Solution Since absolute value signs cannot be included in a linear program, recall that:

$$|x| = \max\ \{x, \ -x\}.$$

With this in mind, the following linear program models the problem:

$$\begin{array}{lrcl}
\text{Minimize} & z & = & (P_1 + P_2) + (P_3 + P_4) + (P_5 + P_6) + (P_7 + P_8) \\
\text{Subject to} & P_1 & \geq & -(x_1 - 3) \\
 & P_1 & \geq & x_1 - 3 \\
 & P_2 & \geq & -(x_2) \\
 & P_2 & \geq & x_2 \\
 & P_3 & \geq & -(x_1 - 1) \\
 & P_3 & \geq & x_1 - 1 \\
 & P_4 & \geq & -(x_2 - 4) \\
 & P_4 & \geq & x_2 - 4 \\
 & P_5 & \geq & -(x_1 + 2) \\
 & P_5 & \geq & x_1 + 2 \\
 & P_6 & \geq & -(x_2 - 1) \\
 & P_6 & \geq & x_2 - 1 \\
 & P_7 & \geq & -(x_1) \\
 & P_7 & \geq & x_1 \\
 & P_8 & \geq & -(x_2 + 3) \\
 & P_8 & \geq & x_2 + 3 \\
 & \text{all variables} & \geq & 0.
\end{array}$$

Each P_{2i-1} represents the horizontal distance between the new machine and the i^{th} old machine for $i = 1, 2, 3, 4$. Also for $i = 1, 2, 3, 4$, P_{2i} represents the vertical distance between the new machine and the i^{th} old machine. The objective function reflects the desire to minimize total distance between the new machine and all the others. The constraints relate the P variables to the distances in terms of x_1 and x_2. Two constraints for each P variable allow each P_i $(i = 1, 2, \ldots, 8)$ to equal the maximum of $x_j - c_j$ and $-(x_j - c_j)$ (for $j = 1, 2$ and where c is the j^{th} component of the position of one of the old machines). Since this program is a minimization problem and the smallest any of the variables can be is $\max\{(x_j - c_j),\ -(x_j - c_j)\}$, each P_i will naturally equal its least possible value. This value will be the absolute value of $x_j - c_j$.

In the next problem we will also interpret a "real-world" situation as a linear program. Perhaps the most notable aspect of this problem is the concept of inventory and recursion in constraints.

Solution In order to minimize the cost per year, decision variables are defined. If we let

P_t – number of pairs of shoes during quarter t, $t = 1, 2, 3, 4$

W_t – number of workers starting work in quarter t, $t = 1, 2, 3, 4$

I_t – number of pairs of shoes in inventory after quarter t, $t = 1, 2, 3, 4$,

the objective function is therefore

$$\min z = 50I_1 + 50I_2 + 50I_3 + 1500W_1 + 1500W_2 + 1500W_3 + 1500W_4.$$

Since each worker works three quarters they must be payed three times their quarterly rate. The objective function takes into account the salary paid to the workers, as well as inventory costs. Next, demand for shoes must be considered. The following constraints account for demand:

$$
\begin{aligned}
P_1 &\geq 600 \\
I_1 &= P_1 - 600 \\
I_2 &= I_1 + P_2 - 300 \\
I_3 &= I_2 + P_3 - 800 \\
I_4 &= I_3 + P_4 - 100 = 0.
\end{aligned}
$$

Note these constraints are recursive, meaning they can be defined by the expression:

$$I_n = I_{n_1} + P_n - D_n$$

where D_n is just a number symbolizing demand for that quarter.

Also, the workers can only make 50 pairs of shoes per quarter, which gives us the constraints

$$
\begin{aligned}
P_1 &\leq 50W_1 + 50W_3 + 50W_4 \\
P_2 &\leq 50W_2 + 50W_4 + 50W_1 \\
P_3 &\leq 50W_3 + 50W_1 + 50W_2 \\
P_4 &\leq 50W_4 + 50W_3 + 50W_2
\end{aligned}
$$

This linear program is somewhat cyclical, since workers starting work in quarter 4 can produce shoes during quarters 1 and 2. It is set up this way in order to promote long-term minimization of cost and to emphasize that the number of workers starting work during each quarter should always be the optimal value, and not vary year to year.

The next example is a similar type of problem. Again, decision variable must be identified and constraints formed to meet a certain objective. However, this problem deals with a different real-world situation and it is interesting to see the difference in the structure of the program.

In the previous problem, the concepts of inventory and scheduling were key, in the next problem, the most crucial aspect is conservation of matter. This means that several options will be provided concerning how to dispose of waste and all of the waste must be accounted for by the linear program.

(3)

$\text{Min } z \Rightarrow 2x_1 + 2x_2 - 5x_3$

$\text{s.t} \Rightarrow 3x_1 + 2x_2 - 4x_3 = 7$

$x_1 - x_2 + 3x_3 = 2$

$x_1, x_2, x_3 \geq 0$

$\text{Max } z \Rightarrow -2x_1 - 2x_2 + 5x_3$

s.t $3x_1 + 2x_2 - 4x_3 = 7$

$x_1 - x_2 + 3x_3 = 2$

z	x_1	x_2	x_3	b
1	2	2	-5	0
0	3	2	-4	7
0	1	-1	3	2

→ Creat artificial variables

W	x_1	x_2	x_3	A_1	A_2	b
1	0	0	0	1	1	0
0	3	2	-4	1	0	7
0		-1	3	0	1	2

subtract →

basics

	W	x_1	x_2	x_3	A_1	A_2	b
R_0	1	-4	-1	1	0	0	-9
R_1	0	3	2	-4	1	0	$7 \to \frac{7}{3}$
R_2	0	1	-1	3	0	1	$2 \to \frac{2}{1}$

$\begin{cases} -3R_2 + R_1 \to R_1 \\ 4R_2 + R_0 \to R_0 \end{cases}$

$X_1 \to$ Enter $A_2 \to$ leave

	W	x_1	x_2	x_3	A_1	A_2	b
R_0	1	0	-5	13	0	4	-1
R_1	0	0	$\frac{5}{1}$	-13 $-\frac{13}{5}$	1 $\frac{1}{5}$	-3 $-\frac{3}{5}$	1 $\frac{1}{5}$
R_2	0	1	-1	3	0	1	2

$X_2 \to$ Enter $A_1 \to$ leave

$\begin{cases} \frac{R_1}{5} \to R_1 \\ 5R_1 + R_0 \to R_0 \\ R_1 + R_2 \to R_2 \end{cases}$

(3)

W	x_1	x_2	x_3	A_1	A_2	b
1	0	0	0	1	1	0
0	0	1	$-\frac{15}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{1}{5}$
0	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{11}{5}$

- $W=0$
- No negative c_i in Z row

Phase 1 solved,

	Z	x_1	x_2	x_3	b
R_0	1	2	2	-5	0
R_1	0	0	1	$-\frac{13}{5}$	$\frac{1}{5}$
R_2	0	1	0	$\frac{2}{5}$	$\frac{11}{5}$

2 2 $\frac{-22}{5}$ $\frac{24}{5}$

Substract 2 times Constraint (1)

subtract 2 times Constraint (2)

↓

Z	x_1	x_2	x_3	b
1	0	0	$-\frac{3}{5}$	$-\frac{24}{5}$
0	0	1	$-\frac{15}{5}$	$\frac{1}{5}$
0	1	0	$\frac{2}{5}$	$\frac{11}{5}$

✓

Big M =>

Max $-2x_1 - 2x_2 + 5x_3$

s.t $3x_1 + 2x_2 - 4x_3 + A_1 = 7$

$x_1 - x_2 + 3x_3 + A_2 = 2$

Z	x_1	x_2	x_3	A_1	A_2	b
1	2	2	-5	M	M	0
0	3	2	-4	1	0	7
0	1	-1	3	0	1	2

$-MR_2 + R_0 \to R_0$

Z	x_1	x_2	x_3	A_1	A_2	b
1	$-M+2$	$M+2$	$-3M-5$	M	0	$-2M$
0	3	2	-9	1	0	7
0	1	-1	3	0	1	2

$-MR_1 + R_0 \to R_0$

→

	Z	x_1	x_2	x_3	A_1	A_2	b
Z_1	1	$-4M+2$	$-M+2$	$M-5$	0	0	$-9M$
A_1	0	3	2	-9	1	0	7
A_2	0	1	-1	3	0	1	2

$$\begin{cases} -3R_2 + R_1 \to R_1 \\ (4M-2)R_2 + R_0 \to R_0 \end{cases}$$

x_1 Enter, A_2 leave

	Z	x_1	x_2	x_3	A_1	A_2	b
Z_1	1	0	$5M+4$	$13M-11$	0	$4M-2$	$M+4$
A_1	0	0	5 $\frac{1}{5}$	-13 $-\frac{13}{5}$	1 $\frac{1}{5}$	-3 $-\frac{3}{5}$	1 $\frac{1}{5}$
x_1	0	1	-1	3	0	1	2

x_2 Enter, A_1 leave

Z	x_1	x_2	x_3	A_1	A_2	b
1	0	0	$-\frac{3}{5}$	$M-\frac{4}{5}$	$M+\frac{2}{5}$	$\frac{24}{5}$
0	0	1	$-\frac{13}{5}$	$\frac{1}{5}$	$-\frac{3}{5}$	$\frac{1}{5}$
0	1	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{11}{5}$

$$\begin{cases} \frac{R_1}{5} \to R_1 \\ (5M + 4)R_1 + R_0 \to R_0 \end{cases}$$

Z	x_1	x_2	x_3	b
1	$\frac{31}{2}$	0	0	$\frac{3}{2}$
0	$\frac{13}{2}$	1	0	$\frac{145}{10}$
	$\frac{5}{2}$	0	1	$\frac{1}{2}$

All variables ≥ 0

Solution:

$Z = \frac{3}{2}$

$x_1 = 0,\ x_2 = 14.5,\ x_3 = 0.5$

* تابع هدف : $\text{minimize } -2x_4 - x_5 - 2x_6$

$\rightarrow c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ c_4 = -2,\ c_5 = -1,\ c_6 = -2$

$Z = -12 \Rightarrow Z = c_B \bar{b} = (-2, 0, -2) \cdot (a, 5, 0) = -2a + 0 \times 5 + -2 \times 0$

$\rightarrow -2a = -12 \rightarrow \boxed{a = 6}$

$\therefore$

$B = [a_6\ \ a_2\ \ a_4] \rightarrow a_2 = (0, 1, 0) = (0, d, e) \rightarrow \boxed{d = 1},\ \boxed{e = 0}$

$b = z_1 - c_1 = c_B y_1 - c_1 \rightarrow b = (-2, 0, -2) \cdot (2, 3, 0) - 0 = -4 + 0 \times 3 + -2 \times 0$

$\rightarrow \boxed{b = -4}$

$c = z_2 - c_2 \rightarrow \boxed{c = 0}$

$z_3 - c_3 = 0 \rightarrow (-2, 0, -2) \cdot (-14/3, 2, f) = \frac{28}{3} - 2f = 0 \rightarrow 2f = \frac{28}{3}$

$\rightarrow f = \frac{28}{6} = 4.\bar{6}$

$g = z_2 - c_2 = 0 \rightarrow \boxed{g = 0}$

$h = z_5 - c_5 = c_B y_5 - c_5 = (-2, 0, -2) \cdot (1, 5/2, 2) + 1 = -2 + \frac{5}{2} \times 0 + (-4) + 1$

$\rightarrow \boxed{h = -5}$

الف) مطابق نتیجه Exel، جواب بهینه و سود بهینه:

$x_1 = 0$ ، $x_2 = 0$ ، $x_3 = 500$ ، $x_4 = 0$ ، سود بهینه = ۷۰۰۰ دلار

ب) براساس گزارش حساسیت نمی‌شود به این سؤال پاسخ داد چون با وجود اینکه Shadow Price آهن ۲۵ دلار است اما Allowable Increase آن صفر است. این بدین معنی است که درباره افزایش ظرفیت آهن نمی‌توان بدون اجرای دوباره برنامه نظری داد. لو آهن را می‌شود حداکثر صفر کیلو اضافه کرد در حالی که جواب به دست آمده صادق باشد

ج) خیر Shadow Price آلومینیوم صفر است (با Allowable Increase بی‌نهایت) یعنی با افزایش ظرفیت آن، هیچ سودی به دست نمی‌آید (در واقع جواب نهایی هم ۲۵۰۰ کیلو آلومینیوم استفاده می‌کند تا حداکثر آن که ۵۰۰۰ کیلو است فاصله دارد)

د) Allowable Increase برای ضریب محصول دوم ۵ است. این یعنی با افزایش حداکثر ۵ دلاری سود (قیمت)، باز هم به صرفه است که از این محصول تولید نشود. بنابراین برای آنکه تولید این محصول سودآور باشد باید حداقل بیش از ۵ دلار قیمت آن افزایش پیدا کند.