# Correction of Midterm exam 1

#### Exercise 1 (2 points)

1.  $\cos(x)^{\sin(x)} = e^{\sin(x)\ln(\cos(x))}$ 

 $\operatorname{But}$ 

$$\sin(x)\ln(\cos(x)) = \left(x - \frac{x^3}{6} + o(x^3)\right)\ln\left(1 - \frac{x^2}{2} + o(x^3)\right) = \left(x - \frac{x^3}{6} + o(x^3)\right)\left(-\frac{x^2}{2} + o(x^3)\right)$$

so 
$$\cos(x)^{\sin(x)} = e^{-x^3/2 + o(x^3)} = 1 - \frac{x^3}{2} + o(x^3).$$

2. 
$$\ln(1+\sin(x)) = \ln\left(1+x-\frac{x^3}{6}+o(x^4)\right) = \left(x-\frac{x^3}{6}\right) - \frac{1}{2}\left(x-\frac{x^3}{6}\right)^2 + \frac{1}{3}\left(x-\frac{x^3}{6}\right)^3 - \frac{1}{4}\left(x-\frac{x^3}{6}\right)^4 + o(x^4)$$
so
$$\ln(1+\sin(x)) = x - \frac{x^3}{6} - \frac{1}{2}\left(x^2 - \frac{x^4}{3}\right) + \frac{1}{3}x^3 - \frac{1}{4}x^4 + o(x^4)$$

Thus

$$\ln(1+\sin(x)) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + o(x^4)$$

$$\sin(\ln(1+x)) = \sin\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^4)\right) = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right) - \frac{1}{6}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}\right)^3 + o(x^4)$$
so
$$\sin(\ln(1+x)) = x - \frac{x^2}{2} + \frac{x^3}{6} + o(x^4)$$

Thus

$$\ln(1+\sin(x)) - \sin(\ln(1+x)) = -\frac{x^4}{12} + o(x^4)$$

and

$$\ln(1+\sin(x)) - \sin(\ln(1+x)) \sim -\frac{x^4}{12}$$

On the other hand,  $x^2 \sin(x^2) \sim x^4$ . Thus

$$\frac{\ln(1+\sin(x)) - \sin(\ln(1+x))}{x^2 \sin(x^2)} \sim \frac{-\frac{x^4}{12}}{x^4} = -\frac{1}{12}$$

and

$$\lim_{x \to 0} \left\lceil \frac{\ln(1+\sin(x)) - \sin(\ln(1+x))}{x^2 \sin(x^2)} \right\rceil = -\frac{1}{12}$$

## Exercice 2 (4,5 points)

1. Let's call 
$$u_n = \frac{2n}{n+2^n}$$
.

$$\begin{array}{rcl} \frac{u_{n+1}}{u_n} & = & \frac{2n+2}{n+1+2^{n+1}} \times \frac{n+2^n}{2n} \\ \\ & = & \frac{n+1}{n} \times \frac{2^n}{2^{n+1}} \times \frac{\frac{n}{2^n}+1}{\frac{n+1}{2n+1}+1} \xrightarrow[n \to +\infty]{} \frac{1}{2} \end{array}$$

But  $\frac{1}{2} < 1$  so, via the rule of d'Alembert,  $\sum u_n$  is convergent.

2. Let's call 
$$v_n = \frac{1+n^2}{n!}$$
.

$$\frac{v_{n+1}}{v_n} = \frac{1 + (n+1)^2}{(n+1)!} \times \frac{n!}{1+n^2} = \frac{1}{n+1} \times \frac{n^2 + 2n + 2}{n^2 + 1} \xrightarrow[n \to +\infty]{} 0.$$

But 0 < 1 so  $\sum v_n$  is convergent.

- 3.  $\left| \frac{\sin(\sqrt{n}+1)}{n^2} \right| \le \frac{1}{n^2}$  and  $\sum \frac{1}{n^2}$  is convergent so  $\sum \frac{\sin(\sqrt{n}+1)}{n^2}$  is absolutely convergent. Therefore it is convergent.
- 4. If  $\alpha \leq 0$ , the term of the series does not tend to 0 so  $\sum \frac{(-1)^n}{n^{\alpha}}$  is divergent.

If  $\alpha > 1$ ,  $\sum \frac{(-1)^n}{n^{\alpha}}$  is absolutely convergent so it is convergent.

If  $0 < \alpha \le 1$ , the series  $\sum \frac{(-1)^n}{n^{\alpha}}$  is alternating and satisfies the conditions of the alternating series criterion: the sequence  $\left(\frac{1}{n^{\alpha}}\right)$  is decreasing and tends to 0 so  $\sum \frac{(-1)^n}{n^{\alpha}}$  is convergent.

#### Exercise 3 (8 points)

- 1. a.  $\sum \frac{1}{\sqrt{n}}$  is a divergent Riemann series because  $\frac{1}{2} < 1$ .
  - b.  $(a_n)$  is the sequence of partial sums associated to the series  $\sum \frac{1}{\sqrt{n}}$ . It is increasing because the series has positive terms. Via the previous question,  $\sum \frac{1}{\sqrt{n}}$  is divergent so  $a_n$  diverges to  $+\infty$  when n tends to  $+\infty$ .
- 2. a. We have

$$u_n = \frac{(-1)^n}{a_n} \left( 1 + \frac{(-1)^n}{a_n} \right)^{-1}$$

$$= \frac{(-1)^n}{a_n} \left( 1 - \frac{(-1)^n}{a_n} + o\left(\frac{1}{a_n}\right) \right)$$

$$= \frac{(-1)^n}{a_n} - \frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right)$$

b. 
$$\frac{1}{a_{n+1}} - \frac{1}{a_n} = \frac{a_n - a_{n+1}}{a_n a_{n+1}}$$

but

$$a_n - a_{n+1} = -\frac{1}{\sqrt{n+1}} < 0$$

so  $\frac{1}{a_{n+1}} - \frac{1}{a_n} < 0$ . Thus,  $\left(\frac{1}{a_n}\right)_{n \in \mathbb{N}^*}$  is decreasing and, according to 1.b, converges to 0.

- c. The series  $\sum \frac{(-1)^n}{a_n}$  is convergent according the alternating series criterion.
- d. We have

$$2\sqrt{2} - 2 \leqslant a_1 = 1 \leqslant 2\sqrt{1} - 1 = 1$$

so the property is true for n = 1. Suppose now that it is true for a given value  $n \ge 1$ . Then, since

$$a_{n+1} = a_n + \frac{1}{\sqrt{n+1}}$$

we have, via the induction hypothesis and the remark in N.B.,

$$2\sqrt{n+1} - 2 + 2\sqrt{n+2} - 2\sqrt{n+1} \leqslant a_{n+1} \leqslant 2\sqrt{n} - 1 + 2\sqrt{n+1} - 2\sqrt{n}$$

that is

$$2\sqrt{n+2} - 2 \leqslant a_{n+1} \leqslant 2\sqrt{n+1} - 1$$

Finally, the property is also true for n + 1.

e. Referring to the previous question, we have

$$\frac{\sqrt{n+1}-1}{\sqrt{n}} \leqslant \frac{a_n}{2\sqrt{n}} \leqslant \frac{2\sqrt{n}-1}{2\sqrt{n}}$$

which means that

$$\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\leqslant \frac{a_n}{2\sqrt{n}}\leqslant 1-\frac{1}{2\sqrt{n}}$$

Using the squeeze theorem, we deduce that  $\frac{a_n}{2\sqrt{n}} \longrightarrow 1$ . Hence

$$a_n \underset{+\infty}{\sim} 2\sqrt{n}$$

f. According to the previous question,

$$-\frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right) \underset{+\infty}{\sim} -\frac{1}{a_n^2} \underset{+\infty}{\sim} -\frac{1}{4n}$$

but 
$$\sum \frac{1}{n}$$
 is divergent so  $\sum \left(-\frac{1}{a_n^2} + o\left(\frac{1}{a_n^2}\right)\right)$  is divergent.

3.  $\sum u_n$  is the sum of a convergent series and a divergent one. It is therefore divergent.

### Exercise 4 (4,5 points)

1. 
$$u_n = \ln\left(\frac{\sqrt{n} + (-1)^n}{\sqrt{n+1}}\right) = \ln\left(\frac{\sqrt{n}}{\sqrt{n+1}} + \frac{(-1)^n}{\sqrt{n+1}}\right) = \ln\left(\frac{\sqrt{n}}{\sqrt{n+1}}\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)\right)$$

$$= \ln\left(\frac{\sqrt{n}}{\sqrt{n+1}}\right) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

$$= \ln(v_n) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

2. 
$$v_n = \sqrt{\frac{n}{n+1}} = \left(1 + \frac{1}{n}\right)^{-\frac{1}{2}}$$
$$= 1 - \frac{1}{2n} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right)$$
So  $\alpha = \frac{1}{2}$  and  $\beta = \frac{3}{8}$ .

3. 
$$\ln(v_n) = \ln\left(1 - \frac{1}{2n} + \frac{3}{8n^2} + o\left(\frac{1}{n^2}\right)\right) = -\frac{1}{2n} + \frac{3}{8n^2} - \frac{1}{2}\left(\frac{1}{2n} - \frac{3}{8n^2}\right)^2 + o\left(\frac{1}{n^2}\right)$$

$$= -\frac{1}{2n} + \frac{3}{8n^2} - \frac{1}{8n^2} + o\left(\frac{1}{n^2}\right)$$

$$= -\frac{1}{2n} + \frac{1}{4n^2} + o\left(\frac{1}{n^2}\right)$$

So 
$$\gamma = \frac{1}{4}$$
.

4. 
$$u_n = \ln(v_n) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = -\frac{1}{2n} + \frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) + \ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right)$$

But

$$\ln\left(1 + \frac{(-1)^n}{\sqrt{n}}\right) = \frac{(-1)^n}{\sqrt{n}} - \frac{(-1)^{2n}}{2n} + \frac{(-1)^{3n}}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
$$= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

Furthermore

$$\frac{1}{4n^2} + o\left(\frac{1}{n^2}\right) = o\left(\frac{1}{n\sqrt{n}}\right)$$

Finally

$$u_n = -\frac{1}{2n} + \frac{(-1)^n}{\sqrt{n}} - \frac{1}{2n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$
$$= \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

5. 
$$u_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

But  $\sum \frac{(-1)^n}{\sqrt{n}}$  is alternating and convergent, because  $\left(\left|\frac{(-1)^n}{\sqrt{n}}\right|\right)$  is decreasing and tends to 0.

Let's call

$$w_n = \frac{(-1)^n}{3n\sqrt{n}} + o\left(\frac{1}{n\sqrt{n}}\right)$$

Then

$$|w_n| \underset{+\infty}{\sim} \frac{1}{3n\sqrt{n}} = \frac{1}{3n^{\frac{3}{2}}}$$

and  $\sum \frac{1}{n^{\frac{3}{2}}}$  is convergent. So  $\sum w_n$  is absolutely convergent and is therefore convergent.

On the other hand,  $\sum \frac{1}{n}$  is divergent.

Finally  $\sum u_n$ , being the sum of a divergent series and a convergent one, is divergent.

#### Exercise 5 (2 points)

1. 
$$\sum (u_{n+1} - u_n)$$
 convergent  $\iff \left(\sum_{k=0}^n (u_{k+1} - u_k)\right)$  convergent  $\iff (u_n - u_0)$  convergent so

$$\sum (u_{n+1} - u_n)$$
 convergent  $\iff (u_n) = (u_n - u_0 + u_0)$  convergent

2.

Suppose that  $\sum a_n$  is convergent. Then

$$\forall n \in \mathbb{N}, \ 0 < u_{n+1} - u_n = \frac{a_n}{u_n} < \frac{a_n}{u_0}$$

so  $\sum (u_{n+1} - u_n)$  is convergent by comparaison and, according to the question 1,  $(u_n)$  is convergent.

 $\Longrightarrow$ 

 $\overline{\text{Supp}}$  ose now that  $(u_n)$  converges to  $\ell$ . Since  $(u_n)$  is strictly increasing and strictly positive,  $\ell > 0$ . Furtheremore

$$0 < u_{n+1} - u_n = \frac{a_n}{u_n} \underset{+\infty}{\sim} \frac{a_n}{\ell}$$

which implies that

$$a_n \underset{+\infty}{\sim} \ell(u_{n+1} - u_n)$$

But  $(u_n)$  is convergent so, according to the question 1,  $\sum (u_{n+1} - u_n)$  is convergent.

Thus,  $\sum a_n$  is convergent by comparaison.