt'));
    end
    xana=toro(:,1);
    rhoana=toro(:,4);
    uana=toro(:,5);
    Pana=toro(:,6);

    figure(1)
    plot(-L4+delx:delx:L1,q0(ct,:))
    grid on
    hold on
    plot(xana,rhoana)
    xlabel('x')
    ylabel('rho')
    legend('Numerical solution','Toro')

    figure(2)
    plot(-L4+delx:delx:L1,q1(ct,:)./q0(ct,:))
    grid on
    hold on
    plot(xana,uana)
    xlabel('x')
    ylabel('u')
    legend('Numerical solution','Toro')

    figure(3)
```

```
    plot(-L4+delx:delx:L1,(gama-1)*(q2(ct,:)-
(0.5*(q1(ct,:).^2)./q0(ct,:))))
    grid on
    hold on
    plot(xana,Pana)
    xlabel('x')
    ylabel('P')
    legend('Numerical solution','Toro')
end

if ch == 3 || ch == 4
    figure(4)
    contourf(-L4+delx:delx:L1,t(1:10:ct),(gama-
1)*(q2(1:10:ct,:)-
(0.5*(q1(1:10:ct,:).^2)./q0(1:10:ct,:))),100,'edgecolor',
'none')
    a=colorbar;
    ylabel(a,'P (Pa)','Rotation',90);
    xlabel('x(m)')
    ylabel('t(sec)')

    figure(5)
    contourf(-
L4+delx:delx:L1,t(1:10:ct),q0(1:10:ct,:),100,'edgecolor',
'none')
    a=colorbar;
    ylabel(a,'\rho (kg/m^{3})','Rotation',90);
    xlabel('x(m)')
    ylabel('t(sec)')

    figure(6)
    contourf(-
L4+delx:delx:L1,t(1:10:ct),q1(1:10:ct,:)./q0(1:10:ct,:),1
00,'edgecolor','none')
    a=colorbar;
    ylabel(a,'U (m/s)','Rotation',90);
    xlabel('x(m)')
    ylabel('t(sec)')

    figure(7)
    contourf(-L4+delx:delx:L1,t(1:10:ct),((gama-
1)*(q2(1:10:ct,:)-
(0.5*(q1(1:10:ct,:).^2)./q0(1:10:ct,:)))./(R*q0(1:10:ct,
:)),100,'edgecolor','none')
    a=colorbar;
    ylabel(a,'T (K)','Rotation',90);
    xlabel('x(m)')
```

```
    ylabel('t(sec)')
end
end
```

Appendix 3: Function FUN_Godunov_flux_recon

```
function [Q] = FUN_Godunov_flux_recon(q1,q2,gama,xt)

% Author    : Abhyudaya Singh
% UIN       : 655691216
% NetID     : singh124@illinois.edu

% This function constructs the fluxes on the cell faces
by using the Riemann solver

%converting vectors from [rho, rhou, rhoE] to [rho, u, P]
qL=[q1(1), q1(2)/q1(1), (gama-1)*(q1(3)-(0.5*q1(2)^2
/q1(1)))];
qR=[q2(1), q2(2)/q2(1), (gama-1)*(q2(3)-(0.5*q2(2)^2
/q2(1)))];

cL=sqrt(gama*qL(3)/qL(1));  % sound speed on left
cR=sqrt(gama*qR(3)/qR(1));  % sound speed on right

%% solving for P*
[pstar,ustar,rhoLstar,cLstar,rhoRstar,cRstar] =
FUN_Root_Bisection(qL,qR,gama);


%% Calculating the fluxes

sL = qL(2)-cL*sqrt(((gama+1)*pstar)/(2*gama*qL(3)) +
((gama-1)/(2*gama)));
sR = qR(2)+cR*sqrt(((gama+1)*pstar)/(2*gama*qR(3)) +
((gama-1)/(2*gama)));

if xt<ustar
    if qL(3)>=pstar
        if xt<=qL(2)-cL
            q=qL;
        elseif xt<=ustar-cLstar
            rhofL = qL(1)*((2/(gama+1)) + ((gama-
1)/(gama+1))*((qL(2)-xt)/cL))^(2/(gama-1));
            ufL = ((gama-1)/(gama+1)*(qL(2) +
2*((xt+cL)/(gama-1))));
            PfL = qL(3)*((2/(gama+1)) + ((gama-
1)/(gama+1))*((qL(2)-xt)/cL))^(2*gama/(gama-1));
            q=[rhofL,ufL,PfL];
        else%if xt>ustar-cLstar && xt<=ustar
            q=[rhoLstar,ustar,pstar];
```

```
        end
    else
        if xt<=sL
            q=qL;
        else%if xt>sL && xt<=ustar
            q=[rhoLstar,ustar,pstar];
        end
    end

else
    if qR(3)>=pstar
        if xt<=ustar+cRstar
            q=[rhoRstar,ustar,pstar];
        elseif xt<=qR(2)+cR
            rhofR = qR(1)*((2/(gama+1)) + ((gama-
1)/(gama+1))*((qR(2)-xt)/cR))^(2/(gama-1));
            ufR = ((gama-1)/(gama+1)*(qR(2) + 2*((xt-
cR)/(gama-1))));
            PfR = qR(3)*((2/(gama+1)) + ((gama-
1)/(gama+1))*((qR(2)-xt)/cR))^(2*gama/(gama-1));
            q=[rhofR,ufR,PfR];
        else
            q=qR;
        end
    else
        if xt<=sR
            q=[rhoRstar,ustar,pstar];
        else%if xt>sR
            q=qR;
        end
    end
end
Q=[q(1), q(1)*q(2), q(3)/(gama-1) + 0.5*q(1)*q(2)^2];
%[rho,rhou,rhoE]
end
```

Appendix 4: Function FUN_Root_Bisection

```
function [pstar,ustar,rhoLstar,cLstar,rhoRstar,cRstar] =
FUN_Root_Bisection(qL,qR,gama)

% Author    : Abhyudaya Singh
% UIN       : 655691216
% NetID     : singh124@illinois.edu

% this function finds the root using the Bisection method
% the qk passed in this function should have qk=[rhok,
uk, Pk];

itmax=100;
a = 0.9 * (qL(3) + qR(3))/2;
b = 1.1 * (qL(3) + qR(3))/2;
it = 0;

% Make sure f(a) is negative
while qR(2) - qL(2) + FUN_fk(a,qL,gama) +
FUN_fk(a,qR,gama) > 0 && it < itmax
    a = a/2;
    it = it + 1;
end

it = 0;

% Make sure f(b) is positive
while qR(2) - qL(2) + FUN_fk(b,qL,gama) +
FUN_fk(b,qR,gama) < 0 && it < itmax
    b = b*2;
    it = it + 1;
end

% Bisection algorithm

it = 0;
e = 1e-6;
pstar = (a + b)/2;
fpstar = qR(2) - qL(2) + FUN_fk(pstar,qL,gama) +
FUN_fk(pstar,qR,gama);

while abs(fpstar) > e && (b - a)/2 > e && it < itmax
    pstar = (a + b)/2;
    fpstar = qR(2) - qL(2) + FUN_fk(pstar,qL,gama) +
FUN_fk(pstar,qR,gama);
```

```
    if fpstar > 0
        b = pstar;
    else
        a = pstar;
    end
    it = it + 1;
end

ustar = (qR(2) + qL(2) + FUN_fk(pstar,qR,gama) -
FUN_fk(pstar,qL,gama))/2;
rhoLstar = qL(1)*(pstar/qL(3))^(1/gama);
cLstar = sqrt(gama*pstar/rhoLstar);
rhoRstar = qR(1)*(pstar/qR(3))^(1/gama);
cRstar = sqrt(gama*pstar/rhoRstar);
end
```

Appendix 5: Function FUN_fK

```
function fK = FUN_fk(pstar,qK,gamma)

% Author    : Abhyudaya Singh
% UIN       : 655691216
% NetID     : singh124@illinois.edu

    % First extract density, velocity, and pressure from
q
    rhoK = qK(1);
    pK = qK(3);

    if pstar <= pK
        % Rarefaction wave
        % Calculate speed of sound
        cK = sqrt(gamma * pK / rhoK);
        % Return fK
        fK = 2 * cK / (gamma - 1) * ((pstar / pK)^((gamma
- 1)/(2 * gamma)) - 1);
    else
        % Shock wave
        % Calculate A and B
        AK = 2 / (rhoK * (gamma + 1));
        BK = (gamma - 1) / (gamma + 1) * pK;
        % Return fK
        fK = (pstar - pK) * sqrt(AK / (BK + pstar));
    end
end
```