Corrigé du contrôle TD blanc

Question de cours

Give an accurate statement of the n^{th}-term test for divergence of series, the convergence rule for Riemann series, the comparison rules for series of positive terms, the ratio test (d'Alembert's test) and the root test (Cauchy's test).
I trust you with this guys :)

Exercise 1

— **Using Cauchy's root test, determine the nature of the series $\sum \frac{n^\alpha}{\alpha^n}$ depending on the values of $\alpha \in \mathbb{R}_+^*$**

Let us set $u_n = \frac{n^\alpha}{\alpha^n}$

$$(u_n)^{\frac{1}{n}} = \frac{n^{\frac{\alpha}{n}}}{\alpha} = \frac{e^{\alpha \frac{\ln(n)}{n}}}{\alpha}$$

Using a result of compared growth, $\frac{\ln(n)}{n} \underset{n\to+\infty}{\longrightarrow} 0$ thus $e^{\alpha\frac{\ln(n)}{n}} \underset{n\to+\infty}{\longrightarrow} 1$.

Then, $(u_n)^{\frac{1}{n}} \underset{n\to+\infty}{\longrightarrow} \frac{1}{\alpha}$. We distinguish the three following cases :

— if $\alpha > 1$, $\frac{1}{\alpha} < 1$ thus, according to the root test $\sum u_n$ is convergent ;

— if $\alpha < 1$, $\frac{1}{\alpha} > 1$ thus, according to the root test $\sum u_n$ is divergent ;

— if $\alpha = 1$ we cannot reach any conclusion using the root test.

However, in this last case $\alpha = 1$ let us remark that $u_n = n$, which is associated with a divergent series ; thus :

$$\boxed{\text{the series } \sum u_n \text{ converges iff } \alpha > 1}$$

— **Determine the nature of the series $\sum \frac{\sin(\beta n)}{n^2}$ depending on the values of $\beta \in \mathbb{R}$**

$$\forall \beta \in \mathbb{R}, \forall n \in \mathbb{N}, \quad \left|\sin(\beta n)\right| \leqslant 1$$

From this we deduce :

$$\forall \beta \in \mathbb{R}, \forall n \in \mathbb{N}, \quad \left|\frac{\sin(\beta n)}{n^2}\right| \leqslant \frac{1}{n^2}$$

As the Riemann series $\sum \frac{1}{n^2}$ is convergent, by comparison of series with positive terms, the series $\sum \left|\frac{\sin(\beta n)}{n^2}\right|$ is convergent.

Hence :

$$\boxed{\text{the series } \sum \frac{\sin(\beta n)}{n^2} \text{ is absolutely convergent, therefore convergent for every } \beta \in \mathbb{R}}$$

Exercise 2

Determine $\lim\limits_{n\longrightarrow+\infty} \frac{\cos\left(\frac{1}{n}\right) - \frac{n^2}{n^2+1}}{\ln(n^2+1) - \ln(n^2)}$

$$\ln(n^2+1) - \ln(n^2) = \ln\left(\frac{n^2+1}{n^2}\right) = \ln\left(1 + \frac{1}{n^2}\right) \underset{+\infty}{\sim} \frac{1}{n^2}$$

$$\begin{aligned}
\cos\left(\frac{1}{n}\right) - \frac{n^2}{n^2+1} &= \cos\left(\frac{1}{n}\right) - \frac{1}{1+\frac{1}{n^2}} \\
&= 1 - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) - \left(1 - \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right) \\
&= \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \underset{+\infty}{\sim} \frac{1}{2n^2}
\end{aligned}$$

Thus :

$$\frac{\cos\left(\frac{1}{n}\right) - \frac{n^2}{n^2+1}}{\ln(n^2+1) - \ln(n^2)} \underset{+\infty}{\sim} \frac{\frac{1}{2n^2}}{\frac{1}{n^2}} \underset{+\infty}{\sim} \frac{1}{2}$$

As a conclusion :

$$\boxed{\lim_{n \longrightarrow +\infty} \frac{\cos\left(\frac{1}{n}\right) - \frac{n^2}{n^2+1}}{\ln(n^2+1) - \ln(n^2)} = \frac{1}{2}}$$

Exercise 3

Using a Taylor expansion, determine the nature of $\sum n\sin\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n}\right)$.

$$\begin{aligned}
n\sin\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n}\right) &= n\left(\frac{1}{n} - \frac{1}{6n^3} + o\left(\frac{1}{n^3}\right)\right) - \left(1 - \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right)\right) \\
&= 1 - \frac{1}{6n^2} - 1 + \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \\
&= \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right) \underset{+\infty}{\sim} \frac{1}{3n^2}
\end{aligned}$$

The series $\sum \frac{1}{3n^2}$ is a convergent Riemann series ; by comparison of series with positive terms, $\sum n\sin\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n}\right)$ converges.

$$\boxed{\sum n\sin\left(\frac{1}{n}\right) - \cos\left(\frac{1}{n}\right) \quad \text{converges.}}$$