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Enhancing Business System Performance Through Confidence-Based Algebraic Aggregation and the p, q, r-Fraction Fuzzy Model for Robust Decision-Making

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Abstract: Effective business system management necessitates strategic planning, efficient resource monitoring, and consistent team coordination. In practice, decision-making (DM) processes are frequently challenged by uncertainty, imprecision, and the need to aggregate diverse information sources. To address these complexities, a confidence-based algebraic aggregation framework incorporating the p, q, r-Fraction Fuzzy model has been proposed to enhance decision accuracy under uncertain environments. Within this framework, four novel aggregation operators are introduced: the Confidence p, q, r-Fraction Fuzzy Weighted Averaging Aggregation ($Cpqr$-FFWAA) operator, the Confidence p, q, r-Fraction Fuzzy Ordered Weighted Averaging Aggregation ($Cpqr$-FFOWAA) operator, the Confidence p, q, r-Fraction Fuzzy Weighted Geometric Aggregation ($Cpqr$-FFWGA) operator, and the Confidence p, q, r-Fraction Fuzzy Ordered Weighted Geometric Aggregation ($Cpqr$-FFOWGA) operator. These operators are designed to capture the inherent vagueness and subjectivity in business-related decision inputs, thereby facilitating robust assessments. The theoretical properties of the proposed operators—such as idempotency, boundedness, and monotonicity—are rigorously analyzed to ensure mathematical soundness and operational reliability. To illustrate the practical applicability of the model, a detailed case study is provided, demonstrating its effectiveness in maintaining resource sufficiency, preventing financial disruptions, and ensuring organizational coherence. The use of these aggregation mechanisms allows for systematic integration of expert confidence levels with varying degrees of fuzzy information, resulting in optimized decisions that are both data-informed and uncertainty-resilient. The methodological contributions are positioned to support real-world business contexts where dynamic inputs, incomplete data, and human judgment intersect. Consequently, the proposed approach offers a substantial advancement in intelligent decision-support systems, providing a scalable and interpretable tool for business performance enhancement.

Keywords: Business system optimization; Fuzzy decision-making (DM); Confidence level; Algebraic aggregation operators; Uncertainty modelling; Intelligent decision-support

1 Introduction and Literature Review

Businesses are recognized as key drivers of national economic growth through their contributions to job creation, income generation, and the continuous provision of essential goods and services. By fulfilling tax obligations, enterprises play a critical role in enabling public investment in infrastructure such as schools, hospitals, and transportation networks. The effective management of a business necessitates strategic planning, strong leadership, and a clearly defined vision. It is essential that measurable objectives are established, progress is regularly monitored, and timely adjustments are implemented to ensure alignment with long-term goals. Organizational success is further supported by the development of cohesive teams, which is facilitated through effective communication, the delegation of responsibilities, and the empowerment of employees. Such practices cultivate a collaborative environment in which collective contribution is maximized. In addition, robust financial management—encompassing budgeting, forecasting, and resource allocation—is vital for ensuring operational sustainability and enabling long-term growth.

Sustained competitiveness and organizational resilience are further achieved through the continuous monitoring of industry trends and the proactive adaptation to evolving market conditions.

Zadeh's fuzzy set (FS) theory [1] presented the idea of a mathematical framework for handling uncertainty by allowing elements to have degrees of membership between 0 and 1. This flexibility enhances DM, particularly in control systems and expert systems, where precise data may be unavailable. Fuzzy logic models human reasoning more effectively than traditional crisp logic by incorporating varying levels of truth. It is widely used in applications such as medical diagnosis, risk assessment, and automated control. By accommodating imprecision, FS theory significantly improves problem-solving in complex, real-world scenarios. Although FS theory has proven valuable in modelling uncertainty, it exhibits limitations when applied to data characterized by both membership and non-membership degrees. To address these limitations, Atanassov [2] introduced an intuitionistic fuzzy set (IFS), which contains both membership and non-membership functions. This approach provides a richer framework for expressing uncertainty. By incorporating hesitation, IFSs offer a more flexible and precise representation of vague information. IFSs ensure that the sum of membership and non-membership does not exceed one, enhancing decision reliability. This approach is particularly useful in scenarios with incomplete or ambiguous information, improving the flexibility of decision support systems. Xu and Yager [3], Xu [4], and Wang and Liu [5, 6] introduced many aggregation operators (AOs) using algebraic and Einstein operational laws to enhance uncertainty management in DM. Rahman et al. [7] introduced some generalized hybrids and presented their applications. Seikh and Mandal [8] introduced Dombi aggregation operators, which offer higher computational efficiency and improved DM accuracy. These advancements have significantly strengthened aggregation operators, making them more reliable. As a result, modern DM processes benefit from enhanced precision and flexibility. Yager [9] introduced the Pythagorean fuzzy set (PyFS) to extend IFS by relaxing their conditions. PyFS handles uncertainty more effectively and efficiently. Garg [10, 11], Rahman et al. [12] and Rahman and Ali [13] introduced advanced Einstein aggregation operators. These methods effectively handle imprecise data, enhancing computational accuracy and reliability. By integrating these approaches, decision processes become more robust and efficient. These contributions significantly advance the field of fuzzy decision analysis. Rahman et al. [14] introduced induced averaging operators and applied them to DM problems. Shakeel et al. [15] introduced Pythagorean trapezoidal fuzzy approaches based on inducing variables and their application on DM analysis. Khan et al. [16] developed a MAGDM problem based on hesitant fuzzy information. Later on, Yager [17] introduced the q-rung orthopair fuzzy set (q-ROFS), extending PyFS by relaxing their condition. Liu and Wang [18], Peng and Liu [19], and Qiyas et al. [20] developed some new methods to combine information, making fuzzy DM analysis more accurate and faster. These improved techniques help in making better choices when dealing with uncertain or unclear data.

Existing models have primarily focused on membership and non-membership, neglecting the vital role of the neutral degree in DM. However, neutrality plays a crucial role in balancing opposing factors, leading to more precise and comprehensive analysis. To overcome these limitations, Cuong and Kreinovich [21] introduced the notion of picture fuzzy set (PFS), incorporating three membership grades: membership, neutral, and non-membership. This refined approach enables more nuanced DM by capturing the inherent uncertainty in choices. By integrating neutrality, PFS enhances the accuracy and depth of decision analysis. However, in many practical scenarios, this condition may not always be satisfied, reducing its overall usefulness. As a result, its real-world applicability can be significantly constrained. To address these limitations, Ashraf et al. [22, 23] introduced the spherical fuzzy set (SPS), offering a more flexible and comprehensive approach. This advancement enhances DM by accommodating greater uncertainty and imprecision. Mahmood et al. [24] introduced T-spherical fuzzy sets (T-SpFS), an extension of SFS that effectively addresses their limitations. This novel approach enhances the flexibility and accuracy of uncertainty modeling in DM and computational intelligence. By incorporating an additional parameter, T-SFSs provide a more comprehensive framework for handling imprecise and vague information. The aforementioned models, PFS, SPS, and T-SFSs, are widely applied to real-life problems (Figure 1).

Due to grade limitations, these systems cannot reach their maximum value of 1, which hinders their full potential and limits their effectiveness. Overcoming this constraint would significantly improve their accuracy, allowing for more precise results. Expanding their capabilities could enhance their applicability across diverse domains, making them more versatile and impactful. For instance, if, $\mathbb{Z} = (1.0, 0.8, 0.7)$, then the existing models fall short. To overcome the limitations, Gulistan et al. [25] introduced the p, q, r-Fraction Fuzzy set (p, q, r-FFS). This new approach enhances the flexibility and precision of fuzzy systems by incorporating three parameters, p, q, and r, which define the degree of membership in a more granular manner. The p, q, r-FFS offers a more accurate representation of uncertainty, making it a valuable tool for complex DM and modeling scenarios.

Keeping the application of the above-mentioned models and their corresponding operators, in this paper, we introduced several operators, namely $Cpqr$-FFWAA, $Cpqr$-FFOWAA, $Cpqr$-FFWGA, and $Cpqr$-FFOWGA. These operators were developed to extend the applicability of traditional methods by incorporating different parameters. We also examined fundamental structural characteristics of these operators, such as idempotency, boundedness, and monotonicity, to understand their properties. These attributes were analyzed to assess their functional behavior. This exploration provided deeper insight into their mathematical framework.

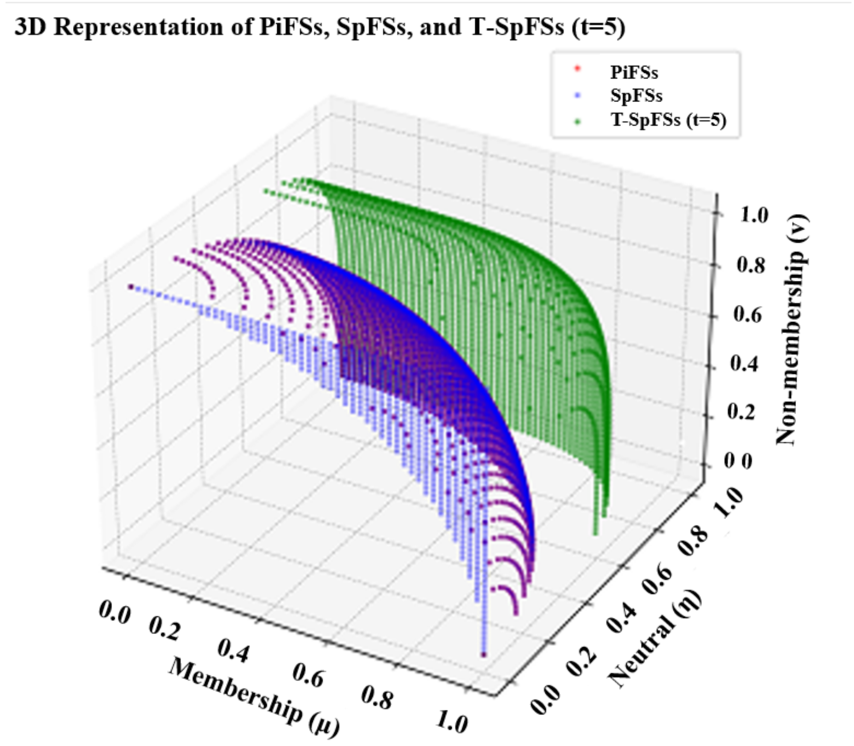

Figure 1. Space of PFSs, SFSs, T-SFSs

This paper is structured as follows: Section 2 provides a comprehensive review of existing studies, establishing the theoretical foundation for the new study. In Section 3, we introduce innovative aggregation operators, including $Cpqr$-FFWAA, $Cpqr$-FFOWAA, $Cpqr$-FFWGA, and $Cpqr$-FFOWGA, which enhance DM processes. Section 4 explores their practical applications, demonstrating their relevance in real-world problems. Section 5 validates their effectiveness through a detailed example, offering a clear comparative analysis. Section 6 explores the comparative and sensitivity analysis. Section 7 highlights the advantages of these proposed methods, emphasizing their superiority over existing techniques. Finally, Section 8 summarizes the key contributions and broader implications of this study, showcasing its significance in advancing aggregation theory.

2 Preliminaries

This section introduces the key ideas, including PFS, SFS, and T-SFS, p, q, r-FFS with score and accuracy functions. These concepts help improve analysis and DM by handling uncertainty better. Using these FS extensions makes the method more precise and flexible.

Definition 1: Let X be a universal set and P be a PFS, then P is written as: $P = \{x, \xi_P(x), \varsigma_P(x), \zeta_P(x) | x \in X\}$, where $\xi_P(x), \varsigma_P(x), \zeta_P(x) \in [0, 1]$ are called the membership, the neutral, and the non-membership degrees with $\xi_P(x) + \varsigma_P(x) + \zeta_P(x) \leq 1$ [21].

Definition 2: Let X be a universal set and S be SFS, then S is mathematically expressed as [22]:

$S = \{x, \xi_S(x), \varsigma_S(x), \zeta_S(x) | x \in X\}$, where $\xi_S(x), \varsigma_S(x), \zeta_S(x) \in [0, 1]$ with condition: $(\xi_P(x))^2 + (\varsigma_P(x))^2 + (\zeta_P(x))^2 \leq 1$ are called the membership, the neutral, and the non-membership degrees, respectively.

Definition 3: Let X be a universal set and T be a TS-FS, then T is defined as: $T = \{x, \xi_T(x), \varsigma_T(x), \zeta_T(x) | x \in X\}$, where $\xi_T(x), \varsigma_T(x), \zeta_T(x) \in [0, 1]$ are called the membership, the neutral, and the non-membership degrees, respectively, with the condition: $(\xi_P(x))^t + (\varsigma_P(x))^t + (\zeta_P(x))^t \leq 1$ [24].

Definition 4: Let X be a universal set and F be a p, q, r-FFS, then F can be defined as: $F = \{x, \langle \xi_F(x), \varsigma_F(x), \zeta_F (x) \rangle_{p,r,q} | x \in X\}$, where $\xi_F(x), \varsigma_F(x), \zeta_F(x) \in [0, 1]$ are called the membership, the neutral, and the non-membership degrees, respectively, under the conditions [25]:

i) $\frac{1}{p}\xi_F(x) + \frac{1}{r}\varsigma_F(x) + \frac{1}{q}\zeta_F(x) \leq 1$.

ii) p and q both are positive integers with $p, q \geq 2$.

iii) $p \leq q, p \geq q, p = q$ and $r = LCM(p, q)$.

Definition 5: Let $\mathbb{Z}_j = (\xi_j, \varsigma_j, \zeta_j)_{p,r,q} (1 \leq j \leq 2)$ be p, q, r-FFNs, and $\ell > 0$, then [25]

1. $\mathbb{Z}_1 \oplus \mathbb{Z}_2 = (\frac{1}{p}\xi_1 + \frac{1}{p}\xi_2 - \frac{1}{p}(\xi_1\xi_2), \frac{1}{r}(\varsigma_1\varsigma_2), \frac{1}{q}(\zeta_1\zeta_2))$
2. $\mathbb{Z}_1 \otimes \mathbb{Z}_2 = (\frac{1}{p}(\xi_1\xi_2), \frac{1}{r}\varsigma_1 + \frac{1}{r}\varsigma_2 - \frac{1}{r}(\varsigma_1\varsigma_2), \frac{1}{q}\zeta_1 + \frac{1}{q}\zeta_2 - \frac{1}{q}(\zeta_1\zeta_2))$
3. $\ell(\mathbb{Z}) = (1 - (1 - \frac{1}{p}\xi)^{\ell}, 1 - (1 - \frac{1}{r}\varsigma)^{\ell}, \frac{1}{q}(\zeta)^{\ell})$
4. $(\mathbb{Z})^{\ell} = (\frac{1}{p}(\xi)^{\ell}, \frac{1}{r}(\varsigma)^{\ell}, 1 - (1 - \frac{1}{q}\zeta)^{\ell})$

Score and accuracy degrees are imperative and significant in DM for ranking alternatives and measuring precision. Traditional methods focus only on membership and non-membership values, while modern ones also include a neutral degree for better accuracy. Adjusting parameters p and q changes how much membership and non-membership influence the results.

Definition 6: Let $\mathbb{Z} = (\xi, \varsigma, \zeta)_{p,r,q}$ be a p, q, r-FFN, then their score and accuracy functions are defined mathematically as $S(\mathbb{Z}) = \frac{1}{3}(1 + \frac{1}{p}\xi + \frac{1}{r}\varsigma - \frac{1}{q}\zeta)$ and $A(\mathbb{Z}) = \frac{1}{3}(1 + \frac{1}{p}\xi + \frac{1}{r}\varsigma + \frac{1}{q}\zeta)$ with conditions: $S(\mathbb{Z}) \in [0, 1]$, and $A(\mathbb{Z}) \in [0, 1]$, respectively, and $p, q \geq 2$, and $r = LCM(p, q)$ [25].

3 Algebraic Aggregation Operators and Their Properties

This section introduces a series of aggregation operators: $Cpqr$-FFWAA, $Cpqr$-FFOWAA, $Cpqr$-FFWGA, and $Cpqr$-FFOWGA, along with their structure properties (see Figure 2). These approaches are designed to enhance the effectiveness of aggregation techniques in various applications. Their unique characteristics contribute to improved DM processes and more accurate outcomes across different domains.

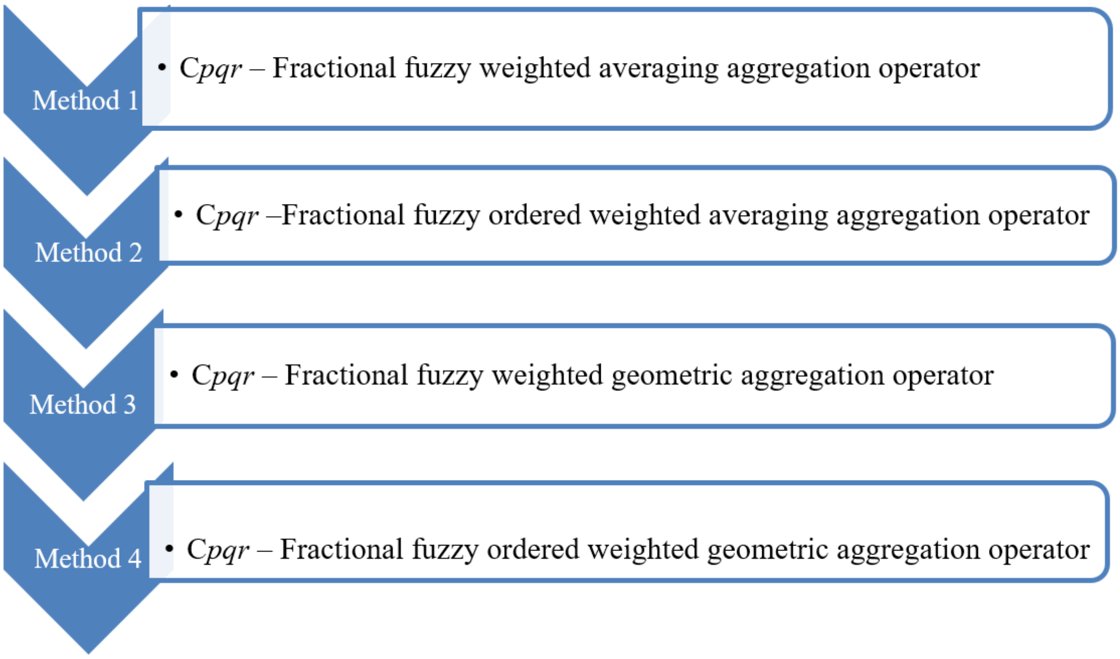

Figure 2. $Cpqr$ Fractional Fuzzy aggregation operators

The concept of p, q, r-Fractional Fuzzy aggregation operators – is essential in DM, as it integrates multiple values with different levels of significance and hierarchy. These operators, based on confidence levels, facilitate more refined and flexible evaluations. Key variations include the $Cpqr$-FFWAA, $Cpqr$-FFOWAA, $Cpqr$-FFWGA, and $Cpqr$-FFOWGA operators, each adhering to distinct mathematical frameworks. They maintain critical properties such as idempotency, boundedness, and monotonicity, ensuring stability and coherence in computations. By preserving these characteristics, the operators enhance reliability in fuzzy logic-based assessments. Their structured approach allows for improved adaptability in handling uncertain or imprecise information. These methods are particularly useful in scenarios where weighted and ordered values play a crucial role. Their application spans various fields, from expert systems to decision analysis. Ultimately, they offer a systematic way to aggregate data while maintaining logical consistency.

Definition 7: Let $(\mathbb{Z}_j, \gamma_j)_{p,r,q} (1 \leq j \leq n)$ be a family of p, q, r-FFNs, with corresponding weights $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ under the limitations $0 \leq \varepsilon_j \leq 1$ and $\sum_{j=1}^{n} \varepsilon_j = 1$; additionally, let $\gamma_j (0 \leq j \leq 1)$ be their confidence level, then $Cpqr$-FFWAA operators can be defined mathematically as:

$$Cpqr-\text{FFWAA}_{\varepsilon}(\langle\mathbb{Z}_1,\gamma_1\rangle,\langle\mathbb{Z}_2,\gamma_2\rangle,...,\langle\mathbb{Z}_n,\gamma_n\rangle)=(1-\prod_{j=1}^{n}(1-\frac{1}{p}\xi_j)^{\gamma_j\varepsilon_j},1-\prod_{j=1}^{n}(1-\frac{1}{r}\varsigma_j)^{\gamma_j\varepsilon_j},\prod_{j=1}^{n}\frac{1}{q}(\zeta_j)^{\gamma_j\varepsilon_j})$$

where, p,q are any positive integers with the conditions $p\leq q$ or $p\geq q$ and r is their LCM.

Example 1: Let $\mathbb{Z}_1=(\langle 0.4,0.8,0.5\rangle,0.3)$, $\mathbb{Z}_2=(\langle 0.9,0.5,0.6\rangle,0.5)$, $\mathbb{Z}_3=(\langle 0.6,0.9,0.7\rangle,0.4)$, $\mathbb{Z}_4=(\langle 0.8,0.9,0.5\rangle,0.6)$ be four p,q,r-FFNs, where $p=3,q=3,r=3$, with the weights vector being $\varepsilon=(0.4,0.2,0.2,0.2)$, then by applying the $Cpqr$- FFWAA operator, we have:

$$\begin{aligned}
&Cpqr-\text{FFWAA}_{\varepsilon}\left((\mathbb{Z}_1,\gamma_1),(\mathbb{Z}_2,\gamma_2),(\mathbb{Z}_3,\gamma_3),(\mathbb{Z}_4,\gamma_4)\right)\\
&=\begin{pmatrix}
1-\left(1-\frac{1}{3}(0.4)\right)^{(0.3)(0.4)}\left(1-\frac{1}{3}(0.9)\right)^{(0.5)(0.2)}\left(1-\frac{1}{3}(0.6)\right)^{(0.4)(0.2)},\left(1-\frac{1}{3}(0.8)\right)^{(0.6)(0.2)}\\
1-\left(1-\frac{1}{3}(0.8)\right)^{(0.3)(0.4)}\left(1-\frac{1}{3}(0.5)\right)^{(0.5)(0.2)}\left(1-\frac{1}{3}(0.9)\right)^{(0.4)(0.2)},\left(1-\frac{1}{3}(0.9)\right)^{(0.6)(0.2)}\\
\frac{1}{3}\left((0.5)^{(0.3)(0.4)}(0.6)^{(0.5)(0.2)}(0.7)^{(0.4)(0.2))}(0.5)^{(0.6)(0.2)}\right)
\end{pmatrix}\\
&=(0.11,12,0.26)
\end{aligned}$$

Theorem 1: Let $\mathbb{Z}_j(1\leq j\leq n)=\mathbb{Z}$, and $\gamma_j(1\leq j\leq n)=\gamma$ be their confidence level, then

$$Cpqr-\text{FFWAA}_{\mathcal{E}}\left(\langle\mathbb{Z}_1,\gamma_1\rangle,\langle\mathbb{Z}_2,\gamma_2\rangle,\ldots,\langle\mathbb{Z}_n,\gamma_n\rangle\right)=\langle\mathbb{Z},\gamma\rangle \tag{1}$$

Proof: Since $\mathbb{Z}_j(1\leq j\leq n)=\mathbb{Z}$ for all j, then we have:

$$\begin{aligned}
&Cpqr-\text{FFWAA}_{\varepsilon}(\langle\mathbb{Z}_1,\gamma_1\rangle,\langle\mathbb{Z}_2,\gamma_2\rangle,...,\langle\mathbb{Z}_n,\gamma_n\rangle)\\
&=(1-\prod_{j=1}^{n}(1-\frac{1}{p}\xi_j)^{\gamma_j\varepsilon_j},1-\prod_{j=1}^{n}(1-\frac{1}{r}\varsigma_j)^{\gamma_j\varepsilon_j},\prod_{j=1}^{n}\frac{1}{q}(\zeta_j)^{\gamma_j\varepsilon_j})\\
&=(1-(1-\frac{1}{p}\xi)^{\gamma_j\sum_{j=1}^{n}\varepsilon_j},1-(1-\frac{1}{r}\varsigma)^{\gamma_j\sum_{j=1}^{n}\varepsilon_j},\frac{1}{q}(\zeta)^{\gamma_j\sum_{j=1}^{n}\varepsilon_j})\\
&=\langle\mathbb{Z},\gamma\rangle_{p,r,q}
\end{aligned}$$

Theorem 2: Let $\mathbb{Z}_j(1\leq j\leq n)$, and $\gamma_j(1\leq j\leq n)$ be their confidence level, with conditions: $\max\{\mathbb{Z}\}=(\xi_{\max},\varsigma_{\max},\zeta_{\min})_{p,r,q}$ and $\min\{\mathbb{Z}\}=(\xi_{\min},\varsigma_{\min},\zeta_{\max})_{p,r,q}$, then we have:

$$\min\{\mathbb{Z},\gamma\}\leq Cpqr-\text{FFWAA}_{\varepsilon}(\langle\mathbb{Z}_1,\gamma_1\rangle,\langle\mathbb{Z}_2,\gamma_2\rangle,...,\langle\mathbb{Z}_n,\gamma_n\rangle)\leq\max\{\mathbb{Z},\gamma\} \tag{2}$$

Proof: Since $\xi_{\min}\leq\xi_j\leq\xi_{\max}$, $\varsigma_{\min}\leq\varsigma_j\leq\varsigma_{\max}$, $\zeta_{\min}\leq\zeta_j\leq\zeta_{\max}$, then we have:

$$\begin{aligned}
&\Leftrightarrow\prod_{j=1}^{n}\left(1-\frac{1}{p}\xi_{\max}\right)^{\gamma_j\varepsilon_j}\leq\prod_{j=1}^{n}\left(1-\frac{1}{p}\xi_j\right)^{\gamma_j\varepsilon_j}\leq\prod_{j=1}^{n}\left(1-\frac{1}{p}\xi_{\min}\right)^{\gamma_j\varepsilon_j}\\
&\Leftrightarrow\left(1-\frac{1}{p}\xi_{\min}\right)^{\gamma_j\varepsilon_j}\leq\left(1-\frac{1}{p}\xi_j\right)^{\gamma_j\varepsilon_j}\leq\left(1-\frac{1}{p}\xi_{\min}\right)^{\gamma_j\varepsilon_j}\\
&\Leftrightarrow\frac{1}{p}\xi_{\max}\leq\frac{1}{p}\xi_j\leq\frac{1}{p}\xi_{\min}
\end{aligned}$$

On the same process we can prove that $\frac{1}{r}\varsigma_{\min}\leq\frac{1}{r}\varsigma\leq\frac{1}{r}\varsigma_{\max}$. Similarly, we have: $\zeta_{\min}\leq\zeta_j\leq\zeta_{\max}$, this implies that:

$$\begin{aligned}
&\Leftrightarrow\prod_{j=1}^{n}\frac{1}{q}(\zeta_{\min})^{\gamma_j\varepsilon_j}\leq\prod_{j=1}^{n}\frac{1}{q}(\zeta_j)^{\gamma_j\varepsilon_j}\leq\prod_{j=1}^{n}\frac{1}{q}(\zeta_{\max})^{\gamma_j\varepsilon_j}\\
&\Leftrightarrow\frac{1}{q}(\zeta_{\min})^{\gamma_j\varepsilon_j}\leq\frac{1}{q}(\zeta_j)^{\gamma_j\varepsilon_j}\leq\frac{1}{q}(\zeta_{\max})^{\gamma_j\varepsilon_j}\\
&\Leftrightarrow\frac{1}{q}\zeta_{\min}\leq\frac{1}{q}\zeta_j\leq\frac{1}{q}\zeta_{\max}
\end{aligned}$$

Thus, from the above process, we have

$$\min\{\mathbb{Z},\gamma\} \leq Cpqr - \text{FFWAA}_{\varepsilon}(\langle \mathbb{Z}_1,\gamma_1\rangle, \langle \mathbb{Z}_2,\gamma_2\rangle, ..., \langle \mathbb{Z}_n,\gamma_n\rangle) \leq \max\{\mathbb{Z},\gamma\}.$$

The proof is completed.

Theorem 3: Let $(\mathbb{Z}_j,\gamma_j)_{p,r,q}(1 \leq j \leq n)$ and $(\mathbb{Z}_j^*,\gamma_j^*)_{p,r,q}(1 \leq j \leq n)$ be any two families of $p, q,$-FFNs, with $\mathbb{Z}_j \leq \mathbb{Z}_j^*$, then we have:

$$Cpqr - \text{FFWAA}_{\varepsilon}(\langle \mathbb{Z}_1,\gamma_1\rangle, ..., \langle \mathbb{Z}_n,\gamma_n\rangle) \leq Cpqr - \text{FFWAA}_{\varepsilon}(\langle \mathbb{Z}_1^*,\gamma_1^*\rangle, ..., \langle \mathbb{Z}_n^*,\gamma_n^*\rangle) \tag{3}$$

Proof: $The\ proof\ is\ similar\ to\ that\ of\ Theorem\ 2.$

Definition 8: Let $(\mathbb{Z}_j,\gamma_j)_{p,r,q}(1 \leq j \leq n)$ be a family of p, q, r-FFNs, with corresponding weights $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ with $0 \leq \varepsilon_j \leq 1$, $\sum\limits_{j=1}^{n} \varepsilon_j = 1$ and $\gamma_j (0 \leq j \leq 1)$ be their confidence level, then the $Cpqr$-FFOWAA operators can be defined mathematically as:

$$\begin{aligned} &Cpqr - \text{FFOWAA}_{\varepsilon}\left(\langle \mathbb{Z}_1,\gamma_1\rangle, \langle \mathbb{Z}_2,\gamma_2\rangle, ..., \langle \mathbb{Z}_n,\gamma_n\rangle\right) \\ &= \left(1 - \prod_{j=1}^{n}\left(1 - \frac{1}{p}\xi_{\sigma(j)}\right)^{\gamma_j\varepsilon_j}, 1 - \prod_{j=1}^{n}\left(1 - \frac{1}{r}\varsigma_{\sigma(j)}\right)^{\gamma_j\varepsilon_j}, \prod_{j=1}^{n}\frac{1}{q}\left(\zeta_{\sigma(j)}\right)^{\gamma_j\varepsilon_j}\right) \end{aligned}$$

where, $\sigma(1), \sigma(2), ..., \sigma(n)$ are the permutation of $(1, 2, ..., n)$ with $\mathbb{Z}_{\sigma(j-1)} \geq \mathbb{Z}_{\sigma(j)}$, p, q are any positive integers under the conditions: $p \leq q$ or $p \geq q$ and r is their LCM.

Example 2: Let $Z_1 = (\langle 0.3, 0.5, 0.3\rangle, 0.1)$, $Z_2 = (\langle 0.6, 0.3, 0.7\rangle, 0.4)$, $Z_3 = (\langle 0.4, 0.2, 0.5\rangle$,0.3) be three p, q, r-FFNs, where $p = 3, q = 2, r = 6$ and their weighted vector $\varepsilon = (0.3, 0.3, 0.4)$. First, we need to calculate the score functions: $S(\mathbb{Z}_1) = \frac{1}{3}(1 + \frac{1}{3}(0.3) + \frac{1}{3}(0.5) - \frac{1}{3}(0.3)) = 0.38$, $S(\mathbb{Z}_2) = \frac{1}{3}(1 + \frac{1}{3}(0.6) + \frac{1}{3}(0.3) - \frac{1}{3}(0.7)) = 0.35$, $S(\mathbb{Z}_3) = \frac{1}{3}(1 + \frac{1}{3}(0.4) + \frac{1}{3}(0.2) - \frac{1}{3}(0.5)) = 0.34$.

Thus, based on the score functions, the ordered values are listed as follows:

$Z_{\sigma(1)} = (\langle 0.3, 0.5, 0.3\rangle, 0.1)$, $Z_{\sigma(2)} = (\langle 0.6, 0.3, 0.7\rangle, 0.4)$, $Z_{\sigma(3)} = (\langle 0.4, 0.2, 0.5\rangle$,0.3). Now by applying the p, q, r- CFFOWAA operator, we have:

$$\begin{aligned} &Cpqr - \text{FFOWAA}_{\varepsilon}\left((\mathbb{Z}_1,\gamma_1), (\mathbb{Z}_2,\gamma_2), (\mathbb{Z}_3,\gamma_3)\right) \\ &= \begin{pmatrix} 1 - \left(1 - \frac{1}{3}(0.3)\right)^{(0.1)(0.3)}\left(1 - \frac{1}{3}(0.6)\right)^{(0.4)(0.3)}\left(1 - \frac{1}{3}(0.4)\right)^{(0.3)(0.4)}, \\ 1 - \left(1 - \frac{1}{6}(0.5)\right)^{(0.1)(0.3)}\left(1 - \frac{1}{6}(0.3)\right)^{(0.4)(0.3)}\left(1 - \frac{1}{6}(0.2)\right)^{(0.3)(0.4)}, \\ \frac{1}{2}\left((0.3)^{(0.1)(0.3)}(0.4)^{(0.4)(0.3)}(0.5)^{(0.3)(0.4)}\right) \end{pmatrix} \\ &= (0.06, 0.03, 0.38) \end{aligned}$$

Definition 9: Let $(\mathbb{Z}_j,\gamma_j)_{p,r,q}(1 \leq j \leq n)$ be a family of p, q, r-FFNs, with weights $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ under the limitations $0 \leq \varepsilon_j \leq 1$ and $\sum\limits_{j=1}^{n} \varepsilon_j = 1$. Moreover, let $\gamma_j(0 \leq j \leq 1)$ be their confidence level, then the $Cpqr$-FFWGA operators can be defined mathematically as:

$$Cpqr - \text{FFWGA}_{\varepsilon}(\langle \mathbb{Z}_1,\gamma_1\rangle, \langle \mathbb{Z}_2,\gamma_2\rangle, ..., \langle \mathbb{Z}_n,\gamma_n\rangle) = (\prod_{j=1}^{n}\frac{1}{p}(\xi_j)^{\gamma_j\varepsilon_j}, \prod_{j=1}^{n}\frac{1}{r}(\varsigma_j)^{\gamma_j\varepsilon_j}, 1 - \prod_{j=1}^{n}(1 - \frac{1}{q}\zeta_j)^{\gamma_j\varepsilon_j})$$

where, p, q are any positive integers with the conditions $p \leq q$ or $p \geq q$ and is their LCM.

Example 3: Let $\mathbb{Z}_1 = (\langle 0.5, 0.8, 0.4\rangle$,0.3), $\mathbb{Z}_2 = (\langle 0.6, 0.5, 0.9\rangle, 0.5)$, $\mathbb{Z}_3 = (\langle 0.7, 0.9, 0.6\rangle, 0.4)$, $\mathbb{Z}_4 = (\langle 0.5, 0.9, 0.8\rangle, 0.6)$ be four p, q, r-FFNs, where $p = 3, q = 3, r = 3$ with weights vector $\varepsilon = (0.4, 0.2, 0.2, 0.2)$, then by applying the $Cpqr$- FFWGA operator, we have:

$$
\begin{aligned}
&Cpqr - \text{FFWGA}_{\varepsilon}\left(\left(\mathbb{Z}_1, \gamma_1\right), \left(\mathbb{Z}_2, \gamma_2\right), \left(\mathbb{Z}_3, \gamma_3\right), \left(\mathbb{Z}_4, \gamma_4\right)\right) \\
&= \begin{pmatrix}
\frac{1}{3}\left((0.5)^{(0.3)(0.4)}(0.6)^{(0.5)(0.2)}(0.7)^{(0.4)(0.2))(0.5)^{(0.6)(0.2)}}\right), \\
\frac{1}{3}\left((0.8)^{(0.3)(0.4)}(0.5)^{(0.5)(0.2)}(0.9)^{(0.4)(0.2)}, (0.9)^{(0.6)(0.2)}\right). \\
1-\left(1-\frac{1}{3}(0.4)\right)^{(0.3)(0.4)}\left(1-\frac{1}{3}(0.9)\right)^{(0.5)(0.2)}\left(1-\frac{1}{3}(0.6)\right)^{(0.4)(0.2)}, \left(1-\frac{1}{3}(0.8)\right)^{(0.6)(0.2)}
\end{pmatrix} \\
&= \left(\frac{1}{3}(0.78), \frac{1}{3}(0.88), 1-(0.96)(0.98)(0.97)(0.95)\right) \\
&= (0.26, 0.29, 0.11)
\end{aligned}
$$

Theorem 4: Let $\mathbb{Z}_j(1 \leq j \leq n) = \mathbb{Z}$, and $\gamma_j(1 \leq j \leq n) = \gamma$ be their confidence level, then:

$$
Cpqr - \text{FFWGA}_{\varepsilon}(\langle\mathbb{Z}_1, \gamma_1\rangle, \langle\mathbb{Z}_2, \gamma_2\rangle, ..., \langle\mathbb{Z}_n, \gamma_n\rangle) = \langle\mathbb{Z}, \gamma\rangle \tag{4}
$$

Proof: $For\ the\ proof\ process,\ please\ see\ the\ proof\ of\ Theorem\ 1.$

Theorem 5: Let $\mathbb{Z}_j(1 \leq j \leq n)$, and $\gamma_j(1 \leq j \leq n)$ be their confidence level, with conditions: $\max\{\mathbb{Z}\} = (\xi_{\max}, \varsigma_{\max}, \zeta_{\min})_{p,r,q}$ and $\min\{\mathbb{Z}\} = (\xi_{\min}, \varsigma_{\min}, \zeta_{\max})_{p,r,q}$, then we have:

$$
\min\{\mathbb{Z}, \gamma\} \leq Cpqr - \text{FFWGA}_{\varepsilon}(\langle\mathbb{Z}_1, \gamma_1\rangle, \langle\mathbb{Z}_2, \gamma_2\rangle, ..., \langle\mathbb{Z}_n, \gamma_n\rangle) \leq \max\{\mathbb{Z}, \gamma\} \tag{5}
$$

Proof: $The\ proof\ is\ similar\ as\ the\ proof\ of\ Theorem\ 2.$

Theorem 6: Let $(\mathbb{Z}_j, \gamma_j)_{p,r,q}(1 \leq j \leq n)$ and $(\mathbb{Z}_j^*, \gamma_j^*)_{p,r,q}(1 \leq j \leq n)$ be any two families of p, q, r-FFNs, with $\mathbb{Z}_j \leq \mathbb{Z}_j^*$, then we have:

$$
Cpqr - \text{FFWGA}_{\varepsilon}(\langle\mathbb{Z}_1, \gamma_1\rangle, ..., \langle\mathbb{Z}_n, \gamma_n\rangle) \leq Cpqr - \text{FFWGA}_{\varepsilon}(\langle\mathbb{Z}_1^*, \gamma_1^*\rangle, ..., \langle\mathbb{Z}_n^*, \gamma_n^*\rangle) \tag{6}
$$

Proof: $The\ proof\ is\ similar\ to\ that\ of\ Theorem\ 2.$

Definition 10: Let $(\mathbb{Z}_j, \gamma_j)_{p,r,q}(1 \leq j \leq n)$ be a family of p, q, r-FFNs, with corresponding weights $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$ with $0 \leq \varepsilon_j \leq 1$, let $\sum_{j=1}^{n} \varepsilon_j = 1$ and $\gamma_j(0 \leq j \leq 1)$ be their confidence level, then the $Cpqr$-FFOWGA operators can be defined mathematically as:

$$
Cpqr\text{-FFOWGA}_{\varepsilon}(\langle\mathbb{Z}_1, \gamma_1\rangle, \langle\mathbb{Z}_2, \gamma_2\rangle, ..., \langle\mathbb{Z}_n, \gamma_n\rangle) = (\prod_{j=1}^{n} \frac{1}{p}(\xi_{\sigma(j)})^{\gamma_j \varepsilon_j}, \prod_{j=1}^{n} \frac{1}{r}(\varsigma_{\sigma(j)})^{\gamma_j \varepsilon_j}, 1-\prod_{j=1}^{n}(1-\frac{1}{q}\zeta_{\sigma(j)})^{\gamma_j \varepsilon_j})
$$

where, $\sigma(1), \sigma(2), ..., \sigma(n)$ are the permutation of $(1, 2, ..., n)$ with $\mathbb{Z}_{\sigma(j-1)} \geq \mathbb{Z}_{\sigma(j)}$, p, q are any positive integers under the conditions: $p \leq q$ or $p \geq q$ and r is their LCM.

Example 4: Let $Z_1 = (\langle 0.6, 0.2, 0.1\rangle, 0.2)$, $Z_2 = (\langle 0.2, 0.8, 0.4\rangle, 0.1)$, $Z_3 = (\langle 0.1, 0.2, 0.3\rangle, 0.4)$, where $p = 3, q = 2, r = 6$ and $\varepsilon = (0.3, 0.3, 0.4)$ are their corresponding weighted vector.

First, we need to calculate the score functions:

$$
S(\mathbb{Z}_1) = \frac{1}{3}(1 + \frac{1}{3}(0.6) + \frac{1}{3}(0.2) - \frac{1}{3}(0.1)) = 0.41
$$

$$
S(\mathbb{Z}_2) = \frac{1}{3}(1 + \frac{1}{3}(0.2) + \frac{1}{3}(0.8) - \frac{1}{3}(0.4)) = 0.40, S(\mathbb{Z}_3) = \frac{1}{3}(1 + \frac{1}{3}(0.1) + \frac{1}{3}(0.2) - \frac{1}{3}(0.3)) = 0.33
$$

Thus, based on the score function, we have the ordered values as:

$$
Z_{\sigma(1)} = (\langle 0.6, 0.2, 0.1\rangle, 0.2), Z_{\sigma(2)} = (\langle 0.2, 0.8, 0.4\rangle, 0.1), Z_{\sigma(3)} = (\langle 0.1, 0.2, 0.3\rangle, 0.4).
$$

By using the $Cpqr$-FFOWGA operator, we have:

$$
\begin{aligned}
&Cpqr-\text{FFOWGA}_{\varepsilon}\left(\langle \mathbb{Z}_1,\gamma_1\rangle,\langle \mathbb{Z}_2,\gamma_2\rangle,\langle \mathbb{Z}_3,\gamma_3\rangle\right)\\
&=\begin{pmatrix}
\frac{1}{3}\left((0.6)^{(0.2)(0.3)}(0.2)^{(0.1)(0.3)}(0.1)^{(0.4)(0.4)}\right),\\
\frac{1}{6}\left((0.2)^{(0.2)(0.3)}(0.8)^{(0.1)(0.3)}(0.2)^{(0.4)(0.4)}\right),\\
1-\left(\left(1-\frac{1}{2}(0.1)\right)^{(0.2)(0.3)}\left(1-\frac{1}{2}(0.4)\right)^{(0.1)(0.3)}\left(1-\frac{1}{2}(0.3)\right)^{(0.4)(0.4)}\right)
\end{pmatrix}\\
&=(0.21,0.12,0.05)
\end{aligned}
$$

4 Real World Application

Making good decisions starts with identifying a problem or opportunity and gathering the right information. After that, different options are considered, and the best one is chosen based on possible outcomes. Good DM requires critical thinking, weighing risks and benefits, and looking at both short-term and long-term effects. Involving others and considering different opinions can lead to better choices. Once a decision is made, clear communication and proper execution are important for success. It is also necessary to track results and make changes if needed. Learning from past decisions helps improve future choices. In complex situations, advanced techniques like $Cpqr$-FFWAA and $Cpqr$-FFWGA help handle detailed fuzzy data. These methods improve accuracy and flexibility in analysis, leading to better outcomes. Using such advanced tools ensures more effective and reliable DM.

Algorithm: We have a set of m options: $Y=\{y_1,y_2,\ldots,y_m\}$, that need to be evaluated based on n criteria: $C=\{c_1,c_2,\ldots,c_n\}$, each with a specific weight: W=$\{w_1,w_2,\ldots,w_n\}$, showing its importance. Additionally, there are k decision-makers: D =$\{d_1,d_2,\ldots,d_k\}$, each with their own weight H = $\{h_1,h_2,\ldots,h_k\}$, representing their influence on the decision. The final evaluation considers both the criteria weights and decision-maker weights, ensuring a fair and balanced decision. This method helps in making better choices by including different perspectives and importance levels. It ensures that more important criteria and influential decision-makers have a greater impact. By combining these factors, we get a structured and fair assessment process. This approach is useful for selecting the best option in a logical way. It is widely used in DM to ensure accuracy and fairness. The following Figure 3 shows the flow chart of the algorithm.

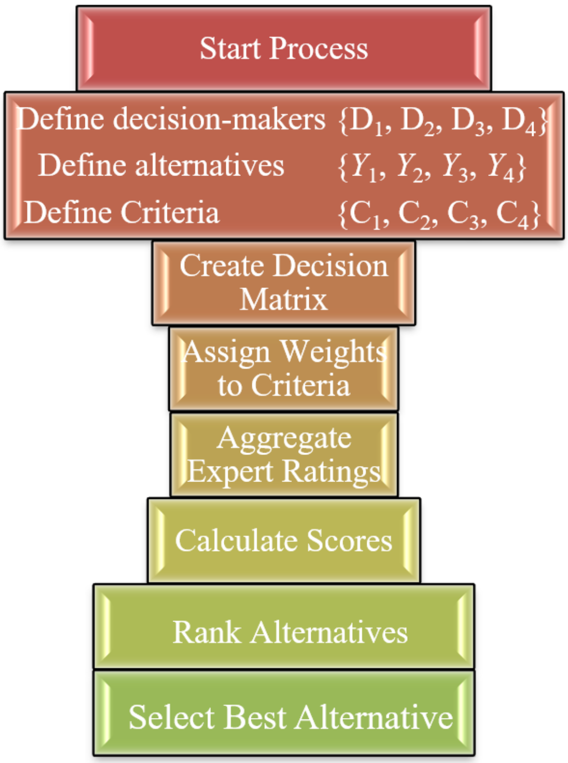

Figure 3. Flow chart of algorithm

5 Descriptive Example

To demonstrate the practical application of the proposed approach, a numerical example is provided below. This example illustrates how the method can be applied in real-life scenarios, showcasing its effectiveness and relevance.

$\boldsymbol{Case\ study}$: Businesses play a crucial role in driving a country's economy by fostering innovation, creating jobs and contributing to overall economic growth. They generate income, stimulate trade, and increase tax revenues, which help fund public services and infrastructure. Additionally, thriving businesses attract investment both domestic and international, which further strengthens the economic foundation. Inventory management is essential for any business to run smoothly. Without proper tracking, a shortage of raw materials can stop production and cause financial losses. For example, a food company ensures smooth operations by closely monitoring its stock levels, preventing shortages and delays in production. By implementing real-time inventory management systems, the company can accurately track stock, ensuring timely replenishments and minimizing waste. This proactive approach enables the business to maintain consistent product availability and meet customer demand without interruptions. Regular audits and forecasting also help in adjusting stock to seasonal fluctuations, optimizing supply chain efficiency. Ultimately, careful stock management enhances overall operational efficiency, supporting customer satisfaction and business growth. This helps them produce their main food products efficiently and avoid waste. Beverages (A_1): Beverages encompass a wide range of drinks, including water, soft drinks, juices, and alcoholic beverages. They play a significant role in daily hydration, nutrition, and social rituals, often catering to various tastes and dietary needs. Bakery Items (A_2): Bakery items include a variety of baked goods such as bread, pastries, cakes, and cookies. They are staples in many diets around the world, celebrated for their versatility, convenience, and the comfort they bring through their freshly baked aroma and taste. Pickles (A_3): Pickles are preserved vegetables or fruits, typically soaked in brine or vinegar, often flavored with various spices. They offer a tangy taste and are popular as side dishes, condiments, or snacks, adding zest and enhancing the flavor profile of meals. Edible Oils (A_4): Edible oils, derived from plants or animals, are essential in cooking and food preparation. They provide flavor, aid in the cooking process, and are a source of essential fatty acids and vitamins, making them crucial in many culinary traditions.

To manufacture these items, stock re-ordering decisions for ingredients must consider the following three key factors: Cost price (C_1): The cost price is the initial amount paid to acquire goods or services, excluding any additional expenses like shipping or taxes. It forms the baseline for calculating profit margins, influencing pricing strategies and overall business profitability. Storage facilities (C_2): Storage facilities are designated spaces for holding inventory, ensuring goods are safely stored and easily accessible. Effective storage facilities minimize loss, streamline operations, and support efficient inventory management, which is crucial for maintaining supply chain efficiency. Staleness level (C_3): Staleness level refers to the degree of decline in quality or value of perishable goods over time. Monitoring staleness levels is vital for inventory control, helping businesses manage stock rotation, reduce waste, and maintain product quality for consumers. The weight vector for these factors is designated as $\omega = (0.3, 0.3, 0.4)$. The given alternatives are evaluated based on these three factors and their values rated in terms of p, q, r-FFNs. Since the company prioritizes maintaining high production quality, the highest priority is assigned to reducing staleness levels. The objective is to identify food items whose ingredient stocks require frequent reordering. To achieve this, the following steps have been executed.

$\boldsymbol{Step}$ **1**: Decision matrices provided by the DMEs are tabulated in Table 1, Table 2, Table 3, and Table 4.

Table 1. Information of expert D_1

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$\langle(0.8, 0.8, 0.9), 0.7\rangle$	$\langle(0.7, 0.8, 0.6), 0.9\rangle$	$\langle(0.9, 0.6, 0.9), 0.3\rangle$
Y_2	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$	$\langle(0.9, 0.8, 0.8), 0.7\rangle$
Y_3	$\langle(0.6, 0.8, 0.9), 0.5\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$	$\langle(0.8, 0.5, 0.9), 0.5\rangle$
Y_4	$\langle(0.7, 0.7, 0.6), 0.4\rangle$	$\langle(0.6, 0.8, 0.7), 0.4\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$

Table 2. Information of expert D_2

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$\langle(0.6, 0.5, 0.9), 0.5\rangle$	$\langle(0.7, 0.5, 0.9), 0.4\rangle$	$\langle(0.7, 0.7, 0.8), 0.5\rangle$
Y_2	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.7, 0.8, 0.6), 0.9\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$
Y_3	$\langle(0.8, 0.7, 0.6), 0.4\rangle$	$\langle(0.6, 0.7, 0.8), 0.7\rangle$	$\langle(0.8, 0.8, 0.6), 0.8\rangle$
Y_4	$\langle(0.6, 0.8, 0.9), 0.5\rangle$	$\langle(0.6, 0.8, 0.7), 0.4\rangle$	$\langle(0.4, 0.7, 0.9), 0.5\rangle$

$\boldsymbol{Step}$ **2**: Table 1, Table 2, Table 3, and Table 4 are combined into a single matrix using $Cpqr$-FFWAA and $Cpqr$-FFWGA operators and $p = q = r = 2$. The resulting matrices are shown in Table 5 and Table 6.

Table 3. Information of expert D_3

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$\langle(0.7, 0.8, 0.6), 0.9\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$
Y_2	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$	$\langle(0.9, 0.8, 0.8), 0.7\rangle$
Y_3	$\langle(0.8, 0.5, 0.9), 0.5\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$	$\langle(0.6, 0.8, 0.9), 0.5\rangle$
Y_4	$\langle(0.8, 0.7, 0.6), 0.4\rangle$	$\langle(0.6, 0.7, 0.8), 0.7\rangle$	$\langle(0.8, 0.8, 0.6), 0.8\rangle$

Table 4. Information of expert D_4

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$\langle(0.8, 0.8, 0.6), 0.8\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$
Y_2	$\langle(0.9, 0.8, 0.8), 0.7\rangle$	$\langle(0.8, 0.5, 0.9), 0.5\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$
Y_3	$\langle(0.8, 0.7, 0.6), 0.4\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$	$\langle(0.6, 0.8, 0.9), 0.5\rangle$
Y_4	$\langle(0.6, 0.8, 0.9), 0.5\rangle$	$\langle(0.8, 0.5, 0.9), 0.5\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$

Table 5. By $Cpqr$-FFWAA operator

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$\langle(0.8, 0.8, 0.6), 0.8\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$	$\langle(0.8, 0.5, 0.9), 0.3\rangle$
Y_2	$\langle(0.9, 0.8, 0.8), 0.7\rangle$	$\langle(0.8, 0.5, 0.9), 0.5\rangle$	$\langle(0.9, 0.7, 0.9), 0.6\rangle$
Y_3	$\langle(0.8, 0.7, 0.6), 0.4\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$	$\langle(0.6, 0.8, 0.9), 0.5\rangle$
Y_4	$\langle(0.6, 0.8, 0.9), 0.5\rangle$	$\langle(0.8, 0.5, 0.9), 0.5\rangle$	$\langle(0.7, 0.4, 0.8), 0.2\rangle$

Table 6. By $Cpqr$-FFWGA operator

	$\mathbf{c_1}$	$\mathbf{c_2}$	$\mathbf{c_3}$
Y_1	$(0.42, 0.32, 0.21)$	$(0.34, 0.35, 0.18)$	$(0.42, 0.36, 0.29)$
Y_2	$(0.37, 0.36, 0.25)$	$(0.35, 0.31, 0.26)$	$(0.37, 0.39, 0.22)$
Y_3	$(0.43, 0.34, 0.28)$	$(0.37, 0.31, 0.25)$	$(0.36, 0.35, 0.23)$
Y_4	$(0.40, 0.26, 0.19)$	$(0.34, 0.35, 0.21)$	$(0.42, 0.30, 0.26)$

***Step* $\mathbf{3(a)}$**: Using $Cpqr$-FFWAA operator with $\omega = (0.2, 0.2, 0.2, 0.4)$, $\phi_1 = (0.45, 0.36, 0.17)$, $\phi_2 = (0.40, 0.38, 0.15)$, $\phi_3 = (0.38, 0.32, 0.28)$, $\phi_4 = (0.37, 0.33, 0.24)$.

***Step* $\mathbf{3(b)}$**: Using the $Cpqr$-FFWGA operator with $\omega = (0.2, 0.2, 0.2, 0.4)$, $\phi_1 = (0.41, 0.36, 0.16)$, $\phi_2 = (0.36, 0.29, 0.13)$, $\phi_3 = (0.29, 0.25, 0.17)$, $\phi_4 = (0.41, 0.27, 0.19)$.

***Step* $\mathbf{4(a)}$**: Using Definition 6, to computing the scores of all preference values of Table 5.

$$S(\phi_1) = \frac{1}{3}(1 + \frac{1}{2}(0.45) + \frac{1}{2}(0.36) - \frac{1}{2}(0.17)) = 0.44$$

$$S(\phi_2) = \frac{1}{3}(1 + \frac{1}{2}(0.40) + \frac{1}{2}(0.38) - \frac{1}{2}(0.15)) = 0.43$$

$$S(\phi_3) = \frac{1}{3}(1 + \frac{1}{2}(0.38) + \frac{1}{2}(0.32) - \frac{1}{2}(0.28)) = 0.40$$

$$S(\phi_4) = \frac{1}{3}(1 + \frac{1}{2}(0.37) + \frac{1}{2}(0.33) - \frac{1}{2}(0.24)) = 0.41$$

***Step* $\mathbf{4(b)}$**: Using Definition 6, to computing the scores of all preference values of Table 6.

$$S(\phi_1) = \frac{1}{3}(1 + \frac{1}{2}(0.41) + \frac{1}{2}(0.36) - \frac{1}{2}(0.16)) = 0.43$$

$$S(\phi_2) = \frac{1}{3}(1 + \frac{1}{2}(0.36) + \frac{1}{2}(0.29) - \frac{1}{2}(0.13)) = 0.42$$

$$S(\phi_3) = \frac{1}{3}(1 + \frac{1}{2}(0.29) + \frac{1}{2}(0.25) - \frac{1}{2}(0.17)) = 0.39$$

$$S(\phi_4) = \frac{1}{3}(1 + \frac{1}{2}(0.41) + \frac{1}{2}(0.27) - \frac{1}{2}(0.19)) = 0.41$$

***Step* $\mathbf{5}$**: Thus, the best option is Beverages.

The proposed methods, shown in Table 7 and Figure 4, are more flexible and adaptable than existing ones. This versatility enhances their effectiveness across various applications.

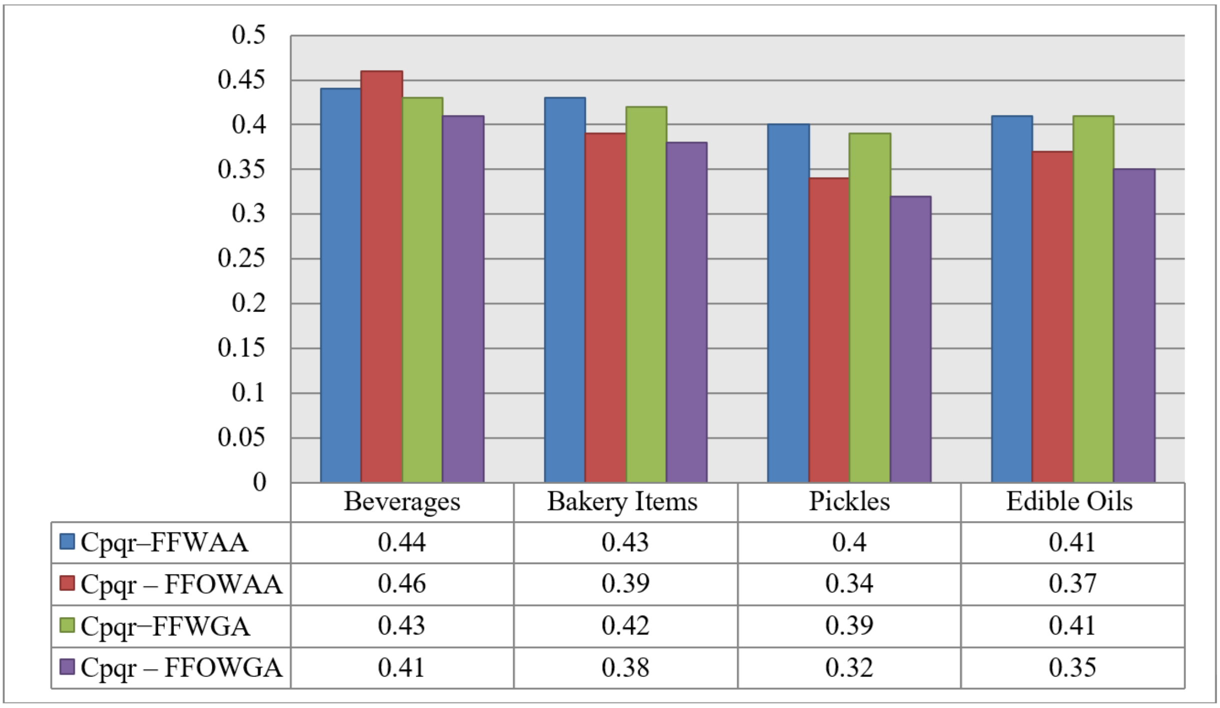

Figure 4. Ranking of all methods

Table 7. Score function of all methods

Alternatives	$Cpqr$-FFWAA	$Cpqr$-FFOWAA	$Cpqr$-FFWGA	$Cpqr$-FFOWGA
Y_1	0.44	0.46	0.43	0.41
Y_2	0.43	0.39	0.42	0.38
Y_3	0.40	0.34	0.39	0.32
Y_4	0.41	0.37	0.41	0.35

6 Comparative and Sensitivity Evaluation

Existing models, namely PFSs, SFSs and T-SFSs, use membership (ξ), neutrality (ς), and non-membership (ζ) functions to present their models. But these models have a limit and cannot reach the highest value of 1 because of certain rules in their grading. This makes them less flexible in many real-life problems. For example, the rating value $\mathbb{Z} = (1.0, 0.8, 0.7)$ fails to satisfy the conditions for these models which are $\xi + \varsigma + \zeta \leq 1$, $\xi^2 + \varsigma^2 + \zeta^2 \leq 1$, and $\xi^t + \varsigma^t + \zeta^t \leq 1, t \geq 1$, i.e., $\xi + \varsigma + \zeta = 1 + 0.8 + 0.7 = 2.5{>}1$, showing that the data cannot be solved using PFSs. $\xi^2 + \varsigma^2 + \zeta^2 = (1)^2 + (0.8)^2 + (0.7)^2 = 2.13{>}1$, showing that the data cannot be solved using SFSs. If $t = 4$, $\xi^t + \varsigma^t + \zeta^t = (1)^4 + (0.8)^4 + (0.7)^4 = 1.64{>}1$, showing that the data cannot be solved using T-SFSs. These limitations make them unsuitable for certain types of data. To overcome these limitations, we introduced the notion of $Cpqr$-FFSs, a groundbreaking framework offering greater flexibility in handling intricate data. By using three parameters p, q, r, this new study can better represent complex data. This model provides a refined way to handle information in a more effective manner. Ranking of all alternatives with different parameters p, q, r shown in Table 8. It also helps in comparing different options more effectively. Reliable methods like this support better planning and strategy.

Table 8 confirms that the ranking of all newly offered options stays the same, no matter the values of p, q, and r. The results show the robustness of the method across different conditions. The stability of rankings, regardless of parameter variations, highlights the method's resilience and reliability in DM. This consistency and reliability show that the best alternatives are consistently identified, reinforcing confidence in the assessment process.

Table 8. Ranking of all methods for different values of p, q, r

p, q, r	$\boldsymbol{Cpqr}$**-FFWAA** Score-values	$\boldsymbol{Cpqr}$**-FFOWAA** Score-values	**Ranking**
p=2, q=2, r=2	0.44, 0.43, 0.40, 0.42	0.46, 0.39, 0.34, 0.37	$Y_1 > Y_2 > Y_4 > Y_3$
p=3, q=3, r=3	0.42, 0.39, 0.35, 0.37	0.43, 0.40, 0.36, 0.38	$Y_1 > Y_2 > Y_4 > Y_3$
p=2, q=3, r=6	0.43, 0.41, 0.36, 0.38	0.41, 0.39, 0.35, 0.37	$Y_1 > Y_2 > Y_4 > Y_3$
p=5, q=5, r=5	0.38, 0.35, 0.32, 0.34	0.37, 0.34, 0.30, 0.32	$Y_1 > Y_2 > Y_4 > Y_3$
p=6, q=4, r=12	0.34, 0.31, 0.27, 0.29	0.33, 0.31, 0.26, 0.28	$Y_1 > Y_2 > Y_4 > Y_3$
p, q, r	$\boldsymbol{Cpqr}$**-FFWGA** Score-values	$\boldsymbol{Cpqr}$**-FFOWGA** Score-values	**Ranking**
p=2, q=2, r=2	0.43, 0.32, 0.39, 0.41	0.41, 0.38, 0.32, 0.35	$Y_1 > Y_2 > Y_4 > Y_3$
p=3, q=3, r=3	0.40, 0.37, 0.33, 0.35	0.45, 0.42, 0.38, 0.40	$Y_1 > Y_2 > Y_4 > Y_3$
p=2, q=3, r=6	0.41, 0.38, 0.34, 0.36	0.39, 0.37, 0.33, 0.35	$Y_1 > Y_2 > Y_4 > Y_3$
p=5, q=5, r=5	0.36, 0.33, 0.29, 0.31	0.38, 0.35, 0.31, 0.33	$Y_1 > Y_2 > Y_4 > Y_3$
p=6, q=4, r=12	0.35, 0.32, 0.28, 0.30	0.30, 0.28, 0.23, 0.25	$Y_1 > Y_2 > Y_4 > Y_3$

7 Advantages of the New Proposed Methods

i) The p, q, r-FFS under confidence level enhances traditional FSs by offering greater flexibility in modeling uncertainty, improved DM precision, and better adaptability to complex, real-world problems through adjustable fractional parameters.

ii) p, q, r-FFS under confidence level with parameters p, q, r offer enhanced flexibility, allowing decision-makers to fine-tune membership functions for more precise adjustments.

iii) $Cpqr$-FFS encompasses PiFS, SpFS and T-SpFS, providing a flexible and precise framework for handling uncertainty in complex DM challenges.

iv) Adjusting parameters p, q, r allows decision-makers to refine the form and properties of membership functions, leading to more precise and targeted preference results.

v) $Cpqr$-FFS level provides a robust framework for handling uncertainty and incomplete information, making them highly valuable across various real-world applications.

8 Conclusion

This paper introduces a series of aggregation operators under confidence levels, including $Cpqr$-FFWAA, $Cpqr$-FFOWAA, $Cpqr$-FFWGA, and $Cpqr$-FFOWGA, with properties such as idempotency, boundedness, and monotonicity to efficiently aggregate p, q, r-Fractional Fuzzy information. A novel MCGDM approach is developed to tackle real-world DM problems, demonstrated through a numerical example for selecting a stable option. By varying the parameters p, q, r, we analyze their impact on the DM process, showcasing the flexibility of the newly proposed method. The approach enhances decision accuracy and adaptability, offering a structured framework for complex evaluations. However, challenges such as computational complexity, parameter sensitivity, and limited applicability may arise. Despite these constraints, the study contributes innovative aggregation operators and methodologies that enrich DM models. Its practical implications extend to diverse fields requiring precise and robust information fusion. Future research can focus on simplifying computational aspects and expanding applicability to broader domains. Overall, this work provides a significant advancement in fractional fuzzy aggregation and DM techniques. The effectiveness of the proposed technique depends on careful parameter selection, which can influence both accuracy and reliability. Additionally, its applicability may be restricted to specific problem domains, limiting its versatility. Achieving optimal results also requires high-quality data, making data integrity a crucial factor.

Data Availability

The data supporting our research results are included within the article or supplementary material.

Conflicts of Interest

The authors declare no conflict of interest.

References

[1] L. A. Zadeh, “Fuzzy sets,” *Inf. Control*, vol. 8, no. 3, pp. 338–353, 1965. https://doi.org/10.1016/S0019-9958(65)90241-X

[2] K. T. Atanassov, “Intuitionistic fuzzy sets,” *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, 1986. https://doi.org/10.1016/S0165-0114(86)80034-3

[3] Z. Xu and R. R. Yager, “Some geometric aggregation operators based on intuitionistic fuzzy sets,” *Int. J. Gen. Syst.*, vol. 35, no. 4, pp. 417–433, 2006. https://doi.org/10.1080/03081070600574353

[4] Z. Xu, “Intuitionistic fuzzy aggregation operators,” *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 6, pp. 1179–1187, 2007. https://doi.org/10.1109/TFUZZ.6.890678

[5] W. Wang and X. Liu, “Intuitionistic fuzzy geometric aggregation operators based on Einstein operations,” *Int. J. Intell. Syst.*, vol. 26, no. 11, pp. 1049–1075, 2011. https://doi.org/10.1002/int.20498

[6] W. Wang and X. Liu, “Intuitionistic fuzzy information aggregation using Einstein operations,” *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 5, pp. 923–938, 2012. https://doi.org/10.1109/TFUZZ.2012.2189405

[7] K. Rahman, S. Abdullah, M. Jamil, and M. Y. Khan, “Some generalized intuitionistic fuzzy Einstein hybrid aggregation operators and their application to multiple attribute group decision making,” *Int. J. Fuzzy Syst.*, vol. 20, no. 5, pp. 1567–1575, 2018. https://doi.org/10.1007/s40815-018-0452-0

[8] M. R. Seikh and U. Mandal, “Intuitionistic fuzzy Dombi aggregation operators and their application to multiple attribute decision-making,” *Granular Comput.*, vol. 6, no. 3, pp. 473–488, 2021. https://doi.org/10.1007/s41066-019-00209-y

[9] R. R. Yager, “Pythagorean membership grades in multicriteria decision making,” *IEEE Trans. Fuzzy Syst.*, vol. 22, no. 4, pp. 958–965, 2013. https://doi.org/10.1109/TFUZZ.2013.2278989

[10] H. Garg, “A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making,” *Int. J. Intell. Syst.*, vol. 31, no. 9, pp. 886–920, 2016. https://doi.org/10.1002/int.21809

[11] H. Garg, “Generalized Pythagorean fuzzy geometric aggregation operators using Einstein t-norm and t-conorm for multicriteria decision-making process,” *Int. J. Intell. Syst.*, vol. 32, no. 6, pp. 597–630, 2017. https://doi.org/10.1002/int.21860

[12] K. Rahman, S. Abdullah, R. Ahmed, and M. Ullah, “Pythagorean fuzzy Einstein weighted geometric aggregation operator and their application to multiple attribute group decision making,” *J. Intell. Fuzzy Syst.*, vol. 33, no. 1, pp. 635–647, 2017. https://doi.org/10.3233/JIFS-16797

[13] K. Rahman and A. Ali, “New approach to multiple attribute group decision-making based on Pythagorean fuzzy Einstein hybrid geometric operator,” *Granular Comput.*, vol. 5, no. 3, pp. 349–359, 2020. https://doi.org/10.1007/s41066-019-00166-6

[14] K. Rahman, S. Abdullah, A. Ali, and F. Amin, “Some induced averaging aggregation operators based on Pythagorean fuzzy numbers,” *Math. Lett.*, vol. 3, no. 4, pp. 40–45, 2017. https://doi.org/10.11648/j.ml.20170304.11

[15] M. Shakeel, K. Rahman, M. S. A. Khan, and M. Ullah, “Induced averaging aggregation operators with interval Pythagorean trapezoidal fuzzy numbers and their application to group decision making,” *Nucleus*, vol. 54, no. 2, pp. 140–154, 2017. https://doi.org/10.71330/thenucleus.2017.112

[16] M. S. A. Khan, S. Abdullah, A. Ali, and K. Rahman, “Pythagorean hesitant fuzzy information aggregation and their application to multi-attribute group decision-making problems,” *J. Intell. Syst.*, vol. 27, no. 4, pp. 1–18, 2018. https://doi.org/10.1515/jisys-2017-0231

[17] R. R. Yager, “Generalized orthopair fuzzy sets,” *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1222–1230, 2016. https://doi.org/10.1109/TFUZZ.2016.2604005

[18] P. Liu and P. Wang, “Some q-rung orthopair fuzzy aggregation operators and their applications to multiple-attribute decision making,” *Int. J. Intell. Syst.*, vol. 33, no. 2, pp. 259–280, 2018. https://doi.org/10.1002/int.21927

[19] X. Peng and L. Liu, “Information measures for q-rung orthopair fuzzy sets,” *Int. J. Intell. Syst.*, vol. 34, no. 8, pp. 1795–1834, 2019. https://doi.org/10.1002/int.22115

[20] M. Qiyas, S. Abdullah, N. Khan, M. Naeem, F. Khan, and Y. Liu, “Case study for hospital-based post-acute care-cerebrovascular disease using sine hyperbolic q-rung orthopair fuzzy Dombi aggregation operators,” *Expert Syst. Appl.*, vol. 215, p. 119224, 2023. https://doi.org/10.1016/j.eswa.2022.119224

[21] B. C. Cuong and V. Kreinovich, “Picture fuzzy sets,” *J. Comput. Sci. Cybern.*, vol. 30, no. 4, pp. 409–420, 2014. https://doi.org/10.15625/1813-9663/30/4/5032

[22] S. Ashraf, S. Abdullah, T. Mahmood, F. Ghani, and T. Mahmood, “Spherical fuzzy sets and their applications in multi-attribute decision making problems,” *J. Intell. Fuzzy Syst.*, vol. 36, no. 3, pp. 2829–2844, 2019. https://doi.org/10.3233/JIFS-17200

[23] S. Ashraf, S. Abdullah, M. Aslam, M. Qiyas, and M. A. Kutbi, “Spherical fuzzy sets and its representation of spherical fuzzy t-norms and t-conorms,” *J. Intell. Fuzzy Syst.*, vol. 36, no. 6, pp. 6089–6102, 2019. https://doi.org/10.3233/JIFS-181941

[24] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, “An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets,” *Neural Comput. Appl.*, vol. 31, no. 10, pp. 7041–7053,

2019. https://doi.org/10.1007/s00521-018-3521-2
[25] M. Gulistan, Y. Hongbin, W. Pedrycz, M. Rahim, F. Amin, and H. A. E. W. Khalifa, “p, q, r-fractional fuzzy sets and their aggregation operators and applications,” *Artif. Intell. Rev.*, vol. 57, no. 3, p. 3466, 2024. https://doi.org/10.1007/s10462-024-10911-2