Mathematical Modeling

Lecture2,3,4

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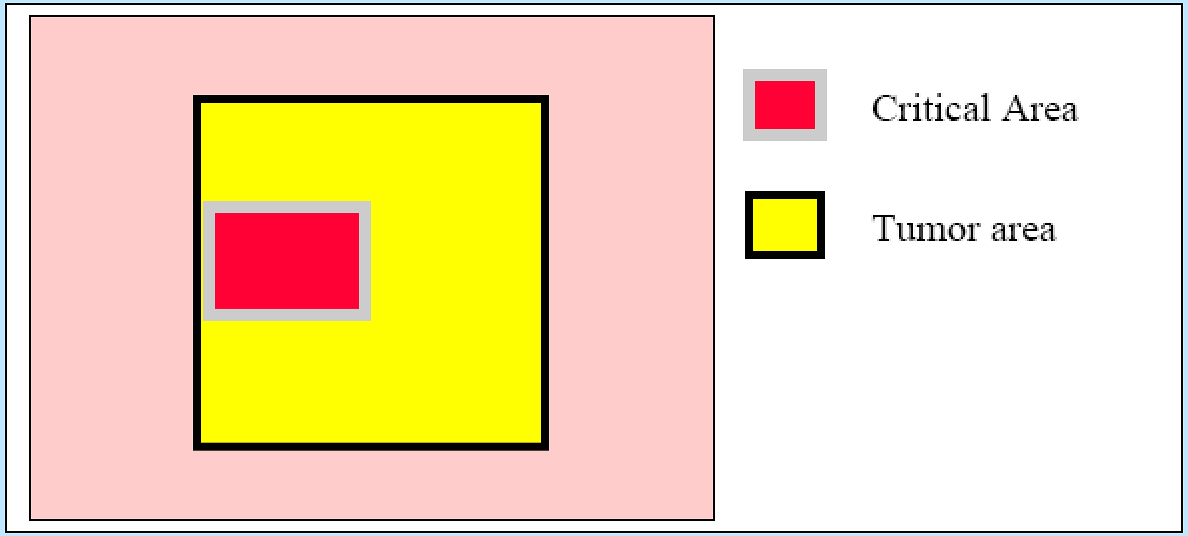
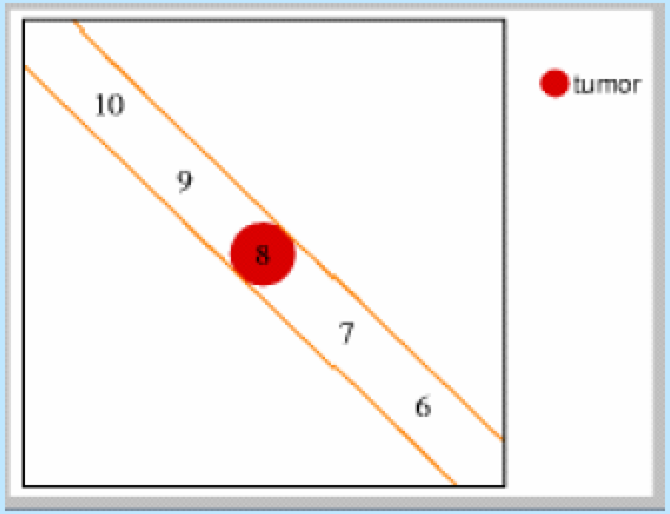
Tomotherapy: a diagram

Goal: maximize the dose to the tumor while minimizing dose to the critical area



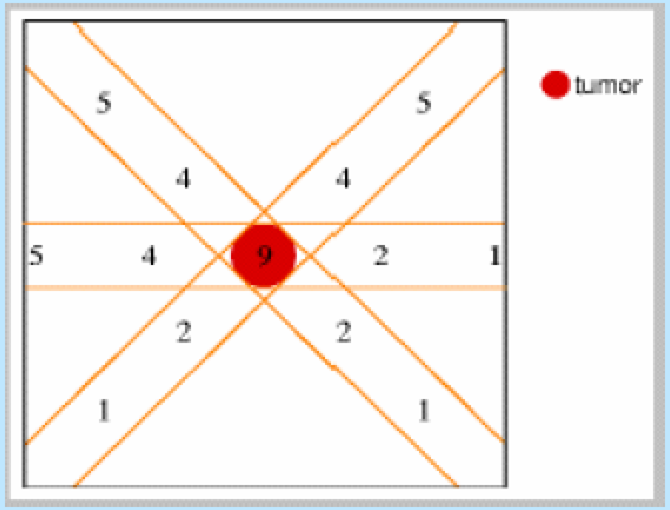
With a small number of beams, it is difficult to achieve these goals.

Conventional Radiotherapy



Relative Intensity of Dose Delivered

Conventional Radiotherapy



Relative Intensity of Dose Delivered

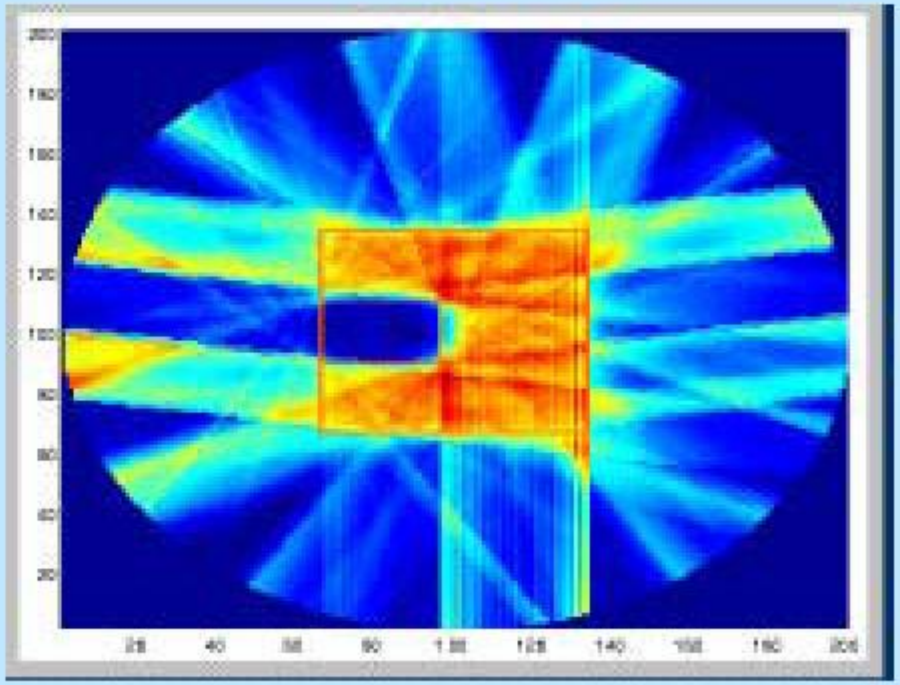
Conventional Radiotherapy

- In conventional radiotherapy
  - 3 to 7 beams of radiation
  - radiation oncologist and physicist work together to determine a set of beam angles and beam intensities
  - determined by manual “trial-and-error” process

Radiation Therapy: Problem Statement

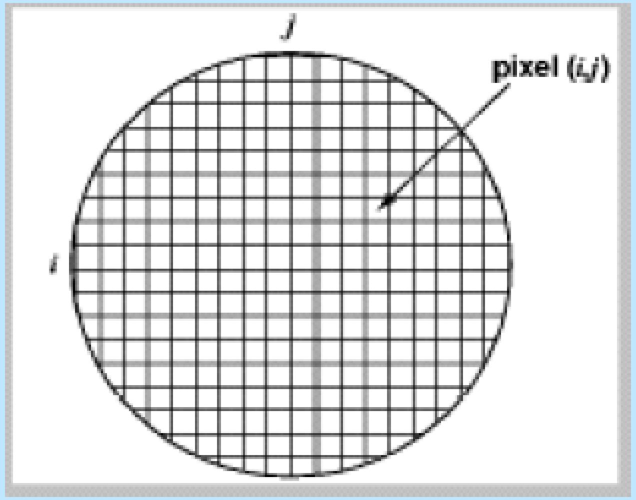
- For a given tumor and given critical areas
- For a given set of possible beamlet origins and angles
- Determine the weight of each beamlet such that:
  - dosage over the tumor area will be at least a target level γ_L .
  - dosage over the critical area will be at most a target level γ_U.

Display of radiation levels



Linear Programming Model

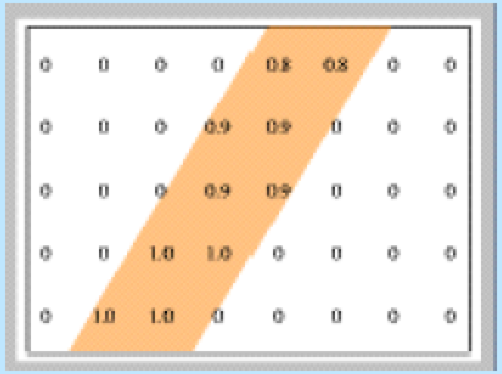
- First, discretize the space
  - Divide up region into a 2D (or 3D) grid of pixels



More on the LP

- Create the beamlet data for each of p = 1, ..., n possible beamlets.
- D^p is the matrix of unit doses delivered by beam p.

D^p_{ij} = unit dose delivered to pixel (i, j) by beamlet p



Linear Program

- Decision variables $w = (w_1, ..., w_p)$
- w_p = intensity weight assigned to beamlet p
  for p = 1 to n;

- D_{ij} = dosage delivered to pixel (i, j)

$$D_{ij} = \sum_{p=1}^{n} D_{ij}^{p} w_p$$

An LP model

took 4 minutes to solve.

minimize $\sum_{(i,j)} D_{ij}$

$$D_{ij} = \sum_{p=1}^{n} D_{ij}^{p} w_p$$

$$D_{ij} \geq \gamma_L \qquad \text{for } (i, j) \in T$$

$$D_{ij} \leq \gamma_U \qquad \text{for } (i, j) \in C$$

$$w_p \geq 0 \qquad \text{for all } p$$

In an example reported in the paper, there were more than 63,000 variables, and more than 94,000 constraints (excluding upper/lower bounds)

What happens if the model is infeasible?

minimize $\sum_{(i,j)} y_{ij}$

Allow the constraint for pixil (i,j) to be violated by an amount y_{ij}, and then minimize the violations.

$$D_{ij} = \sum_{p=1}^{n} D_{ij}^{p} w_p$$

$$D_{ij} + y_{ij} \geq \gamma_L \qquad \text{for } (i,j) \in T$$

$$D_{ij} - y_{ij} \leq \gamma_U \qquad \text{for } (i,j) \in C$$

$$w_p \geq 0 \qquad \text{for all } p$$

$$y_{ij} \geq 0 \qquad \text{for all } (i,j)$$

An even better model

minimize $\sum_{(i,j)} \left(y_{ij}\right)^2$

minimize the sum of squared violations.

$$D_{ij} = \sum_{p=1}^{n} D_{ij}^{p} w_p$$

$$D_{ij} + y_{ij} \geq \gamma_L \qquad \text{for } (i,j) \in T$$

$$D_{ij} - y_{ij} \leq \gamma_U \qquad \text{for } (i,j) \in C$$

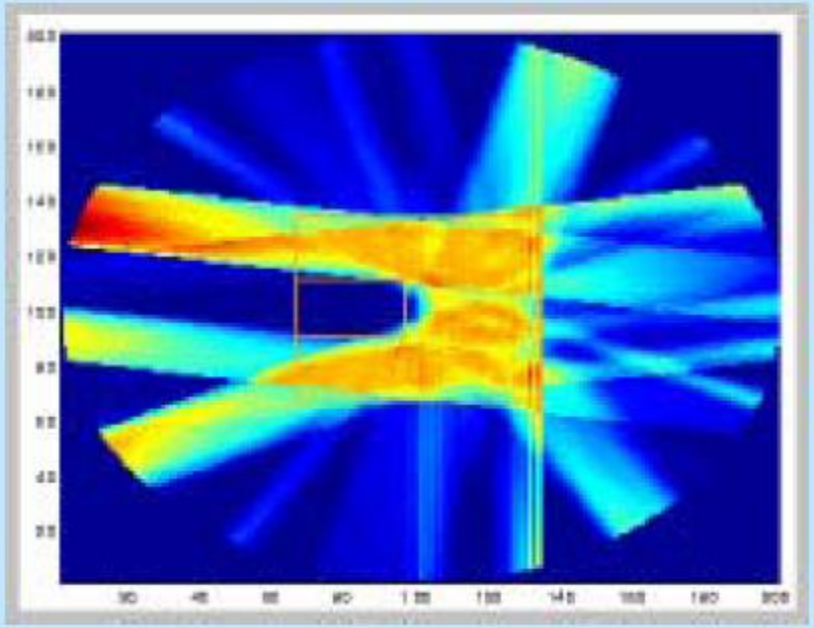
$$w_p \geq 0 \qquad \text{for all } p$$

$$y_{ij} \geq 0 \qquad \text{for all } (i,j)$$

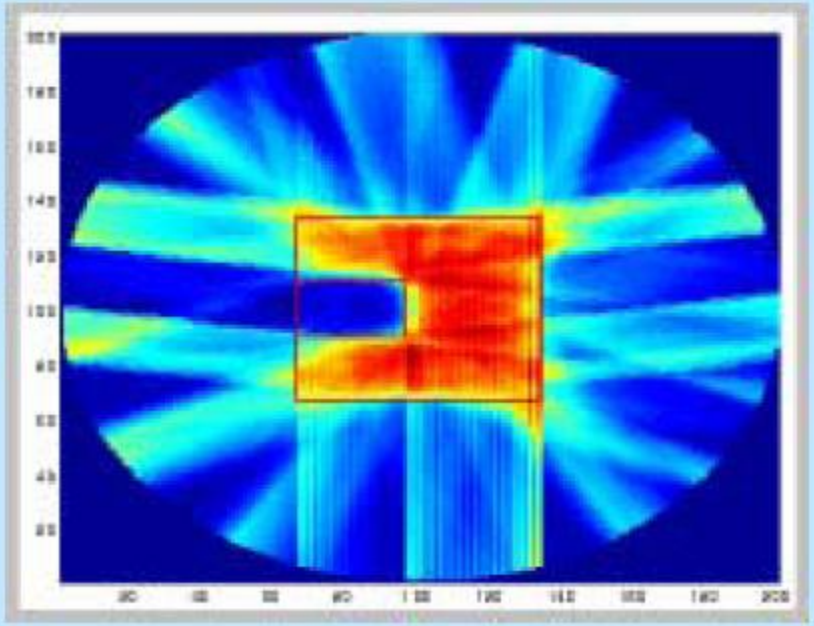
Least squares

This is a nonlinear program (NLP). This one can be solved efficiently.

Optimal Solution for the LP



An Optimal Solution to an NLP



Question: what would be a good diet at McDonalds?

- Suppose that we wanted to design a good 1 week diet at McDonalds. What would we do?
- What data would we need?
- Decision variables?

More on diet problem

- Objective function?

- Constraints?

A simpler problem

- Minimize the cost of a meal
  - just a few choices listed
  - between 600 and 900 calories
  - less than 50% of daily sodium
  - fewer than 40% of the calories are from fat
  - at least 30 grams of protein.
  - fractional meals permitted.

Data from McDonalds
(prices are approximate)

	Hamburger	Big Mac	McChicken	Caesar Salad with Chicken	small French fries
Total Calories	250	770	360	190	230
Fat Calories	81	360	144	45	99
Protein (grams)	31	44	14	27	3
Sodium (mg)	480	1170	800	580	160
Cost	$1.00	$3.00	$2.50	$3.00	$1.00

sodium limit: 2300 mg per day.

LP for McDonalds

$$\text{Minimize} \quad H + 3B + 2.5M + 3C + R$$

$$\text{subject to} \quad 250H + 770B + 360M + 190C + 230R - F = 0$$

$$600 \le F \le 900$$

$$81H + 360B + 144M + 45C + 99R - .4F \le 0$$

$$31H + 44B + 14M + 27C + 3R \ge 30$$

$$480H + 1770B + 800M + 580C + 160R \le 1150$$

$$H, B, M, C, R \ge 0$$

Opt LP Solution: H = 1.13 B = .41 Cost = $2.37

Opt IP Solution: H =1 R = 2 Cost = $3

2.1 TWO-VARIABLE LP MODEL

This section deals with the graphical solution of a two-variable LP. Though two-variable problems hardly exist in practice, the treatment provides concrete foundations for the development of the general simplex algorithm presented in Chapter 3.

Example 2.1-1 (The Reddy Mikks Company)

Reddy Mikks produces both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	Tons of raw material per ton of		Maximum daily availability (tons)
	Exterior paint	*Interior paint*	
Raw material, $M1$	6	4	24
Raw material, $M2$	1	2	6
Profit per ton ($1000)	5	4	

A market survey indicates that the daily demand for interior paint cannot exceed that for exterior paint by more than 1 ton. Also, the maximum daily demand for interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

The LP model, as in any OR model, has three basic components.

1. **Decision variables** that we seek to determine.
2. **Objective** (goal) that we need to optimize (maximize or minimize).
3. **Constraints** that the solution must satisfy.

The proper definition of the decision variables is an essential first step in the development of the model. Once done, the task of constructing the objective function and the constraints becomes more straightforward.

For the Reddy Mikks problem, we need to determine the daily amounts to be produced of exterior and interior paints. Thus the variables of the model are defined as

$$x_1 = \text{Tons produced daily of exterior paint}$$

$$x_2 = \text{Tons produced daily of interior paint}$$

To construct the objective function, note that the company wants to *maximize* (i.e., increase as much as possible) the total daily profit of both paints. Given that the profits per ton of exterior and interior paints are 5 and 4 (thousand) dollars, respectively, it follows that

$$\text{Total profit from exterior paint} = 5x_1 \text{ (thousand) dollars}$$

$$\text{Total profit from interior paint} = 4x_2 \text{ (thousand) dollars}$$

Letting z represent the total daily profit (in thousands of dollars), the objective of the company is

$$\text{Maximize } z = 5x_1 + 4x_2$$

Next, we construct the constraints that restrict raw material usage and product demand. The raw material restrictions are expressed verbally as

$$\begin{pmatrix}\text{Usage of a raw material}\\ \text{by both paints}\end{pmatrix} \leq \begin{pmatrix}\text{Maximum raw material}\\ \text{availability}\end{pmatrix}$$

The daily usage of raw material $M1$ is 6 tons per ton of exterior paint and 4 tons per ton of interior paint. Thus

$$\text{Usage of raw material } M1 \text{ by exterior paint} = 6x_1 \text{ tons/day}$$

$$\text{Usage of raw material } M1 \text{ by interior paint} = 4x_2 \text{ tons/day}$$

Hence

$$\text{Usage of raw material } M1 \text{ by both paints} = 6x_1 + 4x_2 \text{ tons/day}$$

In a similar manner,

$$\text{Usage of raw material } M2 \text{ by both paints} = 1x_1 + 2x_2 \text{ tons/day}$$

Because the daily availabilities of raw materials $M1$ and $M2$ are limited to 24 and 6 tons, respectively, the associated restrictions are given as

$$6x_1 + 4x_2 \leq 24 \quad \text{(Raw material } M1\text{)}$$
$$x_1 + 2x_2 \leq 6 \quad \text{(Raw material } M2\text{)}$$

The first demand restriction stipulates that the excess of the daily production of interior over exterior paint, $x_2 - x_1$, should not exceed 1 ton, which translates to

$$x_2 - x_1 \leq 1 \quad \text{(Market limit)}$$

The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons, which translates to

$$x_2 \leq 2 \quad \text{(Demand limit)}$$

An implicit (or "understood-to-be") restriction is that variables x_1 and x_2 cannot assume negative values. The **nonnegativity restrictions,** $x_1 \geq 0$, $x_2 \geq 0$, account for this requirement.

The complete Reddy Mikks model is

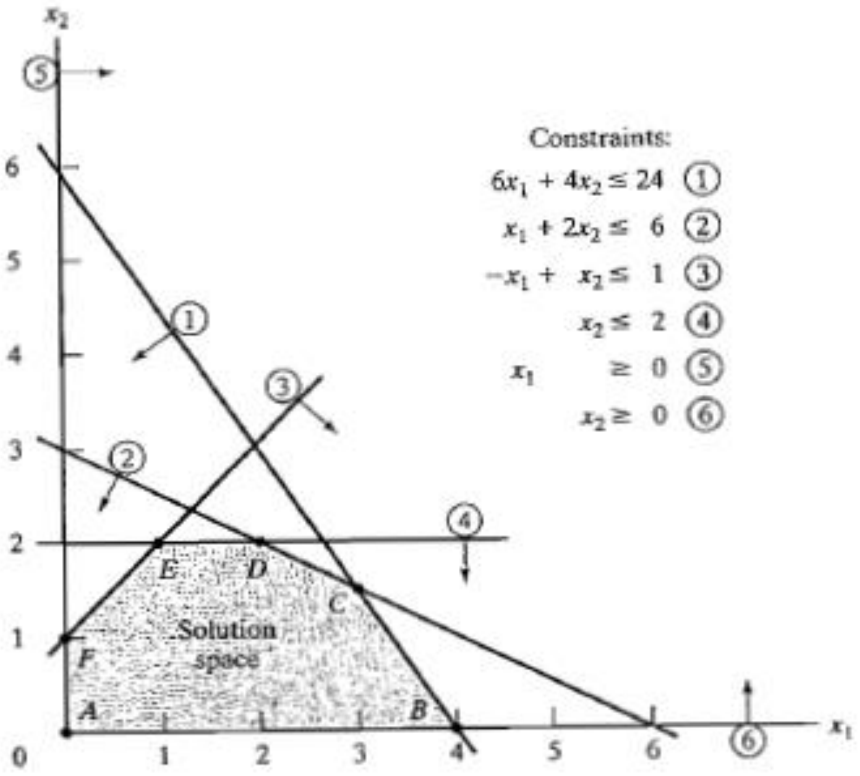
$$\text{Maximize } z = 5x_1 + 4x_2$$

subject to

$$\begin{aligned} 6x_1 + 4x_2 &\leq 24 && (1)\\ x_1 + 2x_2 &\leq 6 && (2)\\ -x_1 + x_2 &\leq 1 && (3)\\ x_2 &\leq 2 && (4)\\ x_1, x_2 &\geq 0 && (5) \end{aligned}$$

FIGURE 2.1

Feasible space of the Reddy Mikks model



Hamdy Taha page 27

2.3 SELECTED LP APPLICATIONS

This section presents realistic LP models in which the definition of the variables and the construction of the objective function and constraints are not as straightforward as in the case of the two-variable model. The areas covered by these applications include the following:

1. Urban planning.
2. Currency arbitrage.
3. Investment.
4. Production planning and inventory control.
5. Blending and oil refining.
6. Manpower planning.

Each model is fully developed and its optimum solution is analyzed and interpreted.

Scheduling Postal Workers

- Each postal worker works for 5 consecutive days, followed by 2 days off, repeated weekly.

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

- Minimize the number of postal workers (for the time being, we will permit fractional workers on each day.)

On the selection of decision variables

- A choice of decision variables that doesn't work
  - Let y_j be the number of workers on day j.
  - No. of Workers on day j is at least d_j. (easy to formulate)
  - Each worker works 5 days on followed by 2 days off (hard).
- Conclusion: sometimes the decision variables incorporate constraints of the problem.
  - Hard to do this well, but worth keeping in mind
  - We will see more of this in integer programming.

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The linear program

Day	Mon	Tues	Wed	Thurs	Fri	Sat	Sun
Demand	17	13	15	19	14	16	11

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$

subject to

$$
\begin{array}{lll}
x_1 + x_4 + x_5 + x_6 + x_7 & \geq 17 & \text{Mon.} \\
x_1 + x_2 + x_5 + x_6 + x_7 & \geq 13 & \text{Tues.} \\
x_1 + x_2 + x_3 + x_6 + x_7 & \geq 15 & \text{Wed.} \\
x_1 + x_2 + x_3 + x_4 + x_7 & \geq 19 & \text{Thurs.} \\
x_1 + x_2 + x_3 + x_4 + x_5 & \geq 14 & \text{Fri.} \\
x_2 + x_3 + x_4 + x_5 + x_6 & \geq 16 & \text{Sat.} \\
x_3 + x_4 + x_5 + x_6 + x_7 & \geq 11 & \text{Sun.}
\end{array}
$$

$x_j \geq 0$ for $j = 1$ to 7

A Modifications of the Model

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- Suppose that there was a pay differential. The cost of each worker who works on day j is c_j. The new objective is to minimize the total cost.

What is the objective coefficient for the shift that starts on Monday for the new problem?

1. c_1
2. $c_1 + c_2 + c_3 + c_4 + c_5$
3. $c_1 + c_4 + c_5 + c_6 + c_7$

A Different Modification of the Model

- Suppose that there is a penalty for understaffing and penalty for overstaffing. If you hire k too few workers on day j, the penalty is 5 k^2. If you hire k too many workers on day j, then the penalty is k^2. How can we model this?

Step 1. Create new decision variables.

Let e_j = "excess workers on day j"

Let d_j = "deficit workers on day j"

Model 2

Minimize $5\sum_{i=1}^{7} d_i^2 + \sum_{i=1}^{7} e_i^2$

$$
\begin{array}{lllllllllll}
x_1 + & & & x_4 + & x_5 + & x_6 + & x_7 + & d_1 - e_1 & = 17 \\
x_1 + & x_2 + & & & x_5 + & x_6 + & x_7 + & d_2 - e_2 & = 13 \\
x_1 + & x_2 + & x_3 + & & & x_6 + & x_7 + & d_3 - e_3 & = 15 \\
x_1 + & x_2 + & x_3 + & x_4 + & & & x_7 + & d_4 - e_4 & = 19 \\
x_1 + & x_2 + & x_3 + & x_4 + & x_5 & & + & d_5 - e_5 & = 14 \\
 & x_2 + & x_3 + & x_4 + & x_5 + & x_6 & + & d_6 - e_6 & = 16 \\
 & & x_3 + & x_4 + & x_5 + & x_6 + & x_7 + & d_7 - e_7 & = 11
\end{array}
$$

$x_j \geq 0, d_j \geq 0, e_j \geq 0$ for $j = 1$ to 7

What is wrong with this model, other than the fact that variables should be required to be integer valued?

What is wrong with Model 2?

1. The constraints should have inequalities.
2. The constraints don't make sense.
3. The objective is incorrect. (Note: it is OK that it is nonlinear)
4. It's possible that e_j and d_j are both positive.
5. Nothing is wrong.

More Comments on Model 2.

Difficulty: The feasible region permits feasible solutions that do not correctly model our intended constraints. Let us call these ***bad feasible solutions***.
The ***good feasible solutions*** are ones in which $d_1 = 0$ or $e_1 = 0$ or both. They correctly model the scenario.

Resolution: All optimal solutions are good.

Illustration of why it works:

$$10 + 10 + 0 + 0 + 0 + d_1 - e_1 = 17$$

$e_1 = 4$ and $d_1 = 1$ is a bad feasible solution.
$e_1 = 3$ and $d_1 = 0$ are good feasible solution.

For every bad feasible solution, there is a good feasible solution whose objective is better.

More on the model

- Summary: the model permits too many feasible solutions.
- All of the optimal solutions are good.
- We will see this technique more in this lecture, and in other lectures as well.

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On the practicality of these models

- In modeling in practice, one needs to capture a lot of reality (but not too much).
- Workforce scheduling is typically much more complex.
- These models are designed to help in thinking about real workforce scheduling models.

Example 1: The Burroughs garment company manufactures men's shirts and women’s blouses for Walmark Discount stores. Walmark will accept all the production supplied by Burroughs. The production process includes cutting, sewing and packaging. Burroughs employs 25 workers in the cutting department, 35 in the sewing department and 5 in the packaging department. The factory works one 8-hour shift, 5 days a week. The following table gives the time requirements and the profits per unit for the two garments:

Minutes per unit

Garment	**Cutting**	**Sewing**	**Packaging**	**Unit profit($)**
Shirts	20	70	12	8.00
Blouses	60	60	4	12.00

Determine the optimal weekly production schedule for Burroughs.

Solution: Assume that Burroughs produces x_1 shirts and x_2 blouses per week.

$$\text{Profit got} = 8x_1 + 12x_2$$

Time spent on cutting = $20x_1 + 60x_2$ mts

Time spent on sewing = $70x_1 + 60x_2$ mts

Time spent on packaging = $12x_1 + 4x_2$ mts

The **objective** is to find x_1, x_2 so as to

maximize the profit $z = 8\, x_1 + 12\, x_2$

satisfying the **constraints**:

$$20\, x_1 + 60\, x_2 \leq 25 \times 40 \times 60$$

$$70\, x_1 + 60\, x_2 \leq 35 \times 40 \times 60$$

$$12\, x_1 + 4\, x_2 \leq 5 \times 40 \times 60$$

$$x_1, x_2 \geq 0, \text{ integers}$$

Example 2: Wild West produces two types of cowboy hats. Type I hat requires twice as much labor as a Type II. If all the available labor time is dedicated to Type II alone, the company can produce a total of 400 Type II hats a day. The respective market limits for the two types of hats are 150 and 200 hats per day. The profit is $8 per Type I hat and $5 per Type II hat. Formulate the problem as an LPP so as to maximize the profit.

Solution: Assume that Wild West produces x_1 Type I hats and x_2 Type II hats per day.

Per day Profit got = $8x_1 + 5x_2$

Assume the time spent in producing one type II hat is c minutes.

Labour Time spent is $(2x_1 + x_2)c$ minutes

The **objective** is to find x_1, x_2 so as to

maximise the profit $z = 8\, x_1 + 5\, x_2$

satisfying the **constraints**:

$$
\begin{aligned}
(2\, x_1 + x_2)\, c &\le 400\, c \\
x_1 \qquad\quad &\le 150 \\
x_2 &\le 200 \\
x_1, x_2 \ge 0,&\ \text{integers}
\end{aligned}
$$

The **objective** is to find x_1, x_2 so as to

maximise the profit $z = 8\,x_1 + 5\,x_2$

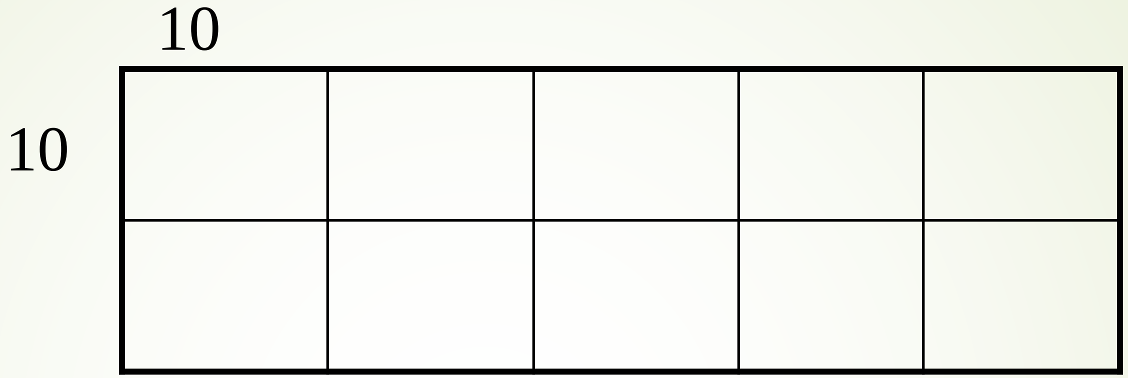
satisfying the **constraints**:

$$
\begin{aligned}
(2\,x_1 + x_2) &\leq 400 \\
x_1 &\leq 150 \\
x_2 &\leq 200 \\
x_1, x_2 &\geq 0, \ \text{integers}
\end{aligned}
$$

Example 3: Trim Loss problem, A company has to manufacture the circular tops of cans. Two sizes, one of diameter 10 cm and the other of diameter 20 cm are required. They are to be cut from metal sheets of dimensions 20 cm by 50 cm. The requirement of smaller size is 20,000 and of larger size is 15,000. The problem is : how to cut the tops from the metal sheets so that the number of sheets used is a minimum. Formulate the problem as a LPP.

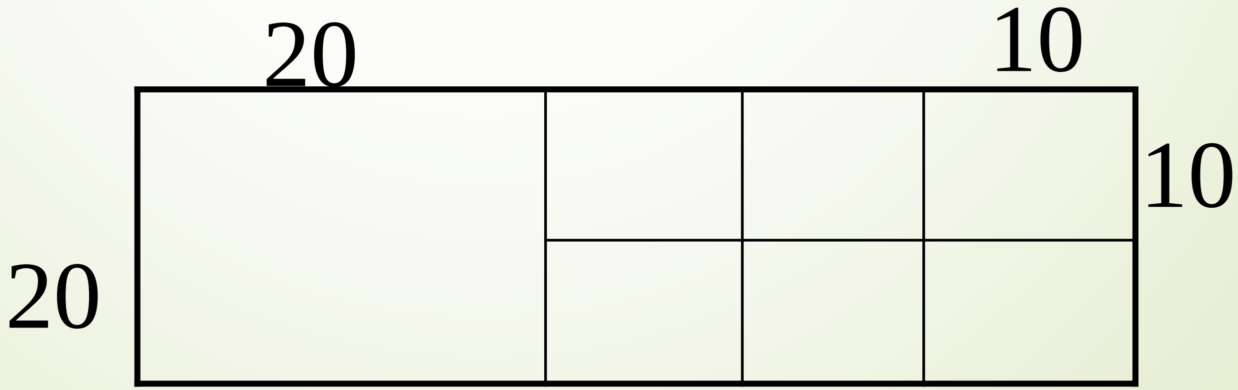
A sheet can be cut into one of the following three patterns:

Pattern I



Pattern II

Pattern III



Pattern I: cut into 10 pieces of size 10 by 10 so as to make 10 tops of size 1

Pattern II: cut into 2 pieces of size 20 by 20 and 2 pieces of size 10 by 10 so as to make 2 tops of size 2 and 2 tops of size 1

Pattern III: cut into 1 piece of size 20 by 20 and 6 pieces of size 10 by 10 so as to make 1 top of size 2 and 6 tops of size 1

So assume that x_1 sheets are cut according to pattern I, x_2 according to pattern II, x_3 according to pattern III

The problem is to

Minimize $z = x_1 + x_2 + x_3$

Subject to $10\,x_1 + 2\,x_2 + 6\,x_3 \geq 20{,}000$

$$2\,x_2 + \quad x_3 \geq 15{,}000$$

$$x_1, x_2, x_3 \geq 0, \text{ integers}$$

Example 5 :BITS wants to host a Seminar for five days. For the delegates there is an arrangement of dinner every day. The requirement of napkins during the 5 days is as follows:

Day	1	2	3	4	5
Napkins Needed	80	50	100	80	150

Institute does not have any napkins in the beginning. After 5 days, the Institute has no more use of napkins. A new napkin costs Rs. 2.00. The washing charges for a used one are Rs. 0.50. A napkin given for washing after dinner is returned the third day before dinner. The Institute decides to accumulate the used napkins and send them for washing just in time to be used when they return. How shall the Institute meet the requirements so that the total cost is minimized ? Formulate as a LPP.

Solution Let x_j be the number of napkins purchased on day j, j=1,2,..,5

Let y_j be the number of napkins given for washing after dinner on day j, j=1,2,3

Thus we must have

$x_1 = 80,\ x_2 = 50,\ x_3 + y_1 = 100,\ x_4 + y_2 = 80$
$x_5 + y_3 = 150$

Also we have

$$y_1 \le 80,\ y_2 \le (80 - y_1) + 50$$

$$y_3 \le (80 - y_1) + (50 - y_2) + 100$$

Thus we have to **Minimize** $z = 2(x_1+x_2+x_3+x_4+x_5)+0.5(y_1+y_2+y_3)$

Subject to

$x_1 = 80$, $x_2 = 50$, $x_3 + y_1 = 100$,

$x_4 + y_2 = 80$, $x_5 + y_3 = 150$,

$y_1 \leq 80$, $y_1+y_2 \leq 130$, $y_1+y_2+y_3 \leq 230$,

all variables ≥ 0, integers

(Simple) Linear regression

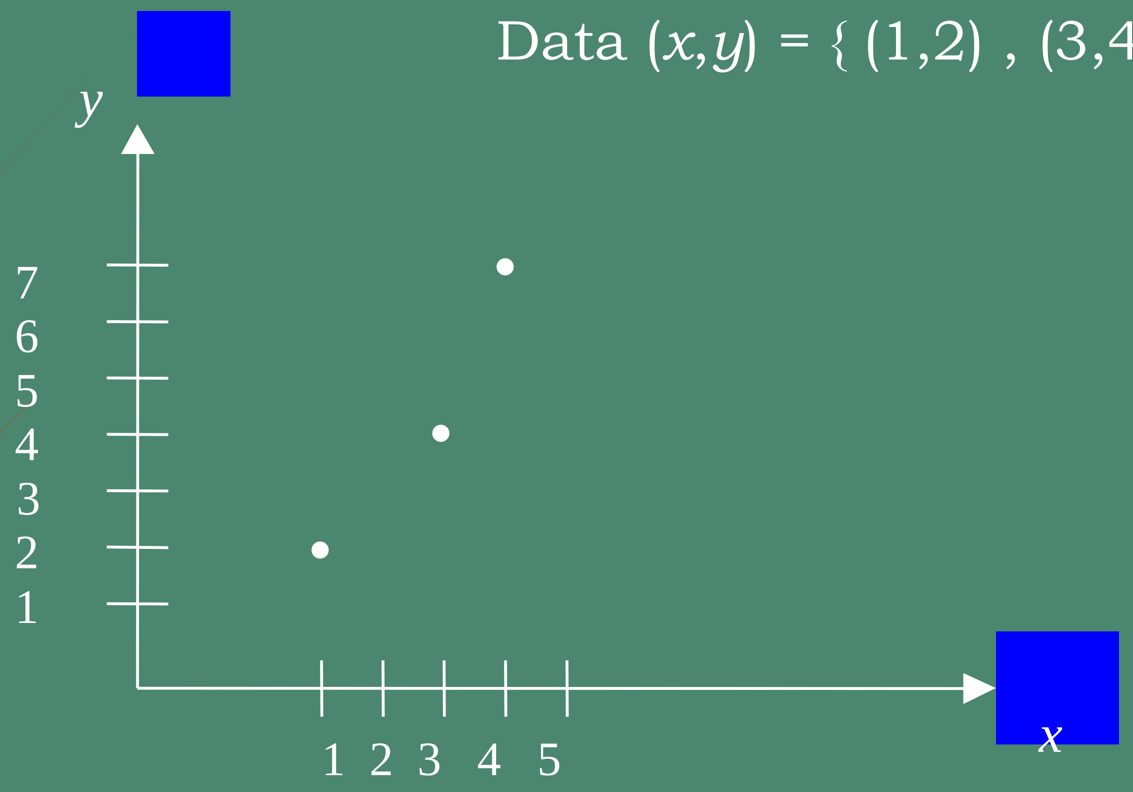
Given a set of datapoints $\{(1,2),(3,4),(4,7)\}$ we want to find a line that most closely represents the datapoints. There are various ways to measure what it means "closely represent". We may, for instance, minimize the average distance (deviation) of the datapoints from the line, or minimize the sum of distances, or the sum of squares of distances, or minimize the maximum distance of a datapoint from the line. Here the distance can be either Euclidean distance, or vertical distance, or Manhattan distance (vertical+horizontal), or other.

We choose to minimize the maximum vertical distance of a point from the line. A general equation of a line with finite slope has form $y = ax + c$ where a and c are parameters. For a point (p,q), the vertical distance of the point from the line $y = ax + c$ can be written as $|q - ap - c|$. Thus we want

Problem: Find constants a, c such that the largest of the three values $|2 - a - c|$, $|4 - 3a - c|$, $|7 - 4a - c|$ is as small as possible.

$$\min \max \left\{ |2 - a - c|,\ |4 - 3a - c|,\ |7 - 4a - c| \right\}$$

Data $(x,y) = \{ (1,2) , (3,4) , (4,7) \}$



We want to "fit" a linear function $y = ax + b$ to these data points; i.e., we have to choose optimal values for a and b.

Objective: Find parameters a and b that minimize the maximum absolute deviation between the data y_i and the fitted line $\hat{y}_i = ax_i + b$.

$$y_i \quad \text{and} \quad \hat{y}_i$$

observed value

Predicted value

In addition, we're going to impose a priori knowledge that the slope of the line must be positive. (We don't know about the intercept.)

Decision variables

a = slope of line ← known to be positive

b = y-intercept ← positive or negative

$$b = b^+ - b^-, \quad b^+ \geq 0, \quad b^- \geq 0$$

Objective function:

Let $w = \max \{ |\hat{y}_i - y_i| : i = 1, 2, 3 \}$

where $\hat{y}_i = ax_i + b$

Optimization model:

$$\text{Min} \quad w$$

$$\text{s.t.} \quad w \geq |\hat{y}_i - y_i| \qquad i = 1, 2, 3$$

Nonlinear constraints:

$$w \geq | \hat{\mathbf{y}}_1 - \mathbf{y}_1 | = | a(1) + b - 2 |$$

$$w \geq | \hat{\mathbf{y}}_2 - \mathbf{y}_2 | = | a(3) + b - 4 |$$

$$w \geq | \hat{\mathbf{y}}_3 - \mathbf{y}_3 | = | a(4) + b - 7 |$$

Convert absolute value terms to linear terms:

Note: $2 \geq |x|$ iff $2 \geq x$ and $2 \geq -x$

Thus $w \geq | \hat{y}_i - y_i |$ is equivalent to

$$w \geq ax_i + b - y_i \quad \text{and} \quad w \geq -ax_i - b + y_i$$

so finally …

$$
\begin{array}{ll}
\text{Min} & w \\
\text{s.t.} & a + b^{+} - b^{-} - w \le 2 \\
& -a - b^{+} + b^{-} - w \le -2 \\
& 3a + b^{+} - b^{-} - w \le 4 \\
& -3a - b^{+} + b^{-} - w \le -4 \\
& 4a + b^{+} - b^{-} - w \le 7 \\
& -4a - b^{+} + b^{-} - w \le -7 \\
& a,\ b^{+},\ b^{-},\ w \ge 0
\end{array}
$$

5.4.3 Financial Portfolio

Problem Definition.

In your finance courses, you will learn a number of techniques for creating "optimal" portfolios. The optimality of a portfolio depends heavily on the model used for defining risk and other aspects of financial instruments. Here is a particularly simple model that is amenable to linear programming techniques.

Consider a mortgage team with $100,000,000 to finance various investments. There are five categories of loans, each with an associated return and risk (1-10, 1 best):

Loan/investment	Return (%)	Risk
First Mortgages	9	3
Second Mortgages	12	6
Personal Loans	15	8
Commercial Loans	8	2
Government Securities	6	1

Any uninvested money goes into a savings account with no risk and 3% return. The goal for the mortgage team is to allocate the money to the categories so as to:

(a) Maximize the average return per dollar

(b) Have an average risk of no more than 5 (all averages and fractions taken over the invested money (not over the saving account)).

(c) Invest at least 20% in commercial loans

(d) The amount in second mortgages and personal loans combined should be no higher than the amount in first mortgages.

Model

Let the investments be numbered 1...5, and let x_i be the amount invested in investment i. Let x_s be the amount in the savings account. The objective is to maximize

$$9x_1 + 12x_2 + 15x_3 + 8x_4 + 6x_5 + 3x_s$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_s = 100,000,000.$$

Now, let's look at the average risk. Since we want to take the average over only the invested amount, a direct translation of this constraint is

$$\frac{3x_1 + 6x_2 + 8x_3 + 2x_4 + x_5}{x_1 + x_2 + x_3 + x_4 + x_5} \leq 5$$

This constraint is not linear, but we can cross multiply, and simplify to get the equivalent linear constraint:

$$-2x_1 + x_2 + 3x_3 - 3x_4 - 4x_5 \leq 0$$

Similarly we need

$$x_4 \geq 0.2(x_1 + x_2 + x_3 + x_4 + x_5)$$

or

$$-0.2x_1 - 0.2x_2 - 0.2x_3 + 0.8x_4 - 0.2x_5 \geq 0$$

The final constraint is

$$x_2 + x_3 - x_1 \leq 0$$

Together with nonnegativity, this gives the entire formulation.

(More) Examples of LP Formulations

1. Employee Scheduling

Macrosoft has a 24-hour-a-day, 7-days-a-week toll free hotline that is being set up to answer questions regarding a new product. The following table summarizes the number of full-time equivalent employees (FTEs) that must be on duty in each time block.

Shift	Time	FTEs
1	0-4	15
2	4-8	10
3	8-12	40
4	12-16	70
5	16-20	40
6	20-0	35

Constraints for Employee Scheduling

- Macrosoft may hire both full-time and part-time employees. The former work 8-hour shifts and the latter work 4-hour shifts; their respective hourly wages are $15.20 and $12.95. Employees may start work only at the beginning of one of 6 shifts.
- At least two-thirds of the employees working at any one time must be full-time employees.
- Part-time employees can only answer 5 calls in the time a full-time employee can answer 6 calls. (i.e., a part-time employee is only 5/6 of a full-time employee.)

Formulate an LP to determine how to staff the hotline at minimum cost.

Decision Variables

x_t = # of full-time employees that begin work in shift t
y_t = # of part-time employees that work shift t

(8×15.20) (4×12.95)

$$\min \quad 121.6\,(x_1 + \cdots + x_6) + 51.8\,(y_1 + \cdots + y_6)$$

s.t.

$$\begin{aligned}
x_6 + x_1 + \tfrac{5}{6} y_1 &\geq 15 \\
x_1 + x_2 + \tfrac{5}{6} y_2 &\geq 10 \\
x_2 + x_3 + \tfrac{5}{6} y_3 &\geq 40 \\
x_3 + x_4 + \tfrac{5}{6} y_4 &\geq 70 \\
x_4 + x_5 + \tfrac{5}{6} y_5 &\geq 40 \\
x_5 + x_6 + \tfrac{5}{6} y_6 &\geq 35
\end{aligned}$$

All shifts must be covered

PT employee is 5/6 FT employee

More constraints:

$$(x_6 + x_1) \geq \frac{2}{3}(x_6 + x_1 + y_1)$$

$$(x_1 + x_2) \geq \frac{2}{3}(x_1 + x_2 + y_2)$$

$$\vdots$$

$$(x_5 + x_6) \geq \frac{2}{3}(x_5 + x_6 + y_6)$$

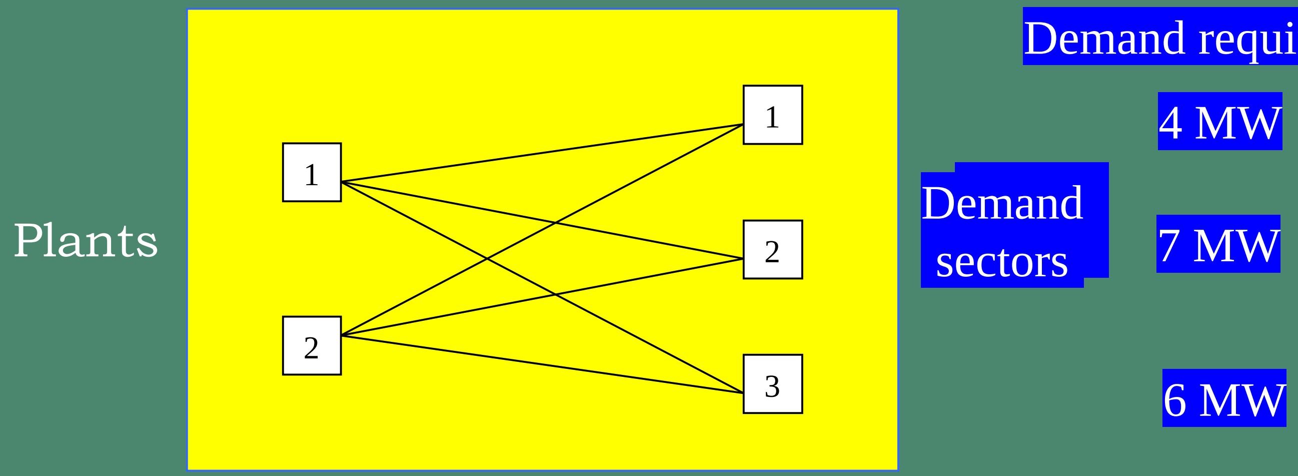
At least 2/3 workers must be full time

$$x_t \geq 0,\ \ y_t \geq 0 \quad t = 1,2,\ \ldots,6$$

Nonnegativity

2. Energy Generation Problem (with piecewise linear objective)

Austin Municipal Power and Light (AMPL) would like to determine optimal operating levels for their electric generators and associated distribution patterns that will satisfy customer demand. Consider the following prototype system



The two plants (generators) have the following (nonlinear) efficiencies:

Plant 1	[0, 6 MW]	[6MW, 10MW]
Unit cost ($/MW)	$10	$25
Plant 2	[0, 5 MW]	[5MW, 11MW]
Unit cost ($/MW)	$8	$28

The table is to be read as follows. For plant #1 if you generate at a rate of 8MW then the cost ($/sec) is = 10(6) + 25(2) = 110.

Problem Statement and Notation

Formulate an LP that, when solved, will yield optimal power generation and distribution levels.

Decision Variables

x_{11}	=	power generated at plant 1 at operating level	1
x_{12}		" " " " 1 " " "	2
x_{21}		" " " " 2 " " "	1
x_{22}		" " " " 2 " " "	2

y_{11}	=	power sent from plant 1 to demand sector	1
y_{12}		" " " " 1 " " "	2
y_{13}		" " " " 1 " " "	3
y_{21}		" " " " 2 " " "	1
y_{22}		" " " " 2 " " "	2
y_{23}		" " " " 2 " " "	3

Formulation

$$\text{Min} \quad 10x_{11} + 25x_{12} + 8x_{21} + 28x_{22}$$

$$\begin{aligned}
\text{s.t.} \quad & y_{11} + y_{12} + y_{13} = x_{11} + x_{12} \\
& y_{21} + y_{22} + y_{23} = x_{21} + x_{22} \\
& y_{11} + y_{21} = 4 \\
& y_{12} + y_{22} = 7 \\
& y_{13} + y_{23} = 6 \\
& 0 \le x_{11} \le 6, \quad 0 \le x_{12} \le 4 \\
& 0 \le x_{21} \le 5, \quad 0 \le x_{22} \le 6 \\
& y_{11}, y_{12}, y_{13}, y_{21}, y_{22}, y_{23} \ge 0
\end{aligned}$$

Note that we can model the nonlinear operating costs with an LP only because the efficiencies have the right kind of structure. In particular, the plant is less efficient (more costly) at higher operating levels. Thus the LP solution will automatically select level 1 first.