Correction of final exam n°1

Exercise 1 (5 points)

1. $\dfrac{u_{n+1}}{u_n} = \dfrac{10^{n+1}}{(n+1)4^{2n+3}} \times \dfrac{n4^{2n+1}}{10^n} = \dfrac{10}{16} \times \dfrac{n}{n+1} \xrightarrow[n\to+\infty]{} \dfrac{10}{16} = \dfrac{5}{8}$.

   As $\dfrac{5}{8} < 1$, $\sum u_n$ converges via D'Alembert's test (ratio test).

2. $\sqrt[n]{v_n} = \dfrac{n^{1/n}}{\ln(n)} = \dfrac{e^{1/n\ln(n)}}{\ln(n)} \xrightarrow[n\to+\infty]{} 0$.

   As $0 < 1$, $\sum v_n$ converges via Cauchy's test (root test).

Exercise 2 (4 points)

After developing with respect to the second column, we directly get that

$$P_A(X) = (2-X)\big((1-X)^2 - 4\big) = (2-X)(X-3)(X+1)$$

Thus P_A is split over $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(A) = \{2, 3, -1\}$ with $m(2) = m(3) = m(-1) = 1$, therefore A is diagonalizable.

$$\begin{aligned} E_2 &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} -x+2z=0 \\ 2x-z=0 \end{array} \right. \right\} \\ &= \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} E_3 &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} -2x+2z=0 \\ -y=0 \\ 2x-2z=0 \end{array} \right. \right\} \\ &= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\} \end{aligned}$$

$$\begin{aligned} E_{-1} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} 2x+2z=0 \\ 3y=0 \end{array} \right. \right\} \\ &= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\} \end{aligned}$$

Thus $D = P^{-1}AP$ with $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$.

Via the transformations $C_1 \longleftarrow C_1 + C_2$ then $L_2 \longleftarrow L_2 - L_1$, we find that $P_B(X) = -(2-X)^2(X+4)$.

Thus P_B is split over $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(B) = \{2, -4\}$ with $m(2) = 2$ and $m(-4) = 1$.

$m(-4) = 1$ thus $\dim(E_{-4}) = 1$.

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} x - y + z = 0 \\ 7x - 7y + z = 0 \\ 6x - 6y = 0 \end{array} \right. \right\} = \text{Vect}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$\dim(E_2) = 1 \neq 2 = m(2)$ thus B is not diagonalizable.

Exercise 3 (3,5 points)

1. Via the transformations $C_1 \longleftarrow C_1 + C_2$ then $L_2 \longleftarrow L_2 - L_1$, we get $P_A(X) = (1 - X)(2 - X)(a - X)$.
2. • If $a \notin \{1, 2\}$, then A admits 3 distinct eigenvalues, thus A is diagonalizable.

   • If $a = 1$, then $P_A(X) = (1 - X)^2(2 - X)$. Thus P_A is split over $\mathbb{R}$ and $\text{Sp}_{\mathbb{R}}(A) = \{1, 2\}$ with $m(1) = 2$ and $m(2) = 1$.

   Thus A is diagonalizable iff $\dim(E_1) = 2$.

   $$E_1 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} z = 0 \\ -x + y + z = 0 \\ x - y = 0 \end{array} \right. \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

   Thus $\dim(E_1) = 1 \neq 2$, therefore A is not diagonalizable.

   • if $a = 2$, then $P_A(X) = (1 - X)(2 - X)^2$. Thus P_A is split over $\mathbb{R}$ and $\text{Sp}_{\mathbb{R}}(A) = \{1, 2\}$ with $m(1) = 1$ and $m(2) = 2$.

   Thus A is diagonalisable iff $\dim(E_2) = 2$.

   $$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } -x + z = 0 \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

   thus $\dim(E_2) = 2$, therefore A is diagonalizable.

Exercise 4 (3,5 points)

1. $A = \begin{pmatrix} 1 & 8 \\ 1 & 3 \end{pmatrix}$
2. $P_A(X) = (X + 1)(X - 5)$ thus the characteristic polynomial of A is split over $\mathbb{R}$, $\text{Sp}_{\mathbb{R}}(A) = \{-1, 5\}$ with $m(-1) = m(5) = 1$, therefore A is diagonalizable.

   Then : $E_{-1} = \text{Span}\left\{ \begin{pmatrix} -4 \\ 1 \end{pmatrix} \right\}$ and $E_5 = \text{Span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

   Thus $D = P^{-1}AP$ with $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$ and $P = \begin{pmatrix} -4 & 2 \\ 1 & 1 \end{pmatrix}$.

3. Thus, $X'(t) = PDP^{-1}X(t)$. By setting $Y(t) = P^{-1}X(t)$ we get that

$$P^{-1}X'(t) = DP^{-1}X(t)$$

, that is to say $Y'(t) = DY(t)$.

So, with $Y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$, we get $y_1'(t) = -y_1(t)$ and $y_2'(t) = 5y_2(t)$

thus $y_1(t) = C_1e^{-t}$ and $y_2(t) = C_2e^{5t}$ where $(C_1, C_2) \in \mathbb{R}^2$.

Therefore : $X(t) = PY(t) = \begin{pmatrix} -4C_1e^{-t} + 2C_2e^{5t} \\ C_1e^{-t} + C_2e^{5t} \end{pmatrix}$

Thus $x(t) = -4C_1e^{-t} + 2C_2e^{5t}$ and $y(t) = C_1e^{-t} + C_2e^{5t}$.

Exercise 5 (3 points)

1. $f(E_{11}) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix} = -E_{11} + E_{21}$

$f(E_{12}) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} = -E_{12} + E_{22}$

$f(E_{21}) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} = 2E_{11}$

$f(E_{22}) = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2E_{12}$

Thus $\mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

2. $\Delta(E_{11}) = X^2$; $\Delta(E_{12}) = X$; $\Delta(E_{21}) = X - 1$ and $\Delta(E_{22}) = X^2 + 1$.

Hence, the matrix of $\Delta$with respect to the standard bases is $\begin{pmatrix} 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$.

Let us denote $\Delta = \begin{vmatrix} a_1 & a_1 & a_1 & \dots & \dots & a_1 \\ a_1 & a_2 & a_2 & \dots & \dots & a_2 \\ a_1 & a_2 & a_3 & \dots & \dots & a_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \dots & \dots & a_n \end{vmatrix}$.

$$\begin{aligned} \Delta &= \begin{vmatrix} a_1 & a_1 & a_1 & \dots & \dots & a_1 \\ 0 & a_2 - a_1 & a_2 - a_1 & \dots & \dots & a_2 - a_1 \\ 0 & 0 & a_3 - a_2 & \dots & \dots & a_3 - a_2 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 0 & a_n - a_{n-1} \end{vmatrix} \quad \begin{matrix} L_n \leftarrow L_n - L_{n-1} \\ \vdots \\ L_2 \leftarrow L_2 - L_1 \end{matrix} \\ &= a_1(a_2 - a_1)...(a_n - a_{n-1}) \end{aligned}$$