MAT216

Assignment 2

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Section: 17

1. $A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

we know, $|A - \lambda I| = 0$, for $Ax = \lambda x$, where x is an eigenvector corresp to λ, where scalar λ is an eigenvalue

$$|A - \lambda I| = \left| \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} -2-\lambda & 0 & 0 & 0 \\ 0 & -2-\lambda & 5 & -5 \\ 0 & 0 & 3-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} \right|$$

$$(-2-\lambda) \left| \begin{bmatrix} -2-\lambda & 5 & -5 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix} \right|$$

$$= (-2-\lambda)(-2-\lambda) \left| \begin{bmatrix} 3-\lambda & 0 \\ 0 & 3-\lambda \end{bmatrix} \right|$$

$$= (-2-\lambda)(-2-\lambda)(3-\lambda)(3-\lambda) = 0$$

$\lambda = -2, 3, -2, 3$

For $\lambda = -2$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -5 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \quad R_2' = R_2 - R_3 + R_4$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix} \Rightarrow \begin{aligned} 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \\ 0x_1 + 0x_2 + 5x_3 + 0x_4 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 5x_4 &= 0 \end{aligned}$$

$5x_3 = 0$

$x_3 = 0$

$5x_4 = 0$

$x_4 = 0$

let $x_1 = s$

$x_2 = t$, so $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} s \\ t \\ 0 \\ 0 \end{pmatrix}$

$$s \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1} + t \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2}$$

eigenvectors with respect to $\lambda = -2$

$$E_{-2} = \text{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{v_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}}_{v_2} \right\}$$

For $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} -5 & 0 & 0 & 0 \\ 0 & -5 & 5 & -5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{aligned} -5x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \\ 0x_1 - 5x_2 + 5x_3 - 5x_4 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

$-5x_1 = 0$ let $x_3 = s$, $x_4 = t$

$x_1 = 0$

$-5x_2 + 5x_3 - 5x_4 = 0$

$-5x_3 + 5s - 5t = 0$

$x_2 = s - t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ s-t \\ s \\ t \end{pmatrix}$$

$$s \overbrace{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}^{v_3} + t \overbrace{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}}^{v_4}$$

eigenvector with respect to $\lambda = 3$

$$E_3 = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

2. $f(x) = x\sin x, \ -\pi < x < \pi \Rightarrow 2L = 2\pi \Rightarrow L = \pi$

For fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos nx\,dx \quad \Rightarrow \quad \begin{aligned} u &= x \\ du &= 1 \\ dv &= \sin x\cos nx \\ v &= \int \sin x \cos nx\,dx \\ &= \tfrac{1}{2}\int \sin(x+nx) + \sin(x-nx)\,dx \\ &= \tfrac{1}{2}\int \sin(1+n)x + \sin(1-n)x\,dx \\ &= \tfrac{1}{2}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right] \end{aligned}$$

we know

$$\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

$, A = x$

$B = nx$

$$a_n = \frac{1}{\pi}\left[\frac{x}{2}\left(-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right) - \frac{1}{2}\int -\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\,dx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x}{2}\left(-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right) + \frac{1}{2}\left(+\frac{\sin(1+n)x}{(1+n)^2} + \frac{\sin(1-n)x}{(1-n)^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\left(-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n}\right) + \frac{1}{2}\frac{\sin(1+n)\pi}{(1+n)^2} + \frac{1}{2}\frac{\sin(1-n)\pi}{(1-n)^2} + \right.$$

$$\left.\frac{\pi}{2}\left(-\frac{\cos(1+n)(-\pi)}{1+n} - \frac{\cos(1-n)(-\pi)}{1-n}\right) - \frac{1}{2}\frac{\sin(1+n)\pi}{(1+n)^2} - \frac{1}{2}\frac{\sin(1-n)\pi}{(1-n)^2}\right)$$

we know, $\sin k\pi = 0$, $\cos A = \cos(-A)$

so,

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\left(\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n}\right) + \frac{1}{2}\left(-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n}\right)\right)$$

$$= -\frac{\frac{1}{2}\cos(1+n)\pi}{1+n} - \frac{\frac{1}{2}\cos(1-n)\pi}{1-n} - \frac{\frac{1}{2}\cos(1+n)\pi}{1+n} - \frac{\frac{1}{2}\cos(1-n)\pi}{1-n}$$

$$= -\frac{\cos \pi + n\pi}{1+n} - \frac{\cos \pi - n\pi}{1-n} \qquad \left[\cos(A+B) = \cos A\cos B - \sin A\sin B\right]$$

$$= -\frac{\cos\pi\cos n\pi - \sin\pi\sin n\pi}{n+1} - \frac{\cos\pi\cos n\pi - \sin\pi\sin n\pi}{1-n}$$

$$\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n}$$

$$\frac{(-1)^n(1+n+1-n)}{1-n^2} \qquad \frac{2(-1)^n}{1-n^2} = a_n$$

$$a_0 = \frac{2(-1)^0}{1-0^2} = 2$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x \sin nx\, dx \Rightarrow$$

$u = x$
$du = 1$
$dv = \sin x \sin nx$
$v = \int \sin x \sin nx\, dx$
$= \frac{1}{2}\int \cos(x - nx) - \cos(x + nx)$
$= \frac{1}{2}\int \cos(1-n)x - \cos(1+n)x$
$\frac{1}{2}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]$

we know,

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\cos(A-B) - \cos(A+B) = 2\sin A\sin B$

$\frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B) = \sin A \sin B$

$A = x,\ B = nx$

$$b_n = \frac{1}{\pi}\left[\frac{x}{2}\left(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right) - \frac{1}{2}\int \frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{x}{2}\left(\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right) + \frac{1}{2}\frac{\cos(1-n)x}{(1-n)^2} - \frac{1}{2}\frac{\cos(1+n)x}{(1+n)^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2}\frac{\cos(1-n)\pi}{(1-n)^2} - \frac{\pi}{2}\frac{\cos(1+n)\pi}{(1+n)^2} - \frac{1}{2}\frac{\cos(1-n)(-\pi)}{(1-n)^2} + \frac{1}{2}\frac{\cos(1+n)(-\pi)}{(1+n)^2}\right)$$

we know, $\cos(A) = \cos(-A)$

$$= \left(\frac{1}{2}\frac{\cos(1-n)\pi}{(1-n)^2} - \frac{1}{2}\frac{\cos(1-n)\pi}{(1-n)^2} - \frac{1}{2}\frac{\cos(1+n)\pi}{(1+n)^2} + \frac{1}{2}\frac{\cos(1+n)(\pi)}{(1+n)^2}\right)$$

$b_n = 0$

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{2(-1)^n}{1-n^2}\cos nx \quad \text{(Ans)}$$

3. $f(x) = \begin{cases} x & 0<x<4 \\ 8-x, & 4<x<8 \end{cases}$ $\quad L=8$

$a_0 = 0$
$a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\int_0^4 x\sin\frac{n\pi x}{8}\,dx + \frac{2}{L}\int_4^8 (8-x)\sin\frac{n\pi x}{8}\,dx$$

$u = x$
$du = 1$
$dv = \sin\frac{n\pi}{8}x$
$v = -\frac{\cos\frac{n\pi}{8}x}{\frac{n\pi}{8}}$

$$\frac{2}{8}\int_0^4 x\sin\frac{n\pi x}{8}\,dx$$

$$= \frac{1}{4}\left[x\left(\frac{-8}{n\pi}\cos\frac{n\pi}{8}x\right) + \frac{8}{n\pi}\int\cos\frac{n\pi}{8}x\,dx\right]_0^4$$

$$= \frac{1}{4}\left[x\left(-\frac{8}{n\pi}\cos\frac{n\pi}{8}x\right) + \frac{64}{n^2\pi^2}\sin\frac{n\pi}{8}x\right]_0^4$$

$$= \frac{1}{4}\left(-\frac{32}{n\pi}\cos\tfrac{1}{2}n\pi + \frac{64}{n^2\pi^2}\sin\tfrac{1}{2}n\pi + 0 - 0\right)$$

$$= -\frac{8}{n\pi}\cos\tfrac{1}{2}n\pi + \frac{16}{n^2\pi^2}\sin\tfrac{1}{2}n\pi \quad \text{—(1)}$$

$$\frac{1}{4}\int_4^8 (8-x)\sin\frac{n\pi x}{8}dx$$

$$=\frac{1}{4}\int_4^8 8\sin\frac{n\pi}{8}x - \frac{1}{4}\int_4^8 x\sin\frac{n\pi x}{8}dx$$

$$=2\left[\frac{-8}{n\pi}\cos\frac{n\pi x}{8}\right]_4^8 - \frac{1}{4}\left[x\left(-\frac{8}{n\pi}\cos\frac{n\pi x}{8}\right)+\frac{64}{n^2\pi^2}\sin\frac{n\pi}{8}x\right]_4^8$$

$$=\frac{-16}{n\pi}\left(\cos n\pi - \cos\tfrac{1}{2}n\pi\right) - \frac{1}{4}\left(-\frac{64}{n\pi}\cos n\pi + \frac{64}{n^2\pi^2}\sin n\pi + \frac{32}{n\pi}\cos\tfrac{1}{2}n\pi - \frac{64}{n^2\pi^2}\sin\tfrac{1}{2}n\pi\right)$$

$$=-\cancel{\frac{16}{n\pi}\cos n\pi} + \frac{16}{n\pi}\cos\tfrac{1}{2}n\pi + \cancel{\frac{16}{n\pi}\cos n\pi}\ \frac{16}{n^2\pi^2}\sin n\pi - \frac{8}{n\pi}\cos\tfrac{1}{2}n\pi + \frac{16}{n^2\pi^2}\sin\tfrac{1}{2}n\pi$$

$$=\frac{8}{n\pi}\cos\tfrac{1}{2}n\pi - \frac{16}{n^2\pi^2}\sin n\pi + \frac{16}{n^2\pi^2}\sin\tfrac{1}{2}n\pi \quad \text{— (ii)}$$

(i) + (ii)

$$\Rightarrow \cancel{-\frac{8}{n\pi}\cos\tfrac{1}{2}n\pi} + \frac{16}{n^2\pi^2}\sin\tfrac{1}{2}n\pi + \cancel{\frac{8}{n\pi}\cos\tfrac{1}{2}n\pi} - \frac{16}{n^2\pi^2}\sin n\pi + \frac{16}{n^2\pi^2}\sin\tfrac{1}{2}n\pi$$

$$\Rightarrow \frac{32}{n^2\pi^2}\sin\tfrac{1}{2}n\pi = b_n$$

so,

$$f(x) = \sum_{n=1}^{\infty}\frac{32}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{8} \quad \text{(Ans)}$$

4. $f(x)=|x|, \ -\pi<x<\pi \qquad 2L=2\pi, L=\pi$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|x|\sin(nx)\,dx$$

$$\Rightarrow \frac{1}{\pi}\left[-\int_{-\pi}^{0}x\sin(nx)dx+\int_0^{\pi}x\sin(nx)dx\right]$$

$$\int x\sin(nx)dx \quad , u=x$$
$$du=1$$

$$=\frac{-x\cos nx}{n}+\int\frac{\cos nx}{n} \qquad dv=\sin(nx)$$
$$v=-\frac{\cos nx}{n}$$

$$=\frac{-x\cos nx}{n}+\frac{\sin(nx)}{n^2}$$

$$\Rightarrow \frac{1}{\pi}\left[-(0-0)+\left(\frac{\pi(-1)^n}{n}\right)+0+0-\frac{\pi(-1)^n}{n}+0+0-0\right]$$

$b_n \Rightarrow 0$

$b_n=0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\,dx$$

$$\Rightarrow \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\int_{-\pi}^{0} x\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right]$$

$$\int x\cos nx \qquad u = x,\quad du = 1,\quad dv = \cos nx,\quad v = \frac{\sin nx}{n}$$

$$= \frac{x\sin nx}{n} - \int \frac{\sin nx}{n}$$

$$= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$$

$$\Rightarrow \frac{1}{\pi}\left[-\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\left(0 + \frac{1}{n^2} - 0 - \frac{(-1)^n}{n^2}\right) + 0 + \frac{(-1)^n}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$\Rightarrow \frac{1}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \cancel{\frac{1}{\pi}\left[\quad 2(-1)\right.} \quad \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}$$

$$a_0 = \cancel{\frac{2}{\pi}\,\frac{(-1)^0 - 1}{0^2}} \quad \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[-\frac{x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left(-\frac{0}{2} + \frac{\pi^2}{2} + \frac{\pi^2}{2} - 0\right)$$

$$a_0 = \pi$$

$$|x| = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{2}{\pi}\,\frac{(-1)^n - 1}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2}(-1)^n$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n\left((-1)^n - 1\right)}{n^2}$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1} - (-1)^n}{n^2}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}|x|^2\,dx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi} x^2\,dx+\int_{-\pi}^{0}(-x)^2\,dx\right]$$

$$=\frac{1}{\pi}\left[\left[\frac{x^3}{3}\right]_0^{\pi}+\left[\frac{x^3}{3}\right]_{-\pi}^{0}\right]$$

$$=\frac{1}{\pi}\left(\frac{\pi^3}{3}+\frac{\pi^3}{3}\right)$$

$$=\frac{2\pi^2}{3}$$

$$\frac{a_0^2}{2}+\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

$$\frac{\pi^2}{2}+\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\,\frac{(-1)^n-1}{n^2}\right]^2+0$$

$$\frac{\pi^2}{2}+\underbrace{\sum_{n=1}^{\infty}\frac{4}{\pi^2 n^4}\left((-1)^n-1\right)^2}$$

when n is even

$$\frac{4}{\pi^2 n^4}(1-1)^2=0$$

when n is odd

$$\frac{4}{\pi^2 n^4}(-2)^2=\frac{16}{\pi^2 n^4}$$

$$\frac{\pi^2}{2}+\frac{16}{\pi^2(1)^4}+\frac{16}{\pi^2(3)^4}+\cdots$$

$$\frac{\pi^2}{2}+\frac{16}{\pi^2}\left(\frac{1}{(1)^4}+\frac{1}{3^4}+\frac{1}{5^4}\cdots\right)$$

So final eqn

$$\frac{2}{3}\pi^2=\frac{\pi^2}{2}+\frac{16}{\pi^2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$$

$$\frac{\pi^2}{6}=\frac{16}{\pi^2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots\right)$$

$$\frac{\pi^4}{96}=1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots$$

$$\therefore\ 1+\frac{1}{3^4}+\frac{1}{5^4}+\cdots=\frac{\pi^4}{96}\quad\text{(Ans)}$$