Operations Research

Lecture Introduction III

Lecture 1: linear optimization: introduction

- Definition of cost / objective function
- Example of cost functions, affine functions, linear functions
- Definition of constraints
- Example of constraints, linear constraints
- Linear programs
- General form of a linear program
- Sigma notation
- Extended example 1: the transportation problem
- Google maps
- Extended example 2: the shortest path problem

What is optimization?

Minimize a cost or objective function (for ex. cost of production)

or

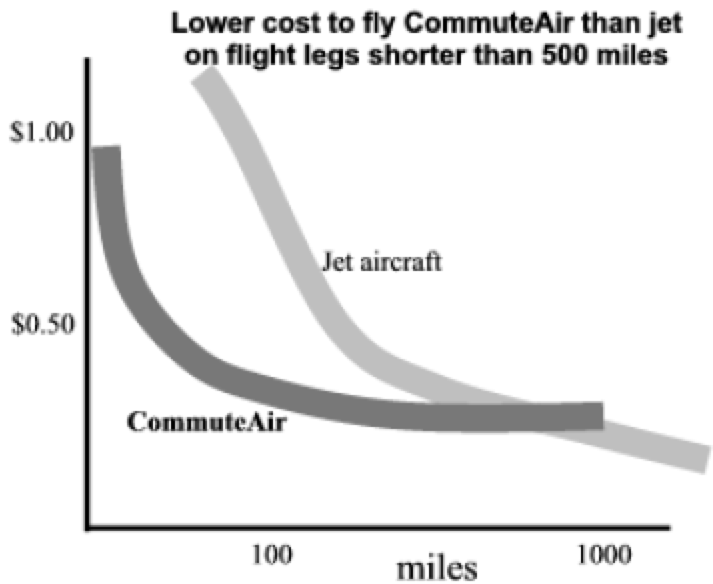
Maximize a cost or objective function (for ex. profit)

with respect to constraints

- Employee cannot work more than x hours a day
- Only three people can use the same machine at a time
- The pipeline's maximal fuel throughput is y

i.e. find a solution that is optimal within limits given

What is a cost function?

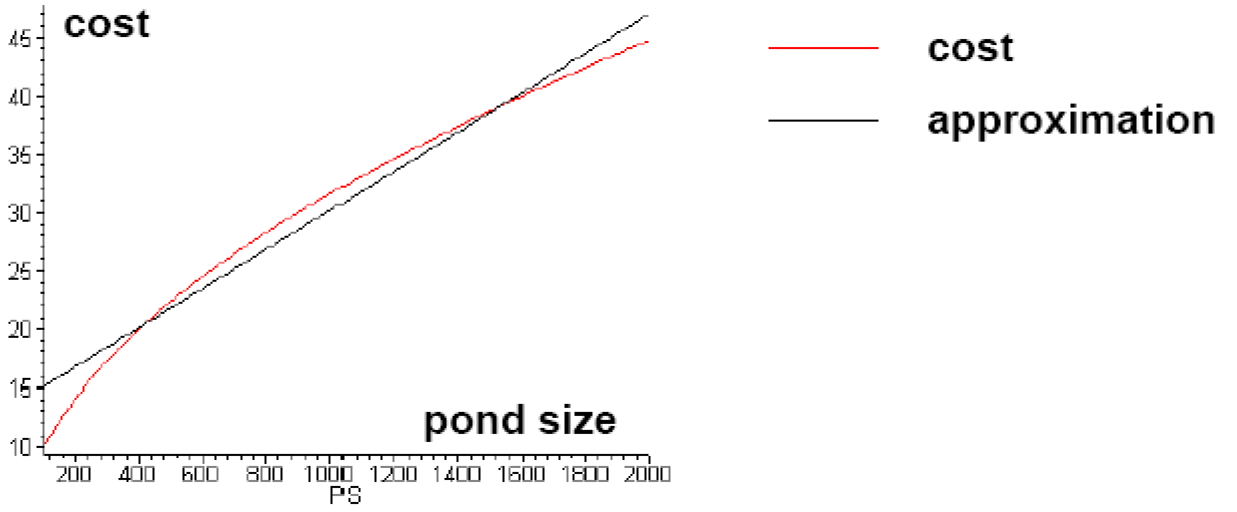


$$\text{cost} = f(\text{variables})$$

Example: cost of a mile as a function of the distance

[http://www.skyaid.org]

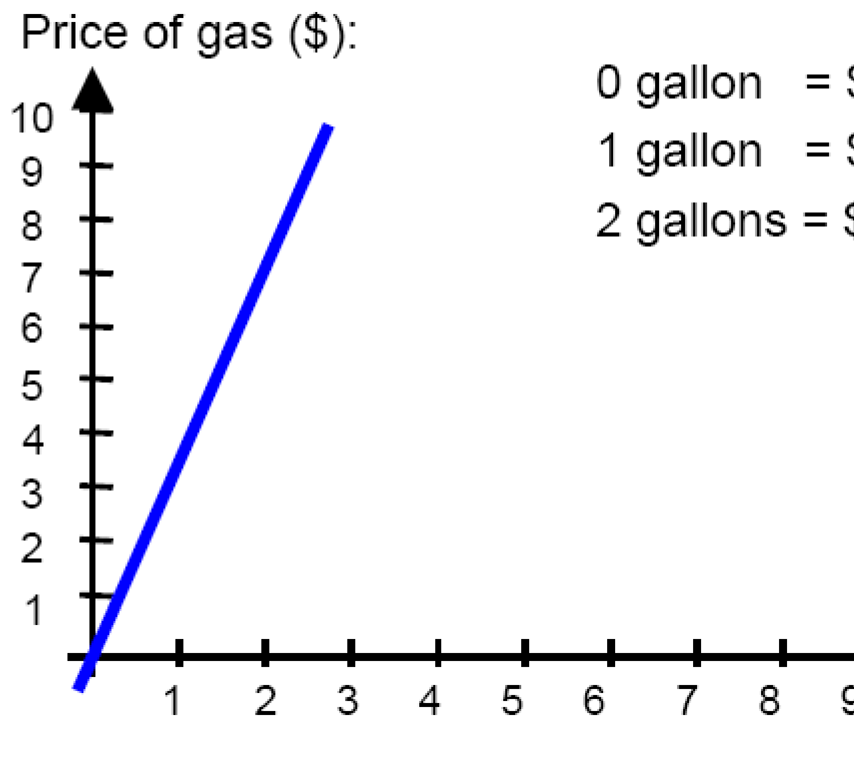
Linear or affine cost functions



For some application, some cost functions look almost like "lines", i.e. are linear or affine. Example here: cost of building a dam as a function of the size of the pond

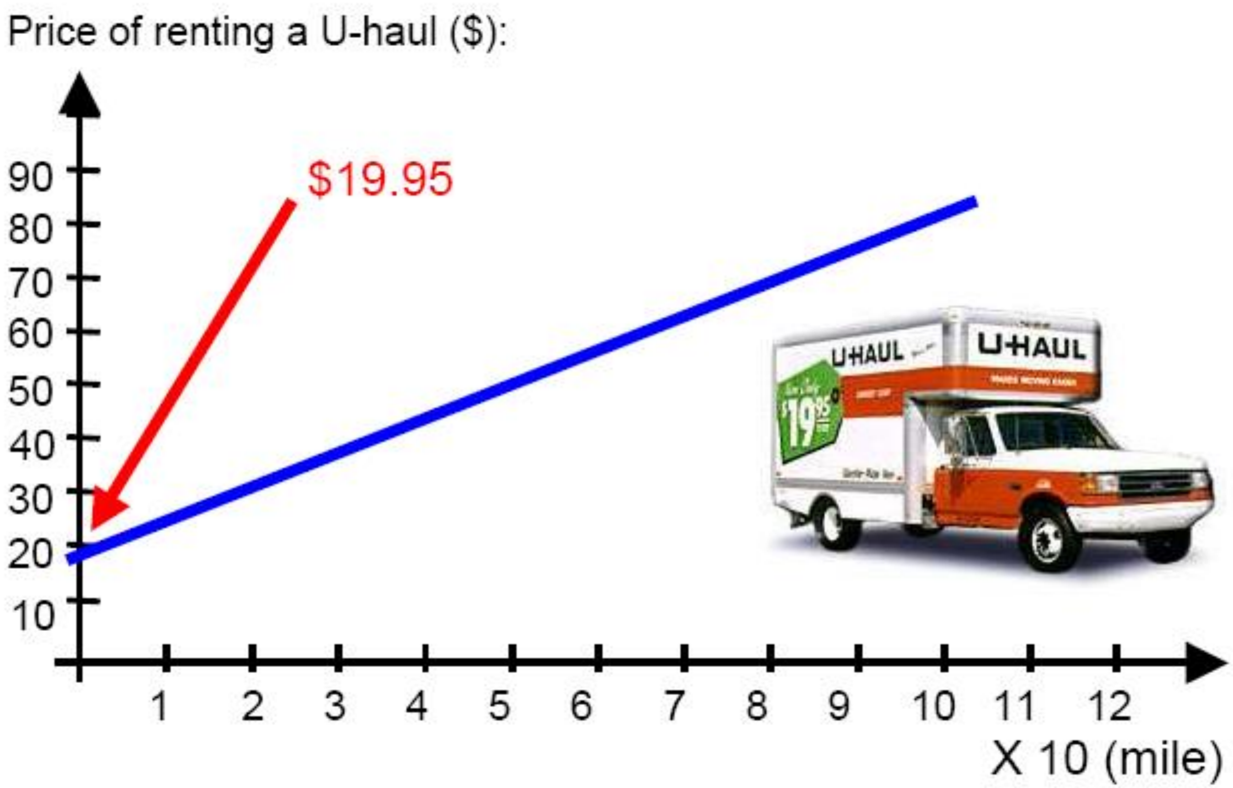
[http://home.hetnet.nl/~krekelberg/chapter3.htm]

Linar functions



0 gallon = $ 0
1 gallon = $ 2.25
2 gallons = $ 4.50

Affine functions



Linear or affine cost functions: formal definition

Minimizing the affine cost function

$$c(x_1, x_2) = 5 + 2x_1 + 3x_2$$

is the same as minimizing the linear cost function

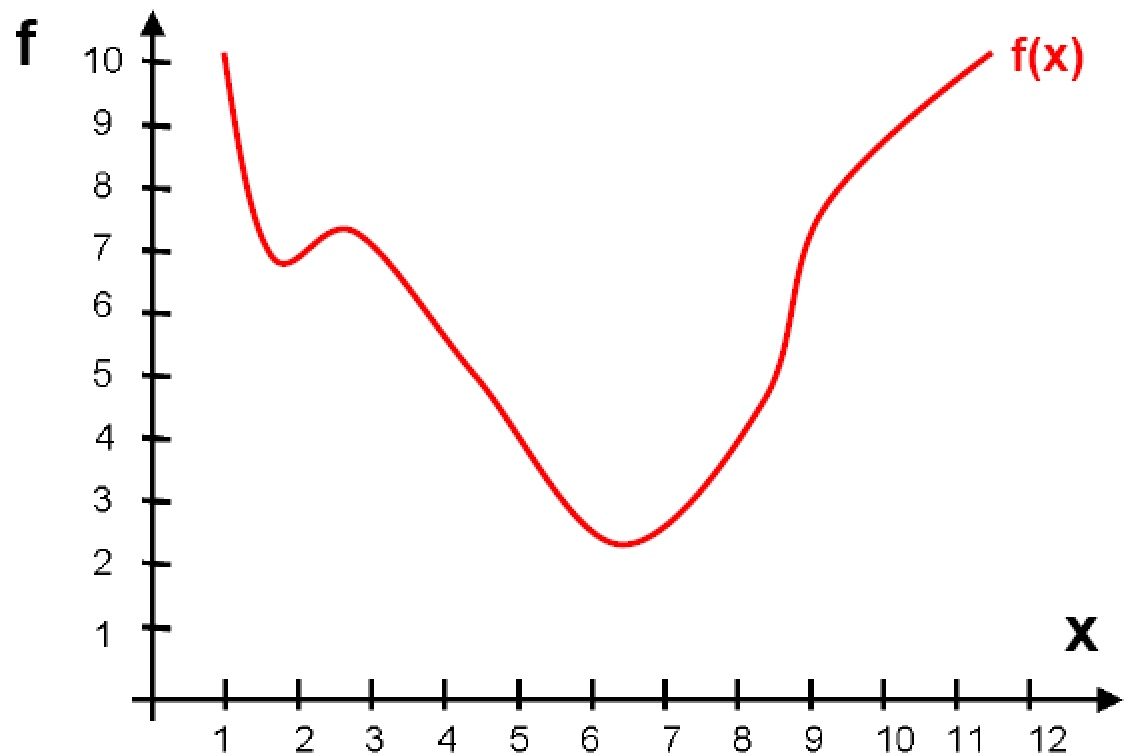
$$c(x_1, x_2) = 2x_1 + 3x_2$$

A more general expression of the cost function:

$$c(x_1, x_2, \cdots, x_n) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

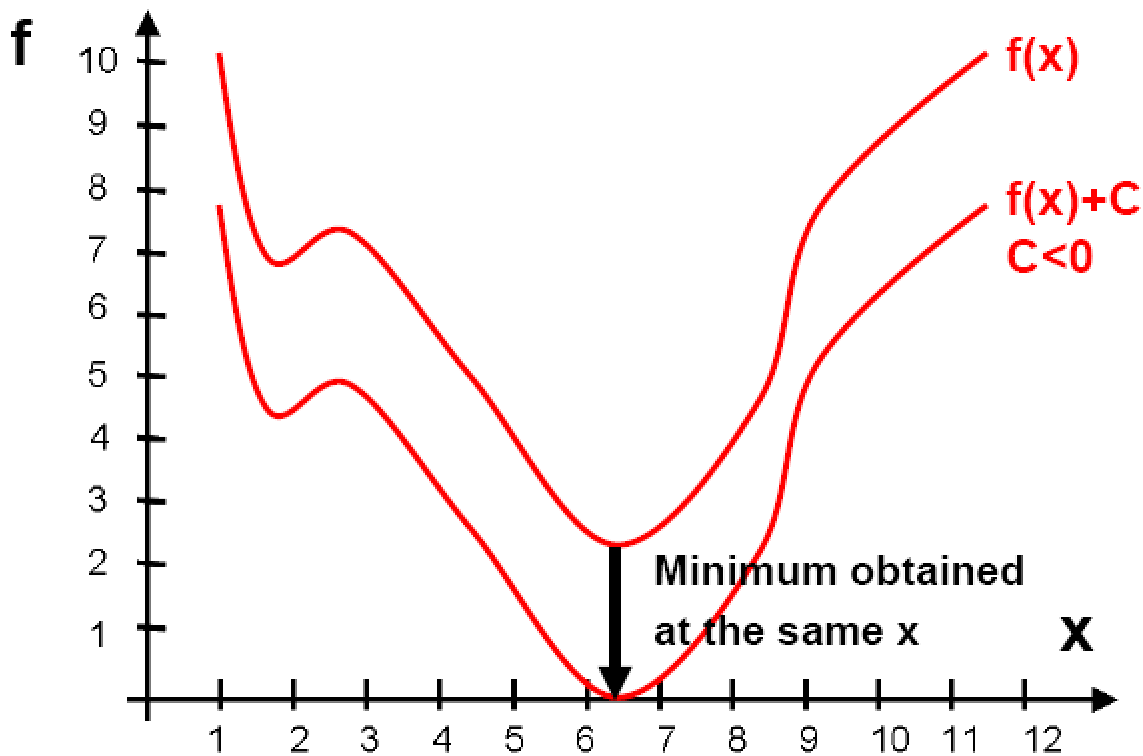
Minimizing affine or linear function is the same

Minimizing a function f(x)



Minimizing affine or linear function is the same

Minimizing a function f(x) or f(x)+c is the same



What is a constraint?

A constraint is a condition on variables which restricts the values they can take

Your maximal budget for cement is $c_{\max}$

$$a_1 x_1 \leq c_{\max}$$

Your minimal budget for steel is $s_{\min}$

$$a_2 x_2 \geq s_{min}$$

You want to spend twice as much for steel as for cement

$$a_2 x_2 \geq 2 a_1 x_1$$

You want to spend a given minimum amount for the wall $a_{\min}$

$$a_1 x_1 + a_2 x_2 \geq a_{\min}$$

Summary

Your optimization program incorporating all your constraints can be formulated as follows.

$$
\begin{array}{ll}
\textbf{Minimize:} & c(x_1, x_2) = a_1 x_1 + a_2 x_2 \\
\textbf{Subject to:} & a_1 x_1 \leq c_{max} \\
 & a_2 x_2 \geq s_{min} \\
 & a_1 x_1 + a_2 x_2 \geq a_{min} \\
 & a_2 x_2 \geq 2 a_1 x_1
\end{array}
$$

Constraints in the form of equalities (I)

Sometimes, constraints are given in the form of equalities

Example: you want to spend exactly twice as much for steel as for cement:

$$a_2 x_2 = 2 a_1 x_1$$

This is exactly the same as

$$a_2 x_2 \geq 2 a_1 x_1 \quad \textbf{and} \quad a_2 x_2 \leq 2 a_1 x_1$$

Constraints in the form of equalities (II)

So you could rewrite the program in the following form:

$$
\begin{array}{ll}
\textbf{Minimize:} & c(x_1, x_2) = a_1 x_1 + a_2 x_2 \\
\textbf{Subject to:} & a_1 x_1 \leq c_{max} \\
 & a_2 x_2 \geq s_{min} \\
 & a_1 x_1 + a_2 x_2 \geq a_{min} \\
 & a_2 x_2 \geq 2 a_1 x_1 \\
 & a_2 x_2 \leq 2 a_1 x_1
\end{array}
$$

One can thus assume that all constraints are always given in the form of inequalities.

General form for a linear program

So you could rewrite the program in the following form:

$$
\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nx_N & \\
\textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\
& \vdots & \vdots \\
& a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M
\end{array}
$$

Sigma notation

So you could rewrite the program in the following form:

$$
\begin{array}{lll}
\textbf{min:} & \sum_{j=1}^{N} c_j x_j & \\
\textbf{s.t.:} & \sum_{j=1}^{N} a_{1,j} x_j & \leq b_1 \\
 & \sum_{j=1}^{N} a_{2,j} x_j & \leq b_2 \\
 & \vdots & \vdots \\
 & \sum_{j=1}^{N} a_{M,j} x_j & \leq b_M
\end{array}
$$

Example: the transportation problem (I)

Paul's farm produces 4 tons of apples per day	$s_p = 4$
Ron's farm produces 2 tons of apples per day	$s_r = 2$
Max's factory needs 1 ton of apples per day	$d_m = 1$
Bob's factory needs 5 tons of apples per day	$d_b = 5$

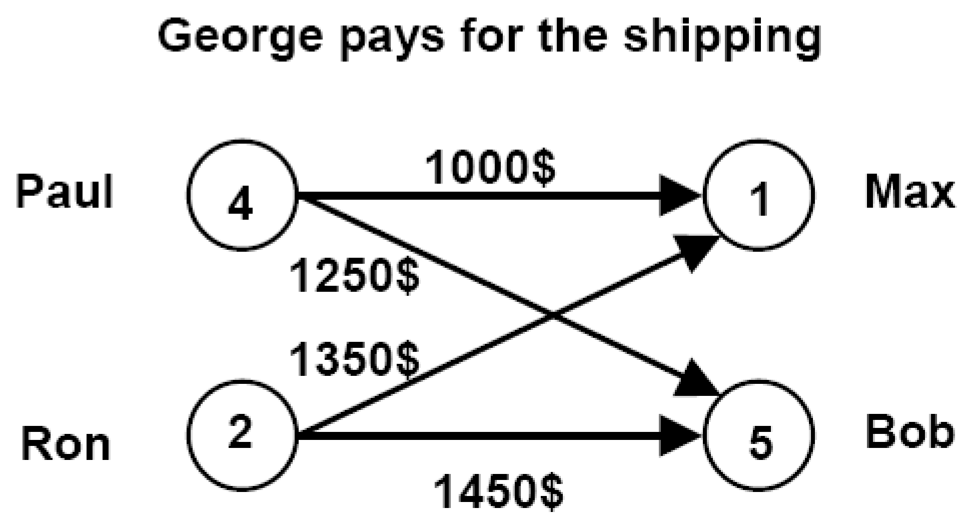
George owns both farms and factories. He is paying the cost of shipping all the apples from the farms to the factories.

The shipping costs for George are:

Paul →Max: 1000$ per ton	$c_{pm} = 1000$	x_{pm}
Ron →Max: 1350$ per ton	$c_{rm} = 1350$	x_{rm}
Paul →Bob: 1250$ per ton	$c_{pb} = 1250$	x_{pb}
Ron →Bob: 1450$ per ton	$c_{rb} = 1450$	x_{rb}

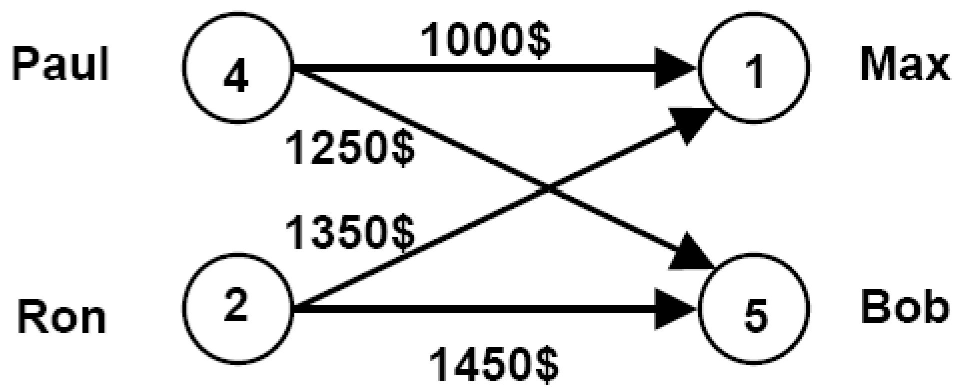
What is the best way to ship the apples?

Example: the transportation problem (II)



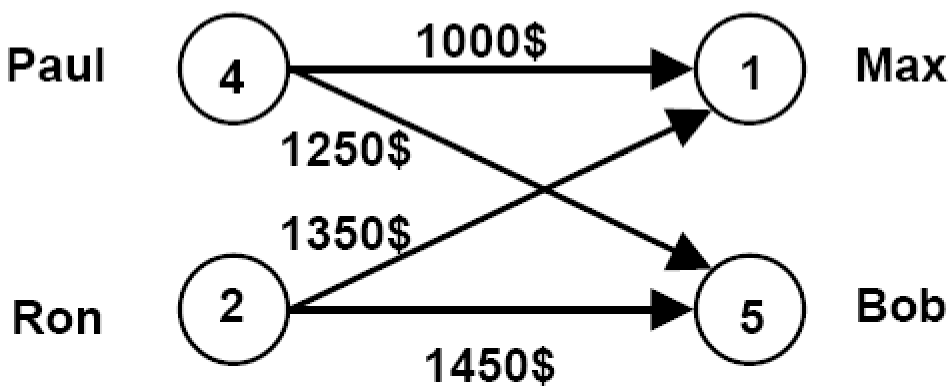
Example: the transportation problem (III)

$$
\begin{aligned}
\textbf{min:} \quad & 1000x_{pm} + 1350x_{rm} + 1250x_{pb} + 1450x_{rb} \\
\textbf{Subject to:} \quad & x_{pm} + x_{rm} = 1 \\
& x_{pb} + x_{rb} = 5 \\
& x_{pm} + x_{pb} = 4 \\
& x_{rm} + x_{rb} = 2 \\
& x_{pm} \geq 0,\ x_{rm} \geq 0,\ x_{pb} \geq 0,\ x_{rb} \geq 0
\end{aligned}
$$



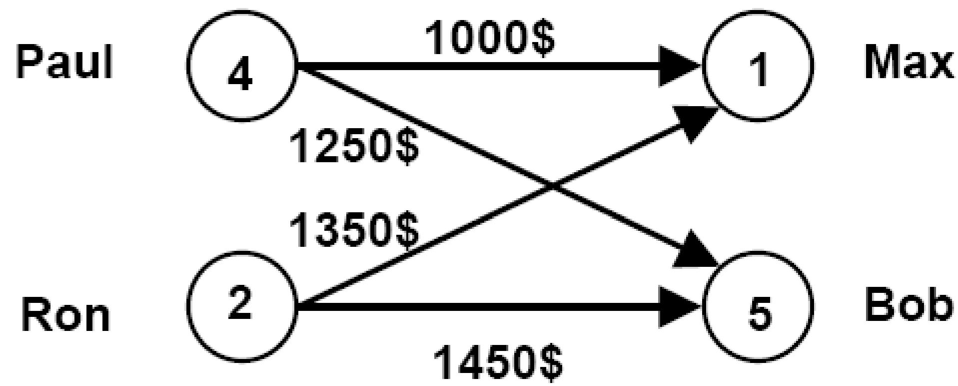
Example: the transportation problem (IV)

$$
\begin{array}{ll}
\textbf{min:} & x_{pm}c_{pm} + x_{rm}c_{rm} + x_{pb}c_{pb} + x_{rb}c_{rb} \\
\textbf{Subject to:} & x_{pm} + x_{rm} = d_m \\
 & x_{pb} + x_{rb} = d_b \\
 & x_{pm} + x_{pb} = s_p \\
 & x_{rm} + x_{rb} = s_r \\
 & x_{pm} \geq 0 \ x_{rm} \geq 0, \ \ x_{pb} \geq 0, \ \ x_{rb} \geq 0
\end{array}
$$



General form of the transportation problem

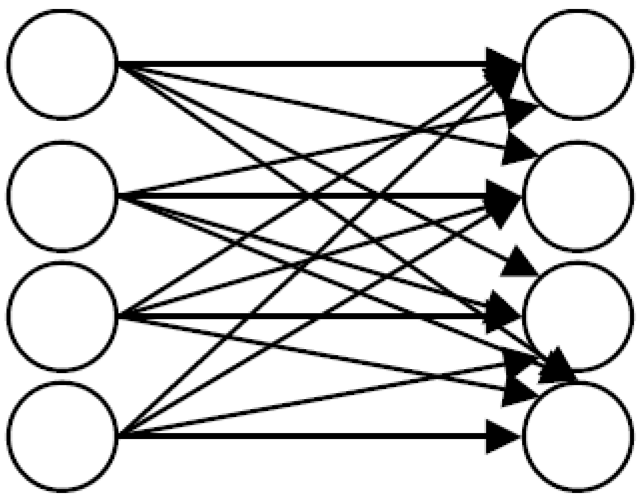
$$
\begin{array}{lll}
\textbf{min:} & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\textbf{Subject to:} & \sum_{i=1}^{m} x_{ij} = d_j & j = 1,\cdots,n \\
 & \sum_{j=1}^{n} x_{ij} = s_i & i = 1,\cdots,m \\
 & x_{ij} \geq 0 & \textit{for all } i,j
\end{array}
$$



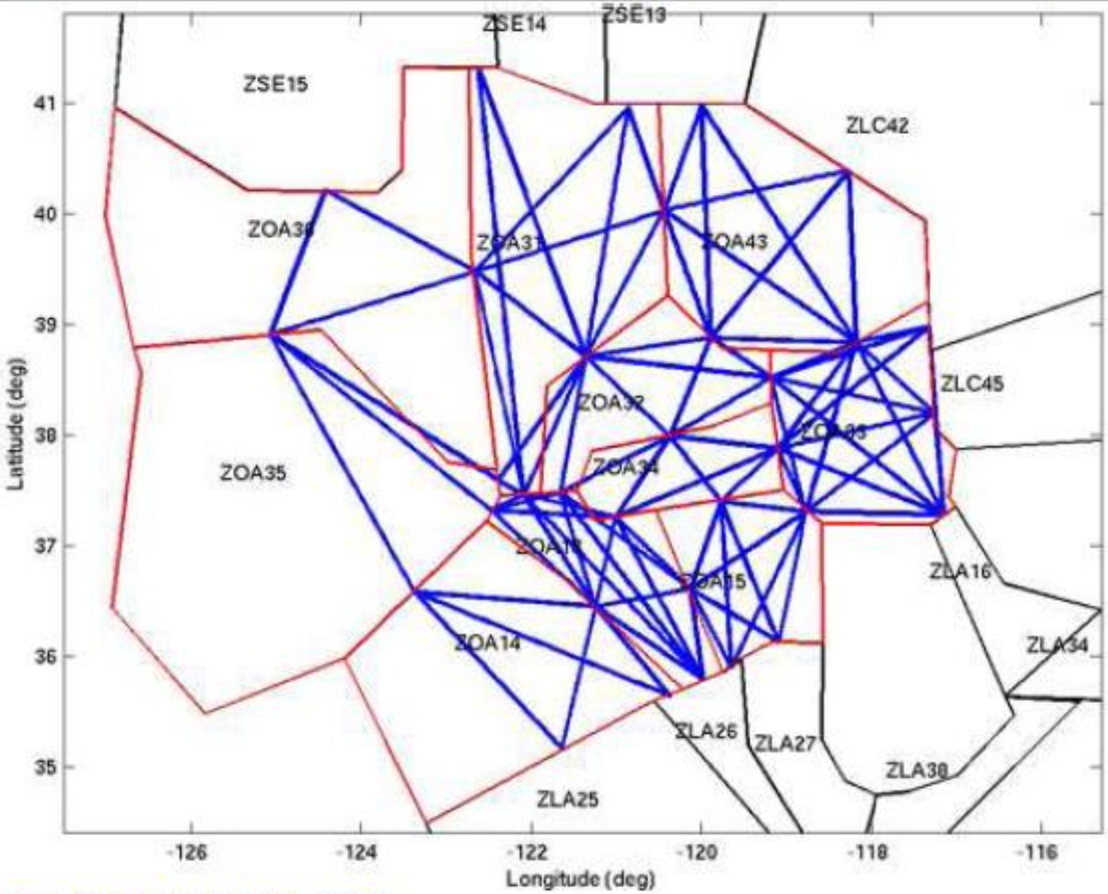
Please, be lazy, do not write pages of equations…

Use summations, they leave you more time to go to the movies

$$
\begin{array}{lll}
\textbf{min:} & \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}x_{ij} & \\
\textbf{Subject to:} & \sum_{i=1}^{m} x_{ij} = d_j & j = 1, \cdots, n \\
 & \sum_{j=1}^{n} x_{ij} = s_i & i = 1, \cdots, m \\
 & x_{ij} \geq 0 & for\ all\ i, j
\end{array}
$$

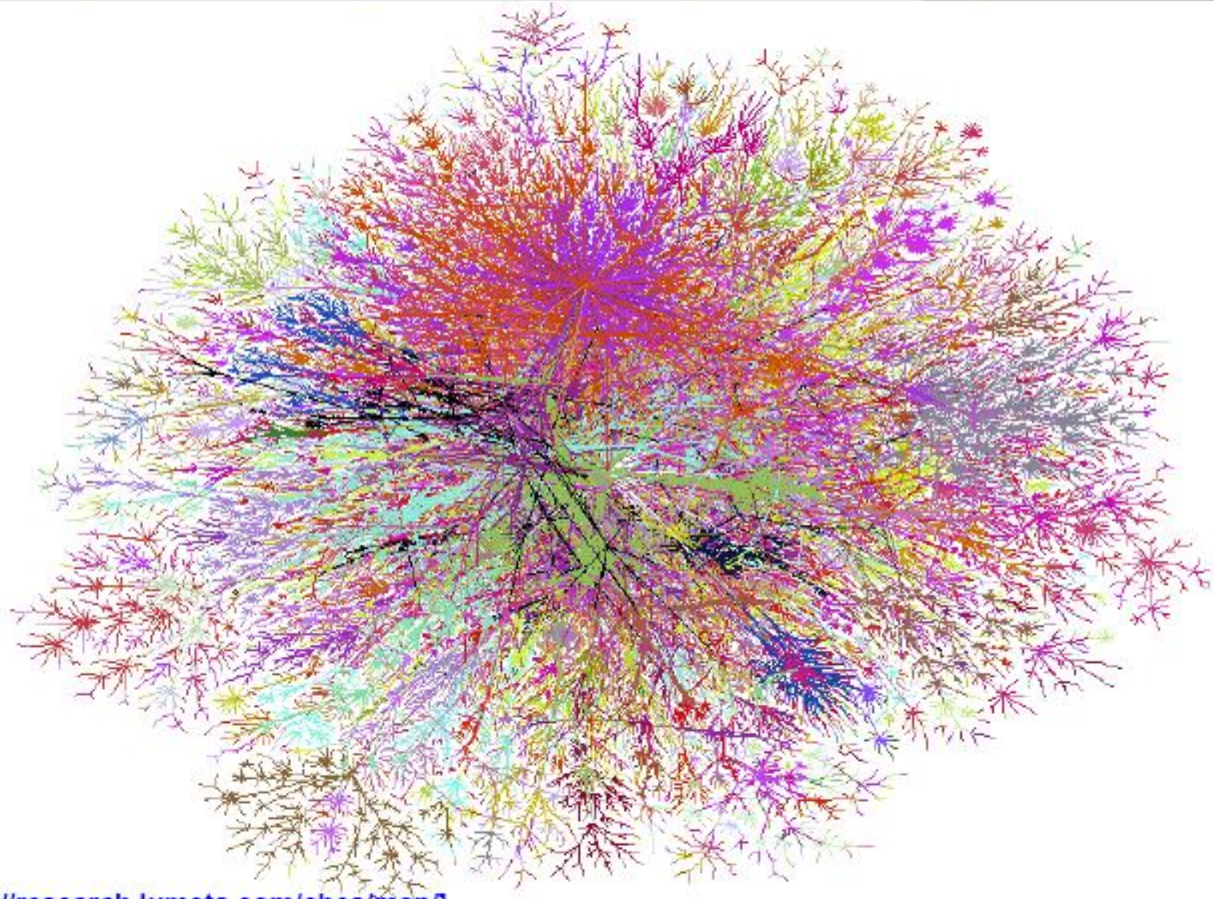


Example: a « small » network (air traffic control)



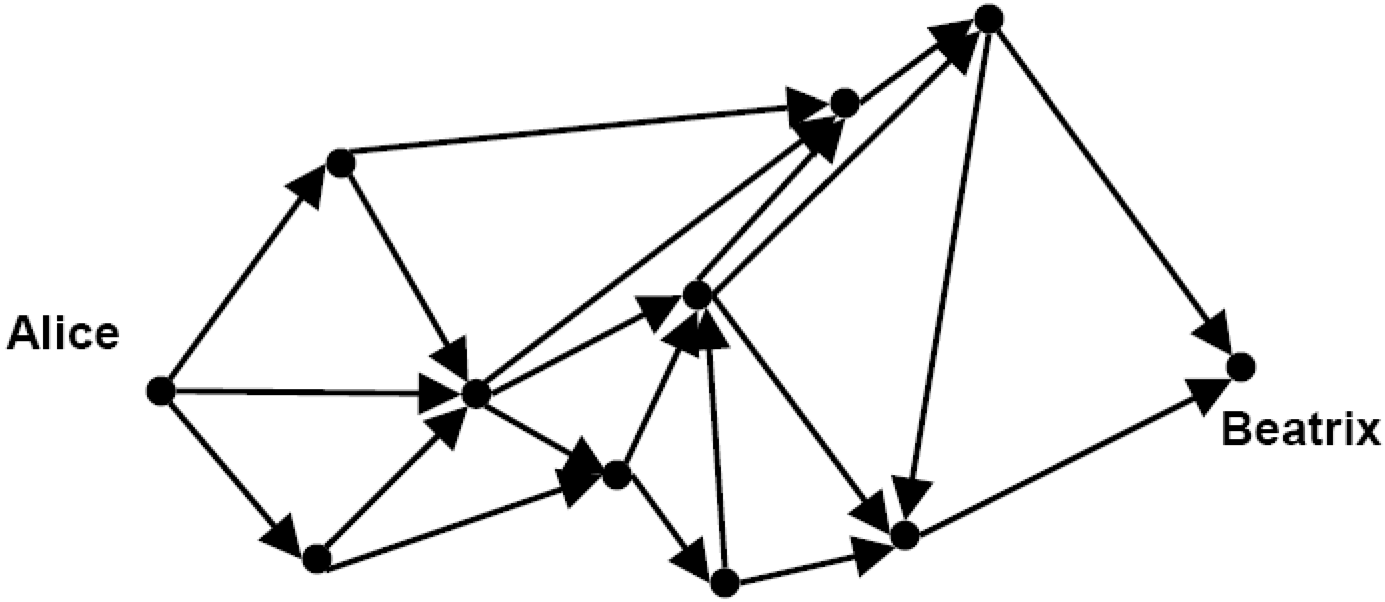
[Robelin, Sun, Bayen, tech. rep., 2005]

Example: a « large » network (the internet)

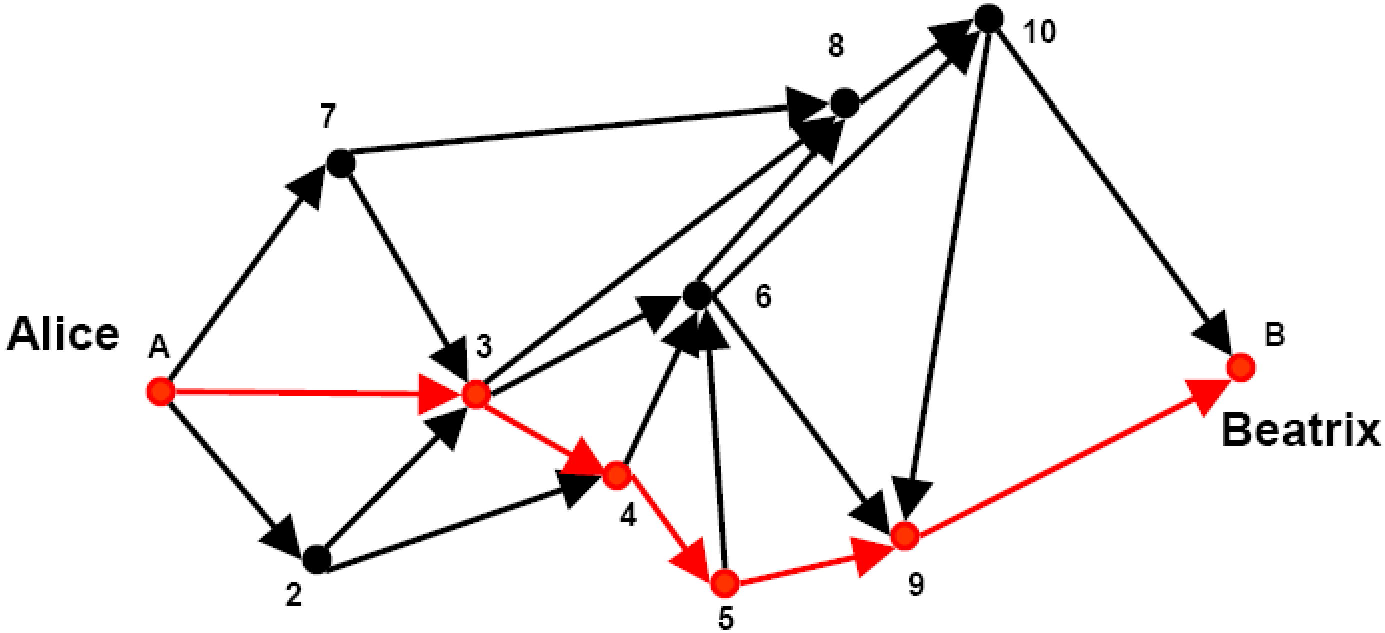


[http://research.lumeta.com/ches/map/]

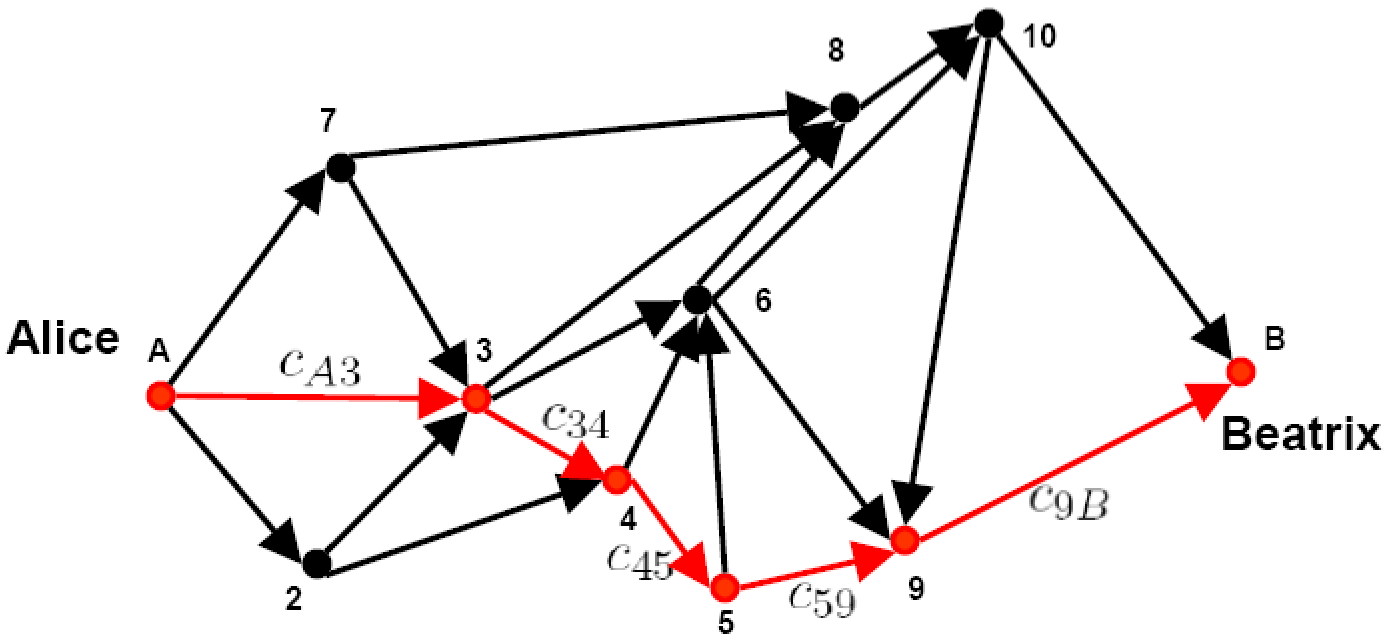
Example: shortest path



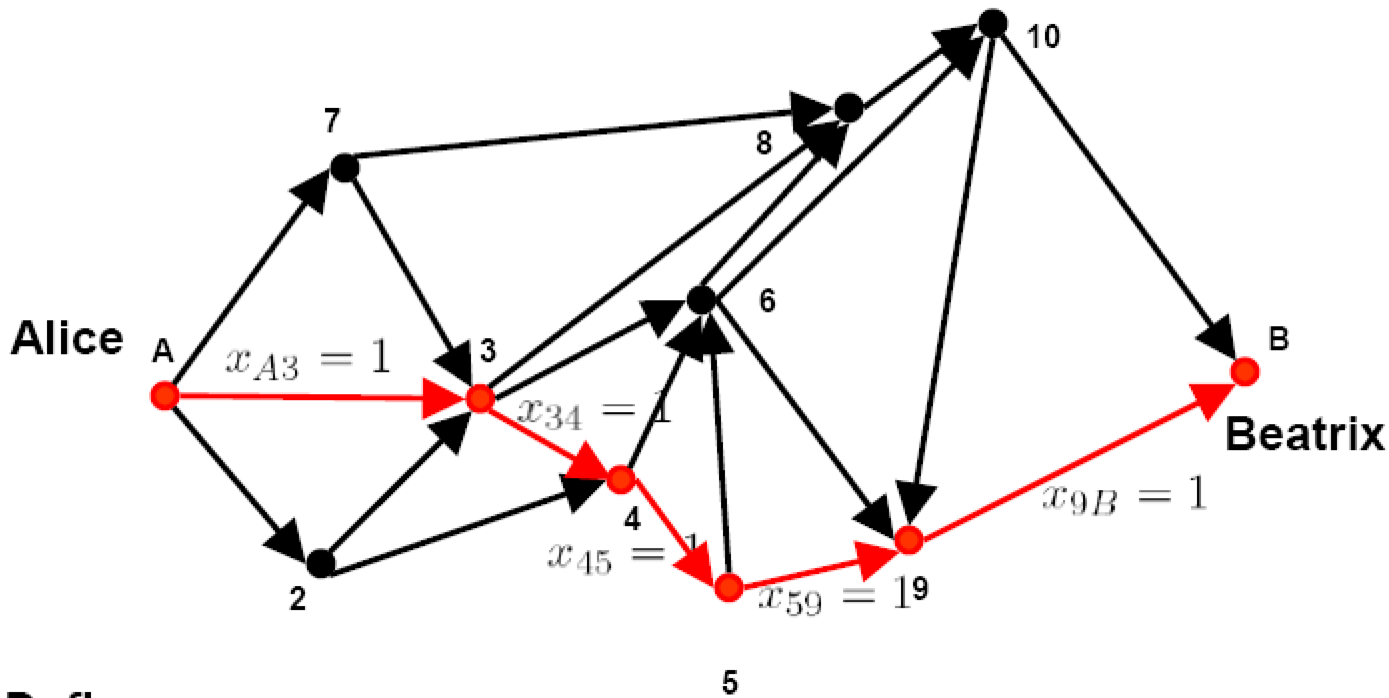
Example: shortest path



Example: shortest path: length of the shortest path



Example: shortest path: length of the shortest path

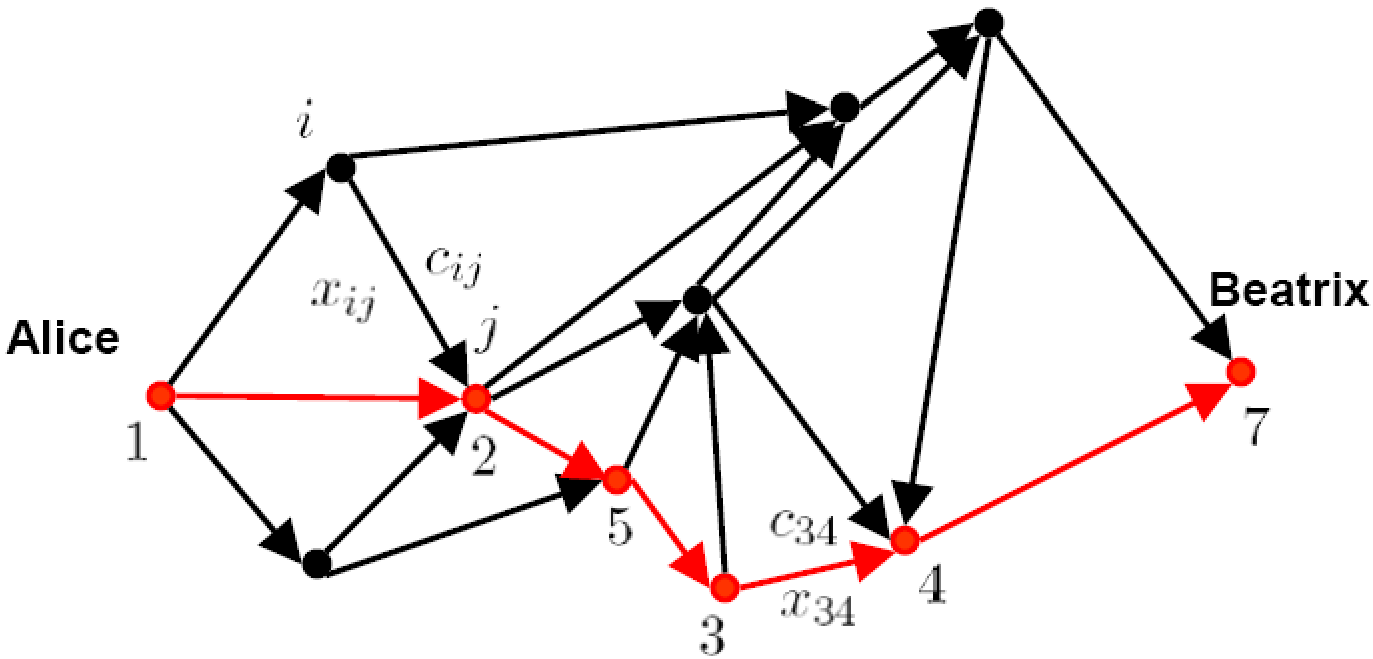


Define

$x_{ij} = 1$ **For every (i,j) on the shortest path**

$x_{ij} = 0$ **For every (i,j) not on the shortest path**

Example: shortest path

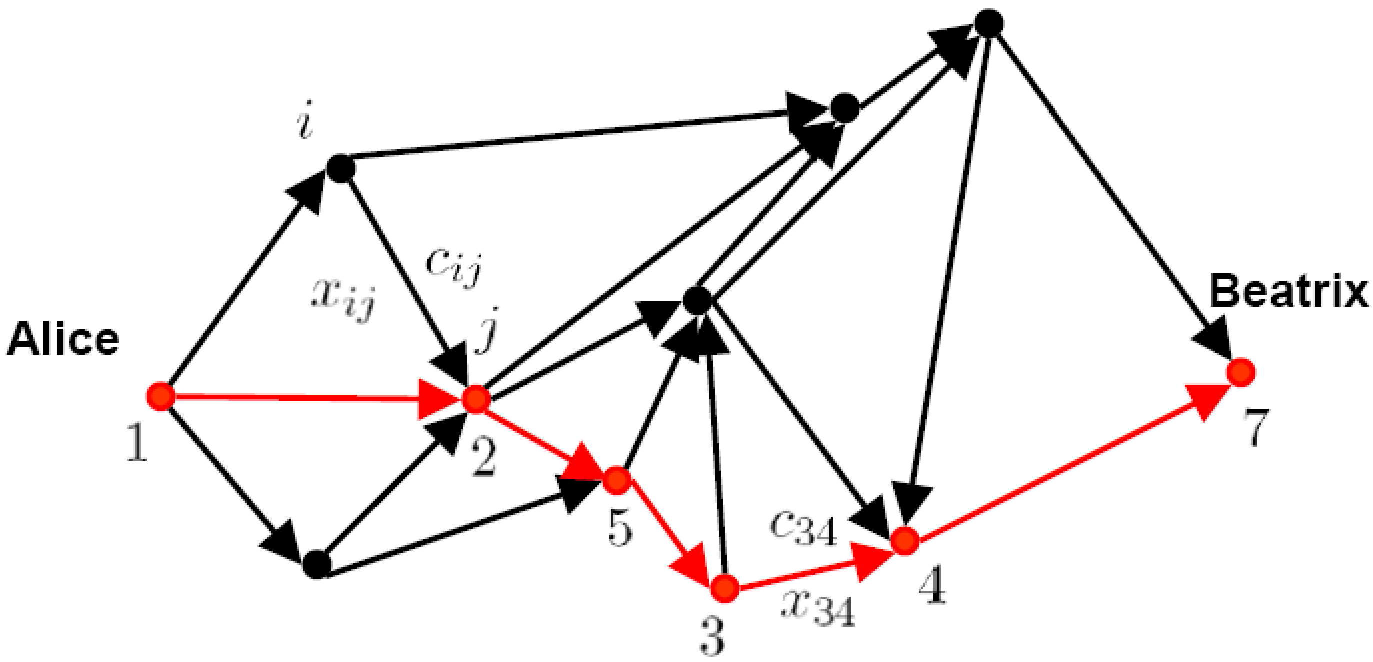


$x_{12} = x_{25} = x_{53} = x_{34} = x_{47} = 1$

All other x_{ij} **are zero**

Total length of this path: $c_{12} + c_{25} + c_{53} + c_{34} + c_{47}$

Example: shortest path



Total length:

$$\sum_{(i,j)\text{ chosen on path}} c_{ij} = \sum_{(i,j)\text{ chosen on path}} x_{ij}c_{ij} = \sum_{\text{all }(i,j)} x_{ij}c_{ij}$$

Example: shortest path

Minimize: $$Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$$

Total length

Cost of arc (i,j)

1 if link (i,j) chosen
0 otherwise

N_i **set of nodes j with direct connections to node i**

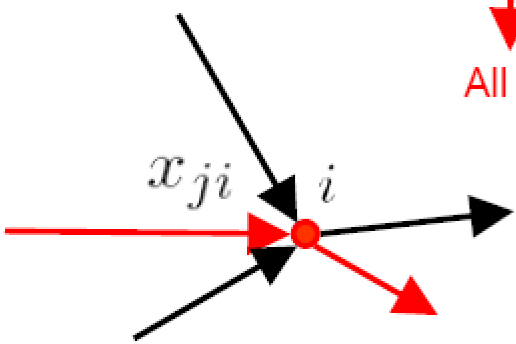
Example: shortest path

minimize: $$Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$$

such that: $$\sum_{j \in N_i} x_{ji} = \sum_{j \in N_i} x_{ij}, \quad i = 1, \cdots, 10$$

All links leaving node i

All links arriving at node i



N_i **set of nodes j with direct connections to node i**

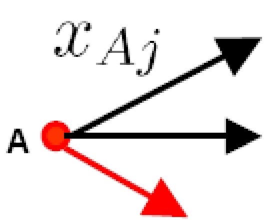
Example: shortest path

minimize: $$Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$$

such that: $$\sum_{j \in N_i} x_{ji} = \sum_{j \in N_i} x_{ij}, \quad i = 1, \cdots, 10$$

$$\sum_{j \in N_A} x_{Aj} = 1$$

Starting from A, Alice can only take one path



N_A **set of nodes j with direct connections to node A**

Example: shortest path

minimize: $Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$

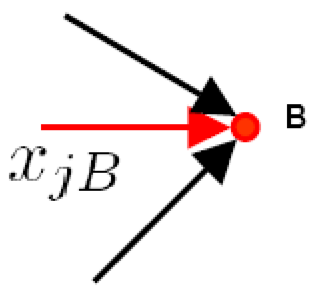
such that:

$$\sum_{j \in N_i} x_{ji} = \sum_{j \in N_i} x_{ij}, \quad i = 1, \cdots, 10$$

$$\sum_{j \in N_A} x_{Aj} = 1$$

$$\sum_{j \in N_B} x_{jB} = 1$$

Arriving at B, one can only take one path



N_B **set of nodes j with direct connections to node B**

Example: shortest path

minimize: $$Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$$

such that: $$\sum_{j \in N_i} x_{ji} = \sum_{j \in N_i} x_{ij}, \quad i = 1, \cdots, 10$$

$$\sum_{j \in N_A} x_{Aj} = 1$$

$$\sum_{j \in N_B} x_{jB} = 1$$

$$x_{ij} \geq 0, \; x_{jB} \geq 0, \quad x_{Aj} \geq 0$$

N_i **set of nodes j with direct connections to node i**

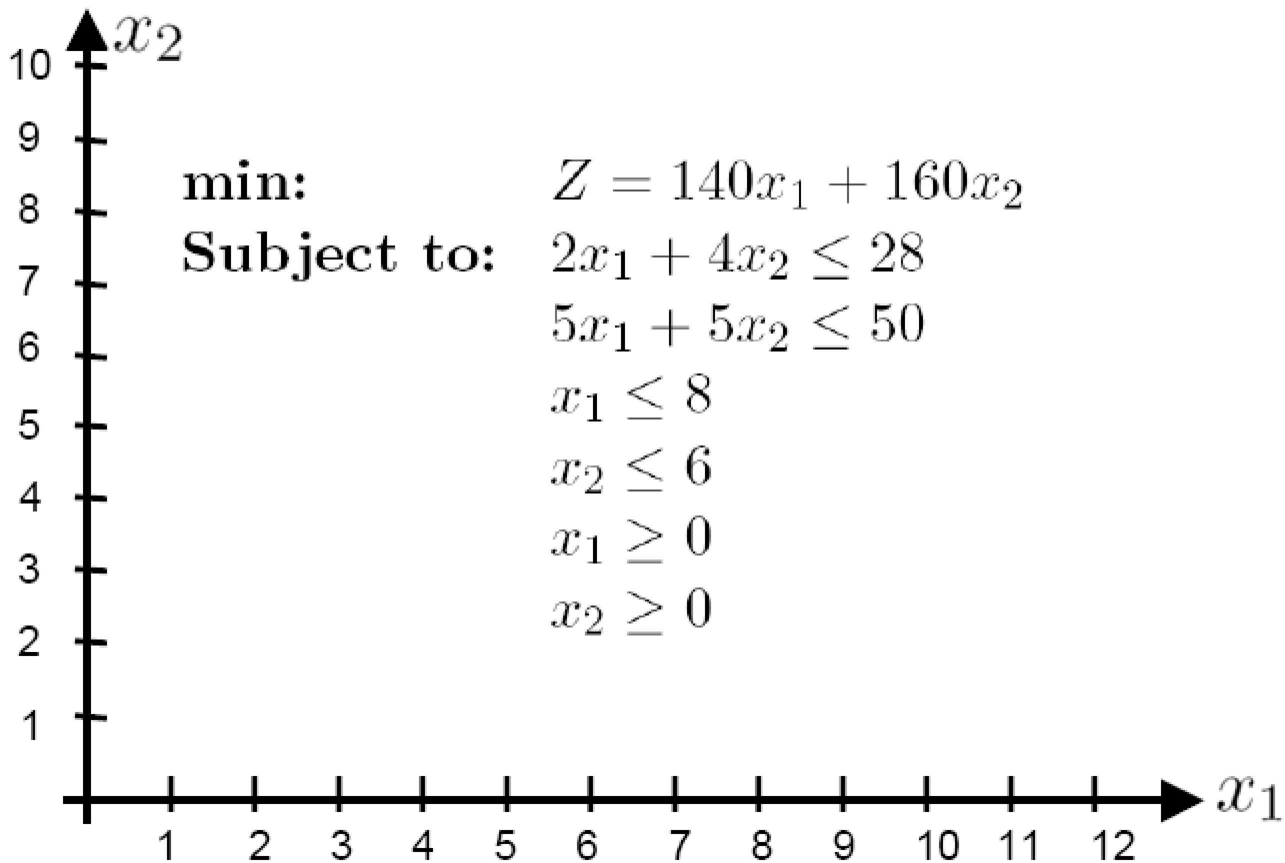
Lecture 2: graphical solutions of LPs

- Graphical interpretation of constraints
- Feasible set
- Gradient of the cost function
- Unbounded feasible set
- Unbounded cost function
- Infeasibility

Graphical solutions of linear programs

$$\begin{aligned}
&\textbf{min:} && Z = 140x_1 + 160x_2 \\
&\textbf{Subject to:} && 2x_1 + 4x_2 \leq 28 \\
& && 5x_1 + 5x_2 \leq 50 \\
& && x_1 \leq 8 \\
& && x_2 \leq 6 \\
& && x_1 \geq 0 \\
& && x_2 \geq 0
\end{aligned}$$

Construction of the feasible set

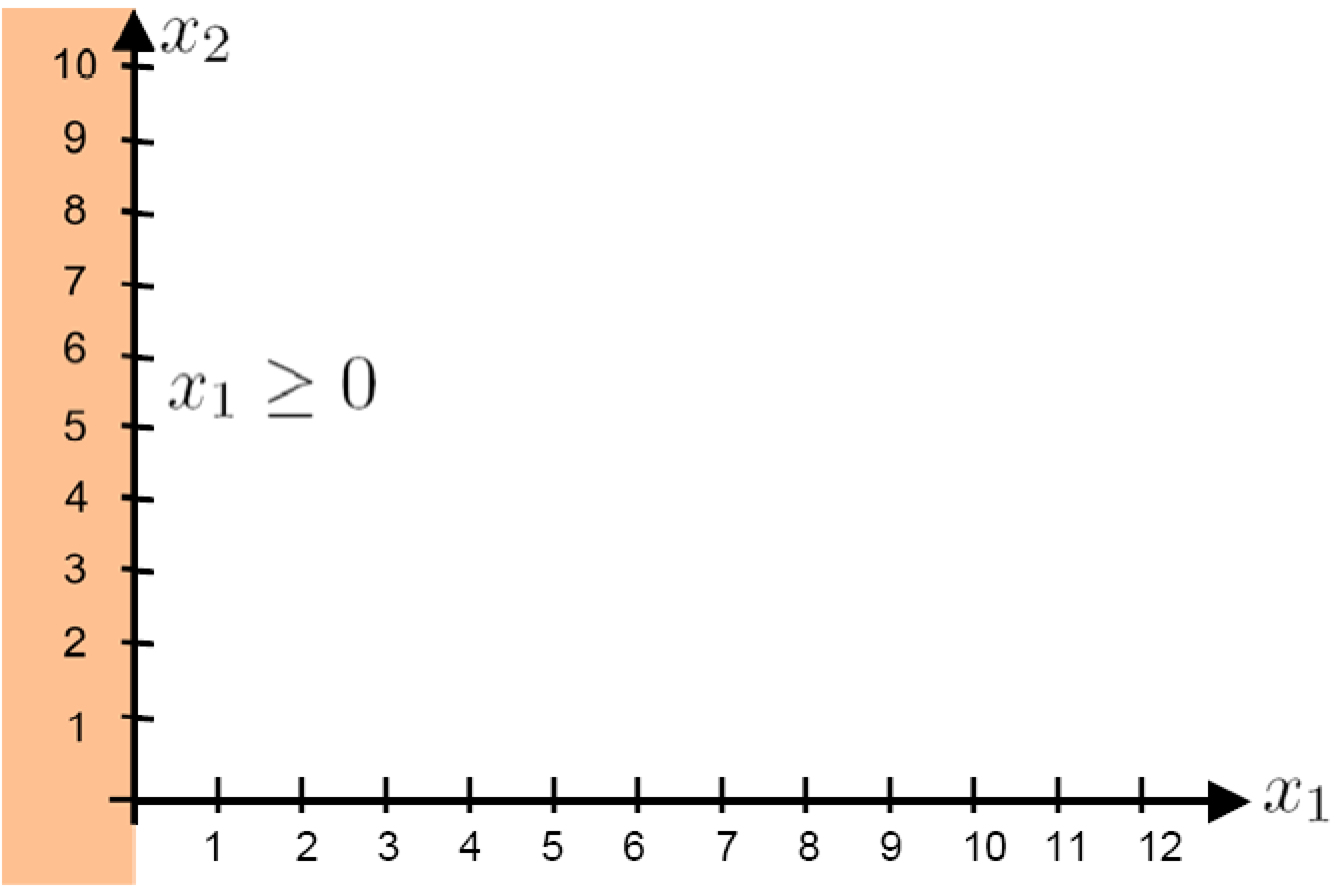


min: $Z = 140x_1 + 160x_2$

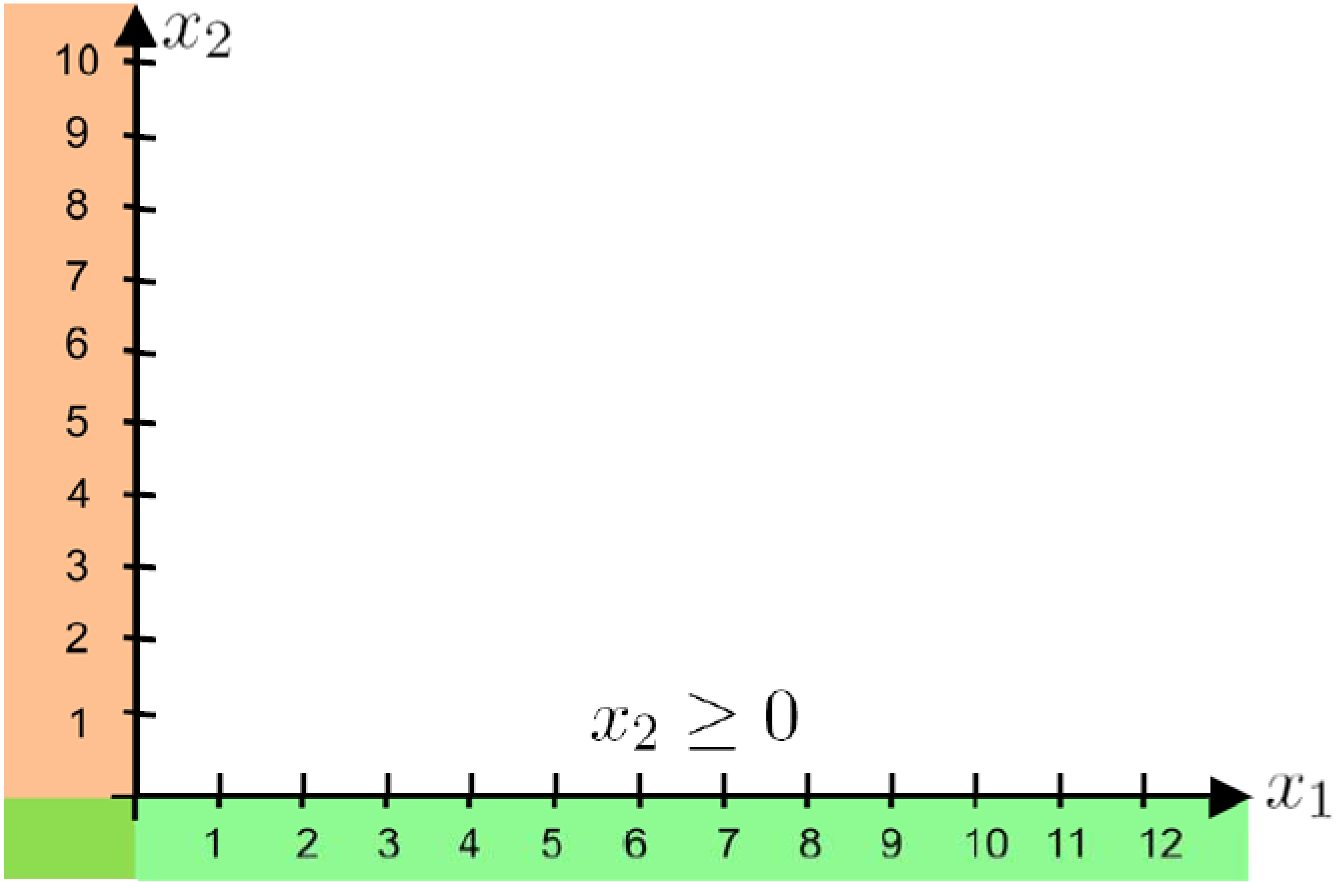
Subject to:

$$2x_1 + 4x_2 \leq 28$$
$$5x_1 + 5x_2 \leq 50$$
$$x_1 \leq 8$$
$$x_2 \leq 6$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

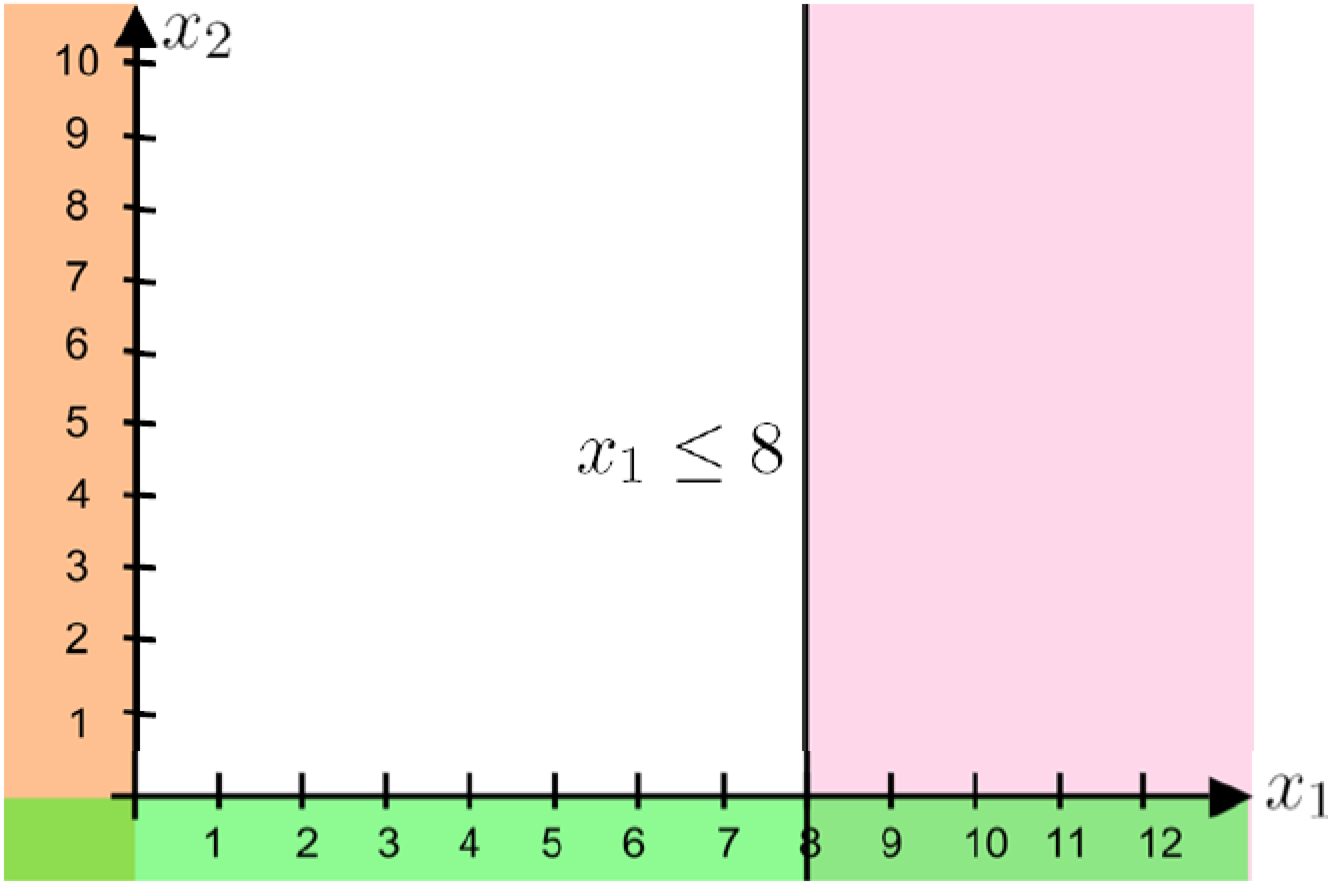
Construction of the feasible set



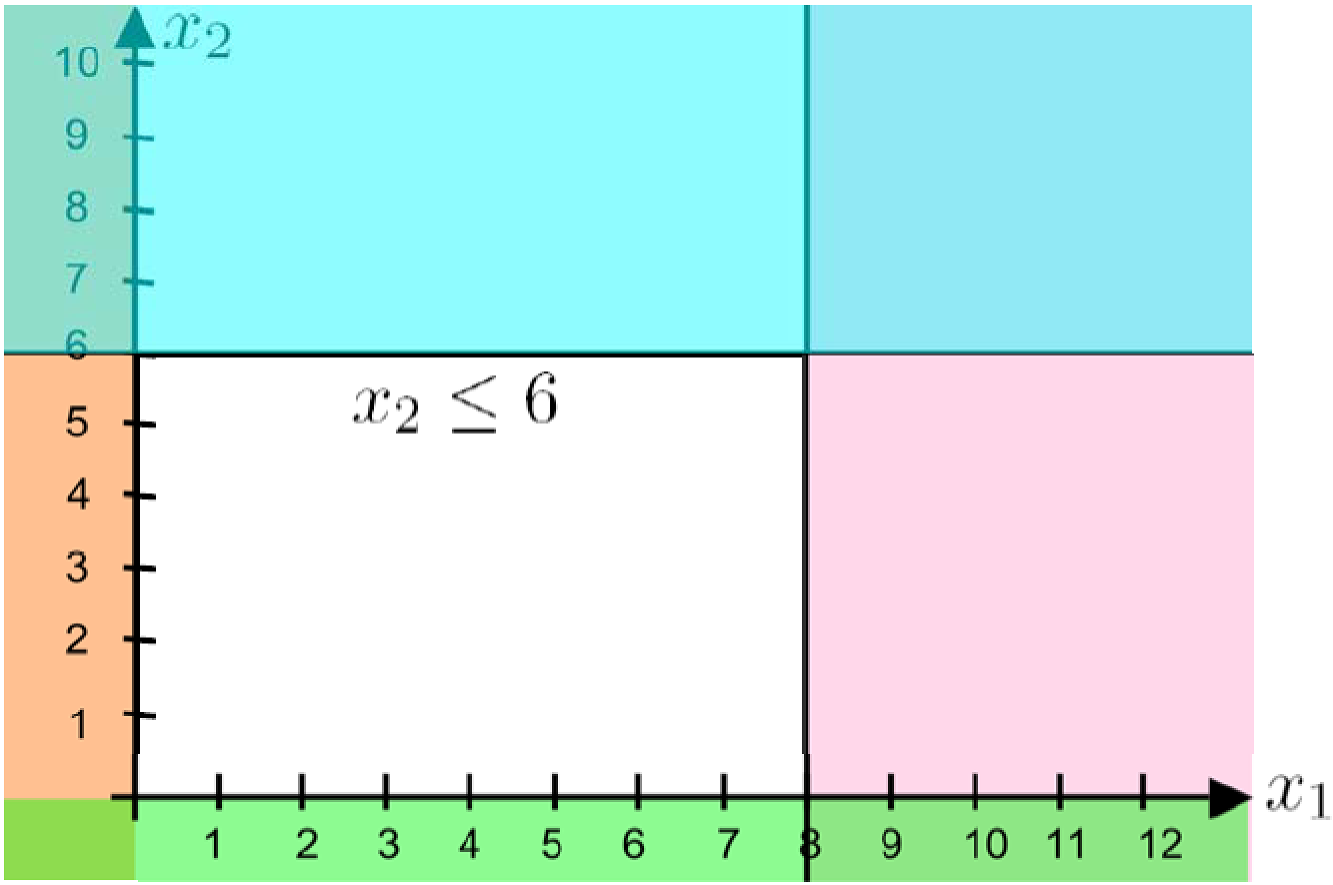
Construction of the feasible set



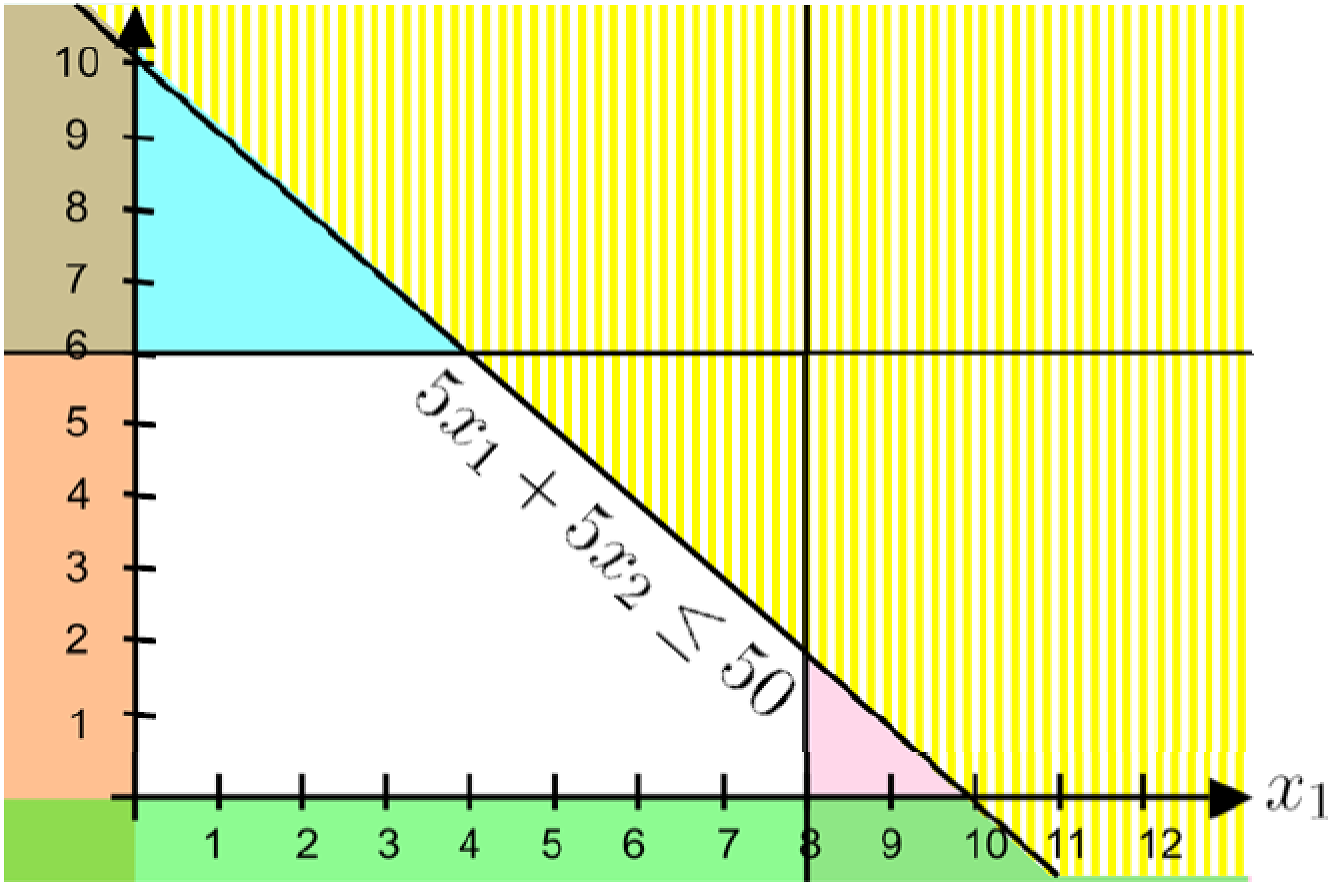
Construction of the feasible set



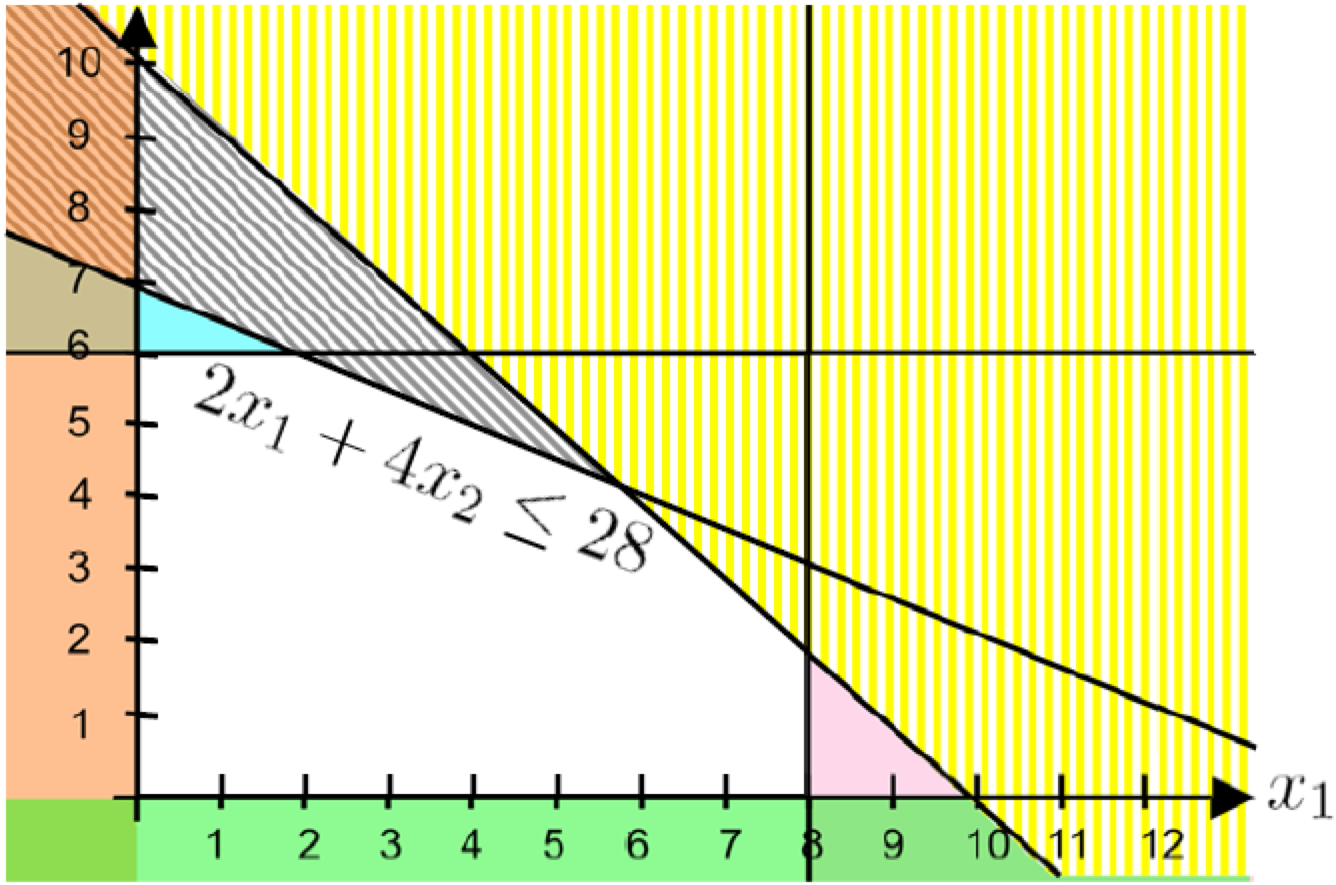
Construction of the feasible set



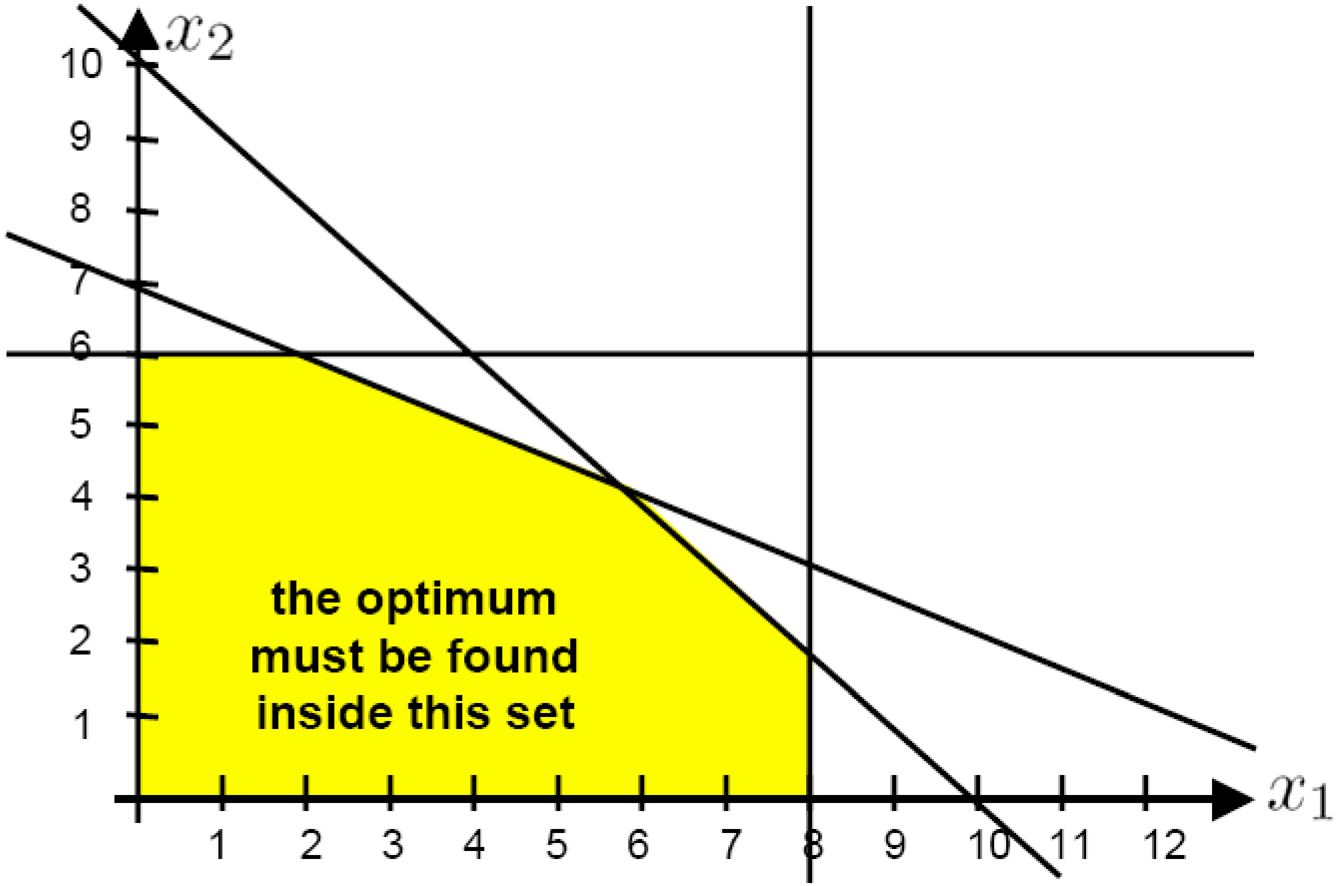
Construction of the feasible set



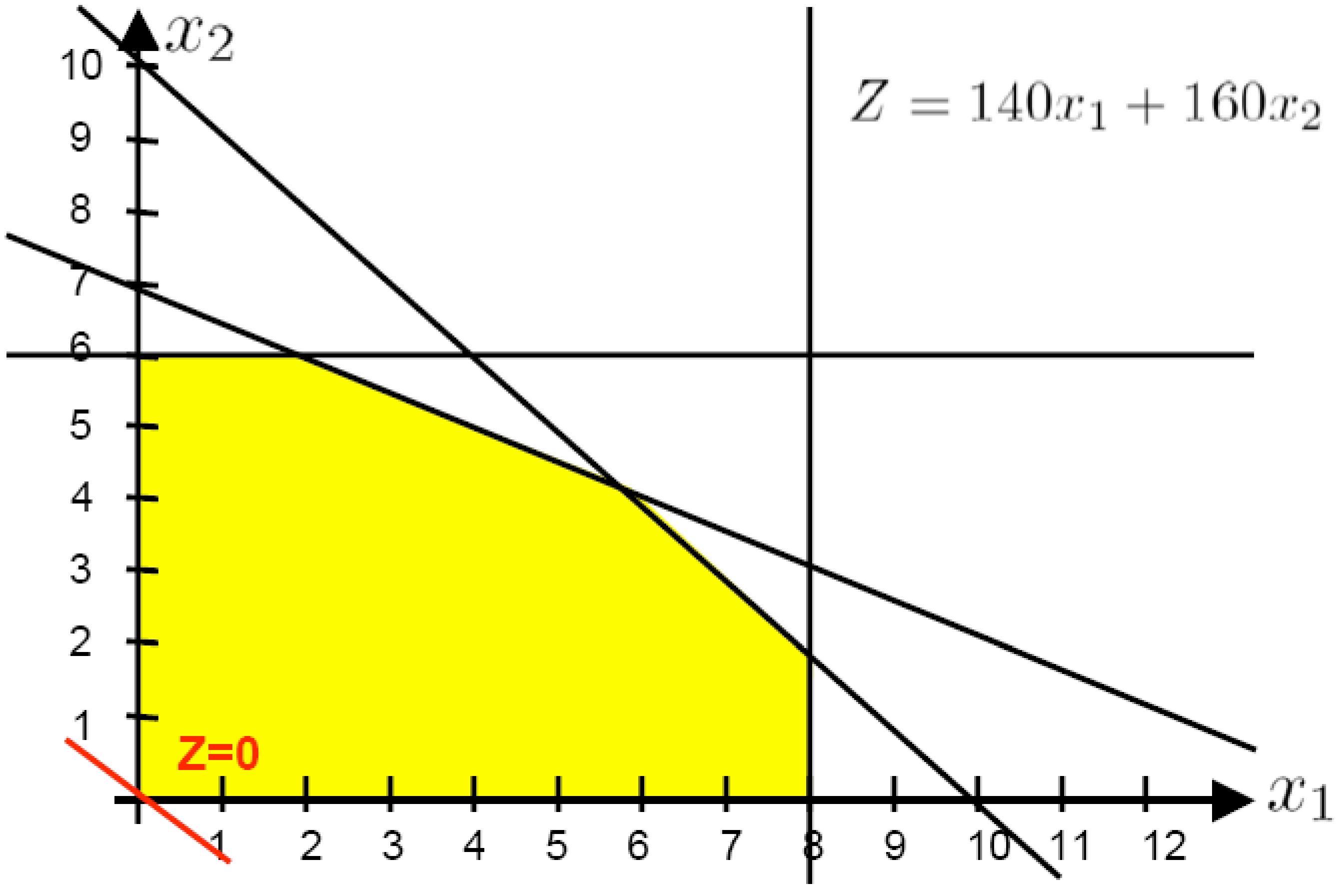
Construction of the feasible set



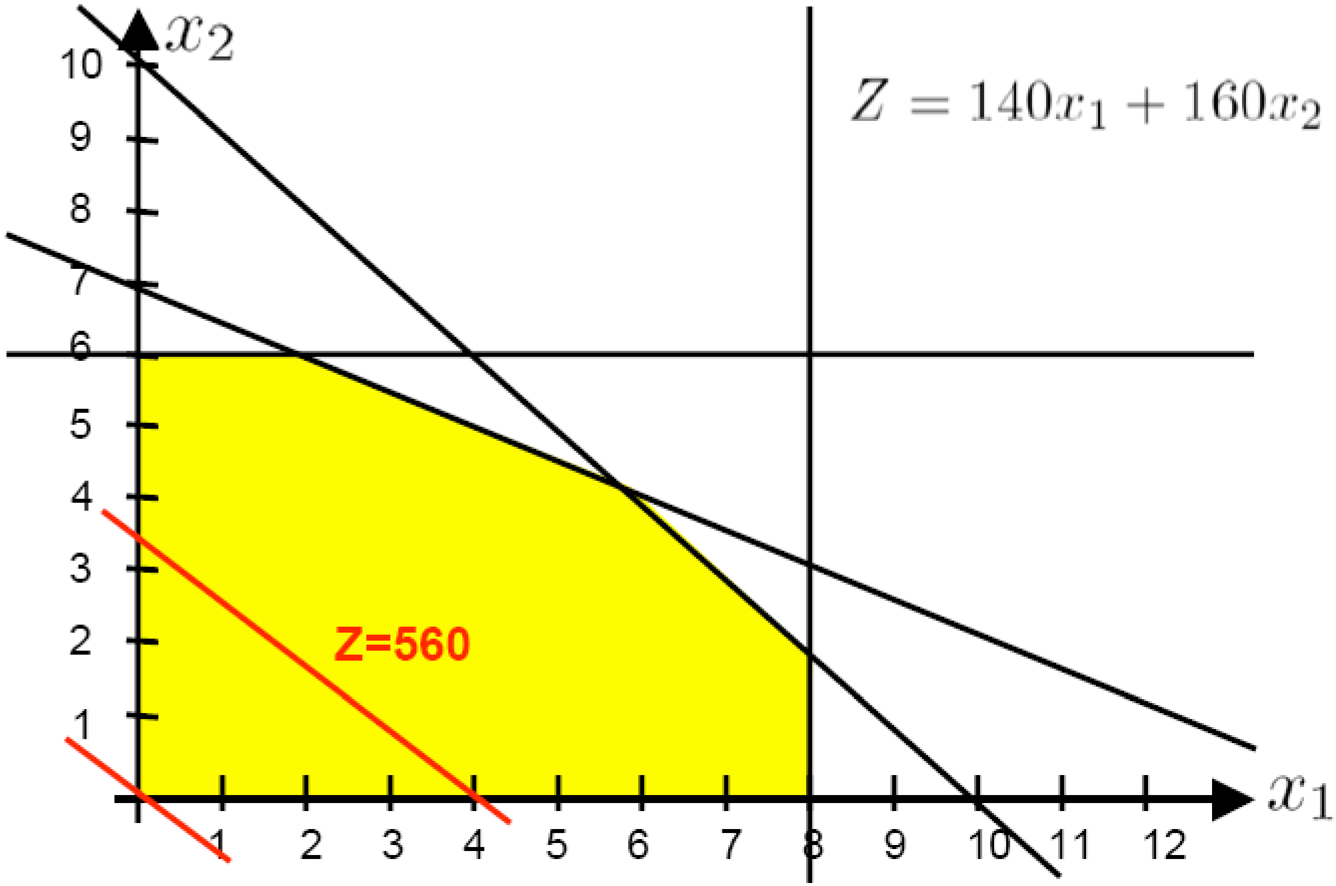
Feasible set: result



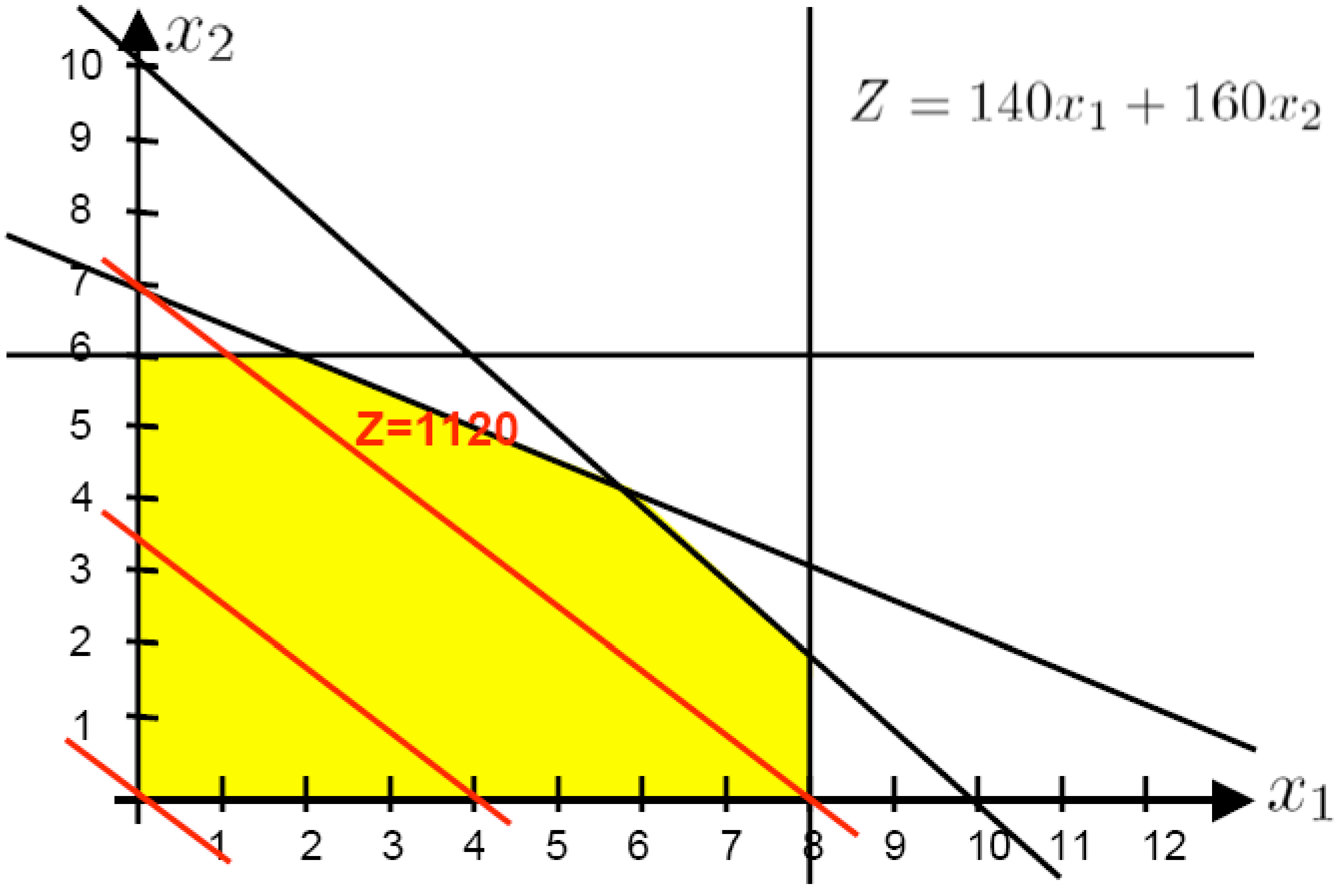
Optimal solution: isolines



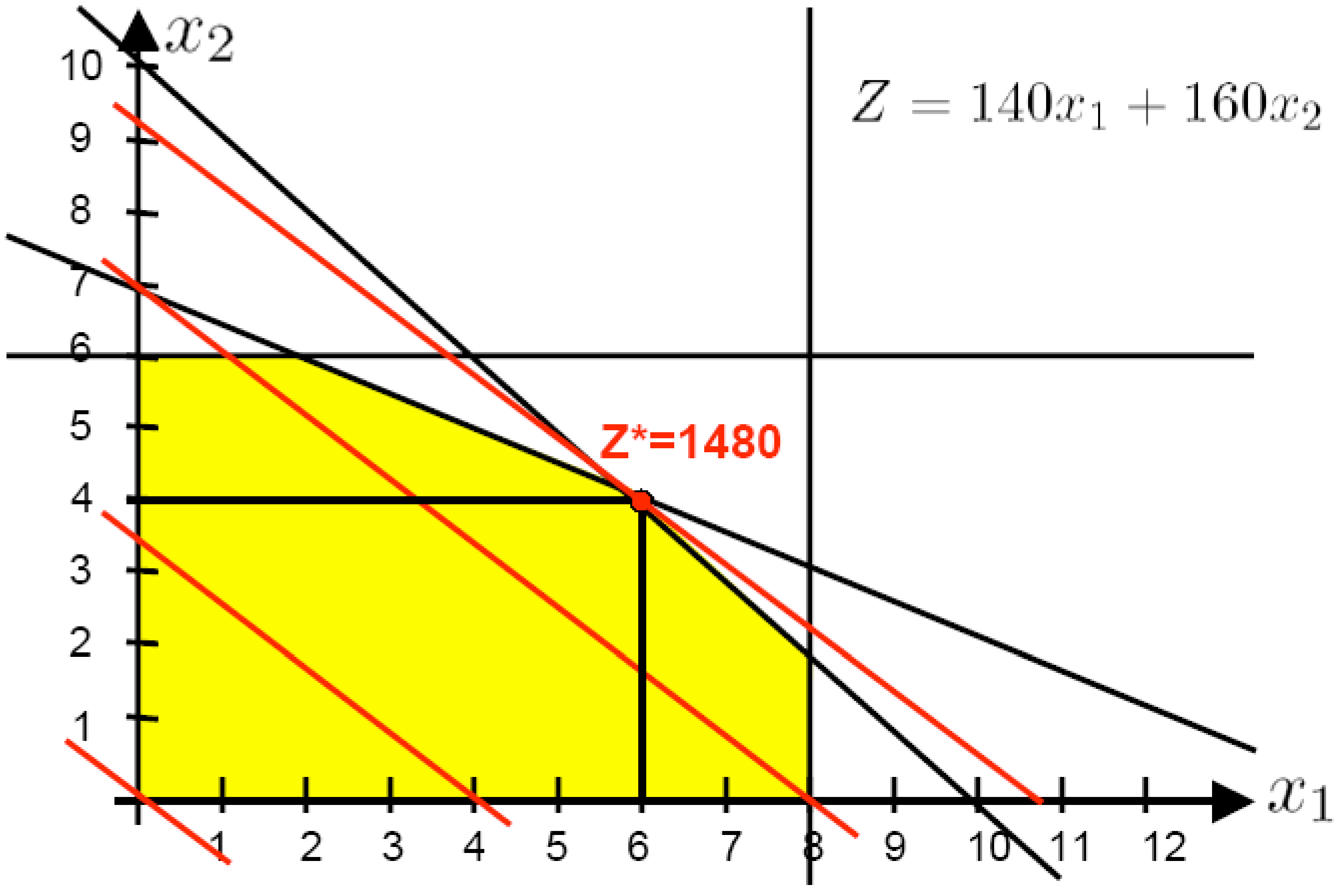
Optimal solution: isolines



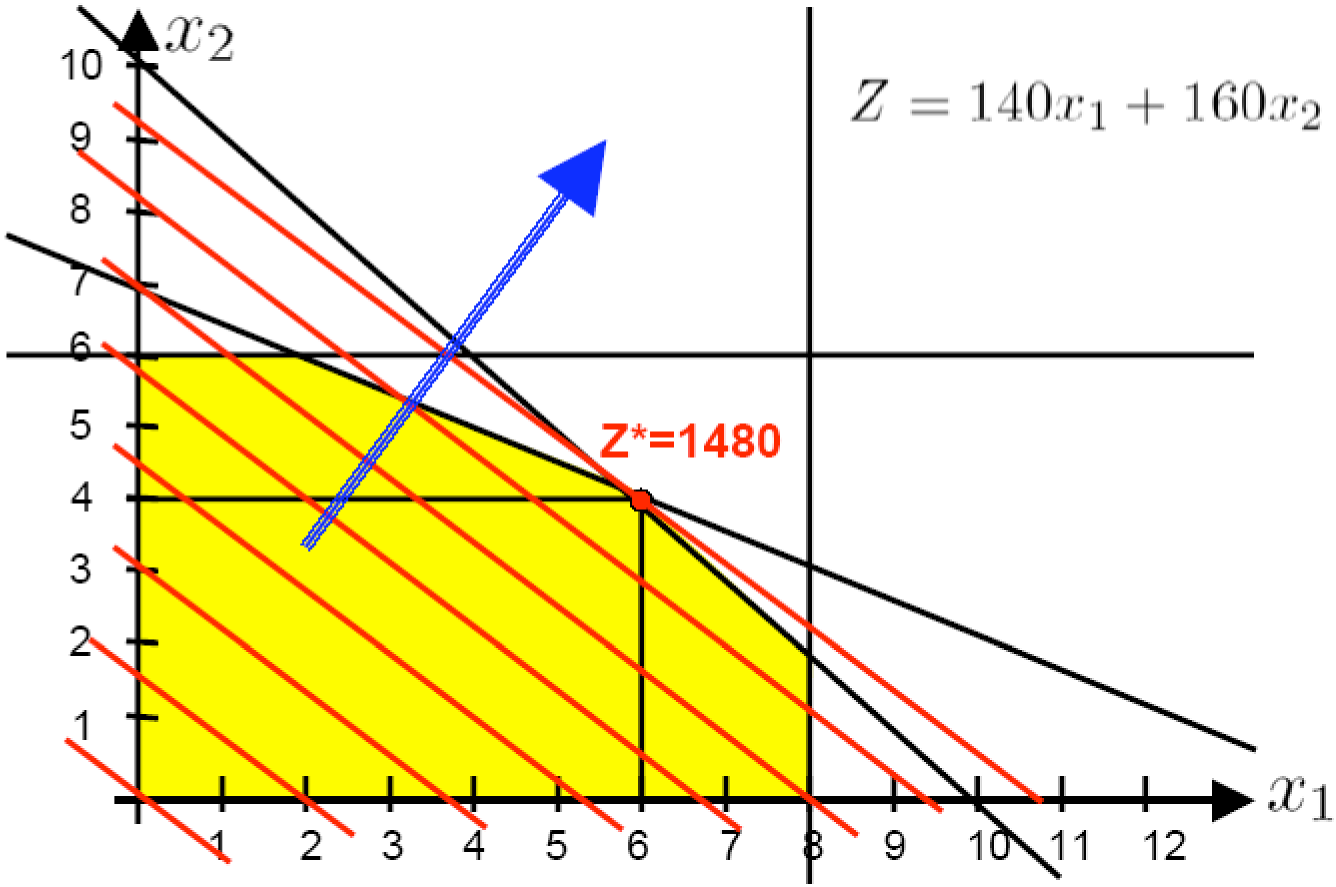
Optimal solution: isolines



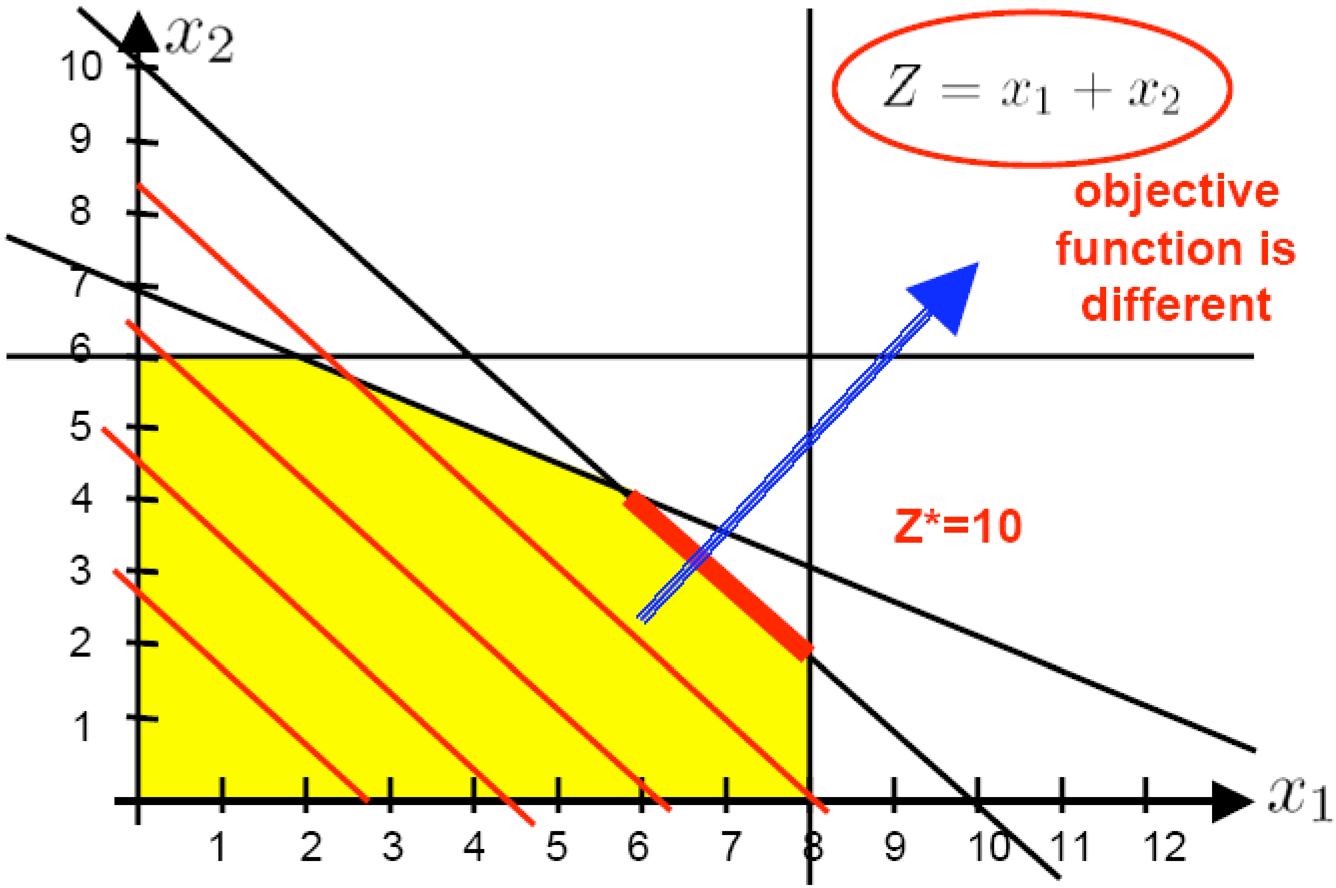
Optimal solution: isolines



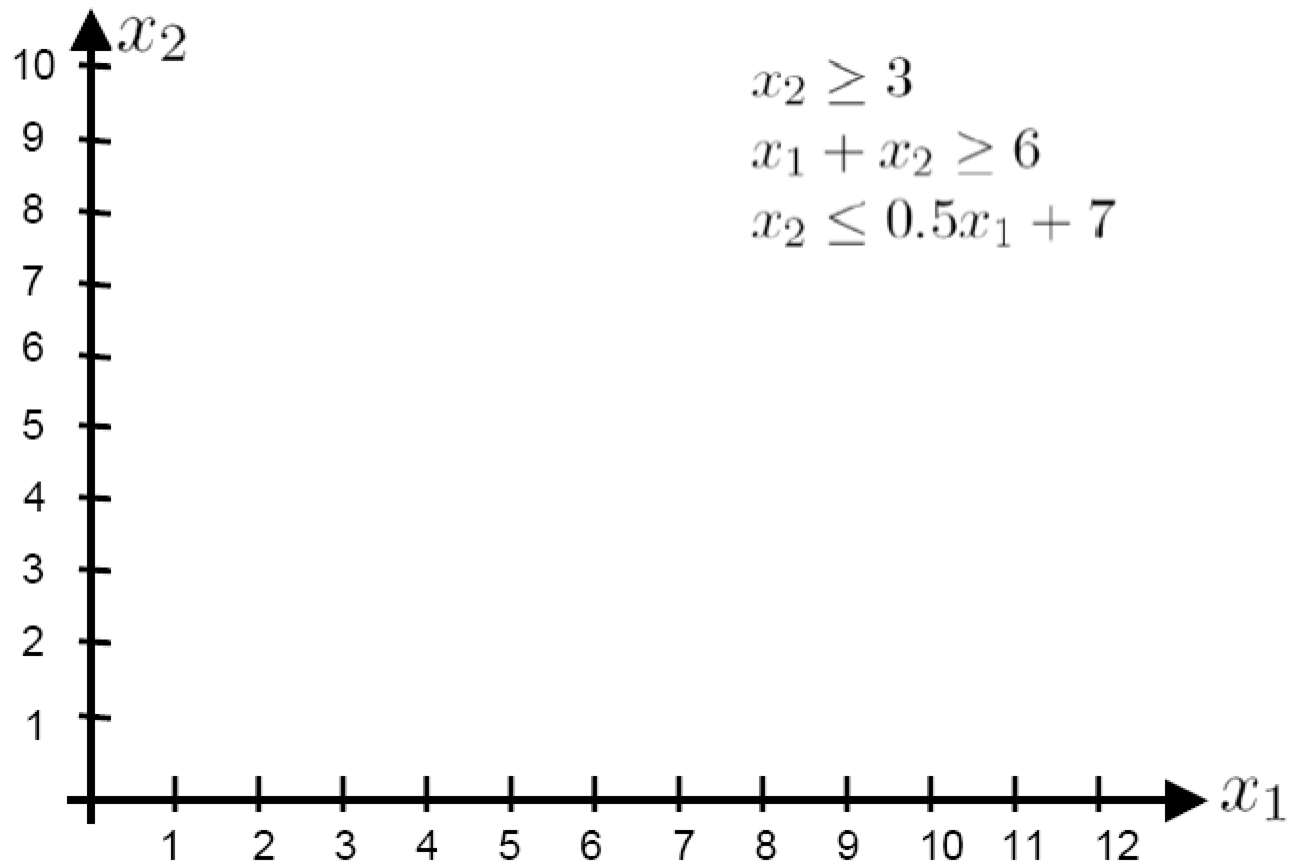
Gradient of the cost function



Uniqueness (or not) of the optimum



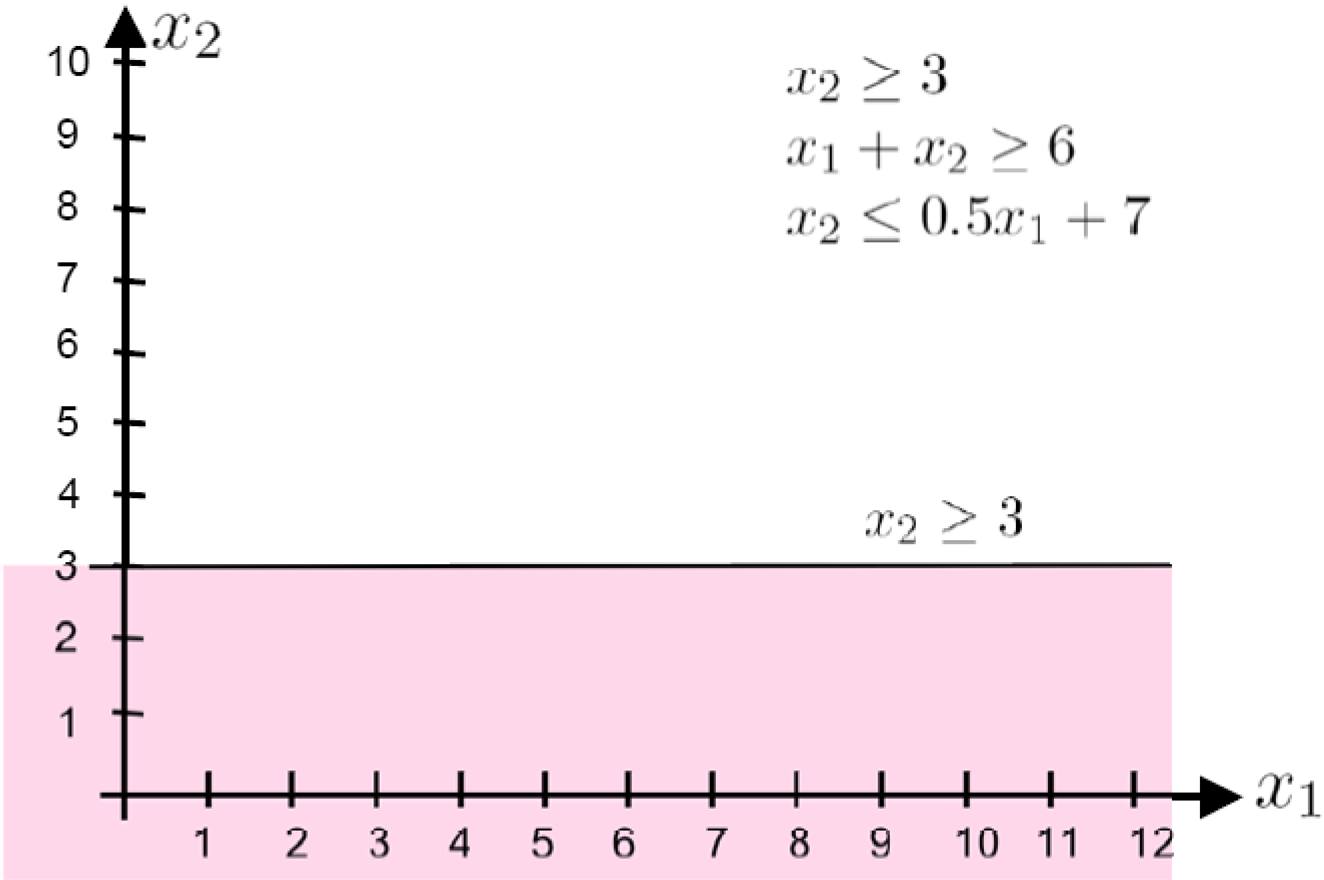
Features of the feasible set



$$x_2 \geq 3$$
$$x_1 + x_2 \geq 6$$
$$x_2 \leq 0.5x_1 + 7$$

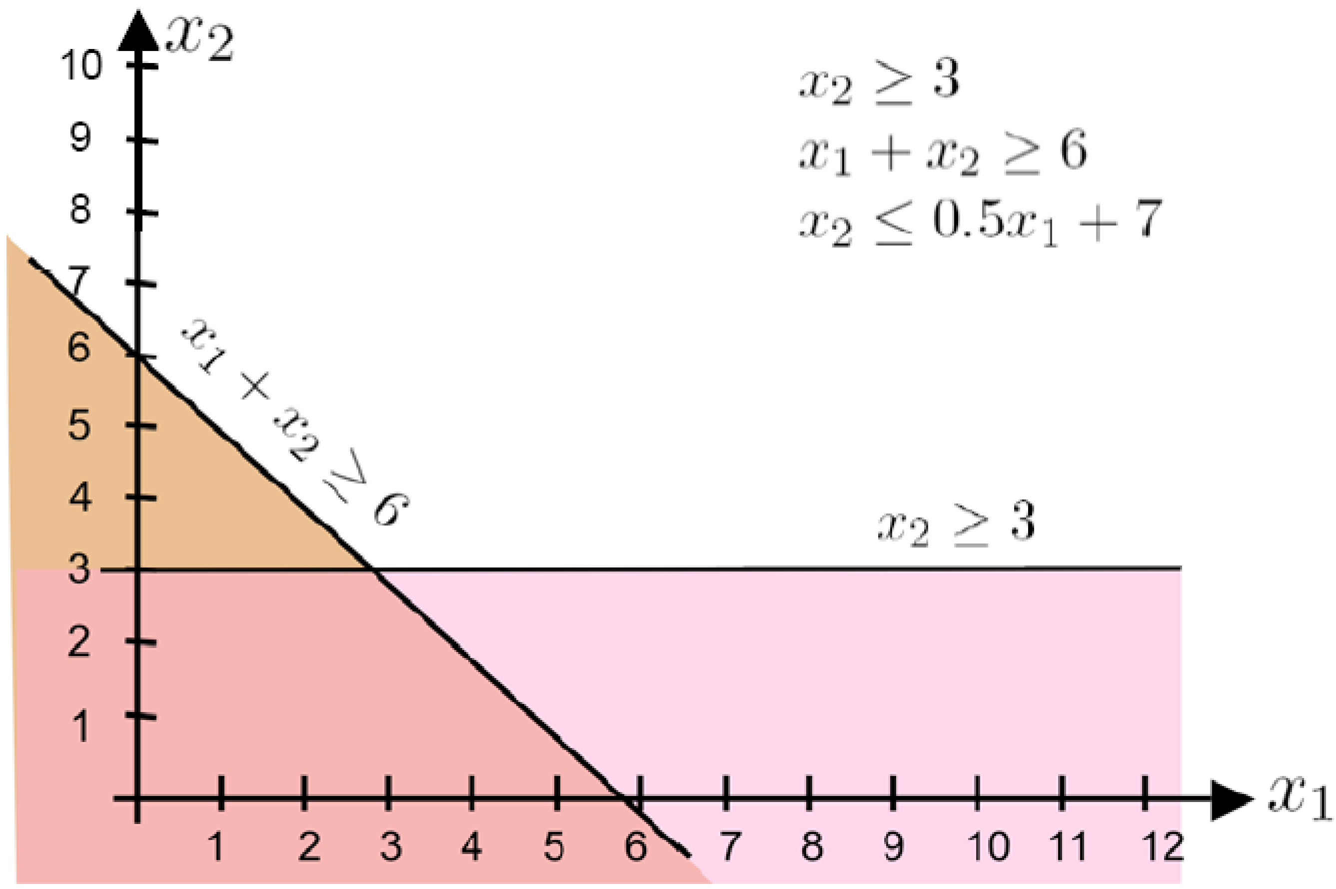
Features of the feasible set

$$x_2 \geq 3$$
$$x_1 + x_2 \geq 6$$
$$x_2 \leq 0.5x_1 + 7$$

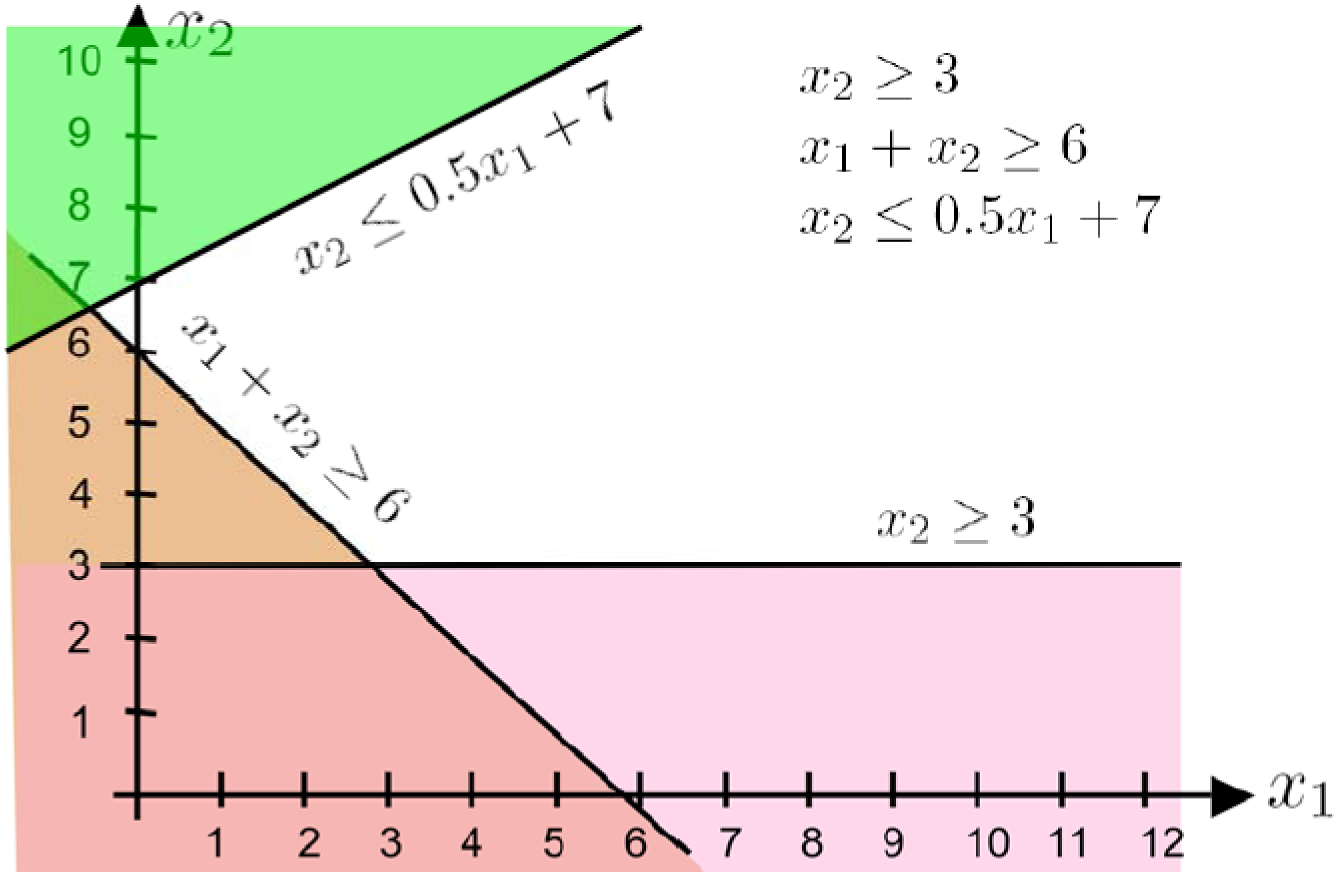


Features of the feasible set

$$x_2 \geq 3$$
$$x_1 + x_2 \geq 6$$
$$x_2 \leq 0.5x_1 + 7$$

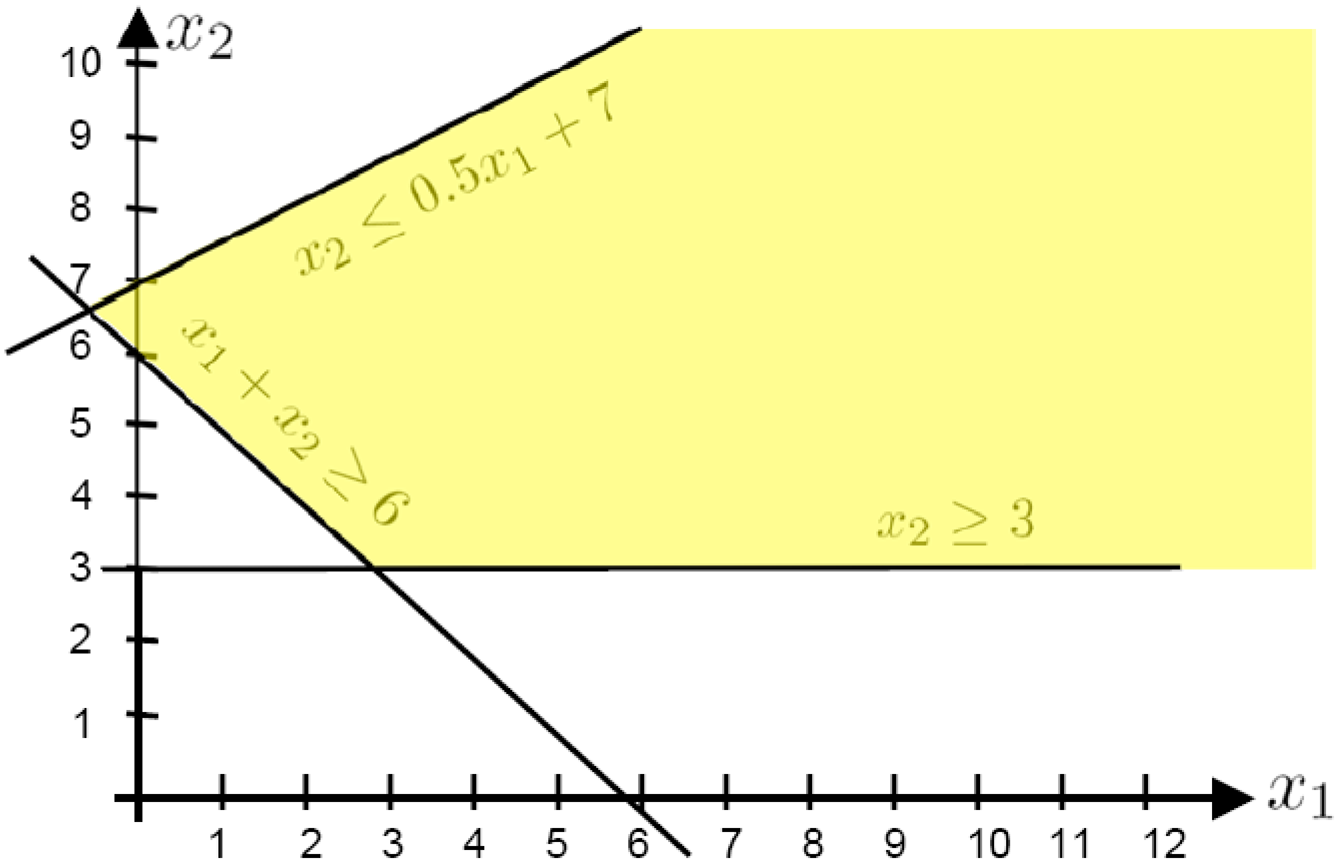


Features of the feasible set

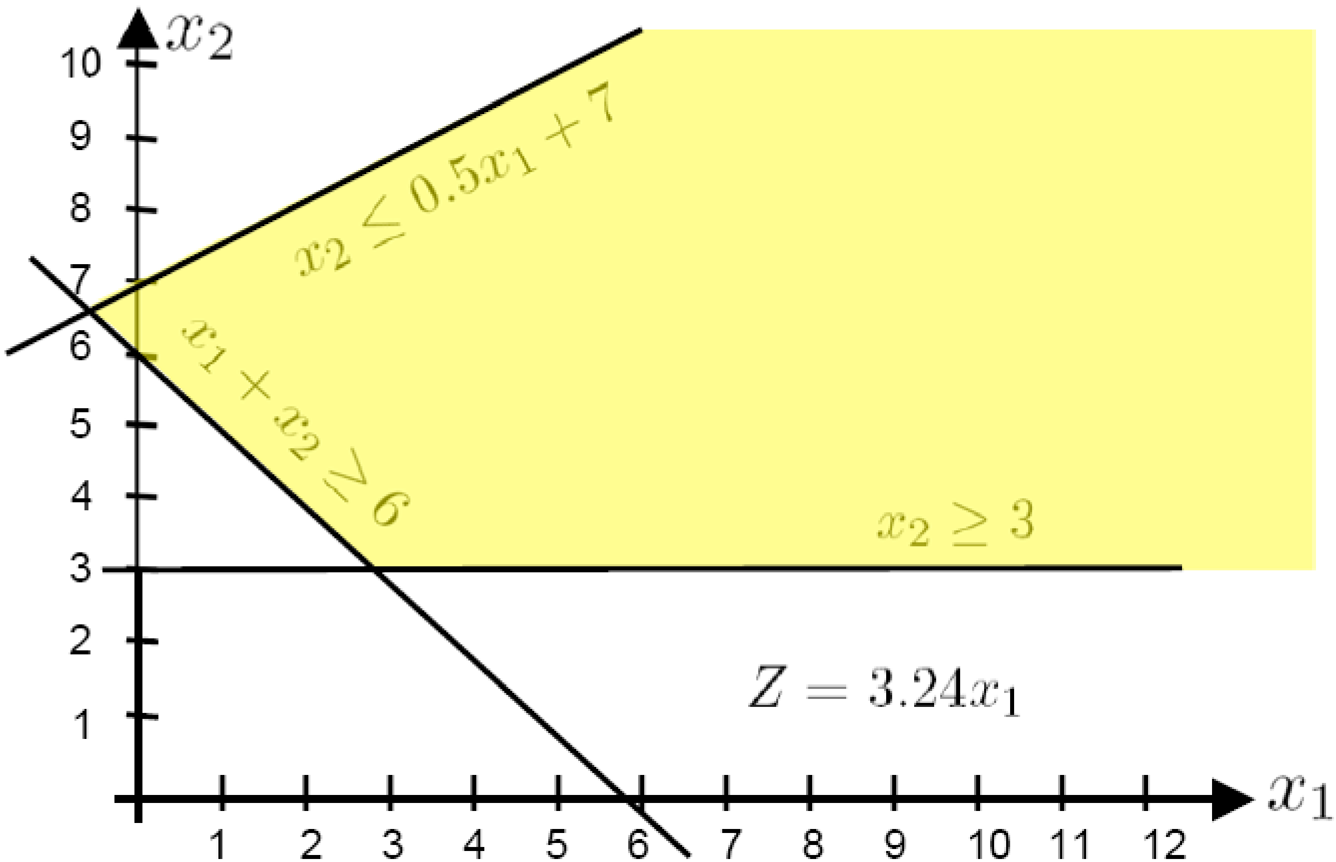


$$x_2 \geq 3$$
$$x_1 + x_2 \geq 6$$
$$x_2 \leq 0.5x_1 + 7$$

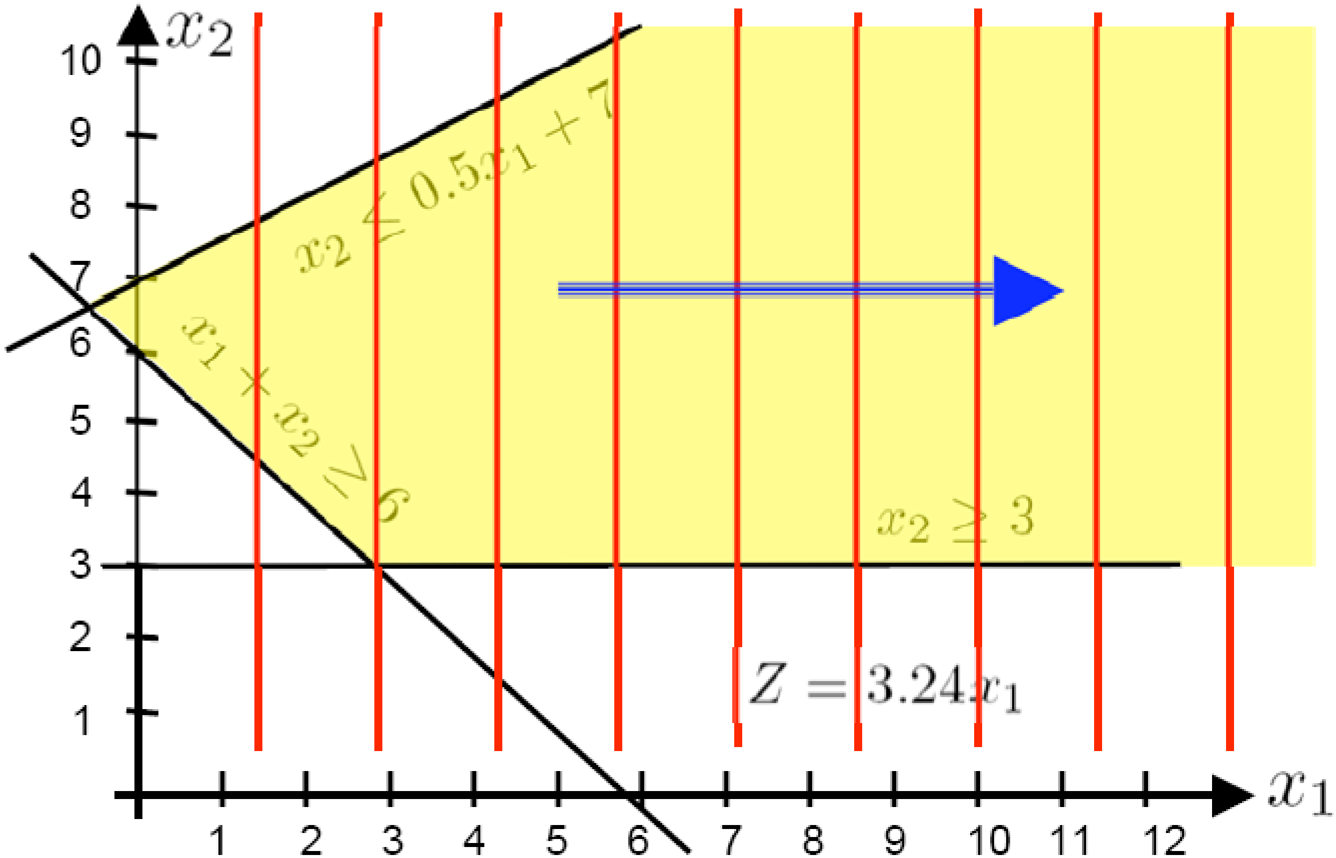
Feasible set is unbounded



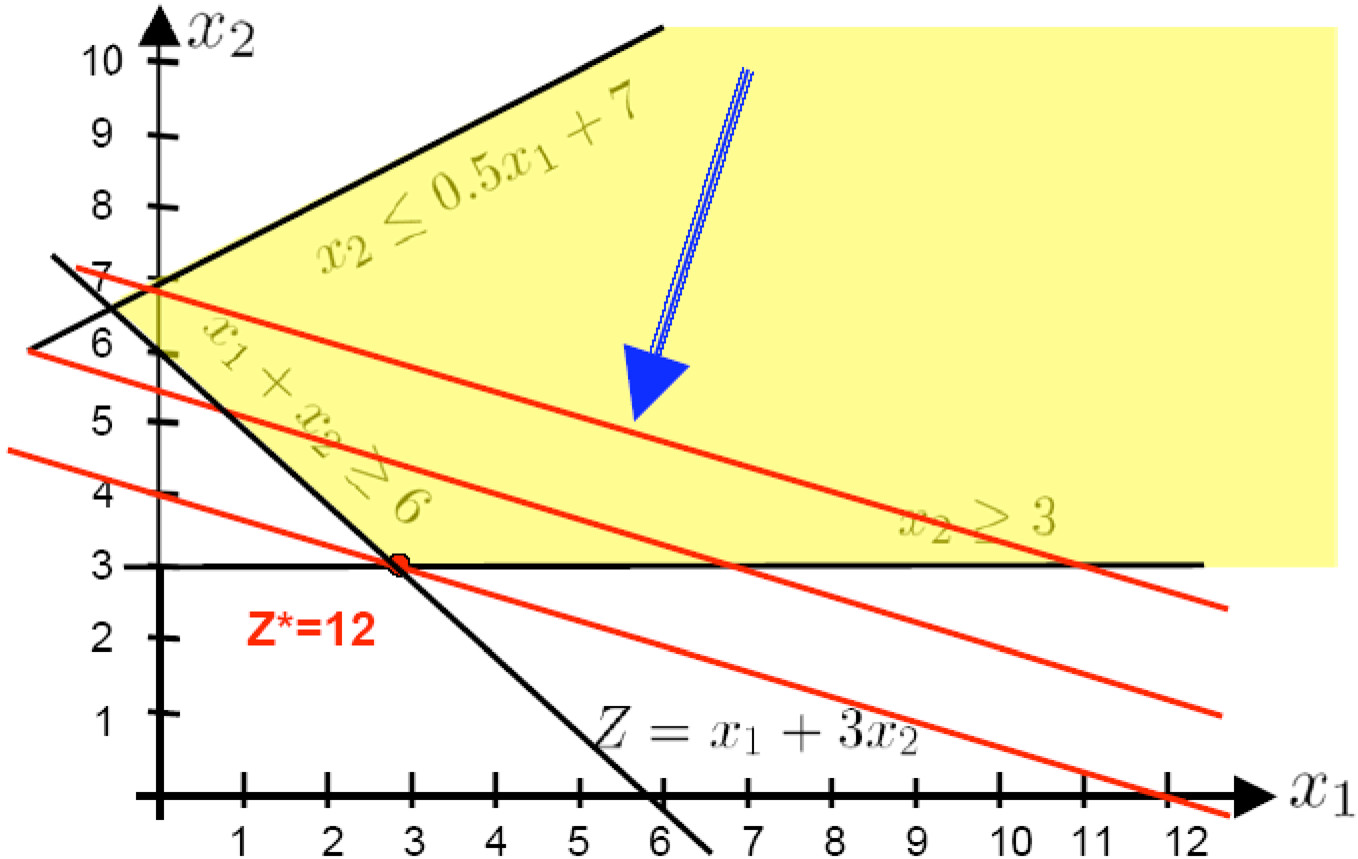
Objective function might be unbounded too



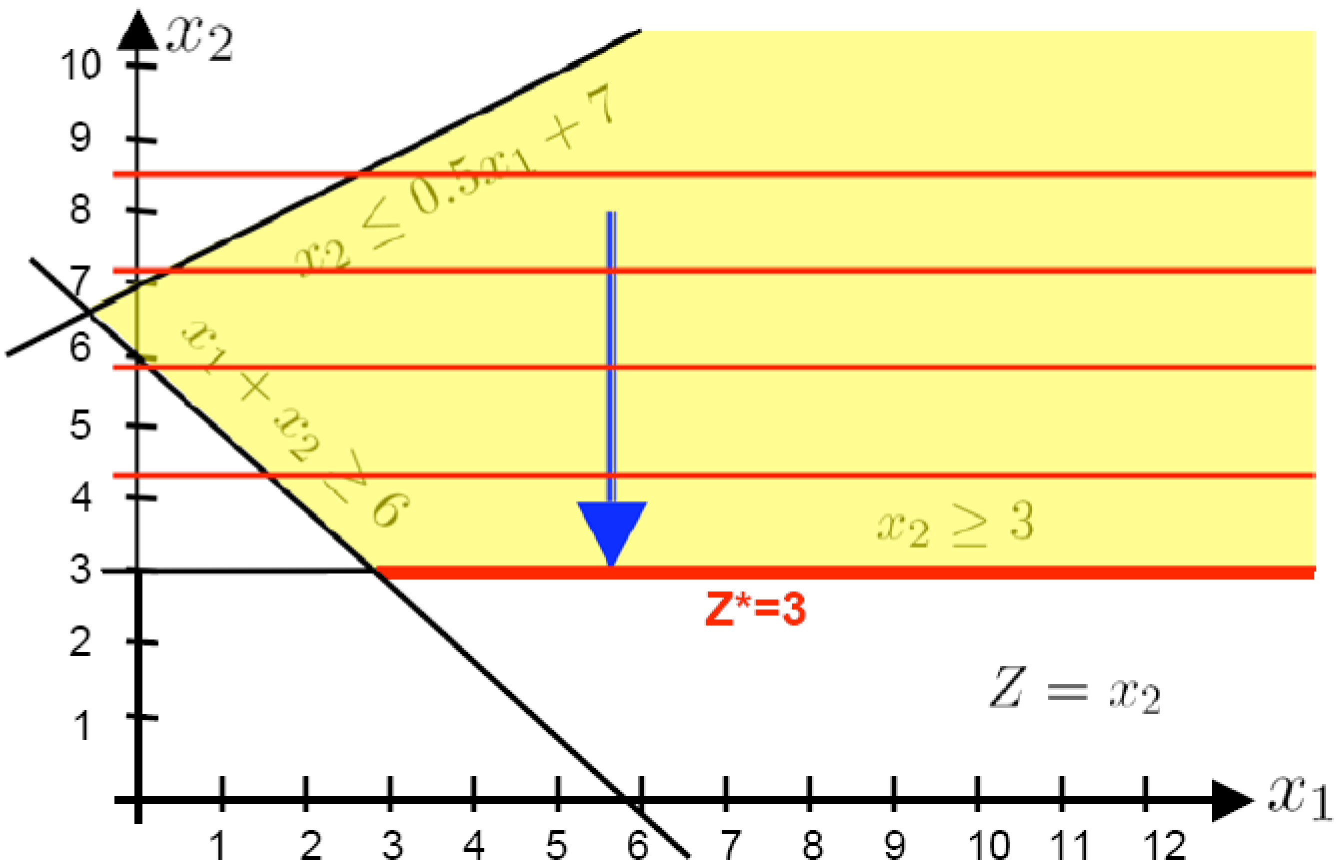
Objective function might be unbounded too



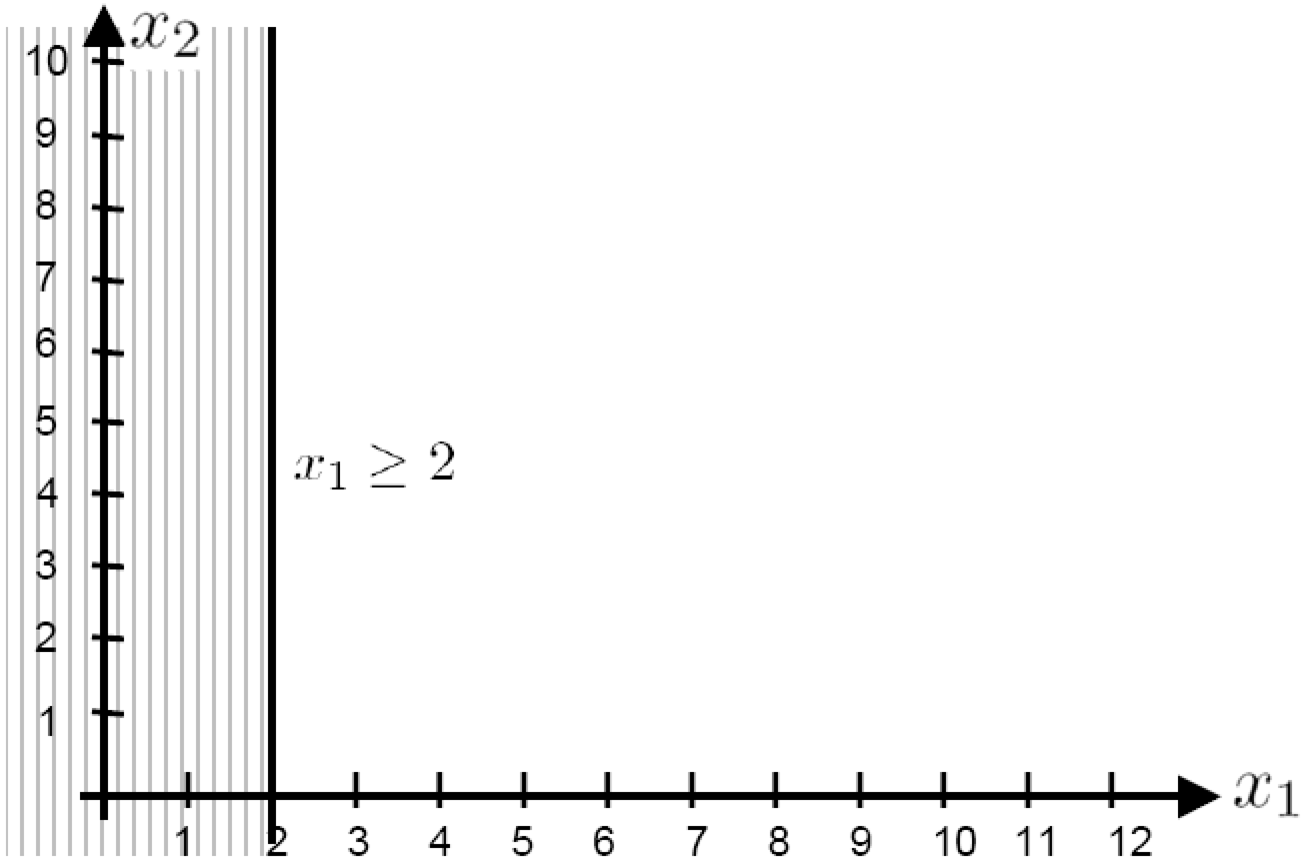
Objective might be bounded



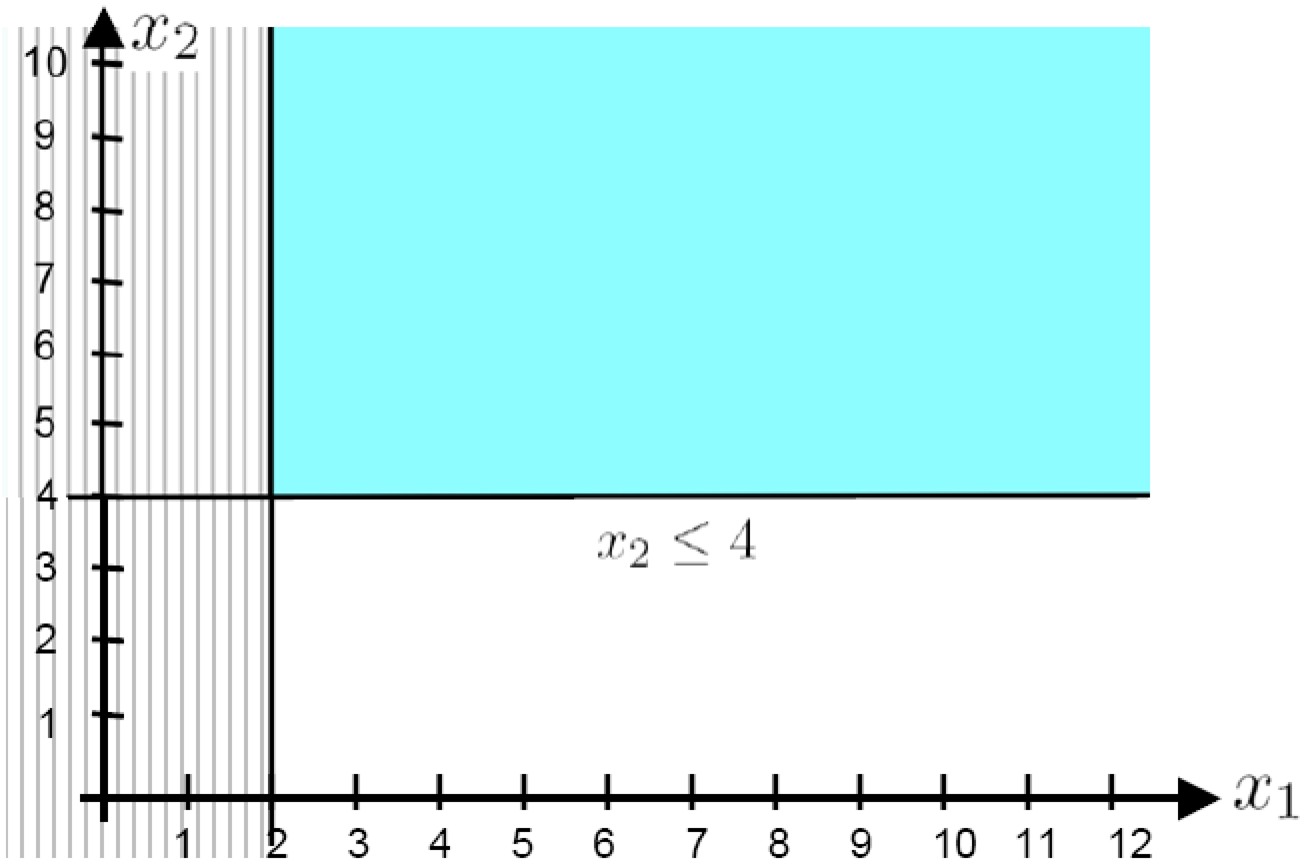
Objective might not be unique



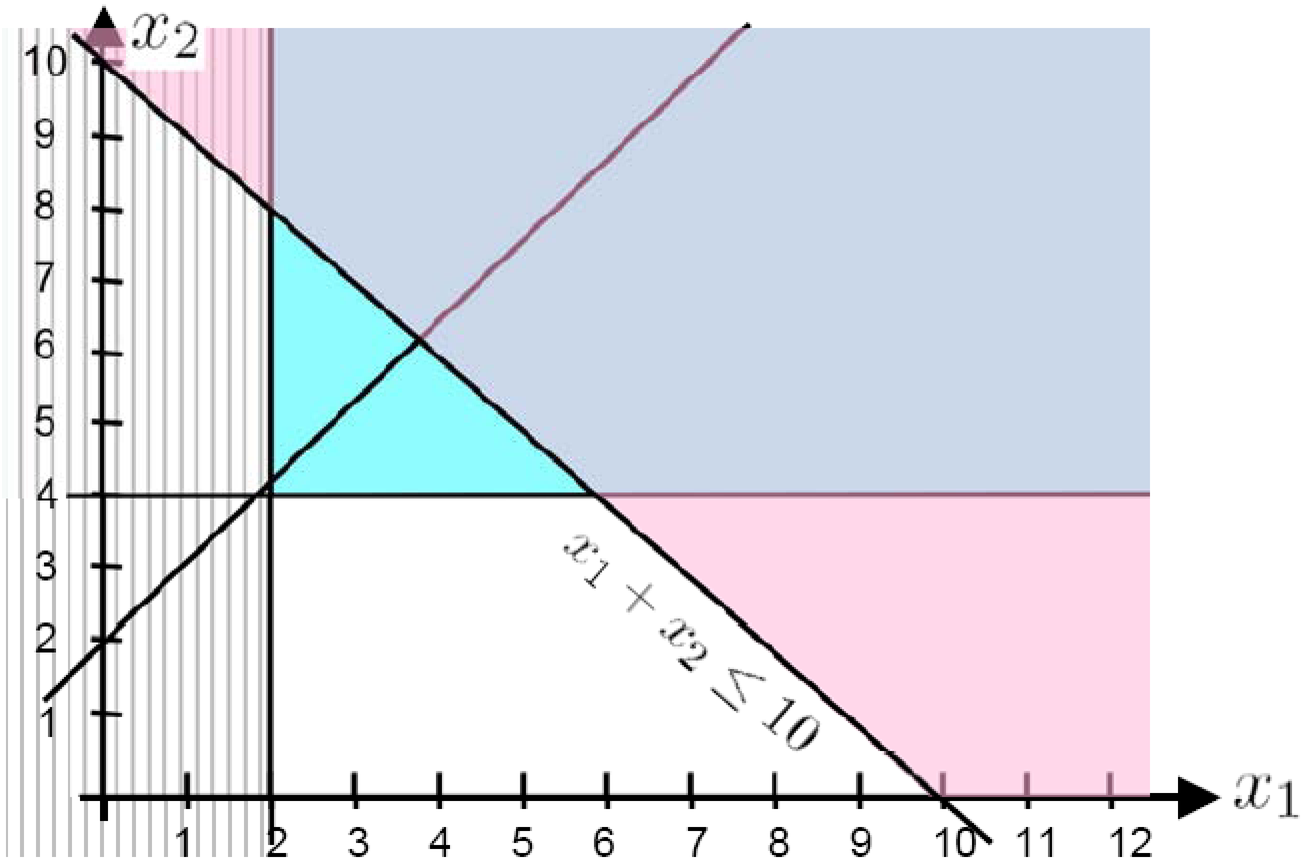
Feasible set might be empty



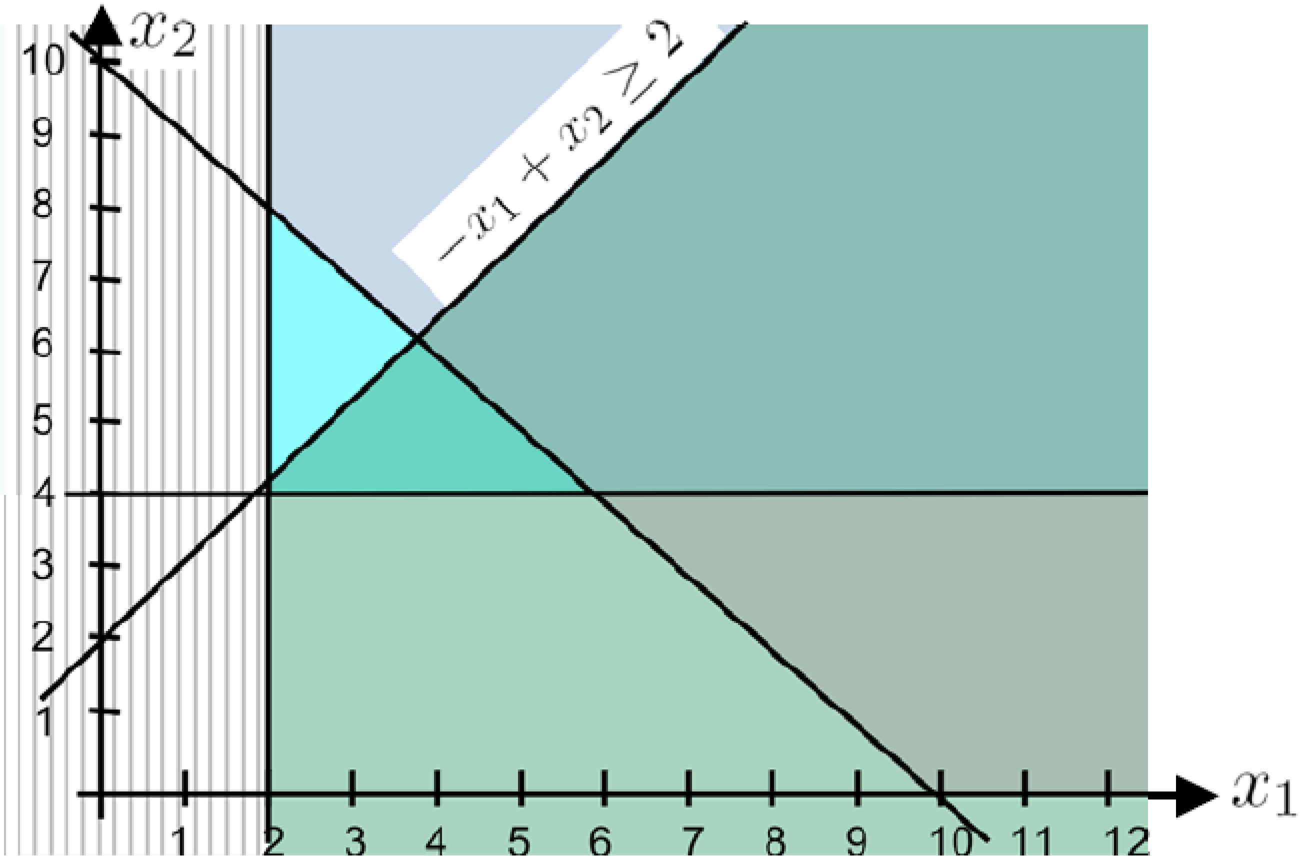
Feasible set might be empty



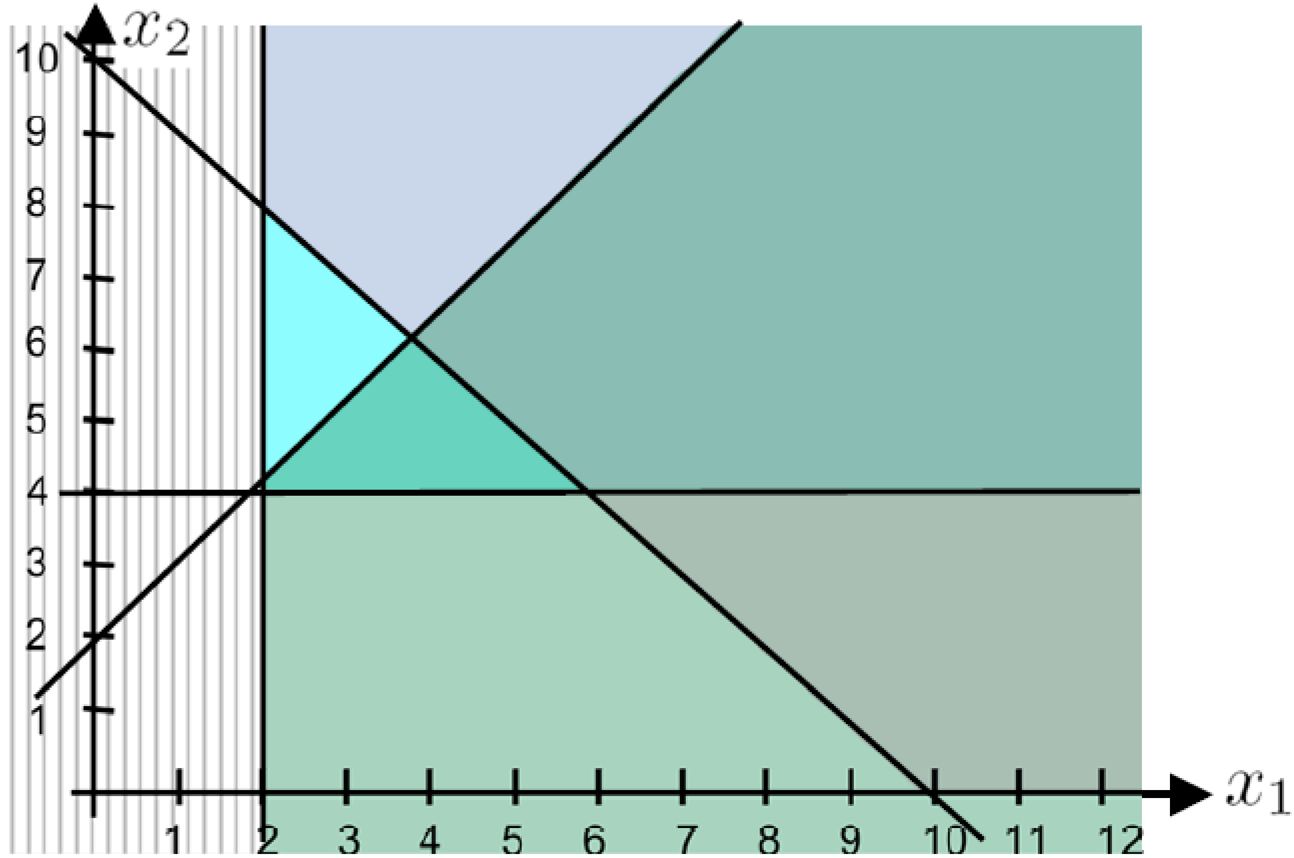
Feasible set might be empty



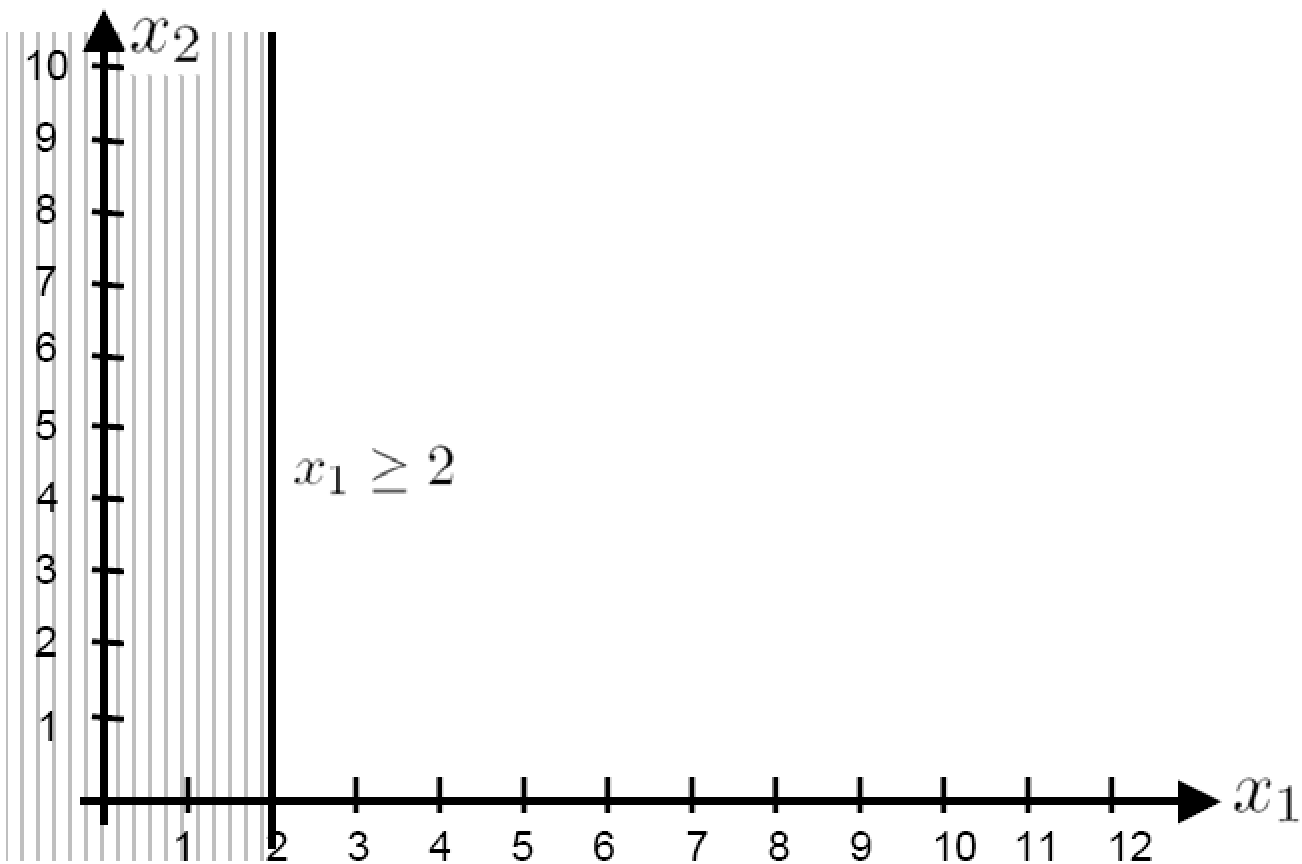
Feasible set might be empty



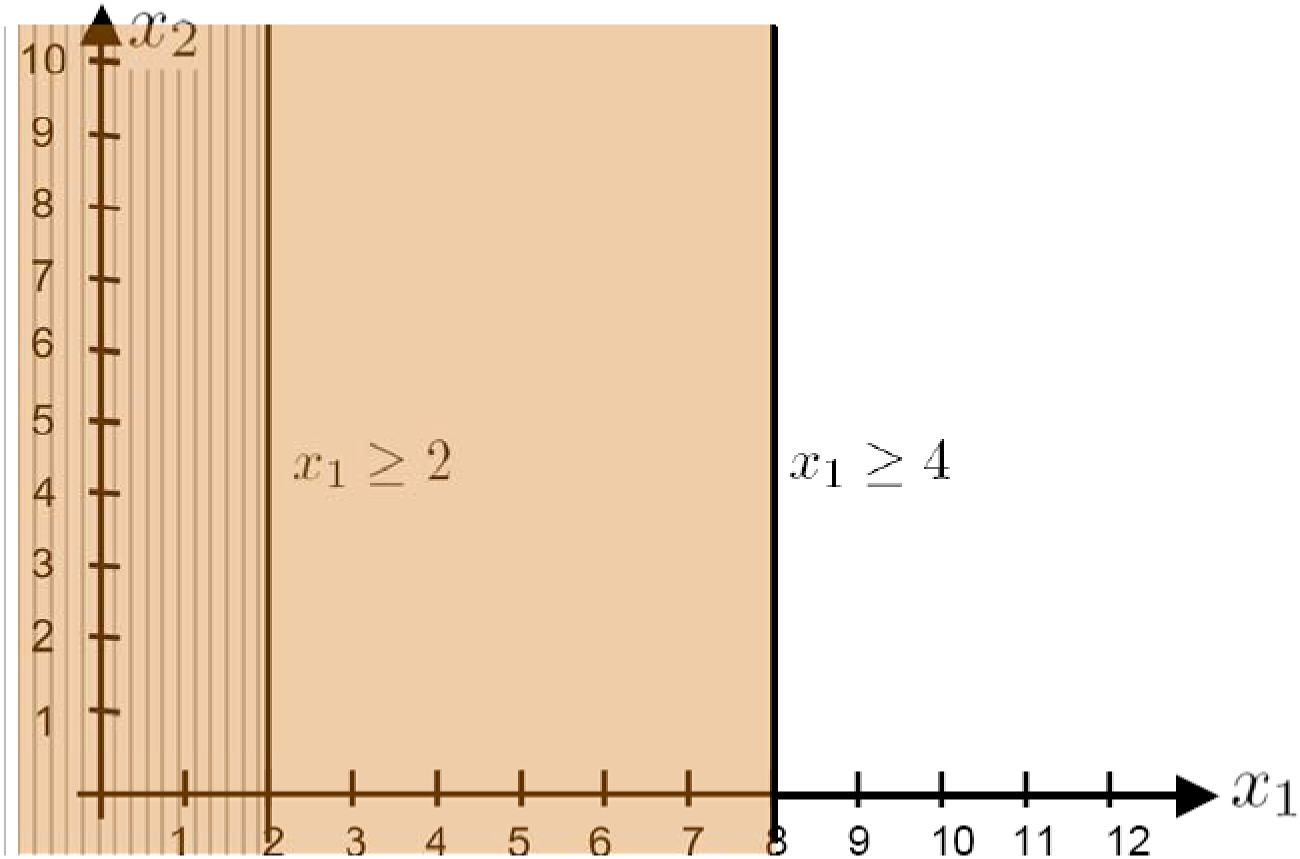
Feasible set might be empty



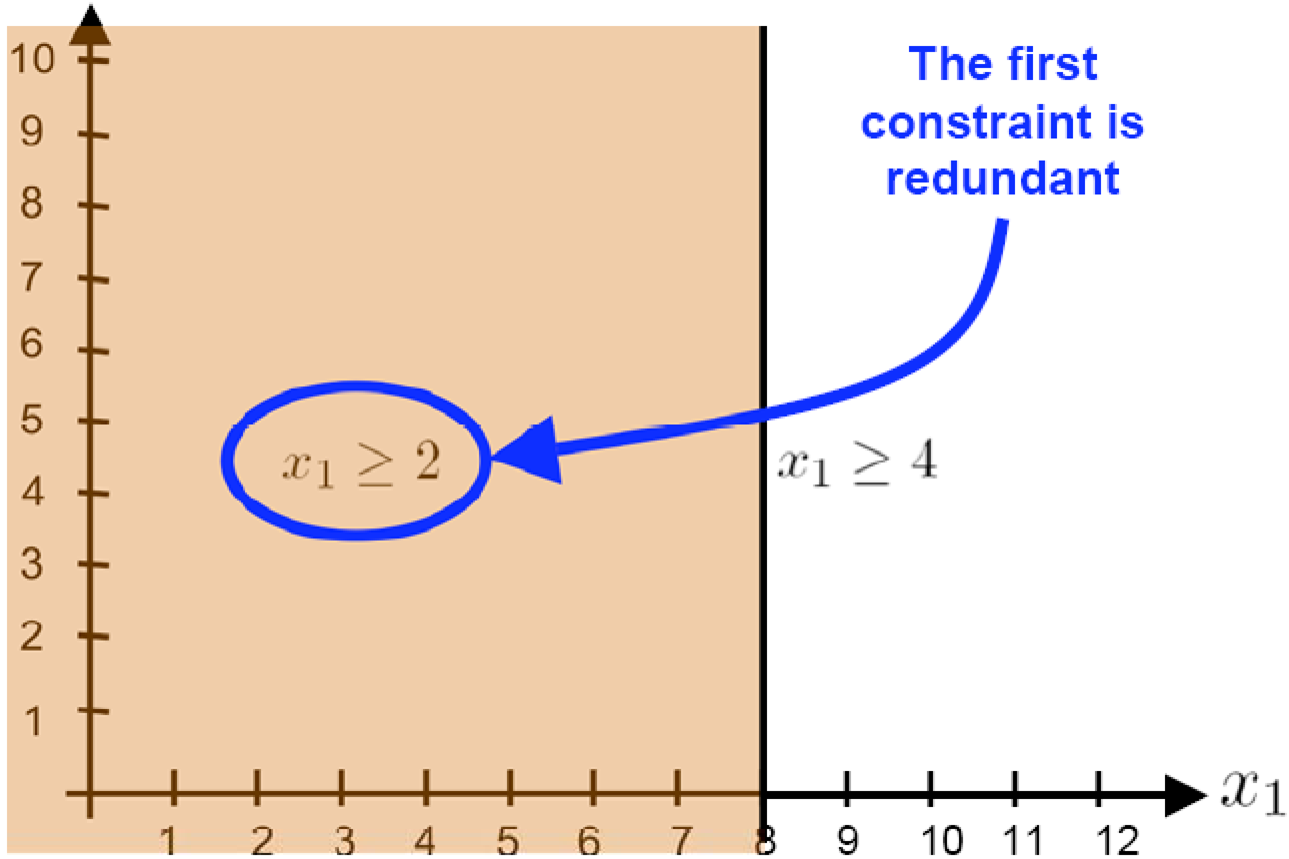
Redundant constraints



Redundant constraints



Redundant constraints



Graphical solution of LPs: general method

- Write your LP
- Successively eliminate half spaces corresponding to your constraints
- Is the feasible set empty?
  - YES → problem infeasible
  - NO → is the feasible set bounded?
    - NO → is solution finite?
      - NO: → finished
      - YES → is there a unique solution?
        - YES → corner point → finished
        - NO → face →finished
    - YES → is there a unique solution?
      - YES → corner point → finished
      - NO → face →finished