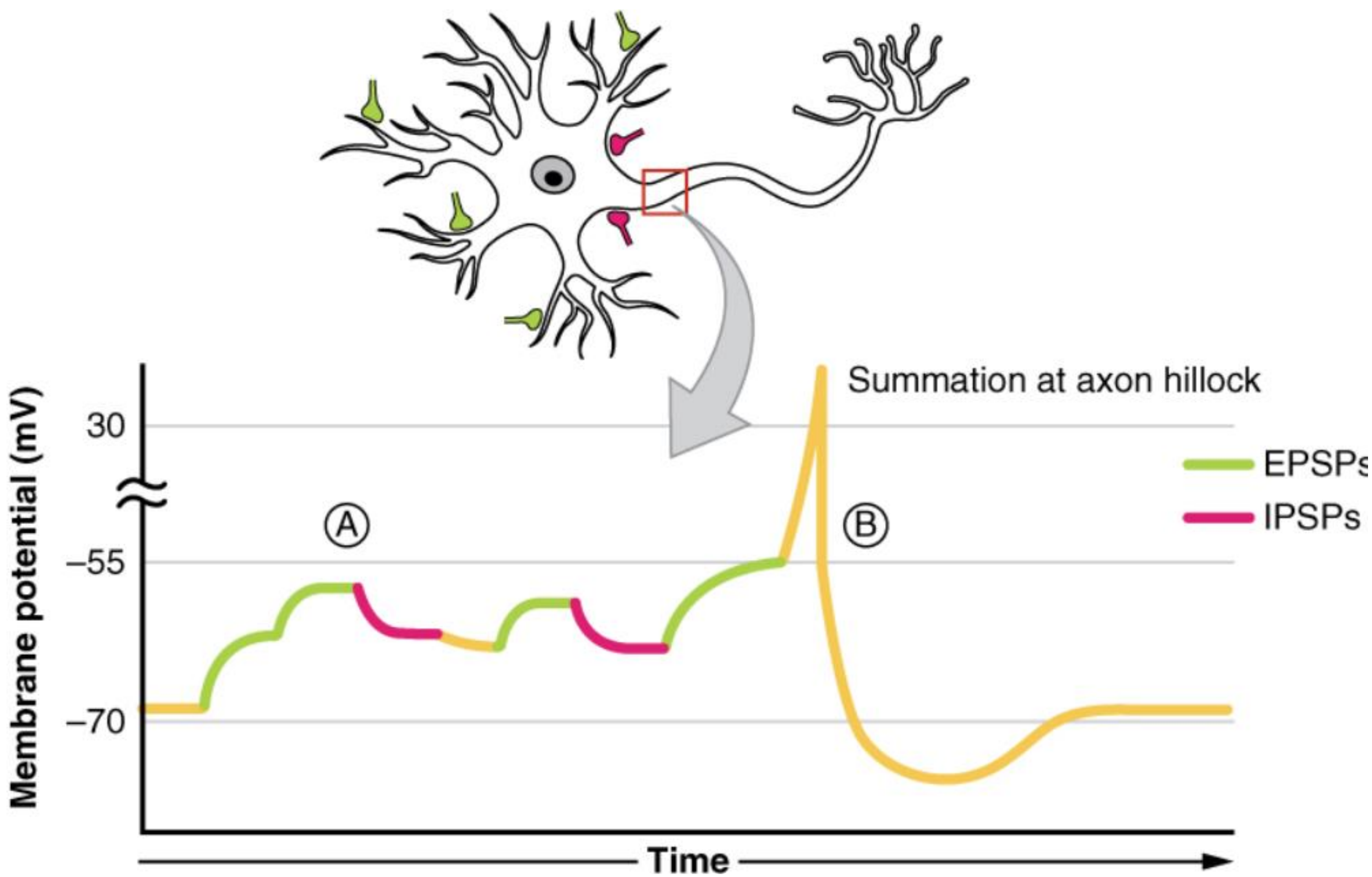
Stochastic optimal control

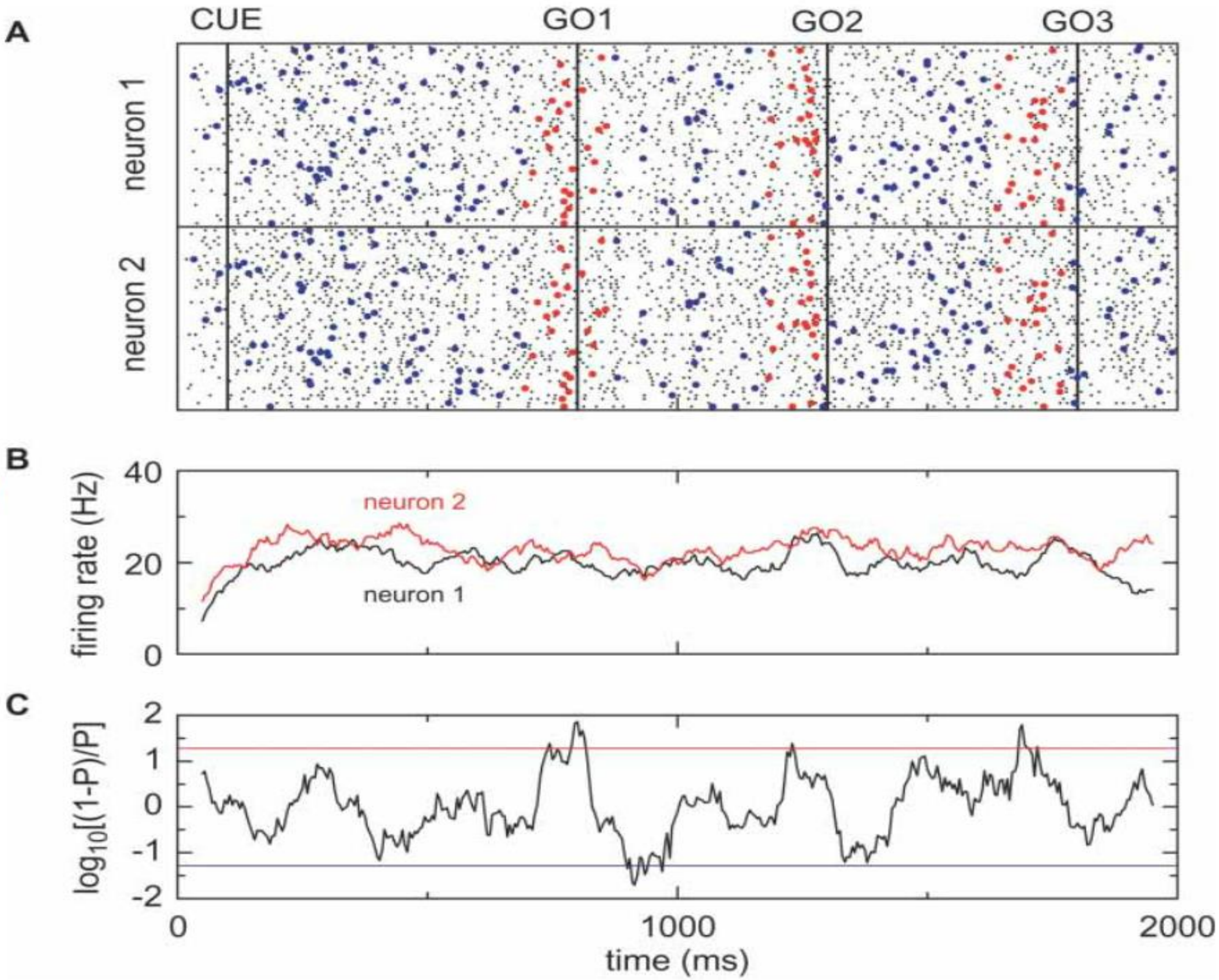
A. De Comité

LGBIO2072 – Practical Assignements

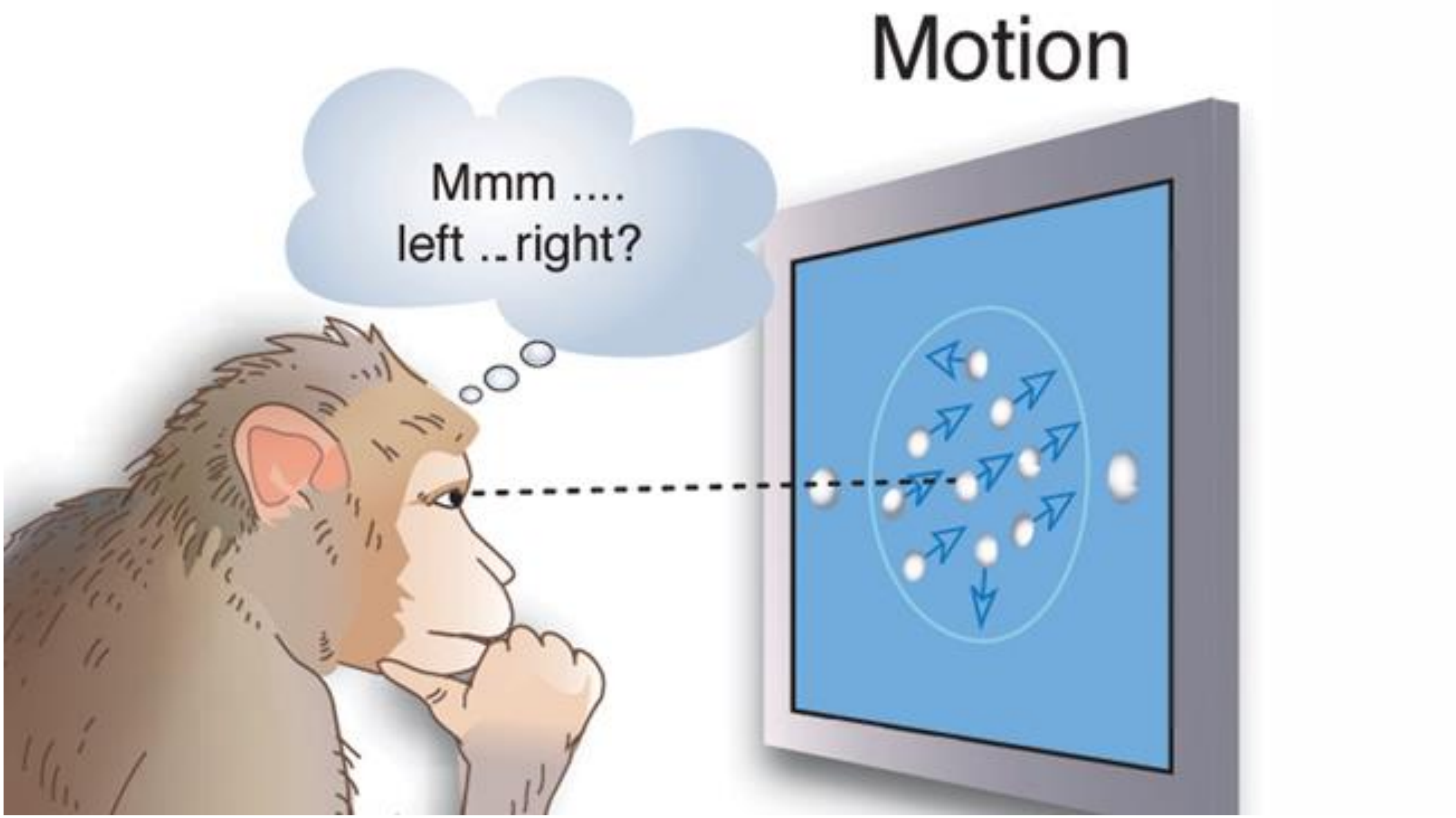
Introduction : Previous sessions



Introduction : Previous sessions



Introduction : Previous sessions



Introduction : Reaching movements

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How does the brain implement this?

Determine the task goal

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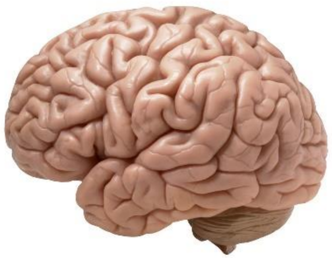
Determine the task goal

Model the system



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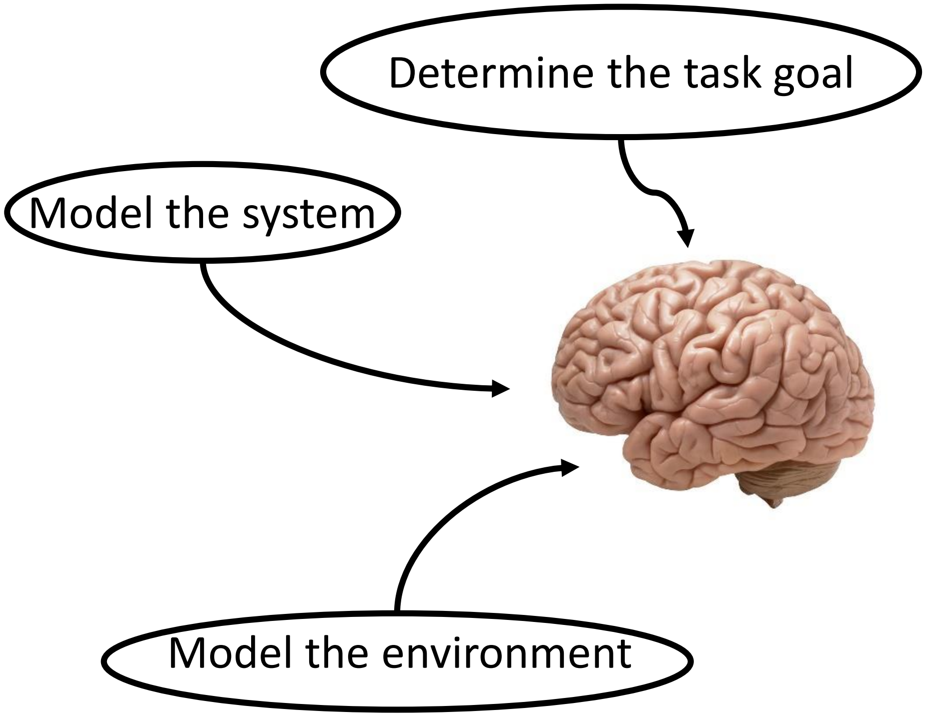

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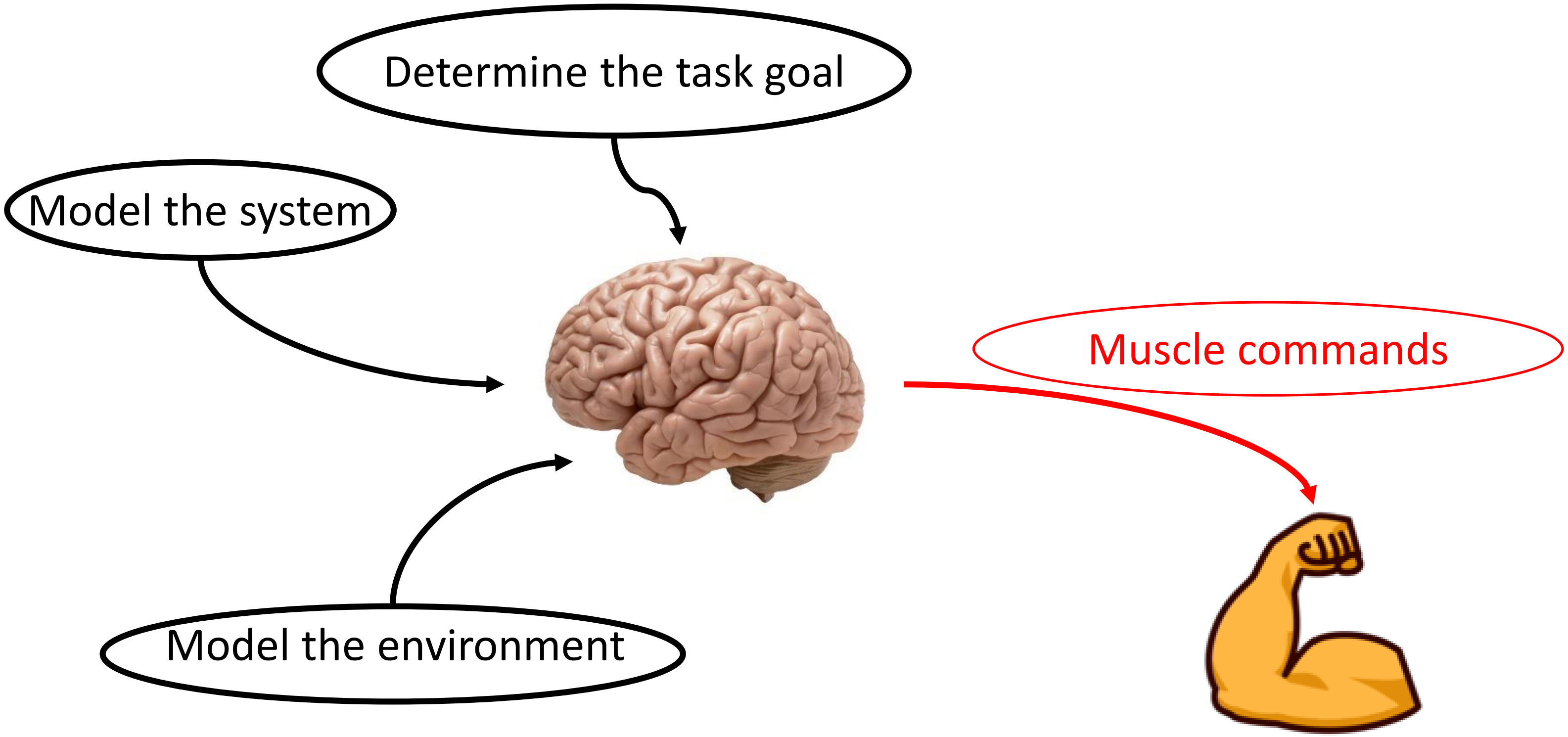
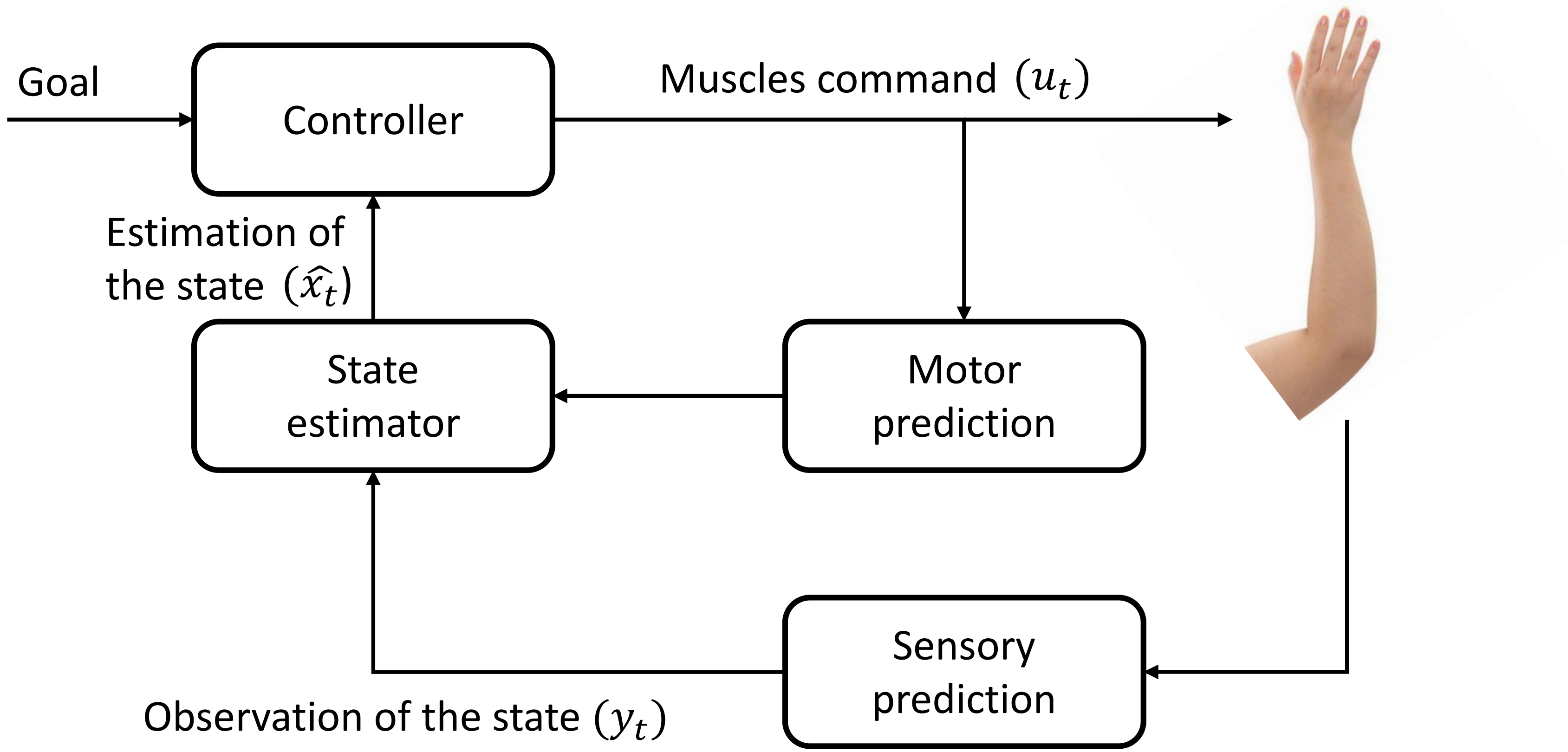


Introduction : What is optimal control?



The muscle commands are optimal (they minimise the cost function)

Outline

- How to model the system and the environment
- How to implement the task goal?
- How to compute the optimal serie of commands?
- Presentation of the project

How to model the system?

System = Hand and forearm

Dynamics defined by :

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

State vector

Command vector

How to model the system?

System = Hand and forearm

Dynamics defined by :

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u}$$

State vector
- State of the system
- State of the environment
- State of the target

Command vector
- Contains the muscle command sent by the brain to the arms

How to model the system?

System = Hand and forearm

Dynamics defined by :

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t)$$

State vector
- State of the system
- State of the environment
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Command vector
- Contains the muscle command sent by the brain to the arms

How to obtain this equation?

Equations of movement

Using Newton's law… (mainly the second)

$$\sum \vec{F} = m\,\vec{a}$$

In both directions!!! (if working in 2D)

$$\sum \vec{T} = I\,\vec{a}_R \quad \Leftrightarrow \quad I\,\ddot{\theta} = -G\,\dot{\theta} + T$$

Filter on the muscle : $\tau\,\dot{T} = u - T$

Equations of movement

Using Newton's law… (mainly the second)

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$$\sum \vec{T} = I\,\vec{a}_R \quad \Leftrightarrow \quad I\,\ddot{\theta} = -G\,\dot{\theta} + T$$

Filter on the muscle : $\tau\,\dot{T} = u - T$

To obtain this equation : $\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}\,(t) + B\boldsymbol{u}\,(t)$

1. Write all the movement equations
2. Put all the variables describing the system in the state vector
3. Fill the A and B matrices

You should obtain a continuous time matricial differential equation

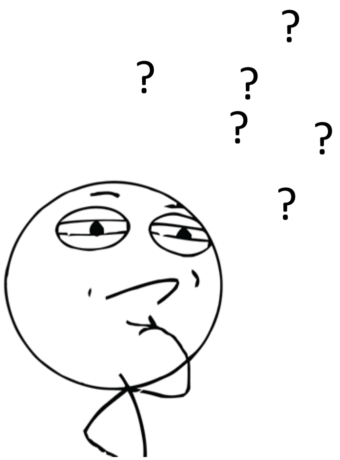
Equations of movement

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}\,(\boldsymbol{t}) + B\boldsymbol{u}\,(t)$$

Continuous

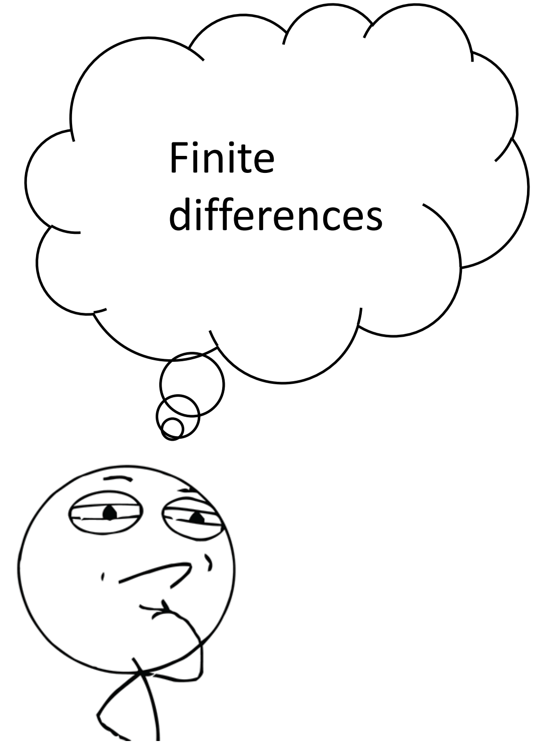
Finite difference schemes

Discrete



We need discrete-time representation because the controller is designed to work on finite horizon.

Equations of movement



$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}\,(\boldsymbol{t}) + B\boldsymbol{u}\,(t)$$ Continuous

Finite difference schemes

Discrete

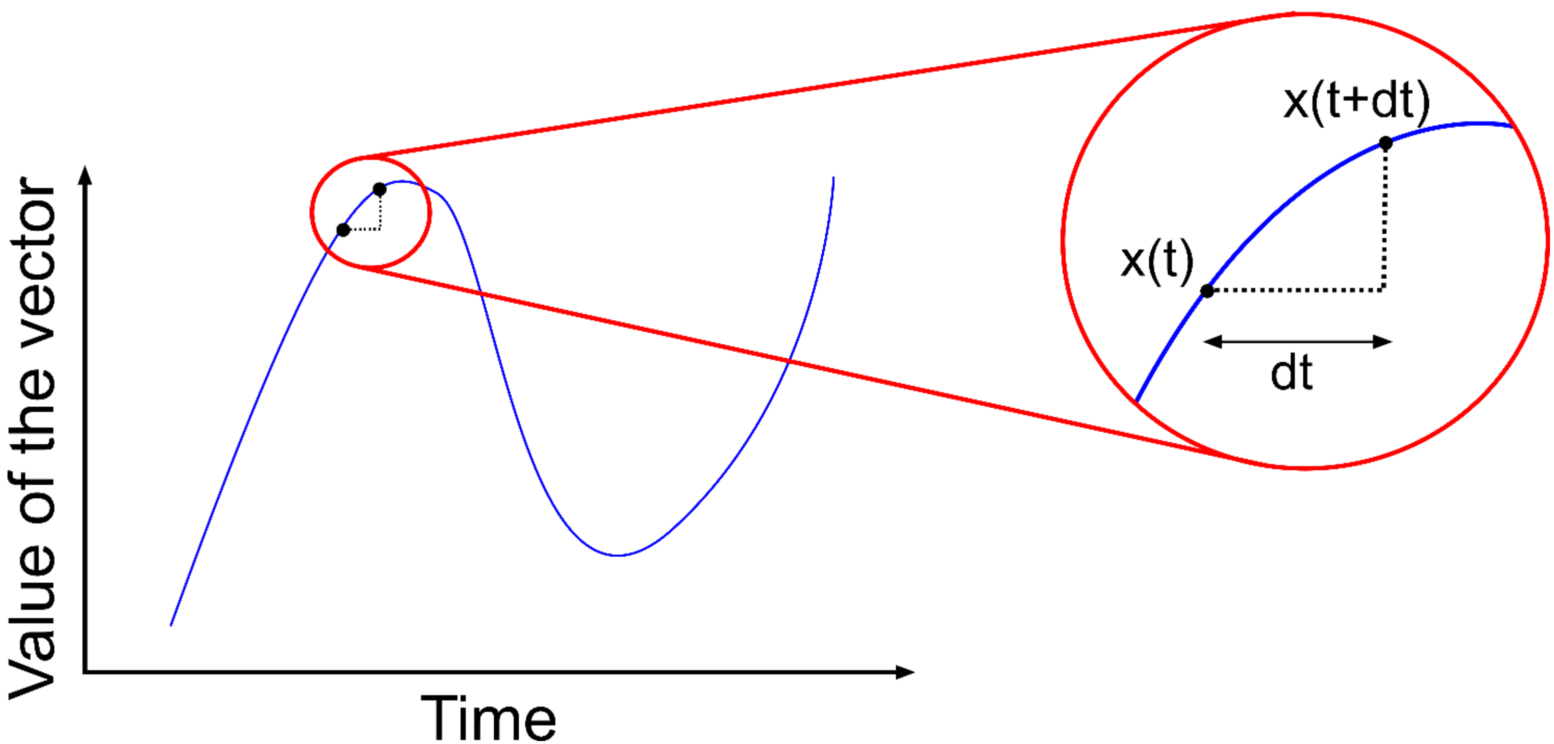
$$\boldsymbol{x}[k+1] = A\boldsymbol{x}\,[k] + B\boldsymbol{u}\,[k]$$

We need discrete-time representation because the controller is designed to work on finite horizon.

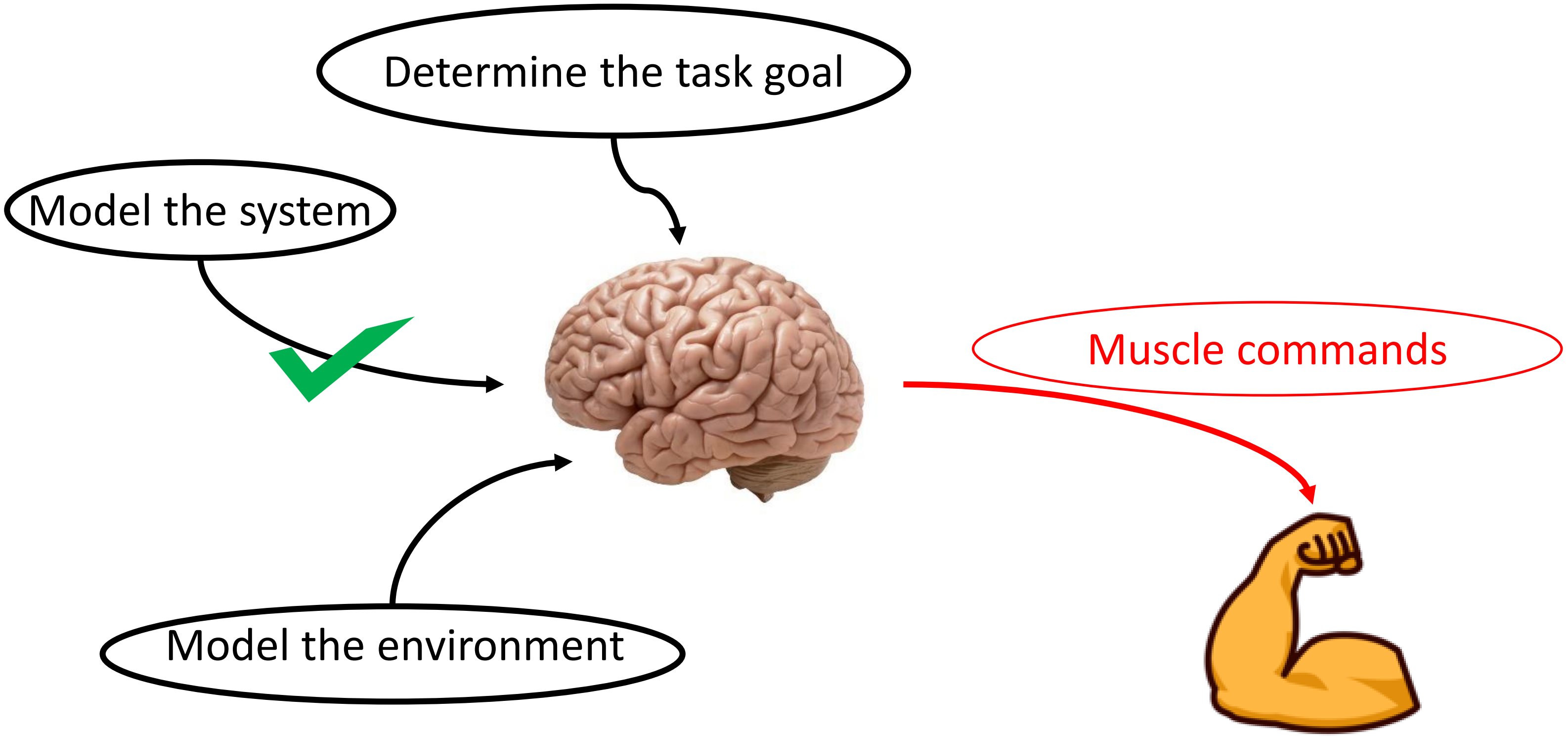
Reminder : Euler implicit (one integration method for more see LFSAB1104)

Allows to approximate a derivative by a finite difference

$$\dot{x}(k) \cong \frac{x(k+1) - x(k)}{\delta t}$$



How does the brain implement this?



How to determine and model the task goal ?

Task goal is associated by cost function :

Cost function = end-point error + energetic cost

$$J(x,u) = x_N^T Q_N x_N + \sum_{k=1}^{N-1} (x_k^T Q_k x_k + u_k^T R u_k)$$

How does this take the position of the target into account?

What's inside the state?

Let's take a look at the state of the system…

Angular position and speed of the hand

Forces exerted by the hand

x_k

What’s inside the state?

Let’s take a look at the state of the system…

Position and speed of the hand

Forces exerted by the hand

x_k

Angular position and speed of the target

Forces at the target

…

What’s inside the state?

Let’s take a look at the state of the system…

x_k

Position and speed of the hand

Forces exerted by the hand

Forces exerted on the hand

Position and speed of the target

Forces at the target

…

$$x_k = \begin{bmatrix} x_{hand} \\ y_{hand} \\ v_{hand}^{x} \\ v_{hand}^{y} \\ \vdots \\ x_{target} \\ y_{target} \\ v_{target}^{x} \\ v_{target}^{y} \end{bmatrix}$$

Extension of the state!!! ➔ Extension of the A and B matrices to match the dimensions!!!

Objective function?

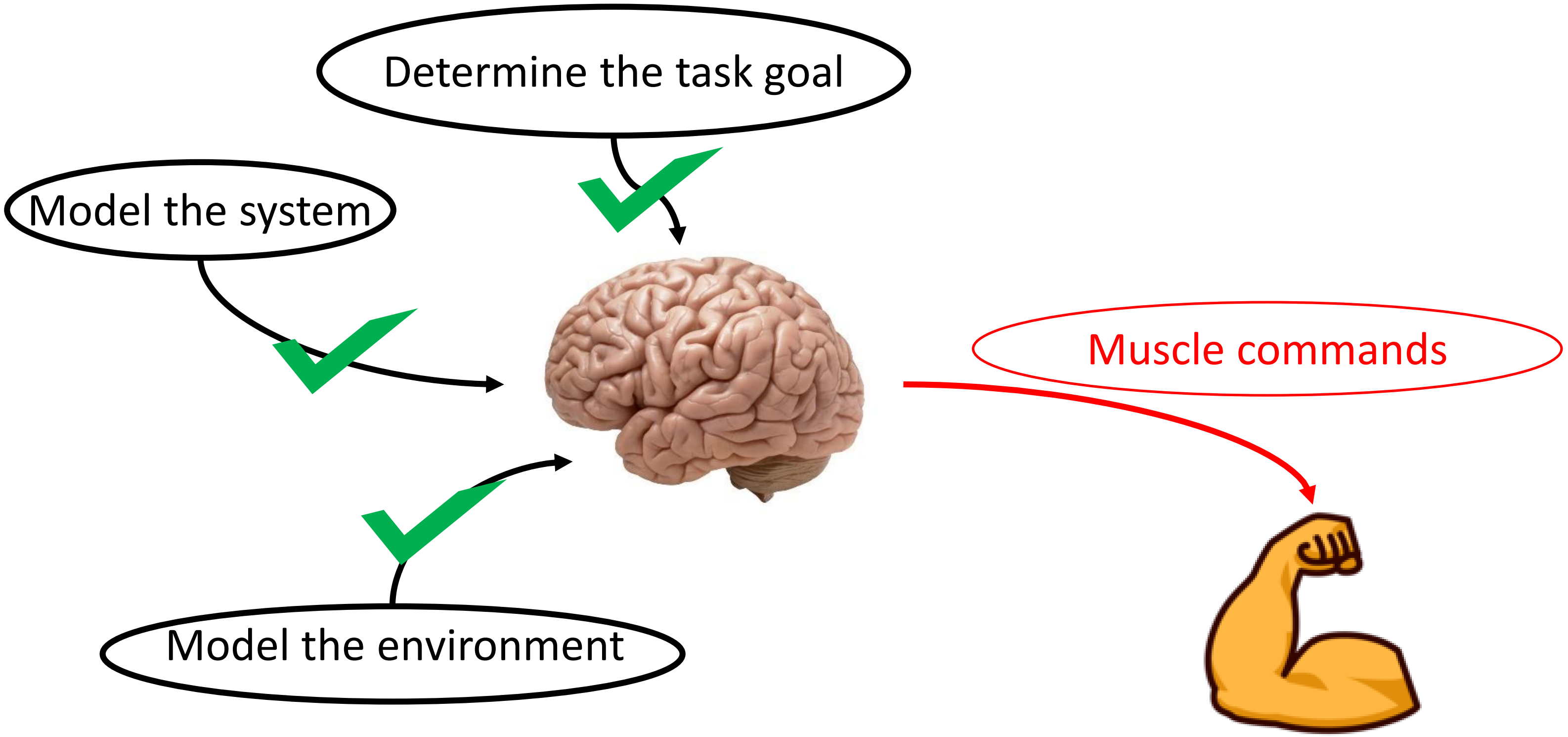
$$J(\boldsymbol{x}) = w_1(\theta_k - \theta_k^*)^2 + w_2\left(\dot{\theta}_k - \dot{\theta}_k^*\right)^2 + w_3(T_k - T_k^*)^2$$

$$\Updownarrow$$

$$J(\boldsymbol{x}) = \boldsymbol{x}[k]^T Q_N\, \boldsymbol{x}[k]$$

Goal : Find the entries of the Q_N matrix

How does the brain implement this?



How to compute the optimal serie of command?

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}\,[k] + B\boldsymbol{u}\,[k]$$

Dynamics of the system
(= how it can evolve)

$$J(x,u) = x_N^T Q_N x_N + \sum_{k=1}^{N-1} (x_k^T Q_k x_k + u_k^T R u_k)$$

Where and how we want to perform the task?

How to compute the optimal serie of command?

$$\boldsymbol{x}[k+1] = A\boldsymbol{x}\,[k] + B\boldsymbol{u}\,[k]$$

Dynamics of the system
(= how it can evolve)

$$J(x,u) = x_N^T Q_N x_N + \sum_{k=1}^{N-1} (x_k^T Q_k x_k + u_k^T R u_k)$$

Where and how we want to perform the task?

$$\min_{u_1,u_2,\ldots,u_N} J(x,u)$$

How to compute the optimal serie of command?

The solution is optimal in the sense that it **minimises the objective function**

In the fully observable case, we have the following recurrence

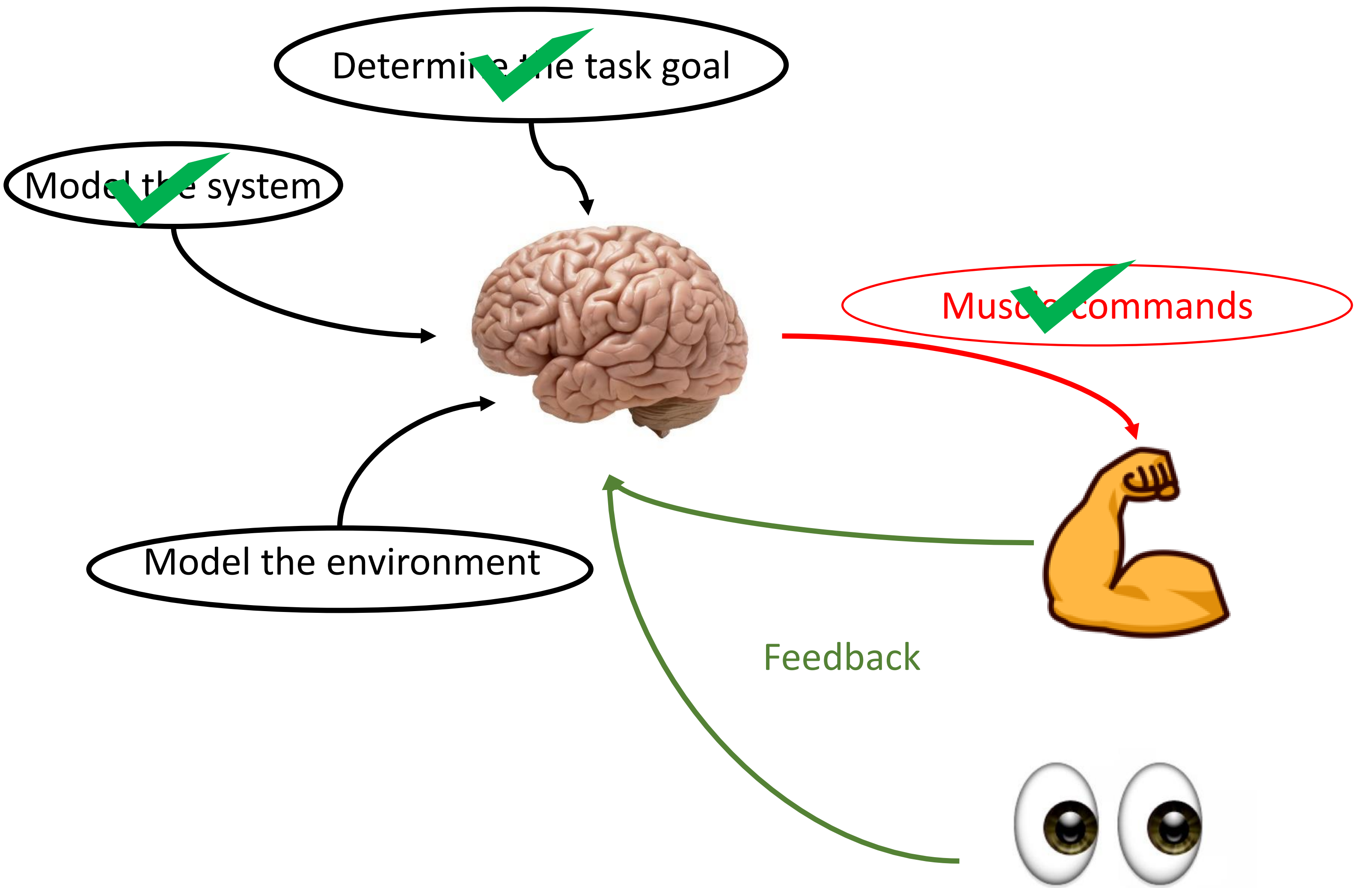
$$
\begin{gathered}
L_k = (R + B^T S_{k+1} B)^{-1} B^T S_{k+1} A \\
S_k = Q_k + A^T S_{k+1} (A - B L_k) \\
S_N = Q_N, \qquad\qquad s_N = 0 \\
s_k = s_{k+1} + tr\left(S_{k+1} \Omega_\xi\right)
\end{gathered}
$$

Simulation of the system

$$x_{k+1} = (A - B L_k) x_k + \xi_k$$

$$u_k = -L_k x_k$$

How does the brain implement this?



Problem?

No feedback…

Perturbations?

Modification of target?

…

Closed loop control = Blind control

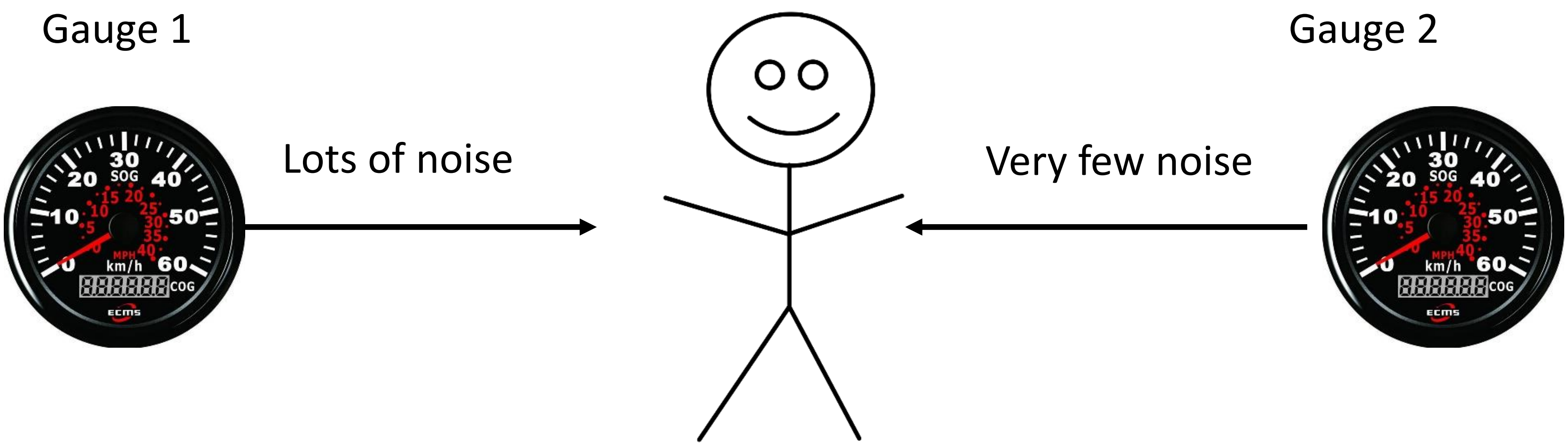
Human body has feedbacks…

Feedbacks from the human body

Eyes and proprioception

$$y_k = Hx_k + \omega_k$$

Noisy observations of the state



Feedbacks from the human body

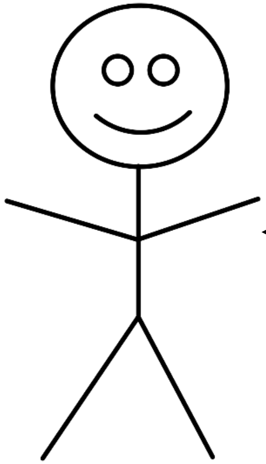
Eyes and proprioception

$$y_k = Hx_k + \omega_k$$

Noisy observations of the state

Gauge 1

Lots of noise



Very few noise

Gauge 2

Prior (last know speed)

Feedback (vision, proprioception

Feedbacks from the human body

Eyes and proprioception

$$y_k = Hx_k + \omega_k$$

Noisy observations of the state

Combination of priors and feedback :

$$\hat{x}_{k+1} = (1 - K) \ * prior + K * feedback$$
$$\hat{x}_{k+1} = A\,\hat{x}_k + B\,u_k + K(y_k - H\,\hat{x}_k)$$

How to find the coefficient K?

Feedbacks from the human body

Eyes and proprioception

$$y_k = Hx_k + \omega_k$$

Noisy observations of the state

Combination of priors and feedback :

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How to find the coefficient K?

Optimal estimation of the state

Ponderate source by their 'accuracy'

Computation of the different gains

$$\hat{x}_{k+1} = A\,\hat{x}_k + Bu_k + K_k(y_k - H\ \hat{x}_k)$$
$$K_k = A\,\Sigma_k H^T(H\Sigma_k H^T + \Omega_\omega)^{-1}$$
$$\Sigma_{k+1} = \Omega_\xi + (A - K_k H)\Sigma_\mathrm{k}\mathrm{A}^\mathrm{T}$$

The command becomes :

$$u_k = -L_k\,\hat{x}_k$$

Project

- See instructions on Moodle
- Groups of 2 students
- Write a report of 6 pages
- Due date : **17th of November 2020**