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A Mathematical Modeling Framework for Analyzing and Optimizing Education Systems

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Abstract: The role of cognitive and affective dimensions in mathematical modeling education was investigated, with a particular emphasis on the influence of mathematical values on student engagement and learning outcomes. A mixed-methods approach was employed, incorporating surveys and interviews with students from diverse educational contexts. This approach enabled the quantitative assessment of cognitive competencies alongside qualitative exploration of attitudes and values toward mathematics. It was found that values associated with realism and cognitive engagement significantly shape students' approaches to mathematical modeling, with cultural differences playing a critical role in the expression and prioritization of these values. The study highlights the necessity of cross-cultural research to elucidate how these values are cultivated across varying educational settings. Furthermore, it is argued that mathematical modeling instruction must be culturally responsive and attuned to the affective dimensions of learning to promote deeper engagement and enhance educational outcomes. The findings underscore the importance of integrating both cognitive and affective factors in the design of educational frameworks, offering insights for optimizing teaching strategies and fostering more effective learning environments.

Keywords: Deterministic EPRTD mathematical model; Results for validation of the model; Stability analysis; Sensitivity analysis; Optimization bifurcation; Numerical scheme and analysis

1. Introduction

Over the past few decades, numerous studies have focused on mathematical modeling as a powerful pedagogical tool in mathematics education (Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2015). These studies primarily emphasize the cognitive dimensions of students, particularly examining their mathematical modeling competencies. However, while cognitive factors are crucial to students' success, recent research has highlighted the importance of affective dimensions, such as attitudes, beliefs, and values, in shaping their approach to mathematics and science (Bishop et al., 2008; Zakaria & Maat, 2012). These affective factors can significantly influence students' engagement with mathematics, impacting their willingness to engage with mathematical problems and their persistence in solving them (Bishop et al., 2006).

Despite the well-documented impact of affective dimensions on learning, educational research has often neglected this aspect due to its more challenging nature of measurement and analysis (Bishop et al., 2003). Furthermore, there is a notable gap in studies that examine both cognitive and affective dimensions simultaneously, particularly in the context of mathematical modeling. Although affective factors, especially mathematical values, play a vital role in shaping students' mathematical experiences, they have not been extensively studied within the framework of mathematical modeling tasks.

Mathematical values such as the attitudes towards mathematics and its real-world applications are often implicitly conveyed during the teaching and learning process. These values may influence how students perceive the relevance of mathematics in their daily lives and how they approach mathematical problems. However, not all learning environments succeed in transmitting positive mathematical values, and in many cases, these values are transferred unconsciously or unintentionally (FitzSimons et al., 2001; Gellert, 2000). Therefore, understanding the role of mathematical values in mathematical modeling tasks, especially through different modeling perspectives (e.g., realistic/applied, cognitive, model-eliciting, and socio-critical approaches), is crucial for improving mathematical modeling instruction).

Moreover, there remains a gap in the literature regarding cross-cultural investigations of mathematical values in mathematical modeling tasks. Different cultural contexts may present unique perspectives on how values are integrated into mathematical modeling, and understanding these differences can lead to more culturally responsive and effective teaching strategies. This study aims to fill this gap by investigating the cognitive and affective dimensions of mathematical modeling, with a particular focus on the mathematical values embedded in various mathematical modeling tasks across different cultures. By examining these dimensions, the study seeks to contribute to the development of more holistic and effective mathematical modeling education that considers both cognitive skills and affective factors. The study of cultural differences in mathematics education between the U.S. and China reveals varying teacher beliefs that influence the teaching and learning process (An et al., 2006). Mathematics education is deeply shaped by the cultural contexts in which it is practiced, affecting both the content and the pedagogical approaches employed in classrooms (Andrews, 2016). Socially open-ended problems in mathematics education offer an opportunity to integrate mathematical models with societal values, enriching student learning (Baba & Shimada, 2019). The values embedded in Japanese mathematics education have evolved through historical shifts, reflecting societal and educational changes (Baba et al., 2012). Mathematical modeling in the classroom, when approached from a socio-critical and discursive perspective, enables students to engage with mathematics in socially meaningful ways (Barbosa, 2006). Mathematical modeling serves as a powerful tool for connecting abstract mathematical concepts to real-world problems, promoting deeper understanding among students (Berry & Houston, 1995). Mathematical enculturation, the process by which students are introduced to the culture of mathematics, plays a central role in shaping both individual and collective mathematical identities (Bishop, 1991). Mathematical enculturation emphasizes the importance of understanding how students' cultural backgrounds influence their engagement with and interpretation of mathematical concepts (Bishop, 1991). The quality teaching of mathematical modeling involves not only understanding the mathematical concepts but also fostering students' ability to apply these concepts to solve real-world problems in a meaningful way (Blum, 2015). Exploring values and valuing in mathematics education provides a framework for understanding how educational contexts and individual beliefs influence the teaching and learning of mathematics (Clarkson et al., 2019). Research methods in education play a crucial role in shaping how educational practices are analyzed and developed, offering both qualitative and quantitative tools for studying educational phenomena (Cohen et al., 2002). Intercultural communication theories are essential in understanding how cultural differences impact the transmission and reception of educational content, including mathematics (Cooper et al., 2007). The choice of research design whether qualitative, quantitative, or mixed methods greatly influences how we approach the study of teaching and learning processes in mathematics education (Creswell & Creswell, 2017). Mathematical literacy is a critical component of modern education, as it integrates mathematical thinking with real-world problem-solving, fostering not only cognitive skills but also societal participation (D'Ambrosio, 1999). Analyzing values in Turkish middle school mathematics textbooks reveals how cultural and societal norms shape the content and presentation of mathematical knowledge **(**Dede, 2006). A comparison of Turkish and German mathematics teachers reveals differing perceptions of the value of mathematics, shaped by distinct educational philosophies and cultural contexts (Dede, 2012). The decision-making processes of mathematics teachers in Turkey and Germany are influenced by underlying values, which guide their instructional choices in group study settings (Dede, 2013). Gender differences in mathematics education values are examined by comparing the perspectives of Turkish and German mathematics teachers, highlighting the role of social and cultural factors (Dede, 2014). Investigating the mathematical modeling competencies of pre-service teachers by gender helps reveal how different educational experiences shape their skills and attitudes toward mathematical modeling (Dede et al., 2018). Theoretical and empirical investigations of the phases in the modeling process contribute to a deeper understanding of how students transition from problem interpretation to mathematical solution (Ferri, 2006). Different models of mathematical modeling whether viewed as genres, purposes, or perspectives offer diverse frameworks for understanding how mathematical concepts can be applied to real-world situations (Galbraith, 2012). Document analysis is a valuable method for examining educational texts and materials, providing insight into the values, norms, and educational practices embedded within them (Gross, 2018). Pedagogical content knowledge, when viewed through the lens of values, emphasizes the role of teachers' beliefs and cultural context in shaping how content is delivered in the classroom (Gudmundsdottir, 1990). Hofstede's cultural dimensions theory provides a framework for understanding how cultural values influence behaviors, communication, and educational practices across different societies (Hofstede, 2009).

2. Model Formulation

The model incorporates five primary components: student enrollment, student performance, resource allocation, teacher quality, and dropout rates. These components are influenced by external factors, such as funding and socio-economic conditions, and internal factors, such as the distribution of resources and quality of education provided. By formulating these relationships as differential equations, the model enables the simulation of various scenarios and the evaluation of policy interventions to optimize the performance of the education system.

The system of equations reflects the following dynamics:

$$
\begin{aligned}
\frac{dE}{dt} &= \lambda - \delta E - DE \\
\frac{dP}{dt} &= \alpha_1 \frac{E}{R} + \alpha_2 T - \beta_1 P \\
\frac{dR}{dt} &= \eta - \gamma E - \rho R \\
\frac{dT}{dt} &= \kappa - \sigma T + \xi \frac{T}{R} \\
\frac{dD}{dt} &= \delta_1 - \delta_2 \frac{E}{P} + \delta_3
\end{aligned}
\tag{1}
$$

With the following initial conditions:

$$E(0) = E_0 \geq 0, \quad P(0) = P_0 \geq 0, \quad R(0) = R_0 \geq 0, \quad T(0) = T_0 \geq 0, \quad D(0) = D_0 \geq 0$$

3. Analysis of the Proposed Model

Considering these properties ensures that the educational model is feasible, realistic, and meaningful for understanding system dynamics.

Theorem 1: If the initial conditions of the model equations are non-negative, then the future solutions are also non-negative.

Proof: Consider the differential equation for $E(t)$:

$$\frac{dE}{dt} = \lambda - (\delta + D)E \tag{2}$$

Now, multiply both sides of the differential equation by the integrating factor $e^{(\delta+D)t}$:

$$e^{(\delta+D)t}\frac{dE}{dt} + e^{(\delta+D)t}(\delta + D)E = e^{(\delta+D)t}\lambda \tag{3}$$

This integrates to give the following solution:

$$\frac{\mathrm{dE}}{\mathrm{dt}}\left(\mathrm{e}^{(\delta+\mathrm{D})\mathrm{t}}\mathrm{E}\right) = \mathrm{e}^{(\delta+\mathrm{D})\mathrm{t}}\frac{\mathrm{dE}}{\mathrm{dt}} + \mathrm{e}^{(\delta+\mathrm{D})\mathrm{t}}(\delta + \mathrm{D})\mathrm{E} = \mathrm{e}^{(\delta+\mathrm{D})\mathrm{t}}\lambda \tag{4}$$

$$E(t) = \frac{\lambda}{\delta + D} + Ce^{-(\delta+D)t} \tag{5}$$

where, $C = E_0 - \frac{\lambda}{\delta+D}$.

As $t \to \infty$, the exponential term vanishes, which leads to:

$$E(t) \geq 0 \quad \text{for all} \quad t \geq 0$$

The solution for $E(t)$ is non-negative, and similarly, for $P(t)$, $R(t)$, $T(t)$, and $D(t)$, the solutions remain non-negative for all $t \geq 0$.

Theorem 2: All the solutions of the model equations are uniformly bounded and contained in a feasible region for all $t \geq 0$.

Proof: Let $\Gamma = \left\{(E(t), P(t), R(t), T(t), D(t),) \in \mathbb{R}_+^5 \mid 0 \leq N(t) \leq \frac{\Lambda}{k}\right\}$ be the positive invariant set and $N = E + P + R + T + D$ be the total population of the animals, where E, P, R, T, and D represent different compartments. Then,

$$\frac{dN}{dt} = \frac{dE}{dt} + \frac{dP}{dt} + \frac{dR}{dt} + \frac{dT}{dt} + \frac{dD}{dt} = \Lambda - kN \tag{6}$$

$$\frac{dN}{dt} = \Lambda - kN \tag{7}$$

To prove that $N(t)$ is bounded, the ordinary differential equation (ODE) needs to be solved.

$$\frac{dN}{dt} = \Lambda - kN \Longrightarrow \frac{dN}{dt} \geq -kN \quad \text{with initial condition} \quad N(0) = N_0$$

First, the corresponding ODE is solved by using the basic method of integration:

$$\frac{dN}{dt} \geq -kN \Longrightarrow \ln|N| \geq -kt + C \tag{8}$$

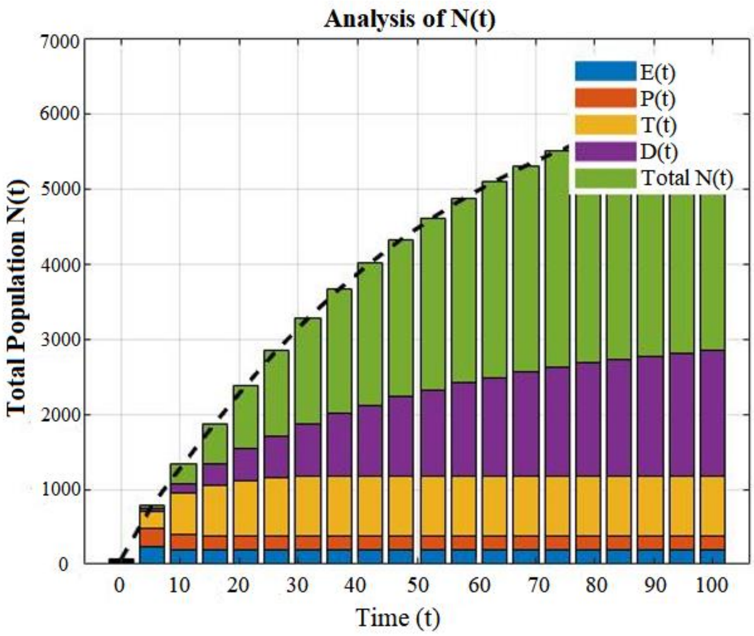

Figure 1. Analysis of $N(t)$

where, C is any constant of integration. At $t = 0$, then $N(t)$ becomes as $N(t) = N_0 e^{-kt}$, which lead to $N(t) \geq N_0 e^{-kt}$, showing that $N(t)$ is a decreasing function. Therefore, $N(t)$ is bounded below. Now to show that $N(t)$ is also bounded above, the inequality$\frac{dN}{dt} \geq -kN$ is considered. Multiplying e^{kt} on both sides and integrating from 0 to t, obtaining $e^{kt}\frac{dN}{dt} + ke^{kt}N \geq 0$, which shows that $N(t) \leq N_0 e^{kt}$ is bounded above. Combining both bounds, it can be concluded that $N_0 e^{-kt} \leq N(t) \leq N_0 e^{kt}$. Therefore, $N(t)$ is bounded for all $t \geq 0$. Figure 1 shows an analysis of $N(t)$.

3.1 Basic Reproduction Number and Equilibrium (DFE) Point

This section establishes the existence of the equilibrium (DFE) point and calculates the basic reproduction number, R_0, for the proposed model. The DFE represents a state where the population is free from infection, with no individuals in the exposed or infected compartments. The basic reproduction number, R_0, was calculated using the next-generation matrix approach or an equivalent method, depending on the structure of the model. This parameter quantifies the average number of secondary infections caused by a single infected individual in a fully susceptible population. The value of R_0 determines the threshold for spread of education. If $R_0 < 1$, the system may eventually stabilize or decline (e.g., students may drop out faster than new students are admitted, or performance may decay). If $R_0 > 1$, the education system can experience growth, potentially leading to a "boom" in enrollment, resources, or performance.

To evaluate the basic reproduction number R_0 for the given proposed mode, the next-generation matrix

technique was used. Let F_i represent the newly enrolled students in the class i and V_i represent the transfer of animals in each class i, where $i \in \{E, P, I, D, R\}$. To find the basic reproductive number of the proposed model, the formulation below can be followed:
where, $X = (E, P, I, D, R)$

$$\frac{dX}{dt} = F(X) - V(X)$$

$$F = \begin{pmatrix} \lambda - \delta E - DE \\ \frac{\alpha_1}{R} E + \alpha_2 T \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad V = \begin{pmatrix} \gamma E + DE \\ \beta_1 P \\ \rho R \\ \sigma T \\ \frac{\delta_2 P}{E} \end{pmatrix}$$

$$F' = \frac{\partial F_i}{\partial x_i}|_{E_0} = \begin{pmatrix} -\delta - D & 0 & 0 & 0 & 0 \\ \frac{\alpha_1}{R} & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad V' = \frac{\partial V_i}{\partial x_i}|_{E_0} = \begin{pmatrix} \gamma + D & 0 & 0 & 0 & 0 \\ 0 & \beta_1 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 & \frac{P}{\delta_2} \end{pmatrix}$$

$$(V')^{-1} = \begin{pmatrix} \frac{1}{\gamma + D} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\beta_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma} & 0 \\ 0 & 0 & 0 & 0 & \frac{P}{\delta_2} \end{pmatrix}$$

$$F' \cdot (V')^{-1} = \begin{pmatrix} -\delta - D & 0 & 0 & 0 & 0 \\ \frac{\alpha_1}{R} & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\gamma + D} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\beta_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sigma} & 0 \\ 0 & 0 & 0 & 0 & \frac{P}{\delta_2} \end{pmatrix}$$

The maximum eigenvalue of $F' \cdot (V')^{-1}$ is called the basic reproductive number, which is given by:
where, $X = (E, P, I, D, R)$

$$\frac{dX}{dt} = F(X) - V(X)$$

$$R_0 = \left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2} \right)$$

As shown in Figure 2, it is clear that the sensitivity index of R_0 with respect to R, β_1, γ and D is directly proportional to R_0. On the other hand, the sensitivity index of R_0 with respect to α_1 and α_2 shows an inverse proportionality to R_0, and it is also clear that the parameter is more sensitive compared with others.

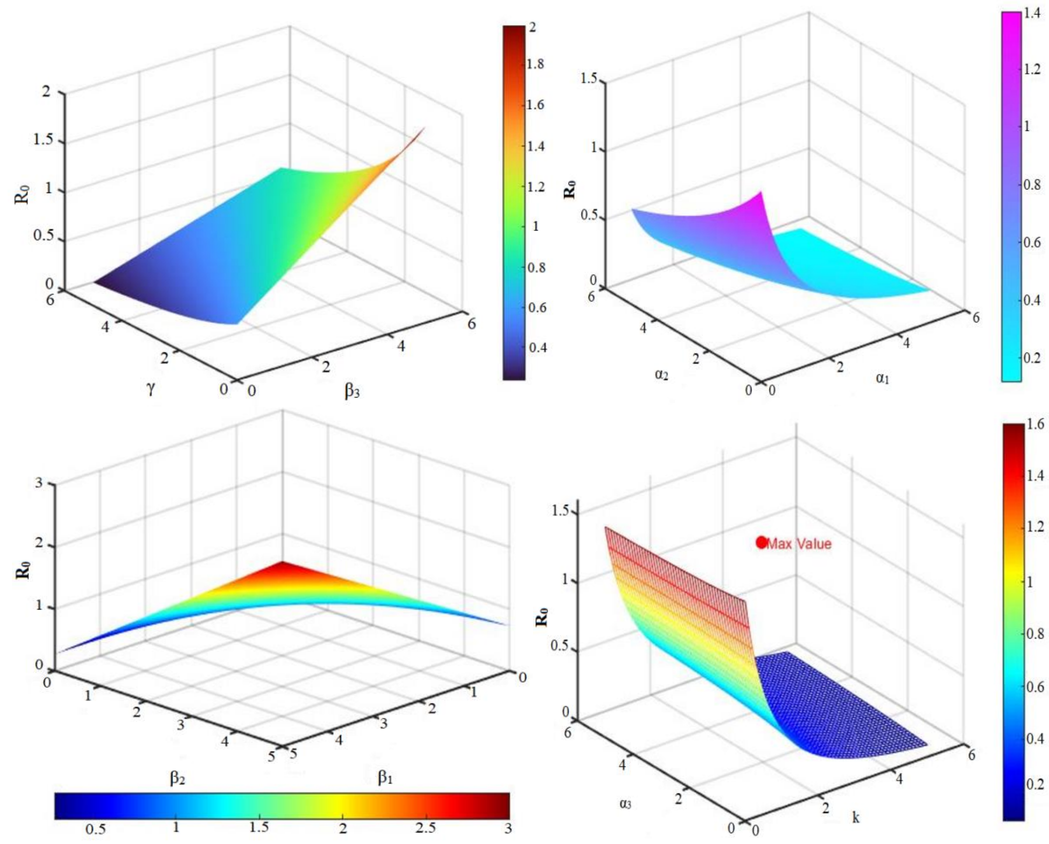

Figure 2. Analysis of R_0

3.2 Free Equilibrium Points

To determine the equilibrium point, the right-hand side of the system of equations in the proposed model was set to zero. The equilibrium point is given by:

$$(E, P, R, T, D) = \left(\frac{\lambda}{\delta + D}, \frac{\alpha_1\left(\frac{E}{R}\right) + \alpha_2 T}{\beta_1}, \frac{\eta - \gamma E}{\rho}, \frac{\kappa}{\sigma + \frac{\xi}{R}}, \frac{\delta_1 + \delta_3}{\delta_2\left(\frac{E}{P}\right)}\right) \tag{9}$$

3.3 Local Stability

Theorem 3: The equilibrium points is locally asymptotically stable if $R_0 < 1$.

Proof: To show that the equilibrium point is asymptotically stable, the Jacobin matrix J for the system at the equilibrium point can be calculated:

$$J = \begin{pmatrix} -\delta - D & R\alpha_1 & -\gamma & 0 & -P\delta_2 \\ 0 & -\beta_1 & 0 & 0 & 0 \\ P^2\delta_2 E & -R^2\alpha_1 E & -\rho & -R^2\xi T & 0 \\ 0 & 0 & \alpha_2 & 0 & -\sigma + R\xi \\ 0 & 0 & 0 & 0 & -E \end{pmatrix} \tag{10}$$

The characteristic equation is solved to look for the eigenvalues:

$$\det(J - \lambda I) = 0 \tag{11}$$

where, λ is the eigenvalue, and I is the identity matrix. Subtracting λ from the diagonal elements of the Jacobian matrix leads to:

$$J-\lambda I=\begin{pmatrix} -\delta-D-\lambda & R\alpha_1 & -\gamma & 0 & -P\delta_2 \\ 0 & -\beta_1-\lambda & 0 & 0 & 0 \\ P^2\delta_2 E & -R^2\alpha_1 E & -\rho-\lambda & -R^2\xi T & 0 \\ 0 & 0 & \alpha_2 & -\sigma+R\xi-\lambda & 0 \\ 0 & 0 & 0 & -E & -\lambda \end{pmatrix} \tag{12}$$

By solving the determinant equation, the eigenvalues are:

$$\lambda_1=R\alpha_1, \quad \lambda_2=R\xi-\sigma, \quad \lambda_3=-\beta_1, \quad \lambda_4=-\rho$$

Since all the eigenvalues are negative (except for $\lambda_1=R\alpha_1$), for stability, $R\alpha_1<1$ is required. This condition is satisfied when $R_0<1$.

3.4 Global Stability Analysis

Theorem 4: The proposed model is globally asymptotically stable at the equilibrium point if $R_0\leq 1$; otherwise, it is unstable.

Proof: To show that the model is globally asymptotically stable when $R_0\leq 1$, the following Lyapunov function is used:

$$H(E,P,R,T,D)=\frac{1}{2}((E-E_0)^2+(P-P_0)^2+(R-R_0)^2+(T-T_0)^2+(D-D_0)^2) \tag{13}$$

To compute the time derivative of H, each term can be differentiated:

$$\frac{d}{dt}((E-E_0)^2+(P-P_0)^2+(R-R_0)^2+(T-T_0)^2+(D-D_0)^2)$$

This simplifies to:

$$\frac{d}{dt}(E+P+R+T+D)=\delta E+\delta P-(\alpha_2 R+\alpha_1 T+\beta_3)$$

Defining $p=(\delta E+\delta P)$ and $q=(\delta R+\alpha_2 E+\alpha_1 T+\beta_1 P)$ leads to:

$$\frac{dH}{dt}=p-q \tag{14}$$

If $p>q$, then $\frac{dH}{dt}<0$, indicating that H is decreasing over time, and the system is globally asymptotically stable at the equilibrium point if $R_0\leq 1$. Otherwise, if $R_0>1$, the system is unstable.

3.5 Global Asymptotic Stability of the Equilibrium Points

Theorem 5: The proposed model is globally asymptotically stable at the endemic equilibrium point if $R_0>1$.

Proof: To show that the model is globally asymptotically stable, the following Lyapunov function is considered:

$$G(x_1,x_2,\dots,x_n)=\frac{1}{2}\sum_{i=1}^{n}(x_i-x_i^*)^2 \tag{15}$$

where, $x_i=(E,P,R,T,D)$ and $x_i^*=(E^*,P^*,R^*,T^*,D^*)$.

First, the time derivative of G is computed:

$$\frac{dG}{dt}=\sum_{i=1}^{n}(x_i-x_i^*)\frac{dx_i}{dt} \tag{16}$$

Let $N(t)=E+P+R+T+D$, and suppose.

$$\frac{dG}{dt} = [N(t) - (E^* + P^* + R^* + T^* + D^*)]\frac{dG}{dt} \tag{17}$$

This expression needs clarification, as it is not clearly representing the system's time derivatives. The equation should instead involve the dynamics of the system.

Rewriting the time derivative expression for G as:

$$\frac{dG}{dt} = \sum_{i=1}^{n} (x_i - x_i^*)\frac{dx_i}{dt} \tag{18}$$

Next, the dynamics of each variable in the system can be substituted. For simplicity, assume that the model is described by the following differential equations:

$$\frac{dE}{dt} = f_1(E,P,R,T,D), \quad \frac{dP}{dt} = f_2(E,P,R,T,D), \quad \frac{dR}{dt} = f_3(E,P,R,T,D),$$

$$\frac{dT}{dt} = f_4(E,P,R,T,D), \quad \frac{dD}{dt} = f_5(E,P,R,T,D)$$

Now consider the following simplification for the Lyapunov function's time derivative:

$$\frac{dG}{dt} = [N(t) - (E^* + P^* + R^* + T^* + D^*)]\left[-\delta N(t) - \frac{k_1 P^* - \wedge}{k}\right]$$

Substitute further for $N(t)$:

$$\frac{dG}{dt} = \left[N(t) - \frac{k_1 P^* - \wedge}{\delta}\right]\left[-\delta N(t) - \frac{\delta_1 P^* - \wedge}{\delta}\right] \tag{19}$$

Finally, the following can be derived:

$$\frac{dG}{dt} = -\delta\left[N(t) + \frac{\wedge}{\delta}\right]^2 \tag{20}$$

Hence, the following can be obtained:

$$\frac{dG}{dt} \leq -\delta\left[N(t) + \frac{\wedge}{\delta}\right]^2 < 0 \tag{21}$$

Since $\frac{dG}{dt} < 0$, all conditions of the Lyapunov function are satisfied. Therefore, the model is globally asymptotically stable at equilibrium point if $R_0 > 1$.

3.6 Bifurcation

Bifurcation explains a qualitative analysis of the nature of a system due to changes in specific parameters. In particular, a bifurcation occurs when a critical parameter, for which $R_0 = 0$, reaches a threshold, triggering a sudden or substantial shift in the system's dynamics, stability, or equilibrium points. This phenomenon is vital in the analysis of nonlinear systems, as it can give rise to new patterns, the coexistence of multiple stable states, or even chaotic dynamics.

$$R_0 = \max\left(\frac{R(\gamma + D)}{\alpha_1}, \frac{\beta_1}{\alpha_2}\right) \Rightarrow \max\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) = 1 \Rightarrow \alpha_2\left(1 - \frac{R(\gamma + D)}{\alpha_1}\right) = \beta_1$$

$$\lambda^2 + \lambda\left(\frac{\beta_2}{\alpha_1 + k} - (\gamma + \alpha_2 + k) + \beta_3 + \delta\right) - \left[\frac{\beta_2(\beta_3 + \delta) + \alpha_3\beta_3 - (\beta_3 + \delta)(\alpha_1 + k)(\gamma + \alpha_2 + k)}{\alpha_1 + k}\right]$$

$$\lambda^2 + \lambda\left[\frac{\wedge\beta_2}{\alpha_1 + k} - (\gamma + \alpha_2 + k) + \beta_3 + \delta\right] - \left[\frac{(\beta_3 + \delta)(\alpha_1 + k)(\gamma + \alpha_2 + k) - \wedge\alpha_3\beta_3}{\wedge(\beta_3 + \delta)} \wedge(\beta_3 + \delta)\right] + \frac{\wedge\alpha_3\beta_3 - (\beta_3 + \delta)(\alpha_1 + k)(\gamma + \alpha_2 + k)}{\alpha_1 + k}$$

Given the equation:

$$\lambda^2 + \lambda\left[\frac{\wedge\beta_2}{\alpha_1 + k} - (\gamma + \alpha_2 + k) + \beta_3 + \delta\right] = 0 \tag{22}$$

Since the constant term in the equation is zero, this implies that one of the roots of the equation is zero. Consequently, the presence of a zero root indicates the potential for bifurcation.

To determine the direction, the right eigenvectors associated with the zero eigenvalue can be used. These are identified by setting the constant term in the characteristic equation to zero. This implies that zero must be one of the roots of the equation. Consequently, the presence of a zero root indicates the possibility of bifurcation. The right eigenvectors corresponding to the zero eigenvalue are solutions to the equation:

$$\begin{pmatrix} -\delta - D & R\alpha_1 & -\gamma & 0 & -P\delta_2 \\ 0 & -\beta_1 & 0 & 0 & 0 \\ P^2\delta_2 E & -R^2\alpha_1 E & -\rho & -R^2\xi T & 0 \\ 0 & 0 & \alpha_2 & 0 & -\sigma + R\xi \\ 0 & 0 & 0 & 0 & -E \end{pmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

This results in the following system of equations:

$$(-\delta - D)w_1 - \beta_1 w_2 = 0 \tag{24}$$

$$(-\delta - D)w_1 + (R\alpha_1)w_2 - \gamma w_3 - P\delta_2 w_5 = 0 \tag{25}$$

$$P^2\delta_2 E w_1 - R^2\alpha_1 E w_2 - \rho w_3 - R^2\xi T w_4 = 0 \tag{26}$$

$$\alpha_2 w_3 - (\sigma - R\xi)w_5 = 0 \tag{27}$$

$$-E w_5 = 0 \tag{28}$$

From the last equation, $w_5 = 0$ can be obtained. Substituting this into Eq. (25) leads to $w_3 = 0$.
Now, substitute $w_3 = 0$ and $w_5 = 0$ into the second equation, and add it to the first equation to get $w_2 = 0$.

$$w_1 = w_2 = w_3 = w_4 = w_5 = 0$$

This means the only solution to this system is the trivial solution, where all the variables $w_1, w_2, w_3, w_4,$ and w_5 are zero.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix}^T \begin{pmatrix} -\delta - D & R\alpha_1 & -\gamma & 0 & -P\delta_2 \\ 0 & -\beta_1 & 0 & 0 & 0 \\ P^2\delta_2 E & -R^2\alpha_1 E & -\rho & -R^2\xi T & 0 \\ 0 & 0 & \alpha_2 & 0 & -\sigma + R\xi \\ 0 & 0 & 0 & 0 & -E \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{29}$$

The matrix system corresponds to the following set of equations:

$$(-\delta - D)v_1 + R\alpha_1 v_2 - \gamma v_3 - P\delta_2 v_5 = 0 \tag{30}$$

$$-\beta_1 v_2 = 0 \tag{31}$$

$$P^2\delta_2 E v_1 - R^2\alpha_1 E v_2 - \rho v_3 - R^2\xi T v_4 = 0 \tag{32}$$

$$\alpha_2 v_3 - (\sigma - R\xi)v_5 = 0 \tag{33}$$

$$-Ev_5 = 0 \tag{34}$$

After solving all the equations, the left eigenvectors can be determined as follows:

$$v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \quad v_4 = 0, \quad v_5 = 0$$

Consequently, both matrices W and V are also zero:

$$W = 0 \quad \text{and} \quad V = 0$$

Since there are no non-zero eigenvectors, there is no direction of bifurcation present in this case. Therefore, this represents a trivial case.

4. Sensitivity Analysis of the Model

The sensitivity analysis of the basic reproductive number R_0 against the model parameters is very important for the presented study. It enables us to identify the most influential parameters that play a role in disease transmission and control. In this section, the sensitivity of various key parameters of the brucellosis model was carried out, as shown in Figure 3.

The sensitivity of the basic reproductive number R_0 with respect to a parameter η is given by:

$$\chi_\eta^{R_0} = \frac{\partial R_0}{\partial \eta} \times \frac{\eta}{R_0}$$

where, η can be any of the parameters in the following set:

$$\eta \in \{\lambda, \delta, \alpha_1, \alpha_2, \beta_1, \gamma, \rho, \kappa, \sigma, \xi, \delta_1, \delta_2, \delta_3\}$$

$$\frac{\partial R_0}{\partial \lambda} \times \frac{\lambda}{R_0} = \frac{1}{\partial \lambda}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\lambda}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx 1$$

$$\frac{\partial R_0}{\partial \delta} \times \frac{\delta}{R_0} = \frac{1}{\partial \delta}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\delta}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx 0.132$$

$$\frac{\partial R_0}{\partial \alpha_1} \times \frac{\alpha_1}{R_0} = \frac{1}{\partial \alpha_1}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\alpha_1}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx 0.423$$

$$\frac{\partial R_0}{\partial \alpha_2} \times \frac{\alpha_2}{R_0} = \frac{1}{\partial \alpha_2}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\delta}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx -0.0035$$

$$\frac{\partial R_0}{\partial \beta_1} \times \frac{\beta_1}{R_0} = \frac{1}{\partial \beta_1}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\beta_1}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx 0.577$$

$$\frac{\partial R_0}{\partial \gamma} \times \frac{\gamma}{R_0} = \frac{1}{\partial \gamma}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\gamma}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx 0.577$$

$$\frac{\partial R_0}{\partial \rho} \times \frac{\rho}{R_0} = \frac{1}{\partial \rho}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\rho}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx -0.0896$$

$$\frac{\partial R_0}{\partial \delta_1} \times \frac{\delta_1}{R_0} = \frac{1}{\partial \delta_1}\left(\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}\right) \times \frac{\delta_1}{\frac{R(\gamma + D)}{\alpha_1} + \frac{\beta_1}{\alpha_2}} \approx -0.0896$$

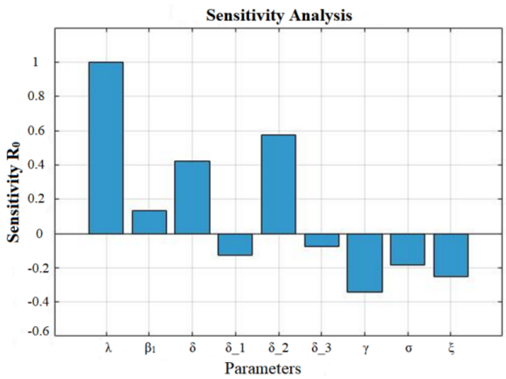

Figure 3. Sensitivity analysis

4.1 Numerical Scheme

This section provides a numerical scheme for the proposed model to describe the dynamics of disease spread. Consider the system of ODEs, which can be written as:

$$f_1(E,P,R,T,D) = \lambda - \delta E - DE \tag{35}$$

$$f_2(E,P,R,T,D) = \alpha_1 \frac{E}{R} + \alpha_2 T - \beta_1 P \tag{36}$$

$$f_3(E,P,R,T,D) = \eta - \gamma E - \rho R \tag{37}$$

$$f_4(E,P,R,T,D) = \kappa - \sigma T + \xi \frac{T}{R} \tag{38}$$

$$f_5(E,P,R,T,D) = \delta_1 - \delta_2 \frac{E}{P} + \delta_3 \tag{39}$$

Let us consider S at $t_1 = t_0 + \Delta t$:

$$k_1^S = f_1(E_t, P_t, R_t, T_t, D_t) \cdot \Delta t \tag{40}$$

$$k_2^S = f_1\left(E_t + \frac{k_1^E}{2}, P_t + \frac{k_1^I}{2}, P_t + \frac{k_1^P}{2}, R_t + \frac{k_1^V}{2}, D_t + \frac{k_1^R}{2}\right) \cdot \Delta \tag{41}$$

$$k_3^S = f_1\left(E_t + \frac{k_2^E}{2}, P_t + \frac{k_2^I}{2}, P_t + \frac{k_2^P}{2}, T_t + \frac{k_2^V}{2}, D_t + \frac{k_2^R}{2}\right) \cdot \Delta t \tag{42}$$

$$k_4^S = f_1(S_t + k_3^S, E_t + k_3^E, I_t + k_3^I, P_t + k_3^P, V_t + k_3^V, R_t + k_3^R) \cdot \Delta t \tag{43}$$

The the Runge-Kutta 4th-order (RK4) method for updating E is as follows:

$$k_1^E = f_2(E_t, P_t, R_t, T_t, D_t) \cdot \Delta t \tag{44}$$

$$k_2^E = f_2\left(E_t + \frac{k_1^S}{2}, P_t + \frac{k_1^I}{2}, P_t + \frac{k_1^P}{2}, R_t + \frac{k_1^V}{2}, D_t + \frac{k_1^R}{2}\right) \cdot \Delta t \tag{45}$$

$$k_3^E = f_2\left(E_t + \frac{k_2^S}{2}, P_t + \frac{k_2^I}{2}, P_t + \frac{k_2^P}{2}, R_t + \frac{k_2^V}{2}, D_t + \frac{k_2^R}{2}\right) \cdot \Delta t \tag{46}$$

$$k_4^E = f_2(E_t + k_3^S, P_t + k_3^I, P_t + k_3^P, R_t + k_3^V, D_t + k_3^R) \cdot \Delta t \tag{47}$$

Similarly, the RK4 method can be applied to all the differential equations. The above system can then be written as:

$$E_{t+1} = E_t + \frac{1}{6}(k_1^E + 2k_2^E + 2k_3^E + k_4^E) \tag{48}$$

$$P_{t+1} = P_t + \frac{1}{6}(k_1^P + 2k_2^P + 2k_3^P + k_4^P) \tag{49}$$

$$D_{t+1} = D_t + \frac{1}{6}(k_1^D + 2k_2^D + 2k_3^D + k_4^D) \tag{50}$$

$$R_{t+1} = R_t + \frac{1}{6}(k_1^R + 2k_2^R + 2k_3^R + k_4^R) \tag{51}$$

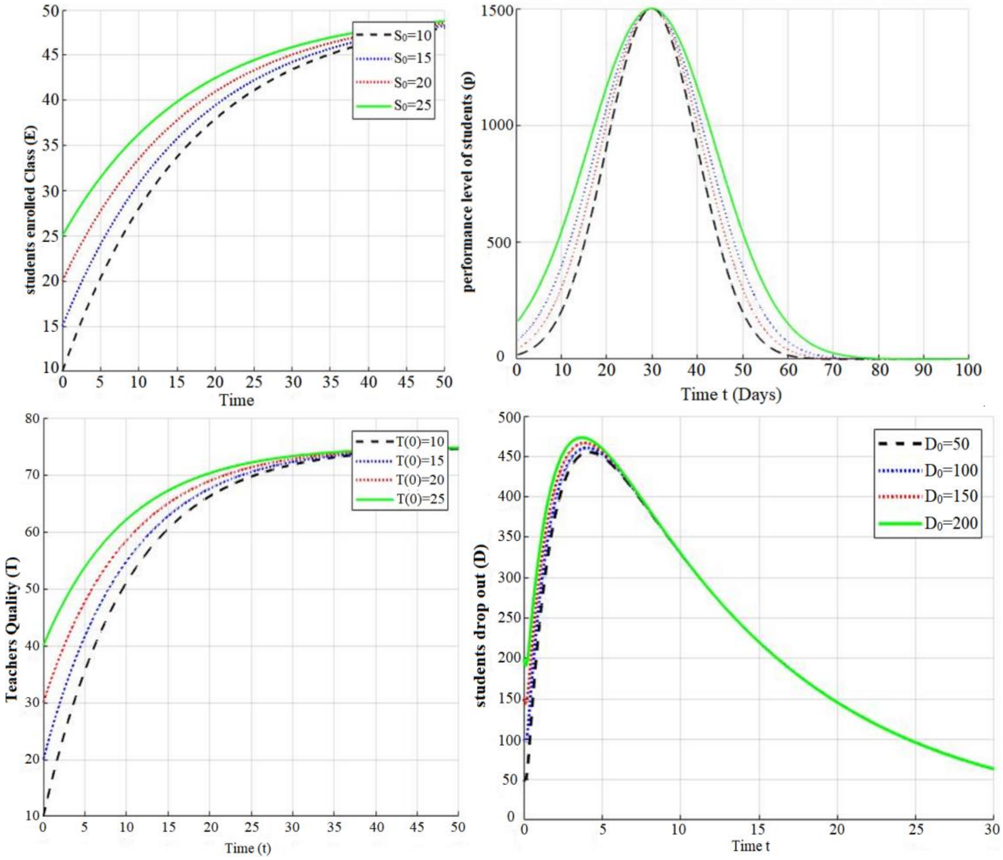

Figure 4. Numerical data analysis

Applying this numerical scheme to analyze the dynamics of the education system, as described by the differential equations, shows how different population classes (students, teachers, resources, etc.) evolve over time under various initial conditions and control measures. The numerical data shown in Figure 4 was obtained. In the figure, it can be observed that $E(t)$ increases with time. This increase is due to new students entering the education system. $P(t)$ is influenced by various factors such as the number of teachers, the availability of resources, and the influx of new students. In addition, it suggests that to improve the education system and avoid inefficiencies, it is crucial to focus on optimizing resources, increasing teacher availability, and enhancing student retention. These interventions can push the system toward a more balanced and ideal state. Teacher quality is incorporated as a parameter influencing student performance, retention rates, and the efficiency of resource utilization within the system. Furthermore, it shows that interventions aimed at improving teacher quality or increasing resource allocation can lower the dropout rate, leading to better student retention and improved educational outcomes.

For the computational results, the following numerical values were used for the parameters:

$$\lambda = 0.15, \quad \delta_1 = 0.04, \quad \delta_2 = 0.03, \quad \delta_3 = 0.005, \quad \alpha_1 = 0.03, \quad \alpha_2 = 0.2, \quad \rho = 0.002, \quad \sigma = 0.0015, \quad k = 0.0003, \quad \gamma = 0.0055, \quad \xi = 0.005$$

$$\lambda = 0.0015, \quad \delta_1 = 0.0074, \quad \delta_2 = 0.0003, \quad \delta_3 = 0.5, \quad \alpha_1 = 0.03, \quad \alpha_2 = 0.2, \quad \sigma = 0.002, \quad \rho = 0.0015, \quad k = 0.0893, \quad \gamma = 0.0055, \quad \xi = 0.005$$

5. Conclusion

In this study, a numerical scheme was developed and applied to model the dynamics of an education system by using a system of ODEs and applying the RK4 method. The evolution of various population classes over time, such as students, teachers, resources, and dropout rates, was simulated and analyzed. The numerical simulations yielded insightful results. The increase in the number of students entering the education system was reflected in the growth of $E(t)$ over time. This increase in students directly influenced the system's dynamics, including the need for a balanced teacher-to-student ratio and adequate resource allocation. The influence of these factors on student performance and retention was highlighted, showing that optimizing resource allocation and teacher availability is key to preventing inefficiencies within the education system. Furthermore, the analysis of teacher quality emphasized the significant impact of this parameter on student retention and the overall efficiency of the education system. The results suggest that improving teacher quality can lead to enhanced student performance and lower dropout rates. Interventions designed to improve teacher quality or resource allocation were found to effectively reduce dropout rates, which, in turn, improves overall educational outcomes and equity within the system. The computational results, based on specific parameter values, confirm the system's sensitivity to various factors. For example, increasing the teacher quality (α_1) or adjusting resource allocation (k) can substantially alter the trajectory of the education system's dynamics. These findings underline the importance of targeted interventions in optimizing education systems and ensuring better retention, resource utilization, and overall efficiency. In conclusion, the proposed model and numerical scheme provide a valuable tool for understanding and improving the dynamics of the education system. By simulating different scenarios and adjusting key parameters, policymakers and educators can make informed decisions that enhance educational outcomes, promote system efficiency, and address challenges related to resource allocation and teacher quality.

This study emphasizes how cognitive and affective factors influence students' engagement with mathematical modeling, and how cultural differences shape the expression of mathematical values. It also discusses the implications of these findings for teaching strategies in mathematical modeling education. Additionally, future research directions have been outlined, particularly focusing on cross-cultural comparisons and the integration of affective dimensions in mathematical modeling tasks. Future studies could explore how different cultural contexts impact the effectiveness of various instructional strategies, as well as how cognitive and affective factors can be measured and incorporated into educational models.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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