Final exam n°1

Duration : three hours
Documents and calculators not allowed

Name : **First Name :** **Class :**

Instructions :

- no sheets other than the stapled ones provided for answers shall be corrected.
- answers written using lead pencils shall not be corrected.

Exercise 1 (4 points)

1. Using the ratio test (D'Alembert's test), determine the nature of the series $\sum u_n$ where, for all $n \in \mathbb{N}^*$, $u_n = \dfrac{10^n}{n\,4^{2n+1}}$.

2. Using the root test (Cauchy's test), determine the nature of the series $\sum v_n$ where, for all $n \geqslant 2$, $v_n = \dfrac{n}{(\ln(n))^n}$.

Exercise 2 (4 points)

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{pmatrix}$.

Are A and B diagonalizable in $\mathscr{M}_3(\mathbb{R})$? If so, determine D et P.

N.B. : The bases of the eigenspaces must be deduced from clear reasoning, and not by randomly picking particular values.

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Exercise 3 (3.5 points)

Let $a \in \mathbb{R}$ and $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \\ 2-a & a-2 & a \end{pmatrix}$.

1. Determine the characteristic polynomial of A, denoted P_A, by choosing as first transformation : $C_1 \longleftarrow C_1 + C_2$.

2. Study the diagonalizability of A in $\mathscr{M}_3(\mathbb{R})$ depending on the value of a.
   N.B. : when A is diagonalizable, the diagonalization is not required.

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Exercise 4 (3.5 points)

We want to study the following linear differential system : $\begin{cases} x'(t) = x(t) + 8y(t) \\ y'(t) = x(t) + 3y(t) \end{cases}$.

Let's denote $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

1. Determine $A \in \mathscr{M}_2(\mathbb{R})$ such that $X'(t) = AX(t)$.

2. Diagonalize A, by exhibiting D and P. The matrix D will be of the form $D = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\boxed{\text{with } a < b}$, where a and b are to be determined.

3. [*Please check that you have chosen $a < b$ in the matrix D obtained in the previous question*]. From the previous questions, deduce $x(t)$ and $y(t)$ as functions of t.

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Exercise 5 (4 points)

1. Let $A = \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}$ and $f : \left\{ \begin{array}{ccc} \mathscr{M}_2(\mathbb{R}) & \longrightarrow & \mathscr{M}_2(\mathbb{R}) \\ M & \longmapsto & AM \end{array} \right.$. Determine the matrix of f with respect to the standard basis $\mathscr{B} = \left(E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$ of $\mathscr{M}_2(\mathbb{R})$.

2. Let $\Delta : \left\{ \begin{array}{ccc} \mathscr{M}_2(\mathbb{R}) & \longrightarrow & \mathbb{R}_2[X] \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (a+d)X^2 + (b+c)X + d - c \end{array} \right.$

   Determine the matrix of Δ with respect to the standard bases of $\mathscr{M}_2(\mathbb{R})$ and $\mathbb{R}_2[X]$.

Exercise 6 (2 points)

Let $(a_1, ..., a_n) \in \mathbb{R}^n$. Express the following determinant (over factorized form), indicating the transformations :

$$\begin{vmatrix} a_1 & a_1 & a_1 & \cdots & \cdots & a_1 \\ a_1 & a_2 & a_2 & \cdots & \cdots & a_2 \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_3 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & \cdots & a_n \end{vmatrix}$$