Journal of Operational and Strategic Analytics
https://www.acadlore.com/journals/JOSA

Production Optimization in Manufacturing Industries Using Cobb-Douglas Production Function

Fazal Hanan[1], Rashid Ali[2*]

[1] Department of Mathematics, Abdul Wali Khan University Mardan, 23200 Khyber Pakhtunkhwa, Pakistan
[2] School of Mathematics and Statistics, Central South University, 410083 Changsha, Hunan, China

[*] Correspondence: Rashid Ali (rashidali@csu.edu.cn)

Received: 08-13-2023 **Revised:** 09-16-2023 **Accepted:** 09-22-2023

Citation: F. Hanan and R. Ali, "Production optimization in manufacturing industries using Cobb-Douglas production function," *J. Oper. Strateg Anal.*, vol. 1, no. 3, pp. 131–139, 2023. https://doi.org/10.56578/josa010304.

Abstract: In the rapidly evolving industrial landscape, the decision-making process concerning which products to manufacture, their quantity, and the methods of their production has become pivotal. This study endeavors to address this need by advocating the most apt functional form of the production process for predominant manufacturing sectors. The central objective has been the maximization of output through the application of the Cobb-Douglas production function, investigated separately for both two-input and three-input scenarios. It is ascertained which of the two models exhibits greater efficacy. Subsequently, parameters of the production function are estimated utilizing advanced optimization subroutines.

Keywords: Cobb-Douglas production function; Nonlinear programming; Cost function; Human labor and capital

1 Introduction

Mathematical optimization, also referred to as mathematical programming, entails the selection of the most fitting element based on specific criteria from an array of available alternatives [1]. Such optimization dilemmas are encountered across various quantitative domains, encompassing computer science, engineering, operations research, and economics. Historical scrutiny reveals a sustained interest in the pursuit of optimal solutions within the mathematical domain [2]. At its core, an optimization problem might involve the maximization or minimization of a real function, achieved by methodically determining input values from a permissible set and evaluating the function's outcome. Expanding optimization theories and techniques to encompass other formats is recognized as a significant segment of applied mathematics. In broader terms, optimization seeks the "best available" values of a target function within a specified domain, spanning diverse objective functions and domains.

Optimization has emerged as an indispensable instrument for decision-making and the scrutiny of physical systems. Mathematically, an optimization issue is articulated as the quest for an optimal solution among a plethora of feasible alternatives. In essence, optimization seeks the judicious selection of inputs to yield the most proficient output. In the presented context, adjustments to various inputs in the production process aim to craft a more efficient output paradigm. Relevantly, an optimization methodology was incorporated into a mathematical model [3]. Paramount in this context is the identification of the most efficacious model tailored for an optimization challenge. Post modelling, available optimization techniques are employed for resolution. Subsequent to the problem's resolution, the efficacy of the derived solution is then assessed [4].

Typically, three pivotal components are discerned in optimization problems: the objective function awaiting optimization, the undetermined decision variables, and constraints influencing the objective function. Conventionally, optimization challenges are bifurcated into two categories: constrained and unconstrained optimization problems. The former is characterized by restrictions imposed on the objective function, suggesting its applicability for select decision variable values. In contrast, unconstrained problems lack such limitations, implying the objective function's validity across all decision variable values [5, 6].

In mathematical disciplines, nonlinear programming encompasses the resolution of optimization challenges wherein some constraints or the objective function manifest nonlinearity. An optimization dilemma seeks the extrema (be it maxima, minima, or stationary points) of an objective function over a set of indeterminate real variables, all the while adhering to a suite of equalities and inequalities, collectively designated as constraints.

Such dilemmas reside within the realm of mathematical optimization focused explicitly on nonlinear issues. The maximization or minimization of a nonlinear optimization quandary can be approached through established nonlinear programming methodologies. Situations arise wherein the objective or a constraint function deviates from linearity. The objective function is discerned as nonlinear in this study, whilst linear constraints are implemented. Succinctly, nonlinear programming strives to pinpoint the optimal solution to a problem, hemmed in by nonlinear constraints. Occasionally, certain issues resist accurate modelling via linear programming, thus necessitating recourse to nonlinear programming [7, 8].

Attention is herein directed towards a distinct production function known as the Cobb-Douglas production function, introduced initially by Cobb and Douglas in 1928. In economic and econometric realms, this production function is recognized for its distinct functional form, adeptly representing the technological interplay between multiple inputs (notably physical capital and labor) and the resultant output. Historical accounts attribute the development and empirical validation of the Cobb–Douglas function to Charles Cobb and Paul Douglas during 1927–1947, although Douglas acknowledged its prior conceptualization by Philip Wicksteed [9, 10]. Its foundation rests on empirical investigations, with applications spanning entire economies [11]. This function inherently maps various inputs to a problem and subsequently yields a singular output. The number of inputs can vary, hinging on the production factors integral to an industry. Situations entailing more than two factors have been denoted as the Generalized Cobb-Douglas production function [12, 13].

In an initial phase, a problem was tackled utilizing the two-factor Cobb-Douglas production function, yielding outcomes aligned with findings presented in the study [14]. A subsequent phase witnessed an extension to a three-factor Cobb-Douglas production function, with analogous techniques employed for resolution. Observations indicated that the latter phase engendered more efficient production maximization than its predecessor.

2 Production Maximization

The optimization dilemma under examination is predicated upon three foundational factors:

i) Costs incurred by an enterprise during production, which encompass expenses levied on labor, capital, and associated resources.

ii) Volume of production.

iii) Revenue derived from the sale of products based on prevailing market prices.

In the context of the water industry, water undergoes filtration in a series of purification tanks, subsequent chemical addition, packaging, and eventual market distribution. The primary ambition of this sector remains twofold: either to curtail incurred costs or to bolster production. The cumulative expenditure of the industry is articulated by a linear function [14, 15].

$$K = \sum_{j=1}^{n} q_i x_i + R_s = q^T x + R_s. \tag{1}$$

where, n epitomizes the factors of production.

$$x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix},$$

embodies the vector representation of these production factors.

$$q = \begin{bmatrix} q_1 \\ q_2 \\ . \\ . \\ . \\ q_n \end{bmatrix},$$

reflects the vector of production factor prices.

The production volume is elucidated by the subsequent relationship [16],

$$y = p(x) = p(x_1, x_2 ... x_n). \tag{2}$$

Revenue amassed by the enterprise from its sales is denoted by:

$$S(y) = b^T y, \tag{3}$$

where,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ . \\ . \\ . \\ y_n \end{bmatrix},$$

signifies a vector detailing the volume of goods produced.

$$b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix},$$

depicts the vector denoting prices of these manufactured goods.

The organization's profit is discerned by the difference between the amassed revenue and the production costs:

$$Z(x, y) = S(y) - K(x). \tag{4}$$

The objective set forth is the maximization of production at a designated level, resulting in optimal profit realization.

3 Problem under Consideration

Central to economic governance lies the challenge of determining apt quantitative associations amidst factors employed within the production process. Foremost among these influencing factors are human labor and capital. The scale of production is inherently dependent upon the magnitude of these inputs, and their judicious application becomes pivotal for enhancing managerial efficacy. Pertaining to the nexus between production yield and expended effort, optimal decisions predominantly arise in one of two scenarios:

First, maximizing production yield (profits) with stipulated factor inputs.

Second, curtailing expenditures at a pre-determined production output level.

However, simultaneous optimization of both inputs and outputs emerges as a paradoxical endeavor. For deducing the befitting correlation among these production elements within the water purification sector, it becomes imperative to ascertain the functional interdependencies between aggregate production yields, expenditures, and the quantum of production factors implicated. Cobb-Douglas production, initially for two inputs and subsequently for three, has been utilized to navigate this quandary.

3.1 Production Maximization of Chemotronics Industry Using Two-Factor Cobb-Douglas Production Function

In this section, an examination of the Cobb-Douglas production functions for both "two-input" and "three-input" scenarios, in context to the regional water purification sector, is undertaken. The production of an item necessitates three integral inputs: capital, chemicals, and labor. During the primary phase, a two-factor Cobb-Douglas production function was employed, rendering a cost function inclusive of the two production factors as:

$$C(X, Y) = \alpha_1 + \alpha_2 X_+ \alpha_3 Y, \tag{5}$$

where,

α_1 epitomizes the industry's fixed costs,

α_2 denotes the per unit costs of labor hours,

α_3 represents the per unit costs of capital,

X signifies labor hours,

Y quantifies the capital. Given values,

$$\alpha_1 = 75000, \quad \alpha_2 = 1400 \quad and \quad \alpha_3 = 100.$$

Substituting the aforementioned values into Eq. (5) yields:

$$C(X, Y) = 75000 + 1400X + 100Y. \tag{6}$$

For production, the Cobb-Douglas production function was employed, expressed as:

$$P = AX^aY^b, \tag{7}$$

where, P represents production volume.

Industry-imposed constraints on these factors were:

$$200 < X < 360,$$

$$850 < Y < 960.$$

An assessment for the year 2020, focusing on the water industry, revealed data on the two pivotal inputs "human labor and capital", alongside their corresponding production outputs (Table 1).

Table 1. Input-output analysis of the water industry with two inputs

Months	**Labors**(X)	**Capital**(Y)	**Production**(P)	***Ln***(X)	***Ln***(Y)	***Ln***(P)
January	300	900	35000	5.7037	6.8023	10.4631
February	280	860	34000	5.6347	6.7569	10.4341
March	304	906	36700	5.7170	6.8090	10.5105
April	315	922	35600	5.7525	6.8265	10.4801
May	296	862	35900	5.6903	6.7592	10.4884
June	318	916	36400	5.7620	6.8200	10.5023
July	292	888	33600	5.6767	6.7889	10.4222
August	300	934	35500	5.7037	6.8394	10.4772
September	320	914	39000	5.7683	6.8178	10.5713
October	328	910	38200	5.7930	6.8134	10.5505
November	294	876	33600	5.6835	6.7753	10.4222
December	330	880	36500	5.7990	6.7799	10.5050

From this analysis:

$$A = 246.0757, \quad a_1 = 0.6875 \quad and \quad a_2 = 0.1536.$$

Within this framework, the production function served as the objective function, whilst the cost function acted as the constraining parameter.

$$L(X, Y, \lambda) = P(X, Y) + \lambda H(X, Y), \tag{8}$$

where,

$$P(X, Y) = 246.0757X^{0.6875}Y^{0.1536}.$$

Incorporating this function into the earlier Eq. (8) gives:

$$L = 246.0757X^{0.6875}Y^{0.1536} + \lambda(435000 - 1400X - 100Y). \tag{9}$$

Derivative calculations of L concerning X, Y, and λ, followed by equating each to zero, yielded several critical relationships. The derivative with respect to X is given by

$$\frac{\partial L}{\partial X} = 0,$$

Further derivatives concerning Y and λ procured:

$$169.1770X^{-0.3125}Y^{0.1536} + \lambda(-1400) = 0,$$

This consequently led to:

$$\lambda = \frac{0.1208Y^{0.1536}}{X^{0.3125}}. \tag{10}$$

For

$$\frac{\partial L}{\partial Y} = 0,$$

This yielded:

$$37.7972X^{0.6875}Y^{-0.8464} + \lambda(-100) = 0.$$

The inference drawn from these equations was delineated as:

$$\lambda = \frac{0.3779X^{0.6875}}{Y^{0.8464}}. \tag{11}$$

Similarly, for

$$\frac{\partial L}{\partial \lambda} = 0,$$

This culminated in:

$$1400X + 100Y = 510000. \tag{12}$$

Upon juxtaposing Eqs. (10) and (11), the following was discerned:

$$\frac{0.1208Y^{0.1536}}{X^{0.3125}} = \frac{0.3779X^{0.6875}}{Y^{0.8464}},$$

This yielded:

$$Y = 3.1283X. \tag{13}$$

Incorporating Eq. (13) into Eq. (12) produced:

$$1400X + 1003.1283X = 510000.$$

From this computation, the value of X was determined:

$$X = 297.7528. \tag{14}$$

Consequently, from Eq. (13), the derivation for Y was:

$$Y = 931.46. \tag{15}$$

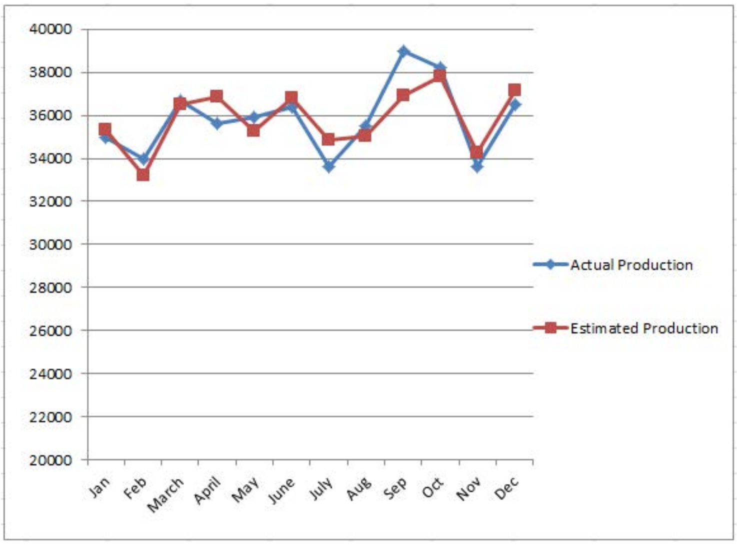

Figure 1. Estimated versus actual production with two inputs for the industry

Figure 1 contrasts the actual versus the theoretical production for the industry during 2020. A noticeable discrepancy between the two production values suggests that potential maximization of production has been substantially achieved.

3.2 Production Maximization of Water Industry Using Three-Factor Cobb-Douglas Function

In the subsequent analysis phase, this study extends to encompass three factors of production to ascertain the cost function of the water industry. Employing a linear cost function defined as:

$$C(X, Y, Z) = \alpha_1 + \alpha_2 X + \alpha_3 Y + \alpha_4 Z \tag{16}$$

where,
α_1 represents the fixed costs of the industry,
α_2 epitomizes the unit costs of labor,
α_3 denotes the unit costs of chemicals,
α_4 represents the unit costs of capital,
X signifies the quantum of labor hours,
Y represents the volume of chemicals,
Z signifies the amount of capital.
Their respective operational ranges are demarcated as:

$$200 < X < 280,$$

$$50 < Y < 80,$$

$$850 < Z < 960.$$

Given constants:

$$\alpha_1 = 75000, \quad \alpha_2 = 1250, \quad \alpha_3 = 2000 \quad and \quad \alpha_4 = 100.$$

Inserting these into Eq. (16) yields:

$$C(X, Y, Z) = 75000 + 1250X + 2000Y + 100Z. \tag{17}$$

The production function, as delineated in the study [16], adopts the form:

$$P = AX^{a_1}Y^{a_2}Z^{a_3}. \tag{18}$$

A comprehensive data analysis concerning the water purifying industry, predicated upon the three inputs for the fiscal year of 2020, is catalogued in Table 2.

Table 2. Input-output analysis of the water industry with three inputs

Months	**Labors(X)**	**Chemicals(Y)**	**Capitals(Z)**	**Production(P)**	***Ln*(X)**	***Ln*(Y)**	***Ln*(Z)**	***Ln*(P)**
Jan	240	60	900	35000	5.4806	4.0943	6.8023	10.4631
Feb	225	55	860	34000	5.4161	4.0073	6.7569	10.4341
March	233	71	906	36700	5.4510	4.2626	6.8090	10.5105
April	251	64	922	35600	5.5254	4.1588	6.8265	10.4801
May	230	66	862	35900	5.4380	4.1896	6.7592	10.4884
June	256	62	916	36400	5.5451	4.1271	6.8200	10.5023
July	230	62	888	33600	5.4380	4.1271	6.7889	10.4222
Aug	245	55	934	35500	5.5012	4.0073	6.8394	10.4772
Sep	258	62	914	39000	5.5529	4.1271	6.8178	10.5713
Oct	262	66	910	38200	5.5683	4.1896	6.8134	10.5505
Nov	239	55	876	33600	5.4764	4.0073	6.7753	10.4222
Dec	270	60	880	36500	5.5984	4.0943	6.7799	10.5050

For the purposes of this investigation, data from the industry spanning January to December 2020 was meticulously collated, with results subsequently computed on a monthly basis. Through rigorous analysis, the coefficients were derived as:

$$A = 283.7124, \quad a_1 = 0.4638, \quad a_2 = 0.2741, \quad and \quad a_3 = 0.1704.$$

Inserting these coefficients into Eq. (18) yielded:

$$P = 283.7124X^{0.4638}Y^{0.2741}Z^{0.1704}. \tag{19}$$

To further the analysis, the Lagrange method of undetermined multipliers was employed, expressed as:

$$L(X, Y, Z, \lambda) = P(X, Y, Z) + \lambda H(X, Y, Z), \tag{20}$$

where,

$$H(X, Y, Z) = 510000 - 1250X - 2000Y - 100Z. \tag{21}$$

The integration of Eqs. (19) and (21) into (20) led to:

$$L = 283.7124X^{0.4638}Y^{0.2741}Z^{0.1704} + \lambda(510000 - 1250X - 2000Y - 100Z). \tag{22}$$

Partial derivatives of L with respect to X, Y, Z, and λ were then calculated, and subsequently set to zero. From these conditions, the following was obtained:

$$\frac{\partial L}{\partial X} = 0,$$

For

$$131.5858X^{-0.5362}Y^{0.2741}Z^{0.1704} + \lambda(-1250) = 0,$$

where, λ was isolated as:

$$\lambda = \frac{131.5858Y^{0.2741}Z^{0.1704}}{1250X^{0.5362}}. \tag{23}$$

Similarly, for

$$\frac{\partial L}{\partial Y} = 0,$$

This culminated in:

$$77.7655X^{0.4638}Y^{-0.7259}Z^{0.1704} + \lambda(-2000) = 0,$$

where, λ was isolated as:

$$\lambda = \frac{77.7655X^{0.4638}Z^{0.1704}}{2000Y^{0.7259}}. \tag{24}$$

For

$$\frac{\partial L}{\partial Z} = 0,$$

Upon execution of partial differentiation, the following was procured:

$$48.3445X^{0.4638}Y^{0.2741}Z^{-0.8296} + \lambda(-100),$$

where, the value of λ was elucidated as:

$$\lambda = \frac{48.3445X^{0.4638}Y^{0.2741}}{100Z^{0.8296}}. \tag{25}$$

Similarly, for

$$\frac{\partial L}{\partial \lambda} = 0,$$

This yielded:

$$510000 - 1250X - 2000Y - 100Z = 0. \tag{26}$$

Eqs. (23) and (24) were juxtaposed, from which the following relation was inferred:

$$\frac{131.5858Y^{0.2741}Z^{0.1704}}{1250X^{0.5362}} = \frac{77.7655X^{0.4638}Z^{0.1704}}{2000Y^{0.7259}}.$$

This consequently led to:

$$X = 2.7073Y. \tag{27}$$

In a similar vein, a comparison between Eqs. (24) and (25) indicated:

$$\frac{77.7655X^{0.4638}Z^{0.1704}}{2000Y^{0.7259}} = \frac{48.3445X^{0.4638}Y^{0.2741}}{100Z^{0.8296}}.$$

This consequently led to:

$$Z = 12.4334Y. \tag{28}$$

By integrating Eqs. (27) and (28) into Eq. (26), it was deduced:

$$510000 - 1250(2.7073Y) - 2000Y - 100(12.4334Y) = 0.$$

This culminated in:

$$6627.465Y = 510000.$$

Thus,

$$Y = 76.9524. \tag{29}$$

Incorporating Eq. (29) into Eqs. (27) and (28) respectively, the values were ascertained as:

$$\begin{aligned} X &= 208.3332 \\ Z &= 956.7799 \end{aligned} \tag{30}$$

For the culmination of this methodology, Figure 2 was produced, juxtaposing both the estimated and the actual production in the water industry throughout the year 2020.

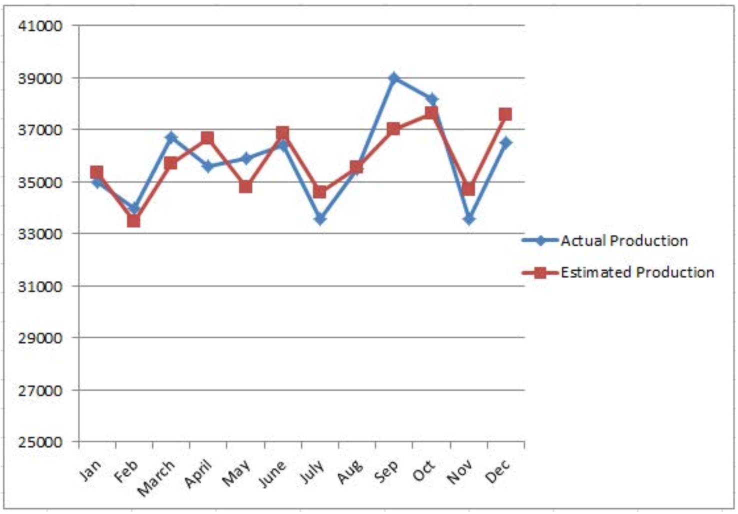

Figure 2. Estimated versus actual production of the water industry for three inputs in 2020

4 Conclusion

From the comprehensive analyses undertaken, it has been deduced that Cobb-Douglas production functions remain instrumental in addressing econometric challenges. Initially, the predicament was approached via the two-input Cobb-Douglas production function, subsequently extending to its three-input counterpart. Maximization of the production value was achieved in both scenarios.

Clear disparities in production values were observed in the presented figures. Such disparities arise, primarily because, in the two-input model, the third parameter is either assimilated as an average across the remaining parameters or perceived as a constant overhead. Conversely, in the three-input variant, all production factors are orchestrated in an optimally refined manner.

The empirical evidence suggests that the three-input Cobb-Douglas production function manifests enhanced efficiency and efficacy compared to its two-input analogue.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] G. B. Dantzig, “The nature of mathematical programming,” Mathematical Programming Glossary, INFORMS Computing Society, archived 2014-03-05 at the Wayback Machine. https://glossary.informs.org/second.php?page=nature.html

[2] D. Z. Du, P. M. Pardalos, and W. Wu, “History of optimization,” in *Encyclopedia of Optimization*, C. Floudas and P. Pardalos, Eds. Springer, Boston, MA., 2008, pp. 1538–1542. https://doi.org/10.1007/978-0-387-74759-0_268

[3] J. A. Snyman and D. N. Wilke, *Practical Mathematical Optimization*. Springer International Publishing, 2018. https://doi.org/10.1007/978-3-319-77586-9

[4] J. Nocedal and J. S. Wright, *Numerical Optimization*. Springer New York, NY, 2006. https://doi.org/10.1007/978-0-387-40065-5

[5] J. E. Dennis Jr and R. B. Schnabel, *Numerical Methods for Unconstrained Optimization and Nonlinear Equations*. Society for Industrial and Applied Mathematics, 1996.

[6] P. C. Pendharkar and A. R. James, “Nonlinear programming and genetic search application for production scheduling in coal mines,” *Ann. Oper. Res.*, vol. 95, pp. 251–267, 2000. https://doi.org/10.1023/a:1018958209290

[7] D. P. Bertsekas, “Nonlinear Programming,” *J. Oper. Res. Soc.*, vol. 48, no. 3, pp. 334–334, 1997. https://doi.org/10.1057/palgrave.jors.2600425

[8] D. G. Luenberger and Y. Ye, *Linear and Nonlinear Programming*. Springer International Publishing, 2021. https://doi.org/10.1007/978-3-030-85450-8

[9] C. W. Cobb and P. H. Douglas, “A theory of production. American Economic Review,” in *Papers and Proceedings of the Fortieth Annual Meeting of the American Economic Association*, 1928, pp. 139–165.

[10] R. J. Barro and X. Sala-i Martin, *Economic Growth (Second Ed.)*. The MIT Press, Cambridge, Massachusetts London, England, 2004.

[11] X. Wang and Y. Fu, “Some characterizations of the Cobb-Douglas and CES production functions in microeconomics,” *Abstr. Appl. Anal.*, vol. 2013, 2013. https://doi.org/10.1155/2013/761832

[12] M. Hossain, T. Basak, and A. K. Majumder, “Application of non-linear Cobb-Douglas production function with autocorrelation problem to selected manufacturing industries in Bangladesh,” *OJS*, vol. 3, no. 3, pp. 173–178, 2013. https://doi.org/10.4236/ojs.2013.33019

[13] K. V. Bhanu Murthy, “Arguing a case for Cobb-Douglas production function,” *Rev. Commer. Stud.*, vol. 20-21, no. 1, 2002.

[14] E. Franik and T. Franik, *Application of NonLinear Programming for Optimization of Factors of Production in Mining Industry*. Alcon Pharmacetutical Ltd anovartis Company. Fribourg, 2009.

[15] T. Franik, “Productivity of hard coal mining during the reform period against the background of changes in the mining and quarrying section,” *Econ. Miner. Mater.*, vol. 21, no. 3, pp. 47–61, 2005.

[16] G. E. Vîlcu, “A geometric perspective on the generalized Cobb–Douglas production functions,” *Appl. Math. Lett.*, vol. 24, no. 5, pp. 777–783, 2011. https://doi.org/10.1016/j.aml.2010.12.038