Credits

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Introduction

This report tackles the analysis of the control and optimization of an Inverted Pendulum, from a theoretical and experimental point of view.

1 Exercise 1

In order to characterize the state-model defined by,

$$\dot{x} = Ax + Bu \tag{1}$$

$$y = Cx + Du \tag{2}$$

where A and B are,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -0.0174 & 20.7861 & -0.0023 & 57.5344 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -0.0174 & 63.8319 & -0.0071 & 57.4388 \\ 0 & -232.0252 & 0 & 0 - 755.4250 & \end{bmatrix} B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 333.337 \end{bmatrix} \tag{3}$$

we must find out the placement of the poles given by the model. These correspond to the eigenvalues of matrix A.

Using *MATLAB*, we imported the matrices from the input file "*fp_lin_matrices_fit3.mat*" and used the command *eig(A)* which gave us the eigenvalues for the matrix A. The real and imaginary part of the eigenvalues are shown below:

$$Re\{|A|^T\} = \begin{bmatrix} 0 & -737.3184 & -19.3673 & -5.7612 & 6.9974 \end{bmatrix} \tag{4}$$

$$Im\{|A|^T\} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

As we can see, there are 5 eigenvalues and, even though every determinant of A sits on the Real axis, there is a positive determinant with $\lambda_5 = 6.9974$, making the system unstable.

Not only that but the first eigenvalue of A is also 0. Since the derivative of x is obtained through the sum of Ax and Bu then $\dot{x}_0 = 0$ which makes sense physically, as $\dot{x}_0$ is the rotation speed, which does not depend on the state, only the tension applied to the motor.

2 Exercise 2

The goal of this exercise is to characterize the open loop system in terms of controllability. So, to obtain the controllability matrix we need to compute the following:

$$\mathtt{C}(A,B) = [B|AB|A^2B|...|A^{n-1}B] \tag{6}$$

As stated before, 3, the matrix A has a size of 5×5. Moreover, our matrix B has size 5×1. Ergo, our $\mathtt{C}$ matrix must be:

$$\mathtt{C}(A,B) = [B|AB|A^2B|A^3B|A^4B] \tag{7}$$

Utilizing the *ctrb(C)* function of *MATLAB* we got:

$$\mathtt{C}(A,B) = \begin{bmatrix} 0 & 0 & 1.92\times10^{4} & -1.45\times10^{7} & 1.07\times10^{10} \\ 0 & 1.92\times10^{4} & -1.45\times10^{7} & 1.07\times10^{10} & -7.88\times10^{12} \\ 0 & 0 & 1.91\times10^{4} & -1.45\times10^{7} & 1.07\times10^{10} \\ 0 & 1.91\times10^{4} & -1.45\times10^{7} & 1.07\times10^{10} & -7.87\times10^{12} \\ 3.33\times10^{2} & -2.52\times10^{5} & 1.86\times10^{8} & -1.37\times10^{11} & 1.01\times10^{14} \end{bmatrix} \tag{8}$$

The controllability criterion states that in order for a system to be controllable, its controllability matrix must have a *rank* equal to the dimension of x, which is 5.

Using the *MATLAB* function *rank*, we were able to confirm that the rank of the controllability matrix $\mathtt{C}$ is indeed 5, thus making the system a controllable one.

3 Exercise 3

Similar to the previous exercise, we need to characterize the open loop system but in terms of observability. In order to obtain the observabillity matrix we need to compute the following:

$$\mathtt{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ ... \\ CA^{n-1} \end{bmatrix} \tag{9}$$

For the case where the output is the pendulum angle x_3, our matrix C is,

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

Therefore, using the same logic for 7, the observability matrix will be:

$$\mathtt{O}(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} \tag{11}$$

Utilizing the *obsv(O)* function of *MATLAB* we got:

$$\mathtt{O}(A,C) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & -1.737\times10^{-2} & 6.383\times10^{1} & 7.113\times10^{-3} & 5.743\times10^{1} \\ 0 & -1.332\times10^{4} & -8.152\times10^{-1} & 6.383\times10^{1} & -4.339\times10^{4} \\ 0 & 1.006\times10^{7} & 2.729\times10^{5} & 2.960\times10^{1} & 3.201\times10^{7} \end{bmatrix} \tag{12}$$

The observability criterion states that in order for a system to be observable, its observability matrix must have a rank equal to the dimension of x, which is 5. So, since the rank of $\mathtt{O}$ matrix is 4, the system is not completely observable when only the x_3 is measured.

However, it was asked to analyze as well the observability where the outputs are both x_3 and x_1 turning our C matrix into:

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{13}$$

Therefore, using the same logic for 7, the observabillity matrix will be:

$$\mathtt{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} \tag{14}$$

Utilizing the *obsv($\mathtt{O}$)* function of *MATLAB* we got:

$$\mathtt{O}(A, C) = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -0.034 & 84.62 & -0.01 & 114.97 \\ 0 & -2.67 \times 10^4 & -1.32 & 84.62 & -8.69 \times 10^4 \\ 0 & 2.02 \times 10^7 & -5.49 \times 10^5 & 59.87 & 6.41 \times 10^7 \end{bmatrix} \tag{15}$$

Making use of the *MATLAB* function *rank*, we confirm that the rank of the $\mathtt{O}$ matrix is 5. So, since the rank of $\mathtt{O}$ is 5 and the dimension of $\boldsymbol{x}$ is 5, the system is observable when both $\boldsymbol{x_3}$ and $\boldsymbol{x_1}$ are measured.

4 Exercise 4

In this exercise, we developed a *MATLAB* macro to plot the Bode diagram of the open loop system. For that, we used the functions *ss2tf* and *bode* to obtain the transfer function from the state model and to plot the Bode diagrams, respectively.

When using the *ss2tf* function, the numerator obtained is of size 2×6. This is because the C and D matrices are 2×5 and 2×1, respectively, which means that our output is, in fact, two outputs, α and β.

We can then analyze the transfer functions for α and β,

$$\frac{\alpha(s)}{u(s)} = \frac{1.918 \cdot 10^4 s^2 + 92.07s - 8.262 \cdot 10^5}{s^5 + 755.4s^4 + 1.33 \cdot 10^4 s^3 - 4.816 \cdot 10^4 s^2 - 5.757 \cdot 10^5 s} \tag{16}$$

$$\frac{\beta(s)}{u(s)} = \frac{1.915 \cdot 10^4 s^2 - 2.07 \cdot 10^{-13} s - 6.636 \cdot 10^{-12}}{s^5 + 755.4s^4 + 1.33 \cdot 10^4 s^3 - 4.816 \cdot 10^4 s^2 - 5.757 \cdot 10^5 s} \tag{17}$$

by observing the bode plots for both of these transfer functions:

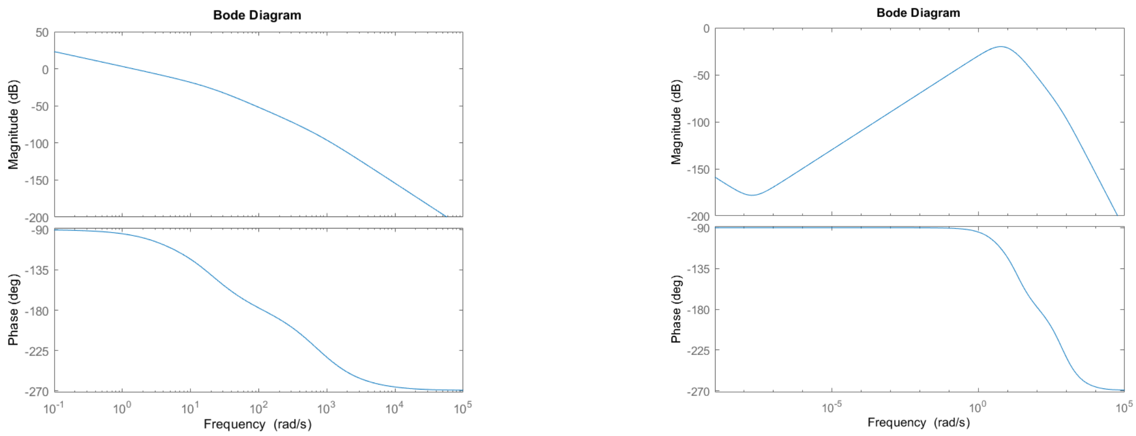

Figure 1: Bode Plots of α (left) and β (right) for the Open Loop System

Looking at the magnitude plot of α, we can observe that in lower frequencies, the response to the input is high. With the increase of the frequency, the value of the magnitude keeps on getting lower, which signifies that as the frequency increases, the influence of the input over α is diminished.

Using the *MATLAB* function `pzplot` we can visualize the location of the poles and zeros of the transfer functions. Looking at the poles and zeros of α, even though an unstable pole exists (i.e. located in the right side of the complex plane), there is a zero close to it which cancels its effect, thus confirming the behaviour observed in the bode plot.

For the magnitude plot of β, we can see that for most of the spectrum of frequencies, the effect of the input is negligible. The alternating slopes of the plot reflect the position of the zeros and poles of the transfer function. Looking at the poles and zeros of β, there is an unstable pole which does not have a zero near it, therefore explaining the growing magnitude followed by its rapid decrease.

Lastly we calculate the gain/phase margins. For an unstable system, the gain margin shows how much the gain must be decreased to make the system stable, while the phase margin shows how much the phase should be increased for the system to become stable.

Firstly, α has a gain margin of $-55dB$ and a phase margin of 83 degrees. Secondly, β has margins of $-55dB/\infty$.

5 Exercise 5

The goal of this exercise is to find the vector of gains of the controller. Coupling the plant model defined by equation 1 and the state feedback defined by, $u(t) = -Kx(t)$, we get the closed-loop model $\dot{x} = (A - BK)x$, where K is the vector of gains.

As stated in **Question 2**, the pair (A,B) is controllable, which means that it's possible to find the vector of controller gains K such that the closed dynamics $(A - BK)$ has its eigenvalues at the specified eigenvalues, wherever they might be.

So, in order to find the vector of controller gains K, we compute K to minimize the quadratic cost J defined by:

$$J = \int_0^\infty (x^T Q_r x + R_r u^2)\, dt \tag{18}$$

To achieve this, the matrix P that verifies the Algebraic Riccati equation:

$$A^T P + PA^T - PBR_r^{-1}B^T P + Q_r = 0 \tag{19}$$

should be found, and finally, the K computation can be made from:

$$K = R_r^{-1}BP \tag{20}$$

To make all of these computations, the *MATLAB lqr* function was used. Since this function take as inputs the A, B, Q matrices and the scalar R, some values of Q (positive semidefinite matrix) and R (positive scalar) were tried out.

We defined Q and R as:

- $Q = \text{diag}(\begin{bmatrix}10 & 0 & 1 & 0 & 0\end{bmatrix})$
- $R = 1$

and the vector of gains, K, computed was:

$$K = \begin{bmatrix}-3.16227 & -2.53239 & 22.256411 & 3.25706 & 0.05413\end{bmatrix}$$

Furthermore, the closed-loop poles were determined by calculating the eigenvalues of $A - BK$ using the *eig* function. The real and imaginary part of the obtained eigenvalues are shown below:

$$Real : \begin{bmatrix} -737.318 & -18.808 & -7.555 & -4.905 & -4.905 \end{bmatrix} \tag{21}$$

$$Im : \begin{bmatrix} 0 & 0 & 0 & 0.931 & -0.931 \end{bmatrix} \tag{22}$$

There are 5 closed-loop poles, which is expected. Looking at the real part and the correspondent imaginary part of each pole, it is possible to visualize that there are two conjugated poles and, mainly, that the real part of all the poles are negative.

This leads us to conclude that this system (with the controller) is stable, in contrast with the system without a controller.

6 Exercise 6

For this exercise, the group designed the SIMULINK model, just as described in the problem sheet.

To run such model, and since the reference is zero, we considered the initial conditions $x_0 = \begin{bmatrix}0.1 & 0 & 0 & 0 & 0\end{bmatrix}$ (where the initial angular velocities are 0, $\beta = 0$ and $\alpha = 0.1$).

After running the *MATLAB* macro, also as described in the problem sheet, the group obtained the effect of the controller on the states, input current and output voltage:

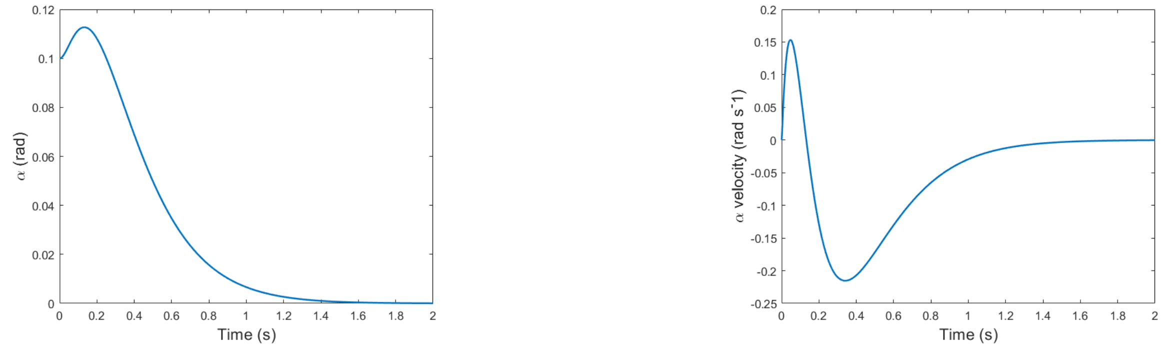

Figure 2: α and $\dot{\alpha}$ Evolution throughout 2 Seconds

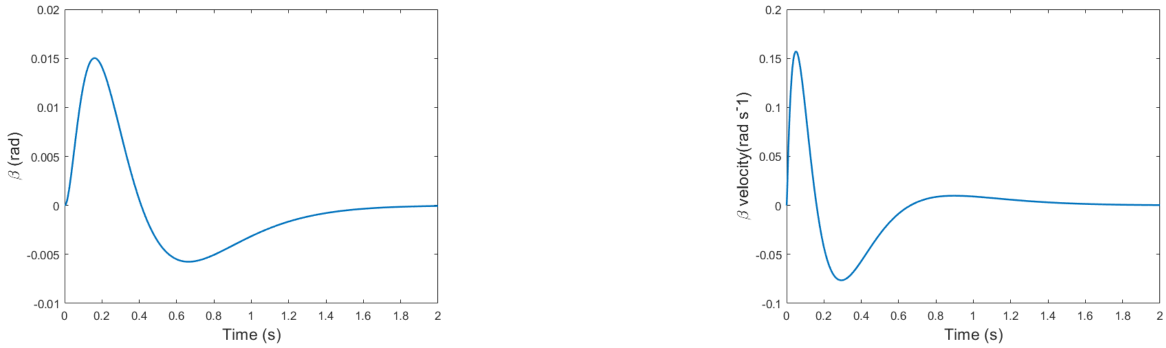

Figure 3: β and $\dot{\beta}$ Evolution throughout 2 Seconds

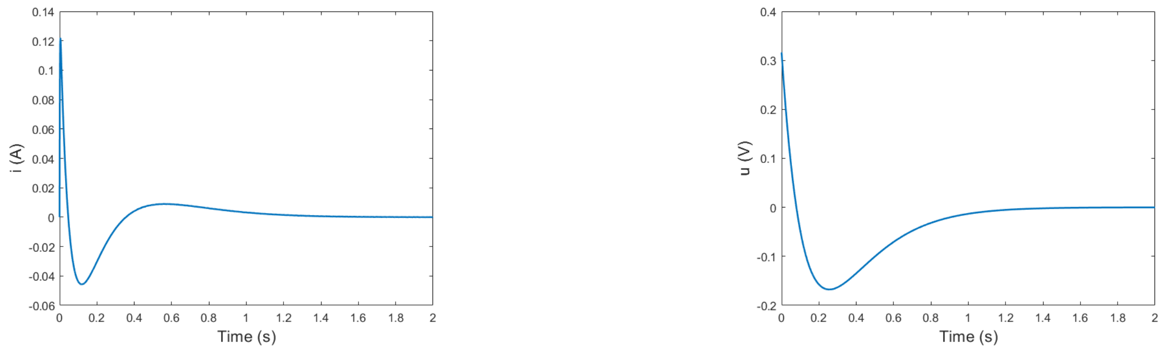

Figure 4: Input Current and Output Voltage Evolution throughout 2 Seconds

Starting with α, we can see a slight overshoot at 0.11 rad, but a rapid stabilization after, reaching 0 rad before the 2 second mark. With $\dot{\alpha}$ however, we can see 2 overshoots before the 1 second mark. They are at the 0.15 rad and -0.2 rad amplituudes.

The same leads us to β that, contrary to α, has 2 overshoots at the 0.015 rad and 0.6 rad amplitudes, stabilizing soon after, also before the 2 second mark. With $\dot{\beta}$ we also have 2 overshoots similar to $\dot{\alpha}$ at the 0.15 rad and -0.075 rad amplitudes, stabilizing also before the 2 second mark, at around 1.8 seconds.

Finally, with our input current and output voltage we can see that they both have 2 overshoots, 0.12/-0.04 rad and 0.3/-0.15 rad respectively, rapidly stabilizing after that before the 2 second mark as well.

This leads us to conclude that our controller is well designed for this task, since all of our states stabilize, rather quickly.

7 Exercise 7

In this exercise, the goal is to design an observer, and acquire its gains vector. The vector of observer gains must be such that the estimation error converges to zero.

Below we can see the general formula for an observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{23}$$

$$\hat{y} = C\hat{x} \tag{24}$$

$$\dot{\hat{x}} = (A - LC)\hat{x} + Bu + Ly \tag{25}$$

In order to employ our observer, a Kalman filter was employed on our *MATLAB* macro, with the aid of the *lqe* function who gave us the estimator gains:

$$L = \begin{bmatrix} 3.267 & 1.137 \\ 0.982 & 11.873 \\ 1.137 & 14.279 \\ 8.078 & 97.589 \\ -0.299 & -3.611 \end{bmatrix} \tag{26}$$

To determine the new (25) system's poles, we calculated the determinants of $|sI - A + LC|$ which gave us the following:

$$\lambda = \begin{bmatrix} -737.318 & -3.157 & -5.039 & -8.071 & -19.410 \end{bmatrix} \tag{27}$$

As we can see, with the observer added, the system has 5 poles and it is stable since every pole resides on the left complex semiplane.

8 Exercise 8

Finally, for this exercise, we add an observer to the designed controller. Due to an immeasurable state we include $u = -K\hat{x}(t)$, where K is the controller gains computed before. Hence our system follows:

$$\begin{cases} \dot{\hat{x}} = (A - BK - LC)\hat{x}(t) + Ly(t) \\ u = -K\hat{x}(t) \end{cases} \tag{28}$$

With this we find a new system dynamic, where our new "A" matrix is $(A-BK-LC)$, our new "B" matrix is L, the input matrix, and our new "C" matrix is $-K$, the output matrix.

After running the *MATLAB* macro, the group obtained the progression of the 2 outputs, with the designed controller and observer:

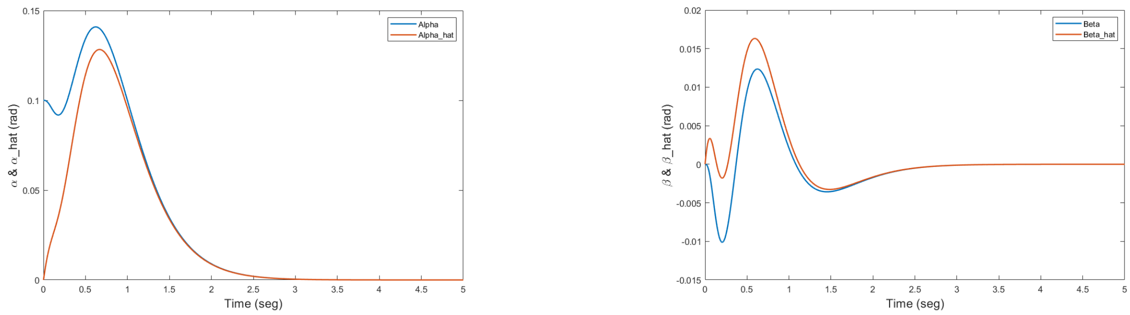

Figure 5: Comparison between Observed - α/β - and Estimated - $\hat{\alpha}/\hat{\beta}$ - throughout 3 Seconds

If we look at the above figures we can see that the observer fulfills its objectives since the observed states tend to the estimated states, although the estimation error only appears to be close to zero after the first second of the simulation.

To be more specific, the estimation starts with a large estimation error, which is to be expected in the case of α, given that its initial condition is 0.1.

For the case of β, we can detail that the approximation tries to always follow the real value, but it's too slow in doing so, resulting in the observed gaps when the slope of the signal alternates.

The behaviour of the observer can be improved, i.e., the estimation error can be diminished, by altering the values of the gains used to compute the Kalman Filter.

To illustrate this, simulations were ran where the values of the variables G, Qe and Re were regulated. By doing that, it was noticed that by increasing the values of G and Qe, the estimation error diminished. We take $Qe = 1000 \cdot \texttt{eye(size(A))}$ and $G = 10 \cdot \texttt{eye(size(A))}$.

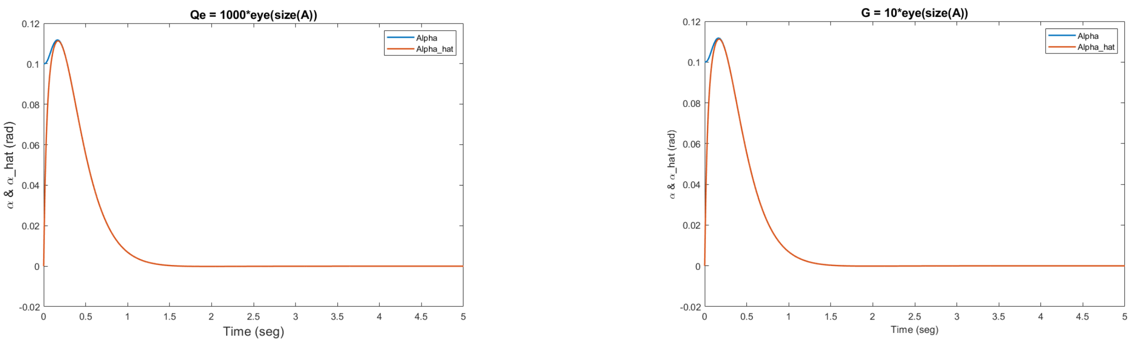

Figure 6: Comparisons between Observed α and Estimated $\hat{\alpha}$ with Qe and G

9 Lab session 1

With the beginning of the lab sessions, the group was tasked to run its *MATLAB* code in class.

After successfully running the code developed for **Question 8**, the group tried to understand the effects of changing the values of Q_{ri} and R_r on the states α and β.

Taking into consideration that the function of the quadratic error goes as:

$$J = \int_0^\infty (x^T Q_r x + R_r u^2)\, dt \tag{29}$$

we know that:

- If we increase Q_{ri} for a constant R_r, x_i must decrease
- If we increase R_r for a constant Q_{ri}, u decreases which leads to a slower response in the closed-system

To simulate these observations, a *MATLAB* script was created, which overlays the effect of the variation of these parameters in the same figure. To see the reaction of the system to a wide range of values, the `logspace` function was used to obtain equally spaced values in logarithmic space. This script was ran for the different variables of interest and the figures produced by this script can be found below:

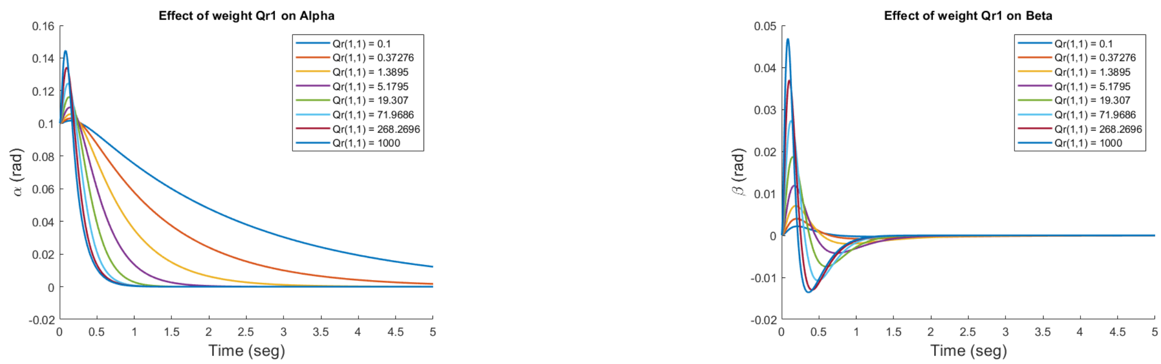

Figure 7: Evolution of α/β for a variation of Q_{r1}

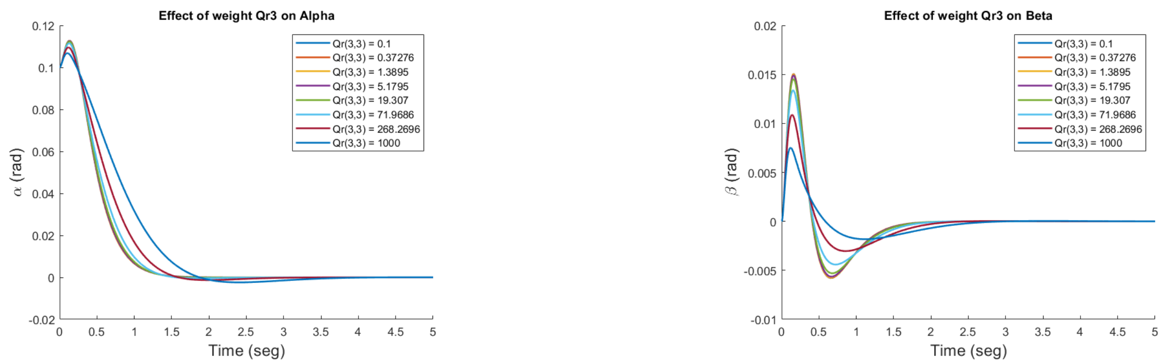

Figure 8: Evolution of α/β for a variation of Q_{r3}

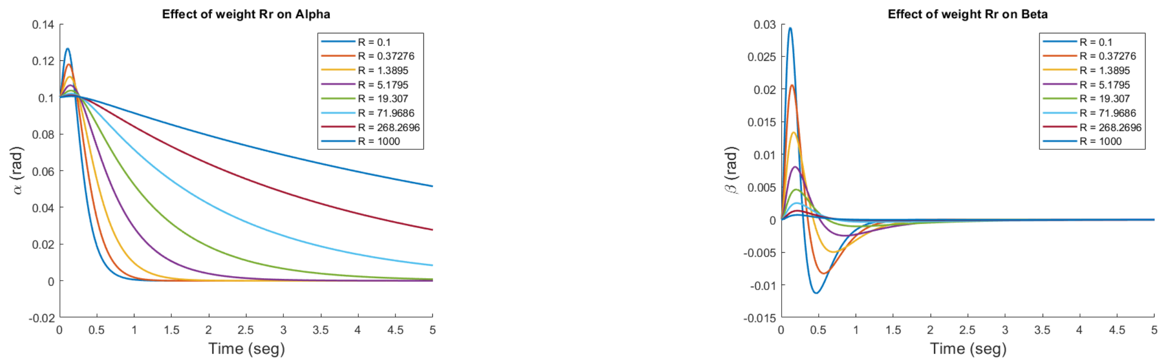

Figure 9: Evolution of α/β for a variation of R_r

As we can see both α and β gradually and predictably change for variations in Q_{ri} and R_r.

10 Lab session 2

In this question, the test on the real system of the inverted pendulum was performed. For that, the group followed the steps described in the lab statement.

Carefully, the group looked at the file that contained: the weight matrix, Q_r , the weight for the input variable, R_r , the calculation of the feedback gain from the function lqr, the weight of the process noise, G , the variance of process errors, Q_e , the variance of measurement errors, R_e and the computation of the estimator gains, and compared the default parameter values for the Q_r and R_r that were given by the professors with the ones obtained in the simulation (shown in section above).

Although the simulation and actual testing environment is not completely the same, this comparison of parameters gave us an idea of what the pendulum could behave like, when we run the **SIMULINK**.

The default values for the Q_r matrix and the R_r weight were:

- $Q_r = \text{diag}(\begin{bmatrix} 10 & 0 & 1 & 0 & 0 \end{bmatrix})$
- $R_r = 1$

and the plots obtained, on the real system run, for α and β in function of the time (seconds), as well as the motor voltage were:

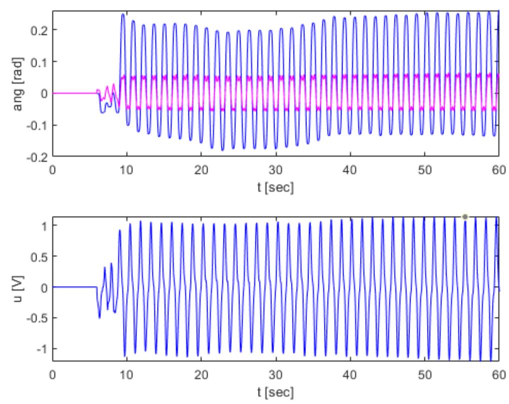

Figure 10: 1) α and β in function of the t. 2)Motor voltage, u, in function of the t.

In the first plot, the blue color represents the variation of the angle α and the pink color the variation of angle β, that is, the angle of the pendulum with respect to the vertical. Analyzing this plot, it is possible to notice, first, for t between 6 and 9 seconds, the angles trying to stabilize themselves (until reaching the tenth second), and later, the oscillations that both angles have when the pendulum tries to balance itself.

For α, the range of values it takes is very similar, during the 60 seconds, and the difference, on average, of the maximum point with the minimum point is 0.4 radians, which means that starting from the initial point of the pendulum, the horizontal bar angle varies on average by 0.2 radians to the left and to the right. The angle β has a similar behavior to the angle alpha in terms of the variation, but the range of values that beta takes is much smaller. The difference between the maximum point and the minimum point is approximately 0.1 radians, which means that the vertical bar balance approx 0.05 radians to the left and to the right starting from its initial point.

Looking now at the second plot, it is visible that the motor voltage, u, takes a maximum absolute voltage of 1 volt, and it varies according to the movement of the horizontal bar.

For the same Q_r and R_r parameters, the plots obtained for α and β in function of the time, due to the simulation phase, were:

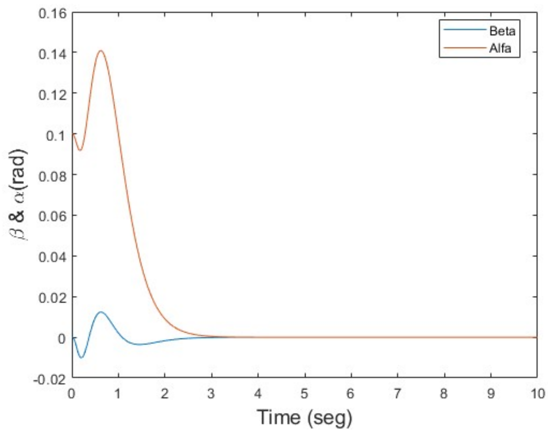

Figure 11: Simulation result for α and β in function of the time.

To conclude, firstly, looking at the time responses obtained using the real system and the ones obtained in simulation, it is easy to notice that, in the simulation plot, along the time, both α and β completely stabilize to zero with much fewer oscillations, which is not the case in the experimentation plot, mainly due to factors caused by the difference in the test environment. As the name indicates, one is in a simulation environment, and the other is in a real test environment with the real pendulum and the motor. These results and this difference was as expected.

Secondly, the values of variations for α and β obtained in the real system for this Q_r and R_r are not the best, and can be optimized in order to minimize the oscillations and to make this system more stable. This optimization and fine-tuning go along with the **Question 11**.

11 Lab session 3

For the last question the group was tasked to optimize its controller/observer system. This had to be made through the state variable β.

In order to control the value of β the group tried to neglect α as much as possible, which would result in a slow and cyclic movement of the pendulum, with a great amplitude for α and a small one for β:

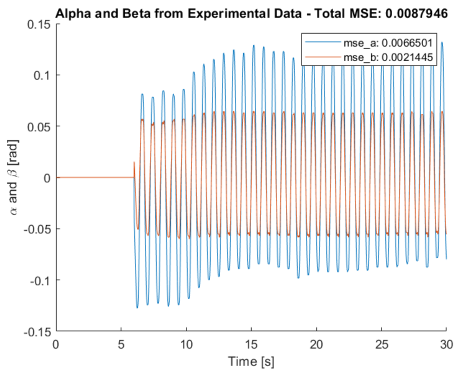

Figure 12: Best Figure of Merit in Simulation

In order to obtain the best possible value for the control of the oscillation of β the group used the *Figure of Merit* parameter, through the *MSE* function. The lower the *MSE* the better optimized our system was. We obtained:

$$\begin{aligned} MSE_\alpha &= 0.0066501 \\ MSE_\beta &= 0.021445 \end{aligned} \tag{30}$$

It was predictable the fact that $MSE_\beta = 0$ was impossible on an experimental level, due to the fact that the equilibrium point at the top of the pendulum is unstable. This makes it impossible to steady the pendulum at an upright position.

In order to better simulate the effect of the pendulum standing upright, we could manually introduce in our **SIMULINK** model a sinusoidal disturbance to β, with a high frequency of oscillation and small amplitude, to counter-act the weight oscillation caused by β.

Conclusion

With all these laboratory sessions completed, the group acquired a better grasping of the concepts of optimal control, observers and controllers.