Correction of final exam 1

Exercise 1 (5 points)

1. Let $(u_n) = \left(\dfrac{(n!)^2}{(3n)!}\right)$.

$$\frac{u_{n+1}}{u_n} = \frac{\big((n+1)!\big)^2}{\big(3n+3\big)!} \times \frac{(3n)!}{(n!)^2} = \frac{(n+1)^2}{(3n+1)(3n+2)(3n+3)} = \frac{n+1}{3(3n+1)(3n+2)} \xrightarrow[n\to+\infty]{} 0 < 1.$$

So $\sum u_n$ is convergent according to the d'Alembert's rule.

2. Let $(v_n) = \left(\dfrac{(n!)^2}{(kn)!}\right)$.

$$\frac{v_{n+1}}{v_n} = \frac{\big((n+1)!\big)^2}{\big(k(n+1)\big)!} \times \frac{(kn)!}{(n!)^2} = \frac{(n+1)^2}{(kn+1)(kn+2)\dots(kn+k)} \underset{+\infty}{\sim} \frac{1}{k^k}\, n^{2-k}.$$

If $k = 2$, $\dfrac{v_{n+1}}{v_n} \xrightarrow[n\to+\infty]{} \dfrac{1}{4} < 1$ so $\sum v_n$ converges according to the d'Alembert's rule.

If $k > 2$, $\dfrac{v_{n+1}}{v_n} \xrightarrow[n\to+\infty]{} 0 < 1$ so $\sum v_n$ converges according to the d'Alembert's rule.

If $k < 2$, $\dfrac{v_{n+1}}{v_n} \xrightarrow[n\to+\infty]{} +\infty$ so $\sum v_n$ diverges according to the d'Alembert's rule.

3. Let $(w_n) = \left(\left(\dfrac{n}{n+a}\right)^{n^2}\right)$.

$$\sqrt[n]{w_n} = \left(\tfrac{n}{n+a}\right)^n = e^{-n\ln(1+a/n)} = e^{-n(a/n+o(1/n))} = e^{-a+o(1)} \xrightarrow[n\to+\infty]{} e^{-a}.$$

If $e^{-a} < 1$ i.e. $a > 0$, $\sum w_n$ converges according to the Cauchy's rule.

If $e^{-a} > 1$ i.e. $a < 0$, $\sum w_n$ diverges according to the Cauchy's rule.

If $e^{-a} = 1$ i.e. $a = 0$, then $(w_n) = (1)$ which does no tend to 0 so $\sum w_n$ is divergent.

Exercise 2 (4 points)

Via the transformations $C_1 \longleftarrow C_1 + C_2 + C_3$ then $L_2 \longleftarrow L_2 - L_1$ and $L_3 \longleftarrow L_3 - L_1$, we find that $P_A(X) = (3 - X)(X + 1)(X + 3)$.

So P_A is split in $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(A) = \{3, -1, -3\}$ with $m(3) = m(-1) = m(-3) = 1$. Thus, A is diagonalizable.

$$\begin{aligned}
E_3 &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} -3x + 3y = 0 \\ x - 5y + 4z = 0 \\ x + y - 2z = 0 \end{array} \right. \right\} \\
&= \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
E_{-1} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} x + 3y = 0 \\ x - y + 4z = 0 \\ x + y + 2z = 0 \end{array} \right. \right\} \\
&= \mathrm{Span}\left\{ \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
E_{-3} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} 3x+3y=0 \\ x+y+4z=0 \end{array} \right. \right\} \\
&= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}
\end{aligned}$$

So we have $D = P^{-1}AP$ with $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & -3 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$.

Via the transformations $C_1 \longleftarrow C_1 - C_2$ and $L_2 \longleftarrow L_2 + L_1$, we find that $P_B(X) = (1-X)(X+1)^2$.

So P_B is split in $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(B) = \{1, -1\}$ with $m(-1) = 2$ and $m(1) = 1$.

$m(1) = 1$ so $\dim(E_1) = 1$.

$$\begin{aligned}
E_{-1} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} x-y=0 \\ x+3y-4z=0 \\ x+y-2z=0 \end{array} \right. \right\} \\
&= \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}
\end{aligned}$$

$\dim(E_{-1}) = 1 \neq 2 = m(-1)$. Therefore, B is not diagonalizable.

Exercise 3 (4 points)

Via the transformations $C_1 \longleftarrow C_1 + C_2 + C_3$ then $L_2 \longleftarrow L_2 - L_1$ and $L_3 \longleftarrow L_3 - L_1$, we find that $P_A(X) = -(X+1)(X+2)^2$.

So P_A is split in $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(A) = \{-1, -2\}$ with $m(-2) = 2$ and $m(-1) = 1$.

Thus, A is diagonalizable iff $\dim(E_{-2}) = 2$.

$$\begin{aligned}
E_{-2} &= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} -x+y=0 \\ (a-3)x+2y+(1-a)z=0 \end{array} \right. \right\} \\
&= \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \text{ such that } \left| \begin{array}{l} x=y \\ (a-1)x=(a-1)z \end{array} \right. \right\}
\end{aligned}$$

If $a = 1$, $E_{-2} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and A is diagonalizable.

If $a \neq 1$, $E_{-2} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and A is not diagonalizable.

Exercise 4 (4 points)

1. a. We have $f(1) = 3X$; $f(X) = 2X^2 + 1$; $f(X^2) = X^3 + 2X$ et $f(X^3) = 3X^2$, which leads to

$$\mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

b. The determinant of this matrix is equal to 9. Thus, it is invertible and the map f is bijective.

2. $f(E_{11}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} = -bE_{12} + cE_{21}$.

$f(E_{12}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -c & a-d \\ 0 & c \end{pmatrix} = -cE_{11} + (a-d)E_{12} + cE_{22}$.

$f(E_{21}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} b & 0 \\ d-a & -b \end{pmatrix} = bE_{11} + (d-a)E_{21} - bE_{22}$.

$f(E_{22}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} = bE_{12} - cE_{21}$.

This leads to

$$\mathrm{Mat}_{\mathscr{B}}(f) = \begin{pmatrix} 0 & -c & b & 0 \\ -b & a-d & 0 & b \\ c & 0 & d-a & -c \\ 0 & c & -b & 0 \end{pmatrix}$$

Exercise 5 (4 points)

We find immediately that $P_A(X) = (1-X)(2-X)^3$.

P_A is split in $\mathbb{R}$ and $\mathrm{Sp}_{\mathbb{R}}(A) = \{1, 2\}$ with $m(2) = 3$ and $m(1) = 1$.

Thus, A is diagonalizable iff the dimension of E_2 is 3.

$$E_2 = \left\{ \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathbb{R}^4 \text{ such that } \left| \begin{array}{r} -x + ay + bz + ct = 0 \\ dz + et = 0 \\ ft = 0 \end{array} \right. \right\}$$

- If $f \neq 0$, we have $t = 0$, then $dz = 0$.
  - If $d \neq 0$, then $z = 0$ and $x = ay$. So $E_2 = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ and A is not diagonalizable.
  - If $d = 0$, then $x = ay + bz$. So $E_2 = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ and A is not diagonalizable.
- If $f = 0$, then $dz + et = 0$.
  - If $e = 0$, then $dz = 0$
    - If $d = 0$, we have $x = ay + bz + ct$. So $E_2 = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and A is diagonalizable.
    - If $d \neq 0$, then $z = 0$ and $x = ay + ct$. We find $E_2 = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} c \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and A is not diagonalizable.
  - If $e \neq 0$, then $t = -\dfrac{d}{e}z$ and $x = ay + \left(b - \dfrac{cd}{e}\right)z$. So $E_2 = \mathrm{Span}\left\{ \begin{pmatrix} a \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} b - cd/e \\ 0 \\ 1 \\ -d/e \end{pmatrix} \right\}$ and A is not diagonalizable.

Conclusion : A is diagonalizable iff $d = e = f = 0$.