International Journal of Knowledge and Innovation Studies
https://www.acadlore.com/journals/IJKIS

Generalized and Group-Generalized Parameter Based Fermatean Fuzzy Aggregation Operators with Application to Decision-Making

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Received: 08-10-2023 **Revised:** 09-16-2023 **Accepted:** 09-22-2023

Citation: A. A. Khan and L. Wang, "Generalized and group-generalized parameter based Fermatean fuzzy aggregation operators with application to decision-making," *Int J. Knowl. Innov Stud.*, vol. 1, no. 1, pp. 10–29, 2023. https://doi.org/10.56578/ijkis010102.

Abstract: Fermatean fuzzy set (FRFS) is very helpful in representing vague information that occurs in real world circumstances. Their eminent characteristic of FRFS is that the degree of membership $\Im^{\ell}$ and degree of non-membership $\daleth^{\gamma}$ satisfy the condition $0 \leq \Im^{\ell^3}(x) + \Im^{\ell^3}(x) \leq 1$, so the space of vague information they can describe is broader. This study introduces the concept of generalized parameters into the FRFS framework and proposes a set of generalized Fermatean fuzzy average aggregation operators for the purpose of information aggregation. Subsequently, the operators are expanded to encompass a generalized parameter based on group consensus, which is derived from the perspectives of numerous experienced senior experts and observers. The present study offers a multi-criteria decision-making (MCDM) methodology, which is demonstrated using a numerical example to successfully showcase the suggested technique. In conclusion, a comparative study is undertaken to validate the efficacy of the suggested technique in relation to existing methodologies.

Keywords: Generalized Fermatean fuzzy set; Aggregation operators; Group-based generalized parameter

1 Introduction

The process of decision-making encompasses several dimensions, including problem identification, alternative evaluation, aim and objective consideration, uncertainty and risk management, and option determination. The ability to effectively handle various circumstances in both personal and professional contexts is considered a vital expertise utilized by individuals and organizations [1]. The process of decision-making is often characterized by an organized approach, encompassing many sequential stages. These stages include the identification of a problem or opportunity, the collection of relevant information, the generation of alternative solutions, the evaluation of available options, the selection of a preferred choice, and the subsequent implementation and monitoring of the selected course of action.

The decision-making process can be influenced by a variety of contextual factors. The spectrum of decision-making encompasses a variety of scenarios, ranging from individual choices such as selecting a professional trajectory or engaging in financial investments, to intricate organizational decisions such as introducing a novel product, entering a fresh market, or engaging in strategic planning. The process of decision-making also encompasses ethical concerns, as the decisions made may carry moral or societal ramifications. The process of ethical decision-making entails the assessment of the moral correctness or incorrectness of actions and their subsequent outcomes. The process of decision-making can be carried out either individually or collectively. Within organizational settings, it is common for teams or committees to engage in collaborative decision-making processes, which entail the incorporation of many viewpoints and areas of expertise. The process of decision-making is characterized by its effectiveness through iterative steps [2]. Following the implementation of a decision, it is important to engage in the process of monitoring its consequences, collecting feedback, and then adjusting or revising the decision as deemed appropriate.

The process of making decisions is a fundamental aspect of achieving success and advancing in various domains. It has significant importance in influencing results and establishing the trajectory of initiatives. The relevance of this phenomenon is emphasized by the many obstacles and potential advantages that individuals and organizations encounter in diverse fields. The field of environmental science largely relies on the process of decision-making in order to effectively solve urgent ecological concerns. Environmental scientists and politicians are faced with the task of making decisions on conservation initiatives, sustainable resource management, and methods for mitigating climate change [3]. Environmental impact assessments (EIAs) play a crucial role in informing choices on the possible effects of development projects, hence aiding in the preservation and safeguarding of delicate ecosystems [4].

In the fields of engineering and technology, the process of decision-making plays a pivotal role in driving the advancement of innovation and overall growth. Engineers and technologists have a pivotal role in shaping our contemporary society through their decision-making processes on product design, technological development, and infrastructure building. The importance of making effective risk management decisions cannot be overstated in industries such as aircraft and nuclear power, since they play a critical role in maintaining safety and dependability [5].

In the realm of GSCM, the act of decision-making assumes a position of utmost significance, since it entails the amalgamation of ecologically friendly practices across the entirety of the supply chain process. The primary objective of GSCM is to mitigate the ecological consequences associated with the supply chain, while concurrently optimizing operational effectiveness and satisfying customer requirements [6].

Fuzzy logic assumes a prominent role in the realm of decision-making, particularly in scenarios characterized by ambiguous borders between various alternatives or states, as well as in the handling of imprecise and unclear data. Fuzzy logic is a theoretical framework that enables the effective management of ambiguity and vagueness, hence empowering decision-makers to arrive at conclusions that are characterized by increased nuance and flexibility.

In 1965, Zadeh [7] introduced the notion of fuzzy sets (FSs) as a means to address uncertain situations. Fuzzy sets are capable of managing imprecise and uncertain data by giving membership grades to components inside a set, often within the range of [0,1]. The contributions made by Zadeh in this particular field were very innovative, since some set theoretic characteristics that were originally applicable only to crisp situations were successfully expanded to encompass fuzzy sets. This notion has been extensively applied in several domains, as evidenced by a multitude of research. While the membership function is a fundamental aspect of a fuzzy system, it can be difficult to accurately characterize complicated fuzzy data. In order to address this concern, Atanassov [8] proposed the concept of the "intuitionistic fuzzy set" (IFS), which encompasses both degrees of membership and non-membership. Over the course of recent decades, it has been demonstrated that IFSs have exhibited efficacy in aiding scholars in managing data that exhibits imprecision and unreliability. Pythagorean fuzzy set (PFS), an extension of the IFS, was developed by Yager [9].The IFS and PFS were unable to adequately illustrate this occurrence. Senapati and Yager [10] established the notion of FRFS as an extension of IFS and PFS to tackle this problem. The theory of FRFS is well recognized for its significant relevance in many fields, owing to its comprehensive conceptual framework that effectively addresses conflicting and erroneous data within a FRF framework.

AOs refer to mathematical functions that are employed in the process of decision-making. Their purpose is to merge or aggregate several criteria, preferences, or pieces of information into a unified judgment or result. The AOs play a vital role in diverse decision support systems and MCDM approaches. The process of data aggregation is of great significance in several industries, such as business, management, social sciences, medicine, technology, mental health, and artificial intelligence. This process plays a vital role in enabling well-informed decision-making. Throughout history, the notion of dual consciousness has been perceived as a discrete being or a verbal number. Nevertheless, the aggregation of the data is a complex undertaking due to the considerable amount of uncertainty connected with it. Undoubtedly, it is apparent that individuals in positions of authority, sometimes referred to as AOs, play a crucial and influential role in the domain of MCDM issues. A considerable body of academic study has focused primarily on FrFSs. The consideration of AOs becomes essential in situations when numerous viable solutions are available for a particular problem, as it aids in identifying the most advantageous choice. Considerable scholarly study has been undertaken, revealing substantial advancements in the domain of FrFSs.

Chen et al. [11] proposed a conceptual framework for MCDM within the domain of sustainable building material selection. Chen et al. [12] proposed a unique methodology for evaluating passenger preferences and quantifying passenger satisfaction through the utilization of online-review analysis. Wei and Lu [13] introduced the notion of "Pythagorean fuzzy power AOs" in their study. Wu and Wei [14] introduced the concept of "Pythagorean fuzzy Hamacher aggregation operators" as described in their publication. Similarly, Garg et al. [15] put out the idea of "confidence levels based Pythagorean fuzzy aggregation operators" within the framework of its application to MCDM. Qiyas et al. [16] proposed the notion of Yager operators within the context of a picture fuzzy set environment, and explored its potential use in emergency program selection. Senapati and Yager [17] provided the fundamental AOs, Rani and Mishra [18] introduced the Einstein AOs, Jeevaraj presented the concept of interval-valued fuzzy rough set [19], Garg et al. [20] proposed the notion of Yager AOs and Shahzadi et al. [21] initiated the concept of Hamacher Interactive AOs for FRFSs. Work related to proposed work can be seen in literatures [22–24]. In their study, Simic et al. [25] put out a proposition on the sustainable selection of routes for petroleum transportation. The objective of this research work is to expand upon the concept of aggregating operators that rely on generalized and group-generalized parameters for Fermatean fuzzy sets. These operators are deemed more efficient in handling imprecise and uncertain data. Additionally, this study aims to establish a MCDM approach that is founded on the aforementioned operators.

The subsequent sections of this work are structured in the following manner. Section 2 provides a concise overview of fundamental terminologies pertaining to FRFSs, which will serve as the foundation for the subsequent analysis conducted in this study. In Section 3, the main results concerning the generalized Fermatean fuzzy set (GFRFS) are given. In Section 4, some generalized Fermatean fuzzy averaging operators are developed. In Section 5, some Fermatean fuzzy operators based on group-generalized parameter are presented. In Section 6, the applications of material selection of high speed naval craft with generalized Fermatean fuzzy average aggregation operators and group-generalized parameter are shown. The paper is concluded in Section 7.

2 Preliminaries

This section provides a quick overview of the definitions that will be utilized throughout the remainder of the paper.

Definition 2.1 [10] A FRFS in a finite universe $\mho$ is of the form

$$\mathfrak{M} = \{< \aleph, \Im^{\ell}{}_{\mathfrak{M}}(\aleph), \daleth^{\gamma}{}_{\mathfrak{M}}(\aleph) >: \aleph \in \mho\}$$

where, $\Im^{\ell}{}_{\mathfrak{M}}(\aleph) : \mho \to [0,1]$ represents the degree of membership and $\daleth^{\gamma}{}_{\mathfrak{M}}(\aleph) : \mho \to [0,1]$ represent the degree of non-membership of the element $\aleph \in \mho$ to the set $\mathfrak{M}$, respectively, with the condition that

$$0 \leq \Im^{\ell}{}_{\mathfrak{M}}(\aleph)^3 + \daleth^{\gamma}{}_{\mathfrak{M}}(\aleph)^3 \leq 1$$

and the degree of indeterminacy is given as

$$\pi_{\mathfrak{M}}(\aleph) = \sqrt[3]{(\Im^{\ell}{}_{\mathfrak{M}}(\aleph)^3 + \daleth^{\gamma}{}_{\mathfrak{M}}(\aleph)^3 - \Im^{\ell}{}_{\mathfrak{M}}(\aleph)^3 \daleth^{\gamma}{}_{\mathfrak{M}}(\aleph)^3)}$$

For each $\aleph \in \mho$, a basic element of the form $\langle \Im^{\ell}{}_{\mathfrak{M}}(\aleph), \daleth^{\gamma}{}_{\mathfrak{M}}(\aleph) \rangle$ in a FRFS $\mathfrak{M}$ is called Fermatean fuzzy number (FRFN). It can be shortly denoted by $\eth = \langle \Im^{\ell}{}_{\mathfrak{M}}, \daleth^{\gamma}{}_{\mathfrak{M}} \rangle$.

2.1 Operational Laws on Fermatean Fuzzy Numbers (FRFNs)

Definition 2.2 [17] Let $\eth_1 = \langle \Im^{\ell}{}_1, \daleth^{\gamma}{}_1 \rangle$ and $\eth_2 = \langle \Im^{\ell}{}_2, \daleth^{\gamma}{}_2 \rangle$ be FRFNs. Then

(1) $\bar{\eth}_1 = \langle \daleth^{\gamma}{}_1, \Im^{\ell}{}_1 \rangle$

(2) $\eth_1 \vee \eth_2 = \langle max\{\Im^{\ell}{}_1, \daleth^{\gamma}{}_1\}, min\{\Im^{\ell}{}_2, \daleth^{\gamma}{}_2\} \rangle$

(3) $\eth_1 \wedge \eth_2 = \langle min\{\Im^{\ell}{}_1, \daleth^{\gamma}{}_1\}, max\{\Im^{\ell}{}_2, \daleth^{\gamma}{}_2\} \rangle$

(4) $\eth_1 \oplus \eth_2 = \langle \sqrt[3]{(\Im^{\ell 3}_1 + \Im^{\ell 3}_2 - \Im^{\ell 3}_1 \Im^{\ell 3}_2)}, \daleth^{\gamma}{}_1 \daleth^{\gamma}{}_2 \rangle$

(5) $\eth_1 \otimes \eth_2 = \langle \Im^{\ell}{}_1 \Im^{\ell}{}_2, \sqrt[3]{(\daleth^{\gamma 3}_1 + \daleth^{\gamma 3}_2 - \daleth^{\gamma 3}_1 \daleth^{\gamma 3}_2)} \rangle$

(6) $\sigma \eth_1 = \langle \sqrt[3]{(1 - (1 - \Im^{\ell 3}_1)^{\sigma})}, \daleth^{\gamma \sigma}_1 \rangle$

(7) $\eth_1^{\eth_1} = \langle \Im^{\ell \sigma}_1, \sqrt[3]{(1 - (1 - \daleth^{\gamma 3}_1)^{\sigma})} \rangle$

Definition 2.3 [17] Assume that $\breve{\eth}_k = \langle \Im^{\ell}{}_k, \daleth^{\gamma}{}_k \rangle$ is the assemblage of FRFNs, and FRFWA: $\Lambda^n \to \Lambda$, if

$$FRFWA(\breve{\eth}_1, \breve{\eth}_2, \ldots \breve{\eth}_n) = \sum_{k=1}^{n} \mathscr{W}_k \breve{\eth}_k = \mathscr{W}_1 \breve{\eth}_1 \oplus \mathscr{W}_2 \breve{\eth}_2 \oplus \ldots, \mathscr{W}_n \breve{\eth}_n$$

where, Λ^n is the set of all FRFNs, and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ is the weight vector (WV) of $(\breve{\eth}_1, \breve{\eth}_2, \ldots, \breve{\eth}_n)$, such that $0 \leqslant \mathscr{W}_k \leqslant 1$ and $\sum_{k=1}^{n} \mathscr{W}_k = 1$. Then, the FRFWA is called the Fermatean fuzzy weighted average operator.

Definition 2.4 [17] Let $\breve{\eth}_k = \langle \Im^{\ell}{}_k, \daleth^{\gamma}{}_k \rangle$ be the assemblage of FRFNs,we can find $FRFWG$ by

$$FRFWA(\breve{\eth}_1, \breve{\eth}_2, \ldots \breve{\eth}_n) = \left\langle \sqrt[3]{(1 - \prod_{k=1}^{n} (1 - \Im^{\ell 3}_k)^{\mathscr{W}_k})}, \prod_{k=1}^{n} \daleth^{\gamma \mathscr{W}_k}_k \right\rangle$$

Definition 2.5 [17] Assume that $\breve{\eth}_k = \langle \Im^{\ell}{}_k, \daleth^{\gamma}{}_k \rangle$ is the assemblage of $FRFN$, and $FRFWG : \Lambda^n \to \Lambda$, if

$$FRFWG(\breve{\eth}_1, \breve{\eth}_2, \ldots \breve{\eth}_n) = \sum_{k=1}^{n} \breve{\eth}_k^{\mathscr{W}_k} = \breve{\eth}_1^{\mathscr{W}_1} \otimes \breve{\eth}_2^{\mathscr{W}_2} \otimes \ldots, \breve{\eth}_n^{\mathscr{W}_n}$$

Then, the $FRFWG$ is called the Fermatean fuzzy weighted geometric operator.

Theorem 2.6 [17] Let $\breve{\eth}_k = \langle \Im^{\ell}{}_k, \daleth^{\gamma}{}_k \rangle$ be the assemblage of FRFNs, we can find $FRFWG$ by

$$FRFWG(\breve{\eth}_1, \breve{\eth}_2, \ldots \breve{\eth}_n) = \left\langle \prod_{k=1}^{n} \Im^{\ell \mathscr{W}_k}_k, \sqrt[3]{(1 - \prod_{k=1}^{n} (1 - \daleth^{\gamma 3}_k)^{\mathscr{W}_k})} \right\rangle$$

Definition 2.7 [17] Suppose $\widetilde{\Re} = \langle \Im^{\ell}, \daleth^{\gamma} \rangle$ is a FRFN, then a score function $\mathfrak{C}$ of $\widetilde{\Re}$ is defined as

$$\mathfrak{C}(\widetilde{\Re}) = \Im^{\ell^3} - \daleth^{\gamma 3}$$

$\mathfrak{C}(\widetilde{\Re}) \in [-1, 1]$. The ranking of a FRFN is determined by its score, whereby a higher score indicates a greater preference for the FRFN. Hence, in order to conduct a comparison of the FRFNs, it is not imperative to depend on the score function. In order to address this issue, we propose the incorporation of an additional approach known as the accuracy function.

Definition 2.8 Suppose $\widetilde{\Re} = \langle \Im^{\ell}, \daleth^{\gamma} \rangle$ is a FRFN, then an accuracy function $\mathfrak{R}$ of $\widetilde{\Re}$ is defined as

$$\mathfrak{R}(\widetilde{\Re}) = \Im^{\ell^3} + \daleth^{\gamma 3}$$

$\mathfrak{R}(\widetilde{\Re}) \in [0, 1]$. The high value of accuracy degree $\mathfrak{R}(\widetilde{\Re})$ defines high preference of $\widetilde{\Re}$.

3 Fermatean Fuzzy Information under Generalized Parameter

Suppose a medical diagnostic problem in which a patient suffers from an unknown disease and presents his/her preferences as FRFNs above the set of symptoms $E = \{d_1, d_2, d_3\}$, where d_1 stands for Rheumatoid arthritis (RA), d_2 stands for Allergies and asthma (AA) and d_3 Liver disease (LD). Let the FRFS $Q = \{(0.34, 0.78)_{RA}, (0.53, 0.88)_{AA}, (0.89, 0.66)_{LD}\}$ represents the preferences of the patient. The data gathered is solely derived from the individual's comprehension, personal encounters, and physical well-being when recording the manifestations. In the event that the physician fails to address the patient's complaints in a timely manner, there is a potential for an inaccurate prognosis and subsequent failure to achieve a complete recovery. This is due to the absence of a secondary assessment by a junior or senior medical professional to verify the information provided by the patient. One option to enhance the realism of the supplied technique is to incorporate a generic parameter that reflects the expert's confidence in the dependability of the information. This addition acknowledges the importance of considering real-life situations and further strengthens the approach. The patient's preferences are evaluated by a senior expert or physician, who provides their information as $h = (0.52, 0.95)$. This information corresponds to the under FRFS generalized parameter, denoted as Q^G, which consists of the following values:

$$Q^G = \{(0.34, 0.78)_{RA}, (0.53, 0.88)_{AA}, (0.89, 0.66)_{LD}(\mathbf{0.52}, \mathbf{0.95})\}$$

The generalized value shown in bold is a FRFN itself, which makes sure that unclear information is shown correctly throughout the system of knowledge representation as much as possible. The universal parameter value can make it easier to improve systems that are already in place, which can lead to more accurate decisions that need to be made. Without the general measure, the initial evaluation is still not clear, which suggests that the test's validity is not certain. So, in the information mapping system, the chance of significant changes to unknown data can only be ruled out based on the opinion of one witness or expert by getting a second opinion from another expert (in the form of the general parameter) when the original FRFNs are put into place. So, the generalized Fermatean fuzzy set (GFRFS) is defined as

Definition 3.1 A GFRFS in a finite universe $\mho$ is of the form

$$\mathfrak{S} = \{(\langle \aleph, \Im^{\ell}{}_{\mathfrak{S}}(\aleph), \daleth^{\gamma}{}_{\mathfrak{S}}(\aleph) \rangle (\Im^{\ell}{}_g, \daleth^{\gamma}{}_g)) : \aleph \in \mho\}$$

where, $\Im^{\ell}{}_{\mathfrak{S}}(\aleph) : \mho \to [0, 1]$ represents the degree of membership and $\daleth^{\gamma}{}_{\mathfrak{S}}(\aleph) : \mho \to [0, 1]$ represents the degree of non-membership of the element $\aleph \in \mho$ to the set $\mathfrak{S}$, respectively, with the condition that

$$0 \le \Im^{\ell}{}_{\mathfrak{S}}(\aleph)^3 + \daleth^{\gamma}{}_{\mathfrak{S}}(\aleph)^3 \le 1$$

$\Im^{\ell}{}_g, \daleth^{\gamma}{}_g \in [0, 1]$ denote the level of truth and falsehood of the GFRFS respectively, with the condition $0 \le \Im^{\ell 3}_{g} + \daleth^{\gamma 3}_{g} \le 1$. Here $(\Im^{\ell}{}_g, \daleth^{\gamma}{}_g)$ is called the generalized parameter (GP) that is the FRFN itself given by some other senior expert / observer showing a preferential evaluation.

4 Fermatean Fuzzy Average Aggregation Operator under GP

In this section we presented the GFRFWA operator, GFRFOWA operator and GFRFHA operator.

4.1 GFRFWA Operator

Definition 4.1 Let $g = (\Im^{\ell}{}_{g}, \gimel^{\gamma}{}_{g})$ be the be the GP for the FRFNs $\eth_{\xi} = (\Im^{\ell}{}_{\xi}, \gimel^{\gamma}{}_{\xi})$, then the GFRFWA operator is defined as

$$GFRFWA((\mathfrak{s}_1, \mathfrak{s}_2, \ldots, \mathfrak{s}_n, g)) = g \otimes \text{FRFWA}(\eth_1, \eth_2, \ldots, \eth_n)$$

Theorem 4.2 Let $\eth_{\xi} = (\Im^{\ell}{}_{\xi}, \gimel^{\gamma}{}_{\xi})$ be the collection of FRFNs and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ is the WV of $\eth_{\xi}$ such that $\mathscr{W}_{\xi} \in [0,1]$ and $\sum_{\xi=1}^{n} \mathscr{W}_{\xi} = 1$. Generalized parameter is $g = (\Im^{\ell}{}_{g}, \gimel^{\gamma}{}_{g})$, then the GFRFWA operator is defined as

$$\begin{aligned} GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) =& g \otimes \text{FRFWA}(\eth_1, \eth_2, \ldots, \eth_n) \\ =& \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{n} (1 - (\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (1 - (\gimel^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_{\xi}})^3} \right) \end{aligned}$$

Proof. In order to demonstrate that this theorem is true, we will employ mathematical induction.
For $n = 2$,

$$GFRFWA((\eth_1, \eth_2), g) = g \otimes (\mathscr{W}_1 \eth_1 \oplus \mathscr{W}_2 \eth_2)$$

First we solve $(\mathscr{W}_1 \eth_1 \oplus \mathscr{W}_2 \eth_2)$ by using the operational law of FRFS, we have

$$\begin{aligned} \mathscr{W}_1 \eth_1 \oplus \mathscr{W}_2 \eth_2 &= \mathscr{W}_1 (\Im^{\ell}{}_{1}, \gimel^{\gamma}{}_{1}) \oplus \mathscr{W}_2 (\Im^{\ell}{}_{2}, \gimel^{\gamma}{}_{2}) \\ &= \left(\sqrt[3]{1 - (1 - \Im^{\ell 3}_{1})^{\mathscr{W}_1}}, \gimel^{\gamma \mathscr{W}_1}_{1} \oplus \sqrt[3]{1 - (1 - \Im^{\ell 3}_{2})^{\mathscr{W}_2}}, \gimel^{\gamma \mathscr{W}_2}_{2} \right) \\ &= \left(\sqrt[3]{1 - (1 - \Im^{\ell 3}_{1})^{\mathscr{W}_1} . (1 - \Im^{\ell 3}_{2})^{\mathscr{W}_2}}, \gimel^{\gamma \mathscr{W}_1}_{1} . \gimel^{\gamma \mathscr{W}_2}_{2} \right) \end{aligned}$$

Now,

$$\begin{aligned} g \otimes (\mathscr{W}_1 \eth_1 \oplus \mathscr{W}_2 \eth_2) &= (\Im^{\ell}{}_{g}, \gimel^{\gamma}{}_{g}) \otimes \left(\sqrt[3]{1 - (1 - \Im^{\ell 3}_{1})^{\mathscr{W}_1} . (1 - \Im^{\ell 3}_{2})^{\mathscr{W}_2}}, \gimel^{\gamma \mathscr{W}_1}_{1} . \gimel^{\gamma \mathscr{W}_2}_{2} \right) \\ &= \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - (1 - \Im^{\ell 3}_{1})^{\mathscr{W}_1} . (1 - \Im^{\ell 3}_{2})^{\mathscr{W}_2}}, \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (\gimel^{\gamma \mathscr{W}_1}_{1} . \gimel^{\gamma \mathscr{W}_2}_{2})^3 - (\gimel^{\gamma}{}_{g})^3 . (\gimel^{\gamma \mathscr{W}_1}_{1} . \gimel^{\gamma \mathscr{W}_2}_{2})^3} \right) \\ &= \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - (1 - \Im^{\ell 3}_{1})^{\mathscr{W}_1} . (1 - \Im^{\ell 3}_{2})^{\mathscr{W}_2}}, \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (1 - (\gimel^{\gamma}{}_{g})^3) . (\gimel^{\gamma \mathscr{W}_1}_{1})^3 (\gimel^{\gamma \mathscr{W}_2}_{2})^3} \right) \\ &= \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{2} (1 - \Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (1 - (\gimel^{\gamma}{}_{g})^3) . \prod_{\xi=1}^{2} (\gimel^{\gamma \mathscr{W}_{\xi}}_{\xi})^3} \right) \end{aligned}$$

We proved for $n = 2$.
Assuming the $n = k$ result is correct, this means

$$\begin{aligned} GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) &= g \otimes \text{FRFWA}(\eth_1, \eth_2, \ldots, \eth_n) \\ &= \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{k} (1 - (\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (1 - (\gimel^{\gamma}{}_{g})^3) \prod_{\xi=1}^{k} (\gimel^{\gamma \mathscr{W}_{\xi}}_{\xi})^3} \right) \end{aligned}$$

Now we will prove for $n = k + 1$,

$$\begin{aligned} GFRFWA((\eth_1, \eth_2, \ldots, \eth_k, \eth_{k+1}), g) &= g \otimes (\mathscr{W}_1 \eth_1 \oplus \ldots, \oplus \mathscr{W}_k \eth_k \oplus \mathscr{W}_{k+1} \eth_{k+1}) \\ &= \left(\Im^{\ell}{}_{g} . \sqrt[3]{1 - (1 - (\Im^{\ell}{}_{k+1})^3)^{\mathscr{W}_{k+1}} \prod_{\xi=1}^{k} (1 - (\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \right. \\ & \left. \sqrt[3]{(\gimel^{\gamma}{}_{g})^3 + (1 - (\gimel^{\gamma}{}_{g})^3)((\gimel^{\gamma}{}_{k+1})^{\mathscr{W}_{k+1}})^3 \prod_{\xi=1}^{k} (\gimel^{\gamma \mathscr{W}_{\xi}}_{\xi})^3} \right) \end{aligned}$$

$$= \Big(\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{k+1} (1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{(\daleth^{\gamma}{}_{g})^3 + (1-(\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{k+1} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3} \Big)$$

This result holds when $n = k+1$. Consequently, the result is valid for any number under a GP.

Theorem 4.3 Aggregated value obtained by GFRFWA operator is also a FRFN.

Proof. For every $\xi = 1,2,...,n$, we have $0 \le \Im^{\ell}{}_{\xi}, \daleth^{\gamma}{}_{\xi} \le 1$ and $0 \le \Im^{\ell}{}_{\xi}^{3} + \daleth^{\gamma}{}_{\xi}^{3} \le 1. \Rightarrow 0 \le 1 - \Im^{\ell}{}_{\xi}^{3} \le 1$. Therefore,

$$0 \le \prod_{\xi=1}^{n} (1-\Im^{\ell}{}_{\xi}^{3})^{\mathscr{W}_\xi} \le 10 \le \Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{n} (1-\Im^{\ell}{}_{\xi}^{3})^{\mathscr{W}_\xi}} \le 1 \quad \text{for} \quad 0 \le \Im^{\ell}{}_{g} \le 1.$$

Also, for $0 \le \daleth^{\gamma}{}_{g} \le 1$, one can write, $0 \le \sqrt[3]{(\daleth^{\gamma}{}_{g})^3 + (1-(\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3} \le 1$.

Now,

$$= \Big(\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{n} (1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}} \Big)^3 + \Big(\sqrt[3]{(\daleth^{\gamma}{}_{g})^3 + (1-(\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3} \Big)^3$$

$$= (\Im^{\ell}{}_{g})^3 \Big(1 - \prod_{\xi=1}^{n} (1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi} \Big) + \Big((\daleth^{\gamma}{}_{g})^3 + (1-(\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 \Big)$$

$$= ((\Im^{\ell}{}_{g})^3 + (\daleth^{\gamma}{}_{g})^3) + \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 - (\Im^{\ell}{}_{g})^3 \prod_{\xi=1}^{n} (1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi} - (\daleth^{\gamma}{}_{g})^3 \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3$$

$$\le ((\Im^{\ell}{}_{g})^3 + (\daleth^{\gamma}{}_{g})^3) + \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 - (\Im^{\ell}{}_{g})^3 \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 - (\daleth^{\gamma}{}_{g})^3 \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3$$

$$as \quad \daleth^{\gamma}{}_{\xi}^{3} \le 1 - \Im^{\ell}{}_{\xi}^{3}$$

$$\le ((\Im^{\ell}{}_{g})^3 + (\daleth^{\gamma}{}_{g})^3) + \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 - ((\Im^{\ell}{}_{g})^3 + (\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3$$

$$\le ((\Im^{\ell}{}_{g})^3 + (\daleth^{\gamma}{}_{g})^3) \Big(1 - \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 \Big) + \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3$$

$$\le 1 - \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 + \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3 \le 1$$

Therefore the GFRFWA operator's aggregated value is q-ROPN.

Example 4.4 Let $g = (0.60, 0.80)$ be the GP of four FRFNs. $\eth_1 = (0.34, 0.78)$, $\eth_2 = (0.53, 0.88)$, $\eth_3 = (0.89, 0.66)$ and $\eth_4 = (0.52, 0.95)$ with associated WV $\mathscr{W} = (0.3, 0.1, 0.4, 0.2)$, here $q = 3$, then

$$\Im^{\ell}{}_{g} . \sqrt[3]{1 - \prod_{\xi=1}^{n} (1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}} = 0.449745$$

Also

$$\sqrt[3]{(\daleth^{\gamma}{}_{g})^3 + (1-(\daleth^{\gamma}{}_{g})^3) \prod_{\xi=1}^{n} (\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3} = 0.901715$$

By Theorem 3.2, we have

$$GFRFWA((\eth_1, \eth_2, \eth_3, \eth_4), g) = g \otimes \text{FRFWA}(\eth_1, \eth_2, \dots, \eth_n)$$

$$= \left(\Im^{\ell}{}_{g}.\sqrt[3]{1-\prod_{\xi=1}^{k}(1-(\Im^{\ell}{}_{\xi})^{3})^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\daleth^{\gamma}{}_{g})^{3}+(1-(\daleth^{\gamma}{}_{g})^{3})\prod_{\xi=1}^{k}(\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_{\xi}})^{3}}\right)$$
$$= (0.449745, 0.901715)$$

Proposition 4.5 Let $\eth_{\xi} = (\Im^{\ell}{}_{\xi}, \daleth^{\gamma}{}_{\xi})$ be the collection of FRFNs and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ is the WV of $\eth_{\xi}$ such that $\mathscr{W}_{\xi} \in [0,1]$ and $\sum_{\xi=1}^{n} \mathscr{W}_1 = 1$. GP is $g = (\Im^{\ell}{}_{g}, \daleth^{\gamma}{}_{g})$, the following properties are available in the GFRFWA operator:

1. (Idempotency) if $\eth_{\xi} = \eth\ \forall i$, then

$$GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) = g \otimes \eth$$

2. (Boundary condition) if $\eth_{\xi}^{-} = (\Im^{\ell}{}^{min}_{g\otimes\eth_{\xi}}, \daleth^{\gamma}{}^{max}_{g\otimes\eth_{\xi}})$ and $\eth_{\xi}^{+} = (\Im^{\ell}{}^{max}_{g\otimes\eth_{\xi}}, \daleth^{\gamma}{}^{min}_{g\otimes\eth_{\xi}})$, then for every $\mathscr{W}_{\xi}$,

$$\eth_{\xi}^{-} \leq GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) \leq \eth_{\xi}^{+}$$

3. (Monotonicity) Let $\eth_{\xi}^{*} = (\Im^{\ell}{}_{\xi}^{*}, \daleth^{\gamma}{}_{\xi}^{*})$ be a collection of FRFNs such that $\Im^{\ell}{}_{\xi} \leq \Im^{\ell}{}_{\xi}^{*}$ and $\daleth^{\gamma}{}_{\xi} \leq \daleth^{\gamma}{}_{\xi}^{*}$ for all i, then for every $\mathscr{W}_{\xi}$,

$$GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) \leq GFRFWA((\eth_1^{*}, \eth_2^{*}, \ldots, \eth_n^{*}), g)$$

4. (Commutativity) Let $\eth_{\xi} = (\Im^{\ell}{}_{\xi}, \daleth^{\gamma}{}_{\xi})$ and $\ddot{\eth}_{\xi} = (\ddot{\Im}^{\ell}{}_{\xi}, \ddot{\daleth}^{\gamma}{}_{\xi})$ be two collection of n FRFNs such that $\ddot{\eth}_{\xi}$ is any permutation of $\eth_{\xi}$, then

$$GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) = GFRFWA((\ddot{\eth}_1, \ddot{\eth}_2, \ldots,, \ddot{\eth}_n), g)$$

Proof. 1. if $\eth_{\xi} = \eth\ \forall i$, then by GFRFWA operator,

$$\begin{aligned}
GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) &= \left(\Im^{\ell}{}_{g}.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^{3})^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\daleth^{\gamma}{}_{g})^{3}+(1-(\daleth^{\gamma}{}_{g})^{3})\prod_{\xi=1}^{n}(\daleth^{\gamma}{}_{\xi}^{\mathscr{W}_{\xi}})^{3}}\right) \\
&= \left(\Im^{\ell}{}_{g}.\sqrt[3]{1-(1-(\Im^{\ell})^{3})^{\sum_{\xi=1}^{n}\mathscr{W}_{\xi}}}, \sqrt[3]{(\daleth^{\gamma}{}_{g})^{3}+(1-(\daleth^{\gamma}{}_{g})^{3})\left(\daleth^{\gamma\sum_{\xi=1}^{n}\mathscr{W}_{\xi}}\right)^{3}}\right) \\
&= \left(\Im^{\ell}{}_{g}.\sqrt[3]{1-(1-(\Im^{\ell})^{3})}, \sqrt[3]{(\daleth^{\gamma}{}_{g})^{3}+(1-(\daleth^{\gamma}{}_{g})^{3})(\daleth^{\gamma})^{3}}\right) \\
&= \left(\Im^{\ell}{}_{g}.\Im^{\ell}, \sqrt[3]{(\daleth^{\gamma}{}_{g})^{3}+\daleth^{\gamma 3}-(\daleth^{\gamma}{}_{g})^{3}(\daleth^{\gamma})^{3}}\right) \\
&= g \otimes \eth
\end{aligned}$$

2. Let $\eth_{\xi}^{+} = (\Im^{\ell}{}^{max}_{g\otimes\eth_{\xi}}, \daleth^{\gamma}{}^{min}_{g\otimes\eth_{\xi}})$ and $\eth_{\xi}^{-} = (\Im^{\ell}{}^{min}_{g\otimes\eth_{\xi}}, \daleth^{\gamma}{}^{max}_{g\otimes\eth_{\xi}})$ where $\Im^{\ell}{}^{min}_{g\otimes\eth_{\xi}} = \Im^{\ell}{}_{g}(min\Im^{\ell}{}_{\xi})$, $\Im^{\ell}{}^{max}_{g\otimes\eth_{\xi}} = \Im^{\ell}{}_{g}(max\Im^{\ell}{}_{\xi})$, $\Im^{\ell}{}^{min}_{g\otimes\eth_{\xi}} = \sqrt[3]{\daleth^{\gamma 3}_{g}+(1-\daleth^{\gamma 3}_{g})(min(\daleth^{\gamma}{}_{\xi}))^{3}}$, and $\Im^{\ell}{}^{max}_{g\otimes\eth_{\xi}} = \sqrt[3]{\daleth^{\gamma 3}_{g}+(1-\daleth^{\gamma 3}_{g})(max(\daleth^{\gamma}{}_{\xi}))^{3}}$ for all i, it is clearly that $min(\Im^{\ell}{}_{\xi}) \leq \Im^{\ell}{}_{\xi} \leq max(\Im^{\ell}{}_{\xi}) \Rightarrow max(1-\Im^{\ell 3}_{\xi}) \leq (1-\Im^{\ell 3}_{\xi}) \leq min(1-\Im^{\ell 3}_{\xi})$, for each $\mathscr{W}$,

$$\begin{aligned}
&\Rightarrow \prod_{\xi=1}^{n}(1-max(\Im^{\ell}{}_{\xi})^{3})^{\mathscr{W}_{\xi}} \leq \prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}} \leq \prod_{\xi=1}^{n}(1-min(\Im^{\ell}{}_{\xi})^{3})^{\mathscr{W}_{\xi}} \\
&\Rightarrow (1-max(\Im^{\ell}{}_{\xi})^{3})^{\sum_{\xi=1}^{n}\mathscr{W}_{\xi}} \leq \prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}} \leq (1-min(\Im^{\ell}{}_{\xi})^{3})^{\sum_{\xi=1}^{n}\mathscr{W}_{\xi}} \\
&\Rightarrow 1-((1-min(\Im^{\ell}{}_{\xi})^{3})) \leq \prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}} \leq 1-((1-max(\Im^{\ell}{}_{\xi})^{3})) \\
&\Rightarrow \sqrt[3]{1-((1-min(\Im^{\ell}{}_{\xi})^{3}))} \leq \sqrt[3]{\prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}}} \leq \sqrt[3]{1-((1-max(\Im^{\ell}{}_{\xi})^{3}))} \\
&\Rightarrow min(\Im^{\ell}{}_{\xi}) \leq \sqrt[3]{\prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}}} \leq max(\Im^{\ell}{}_{\xi})
\end{aligned}$$

As we know, $0 \leq \Im^{\ell}{}_{g} \leq 1$, we can write

$$\Im^{\ell}{}_g.min(\Im^{\ell}{}_{\xi}) \leq \Im^{\ell}{}_g.\sqrt[3]{\prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}}} \leq \Im^{\ell}{}_g.max(\Im^{\ell}{}_{\xi})\Im^{\ell min}_{g\otimes\eth_{\xi}} \leq \Im^{\ell}{}_g.\sqrt[3]{\prod_{\xi=1}^{n}(1-\Im^{\ell 3}_{\xi})^{\mathscr{W}_{\xi}}} \leq \Im^{\ell max}_{g\otimes\eth_{\xi}}$$

Furthermore, $min(\beth^{\gamma}{}_{\xi}) \leq \beth^{\gamma}{}_{\xi} \leq max(\beth^{\gamma}{}_{\xi}) \Longleftrightarrow (min(\beth^{\gamma}{}_{\xi}))^3 \leq \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3 \leq (max(\beth^{\gamma}{}_{\xi}))^3$. Also for $0 \leq \beth^{\gamma}{}_g \leq 1$, we can write

$$\Longrightarrow (1-\beth^{\gamma 3}_{g})(min(\beth^{\gamma}{}_{\xi}))^3 \leq (1-\beth^{\gamma 3}_{g})\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3 \leq (1-\beth^{\gamma 3}_{g})(max(\beth^{\gamma}{}_{\xi}))^3$$

$$\Longrightarrow \beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})(min(\beth^{\gamma}{}_{\xi}))^3 \leq \beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3 \leq \beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})(max(\beth^{\gamma}{}_{\xi}))^3$$

$$\Longrightarrow \sqrt[3]{\beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})(min(\beth^{\gamma}{}_{\xi}))^3} \leq \sqrt[3]{\beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3} \leq \sqrt[3]{\beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})(max(\beth^{\gamma}{}_{\xi}))^3}$$

$$\Longrightarrow \beth^{\gamma max}_{g\otimes\eth_{\xi}} \leq \sqrt[3]{\beth^{\gamma 3}_{g} + (1-\beth^{\gamma 3}_{g})\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3} \leq \beth^{\gamma min}_{g\otimes\eth_{\xi}}$$

$GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) = \eth = (\Im^{\ell}{}_{g\otimes\eth_{\xi}}, \beth^{\gamma}{}_{g\otimes\eth_{\xi}})$, then we have $\Im^{\ell min}_{g\otimes\eth_{\xi}} \leq \Im^{\ell}{}_{g\otimes\eth_{\xi}} \leq \Im^{\ell max}_{g\otimes\eth_{\xi}}$ and $\beth^{\gamma min}_{g\otimes\eth_{\xi}} \leq \beth^{\gamma}{}_{g\otimes\eth_{\xi}} \leq \beth^{\gamma max}_{g\otimes\eth_{\xi}}$. Therefore, by score function, we write

$$\eth^{-}_{\xi} \leq GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g)) \leq \eth^{+}_{\xi}$$

Proposition 4.6 If the senior expert's preference for the evaluated object is viewed to be $g = (1, 0)$, then the GFRFWA operator will be reduced in FRFWA operator.

Proof. If we take $g = (1, 0)$ as given then by Theorem 3.2, we have

$$\begin{aligned} GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) &= \left(\Im^{\ell}{}_g.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\beth^{\gamma}{}_g)^3 + (1-(\beth^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3}\right) \\ &= \left(\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \sqrt[3]{\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3}\right) \\ &= \left(\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3\right) \\ &= \text{FRFWA}(\eth_1, \eth_2, \ldots, \eth_n). \end{aligned}$$

Proposition 4.7 If the senior expert's preference for the evaluated object is viewed to be $g = (0, 1)$, then the GFRFWA operator will give the value $(0, 1)$.

Proof. If we take $g = (0, 1)$ as given then by Theorem 3.2, we have

$$\begin{aligned} GFRFWA((\eth_1, \eth_2, \ldots, \eth_n), g) &= \left(\Im^{\ell}{}_g.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_{\xi}}}, \sqrt[3]{(\beth^{\gamma}{}_g)^3 + (1-(\beth^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3}\right) \\ &= \left(0, \sqrt[3]{1+(1-1)\prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_{\xi}}_{\xi})^3}\right) \\ &= (0, 1). \end{aligned}$$

4.2 GFRFOWA Operator

Definition 4.8 Let $g = (\Im^{\ell}{}_g, \beth^{\gamma}{}_g)$ be the be the GP for the FRFNs $\eth_{\xi} = (\Im^{\ell}{}_{\xi}, \beth^{\gamma}{}_{\xi})$, then the GFRFOWA operator is described as

$$GFRFWA((\eth_1,\eth_2,\ldots,\eth_n,g)) = g \otimes \text{FRFOWA}(\eth_1,\eth_2,\ldots,\eth_n)$$

Theorem 4.9 Let $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$ be the collection of FRFNs and $\mathscr{W} = (\mathscr{W}_1,\mathscr{W}_2,\ldots,\mathscr{W}_n)^T$ is the WV of $\eth_\xi$ such that $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n}\mathscr{W}_\xi = 1$. GP is $g = (\Im^{\ell}{}_g, \daleth^{\gamma}{}_g)$, then the GFRFOWA operator is described as

$$\begin{aligned} GFRFWA((\eth_1,\eth_2,\ldots,\eth_n),g) &= g \otimes \text{FRFOWA}(\eth_1,\eth_2,\ldots,\eth_n) \\ &= \Bigg(\Im^{\ell}{}_g.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}}, \\ &\sqrt[3]{(\daleth^{\gamma}{}_g)^3+(1-(\daleth^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3}\Bigg) \end{aligned}$$

$(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\eth_{\sigma(i-1)} \geq \eth_{\sigma(i)}$ for any i.
*P*roof. The proof can be done as Theorem 3.2.

Example 4.10 Let $g = (0.6, 0.8)$ be the GP of four Fermatean fuzzy numbers. $\eth_1 = (0.34, 0.78)$, $\eth_2 = (0.53, 0.88)$, $\eth_3 = (0.89, 0.66)$ and $\eth_4 = (0.52, 0.95)$ with associated WV $\mathscr{W} = (0.3, 0.1, 0.4, 0.2)$, then firstly

$$\mathfrak{E}(\eth_1) = -0.4352$$
$$\mathfrak{E}(\eth_2) = 0.5326$$
$$\mathfrak{E}(\eth_3) = 0.4175$$
$$\mathfrak{E}(\eth_4) = 0.7168$$

On the behalf of score functions, $\eth_{\sigma(1)} = \eth_3, \eth_{\sigma(2)} = \eth_1, \eth_{\sigma(3)} = \eth_2$ and $\eth_{\sigma(4)} = \eth_4$

$$\Im^{\ell}{}_g.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}} = 0.431465$$

Also

$$\sqrt[3]{(\daleth^{\gamma}{}_g)^3+(1-(\daleth^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} = 0.917045$$

By Theorem 3.9, we have

$$\begin{aligned} GFRFOWA((\eth_1,\eth_2,\eth_3,\eth_4),g) &= g \otimes \text{FRFOWA}(\eth_1,\eth_2,\ldots,\eth_n) \\ &= \Bigg(\Im^{\ell}{}_g.\sqrt[3]{1-\prod_{\xi=1}^{k}(1-(\Im^{\ell}{}_\xi)^3)^{\mathscr{W}_\xi}}, \\ &\sqrt[3]{(\daleth^{\gamma}{}_g)^3+(1-(\daleth^{\gamma}{}_g)^3)\prod_{\xi=1}^{k}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\xi})^3}\Bigg) \\ &= (0.431465, 0.917045) \end{aligned}$$

Proposition 4.11 Let $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$ be the collection of FRFNs and $\mathscr{W} = (\mathscr{W}_1,\mathscr{W}_2,\ldots,\mathscr{W}_n)^T$ is the WV of $\eth_\xi$ such that $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n}\mathscr{W}_\xi = 1$. GP is $g = (\Im^{\ell}{}_g, \daleth^{\gamma}{}_g)$, the following properties are available in the GFRFOWA operator:

1. (Idempotency) if $\eth_\xi = \eth \ \forall i$, then

$$GFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),g) = g \otimes \eth$$

2. (Boundary condition) if $\eth_\xi^- = (\Im^{\ell}{}^{min}_{g\otimes\eth_\xi}, \daleth^{\gamma}{}^{max}_{g\otimes\eth_\xi})$ and $\eth_\xi^+ = (\Im^{\ell}{}^{max}_{g\otimes\eth_\xi}, \daleth^{\gamma}{}^{min}_{g\otimes\eth_\xi})$, then for every $\mathscr{W}_\xi$,

$$\eth_\xi^- \leq GFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),g) \leq \eth_\xi^+$$

3. (Monotonicity) Let $\eth_\xi^* = (\Im^{\ell}{}^*_\xi, \daleth^{\gamma}{}^*_\xi)$ be a collection of FRFNs such that $\Im^{\ell}{}_\xi \leq \Im^{\ell}{}^*_\xi$ and $\daleth^{\gamma}{}_\xi \leq \daleth^{\gamma}{}^*_\xi$ for all i, then for every $\mathscr{W}_\xi$,

$$GFRFOWA((\eth_1,\eth_2,\ldots,,\eth_n),g) \leq GFRFOWA((\eth_1^*,\eth_2^*,\ldots,\eth_n^*),g)$$

4. (Commutativity) Let $\eth_\xi = (\Im^{\ell}{}_\xi, \gimel^{\gamma}{}_\xi)$ and $\ddot{\eth}_\xi = (\ddot{\Im}^{\ell}{}_\xi, \ddot{\gimel}^{\gamma}{}_\xi)$ be two collection of n FRFNs such that $\ddot{\eth}_\xi$ is any permutation of $\eth_\xi$, then

$$GFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),g) = GFRFOWA((\ddot{\eth}_1,\ddot{\eth}_2,\ldots,\ddot{\eth}_n),g)$$

5. If the senior expert considers $g = (1,0)$ to be his or her preference for the evaluated object, the GFRFOWA operator will be replaced by the FRFOWA operator.

6. If the preference of the senior expert for the evaluated object is $g = (0,1)$, then the GFRFOWA operator will return the value $(0,1)$.

Proof. Here we leave proof.

4.3 GFRFHA Operator

Definition 4.12 Let $g = (\Im^{\ell}{}_g, \gimel^{\gamma}{}_g)$ be the be the GP for the FRFNs $\eth_\xi = (\Im^{\ell}{}_\xi, \gimel^{\gamma}{}_\xi)$, then the GFRFHA operator is described as

$$GFRFHA((\eth_1,\eth_2,\ldots,\eth_n,g)) = g \otimes \text{FRFHA}(\eth_1,\eth_2,\ldots,\eth_n)$$

Theorem 4.13 Let $\eth_\xi = (\Im^{\ell}{}_\xi, \gimel^{\gamma}{}_\xi)(\xi = 1,2,\ldots,n)$ be the collection of FRFNs and $\mathscr{W} = (\mathscr{W}_1,\mathscr{W}_2,\ldots,\mathscr{W}_n)^T$ is the WV of $\eth_\xi$ such that $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n}\mathscr{W}_\xi = 1$. GP is $g = (\Im^{\ell}{}_g, \gimel^{\gamma}{}_g)$ and the standard WV is $\mathfrak{d} = (\mathfrak{d}_1,\mathfrak{d}_2,\ldots,\mathfrak{d}_n)^T$ such that $\mathfrak{d}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n}\mathfrak{d}_\xi = 1$. then the GFRFHA operator is defined as

$$\begin{aligned}
GFRFHA((\eth_1,\eth_2,\ldots,\eth_n),g) =& g \otimes \text{FRFHA}(\eth_1,\eth_2,\ldots,\eth_n) \\
=& \left(\Im^{\ell}{}_g . \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\breve{\Im}^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}}, \right. \\
& \left. \sqrt[3]{(\gimel^{\gamma}{}_g)^3 + (1-(\gimel^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\breve{\gimel}^{\gamma}{}_{\sigma(i)}^{\mathscr{W}_\xi})^3} \right)
\end{aligned}$$

where, $\breve{\eth}_\xi = n\mathfrak{d}_\xi\eth_\xi$, n is the number of FRFNs and $\mathfrak{d}_\xi$ standard WV of $\eth_\xi$ and $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\breve{\eth}_{\sigma(i-1)} \geq \breve{\eth}_{\sigma(i)}$ for any i.

Proof. The proof can be done same as Theorem 3.2.

Example 4.14 Let $g = (0.6,0.3)$ be the GP of four Fermatean fuzzy numbers. $\eth_1 = (0.79,0.36)$, $\eth_2 = (0.37,0.61)$ and $\eth_3 = (0.61,0.72)$ with associated WV $\mathscr{W} = (0.5,0.3,0.2)$, here $q = 3$. Standard WV will be $\mathfrak{d} = (0.6,0.2,0.2)$ first we find $\breve{\eth}_\xi = n\mathfrak{d}_\xi\eth_\xi$ for each $\eth_\xi$, then we find score functions of each $\breve{\eth}_\xi$.

$$\breve{\eth}_1 = (0.890261, 0.158981)$$
$$\breve{\eth}_1 = (0.313146, 0.743358)$$
$$\breve{\eth}_1 = (0.523093, 0.821107)$$

The score function will be,

$$\mathfrak{E}\left(\breve{\eth}_1\right) = 0.701571$$
$$\mathfrak{E}\left(\breve{\eth}_2\right) = -0.380058$$
$$\mathfrak{E}\left(\breve{\eth}_3\right) = -0.410471$$

On the behalf of score functions, $\breve{\eth}_{\sigma(1)} = \breve{\eth}_1, \breve{\eth}_{\sigma(2)} = \breve{\eth}_3$ and $\eth_{\sigma(3)} = \breve{\eth}_2$

$$\Im^{\ell}{}_g . \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\breve{\Im}^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}} = 0.421921$$

Also

$$\sqrt[3]{(\gimel^{\gamma}{}_g)^3 + (1-(\gimel^{\gamma}{}_g)^3)\prod_{\xi=1}^{n}(\breve{\gimel}^{\gamma}{}_{\sigma(i)}^{\mathscr{W}_\xi})^3} = 0.523227$$

By Theorem 3.13, we have

$$GFRFOWA((\eth_1,\eth_2,\eth_3),g)) = \left(\Im^{\ell}{}_{g}.\sqrt[3]{1-\prod_{\xi=1}^{k}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}}, \sqrt[3]{(\beth^{\gamma}{}_{g})^3+(1-(\beth^{\gamma}{}_{g})^3)\prod_{\xi=1}^{k}(\beth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\right)$$
$$= (0.421921, 0.523227)$$

5 Fermatean Fuzzy Average AOs Based on Group-Generalized Parameter

This entire section is devoted to enlarging collaborators beyond AOs by incorporating the perspectives of multiple observers/experts on the original data in order to better incorporate the diverse preferences of decision-makers. This can be accomplished by providing GGFRFWA, GGFRFOWA, and GGFRFHA operators.

5.1 GGFRFWA Operator

Definition 5.1 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \beth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_{\xi}, \beth^{\gamma}{}_{\xi})$, then the GGFRFWA operator is described as

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = \text{FRFWA}(g_1,g_2,\ldots,g_q)\otimes\text{FRFWA}(\eth_1,\eth_2,\ldots,\eth_n)$$

Theorem 5.2 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \beth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_{\xi}, \beth^{\gamma}{}_{\xi})$. $\mathscr{W}' = (\mathscr{W}'_1, \mathscr{W}'_2, \ldots,, \mathscr{W}'_q)^T$ and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ are the WVs of experts/observers and $\eth_\xi$ respectively. $\mathscr{W}'_\xi \in [0,1]$, $\sum_{\xi=1}^{3}\mathscr{W}'_\xi = 1$, $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n}\mathscr{W}_\xi = 1$, then the GGFRFWA operator is described as

$$\begin{aligned}
GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) &= \text{FRFWA}(g_1,g_2,\ldots,g_q)\otimes\text{FRFWA}(\eth_1,\eth_2,\ldots,\eth_n)\\
&= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},\\
&\sqrt[3]{\prod_{z=1}^{3}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{n}(\beth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{n}(\beth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\Bigg)
\end{aligned}$$

Proof. We use mathematical induction to prove this theorem
For $n = 2$

$$\begin{aligned}
GGFRFWA((\eth_1,\eth_2),(g_1,g_2,\ldots,g_q)) &= \text{FRFWA}(g_1,g_2,\ldots,g_q)\otimes\text{FRFWA}(\eth_1,\eth_2)\\
&= \left(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}, \prod_{k=1}^{n}\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z}\right)\otimes(\mathscr{W}_1\eth_1\oplus\mathscr{W}_2\eth_2)\\
&= \left(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}, \prod_{k=1}^{n}\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z}\right)\\
&\otimes\left(\sqrt[3]{1-(1-\Im^{\ell}{}_{1}^{3})^{\mathscr{W}_1}.(1-\Im^{\ell}{}_{2}^{3})^{\mathscr{W}_2}}, \beth^{\gamma}{}_{1}^{\mathscr{W}_1}.\beth^{\gamma}{}_{2}^{\mathscr{W}_2}\right)\\
&= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-(1-\Im^{\ell}{}_{1}^{3})^{\mathscr{W}_1}.(1-\Im^{\ell}{}_{2}^{3})^{\mathscr{W}_2}},\\
&\sqrt[3]{\prod_{k=1}^{n}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+(\beth^{\gamma}{}_{1}^{\mathscr{W}_1}.\beth^{\gamma}{}_{2}^{\mathscr{W}_2})^3-\prod_{k=1}^{n}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.(\beth^{\gamma}{}_{1}^{\mathscr{W}_1}.\beth^{\gamma}{}_{2}^{\mathscr{W}_2})^3}\Bigg)\\
&= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{2}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},\\
&\sqrt[3]{\prod_{z=1}^{3}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{2}(\beth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\beth^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{2}(\beth^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\Bigg)
\end{aligned}$$

We proved for $n = 2$.

Assume that result for $n = k$ is true , we have

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_k),(g_1,g_2,\ldots,g_q)) = \mathrm{FRFWA}(g_1,g_2,\ldots,g_q)\otimes \mathrm{FRFWA}(\eth_1,\eth_2,\ldots,\eth_k)$$

$$= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{k}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{k}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{k}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\Bigg)$$

Now we will prove for $n = k+1$,

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_{k+1}),(g_1,g_2,\ldots,g_q)) = \mathrm{FRFWA}(g_1,g_2,\ldots,g_q)\otimes \mathrm{FRFWA}(\eth_1,\eth_2,\ldots,\eth_{k+1})$$

$$= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{k+1}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{k+1}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{k+1}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\Bigg)$$

Result holds for $n = k+1$. In this way we completed proof.

Example 5.3 To validate the FRF details, let $g_z = \{g_1,g_2,g_3\}$ be the set of experts / observers with WV $\mathscr{W}' = \{0.5,0.3,0.2\}$, where $g_1 = (0.8,0.2)$, $g_2 = (0.6,0.8)$, and $g_3 = (0.9,0.5)$. Here we have four FRFNs. $\eth_1 = (0.89,0.56)$, $\eth_2 = (0.43,0.67)$, $\eth_3 = (0.78,0.44)$ and $\eth_4 = (0.98,0.32)$ with WV $\mathscr{W} = (0.4,0.2,0.3,0.1)$, here $q = 3$, then

$$\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}} = 0.67068$$

Also

$$\sqrt[3]{\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{n}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{n}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3} = 0.559272$$

By Theorem 4.2, we have

$$GGFRFWA((\eth_1,\eth_2,\eth_3,\eth_4),(g_1,g_2,g_3)) = \mathrm{FRFWA}(g_1,g_2,g_3)\otimes \mathrm{FRFWA}(\eth_1,\eth_2,\eth_3,\eth_4)$$

$$= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}}.\sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\xi})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3+\prod_{\xi=1}^{n}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3-\prod_{z=1}^{3}(\gimel^{\gamma}{}_{g_z}^{\mathscr{W}'_z})^3.\prod_{\xi=1}^{n}(\gimel^{\gamma}{}_{\xi}^{\mathscr{W}_\xi})^3}\Bigg)$$

$$= (0.67068, 0.559272)$$

Proposition 5.4 Let $\eth_\xi = (\Im^{\ell}{}_{\xi},\gimel^{\gamma}{}_{\xi})$ be the collection of FRFNs, there are q experts/observers to evaluate the FRF information. If $g_z = (\Im^{\ell}{}_{g_z},\gimel^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi$, the following properties are available in the GGFRFWA operator:

1. (Idempotency) if $\eth_\xi = \eth$ and $g_z = g$, for all i and z then

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = g\otimes\eth$$

2. (Monotonicity) Let $\eth^*_\xi = (\Im^{\ell}{}^*_{\xi},\gimel^{\gamma}{}^*_{\xi})$ be a collection of FRFNs such that $\Im^{\ell}{}_{\xi} \le \Im^{\ell}{}^*_{\xi}$ and $\gimel^{\gamma}{}_{\xi} \le \gimel^{\gamma}{}^*_{\xi}$ for all i, then

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) \le GGFRFWA((\eth^*_1,\eth^*_2,\ldots,\eth^*_n),(g_1,g_2,\ldots,g_q))$$

3. (Commutativity) Let $\eth_\xi = (\Im^{\ell}{}_{\xi},\gimel^{\gamma}{}_{\xi})$ and $\ddot{\eth}_\xi = (\ddot{\Im}^{\ell}{}_{\xi},\ddot{\gimel}^{\gamma}{}_{\xi})$ be two collection of n FRFNs such that $\ddot{\eth}_\xi$ is any permutation of $\eth_\xi$, then

$$GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = GGFRFWA((\ddot{\eth}_1,\ddot{\eth}_2,\ldots,\ddot{\eth}_n),(g_1,g_2,\ldots,g_q))$$

4. If the senior expert's preference for the evaluated object is viewed to be $g=(1,0)$ for all z, then the GGFRFWA operator will be reduced in FRFWA operator.

5. If the senior expert's preference for the evaluated object is viewed to be $g=(0,1)$ for all z, then the GGFRFWA operator will give the value $(0,1)$.

*P*roof. Here we leave proof.

5.2 GGFRFOWA Operator

Definition 5.5 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \beth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_{\xi}, \beth^{\gamma}{}_{\xi})$, then the GGFRFOWA operator is described as

$$GGFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = \mathrm{FRFWA}(g_1,g_2,\ldots,g_q) \otimes \mathrm{FRFOWA}(\eth_1,\eth_2,\ldots,\eth_n)$$

Theorem 5.6 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \beth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_{\xi}, \beth^{\gamma}{}_{\xi})$. $\mathscr{W}' = (\mathscr{W}'_1, \mathscr{W}'_2, \ldots, \mathscr{W}'_q)^T$ and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ are the WVs of experts/observers and $\eth_\xi$ respectively. $\mathscr{W}'_\xi \in [0,1]$, $\sum_{\xi=1}^{3} \mathscr{W}'_\xi = 1$, $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n} \mathscr{W}_\xi = 1$, then the GGFRFOWA operator is described as

$$\begin{aligned} GGFRFWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) &= \mathrm{FRFWA}(g_1,g_2,\ldots,g_q) \otimes \mathrm{FRFOWA}(\eth_1,\eth_2,\ldots,\eth_n) \\ &= \Bigg(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} . \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}}, \\ &\quad \sqrt[3]{\prod_{z=1}^{3}(\beth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\beth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 . \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} \Bigg) \end{aligned}$$

$(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\eth_{\sigma(i-1)} \geq \eth_{\sigma(i)}$ for any i.

Proof. The proof is same as Theorem 4.2.

Example 5.7 To validate the FRF details, let $g_z = \{g_1, g_2, g_3\}$ be the set of experts / observers with WV $\mathscr{W}' = \{0.5, 0.3, 0.2\}$, where $g_1 = (0.8, 0.2)$, $g_2 = (0.6, 0.8)$, and $g_3 = (0.9, 0.5)$. Here we have four FRFNs. $\eth_1 = (0.89, 0.56)$, $\eth_2 = (0.43, 0.67)$, $\eth_3 = (0.78, 0.44)$ and $\eth_4 = (0.98, 0.32)$ with associated WV $\mathscr{W} = (0.4, 0.2, 0.3, 0.1)$. Here $q=3$, first we find score functions of all $\eth_\xi$.

$$\mathfrak{E}(\eth_1) = 0.529353$$

$$\mathfrak{E}(\eth_2) = -0.221256$$

$$\mathfrak{E}(\eth_3) = 0.389368$$

$$\mathfrak{E}(\eth_4) = 0.908424$$

On the behalf of score functions, $\eth_{\sigma(1)} = \eth_4, \eth_{\sigma(2)} = \eth_1, \eth_{\sigma(3)} = \eth_3$ and $\eth_{\sigma(4)} = \eth_2$ then

$$\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} . \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}} = 0.738374$$

Also,

$$\sqrt[3]{\prod_{z=1}^{3}(\beth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\beth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 . \prod_{\xi=1}^{n}(\beth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} = 0.494345$$

By Theorem 4.6, we have

$$GGFRFOWA((\eth_1,\eth_2,\eth_3,\eth_4),(g_1,g_2,g_3)) = \mathrm{FRFWA}(g_1,g_2,g_3) \otimes \mathrm{FRFOWA}(\eth_1,\eth_2,\eth_3,\eth_4)$$

$$= \Big(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} \cdot \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{\prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 \cdot \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} \Big)$$

$$= (0.738374, 0.494345)$$

Proposition 5.8 Let $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$ be the collection of FRFNs, there are q experts/observers to certify the FRF information. If $g_z = (\Im^{\ell}{}_{g_z}, \daleth^{\gamma}{}_{g_z})$ $(\xi = 1,2,\ldots,q)$ be the experts/observers for the FRFNs $\eth_\xi$, the following properties are available in the GGFRFOWA operator:

1. (Idempotency) if $\eth_\xi = \eth$ and $g_z = g$, for all i and z then

$$GGFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = g \otimes \eth$$

2. (Monotonicity) Let $\eth^*_\xi = (\Im^{\ell}{}^*_\xi, \daleth^{\gamma}{}^*_\xi)$ be the collection of FRFNs such that $\Im^{\ell}{}_\xi \le \Im^{\ell}{}^*_\xi$ and $\daleth^{\gamma}{}_\xi \le \daleth^{\gamma}{}^*_\xi$ for all i, then

$$GGFRFOWA((\eth_1,\eth_2,\ldots,,\eth_n),(g_1,g_2,\ldots,g_q)) \le GGFRFOWA((\eth^*_1,\eth^*_2,\ldots,\eth^*_n),(g_1,g_2,\ldots,g_q))$$

3. (Commutativity) Let $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$ and $\ddot{\eth}_\xi = (\ddot{\Im}^{\ell}{}_\xi, \ddot{\daleth}^{\gamma}{}_\xi)$ be two collection of n FRFNs such that $\ddot{\eth}_\xi$ is any permutation of $\eth_\xi$, then

$$GGFRFOWA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = GGFRFOWA((\ddot{\eth}_1,\ddot{\eth}_2,\ldots,\ddot{\eth}_n),(g_1,g_2,\ldots,g_q))$$

4. If the senior expert's preference for the evaluated object is viewed to be $g = (1,0)$ for all z, then the GGFRFOWA operator will be reduced in FRFOWA operator.

5. If the senior expert's preference for the evaluated object is viewed to be $g = (0,1)$ for all z, then the GGFRFOWA operator will give the value $(0,1)$.

Proof. Here we leave proof.

5.3 GGFRFHA Operator

Definition 5.9 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \daleth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$, then the GGFRFHA operator is described as

$$GGFRFHA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = \mathrm{FRFWA}(g_1,g_2,\ldots,g_q) \otimes \mathrm{FRFHA}(\eth_1,\eth_2,\ldots,\eth_n)$$

Theorem 5.10 To validate the FRF details, let q be the number of experts / observers. If $g_z = (\Im^{\ell}{}_{g_z}, \daleth^{\gamma}{}_{g_z})$ be the experts/observers for the FRFNs $\eth_\xi = (\Im^{\ell}{}_\xi, \daleth^{\gamma}{}_\xi)$. $\mathscr{W}' = (\mathscr{W}'_1, \mathscr{W}'_2, \ldots, \mathscr{W}'_q)^T$ and $\mathscr{W} = (\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n)^T$ are the WVs of experts/observers and $\eth_\xi$ respectively. $\mathscr{W}'_\xi \in [0,1]$, $\sum_{\xi=1}^{3} \mathscr{W}'_\xi = 1$, $\mathscr{W}_\xi \in [0,1]$ and $\sum_{\xi=1}^{n} \mathscr{W}_\xi = 1$, then the GGFRFHA operator is described as

$$GGFRFHA((\eth_1,\eth_2,\ldots,\eth_n),(g_1,g_2,\ldots,g_q)) = \mathrm{FRFWA}(g_1,g_2,\ldots,g_q) \otimes \mathrm{FRFOHA}(\eth_1,\eth_2,\ldots,\eth_n)$$

$$= \Big(\sqrt[3]{1-\prod_{z=1}^{3}(1-(\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} \cdot \sqrt[3]{1-\prod_{\xi=1}^{n}(1-(\breve{\Im}^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}},$$

$$\sqrt[3]{\prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\breve{\daleth}^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 \cdot \prod_{\xi=1}^{n}(\breve{\daleth}^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} \Big)$$

where, $\breve{\eth}_\xi = n\mathfrak{d}_\xi\eth_\xi$, n is the number of FRFNs and $\mathfrak{d}_\xi$ standard WV of $\eth_\xi$ and $(\sigma(1),\sigma(2),\ldots,\sigma(n))$ is a permutation of $(1,2,\ldots,n)$, such that $\breve{\eth}_{\sigma(i-1)} \ge \breve{\eth}_{\sigma(i)}$ for any i.

*P*roof. Proof is same as Theorem 4.2.

Example 5.11 To validate the FRF details, let $g_z = \{g_1,g_2,g_3\}$ be the set of experts / observers with WV $\mathscr{W}' = \{0.5,0.3,0.2\}$, where $g_1 = (0.8,0.2)$, $g_2 = (0.6,0.8)$, and $g_3 = (0.9,0.5)$. Here we have four FRFNs. $\eth_1 = (0.79,0.46)$,

$\eth_2 = (0.66, 0.89)$ and $\eth_4 = (0.56, 0.72)$ with associated WV $\mathscr{W} = (0.4, 0.2, 0.4)$. Here $q = 3$, standard WV will be $\mathfrak{d}_\xi = (0.5, 0.2, 0.3)$ first we find $\breve{\eth}_\xi = n\mathfrak{d}_\xi \eth_\xi$ for each $\eth_\xi$, then we find score functions of each $\breve{\eth}_\xi$.

$$\breve{\eth}_1 = (0.861342, 0.311987)$$
$$\breve{\eth}_2 = (0.568808, 0.932468)$$
$$\breve{\eth}_3 = (0.542364, 0.744045)$$

The score function will be,

$$\mathfrak{E}(\breve{\eth}_1) = 0.60867 \qquad \mathfrak{E}(\breve{\eth}_2) = -0.626745 \qquad \mathfrak{E}(\breve{\eth}_3) = -0.252365$$

On the behalf of score functions, $\breve{\eth}_{\sigma(1)} = \breve{\eth}_1, \breve{\eth}_{\sigma(2)} = \breve{\eth}_3$ and $\eth_{\sigma(3)} = \breve{\eth}_2$

$$\sqrt[3]{1 - \prod_{z=1}^{3}(1 - (\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} . \sqrt[3]{1 - \prod_{\xi=1}^{n}(1 - (\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}} = 0.56445$$

Also,

$$\sqrt[3]{\prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 . \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} = 0.681039$$

By Theorem 4.10, we have

$$\begin{aligned} GGFRFHA((\eth_1, \eth_2, \eth_3), (g_1, g_2, g_3)) &= \text{FRFWA}(g_1, g_2, g_3) \otimes \text{FRFHA}(\eth_1, \eth_2, \eth_3) \\ &= \Bigg(\sqrt[3]{1 - \prod_{z=1}^{3}(1 - (\Im^{\ell}{}_{g_z})^3)^{\mathscr{W}'_z}} . \sqrt[3]{1 - \prod_{\xi=1}^{n}(1 - (\Im^{\ell}{}_{\sigma(i)})^3)^{\mathscr{W}_\xi}}, \\ &\quad \sqrt[3]{\prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 + \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3 - \prod_{z=1}^{3}(\daleth^{\gamma}{}^{\mathscr{W}'_z}_{g_z})^3 . \prod_{\xi=1}^{n}(\daleth^{\gamma}{}^{\mathscr{W}_\xi}_{\sigma(i)})^3} \Bigg) \\ &= (0.56445, 0.681039) \end{aligned}$$

6 MCDM Approach based on proposed AOs

Let us consider $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \ldots, \mathfrak{B}_m\}$ be the set of alternatives, $\breve{\mathfrak{L}} = \{\breve{\mathfrak{L}}_1, \breve{\mathfrak{L}}_2, \ldots, \breve{\mathfrak{L}}_n\}$ be the set of criteria and $\mathscr{W} = \{\mathscr{W}_1, \mathscr{W}_2, \ldots, \mathscr{W}_n\}$ is the WV, such that $\mathscr{W}_j \in [0,1]$ and $\sum_{j=1}^{n} \mathscr{W}_j = 1$. DMs are evaluating alternatives to criteria and evaluation values must be in FRFNs. Assume that $(\mathfrak{B}_{ij})_{m\times n} = (\Im^{\ell}{}_{ij}, \daleth^{\gamma}{}_{ij})_{m\times n}$ is the decision matrix provided by DM. Where $\Im^{\ell}{}_{ij}$ and $\daleth^{\gamma}{}_{ij}$ indicates the degree of satisfied and unsatisfied respectively corresponding the alternative $\mathfrak{B}_\xi$ to criteria $\breve{\mathfrak{L}}_j$. Let us consider a group of other observers / experts to make the circumstances more credible, $\mathfrak{G} = \{\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_n\}$ with WV $\mathscr{W}' = \{\mathscr{W}'_1, \mathscr{W}'_2, \ldots, \mathscr{W}'_n\}$. These experts give their assessment in the form of FRFNs to the preference of each alternative denoted by $\mathfrak{g}_{\mathfrak{k}} = (\Im^{\ell}{}_{g_k}, \daleth^{\gamma}{}_{g_k})$. Following are the steps of algorithm to solve the MCDM problem.

Algorithm

INPUT

Step 1:

Take the opinion of the expert on each alternative in the form of FRFNs against the various criteria and then obtain a decision matrix $[\mathfrak{B}]_{m\times n} = (\Im^{\ell}{}_{ij}, \daleth^{\gamma}{}_{ij})_{m\times n}$. If normalization of decision matrix is required then normalize the decision matrix. If there are different types of criteria or attributes like cost and benefit, then we normalize the decision matrix by using compliment of criteria like cost. By normalize the decision matrix we deal all criteria or attributes in the same way. Otherwise, different criterion or attributes should be aggregate in different ways.

OUTPUT

Step 2:

Collect the group of other experts / observers choice on each alternative based on the GP principle and then obtain a generalized parameter matrix $[\mathfrak{W}]_{m\times l} = (\Im^{\ell}{}_{ij}, \daleth^{\gamma}{}_{ij})_{m\times l}$.

Step 3:

Combine the matrices acquired in steps 1 and 2 to form a new $[\mathfrak{L}]_{m\times(n+k)}$ matrix structure, which reflects the expert's evaluation of each alternative against the criteria under GPs.

Step 4:

By using GGFRFWA operator, We aggregate the efficiency of each alternative of the $\mathfrak{L}$ row-wise matrix in order to achieve overall performance and $\mathscr{O}_\xi$ is indicated. Here, we also use GGFRFOWA operator and GGFRFHA operator.

Step 5:

Evaluate the score functions for all $\mathscr{O}_\xi$ for the collective overall FRFNs.

Step 6:

Rank all the $\mathscr{O}_\xi$ according to the score values.

7 Case Study

The area of GSCM relies heavily on decision-making, since its major goal is to include environmentally friendly practices across the supply chain process, while simultaneously ensuring operational efficiency and satisfying consumer needs. Decisions made throughout the entirety of the supply chain, encompassing procurement, transportation, manufacturing, distribution, and waste management, possess significant ramifications for sustainability. Within the field of GSCM, the initial pivotal decision frequently centers on the process of selecting and evaluating suppliers [26, 27]. In order to make informed decisions, decision-makers are required to evaluate potential suppliers not just on the basis of cost and quality, but also with regard to their environmental practices and sustainability initiatives. The first selection establishes the basis for an enduring supply chain. In addition, sustainable procurement decisions are a significant factor, as firms make choices to utilize environmentally conscious materials and suppliers, resulting in a reduction of the ecological impact associated with products. During the process of product design, many determinations are undertaken pertaining to the selection of materials, implementation of energy-efficient production techniques, and adoption of packaging strategies aimed at reducing waste and emissions [28]. The decisions pertaining to route optimization and mode selection in the field of transportation and logistics have a direct impact on fuel consumption and emissions. These decisions involve finding the most efficient routes and selecting the appropriate modes of transportation, while also considering the importance of timely delivery and taking into account environmental factors. Effective inventory management decisions play a crucial role in mitigating waste and minimizing energy consumption. By maintaining optimal inventory levels, organizations may effectively curtail overproduction and avoid superfluous resource utilization [29, 30].

Waste management decisions play a critical role in the field of GSCM, with a specific emphasis on practices such as recycling, reusing, and responsible disposal. These decisions aim to reduce environmental impact and advance the concepts of a circular economy. In order to maintain adherence to environmental standards and laws and mitigate any legal complications, supply chain experts are required to make well-informed judgments about regulatory compliance. Furthermore, stakeholder engagement decisions are a crucial aspect of GSCM, as firms are required to effectively express their dedication to sustainability to various stakeholders such as consumers, investors, and advocacy groups. This facilitates the establishment of favorable connections and serves as evidence of their environmental accountability. In the realm of GSCM, the significance of continuous improvement decisions cannot be overstated, since the pursuit of sustainability is an ever-evolving objective. Regular evaluations of the ecological ramifications of supply chain activities, along with subsequent strategic choices aimed at enhancing performance, are crucial for ensuring enduring sustainability [31]. Risk management choices play a significant role in organizational operations, since they include the anticipation and mitigation of environmental hazards. These risks may arise from climatic events or regulatory changes, leading to disruptions in the supply chain. To address these challenges, businesses must adopt proactive methods and develop contingency plans [32].

The process of decision-making in GSCM is complex and involves several dimensions. The process encompasses the assessment of suppliers, the identification and utilization of sustainable materials, the implementation of environmentally-conscious design decisions, the optimization of transportation routes, the prudent management of inventory, the responsible handling of waste, the adherence to regulatory requirements, the active engagement of stakeholders, the ongoing enhancement of sustainability initiatives, and the mitigation of environmental hazards. The actions made by an organization have a significant influence on its environmental footprint, reputation, and capacity to succeed in a society that is increasingly prioritizing sustainability [33].

An Electronics, a prominent multinational corporation in the electronics manufacturing industry, is committed to mitigating its environmental impact and promoting sustainability. The organization is currently confronted with a pivotal choice concerning the designation of a fresh provider for electronic components, a decision that is in accordance with their dedication to GSCM. This case study delves into the examination of the decision-making process involving four alternative suppliers and the full evaluation of these providers by Electronics using four criteria.

Electronics has identified four potential suppliers for electronic components, each with its unique characteristics:

$\mathfrak{B}_1$: Supplier A is a regional supplier that has established a commendable track record in terms of delivering high-quality products and maintaining a consistent level of dependability. Nevertheless, their efforts to embrace sustainability principles have not yielded significant advancements. The organization is renowned for its close closeness and prompt responsiveness to fluctuations in demand.

$\mathfrak{B}_2$: Supplier B is a renowned worldwide supplier renowned for its steadfast dedication to sustainability. The company has successfully integrated environmentally sustainable production practices, obtained relevant environmental certifications, and actively participates in community engagement activities. Nevertheless, the cost offered by the company is somewhat greater when compared to its competitors.

$\mathfrak{B}_3$: Supplier C provides electronic components at a notably reduced price in comparison to the other available options. Although Supplier A's sustainability measures are somewhat modest in comparison to those of Supplier B, their cost is rather attractive.

$\mathfrak{B}_4$: Supplier D is widely recognized for its pioneering and advanced technology, which has the potential to facilitate the creation of components that are more efficient in terms of energy consumption. The organization has made notable progress in the pursuit of sustainability; nonetheless, it lags below Supplier B in terms of advancement in this area. The items they sell may provide advantages in terms of long-term sustainability.

Electronics has established a set of four criteria to assess the various supplier choices, assigning precise weights to each criterion to indicate its relative significance.

$\check{\mathfrak{L}}_1$ Environmental Impact: This criterion evaluates the extent to which the provider demonstrates a dedication to mitigating environmental harm by using sustainable practices. The evaluation encompasses several aspects, including but not limited to energy efficiency, waste reduction, emissions management, and compliance with environmental requirements.

$\check{\mathfrak{L}}_2$ Quality and Reliability: Ensuring quality and dependability is of utmost importance in order to uphold production standards and mitigate interruptions within the supply chain. This criterion assesses the supplier's historical performance in consistently delivering items that are free from defects and meeting the agreed-upon delivery schedule.

$\check{\mathfrak{L}}_3$ Cost: The importance of cost-effectiveness cannot be overstated in terms of expenditure management and maintaining competitiveness within the industry. This criterion evaluates the supplier's price competitiveness.

$\check{\mathfrak{L}}_4$ Innovation: The supplier's capacity to offer cutting-edge technology that coincides with Electronics' dedication to energy efficiency and environmentally friendly goods is exemplified through innovation. This phenomenon covers the progress made in technology and the potential for future advantages in terms of sustainability.

7.1 Numerical Illustration

Here we consider set of alternatives given as above $\mathfrak{B} = \{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3, \mathfrak{B}_4\}$, with WV $\mathscr{W} = (0.40, 0.20, 0.10, 0.30)$ after discussion with experts we select four attributes, $\check{\mathfrak{L}}_1$= environmental impact, $\check{\mathfrak{L}}_2$ = quality and reliability, $\check{\mathfrak{L}}_3$ = cost and $\check{\mathfrak{L}}_4$ = innovation. For verification of decision matrix we have other group of senior experts $\mathfrak{G} = \{\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3\}$ with WV $\mathscr{W}' = \{0.40, 0.30, 0.30\}$.

Step 1:

Evaluate the decision matrix, $[\mathfrak{B}]_{m\times n} = (\Im^{\ell}{}_{ij}, \beth^{\gamma}{}_{ij})_{m\times n}$, given in Table 1.

Table 1. FRF matrix of expert assessment $[\mathfrak{B}]_{4\times 4}$

	$\check{\mathfrak{L}}_1$	$\check{\mathfrak{L}}_2$	$\check{\mathfrak{L}}_3$	$\check{\mathfrak{L}}_4$
$\mathfrak{B}_1$	(0.78, 0.32)	(0.68, 0.21)	(0.97, 0.25)	(0.81, 0.10)
$\mathfrak{B}_2$	(0.46, 0.49)	(0.32, 0.79)	(0.62, 0.77)	(0.38, 0.49)
$\mathfrak{B}_3$	(0.52, 0.17)	(0.46, 0.56)	(0.78, 0.60)	(0.39, 0.89)
$\mathfrak{B}_4$	(0.22, 0.77)	(0.41, 0.20)	(0.46, 0.70)	(0.20, 0.61)

Step 2:

The group of other senior experts / observers' preference for each alternative and the corresponding generalized parameter matrix $[\mathscr{B}]_{m\times l} = (\mathfrak{W}_{ij}, \beth^{\gamma}{}_{ij})_{m\times l}$, given in Table 2.

Table 2. FRF matrix of GP preference $[\mathfrak{W}]_{4\times 3}$

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$\mathfrak{B}_1$	(0.83, 0.16)	(0.79, 0.23)	(0.69, 0.13)
$\mathfrak{B}_2$	(0.32, 0.46)	(0.46, 0.71)	(0.91, 0.17)
$\mathfrak{B}_3$	(0.56, 0.17)	(0.46, 0.82)	(0.748, 0.63)
$\mathfrak{B}_4$	(0.25, 0.71)	(0.36, 0.25)	(0.49, 0.38)

Step 3:
Obtain the matrix $[\mathfrak{L}]_{m\times(n+k)}$, given in Table 3.

Table 3. FRF matrix of GP preference $[\mathfrak{W}]_{4\times 3}$

	$\breve{\mathfrak{L}}_1$	$\breve{\mathfrak{L}}_2$	$\breve{\mathfrak{L}}_3$	$\breve{\mathfrak{L}}_4$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$\mathfrak{B}_1$	(0.78, 0.32)	(0.68, 0.21)	(0.97, 0.25)	(0.81, 0.10)	(0.83, 0.16)	(0.79, 0.23)	(0.69, 0.13)
$\mathfrak{B}_2$	(0.46, 0.49)	(0.32, 0.79)	(0.62, 0.77)	(0.38, 0.49)	(0.32, 0.46)	(0.46, 0.71)	(0.91, 0.17)
$\mathfrak{B}_3$	(0.52, 0.17)	(0.46, 0.56)	(0.78, 0.60)	(0.39, 0.89)	(0.56, 0.17)	(0.46, 0.82)	(0.748, 0.63)
$\mathfrak{B}_4$	(0.22, 0.77)	(0.41, 0.20)	(0.46, 0.70)	(0.20, 0.61)	(0.25, 0.71)	(0.36, 0.25)	(0.49, 0.38)

Step 4:
Calculate $\mathscr{O}_\xi$ for the collective overall Fermatean fuzzy numbers using GqROPWA operator.

$$\mathscr{O}_1 = (0.644469, 0.234939)$$

$$\mathscr{O}_2 = (0.319351, 0.610586)$$

$$\mathscr{O}_3 = (0.272107, 0.502131)$$

$$\mathscr{O}_4 = (0.120209, 0.610157)$$

Step 5:
Calculate the score functions for all $\mathscr{O}_\xi$ for the collective overall Fermatean fuzzy numbers.

$$\mathfrak{E}(\mathscr{O}_1) = 0.254706$$

$$\mathfrak{E}(\mathscr{O}_2) = -0.195067$$

$$\mathfrak{E}(\mathscr{O}_3) = -0.106458$$

$$\mathfrak{E}(\mathscr{O}_4) = -0.225419$$

Step 6:
The preference order of the alternatives, therefore, is $\mathfrak{t}_1 \succ \mathfrak{t}_3 \succ \mathfrak{t}_2 \succ \mathfrak{t}_4$. So, $\mathfrak{t}_1$ select as a best alternative.

7.2 Comparative Analysis

$\mathfrak{t}_1$ which is alternative to the above analysis that it provides the best in some mentioned criteria. If the analysis done on the recommendation of one expert/observer of times, in terms of reliability of the information provided, then we have the following conclusions:

1. If only $\mathfrak{g}_1$ is to be considered, then by the above analysis score functions are $\mathfrak{E}(\mathscr{O}_1) = 0.30357$, $\mathfrak{E}(\mathscr{O}_2) = -0.256442$, $\mathfrak{E}(\mathscr{O}_3) = -0.0431266$ and $\mathfrak{E}(\mathscr{O}_4) = -0.46031$. Thus, $\mathfrak{t}_1 \succ \mathfrak{t}_3 \succ \mathfrak{t}_2 \succ \mathfrak{t}_4$.

2. If only $\mathfrak{g}_2$ is to be considered, then by the above analysis score functions are $\mathfrak{E}(\mathscr{O}_1) = 0.252055$, $\mathfrak{E}(\mathscr{O}_2) = -0.464596$, $\mathfrak{E}(\mathscr{O}_3) = -0.565853$ and $\mathfrak{E}(\mathscr{O}_4) = -0.171914$. Thus, $\mathfrak{t}_1 \succ \mathfrak{t}_4 \succ \mathfrak{t}_2 \succ \mathfrak{t}_3$.

3. If only $\mathfrak{g}_3$ is to be considered, then by the above analysis score functions are $\mathfrak{E}(\mathscr{O}_1) = 0.171035$, $\mathfrak{E}(\mathscr{O}_2) = -0.117394$, $\mathfrak{E}(\mathscr{O}_3) = -0.282135$ and $\mathfrak{E}(\mathscr{O}_4) = -0.202691$. Thus, $\mathfrak{t}_1 \succ \mathfrak{t}_2 \succ \mathfrak{t}_4 \succ \mathfrak{t}_3$.

The ranking acquired by taking into account only one expert / observer at a time in the truthfulness of the information provided is different, but the best possible alternative remains the same, indicating and verifying that each expert has its own priorities and parametric principles due to its own perception, understanding, views, and many more parameters.

8 Conclusion

The topic of MCDM has garnered significant attention from a multitude of scholars, who have conducted extensive study in this area. The methodologies employed for this undertaking primarily rely on the specific nature of the decision problem being examined. The majority of real-world scenarios exhibit characteristics of uncertainty, imperfection, imprecision, and vagueness. Numerous methodologies have been suggested for the aggregation of FRFNs. The current FRF aggregation operators were formulated based on the underlying premise that decision makers possess a comprehensive understanding of the available choices. However, such a scenario is not typically seen in real-world settings, as the assessments made by decision-makers regarding the available options are subjective and influenced by their own perceptions. Hence, the development of novel procedures becomes imperative. In order to tackle this matter, the concept of generalized FRFS is introduced, which involves integrating the notion of a generalized parameter from an external expert or observer within a FRF setting. This approach provides a framework for evaluating the reliability of the information in the original FRFS, with the aim of eliminating any distortion in expert preference. The inclusion of a generalized parameter in the analysis offers a significant advantage in mitigating potential errors resulting from imprecise information. This is achieved by including external expert opinions during the initial evaluations. The practicality and viability of the proposed methodology are demonstrated through the use of a numerical illustration.

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