Quantum Mechanics Exam
Angular Momentum and Rotations

Information that might be useful:

$$J_\pm|jm\rangle = \hbar\sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle, \qquad J_\pm = J_x \pm iJ_y$$

$$d^{(1)}_{1,\pm 1} = \tfrac{1}{2}(1\pm\cos\theta), \qquad d^{(1)}_{1,0} = -\tfrac{1}{\sqrt{2}}\sin\theta, \qquad d^{(1)}_{0,0} = \cos\theta$$

$$T^k_q = \sum_{q_1,q_2} X^{k_1}_{q_1} Y^{k_2}_{q_2} \langle k_1 q_1; k_2 q_2 | kq\rangle$$

1. [30 points total]

(a) [5 points] Write an expression for J_x and J_y in terms of the ladder operators J_+ and J_-.

(b) [5 points] A quantum state is rotated around the y-axis by an angle θ. Write an expression for the rotation operator $U(\hat{y},\theta)$ in terms of the raising and lowering operators $J_\pm$.

(c) [20 points] An angular momentum eigenstate $|j, m=j\rangle$ is rotated around the y-axis by a small angle ϵ. After the rotation, a measurement of j and m is made. Obtain an approximate expression for the probability that the rotated state is measured to be in the same state as the original $|j, m=j\rangle$, keeping terms up to $\mathcal{O}(\epsilon^2)$.

2. [20 points total] Add angular momenta $j_1 = 1$ and $j_2 = \frac{1}{2}$ (e.g., two particles with spin-1 and spin-1/2, or one particle with spin-1/2 in an orbital $l = 1$).

(a) What are the possible values for the total angular momentum j? What are the possible values for m?

(b) Express all possible eigenstates $|j,m\rangle$ in terms of the tensor product basis $|j_1 m_1; j_2 m_2\rangle$. For simplicity, you may omit the values of j_1 and j_2 and write your answer as

$$|j=\#, m=\#\rangle = \#|{+}{+}\rangle + \#|{+}0\rangle + \#|{+}{-}\rangle + \#|0{+}\rangle + \#|{-}0\rangle + \dots$$

where the first symbols $(+,-)$ denote $m_1 = +\frac{1}{2}, -\frac{1}{2}$ respectively, and the second $(+,0,-)$ denote $m_2 = +1, 0, -1$.

3. [15 points total] Consider an eigenstate of orbital angular momentum with $|l = 1, m = 0\rangle$. Suppose this state is rotated by an angle β about the y-axis. Find the probability for the rotated state to be measured with $m = 0, -1, +1$.

4. [45 points] Consider a vector operator $\mathbf{V}$ with Cartesian components (V_x, V_y, V_z).

(a) [5 points] How does this operator transform under a rotation? That is, calculate $UV_iU^\dagger$, where $U = U(\hat{n},\theta)$ is the quantum rotation operator, and R is a 3×3 rotation matrix in Euclidean space.

(b) [5 points] Write $\mathbf{V}$ as a rank-1 $k = 1$ spherical tensor operator $T_q^{(k)}$. Explicitly write all components $q = 1, 0, -1$ in terms of the Cartesian operators $V_{x,y,z}$. How does $T_q^{(1)}$ transform under rotation? (Compute $UT_q^{(1)}U^\dagger$.)

(c) [5 points] Consider the angular momentum eigenstates $|jm\rangle$:

$$J^2|jm\rangle = \hbar^2 j(j+1)|jm\rangle, \qquad J_z|jm\rangle = \hbar m|jm\rangle$$

What relations among j, j', m, m', q are necessary for the following matrix element to be non-zero?

$$\langle j'm'|T_q^{(1)}|jm\rangle \neq 0$$

(d) [15 points] Consider a second vector operator $\mathbf{W} = (W_x, W_y, W_z)$. Construct a rank-1 spherical tensor operator from $\mathbf{W}$ and $\mathbf{V}$, using the notation $Z_q^{(1)}$ for this new operator.

(e) [15 points] Using the Wigner–Eckart theorem, write an expression for the following ratios:

$$\frac{\langle j'm'|T_{+1}^{(1)}|jm\rangle}{\langle j', m'''|T_{-1}^{(1)}|jm''\rangle}, \qquad \frac{\langle j'm'|Z_{+1}^{(1)}|jm\rangle}{\langle j', m'''|Z_{-1}^{(1)}|jm''\rangle}$$

Each matrix element satisfies the selection rules from part (c) and is non-zero. Explain how these ratios are related.

5. [50 points] Consider the spin angular momentum of a system of spin-$\frac{1}{2}$ particles (ignore orbital angular momentum).

(a) [5 points] How many possible spin states can one spin-$\frac{1}{2}$ particle have? How many total spin states can two spin-$\frac{1}{2}$ particles have?

(b) [5 points] What are the possible values of total spin s for a system of two spin-$\frac{1}{2}$ particles? How many states are there with each value of s (i.e., what is the degeneracy)?

(c) [15 points] Explicitly write an expression for each state of total angular momentum $|sm\rangle$ in terms of tensor product states with definite angular momentum of single particles

$$|m_1 m_2\rangle \equiv |s_1 = \tfrac{1}{2}, m_1\rangle \otimes |s_2 = \tfrac{1}{2}, m_2\rangle.$$

(d) [15 points] A third spin-$\frac{1}{2}$ particle is added to the system. What is the maximum value of total spin s for this three-particle state? Write the stretched state $|s_{\max}, m = s_{\max}\rangle$ in terms of product states $|m_1 m_2 m_3\rangle$, and use this to find all possible states with $s = s_{\max}$.

(e) [10 points] What are all the possible values of total spin s of a system of three spin-$\frac{1}{2}$ particles? How many states are there with each value of s, and what is the total number of states?

6. [40 points] Consider a spinless particle inside an empty sphere of radius R. That is, its wave function obeys the free particle Schrödinger equation $V(r) = 0$ for $r < R$, but vanishes outside $\psi(r \geq R) = 0$, where $r = \sqrt{\mathbf{x}^2}$.

$$V(\mathbf{x}) = V(r) = \begin{cases} 0, & r < R, \\ \infty, & r \geq R \end{cases} \tag{1}$$

(a) [5 points] What symmetries does the system have? List as many operators as you can that commute with the Hamiltonian.

(b) [15 points] Write the time-independent Schrödinger equation in configuration space for the particle in the region $r < R$, using spherical coordinates (r, θ, ϕ). For a wave-function with angular momentum ℓ, write an equation for the radial (r) dependence.

(c) [10 points] Solve the radial equation to find the general solution of the (free) Schrödinger equation.

(d) [10 points] For spherically symmetric wavefunctions ($\ell = 0$), enforce the boundary condition at radius $r = R$ to find the energy spectrum and the associated energy eigenfunctions with $\ell = 0$. You do not have to normalize the wave function.

7. [25 points] A spin-1 particle has its spin component along the direction

$$\hat{n} = \frac{1}{\sqrt{2}}(1, 0, 1) \tag{21}$$

measured with result $\hbar$. Subsequently S_z is measured, with probabilities for the three possible outcomes.

Let R be a rotation that maps the $\hat{z}$ axis into $\hat{n}$, that is,

$$R\hat{z} = \hat{n}. \tag{22}$$

Express the probabilities of the three possible measurement outcomes in terms of the rotation matrix

$$D^1_{mm'}(R) \equiv \langle j = 1, m'|U(R)|j = 1, m\rangle. \tag{23}$$

Solve this matrix to find the probabilities explicitly.