EPITA

Mathematics

Midterm exam (S3)

November 2017

Name:
First name:
Class:

MARK:



Midterm exam (S3)

Duration: three hours

Documents and calculators not allowed

Exercise	1	(3)	points)
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1.	1. Determine the Taylor expansion around 0 at order 2 of $e^x \ln(e + ex)$.						
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2. Determine $\lim_{x\to 0} (1+\sin(x))^{1/x}$.

Exercise 2 (5 points)

1. Determine $\lim_{n\to+\infty}\frac{\ln(n+1)}{\ln(n)}$. Then, using d'Alembert's rule, determine the nature of $\sum \frac{\ln(n)}{(n-1)!}$.

2. Determine the nature of the series $\sum u_n$ where $u_n = e - \left(1 + \frac{1}{n}\right)^n$.

3. Determine the nature of $\sum \left(\frac{(-1)^n}{\sqrt{n}} + \frac{1}{n}\right)$.

4. Determine the nature of $\sum \frac{\sin(n!)}{n^2}$.

Exercise 3 (5 points)

The purpose of this exercise is to determine the nature of the series with the general term : $u_n = (-1)^n n^{\alpha} \left(\ln \left(\frac{n+1}{n-1} \right) \right)^{\beta}$ where $(\alpha, \beta) \in \mathbb{R}^2$ and $n \in \mathbb{N} \setminus \{0, 1\}$.

1. Show that $\ln\left(\frac{n+1}{n-1}\right) = \frac{2}{n}\left(1 + \frac{1}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$.

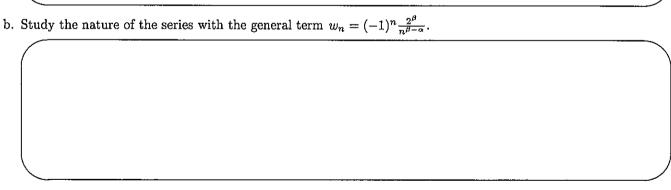
2. Deduce that $u_n = (-1)^n \frac{2^{\beta}}{n^{\beta-\alpha}} \left(1 + \frac{\beta}{3n^2} + o\left(\frac{1}{n^2}\right)\right)$.

3. Show that in case $\beta \leqslant \alpha$, the series $\sum u_n$ diverges.

4. We focus on the case $\beta > \alpha$ and we put

$$u_n = (-1)^n \frac{2^{\beta}}{n^{\beta - \alpha}} + v_n$$
 with $v_n = (-1)^n \frac{\beta 2^{\beta}}{3n^{2+\beta - \alpha}} + o\left(\frac{1}{n^{2+\beta - \alpha}}\right)$

a. Study the nature of the series $\sum v_n$.



c. Deduce the nature of $\sum u_n$.

Exercise 4 (3 points)

Let us consider the series $\sum u_n$ where $u_n = \left(\frac{n^2 - 3n + 1}{n^2 + n + 1}\right)^{n^2}$.

1. Check that $\ln\left(\frac{n^2 - 3n + 1}{n^2 + n + 1}\right) = \ln\left(1 - \frac{3}{n} + \frac{1}{n^2}\right) - \ln\left(1 + \frac{1}{n} + \frac{1}{n^2}\right)$.

2. Determine $a \in \mathbb{R}$ such that $\ln \left(\frac{n^2 - 3n + 1}{n^2 + n + 1} \right) = \frac{a}{n} + o\left(\frac{1}{n} \right)$.

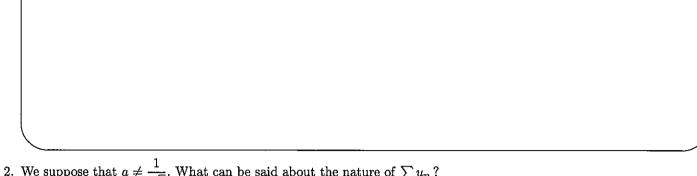
3. Deduce the nature of $\sum u_n$ using Cauchy's rule.

Exercise 5 (5 points)

Let $a \in \mathbb{R}$ and let $\sum u_n$ be the series with the general term $u_n = \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n - a$.

1. Using a Taylor expansion, determine $\lim_{n\to+\infty} \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n$. Then deduce $\lim_{n\to+\infty} \left(\cos\left(\frac{1}{\sqrt{n}}\right)\right)^n - a$.

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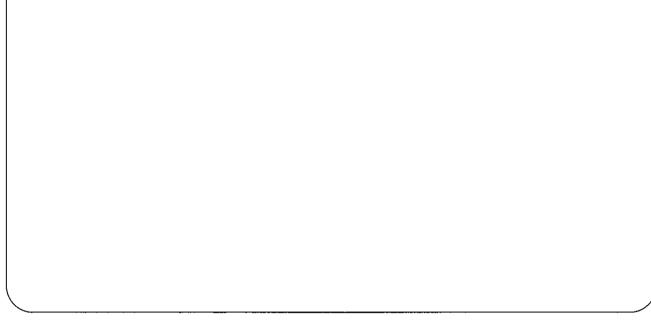


2. We suppose that $a \neq \frac{1}{n}$. What can be said about the nature of $\sum u_n$?

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3. We suppose now that $a = \frac{1}{\sqrt{e}}$.

a. Using a Taylor expansion, show that $e^{n \ln(\cos(1/\sqrt{n}))} = e^{-\frac{1}{2}} e^{-\frac{1}{12n} + o(\frac{1}{n})}$.



b. Deduce an equivalent of u_n and the nature of $\sum u_n$.