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Buckling Characteristics of Exponentially Graded Cylindrical Shells with Clamped Edges Supported by Elastic Medium

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Abstract: The new generation composite materials are widely used in Engineering due to their light weight, high strength, and resistance to corrosion and wear. Two main modeling strategies, the piecewise layered approach and the continuously graded approach were employed in the literature, with the latter offering a more realistic representation. Recent studies have highlighted the importance of analyzing the stability and vibration behavior of exponentially graded cylindrical shells, particularly when embedded in elastic media. Nevertheless, most works were limited to simply supporting boundary conditions and so neglected the foundation effects. To fill this notable gap in the literature, the present study focused on the buckling behavior of exponentially graded cylindrical shells (EGCSs) with clamped edges under external pressure within an elastic medium. A theoretical framework was then established for future design applications in advanced Engineering fields.

Keywords: Exponentially graded material; Boundary conditions; Cylindrical shells; Stability; Critical lateral pressure; Elastic foundation

1 Introduction

Composite materials have consistently attracted extensive scholarly attention due to their superior characteristics, such as low density, high strength, and remarkable resistance to corrosion and wear. These advantageous properties have facilitated their broad utilization across diverse Engineering disciplines. Nevertheless, a major limitation of conventional composites arises from the abrupt discontinuities in mechanical properties at the interfaces of their constituents, which may initiate cracks and foster their propagation. In response to this challenge, a new class of advanced materials, termed Functionally Graded Materials (FGMs), was pioneered by Japanese researchers in 1984 within the scope of a space exploration project. Unlike traditional composites, FGMs are characterized by a smooth and continuous variation of physical, chemical, and mechanical properties including elastic modulus, Poisson's ratio, and density across the material volume [1].

Two principal modeling strategies were predominantly adopted to represent FGMs in the scientific literature. The first approach regards the ceramic-to-metal volume fraction as a piecewise continuous function. In this framework, the structure is discretized into a series of sub-layers, each exhibiting constant phase fractions, thereby approximating the material as semi-homogeneous layers of ceramic and metal. Such approximation preserves uniform properties within each layer while ensuring a gradual transition across the thickness, thus providing a simplified yet efficient modeling tool for Engineering analysis. Conversely, the second approach assumes a continuously varying distribution of constituents through thickness, thus offering a more accurate reflection of material heterogeneity. In this context, the ceramic phase volume fraction is expressed as a smooth function of the thickness coordinate, enabling a seamless gradation profile and eliminating artificial layer discretization [2, 3].

The unique advantages of FGMs have stimulated a growing body of research into their structural behavior, particularly under diverse boundary conditions, loading scenarios, and foundation effects [4]. Among these, considerable emphasis has been placed on the stability and vibration analyses of FGM cylindrical shells [5–14]. In the majority of existing studies, the material properties of structural elements are assumed to vary spatially according to a power-law function. However, the number of investigations focusing on the mechanical behavior of structural elements whose material properties vary exponentially remained relatively limited [15–17].

However, most existing contributions were restricted to simply supporting configurations and neglecting the influence of elastic foundations, with analyses largely confined to analytical and numerical methods. In practical applications, FGM cylindrical shells are frequently embedded in elastic media, where the surrounding foundation exerts a significant influence on their mechanical performance. The Pasternak two-parameter elastic foundation model is particularly favored in this regard, as it simultaneously incorporates the effects of shear interaction and normal stiffness [18–20].

In recent years, extensive research has focused on investigating the mechanical behavior of FGM composite shell structures with various geometries and material configurations resting on elastic foundations. These studies commonly employed Winkler, Pasternak, or extended foundation models to analyze in detail the influence of foundation stiffness on buckling, bending, and vibration responses. The results indicated that increasing the foundation stiffness parameters generally enhanced structural stability and natural frequencies, while the distinction between linear and nonlinear foundation models led to noticeable differences in the overall response. Moreover, factors such as thermal effects, porosity, geometric imperfections, and material gradation, when examined together with the foundation–structure interaction, play a crucial role in understanding the complex mechanical behavior of such systems [21–29].

In the existing body of literature, the material properties of structural elements are predominantly assumed to vary spatially according to a power-law distribution. This functional gradation approach has been widely adopted in the analysis of FGMs and has been extensively studied under various geometrical configurations, boundary conditions, and loading types. However, comparatively fewer studies focused on the structural behavior such as static response, vibration characteristics, and buckling stability of elements whose material properties vary exponentially through the thickness or along the geometry. Despite the potential advantages of exponential gradation in practical engineering applications, its mechanical implications remain underexplored. Therefore, investigating the mechanical response of such structures represents a significant and promising research direction, thus contributing to a comprehensive understanding and optimization of functionally graded systems.

In this study, cylindrical shells composed of exponentially graded materials (EGMs) have been considered to address the research gap and contribute to the growing body of knowledge in the field. In addition, it provides a theoretical modeling framework that could serve as a foundation for the development of design criteria applicable to various Engineering disciplines.

2 Theoretical Development

An exponentially graded cylindrical shell (EGCS) with thickness , radius , and length is considered to be resting on a Pasternak elastic foundation (Figure 1). The Cartesian coordinate system is denoted by $Oxyz$, where the origin is located at the left end of the mid-surface of the cylindrical shell. The Ox and Oy axes are aligned with the longitudinal and circumferential directions, respectively, while the Oz axis is oriented perpendicular to the surface defined by the Ox and Oy directions and points inward toward the center of the shell. According to the selected coordinate system, the EG cylindrical shell is defined as a three-dimensional domain Ω, as follows:

$$\Omega = \{x, y, z : (x, y, z) \in [0, a] \cdot [0, 2\pi r] \cdot [-0.5t, +0.5t]\} \tag{1}$$

The Pasternak foundation model employed in this study is analytically defined as follows [18]:

$$R = k_1 w - k_2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \tag{2}$$

In this equation, R represents the reactive force per unit area exerted by the Pasternak-Winkler type elastic foundation. The coefficient k_1 (N/m^3) denotes the vertical subgrade reaction modulus of the elastic foundation, while k_2 (N/m) is the shear modulus of an intermediate layer that is negligibly thin and allows shear deformation. The term refers to the transverse displacement component, which is small relative to the shell thickness and is measured perpendicular to the reference surface. This model characterizes the elastic foundation effect by accounting for both vertical stiffness (Winkler contribution) and shear layer (Pasternak contribution).

In this study, the ceramic volume fraction is mathematically defined as $V_c = \left(\frac{2z+t}{2t}\right)^N$, where $N(0 \le N \le \infty)$ denotes the grading index that governs the material transition. This functional representation enables a continuous variation of material properties from ceramic to metal or vice versa. Such an approach allows a more accurate modeling of material behavior, particularly under thermal and mechanical loading conditions, thus leading to more precise analytical results.

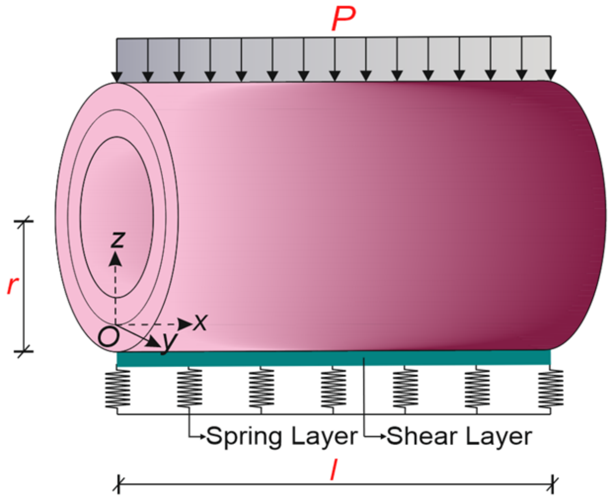


Figure 1. Schematic representation of an EGCS resting on a pasternak foundation under external lateral loading and the corresponding coordinate axes

FGMs are typically modeled using power-law or exponential distribution functions. In this study, cylindrical shells composed of exponentially graded materials were considered. The effective elastic properties of the EGCS, namely the effective Young's modulus and Poisson's ratio, are expressed as follows [30]:

$$E_Z^{EG} = Y_m e^{V_c \ln\left(\frac{Y_c}{Y_m}\right)}, \quad v_Z^{EG} = v_m e^{V_c \ln\left(\frac{Y_c}{Y_m}\right)} \tag{3}$$

It is assumed that the effective Young's modulus E_Z^{EG} and Poisson's ratio v_Z^{EG} vary nonlinearly with temperature, hence reflecting the temperature-dependent behavior of the material [30]:

$$\mathrm{p}_i = p_0\left(p_{-1} + T + p_1 T^2 + p_2 T^3 + p_3 T^4\right) T^{-1} \tag{4}$$

The coefficients $p_j (j = -1, 0, 1, 2, 3)$ are unique to each material and depend on temperature T, measured in Kelvin. Using the mechanical properties of the ceramic constituents (Si_3 N_4 and ZrO_2) and the metallic constituents (SUS304 and Ti-6Al-4V), as provided by Miyamoto et al. [2] and Sofiyev and Avey [29], along with Eq. (4), the mechanical properties of the functionally graded materials (FGMs) formed from their mixtures are determined.

In cylindrical shells composed of FGMs obtained by mixing silicon nitride and stainless steel (Ti-6Al-4V), the ratio of the effective Young's modulus of the exponentially graded material to that of the metallic phase, denoted as E_Z^{EG}/E_m , is assumed to vary exponentially. Using Eq. (4), this ratio is computed over the range $-1/2 \leq Z \leq 1/2$ and $0 \leq X \leq 1$, and the resulting values of $-1/2 \leq Z \leq 1/2$ and $0 \leq X \leq 1$ are used to generate 3D plot for N = 1, as presented in Figure 2.

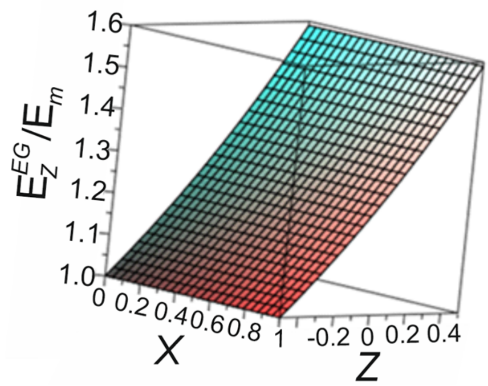


Figure 2. 3D distribution of normalized effective Young's modulus of the E_Z^{EG}/E_m as a function of the thickness coordinate

3 Basic Equations

The fundamental relationships for EGCSs can be expressed within the framework of the Kirchhoff–Love shell theory as follows [30]:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} Q_{11}^{EG} & Q_{12}^{EG} & 0 \\ Q_{12}^{EG} & Q_{11}^{EG} & 0 \\ 0 & 0 & Q_{66}^{EG} \end{bmatrix} \begin{bmatrix} e_{11} - zw,_{xx} \\ e_{22} - zw,_{yy} \\ e_{12} - 2zw,_{xy} \end{bmatrix} \tag{5}$$

where, $\sigma_{ij}(i,j=1,2)$ represent the stress components, $e_{ij}(\mathrm{i,j}=1,2)$ denote the strain components on the mid-surface of the cylindrical shell composed of EGMs, and $Q_{ij}^{EG}(i,j=1,2,,6)$ are the material-dependent stiffness coefficients determined by the properties of the EGCS. These quantities are defined by the following expressions:

$$Q_{11Z}^{EG} = \frac{E_Z^{EG}}{1-\left(v_Z^{EG}\right)^2}, Q_{12Z}^{EG} = \frac{v_Z^{EG} E_Z^{EG}}{1-\left(v_Z^{EG}\right)^2}, Q_{66Z}^{EG} = \frac{E_Z^{EG}}{2\left(1+v_Z^{EG}\right)} \tag{6}$$

The expressions of the force and moment components in terms of the stress components are formulated through the following integrzals [30–33]:

$$[N_{ij}, M_{ij}] = \int_{-t/2}^{t/2} \sigma_{ij}[1,z]\mathrm{d}z \quad (i,j=1,2) \tag{7}$$

The expression of the force components along different directions through the Airy stress function is formulated as follows [30–33]:

$$[N_{11}, N_{22}, N_{12}] = t\left[\Phi,_{yy}, \Phi,_{xx}, -\Phi,_{xy}\right] \tag{8}$$

Under the assumption that the EGCS is exposed to a uniformly distributed lateral pressure, the mathematical formulations of the membrane force components, which are independent of the first moment on the reference surface, are presented as follows:

$$N_{11}^0 = 0, N_{12}^0 = 0, \quad N_{22}^0 = -r \cdot P \tag{9}$$

By simultaneously employing Eq. (2), Eq. (6), and Eqs. (7)–(9), the final form of the governing equations describing the behavior of EGCSs resting on a Pasternak elastic foundation under uniform lateral pressure is obtained. This model aims to provide a comprehensive description of the shell's mechanical response by accounting for both the effects of the elastic foundation and the material heterogeneity:

$$\begin{aligned} &P_1\Phi,_{xxxx} + 2\left(P_2 - P_5\right)\Phi,_{xxyy} + P_1\Phi,_{xyy} + \Phi,_{xx} r^{-1} - P_3 w,_{xxxx} \\ &- 2\left(P_4 + S_6\right) w,_{xxyy} - P_3 w,_{xyy} + w,_{xx} r^{-1} - r \cdot P w_{xy} - k_1 w + k_2\left(w,_{xx} + w,_{xy}\right) = 0 \end{aligned} \tag{10}$$

$$Q_1\Phi,_{xxxx} + 2\left(Q_2 + Q_5\right)\Phi,_{xxyy} + Q_1\Phi,_{yyyy} - Q_4 w,_{xxxx} - 2\left(Q_3 - Q_6\right) w,_{xxyy} - Q_4 w,_{yyy} + w,_{xx} r^{-1} = 0 \tag{11}$$

In this context, the coefficients are parameters determined by the material gradation characteristics of the EG material and the geometric and mechanical properties of the cylindrical shell.

4 Solutions

At both longitudinal ends of the EGCS, clamped boundary conditions are assumed to be valid. Within the framework of linear elasticity theory, these boundary conditions are defined by the following analytical expressions [3, 9]:

$$w = w,_x = 0 \tag{12}$$

These types of boundary conditions can be fulfilled through the use of appropriate shape functions defined for the displacement field and the Airy stress function.

The following approximating functions, corresponding to the deflection and the Airy stress function, are constructed to satisfy the clamped boundary conditions [9]:

$$w = A_1 \sin^2\left(\beta_1 x\right)\cos\left(\beta_2 y\right), \Phi = A_2 \operatorname{in}^2\left(\beta_1 x\right)\cos\left(\beta_2 y\right) \tag{13}$$

where, $\beta_1 = m\pi l^{-1}$ and $\beta_2 = nyr^{-1}$ are parameters, while and denote the longitudinal and circumferential wave numbers, respectively. The coefficients $A_j(j=1,2)$ represent the unknown amplitudes associated with the deflection and the Airy stress function.

For EGCSs resting on a Pasternak foundation and subject to lateral pressure, the strain compatibility and stability equations defined by Eq. (10) and Eq. (11) are multiplied by a weight function over the relevant region $\Lambda = \{(x, y) : 0 \le x \le l, 0 \le y \le 2\pi r\}$, and the Galerkin procedure is applied. As a result of this process, the following analytical expression is obtained for the critical lateral pressure of the EGCSs with clamped-clamped edges:

$$\begin{aligned} P_{wp}^{cr} &= \frac{1}{3\alpha_2^2}\left\{16Q_3\beta_1^4 + 8\beta_1^2\beta_2^2\left(Q_4+Q_6\right) + 3Q_3\beta_2^2 - \right. \\ &- \left[16Q_2\beta_1^4 + 8\beta_1^2\beta_2^2\left(Q_1-Q_5\right) + 3Q_2\beta_2^4 - 4\beta_1^2 r^{-1}\right] \\ &\times \left. \frac{16P_4\beta_1^4 + 4\beta_1^2\beta_2^2\left(2P_3-P_6\right) + 3P_4\beta^4 + 4\beta_1^2 r^{-1}}{16P_1\beta_1^4 + 8\beta_1^2\beta_2^2\left(2P_2+P_5\right) + 3P_1\beta_2^4} + 3k_1 + k_2\left(4\beta_1^2 + 3\beta_2^2\right)\right\} = 0 \end{aligned} \tag{14}$$

The expression given below is employed for evaluating the non-dimensional critical lateral pressure (NCLP) acting on EGCSs with both ends clamped and supported by a Pasternak elastic foundation:

$$P_{1L_{wp}}^{cr} = P_{L_{wp}}^{cr} E_c^{-1} \tag{15}$$

Eq. (15) yields a particular case representing the critically lateral pressure for EGCSs without foundation support when the foundation parameters and are set to zero.

Minimization of Eq. (15) with respect to the mode pairs (m, n) yields the minimum values of the NCLP. For the numerical computation of these values, the Maple computational software was utilized.

5 Numerical Results

In the numerical investigation, two distinct types of functionally graded cylindrical shells, characterized by different exponential gradation profiles designated as EG_1 and EG_2 were analyzed. These exponentially graded configurations consisted of a continuous variation between metallic and ceramic constituents to achieve the desired mechanical performance. In particular, the EG_1 shell is composed of a combination of silicon nitride (Si_3N_4) and stainless steel (SUS304), whereas the EG_2 shell is formed from a mixture of titanium alloy (Ti–6Al–4V) and zirconium dioxide (ZrO_2). Such material gradation allows a smooth transition of mechanical properties across the shell thickness, thus effectively enhancing the structural efficiency under complex loading and thermal environments. The corresponding mechanical and physical properties of the constituent materials were adopted from the reference work of Shen [29]. The geometric parameters of the cylindrical shell are taken as follows: $r/t = 100$ ve $l/r = 2$.

Figure 3 demonstrates the variation of the NCLP as a function of the material grading index N for cylindrical shells with EG_1 and EG_2 profiles with and without elastic foundation. The geometric parameters are taken as $r/t = 100$ ve $l/r = 2$, while the foundation stiffness coefficients are selected as $k_1 = k_2 = 0, k_1 = 3 \times 10^6$, $k_2 = 0$, and $k_1 = 3 \times 10^6$, $k_2 = 5 \times 10^4$. It is observed that increasing the value of N leads to a decrease in NCLP values for both EG_1 and EG_2 profiles. When comparing the EG_1 and EG_2 profiles, the minimum NCELP values occur at N = 3, while the maximum values are obtained at N = 0.5.

It is seen that in the no-soil condition ($k_1 = k_2 = 0$), the EG_1 and EG_2 values are low. EG_2 is always lower than EG_1, thus indicating that the second mode is weaker. When the cylindrical shell is on a Winkler foundation ($k_1 = 3 \times 10^6, k_2 = 0$), the EG_1 and EG_2 values increase compared to the previous case.

The stiffness increases alongside an increase in the critical load, thus improving the stability of the system. The increase for EG_2 is more pronounced than for EG_1.

When the cylindrical shell is on a Winkler foundation ($k_1 = 3 \times 10^6, k_2 = 5 \times 10^4$), the highest NCLP values are obtained. The increase is particularly pronounced in mode EG_2. This demonstrates that the overall stability of the system is significantly increased by the addition of the upper link. In short, as stiffness increases, the system becomes more durable. The EG_2 cylindrical shell is more critical in all cases and is more affected by the increased stiffness.

Figure 4 and Figure 5 are comparative representations of NCLP of (a) ZrO_2 (b) EG_2(N = 1) cylindrical shells with various soil conditions versus r/t at $l/r = 2$. For all soil conditions and material types, increasing the r/t ratio reduces NCLP.

The NCLP value of the EG_2 (N = 1) cylindrical shell is approximately 15% lower than that of the shell, while the decrease reaches 24–25% for the EG_2 (N = 2) cylindrical shell. This indicates that as N increases, the effect of EG_2 on NCLP increases, while the NCLP value decreases.

The NCLP value for the EG_2 (N = 1) cylindrical shell is approximately 34% higher than that of Ti-6Al-4V, while this difference decreases to around 19–20% for EG_2 (N = 2). This suggests that the EG_2 cylindrical shell offers superior mechanical performance, particularly for N = 1, but this effect weakens at N = 2.

At large r/t values, thin shells are more unstable, and the soil stiffness and EGM effect become more dominant. In this case, ground support creates a greater difference, especially r/t at low ratios. To reduce ground effects through material replacement, a material that minimizes ground effects should be selected, especially for thin shells.

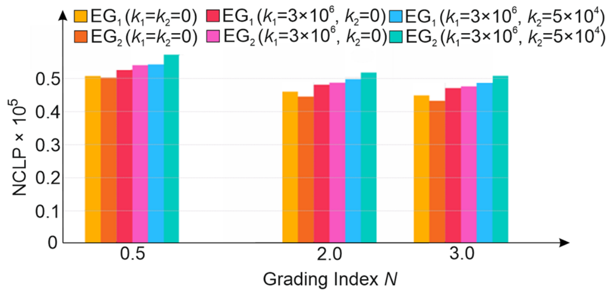
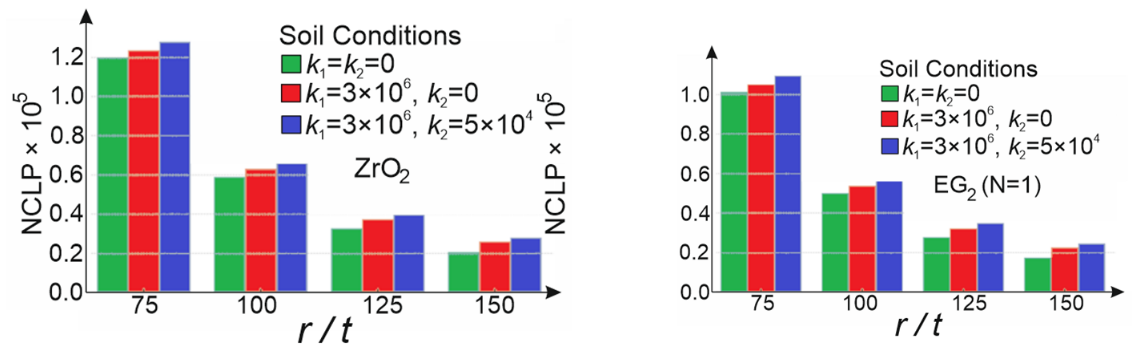


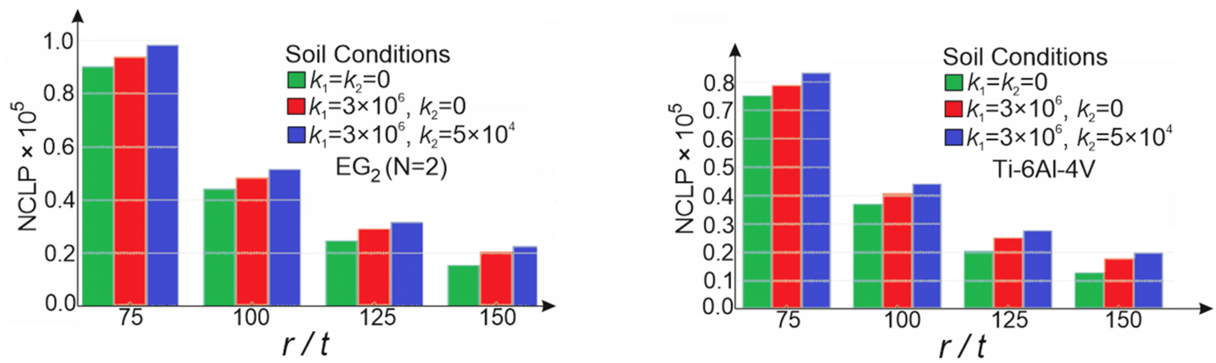
Figure 3. Variation of the NCLP versus the material grading index N for cylindrical shells with EG_1 and EG_2 profiles with and without elastic foundation



(a)

(b)

Figure 4. Comparative representation of NCLP (a) ZrO_2; (b) EG_2 (N = 1) cylindrical shells with various soil conditions versus r/t.



(a)

(b)

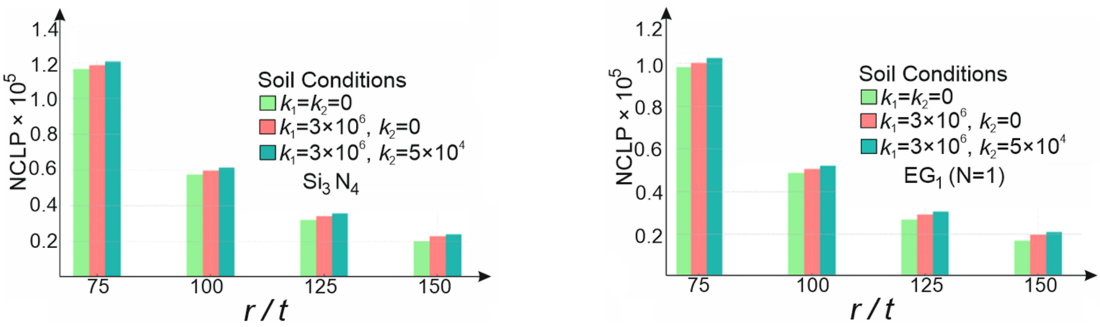
Figure 5. Comparative representation of NCLP (a) EG_2 (N = 2); (b) Ti-6Al-4V cylindrical Shells with various soil conditions versus r/t

As ground hardness (k_1, k_2) increases, NCLP increases for all materials, with the highest values achieved at $(k_1, k_2) = \left(3 \times 10^6, 5 \times 10^4\right)$.

Figure 6 and Figure 7 are plotted comparative representations of NCLP of (a) Si_3N_4 (b) EG_1 (N = 1) cylindrical shells with various soil conditions versus r/t at $l/r = 2$.

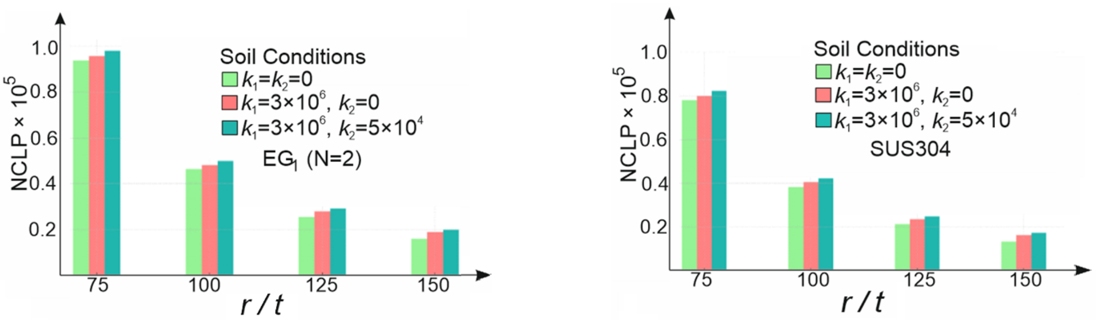
Considering the Si_3N_4 and EG_1 cylindrical shells presented in Figure 6 and Figure 7, increasing the r/t ratio, i.e., thinning the shell, reduces the NCLP values across all soil conditions and material types. This confirmed, as in previous analyses that the stiffness of the shells decreased with thinning while resistance to critical lateral pressure decreased.

The EG_1 (N = 1) shell had an average 14.9% lower NCLP value than the Si_3N_4. This indicated that the lateral stability of the structure decreased as material heterogeneity increased. This difference became even larger in the EG_1 (N = 2) shell, reaching an average of 18.7%. In other words, as the degree of heterogeneity (N) increased, the weakening effect on NCLP increased. The NCLP value of the EG_1 (N = 1) shell was 24.1% higher on average than that of SUS304. This demonstrated that EG_1 offered superior mechanical performance compared to homogeneous metal. For N = 2, this difference decreased to 18.7%, indicating that the mechanical advantage decreased as the degree of heterogeneity increased. Just as in the EG_2 analysis, NCLP values increased for all materials as the soil stiffness parameters and increased, with the highest values being obtained $(k_1, k_2) = \left(3 \times 10^6, 5 \times 10^4\right)$.



(a) (b)

Figure 6. Comparative representation of NCLP (a) Si_3 N_4; (b)EG_1 (N = 1) cylindrical shells with various soil conditions versus r/t



(a) (b)

Figure 7. Comparative representation of NCLP (a) EG_1 (N = 2); (b) Ti-6Al-4V cylindrical shells with various soil conditions versus r/t

For both material groups (EG_1 and EG_2), NCLP values decreased as heterogeneity increased (as N increases), and structural stability weakened as the ratio increased. EG_2 was observed to be more sensitive to increasing heterogeneity (24–25% decrease), while EG_1 exhibited a more moderate decrease (18.7%). Furthermore, EG_2 offered a greater advantage, especially at N = 1, while EG_1 exhibited a more balanced performance (see Figure 4, Figure 5, Figure 6 and Figure 7).

6 Conclusions

In this study, the stability problem of EGM cylindrical shells subject to external lateral pressure was investigated within the framework of the Donnell-type shell theory on an elastic foundation. Initially, the effects of material gradation, elastic foundation, and external lateral pressure were mathematically modeled. Subsequently, considering the two-parameter elastic foundation, the stability and compatibility equations were derived based on the Kirchhoff-Love shell theory. Appropriate admissible functions satisfying the clamped boundary conditions were selected, and analytical solutions were obtained using the Galerkin method. From the resulting algebraic expressions, a dimensionless analytical formula for the DCLP was derived. Particular cases of these expressions were obtained in the absence of the elastic foundation effect. Finally, a series of numerical analyses were conducted by varying the volume fraction profiles, foundation parameters, and geometric properties.

The analysis in this study led to several generalized observations:

The EG_2 profile demonstrated superior performance, particularly at high elastic foundation coefficients. While the EG_1 profile exhibited a more rapid decline in cases where the elastic foundation was absent or weak, it could reduce the difference with foundation support. For applications requiring high stability in the design, the EG_2 profile and ceramic-based structures are preferred, while the EG_1 profile is preferred in applications requiring low weight and moderate stability.

These findings indicated that the selection of appropriate material gradation and foundation parameters was crucial for the design of next generation structural elements, to guarantee an optimal balance among mechanical performance, weight efficiency, and application-specific demands.

Author Contributions

Conceptualization, A.H.S. and E.S.; A.H.S. and A.A.M.; software, A.H.S.; validation, A.H.S., A.A.M. and E.S.; formal analysis, A.H.S.; investigation, A.H.S.; resources, A.H.S. and A.A.M.; data curation, A.A.M.; writing—original draft preparation, A.H.S.; writing—review and editing, A.H.S. and E.S.; visualization, A.A.M.; project administration, A.A.M. All authors have read and agreed to the published version of the manuscript.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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