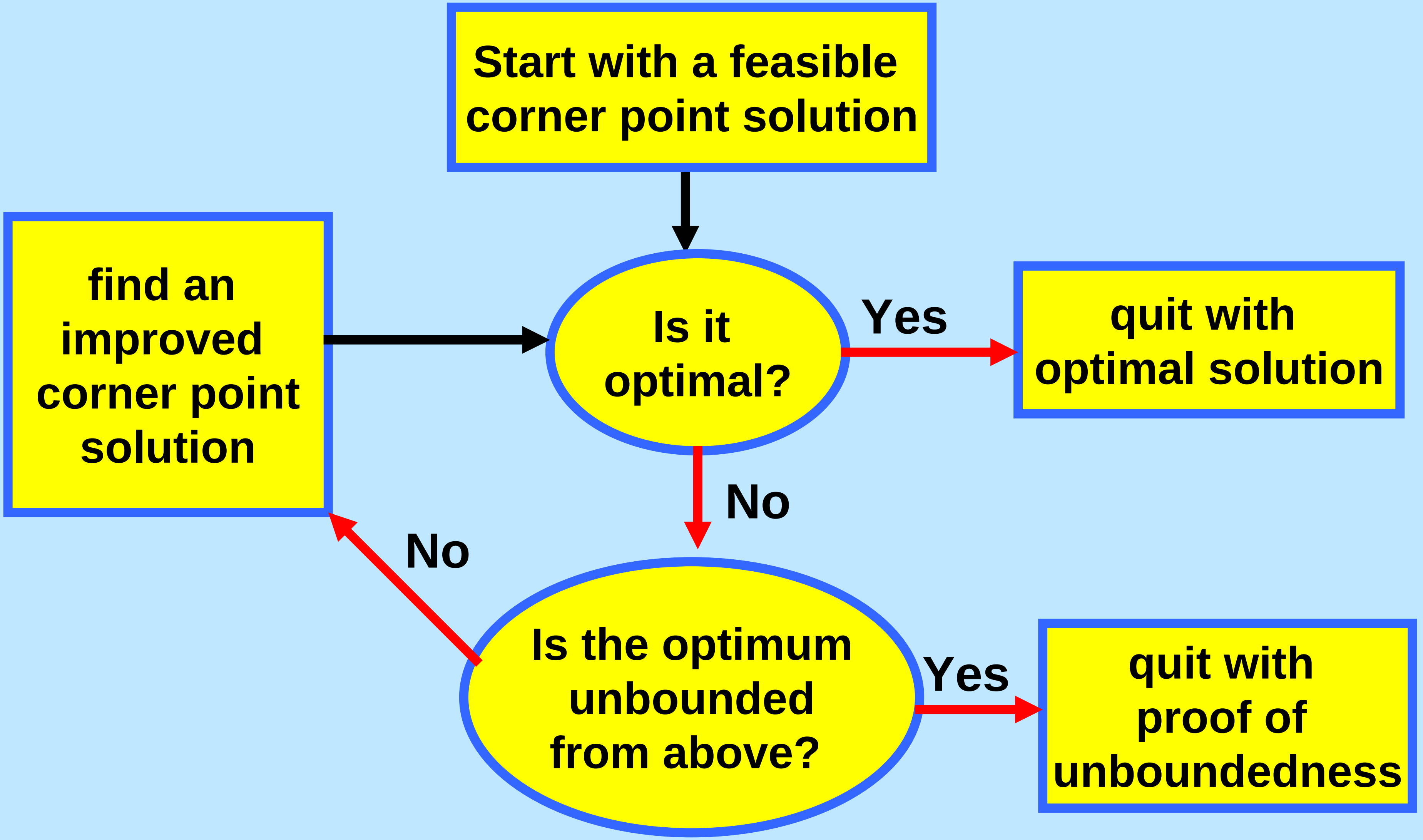
## 15.053 February 17, 2005

- **Simplex Method Continued**

# Today's Lecture

- Review of the simplex algorithm.
- Formalizing the approach
- Alternative Optimal Solutions
- Obtaining an initial bfs
- Is the simplex algorithm finite? (Answer, yes, but only if we are careful)

# The simplex algorithm (for max problems)



# LP Canonical Form iff there is a basis that is feasible.

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | 3 | -2 | 0 | 0 | = | 0 |
| 0 | -3 | 3 | 1 | 0 | = | 6 |
| 0` | -4 | 2 | 0 | 1 | = | 2 |

The **basic variables** are z, $x_3$ and $x_4$.

The **non-basic variables** are $x_1$ and $x_2$.

The **basic feasible solution** (**bfs**) is

$z = 0$, $x_1 = x_2 = 0$, $x_3 = 6$, $x_4 = 2$

# Optimality Conditions: maximization form

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | $-\bar{c}_1$ | $-\bar{c}_2$ | 0 | 0 | = | 0 |
| 0 | -3 | 3 | 1 | 0 | = | 6 |
| 0 | -4 | 2 | 0 | 1 | = | 2 |

**The bar indicates that it is possibly the coefficient after some pivots**

**The bfs is optimal if all coefficients z-row are $\geq 0$.**

**Use $\bar{c}_j$ to denote the cost coefficients.**

***Opt. conditions*: The bfs is optimal if $-\bar{c}_j \geq 0$ for all j.**

# The Simplex Pivot Rule

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | -2 | 0 | = | 3 |
| 0 | 2 | 0 | 1 | 3 | 0 | = | 9 |
| 0 | 1 | 1 | 0 | -1 | 0 | = | 1 |
| 0 | 0 | 0 | 0 | 2 | 1 | = | 4 |

**Pivot in a variable whose cost-row coefficient is negative.**

**If there are multiple choices, then use a "refined" rule.**

**If $-\bar{c}_j \geq 0$ for all j, then the basis is optimal.**

# Review from last lecture: To do with your partner

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | -2 | 0 | = | 3 |
| 0 | 2 | 0 | 1 | 3 | 0 | = | 9 |
| 0 | 1 | 1 | 0 | -1 | 0 | = | 1 |
| 0 | 0 | 0 | 0 | 2 | 1 | = | 4 |

1. Determine the basic variables and the basic solution
2. What will be the entering variable?
3. What will be the variable that leaves the basis?
   Reminder: set the value of the entering variable to $\Delta$.

# The min ratio rule

If $\bar{a}_{is} \leq 0$ for all i, then the objective value is unbounded from above.

| z | $x_1$ | $x_2$ | $x_3$ | $x_s$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | $-\bar{c}_s$ | 0 | = | $z_0$ |
| 0 | 2 | 0 | 1 | $\bar{a}_{1s}$ | 0 | = | $\bar{b}_1$ |
| 0 | 1 | 1 | 0 | $\bar{a}_{2s}$ | 0 | = | $\bar{b}_2$ |
| 0 | 0 | 0 | 0 | $\bar{a}_{3s}$ | 1 | = | $\bar{b}_3$ |

Min Ratio = min{ $\bar{b}_i$ / $\bar{a}_{is}$ : $\bar{a}_{js}$ > 0 and i = 1 to m }.

Usually, the "argmin" value is denoted as r.

In this case, r = 3. Pivot on $\bar{a}_{rs}$

# Minimum Ratio Rule

Pivot out the basic variable in row r, where

$$r = \operatorname{argmin}_i \; \{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}, \text{ and thus}$$

$$\bar{b}_r / \bar{a}_{rs} = \min \{ \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0\}.$$

If $\bar{a}_{is} \leq 0$ for all i, then the solution is unbounded.

# The pivot

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 1 | = | 7 |
| 0 | 2 | 0 | 1 | 0 | -3/2 | = | 3 |
| 0 | 1 | 1 | 0 | 0 | 1/2 | = | 3 |
| 0 | 0 | 0 | 0 | 1 | 1/2 | = | 2 |

1. The objective value strictly improves so long as $\bar{b}_r > 0$.
2. The new (basic feasible solution) bfs is feasible.

# Alternative Optima (maximization)

Recall: $z + 4\ x_2 = 8$

Note that z does not depend on $x_1$ .

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | 0 | 4 | 0 | 0 | = | 8 |
| 0 | 1 | 3 | 1 | 0 | = | 6 |
| 0 | -1 | 2 | 0 | 1 | = | 2 |

This basic feasible solution is optimal! There is an alternative optimum because the non-basic variable $x_1$ has a 0 cost-coefficient.

If $x_1$ enters the basis, what variable leaves?

# Alternative Optima (maximization)

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS |
|---|---|---|---|---|---|---|
| 1 | $-\bar{c}_1$ | $-\bar{c}_2$ | 0 | 0 | = | 8 |
| 0 | 1 | 3 | 1 | 0 | = | 6 |
| 0 | -1 | 2 | 0 | 1 | = | 2 |

There may be alternative optima if $-\bar{c}_j \geq 0$ for all j and $\bar{c}_j = 0$ for some j where $x_j$ is non-basic

Use the min ratio rule to determine which variable leaves the basis.

# Alternative Optima (maximization)

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | 0 | 4 | 0 | 0 | = | 8 |
| 0 | 1 | 3 | 1 | 0 | = | 6 |
| 0 | 0 | 5 | 1 | 1 | = | 8 |

**Perform the pivot.**

**Note: the solution is different, but the objective value is the same.**

# Review of notation

**A basic feasible solution is optimal if**
**-** $\bar{c}_j \geq 0$ **for all j.**

**Assumption: the entering variable is** $x_s$
**(and so** $-\bar{c}_s < 0$**)**

**Pivot out the basic variable in row r, where**

$$r = \text{argmin}_i \ \{ \bar{b}_i / \bar{a}_{is} : \ \bar{a}_{is} > 0\}, \text{ and thus}$$

$$\bar{b}_r / \bar{a}_{rs} = \min \{ \bar{b}_i / \bar{a}_{is} : \ \bar{a}_{is} > 0\}.$$

**If** $\bar{a}_{is} \leq 0$ **for all i, then the solution is unbounded.**

# Simplex Method (Max Form)

**Step 0.** The problem is in canonical form and $\bar{b} \geq 0$.

**Step 1.** If $-\bar{c} \geq 0$ then stop. The solution is optimal. If we continue, then there exists some $-\bar{c}_j < 0$.

**Step 2.** Choose any non-basic variable to pivot in with $-\bar{c}_s < 0$, e.g., $-\bar{c}_s = \min_j \{ -\bar{c}_j \mid -\bar{c}_j < 0 \}$. If $\bar{a}_{is} \leq 0$ for all i, then stop; the LP is unbounded. If we continue, then there exists some $\bar{a}_{is} > 0$.

**Step 3.** Pivot out the basic variable in row r, where r is chosen by the min ratio rule, that is $r = \operatorname{argmin}_i ( \bar{b}_i / \bar{a}_{is} : \bar{a}_{is} > 0 )$.

**Step 4.** Replace the basic variable in row r with variable $x_s$ and re-establish canonical form (i.e., pivot on the coefficient $\bar{a}_{rs}$.)

**Step 5.** Go to Step 1.

# Preview of what is to come

- Obtaining an initial canonical form
- Degeneracy and improving solutions
- Proving finiteness and optimality under no degeneracy
- Handling degeneracy

# The Phase 1 method to obtain an initial bfs

**We need an initial bfs.**

**Approach: create a new LP that is related to the original LP and has the following features:**

1. **It is easy to find a bfs for the new LP**
2. **An optimal bfs for the new LP is a bfs for the original problem.**

# Step 0.  Start with a tableau for the original LP

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | 3 | -2 | -4 | -1 | = | 0 |
| 0 | -3 | 3 | 2 | 5 | = | 6 |
| 0 | -4 | 2 | 1 | 3 | = | 2 |

**Note:  this is different than our running example from the last lecture.**

# Step 1. Write the constraints of the original LP

| $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| -3 | 3 | 2 | 5 | = | 6 |
| -4 | 2 | 1 | 3 | = | 2 |

**FACT: Once we find a basic feasible solution, we can reconsider the original cost coefficients.**

## Step 2. Create new variables called "*artificial variables*" which will be the basic variables for the new LP.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | | |
|---|---|---|---|---|---|---|---|
| -3 | 3 | 2 | 5 | 1 | 0 | = | 6 |
| -4 | 2 | 1 | 3 | 0 | 1 | = | 2 |

**We will choose an objective function soon.**

## Step 3. Choose an objective so that the artificial variables will be 0 in an optimal solution, assuming that there is a bfs for the original problem.

| w | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 1 | 1 | = | 0 |
| 0 | -3 | 3 | 2 | 5 | 1 | 0 | = | 6 |
| 0 | -4 | 2 | 1 | 3 | 0 | 1 | = | 2 |

**The objective function is $w = -x_5 - x_6$**

**max w = 0 if and only if there is a feasible solution for the original problem**

**We use "w" to not confuse it "z".**

## Step 4. Subtract constraints 1, 2 (etc) to the objective constraint so that the tableau becomes canonical

| $w$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 7 | -5 | -3 | -8 | 0 | 0 | = | -8 |
| 0 | -3 | 3 | 2 | 5 | 1 | 0 | = | 6 |
| 0 | -4 | 2 | 1 | 3 | 0 | 1 | = | 2 |

# Step 5. Solve the Phase 1 Problem using the simplex algorithm.

| BV | w | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | | |
|---|---|---|---|---|---|---|---|---|---|
| w | 1 | 0 | 0 | 0 | 0 | -1 | -1 | = | 0 |
| $x_1$ | 0 | 1 | 0 | 1/6 | 1/6 | 1/3 | -1/2 | = | 1 |
| $x_2$ | 0 | 0 | 1 | 5/6 | 11/6 | 2/3 | -1/2 | = | 3 |

# Step 6. Eliminate the artificial variables, and put back in the original objective function.

| BV | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|---|
| z | 1 | 3 | -2 | -4 | -1 | = | 0 |
| $x_1$ | 0 | 1 | 0 | 1/6 | 1/6 | = | 1 |
| $x_2$ | 0 | 0 | 1 | 5/6 | 11/6 | = | 3 |

# Step 7. Add multiples of constraints (1) (2), etc to get the tableau in canonical form

| BV | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS |
|---|---|---|---|---|---|---|---|
| z | 1 | 0 | 0 | -17/6 | 13/6 | = | 3 |
| $x_1$ | 0 | 1 | 0 | 1/6 | 1/6 | = | 1 |
| $x_2$ | 0 | 0 | 1 | 5/6 | 11/6 | = | 3 |

**Subtract 3 times constraint 1 from the objective. Add 2 times constraint 2.**

# Step 8. Solve the original LP starting with the previous bfs

| BV | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | 0 | 3.4 | 0 | 8.4 | = | 13.2 |
| $x_1$ | 0 | 1 | -.2 | 0 | -.2 | = | .4 |
| $x_2$ | 0 | 0 | 1.2 | 1 | 2.2 | = | 3.6 |

**This tableau was reached after one more pivot**

# Phase 1 method: a summary

1. Write the constraints of the original LP
2. Create new variables called "*artificial variables*"
3. New objective: Minimize the sum of the artificials
4. Put the tableau in canonical form
5. Solve the Phase 1 Problem
6. Write the old objective function in the optimal tableau for the phase 1 problem. (delete artificials too)
7. Put the tableau in canonical form
8. Solve the original LP starting with the feasible bfs that you have obtained.

# More on the phase 1 method

What happens if w < 0 at the end of phase 1?

– It means the original problem is infeasible

Is it possible that w = 0 at the end of phase 1 and there are still artificial variables in the basis?

– Yes, for technical reasons; but it is possible to find a new bfs with w = 0 and without the artificial variables. So, there is no real problem.

- The phase 1 method is widely used.
- There are “bells and whistles” to speed it up.
- An alternative approach: the big M method

# The Big M-Method

- Starts off the same as the Phase 1 approach.
- It creates a new problem with artificial variables.

| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
|---|---|---|---|---|---|

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| -3 | 3 | 2 | 5 | 1 | 0 | = | 6 |
| -4 | 2 | 1 | 3 | 0 | 1 | = | 2 |

# The Big M-Method, continued

- **Choose original objective function but penalize artificial variables**

**maximize $z = -3x_1 + 2x_2 + 4x_3 + x_4 - 99x_5 - 99x_6$**

| $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | | |
|---|---|---|---|---|---|---|---|---|
| 1 | 3 | -2 | -4 | -1 | 99 | 99 | = | 0 |
| 0 | -3 | 3 | 2 | 5 | 1 | 0 | = | 6 |
| 0 | -4 | 2 | 1 | 3 | 0 | 1 | = | 2 |

**If the profit is sufficiently negative, then an optimal solution will set $x_5 = x_6 = 0$.**

# Big M Method continued.

- Then one gets the basis into canonical form, and pivots to obtain an optimal solution to the new problem, which will also be optimal for the original problem.

- If some artificial variable is in the basis at the end, then make its profit even more negative and continue.

- If an artificial variable is positive no matter what we do, then there was no feasible solution to the original problem.

# Is the Simplex Algorithm Finite?

- The number of basic feasible solutions is at most n! / (n-m)! m! , which is the number of ways of selecting m basic variables out of n.

- It is finite if we can guarantee that a sequence of pivots cannot lead back to the same bfs

- It is finite if we can guarantee that each bfs has a strictly better objective than the last bfs.

- The only thing preventing objective values from improving is "degeneracy."

# An improving pivot

| BV | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | 3 | -2 | 0 | 0 | = | 2 |
| $x_3$ | 0 | -3 | 3 | 1 | 0 | = | 6 |
| $x_4$ | 0 | -4 | 2 | 0 | 1 | = | 2 |

$z = 2 + 2\Delta$
$x_1 = 0$
$x_2 = \Delta$
$x_3 = 6 - 3\Delta$
$x_4 = 2 - 2\Delta$

**Suppose that variable $x_2$ enters the basis.**

**$\Delta = 1$.      The objective value will increase by 2.**

**If the RHS is strictly positive, the objective will improve.**

# A non-improving (degenerate) pivot

| BV | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS |
|---|---|---|---|---|---|---|---|
| z | 1 | 3 | -2 | 0 | 0 | = | 2 |
| $x_3$ | 0 | -3 | 3 | 1 | 0 | = | 6 |
| $x_4$ | 0 | -4 | 2 | 0 | 1 | = | 0 |

$z = 2 + 2\Delta$
$x_1 = 0$
$x_2 = \Delta$
$x_3 = 6 - 3\Delta$
$x_4 = 0 - 2\Delta$

Suppose that variable $x_2$ enters the basis.

$\Delta = 0.$ The solution stays the same.

But the basis will change.

# Degeneracy

| BV | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|---|
| z | 1 | 3 | -2 | 0 | 0 | = | 2 |
| $x_3$ | 0 | -3 | 3 | 1 | 0 | = | 6 |
| $x_4$ | 0 | -4 | 2 | 0 | 1 | = | 0 |

A bfs is ***degenerate*** if $\bar{b}_j = 0$ for some j. Otherwise, it is ***non-degenerate***.

This bfs is *degenerate*.

In a degenerate pivot, the solution may stay the same.

# A degenerate pivot

| BV | $z$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|---|
| $z$ | 1 | -1 | 0 | 0 | 1 | = | 2 |
| $x_3$ | 0 | 3 | 0 | 1 | -3/2 | = | 6 |
| $x_4$ | 0 | -2 | 1 | 0 | 1/2 | = | 0 |

**The entering variable is $x_2$.**
**The exiting variable is the one in constraint 2.**

**In this case <u>the solution did not change</u>.**
**(But the tableau did.)**

**In a non-degenerate pivot, $z_0$ increases.**

***Theorem***. **If every basic feasible solution is non-degenerate, the simplex method is finite.**

**Fact. If bases can be degenerate, it is possible that the simplex method will cycle, that is repeat bases in a periodic manner.**

# Overview on how to make the simplex method finite.

- “Reduce it to a previously solved problem”

- Replace the LP by a similar LP that has the following features:
  - The RHS of the new problem is nearly the same as the original problem. It differs by an almost infinitesimal amount
  - every feasible basis for the perturbed problem is also a feasible basis for the original problem
  - an optimal basis for the perturbed problem is also an optimal basis the original problem

# Creating the new LP

| BV | z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | RHS | |
|---|---|---|---|---|---|---|---|---|
| z | 1 | 3 | -2 | 0 | 0 | = | 0 | |
| $x_3$ | 0 | -3 | 3 | 1 | 0 | = | 6 | $+\varepsilon$ |
| $x_4$ | 0 | -4 | 2 | 0 | 1 | = | 2 | $+\varepsilon^2$ |

**In theory $\varepsilon$ must be absurdly small (less than $10^{-100}$) for this technique to work.**

**In practice, the simplex algorithm doesn't repeat bases, and so they don't worry about implementing this approach.**

**Perturbations will not be on the quiz, exam, or on homeworks.**

| z | $x_1$ | $x_2$ | $x_3$ | $x_4$ | | |
|---|---|---|---|---|---|---|
| 1 | a | 0 | 0 | 0 | = | 3 |
| 0 | b | 0 | 1 | 0 | = | c |
| 0 | 2 | 1 | 0 | 0 | = | 3 |
| 0 | -1 | 0 | 0 | 1 | = | 5 |

**To do with your partner (4 minutes)**

**Give conditions on a, b and c so that the following are true.**

1. **The current bfs is the unique optimal solution.**
2. **The current bfs is optimal, but there are alternative optima.**
3. **The current bfs is degenerate.**
4. **The current bfs is non-optimal, and $x_3$ is pivoted out at the next iteration.**

# Some Remarks on Degeneracy and Alternative Optima

- It might seem that degeneracy would be rare. After all, why should we expect that the RHS would be 0 for some variable?
  - In reality, degeneracy is incredibly common.

- The simplex method is not necessarily finite unless care is taken in the pivot rule.
  - In reality, almost no one takes care, and the simplex method is not only finite, but it is incredibly efficient.

- The issue of alternative optima arises frequently, and is of importance in practice.

# Overview

- **The simplex method has been a huge success in optimization.**
  - It solves linear programs efficiently
  - We can solve problems with millions of variables
  - It can be a starting point for problems that are not linear

- **The simplex method requires some simple techniques to get started**
  - Transformation into standard form
  - Big M method or variant

- **In practice it requires lots of "engineering"**
  - numerical stability
  - choosing pivot rules that are fast in practice
  - carrying out fast linear algebra
  - CPLEX is one of several good LP codes

# Summary of today's lecture

- **The simplex method starts with an initial bfs and (under non-degeneracy) each pivot improves the objective function.**
  - **It is very effective in practice.**
  - **very good rules for choosing the entering variable**
  - **very good implementations in practice to speed up the linear algebra**
- **Degeneracy: 0's in RHS can lead to no improvement in the objective function.**
- **0's in the "reduced costs" can lead to alternative optima**
- **An artificial variable with a large negative cost can be used to create the first bfs. (Or use phase 1.)**

# Lecture check problems

- **Section 4.6.  Problems 1 and 2**
- **Section 4.7.  Problem 4**
- **Section 4.8.  Problem 1**
- **Section 4.11.  Problem 2**
- **Section 4.12.  Problem 4**