15.053 February 15, 2005

- **Introduction to Simplex Algorithm**

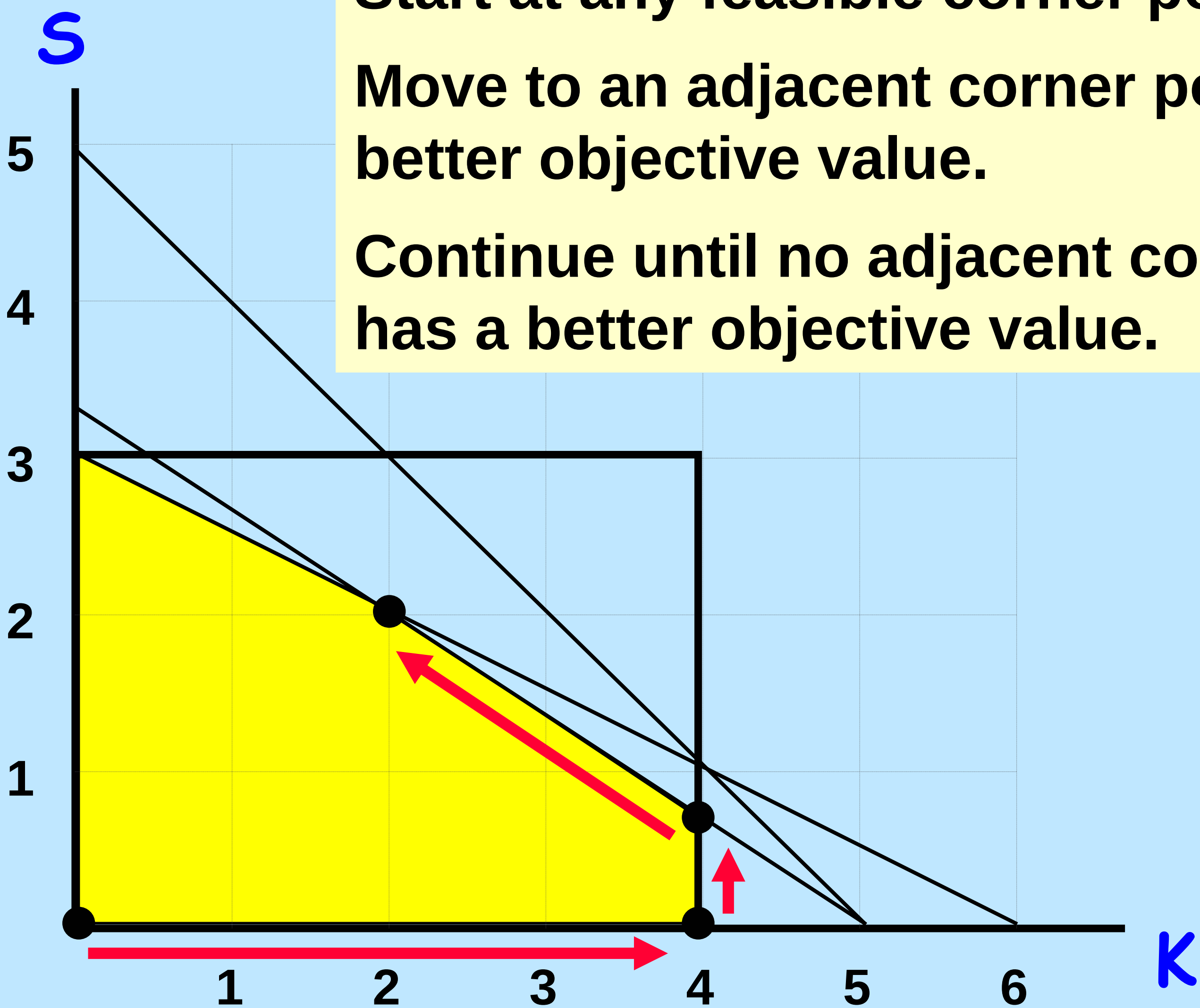
Preview of the Simplex Method

Maximize $z = 3K + 5S$

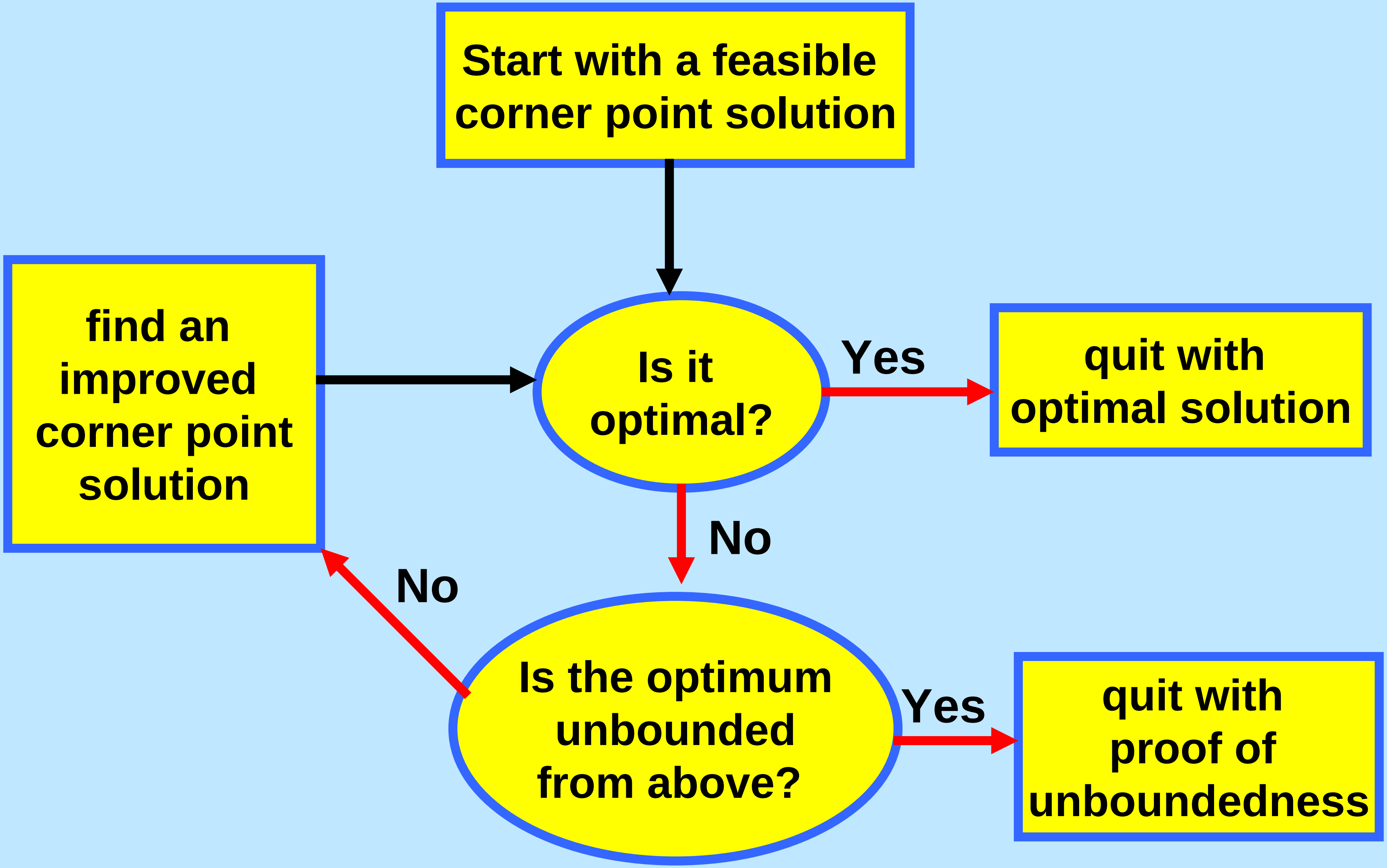
Start at any feasible corner point.

Move to an adjacent corner point with better objective value.

Continue until no adjacent corner point has a better objective value.



The simplex algorithm (for max problems)



Goals for this lecture

Major Issues of the Simplex Algorithm

1. **How does one get the LP into the correct starting form?**
2. **How does one recognize optimality and unboundedness?**
3. **How does one move to the next corner point solution?**

Next: getting a linear program into nearly into the starting form.

Linear Programs in Standard Form

We say that a linear program is *in standard form* if the following are all true:

1. **Non-negativity constraints for all variables.**
2. **All remaining constraints are expressed as equality constraints.**
3. **The right hand side vector, *b*, is non-negative.**

An LP not in Standard Form

maximize $z = 3x_1 + 2x_2 - x_3 + x_4$

$x_1 + 2x_2 + x_3 - x_4 \leq 5$; **not equality**

$-2x_1 - 4x_2 + x_3 + x_4 \leq -1$; **not equality**

$x_1 \geq 0, x_2 \geq 0$ **x_3 and x_4 may be negative**

Converting Inequalities into Equalities Plus Non-negatives

Before

$x_1 + 2x_2 + x_3 - x_4 \leq 5$

After

$x_1 + 2x_2 + x_3 - x_4 + s_1 = 5$

$s_1 \geq 0$

s_1 is called a *slack variable*, which measures the amount of "unused resource."

Note that $s_1 = 5 - x_1 - 2x_2 - x_3 + x_4$.

To convert a "≤" constraint to an equality, add a slack variable.

Converting RHS and "$\geq$" constraints

- Consider the inequality $-2x_1 - 4x_2 + x_3 + x_4 \leq -1$;
- Step 1. Eliminate the negative RHS

$$2x_1 + 4x_2 - x_3 - x_4 \geq 1$$

- Step 2. Convert to an equality

$$2x_1 + 4x_2 - x_3 - x_4 - s_2 = 1$$

$$s_2 \geq 0$$

- The variable added will be called a "***surplus variable***."

To convert a "$\geq$" constraint to an equality, subtract a surplus variable.

More Transformations

How can one convert a minimization problem to a maximization problem?

Example: Minimize $z = 3x_1 + 2x_2$

subject to "constraints"

Has the same optimum solution(s) as

Maximize $v = -3x_1 - 2x_2$

subject to "constraints"

The Last Transformations (for now)

Transforming variables that may take on negative values.

maximize $z = 3x_1 + 4x_2 + 5x_3$

subject to $2x_1 - 5x_2 + 2x_3 = 7$

other constraints

$x_1 \leq 0$, x_2 is unconstrained in sign, $x_3 \geq 0$

Transforming x_1: replace x_1 by $y_1 = -x_1$; $y_1 \geq 0$.

max $z = -3y_1 + 4x_2 + 5x_3$

$-2y_1 - 5x_2 + 2x_3 = 7$

$y_1 \geq 0$, x_2 is unconstrained in sign, $x_3 \geq 0$

One can recover x_1 from y_1.

Transforming variables that may take on negative values.

$$\max z = -3y_1 + 4x_2 + 5x_3$$
$$-2y_1 - 5x_2 + 2x_3 = 7$$
$$y_1 \geq 0,\ x_2 \text{ is unconstrained in sign},\ x_3 \geq 0$$

e.g., $y_1 = 1$, $x_2 = -1$, $x_3 = 2$ is feasible.

Transforming x_2: replace x_2 by $x_2 = y_3 - y_2$; $y_2 \geq 0$, $y_3 \geq 0$.

$$\max z = -3y_1 + 4(y_3 - y_2) + 5x_3$$
$$-2y_1 - 5y_3 + 5y_2 + 2x_3 = 7$$
$$\text{all vars} \geq 0$$

One can recover x_2 from y_2, y_3.

e.g., $y_1 = 1$, $y_2 = 0$, $y_3 = 1$ $x_3 = 2$ is feasible.

A Class Exercise

- Exercise: transform the following to standard form (maximization):

Minimize $v = x_1 + 3x_2$

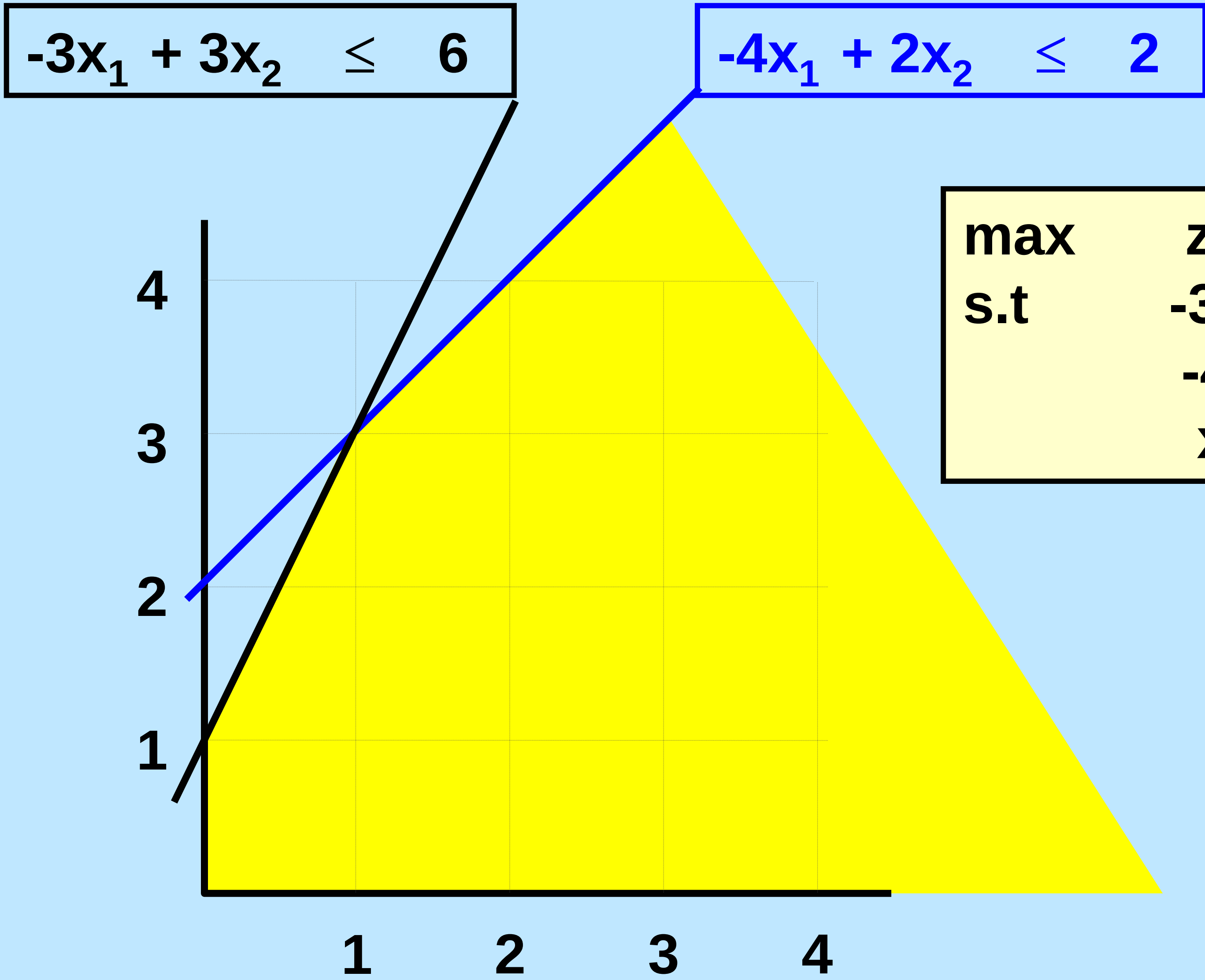
Subject to $2x_1 + 5x_2 \leq 12$

$x_1 + x_2 \geq 1$

$x_1 \geq 0$

Perform the transformation with your partner (2 minutes)

A 2-variable LP: our running example



max	$z =$	$-3x_1 + 2x_2$
s.t		$-3x_1 + 3x_2 \leq 6$
		$-4x_1 + 2x_2 \leq 2$
		$x_1 \geq 0,\ x_2 \geq 0$

A 2-variable LP

maximize $z = -3x_1 + 2x_2$

subject to

$$-3x_1 + 3x_2 \leq 6$$
$$-4x_1 + 2x_2 \leq 2$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

maximize $z = -3x_1 + 2x_2$

subject to

$$-3x_1 + 3x_2 + x_3 = 6$$
$$-4x_1 + 2x_2 + x_4 = 2$$
$$x_1, \quad x_2, \quad x_3, \quad x_4 \quad \geq \quad 0$$

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

LP “canonical form”

An LP is in “*canonical form*” if it is written so that it is in standard form, and such that each constraint has a variable with a coefficient “1” and such that that variable has a 0 in all other constraints.

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The Canonical Form and its Basic Feasible Solution (bfs)

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

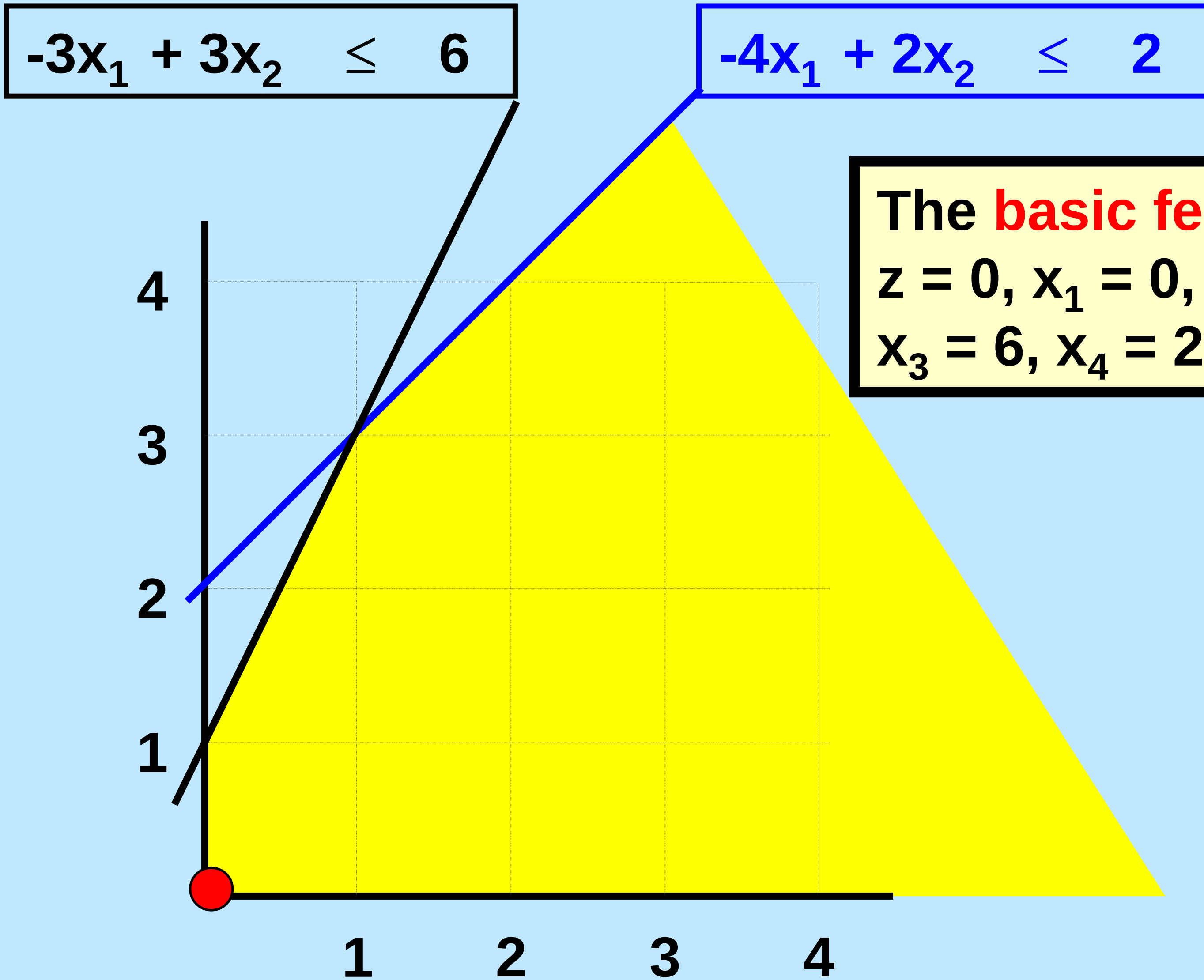
The **basic variables** are z, x_3 and x_4.

The **non-basic variables** are x_1 and x_2.

The **basic feasible solution** (bfs) for this basis is
$z = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 6$, $x_4 = 2$

(set the non-basic variables to 0, and then solve)

Theorem. ***Every basic feasible solution of a linear program in standard form is a corner point solution, and vice versa.***



The **basic feasible solution**

$z = 0$, $x_1 = 0$, $x_2 = 0$,

$x_3 = 6$, $x_4 = 2$

LP Canonical Form and the bfs.

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The text treats z as a basic variable.

The simplex method starts with a *tableau* in *LP canonical form* (or it creates canonical form at a preprocess step.)

The first solution is the bfs for that tableau.

For each constraint there is a basic variable

z	x_1	x_2	x_3	x_4			
1	3	-2	0	0	=	0	Objective function.
0	-3	3	1	0	=	6	Constraint 1
0	-4	2	0	1	=	2	Constraint 2

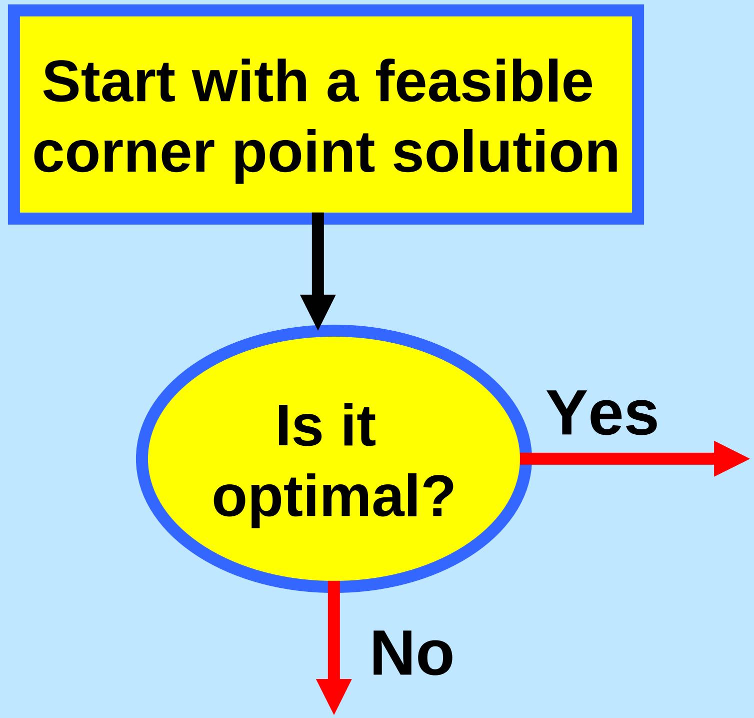
The basic variable for the objective function is always z

Constraint 1: basic variable is x_3

Constraint 2: basic variable is x_4

The basic variables are z , x_3 and x_4

The simplex algorithm (for max problems)



We were lucky to be able to start with a feasible bfs. We now move on to the rest of the algorithm.

Next lecture: how to find a starting bfs

Next: how to recognize optimality.

Recognizing Optimality

(note that the data is different here)

z	x_1	x_2	x_3	x_4		
1	2	4	0	0	=	8
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

Important Fact.

If there is no negative coefficient in the z row, the basic feasible solution is optimal!

$z + 2x_1 + 4x_2 = 8$. And $x_1 \geq 0$ and $x_2 \geq 0$

Therefore, $z \leq 8$.

And z = 8 in the current basic feasible solution

This basic feasible solution is optimal!

Recognizing non-optimality

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

The cost-coefficient of x_2 is -2.

$z + 3x_1 - 2x_2 = 0$

The current bfs can be improved if we can increase x_2 and hold x_1 at 0.

The current basic feasible solution (bfs) is not optimal!

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

If there is a negative coefficient in the z row, the basis is not optimal**

Recall: $z + 3x_1 - 2x_2 = 0$

Increasing x_2 and adjusting x_3 and x_4 and z will improve the solution.

Increase x_2 to $\Delta > 0$. Let x_1 stay at 0.

$z = 2\Delta.$

$x_3 = 6 - 3\Delta.$

$x_4 = 2 - 2\Delta.$

Moving along an edge of the feasible region.

Let $x_2 = \Delta$.

z	x_1	x_2	x_3	x_4		
1	3	-2	0	0	=	0
0	-3	3	1	0	=	6
0	-4	2	0	1	=	2

$z = 2\Delta.$
$x_1 = 0$
$x_2 = \Delta$
$x_3 = 6 - 3\Delta.$
$x_4 = 2 - 2\Delta.$

Choose Δ as large as it can be so that all variables remain non-negative. That is, the solution stays feasible.

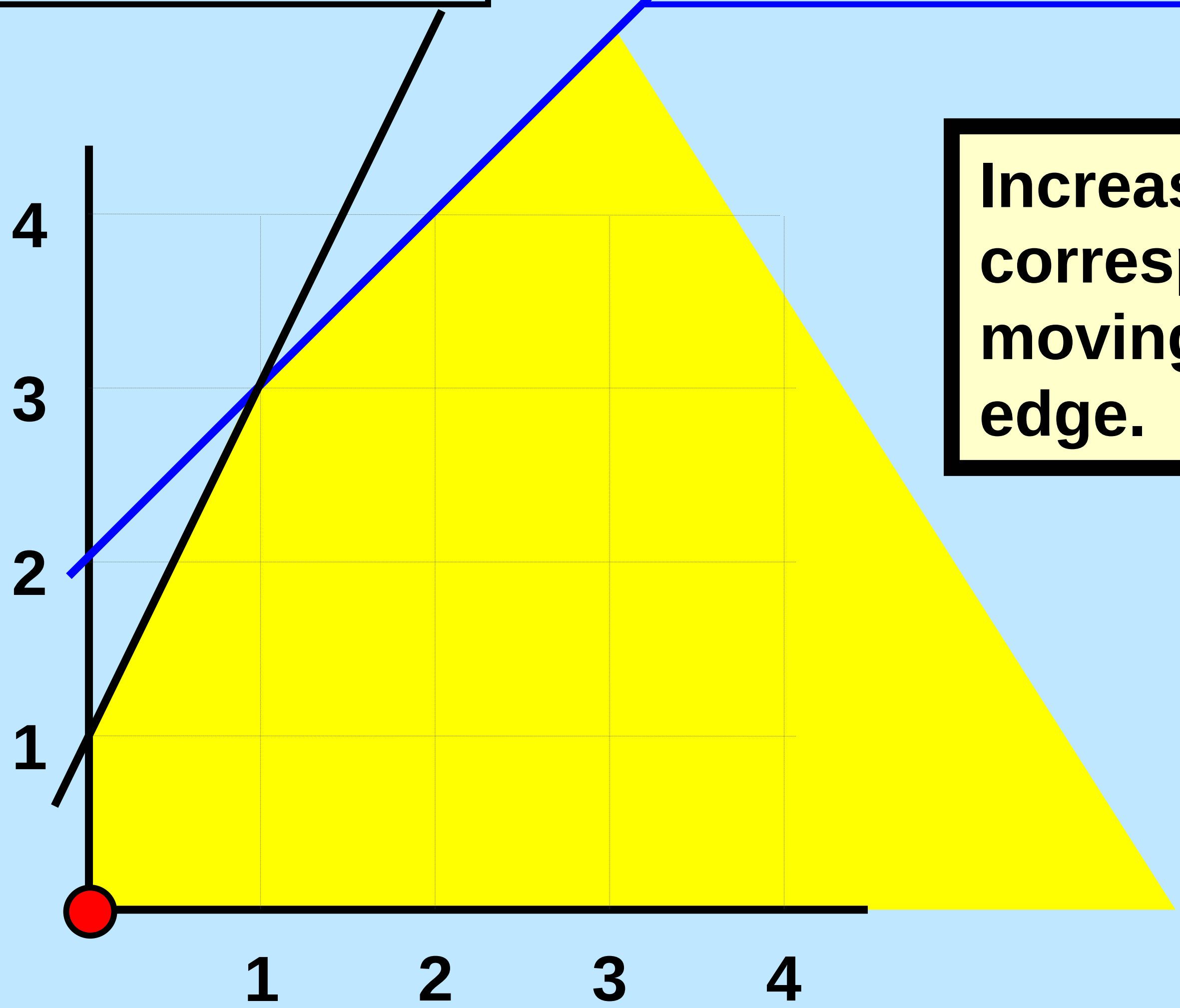
$\Delta = 1.$

Fact. The resulting solution is a basic feasible solution for a different basis.

The two dimensional geometry

$-3x_1 + 3x_2 + x_3 = 6$

$-4x_1 + 2x_2 + x_4 = 2$



Increasing x_2 by Δ corresponds to moving along an edge.

Pivoting to obtain a better solution

New Solution: basic variables z, x_2 and x_3. Nonbasics: x_1 and x_4.

z	x_1	x_2	x_3	x_4		
1	-1	0	0	1	=	2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z = 2$
$x_1 = 0$
$x_2 = 1$
$x_3 = 3$
$x_4 = 0$

Pivot on a coefficient in the matrix. Choose the column of the basic variable that enters. Choose the row of the basic variable that leaves.

If we pivot on the coefficient 2, we obtain the new basic feasible solution.

Summary of Simplex Algorithm

- Start in canonical form with a basic feasible solution

1. Check for optimality conditions
2. If not optimal, determine a non-basic variable that should be made positive
3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs
4. Continue until optimal (or unbounded).

z	x_1	x_2	x_3	x_4		
1	a	0	0	0	=	3
0	b	0	1	0	=	6
0	c	1	0	0	=	3
0	d	0	0	1	=	5

To do with your partner (2 minutes)

The values a, b, c, and d are unknown

1. What are the basic variables? What is the current bfs?

2. Under what condition is the current bfs optimal?

To do with your partner (3 minutes)

z	x_1	x_2	x_3	x_4		
1	a	0	0	0	=	3
0	b	0	1	0	=	6
0	c	1	0	0	=	3
0	d	0	0	1	=	5

1. If we set x_1 to Δ, what are x_2, x_3, and x_4, all expressed in terms of Δ.

2. Assume that $b > 0$ and $d < 0$. Under what condition we will set $\Delta = 3/c$?

3. If $\Delta = 3/c$, what coefficient do we pivot on next?

Recognizing Unboundedness

If the non-cost coefficients in the entering column are ≤ 0, then the solution is *unbounded*

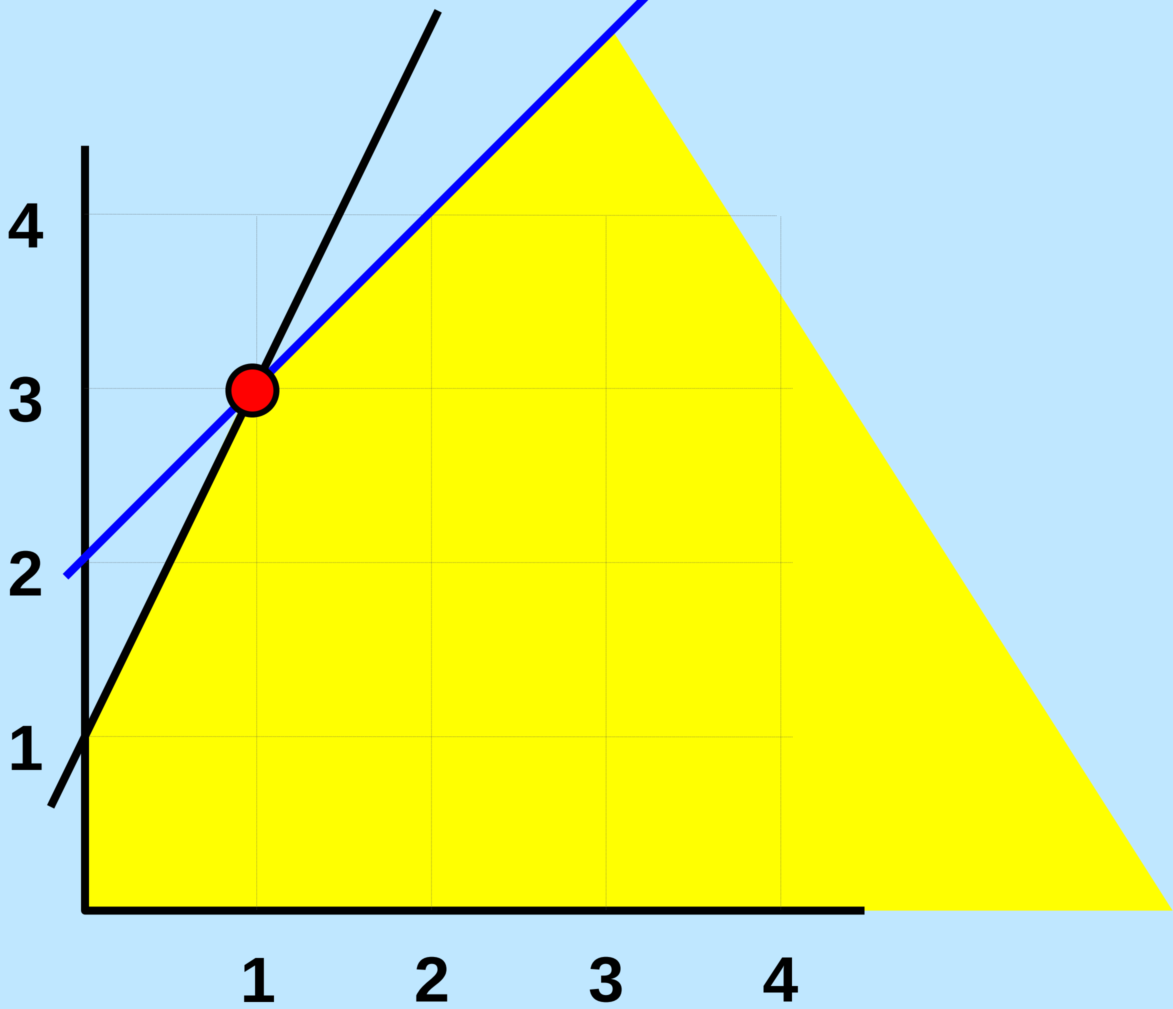
$$z - x_1 + x_4 = 2$$

z	x_1	x_2	x_3	x_4		RHS
1	-1	0	0	1	=	2
0	-3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z = 2 + \Delta$
$x_1 = \Delta$
$x_2 = 1 + 2\Delta$
$x_3 = 3 + 3\Delta.$
$x_4 = 0$

Δ can grow to ∞, and then z goes to ∞.

The two dimensional geometry



Next: two more iterations.
(Then showing the standard shortcut.)

z	x_1	x_2	x_3	x_4		
1	-1	0	0	1	=	2
0	3	0	1	-1.5	=	3
0	-2	1	0	.5	=	1

$z - x_1 + x_4 = 2$

$z = 2 + \Delta$
$x_1 = \Delta$
$x_2 = 1 + 2\Delta$
$x_3 = 3 - 3\Delta.$
$x_4 = 0$

The cost coefficient of x_1 is negative.
Set $x_1 = \Delta$ and $x_4 = 0$.

Then $\Delta = 3/3$.

Another pivot

z	x_1	x_2	x_3	x_4		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	-1/2	=	3

$z = 3$
$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$

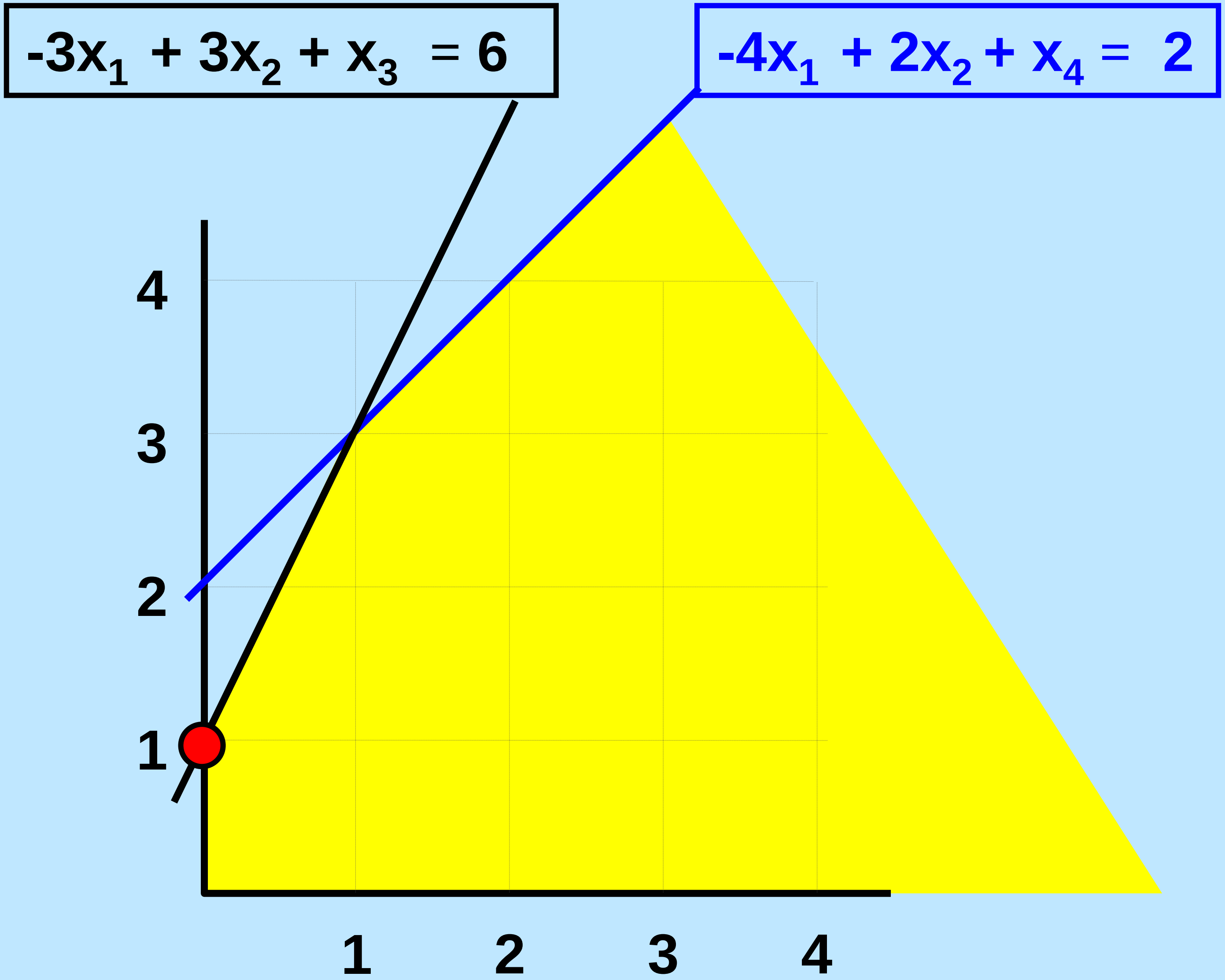
The largest value of Δ is 3/3.

Variable x_1 becomes basis, x_3 becomes nonbasic.

So, x_1 becomes the basic variable for constraint 1.

Pivot on the coefficient with a 3.

The two dimensional geometry



Check for optimality

$$z + x_3/3 + x_4/2 = 3$$

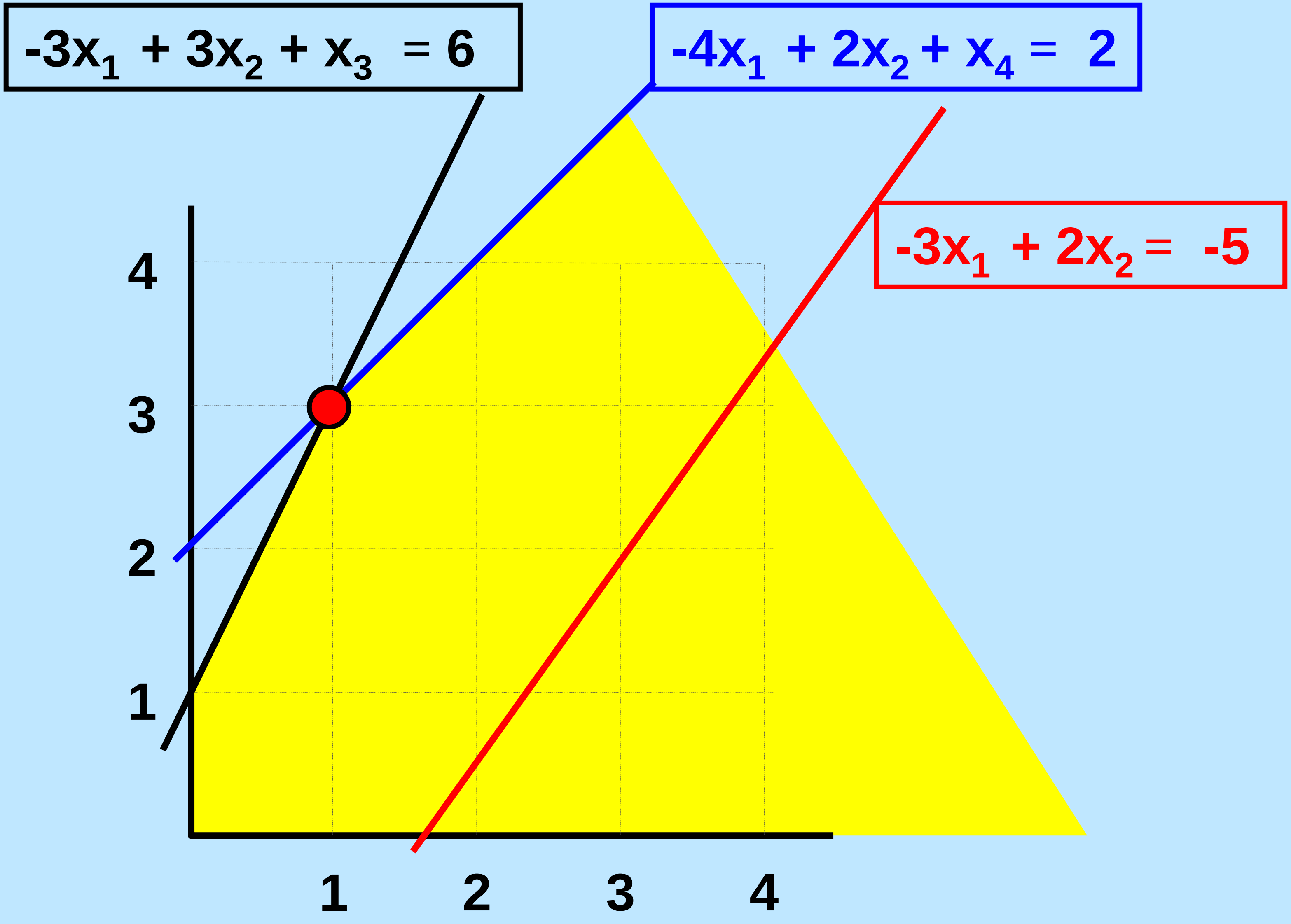
z	x_1	x_2	x_3	x_4		
1	0	0	+1/3	+1/2	=	3
0	1	0	1/3	-1/2	=	1
0	0	1	2/3	1/2	=	3

$z = 3$
$x_1 = 1$
$x_2 = 3$
$x_3 = 0$
$x_4 = 0$

There is no negative coefficient in the z-row.

The current basic feasible solution is optimal!

The two dimensional geometry



Two views of the simplex method

- **Improvement by “moving along an edge.”**
  - **Increase Δ, and increase z.**
  - **An approach used in other algorithms, and that shows what is going on.**

- **Improvement by “moving to an adjacent corner point”**
  - **Move to an adjacent corner point and increase z**
  - **It can be viewed as a “shortcut”**

Summary of Simplex Algorithm Again

- Start in canonical form with a basic feasible solution

1. Check for optimality conditions
   - Is there a negative coefficient in the cost row?
2. If not optimal, determine a non-basic variable that should be made positive
   - Choose a variable with a negative coef. in the cost row.
3. Increase that non-basic variable, and perform a pivot, obtaining a new bfs (or unboundedness)
   - We will review this step, and show a shortcut
4. Continue until optimal (or unbounded).

The Minimum Ratio Rule for determining the leaving variable.

$$z - 3x_1 = 3$$

z	x_1	x_2	x_3	x_4		RHS
1	-3	0	0	0	=	3
0	3	1	0	0	=	6
0	-2	0	1	0	=	1
0	2	0	0	1	=	5

$z = 3 + 2\Delta$

$x_1 = \Delta$

$x_2 = 6 - 3\Delta$

$x_3 = 1 + 2\Delta$

$x_4 = 5 - 2\Delta$

Δ = min (6/3, 5/2). At next iteration, pivot on the 3.

ratio: RHS coefficient/ entering column coefficient

s.t. entering column coefficient is positive

More on performing a pivot

- To determine the column to pivot on, select a variable with a negative cost coefficient

- To determine a row to pivot on, select a coefficient according to a minimum ratio rule

- Carry out a pivot as one does in solving a system of equations.

Next Lecture

- Review of the simplex algorithm
- Formalizing the simplex algorithm
- How to find an initial basic feasible solution, if one exists
- A proof that the simplex algorithm is finite (assuming non-degeneracy)