L11 - Manipulator Control and Dynamics of Constrained Manipulators

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Centralized control

The manipulator's dynamics are

$$B(q)\,\ddot{q} + \underbrace{C(q,\dot{q})\,\dot{q} + g(q)}_{n(q,\dot{q})} = \tau \Rightarrow B\ddot{q} + n = \tau .$$

Let the control action be $\tau = \hat{n} + \hat{B}u$. Then, $B\ddot{q} + n = \hat{n} + \hat{B}u$.
If the estimates $\hat{B}$ and $\hat{n}$ are exact, the dynamics reduce to

$$\ddot{q} = u .$$

The dynamics can now be controlled linearly, through

$$\ddot{e} + K_d\dot{e} + K_p e = 0 \quad , \text{ with } e = q - q_d .$$

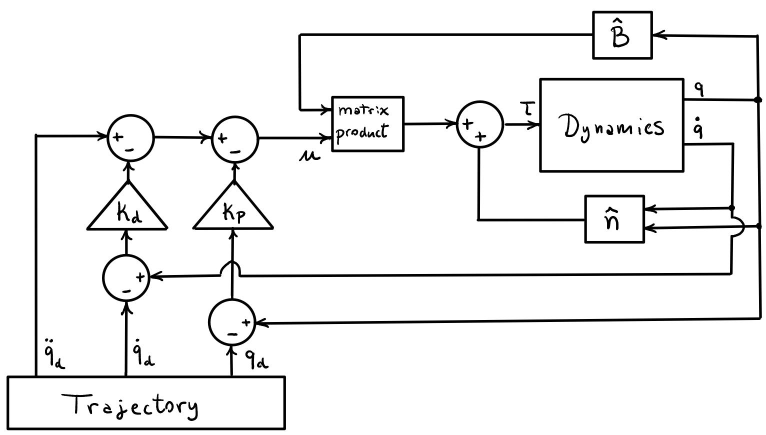
In which case, the input, u, must be

$$u = \ddot{q} = \ddot{q}_d - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d) .$$

Here, $q \in \mathbb{R}^n$, $K_d \in \mathbb{R}^{n\times n}$ and $K_p \in \mathbb{R}^{n\times n}$.

- Example (11th Set of Problems , P1)

In block diagram form, the previous controller takes the following form.



For a 2 degrees of freedom manipulator ($n=2$), the stiffness and damping matrices may be defined as

$$K_p = \begin{pmatrix} K_{p_1} & 0 \\ 0 & K_{p_2} \end{pmatrix} , \quad K_d = \begin{pmatrix} K_{d_1} & 0 \\ 0 & K_{d_2} \end{pmatrix} , \quad e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} q_1 - q_{1d} \\ q_2 - q_{2d} \end{pmatrix} .$$

The resulting linear system is then

$$\ddot{e}_i + k_{d_i}\dot{e}_i + k_{p_i} e_i = 0 \quad , \quad \text{for } i = 1,2.$$

Taking its Laplace transform reveals $E_i(s)\left(s^2 + k_{d_i} s + k_{p_i}\right) = 0$, which corresponds to the dynamics of a Damped Harmonic Oscillator.

This can also be written in terms of frequency and damping ratio, as $E_i(s)\left(s^2 + 2\omega_{n_i}\xi_i s + \omega_{n_i}^2\right) = 0$. Thus,

$$K_p = \begin{pmatrix} \omega_{n_1}^2 & 0 \\ 0 & \omega_{n_2}^2 \end{pmatrix}, \quad K_d = \begin{pmatrix} 2\omega_{n_1}\xi_1 & 0 \\ 0 & 2\omega_{n_2}\xi_2 \end{pmatrix},$$

and

$$\tau = \hat{n} + \hat{B}\left(\ddot{q}_d - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d)\right).$$

Decentralized Control (See the 10th Set of Problems)

In the decentralized control the joints are controlled independently. The coupling between the joints motion is treated as a disturbance, toghether with the variations in the mass distribution. The approach can be summarized through approximation of the manipulator's dynamics, taking here the form of a linear system, as follows.

$$B_{max}\ddot{q} \approx \tau \quad , \quad \text{where} \quad B_{max} = \operatorname{diag}\left(\max(B_{11}) \dots \max(B_{nn})\right).$$

The independent control of the joints can be achieved through a P-D controller, defined as

$$\tau = -K_d\dot{q} - K_p(q - q_d) \quad , \quad \text{with} \quad K_d = \begin{pmatrix} k_{d_1} & & 0 \\ & \ddots & \\ 0 & & k_{d_n} \end{pmatrix}, \; K_p = \begin{pmatrix} k_{p_1} & & 0 \\ & \ddots & \\ 0 & & k_{p_n} \end{pmatrix}.$$

The closed loop becomes

$$B_{max}\ddot{q} + K_d\dot{q} + K_p q \approx K_p q_d ,$$

or, in the frequency domain,

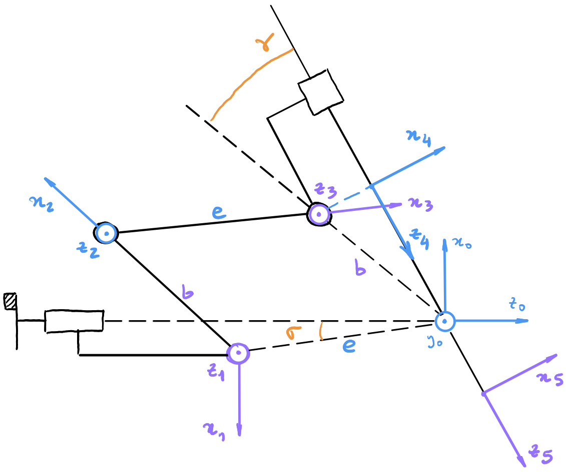
$$\left(s^2 + \frac{k_{d_i}}{B_{max_{ii}}} s + \frac{k_{p_i}}{B_{max_{ii}}}\right) Q_i(s) \approx \frac{k_{p_i}}{B_{max_{ii}}} Q_{d_i}(s) \quad , \quad i = 1, \dots, n.$$

The stiffness and damping are therefore related to the frequency and damping ratio through

$$k_{p_i} = B_{max_{ii}}\omega_{n_i}^2 \quad , \quad k_{d_i} = 2B_{max_{ii}}\xi_i\omega_{n_i}.$$

Da Vinci's Equations of Motion

Recall the kinematics of the Da Vinci Xi manipulator.



d_i	θ_i	a_i	α_i
$-e\cos(\sigma)$	$\theta_1' + \pi$	$e\sin(\sigma)$	$\pi/2$
0	$\theta_2' + \pi + \gamma$	b	0
0	$-\theta_2' - \gamma + \sigma - \frac{\pi}{2}$	e	0
0	$\theta_2' - \sigma$	$b\sin(\gamma)$	$\pi/2$
$d_5' + b\cos\gamma$	0	0	0

$$q = (\theta_1'+\pi \quad \theta_2'+\pi+\gamma \quad -\theta_2'-\gamma+\sigma-\tfrac{\pi}{2} \quad \theta_2'-\sigma \quad d_5'+bc_\gamma)^T$$

$$\ddot{q} = (\ddot{\theta}_1' \quad \ddot{\theta}_2' \quad -\ddot{\theta}_2' \quad \ddot{\theta}_2' \quad \ddot{d}_5')^T$$

$$\bar{q} = (\theta_1'+\pi \quad \theta_2'+\pi+\gamma \quad d_5'+bc_\gamma)^T$$

$$\ddot{\bar{q}} = (\ddot{\theta}_1' \quad \ddot{\theta}_2' \quad \ddot{d}_5')^T$$

Its dynamics may be obtained through Newton-Euler's recursive algorithm,

$$\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \end{pmatrix} = \text{Tau\_NewtonEuler}\left(DH, M, \dot{q}, \ddot{q}, \omega_0^0, \dot{\omega}_0^0, \ddot{p}_0^0(|g|)\right).$$

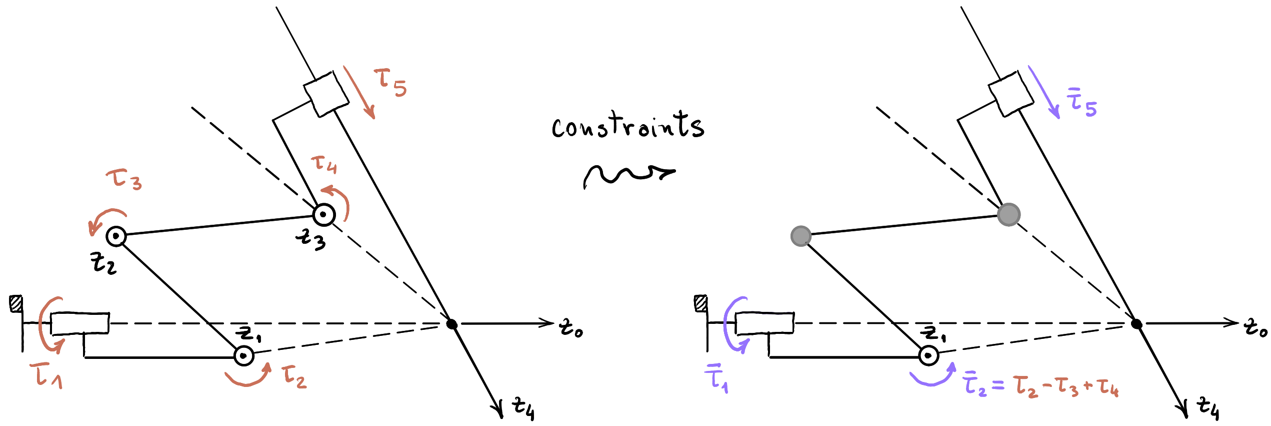
This allows to obtain the equations of motion in the form of

$$B\ddot{\bar{q}} + n = \tau \quad , \quad \text{with} \quad n = C\dot{q} + g .$$

Because of the kinematic constraint, we find that the mass matrix is not square,

$$B(\bar{q}) = \frac{\partial \tau}{\partial \ddot{\bar{q}}} \in \mathbb{R}^{5\times3}.$$

An additional step is therefore required, in order to complete the transition between the q and $\bar{q}$ spaces. This transformation step can be found by noting that the input actions τ_3 and τ_4 have been relinquished. Thus, a corresponding input space $\bar{\tau} \in \mathbb{R}^3$ must also be found.



Suppose an input $\tau = (\tau_1\ \tau_2\ \tau_3\ \tau_4\ \tau_5)^T$ is applied to the joints of the unconstrained manipulator (space q), as illustrated. Then, after the constraint is applied, the inputs τ_1 an τ_5 remain unchanged in the constrained input space, termed $\bar{\tau} = (\bar{\tau}_1\ \bar{\tau}_2\ \bar{\tau}_5)^T$. In other words, $\tau_1 = \bar{\tau}_1$ and $\tau_5 = \bar{\tau}_5$. Now, because the motion in θ_4 is coupled in the same direction of θ_2, and θ_3 is coupled in the opposite direction, it must be true that $\bar{\tau}_2 = \tau_2 - \tau_3 + \tau_4$.

The previous description can be written in matrix form, as

$$\begin{pmatrix} \bar{\tau}_1 \\ \bar{\tau}_2 \\ \bar{\tau}_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \end{pmatrix} \iff \bar{\tau} = S^T \tau \ .$$

Thus, the equations of motion are obtained through

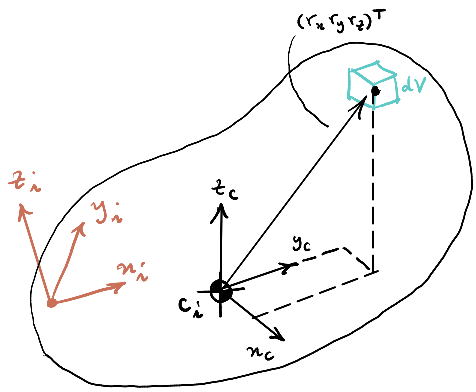
$$S^T B \ddot{\bar{q}} + S^T n = S^T \tau \Rightarrow \bar{B} \ddot{\bar{q}} + \bar{n} = \bar{\tau} \ .$$

This result is equivalent to the application of D'Alembert's principle. Through which it can be found that the generalized forces can be transformed between two spaces of coordinates α and β as $\xi_\alpha = \left(\frac{\partial \beta}{\partial \alpha}\right)^T \xi_\beta$. Thus, in the present example, $\bar{\tau} = \left(\frac{\partial q}{\partial \bar{q}}\right)^T \tau$ and $S = \frac{\partial q}{\partial \bar{q}} = \frac{\partial \dot{q}}{\partial \dot{\bar{q}}}$.

Inertia Tensor

Consider a rigid-body termed link i (or body i). Then, its equations of motion, in decoupled form, may be written as

$$\begin{cases} m_i \ddot{p}_{c_i} = \Sigma f \, , \\ I_i \dot{\omega}_i + \omega_i \times (I_i \omega_i) = \Sigma \mu \ . \end{cases}$$



Here, I_i is the inertia tensor relative to the center of mass of link i. Its coordinates are here expressed in the inertial frame, $I_i = I_i^o$.

This inertia tensor can be evaluated around a body frame fixed to the mass center, as follows.

$$I_i^c = \begin{pmatrix} \int (r_y^c)^2 + (r_z^c)^2 \rho \, dV & -\int r_x^c r_y^c \rho \, dV & -\int r_x^c r_z^c \rho \, dV \\ & \int (r_x^c)^2 + (r_z^c)^2 \rho \, dV & -\int r_y^c r_z^c \rho \, dV \\ \text{sym.} & & \int (r_y^c)^2 + (r_z^c)^2 \rho \, dV \end{pmatrix}.$$

The coordinates of this tensor can be expressed in the inertial frame through $I_i = R_c I_i^c R_c^T$, or in any body-fixed frame through

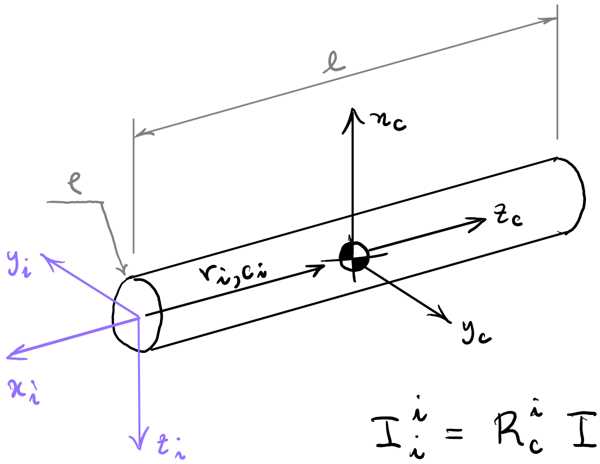
$$I_i^i = R_c^i \, I_i^c (R_c^i)^T.$$

Note that this transformation does not change the fact that the inertia tensor is still expressed relative to the center of mass of the link.

For complex shapes, I_i^c must be evaluated numerically (solidworks, meshlab, etc.).

For simple shapes, I_i^c can usually be found in tables, as follows.

- Example – Cylinder



$$I_i^c = \begin{pmatrix} \frac{1}{12} m_i (3\rho^2 + \ell^2) & 0 & 0 \\ 0 & \frac{1}{12} m_i (3\rho^2 + \ell^2) & 0 \\ 0 & 0 & \frac{1}{2} m_i \rho^2 \end{pmatrix}$$

$$R_c^i = \begin{pmatrix} x_c^i & y_c^i & z_c^i \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$I_i^i = R_c^i \, I_i^c (R_c^i)^T = \begin{pmatrix} \frac{1}{2} m_i \rho^2 & 0 & 0 \\ 0 & \frac{1}{12} m_i (3\rho^2 + \ell^2) & 0 \\ 0 & 0 & \frac{1}{12} m_i (3\rho^2 + \ell^2) \end{pmatrix}$$