Optimizing SMS with a Common Due Date in an Uncertain Environment: A Fuzzy Logic Approach

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Abstract: This investigation explores the scheduling of n jobs on a single machine, where each job possesses a common due date, and processing time is characterized by pentagonal fuzzy numbers (PFNs). The primary objective is to minimize the aggregate of inventory holding and penalty costs, addressing the critical impact of earliness and tardiness on profitability. It is identified that earliness leads to increased inventory carrying costs and potential degradation in product quality, whereas tardiness undermines customer goodwill and inflicts reputational damage through delayed payments. Consequently, the scheduling dilemma that seeks to minimize the combined penalties of earliness and tardiness, whilst adhering to a common due date on a single machine, emerges as a pivotal and challenging endeavor in optimizing goods delivery within production settings. Recognized as a non-deterministic polynomial-time hardness (NP-hard) problem, this task underscores the complexity and competitive nature inherent in manufacturing operations. To navigate the uncertainties embedded in this problem, a fuzzy logic approach, augmented by a heuristic algorithm, is employed. Through this methodology, the problem is addressed in a manner that encapsulates the vagueness and imprecision inherent in processing time, thereby facilitating more resilient and adaptable scheduling decisions. The efficacy of this approach is demonstrated via a computational example, underscoring its potential to enhance decision-making in the realm of job scheduling.

Keywords: Optimization; Job scheduling problem; Common due date; Earliness; Tardiness; Processing time; Pentagonal fuzzy numbers; Closed interval approximation; Decision making

1 Introduction

Single-machine scheduling (SMS) problems require complex computations; analyzing such problems is vital for better understanding them. Among SMS problems, those related to earliness and tardiness are more important. Completing jobs or tasks earlier than their due dates should be discouraged as much as completing jobs or tasks later than their due dates. In the real world, since customer experts receive the product on a specific date, scheduling based on the due date is also an essential task in production planning. Earliness leads to inventory and maintenance carrying costs, while tardiness leads to customer dissatisfaction and loss of goodwill and reputation [1]. Many authors addressed the SMS problem using heuristic and metaheuristic approaches [2–4]. Yazdani et al. [5] studied a scheduling problem with multiple unavailability periods and distinct due dates to minimize the maximum earliness and tardiness of the jobs. Touat et al. [6] studied a new scheduling problem considering both production and flexible preventive maintenance on a single machine with human resources as constraints. They proposed a mathematical formulation of the studied problem expressed in constrained programming as a set of linear constraints. Komaki et al. [7] conducted a joint survey of their templates for assembly flow shops.

Zadeh [8] first proposed the philosophy of fuzzy sets. Zimmermann [9] introduced fuzzy optimization and linear programming with multiple objective functions. Later, several researchers worked on fuzzy set theory [10–16]. Dubois [17] studied the theory and applications of fuzzy sets and systems. Kaufmann and Gupta [18] reviewed

several fuzzy mathematical models and their applications to engineering and management sciences. Xie et al. [19] investigated a SMS problem with fuzzy due dates and precedence constraints. Mahavi Mazdeh et al. [20] developed a model for a single-machine bi-criteria scheduling problem aiming to minimize the total delay and work-in-process cost. Chanas and Kasperski [21] investigated two approaches to the SMS problem: fuzzy processing time and fuzzy due dates. Xie et al. [22] introduced a single-machine problem of (n+1) jobs involving a particular fuzzy time delay structure and developed optimal solutions for the scheduling model in several cases. Khalifa [23] considered an SMS problem involving fuzzy due dates to aim to minimize the total penalty cost. El-Wahed Khalifa et al. [24] applied a closed interval approximation for piecewise quadratic fuzzy numbers to solve the constrained multistage machine flow-shop scheduling problem based on the fuzzy bending approach. Bicakci et al. [25] examined the relationship between setup time and release dates, considering the overall scheduling literature, and also proposed a new mixed-integer linear programming formulation with $O(n^2)$ binary decision variables and $O(n^2)$ constraints for obtaining the optimal solution. Alharbi and El-Wahed Khalifa [26] introduced a flow shop scheduling problem with a pentagonal processing time of jobs to minimize the rental cost of the machine in compliance with the rental policy. Recently, an enormous number of researchers studied scheduling problems in different environments [27–32].

In this study, the n-job SMS problem with fuzzy pentagonal common due date and processing time for minimizing the sum of total inventory and penalty costs, whereas earliness and tardiness are harmful to profitability, is considered. A fuzzy approach with the help of a heuristic approach is proposed for solving the problem.

This study is organized in the following sections: Section 2 introduces some necessary preliminary information. Section 3 formulates n jobs to be processed on a single machine. A heuristic approach is developed for solving the problem under study in Section 4. Section 5 introduces an example to illustrate the proposed approach. Finally, some concluding remarks are reported in Section 6.

2 Preliminaries

To easily discuss the problem, basic rules and findings related to fuzzy numbers are recalled, such as PFNs, arithmetic operations of PFNs, closed interval approximation, and its ranking function.

Definition 1. Proposed by Zadeh [8], a fuzzy set $\widetilde{A}$ is a set of real numbers. R is called fuzzy if its membership function $\mu_{\widetilde{A}}(x) : \mathbb{R} \to [0, 1]$. They have the following properties:
(i) $\mu_{\widetilde{A}}(x)$ is an upper semi-continuous membership function;
(ii) $\widetilde{A}$ is a convex fuzzy set, i.e., $\mu_{\widetilde{A}}(\delta x + (1 - \delta)y) \geq \min \left\{ \mu_{\widetilde{A}}(x), \mu_{\widetilde{A}}(y) \right\}$ for all $x, y \in \mathbb{R}$, and $0 \leq \delta \leq 1$;
(iii) $\widetilde{A}$ is normal, i.e., $\exists x_0 \in \mathbb{R}$, for which $\mu_{\widetilde{A}}(x_0) = 1$;
(iv) $\mathrm{Supp}(\widetilde{A}) = \left\{ x \in \mathbb{R} : \mu_{\widetilde{A}}(x) > 0 \right\}$ is the support of $\widetilde{A}$, and the closure $\mathrm{cl}(\mathrm{Supp}(\widetilde{A}))$ is a compact set.

Definition 2. Proposed by Abbasbandy and Hajiari [33], a fuzzy number $\tilde{A}_P = (a_1, a_2, a_3, a_4, a_5), a_1 \leq a_2 \leq a_3 \leq a_4 \leq a_5$ on $\mathbb{R}$ is said to be a PFN if its membership function is as follows:

$$
\mu_{\widetilde{A}_P} = \begin{cases}
0, & x < a_1, \\
w_1 \left(\frac{x - a_1}{a_2 - a_1} \right), & \text{for } a_1 \leq x \leq a_2, \\
1 - (1 - w_1) \left(\frac{x - a_2}{a_3 - a_2} \right), & \text{for } a_2 \leq x \leq a_3, \\
1, & \text{for } x = a_3, \\
1 - (1 - w_2) \left(\frac{a_4 - x}{a_4 - a_3} \right), & \text{for } a_3 \leq x \leq a_4, \\
w_2 \left(\frac{a_5 - x}{a_5 - a_4} \right), & \text{for } a_4 \leq x \leq a_5, \\
0, & \text{for } x > a_5.
\end{cases}
$$

Figure 1 shows the graphical representation of the PFN $\widetilde{A}_P$.

Definition 3. Let $\widetilde{A}_P = (a_1, a_2, a_3, a_4, a_5)$, and $\widetilde{B}_P = (b_1, b_2, b_3, b_4, b_5)$ be two PFNs and $v \neq 0$. The arithmetic operations on $\widetilde{A}_P$, and $\widetilde{B}_P$ are
(i) Addition: $\widetilde{A}_P \oplus \widetilde{B}_P = (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4, a_5 + b_5)$,
(ii) Subtraction: $\widetilde{A}_P \ominus \widetilde{B}_P = (a_1 - b_5, a_2 - b_4, a_3 - b_3, a_4 - b_2, a_5 - b_1)$,
(iii) Multiplication: $\widetilde{A}_P \otimes \widetilde{B}_P = (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4, a_5 b_5)$,
(iv) Division: $\widetilde{A}_P \oslash \widetilde{B}_P = \left(\frac{a_1}{b_5}, \frac{a_2}{b_4}, \frac{a_3}{b_3}, \frac{a_4}{b_2}, \frac{a_5}{b_1} \right)$,
(v) Inverse: $\widetilde{A}_P^{-1} = \left(\frac{1}{b_5}, \frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1} \right)$,
(vi) Scaler multiplication: $k\widetilde{A}_P = \begin{cases} (ka_1, ka_2, ka_3, ka_4, ka_5) \, mk > 0, \\ (ka_5, ka_4, ka_3, ka_2, ka_1) , k < 0. \end{cases}$
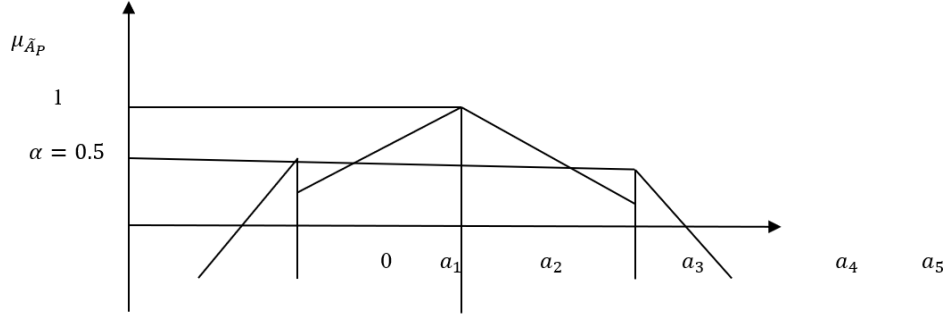
Figure 1. Graphical representation of PFN

Definition 4. The associated ordinary number corresponds to the PFN $\widetilde{A}_P = (a_1, a_2, a_3, a_4, a_5)$, which is defined by $R\left(\widetilde{A}_{\mathrm{P}}\right) = \widehat{A} = \frac{a_1 + a_2 + 2a_3 + a_4 + a_5}{6}$.

Definition 5. An interval approximation $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ of PFN $\widetilde{A}_{\mathrm{P}}$ is said to be a closed interval approximation if $a_\alpha^L = \inf\left\{x \in \mathbb{R} : \mu_{\widetilde{A}_{\mathrm{P}}} \geq \frac{1}{2}\right\}$, and $a_\alpha^U = \mathrm{Sup}\left\{x \in \mathbb{R} : \mu_{\widetilde{A}_{\mathrm{P}}} \geq \frac{1}{2}\right\}$.

Definition 6. If $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ is the closed interval approximation of PFNs, then the associated ordinary number of $[A]$ is denoted by $\mathrm{Re}[A]$ and is defined by $\mathrm{Re}[A] = \frac{a_\alpha^L + a_\alpha^U}{2}$.

Definition 7. Let $[A] = \left[a_\alpha^L, a_\alpha^U\right]$ and $[B] = \left[b_\alpha^L, b_\alpha^U\right]$ be two closed interval approximations of PFNs. Then, the arithmetic operations on $[A]$ and $[B]$ are defined as follows:

(i) Addition: $[A] \oplus [B] = \left[a_\alpha^L + b_\alpha^L, a_\alpha^U + b_\alpha^U\right]$.

(ii) Subtraction: $[A] \ominus [B] = \left[a_\alpha^L - b_\alpha^U, a_\alpha^U - b_\alpha^L\right]$.

(iii) Scalar multiplication: $k[A] = \begin{cases} \left[ka_\alpha^L, ka_\alpha^U\right], k > 0, \\ \left[ka_\alpha^U, ka_\alpha^L\right], k < 0. \end{cases}$

(iv) Multiplication: $[A] \otimes [B] = \left[\frac{1}{2}\left(a_\alpha^U \cdot b_\alpha^L + a_\alpha^L \cdot b_\alpha^U\right), \frac{1}{2}\left(a_\alpha^L \cdot b_\alpha^L + a_\alpha^U \cdot b_\alpha^U\right)\right]$.

(v) Division: $[A] \oslash [B] = \begin{cases} \left[2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right), 2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right)\right], [B] > 0, b_\alpha^L + b_\alpha^U \neq 0, \\ \left[2\left(\frac{a_\alpha^U}{b_\alpha^L + b_\alpha^U}\right), 2\left(\frac{a_\alpha^L}{b_\alpha^L + b_\alpha^U}\right)\right], [B] < 0, b_\alpha^L + b_\alpha^U \neq 0. \end{cases}$

Definition 8. The order of relations between $\widetilde{A}_P$ and $\widetilde{B}_P$ based on the associated ordinary number corresponding to the PFNs is defined as follows:

(i) $\widetilde{A}_P \succcurlyeq \widetilde{B}_P$ if and only if $\mathrm{R}\left(\widetilde{A}_P\right) \geq \mathrm{R}\left(\widetilde{B}_P\right)$,

(ii) $\widetilde{A}_P \preccurlyeq \widetilde{B}_P$ if and only if $\mathrm{R}\left(\widetilde{A}_P\right) \leq \mathrm{R}\left(\widetilde{B}_P\right)$,

(iii) $\widetilde{A}_{\mathrm{P}} \cong \widetilde{B}_{\mathrm{P}}$ if and only if $\mathrm{R}\left(\widetilde{A}_P\right) = \mathrm{R}\left(\widetilde{B}_{\mathrm{P}}\right)$.

3 Problem Formulation

Considering n jobs to be processed on a single machine, the processing time is denoted as $\tilde{t}_{i_P}$ for job $i = \overline{1, n}$. It is assumed that all jobs are ready for processing at time zero and have the same common due date $\widetilde{D}_P$. In addition, no more than one job can be processed at any point. Each job i requires exactly one operation, and its processing time $\tilde{p}_{i_P}$ is a PFN. If a job i is completed before the due date, its earliness is given by $\tilde{E}_{i_P} = \tilde{d}_P - \tilde{c}_{i_P}$, where $\tilde{c}_{i_P}$ is the completion time of the job. On the other hand, if a job i is completed after the desired date, its delay is given by $\widetilde{T}_{l_P} = \tilde{c}_{i_P} - \tilde{d}_P$. Each job i has its own unit earliness penalty γ_i and unit tardiness penalty δ_i. The problem aims to find the order in which these n jobs should be processed to minimize the sum of total earliness and tardiness costs.

The problem can be mathematically formulated as

$$\min \widetilde{F} = \sum_{i=1}^{n} \left\{ \gamma_i \max\left(\tilde{d}_P - \tilde{c}_{i_P}, 0\right) + \delta_i \max\left(\tilde{c}_{i_P} - \tilde{d}_P, 0\right) \right\} \tag{1}$$

where, $\tilde{c}_{i_P}, i = \overline{1, n}$ is the fuzzy completion time of job i, the fuzzy common due date is denoted as $\tilde{d}_P$, and γ_i and δ_i are the unit penalty costs associated with earliness and tardiness, respectively.

4 A Heuristic Approach

In this section, a heuristic approach to minimize the sum of total earliness and tardiness costs can be illustrated in the following steps:

Step 1: After sorting and numbering the n jobs in non-increasing order of processing time $\tilde{t}_{i_P}$, with $i = \overline{1, n}$, it leads to $\operatorname{Re}\left(\tilde{t}_1\right) \geq \operatorname{Re}\left(\tilde{t}_2\right) \ldots \geq \operatorname{Re}\left(\tilde{t}_{i-1}\right) \geq \operatorname{Re}\left(\tilde{t}_i\right) \geq \operatorname{Re}\left(\tilde{t}_{i+1}\right) \ldots \geq \operatorname{Re}\left(\tilde{t}_n\right)$, which can be represented in Table 1.

Table 1. n jobs in non-increasing order of processing time $\tilde{t}_{i_P}$

J_i	J_1	J_2	$\ldots$	J_{i-1}	J_i	J_{i+1}	$\ldots$	J_n
$\operatorname{Re}\left(\tilde{t}_i\right)$	$\operatorname{Re}\left(\tilde{t}_1\right)$	$\operatorname{Re}\left(\tilde{t}_2\right)$	$\ldots$	$\operatorname{Re}\left(\tilde{t}_{i-1}\right)$	$\operatorname{Re}\left(\tilde{t}_i\right)$	$\operatorname{Re}\left(\tilde{t}_{i+1}\right)\ldots$		$\operatorname{Re}\left(\tilde{t}_n\right)$

The calculation is as follows:
1. The total processing time $T = \sum_{i=1}^{n} \operatorname{Re}\left(\tilde{t}_I\right)$,
2. $\widetilde{T}_{0P} = \tilde{T}_P \ominus \widetilde{D}_P$, and $\widetilde{E}_{0P} = \widetilde{D}_P$.
Two empty sets, $S_0^E = \emptyset$ and $S_0^T = \emptyset$, are introduced.

Step 2: Considering a job J_1 with processing time $R\left(\tilde{t}_I\right) = \max\left(Re\left(\tilde{t}_i, i = \overline{1, n}\right)\right)$, then $\widetilde{T}_{1P} = \widetilde{T}_{0P}$, and $\widetilde{E}_{1P} = \widetilde{E}_{0P}$ are set.
  - If $R\left(\widetilde{T}_{1P}\right) < R\left(\widetilde{E}_{1P}\right)$, then $S_1^E = S_0^E + \{J_1\}$ and $S_1^T = S_0^T$ are set.
  - If $R\left(\widetilde{T}_{1P}\right) \geq R\left(\widetilde{E}_{1P}\right)$, then $S_1^T = S_0^T + \{J_1\}$ and $S_1^E = S_0^E$ are set.

Step 3: Considering a job J_i with processing time $\operatorname{Re}\left(\tilde{t}_i, i = \overline{1, n}\right)$,
  - If previous job $J_{i-1} \in S_{i-1}^E$, then $\widetilde{T}_{lP} = \widetilde{T_{l-1}}_P$ and $\widetilde{E}_{lP} = \widetilde{E_{l-1}}_P \ominus \widetilde{t_{l-1}}_P$ are computed.
  - If previous job $J_{i-1} \in S_{i-1}^T$, then $\widetilde{T}_{lP} = \widetilde{T_{l-1}}_P \ominus \widetilde{t_{l-1}}_P$ and $\widetilde{E}_{lP} = \widetilde{E_{l-1}}_P$ are computed.

Step 4: The assignment decision is calculated.
  - If $R\left(\widetilde{T}_{1P}\right) < R\left(\widetilde{E}_{1P}\right)$, then $S_1^E = S_{i-1}^E + \{J_1\}$ and $S_i^T = S_{i-1}^T$ are set.
  - If $R\left(\widetilde{T}_{1P}\right) \geq R\left(\widetilde{E}_{1P}\right)$, then $S_1^E = S_{i-1}^T + \{J_i\}$ and $S_i^E = S_{i-1}^E$ are set.

Step 5: Iteration terminates when all jobs are assigned to either S_n^E or S_n^T.

Step 6: The optimal schedule is $\hat{S} = \left\{S_n^E, S_n^T\right\}$, i.e., jobs are scheduled based on their sequencing in S_n^E, followed by S_n^T. In this step, S_n^E and S_n^T are jobs in non-increasing and non-decreasing order of processing time, respectively.

Remark 1: The procedure only involves n iterations as compared to n ! possible schedules.

5 Numerical Example

Considering a $10-$ job scheduling problem with a common due date, then $\widetilde{D}_P = (30, 35, 45, 50, 60)$ is tabulated in Table 2.

The heuristic approach is applied to determine the optimal scheduling decision as follows:

Step 1: The total processing time is estimated.

$\tilde{T}_P = \sum_{i=1}^{10} \tilde{t}_{i_P} = (71, 87, 102, 122, 135)$,

$\tilde{T}_{0_P} = \widetilde{T}_P \ominus \widetilde{D}_P = (71, 87, 102, 122, 135) \ominus (30, 35, 45, 50, 60) = (11, 37, 57, 87, 132)$, and

$\widetilde{E}_{0P} = \widetilde{D}_P = (30, 35, 45, 50, 60)$.
Now, two empty sets, $S_0^E = \varnothing$ and $S_0^T = \varnothing$, are introduced.

Step 2: Consider a job J_1 with $\tilde{t}_{2_P} = (18, 19, 20, 21, 22)$, then $\widetilde{T}_{1P} = \widetilde{T}_{0P} = (11, 37, 57, 87, 132)$ and $\widetilde{E}_{1P} = \widetilde{E}_{0P} = (30, 35, 45, 50, 60)$, are set.

$$R\left(\widetilde{T}_{1P}\right) > R\left(\widetilde{E}_{1P}\right) \Rightarrow S_1^T = S_0^T + \{J_1\} = \{J_1\} \text{ and } S_1^E = S_0^E = \{\}.$$

Table 2. Pentagonal fuzzy processing time of the 10- job scheduling problem

J_i	$\tilde{t}_{i_P}$
J_1	$(18, 19, 20, 21, 22)$
J_2	$(14, 17, 18, 20, 21)$
J_3	$(11, 13, 15, 18, 20)$
J_4	$(7, 9, 10, 12, 14)$
J_5	$(6, 8, 9, 11, 13)$
J_6	$(5, 7, 9, 10, 11)$
J_7	$(4, 5, 7, 9, 10)$
J_8	$(3, 4, 6, 8, 9)$
J_9	$(2, 3, 5, 7, 8)$
J_{10}	$(1, 2, 4, 6, 7)$
T*otal*	$(71, 87, 102, 122, 135)$

Step 3: Considering a job J_2 with $\tilde{t}_{2_P} = (14, 17, 18, 20, 21)$, and $J_1 \in S_1^T$, then $\tilde{T}_{2_P} = \tilde{T}_{1_P} \ominus \tilde{t}_{1_P} = (11, 37, 57, 87, 132), \ominus(18, 19, 20, 21, 22) = (-11, 16, 37, 68, 114)$ and $\widetilde{E}_{2p} = \widetilde{E}_{1p} = (30, 35, 45, 50, 60)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{2_P}\right) < R\left(\widetilde{E}_{2P}\right) \Rightarrow S_2^E = S_0^E + \{J_2\} = \{J_2\} \text{ and } S_2^T = S_1^T = \{J_1\}.$$

Step 4: Considering a job J_3 with $\tilde{t}_{3_P} = (11, 13, 15, 18, 20)$, and $J_2 \in S_2^E$, then $\tilde{T}_{3_P} = \tilde{T}_{2_P} = (-11, 16, 37, 68, 114)$ and $\widetilde{E}_{3P} = \widetilde{E}_{2P} \ominus \tilde{t}_{i_P} = (30, 35, 45, 50, 60) \ominus (14, 17, 18, 20, 21) = (9, 15, 27, 33, 46)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{3_P}\right) < R\left(\widetilde{E}_{3p}\right) \Rightarrow S_3^T = S_2^T + \{J_3\} = \{J_1, J_3\} \text{ and } S_3^E = S_2^E = \{J_2\}.$$

Step 5: Considering a job J_4 with $\tilde{t}_{4_P} = (7, 9, 10, 12, 14)$, and $J_3 \in S_3^T$, then $\tilde{T}_{4_P} = \tilde{T}_{3_P} \ominus \tilde{t}_{3_P} = (-11, 16, 37, 68, 114) \ominus (11, 13, 15, 18, 20) = (-31, -2, 22, 55, 103)$ and $\widetilde{E}_{4_P} = \widetilde{E}_{3P} = (9, 15, 27, 33, 46)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{4_P}\right) < R\left(\widetilde{E}_{4P}\right) \Rightarrow S_4^E = S_3^E + \{J_4\} = \{J_2, J_{34}\} \text{ and } S_4^T = S_3^T = \{J_1, J_3\}.$$

Step 6: Considering a job J_5 with $\tilde{t}_{5_P} = (6, 8, 9, 11, 13)$, and $J_4 \in S_4^E$, then $\tilde{T}_{5_P} = \tilde{T}_{4_P} = (-31, -2, 22, 55, 103)$ and $\widetilde{E}_{5_P} = \widetilde{E}_{4_P} \ominus \tilde{t}_{4_P} = (9, 15, 27, 33, 46) \ominus (7, 9, 10, 12, 14) = (-15, 3, 17, 24, 37)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{5_P}\right) < R\left(\widetilde{E}_{5P}\right) \Rightarrow S_5^T = S_4^T + \{J_5\} = \{J_1, J_3, J_5\} \text{ and } S_5^E = S_4^E = \{J_2, J_4\}.$$

Step 7: Considering a job J_6 with $\tilde{t}_{6_P} = (5, 7, 8, 10, 11)$, and $J_5 \in S_5^T$, then $\tilde{T}_{6_P} = \tilde{T}_{5_P} \ominus \tilde{t}_{5_P} = (-31, -2, 22, 55, 103) \ominus (6, 8, 9, 11, 13) = (-44, -13, 13, 47, 97)$ and $\widetilde{E}_{6_P} = \widetilde{E}_{5_P} = (-15, 3, 17, 24, 37)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{6_P}\right) < R\left(\widetilde{E}_{6P}\right) \Rightarrow S_6^E = S_5^E + \{J_6\} = \{J_2, J_4, J_6\} \text{ and } S_6^T = S_5^T = \{J_1, J_3, J_5\}.$$

Step 8: Considering a job J_7 with $\tilde{t}_{7_P} = (4, 5, 7, 9, 10)$, and $J_6 \in S_6^E$, then $\tilde{T}_{7_P} = \tilde{T}_{6_P} = (-44, -13, 13, 47, 97)$ and $\widetilde{E}_{7_P} = \widetilde{E}_{6_P} \ominus \tilde{t}_{6_P} = (-15, 3, 17, 24, 37) \ominus (5, 7, 8, 10, 11) = (-26, -7, 9, 17, 32)$ are computed.

The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{7_P}\right) < R\left(\widetilde{E_{7P}}\right) \Rightarrow S_7^T = S_6^T + \{J_7\} = \{J_1, J_3, J_5, J_7\} \text{ and } S_7^E = S_5^E = \{J_2, J_4, J_6\}.$$

Step 9: Considering a job J_8 with $\tilde{t}_{8_P} = (3, 4, 6, 8, 9)$, and $J_7 \in S_7^T$, then $\tilde{T}_{8_P} = \tilde{T}_{7_P} \ominus \tilde{t}_{7_P} = (-44, -13, 13, 47, 97) \ominus (4, 5, 7, 9, 10) = (-54, -22, 6, 42, 95)$ and $\widetilde{E}_{8_P} = \widetilde{E}_{7P} = (-26, -7, 9, 17, 32)$ are computed.
The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{8_P}\right) < R\left(\widetilde{E}_{8P}\right) \Rightarrow S_8^T = S_7^T + \{J_8\} = \{J_2, J_4, J_6, J_8\} \text{ and } S_8^T = S_7^T = \{J_1, J_3, J_5, J_7\}.$$

Step 10: Considering a job J_9 with $\tilde{t}_{9_P} = (2, 3, 5, 7, 8)$, and $J_8 \in S_8^E$, then $\tilde{T}_{9_P} = \tilde{T}_{8_P} = (-54, -22, 6, 42, 95)$ and $\widetilde{E}_{9P} = \widetilde{E}_{8P} \ominus \tilde{t}_{8P} = (-26, -7, 9, 17, 32) \ominus (-54, -22, 6, 42, 95) = (-121, -49, 3, 33, 86)$ are computed.
The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{9_P}\right) < R\left(\widetilde{E_{9_P}}\right) \Rightarrow S_9^T = S_8^T + \{J_9\} = \{J_1, J_3, J_5, J_7, J_9\} \text{ and } S_9^E = S_8^E = \{J_2, J_4, J_6, J_8\}.$$

Step 11: Considering a job J_{10} with $\tilde{t}_{10_P} = (1, 2, 4, 6, 7)$, and $J_9 \in S_9^T$, then $\tilde{T}_{9_P} \ominus \tilde{t}_{9_P} = \tilde{T}_{8_P} = (-54, -22, 6, 42, 95) \ominus (2, 3, 5, 7, 8) = (-62, -29, 1, 71, 93)$ and $\widetilde{E}_{10P} = \widetilde{E}_{9P} = (-121, -49, 3, 33, 86)$ are computed.
The assignment decision is calculated as follows:

$$R\left(\tilde{T}_{10_P}\right) < R\left(\tilde{E}_{10_P}\right) \Rightarrow S_{10}^E = S_9^{TE} + \{J_{10}\} = \{J_2, J_4, J_6, J_8, J_{10}\} \text{ and } S_{10}^T = S_9^T = \{J_1, J_3, J_5, J_7, J_9\}.$$

At the end of the iteration, all jobs are assigned to either S_{10}^E or S_{10}^T.
It is noted that jobs $S_{10}^E = \{J_2, J_4, J_6, J_8, J_{10}\}$ are in the required non-increasing order, while jobs in $S_{10}^T = \{J_1, J_3, J_5, J_7, J_9\}$ are in the non-decreasing order (not required). Thus, the fuzzy optimal schedule is given by $\hat{S} = \{J_2, J_4, J_6, J_8, J_{10}; J_9, J_7, J_5, J_3, J_1\}$, which can be illustrated in Table 3.

Table 3. The fuzzy optimal schedule

J_i	$\tilde{t}_{i_P}$	$\tilde{c}_{i_P}$	$\left\| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right) \right\|$
J_2	$(14, 17, 18, 20, 21)$	$(14, 17, 18, 20, 21)$	27
J_4	$(7, 9, 10, 12, 14)$	$(21, 26, 28, 32, 35)$	17
J_6	$(5, 7, 8, 10, 11)$	$(26, 33, 36, 42, 46)$	9
J_8	$(3, 4, 6, 8, 9)$	$(29, 37, 42, 50, 55)$	3
J_{10}	$(1, 2, 4, 6, 7)$	$(30, 39, 46, 56, 62)$	1
J_9	$(2, 3, 5, 7, 8)$	$(32, 42, 51, 63, 70)$	6
J_7	$(4, 5, 7, 9, 10)$	$(36, 47, 58, 72, 80)$	13
J_5	$(6, 8, 9, 11, 13)$	$(42, 55, 67, 83, 93)$	22
J_3	$(11, 13, 15, 18, 20)$	$(53, 68, 82, 101, 113)$	37
J_1	$(18, 19, 20, 21, 22)$	$(71, 87, 102, 122, 135)$	57
Sum	—	—	192

Therefore, $\hat{V} = \sum_{i=1}^{10} \left| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right) \right| = 192$ (days).
Now, a few selected schedules, $S_1 = \{J_1, J_2 J_3, J_4 J_5, J_6 J_7, J_8 J_9, J_{10}\}$, are computed for comparison.
Hence, $\hat{V}_1 = \sum_{i=1}^{10} \left| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right) \right| = 320$ (days), and $S_2 = \{J_9, J_7 J_5, J_3 J_1, J_2 J_4, J_6 J_8, J_{10}\}$.
Hence, $\hat{V}_2 = \sum_{i=1}^{10} \left| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right) \right| = 342$ (days).
It is evident that all of the schedules illustrated in Tables 4 and 5 give the total earliness and tardiness of more than 192 (days).

Table 4. The first comparison

J_i	$\tilde{t}_{i_P}$	$\tilde{c}_{i_P}$	$\left\| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right)\right\|$
J_1	$(18, 19, 20, 21, 22)$	$(18, 19, 20, 21, 22)$	25
J_2	$(14, 17, 18, 20, 21)$	$(32, 36, 38, 41, 43)$	7
J_3	$(11, 13, 15, 18, 20)$	$(43, 49, 53, 59, 63)$	8
J_4	$(7, 9, 10, 12, 14)$	$(50, 58, 63, 71, 77)$	18
J_5	$(6, 8, 9, 11, 13)$	$(56, 66, 72, 88, 90)$	27
J_6	$(5, 7, 8, 10, 11)$	$(61, 73, 80, 98, 101)$	35
J_7	$(4, 5, 7, 9, 10)$	$(65, 78, 87, 107, 111)$	42
J_8	$(3, 4, 6, 8, 9)$	$(68, 82, 93, 115, 120)$	48
J_9	$(2, 3, 5, 7, 8)$	$(70, 85, 98, 122, 128)$	53
J_{10}	$(1, 2, 4, 6, 7)$	$(71, 87, 102, 122, 135)$	57
Sum	—	—	320

Table 5. The second comparison

J_i	$\tilde{t}_{i_P}$	$\tilde{c}_{i_P}$	$\left\| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right)\right\|$
J_9	$(2, 3, 5, 7, 8)$	$(2, 3, 5, 7, 8)$	40
J_7	$(4, 5, 7, 9, 10)$	$(6, 8, 12, 16, 18)$	33
J_5	$(6, 8, 9, 11, 13)$	$(12, 16, 21, 27, 31)$	24
J_3	$(11, 13, 15, 18, 20)$	$(23, 29, 36, 45, 51)$	9
J_1	$(18, 19, 20, 21, 22)$	$(41, 48, 56, 66, 73)$	9
J_2	$(14, 17, 18, 20, 21)$	$(55, 65, 74, 86, 94)$	29
J_4	$(7, 9, 10, 12, 14)$	$(62, 74, 84, 98, 108)$	39
J_6	$(5, 7, 8, 10, 11)$	$(67, 81, 92, 108, 119)$	47
J_8	$(3, 4, 6, 8, 9)$	$(70, 85, 98, 116, 128)$	53
J_{10}	$(1, 2, 4, 6, 7)$	$(71, 87, 102, 122, 135)$	57
Sum	—	—	342

An Alternative Approach

For individuals requiring elevated precision in their analytical endeavors, an alternative methodology is herein proposed. Initially, optimization is conducted utilizing the associated ordinary number $\mathrm{Re}[A] = \frac{a_\alpha^L + a_\alpha^U}{2}$. Subsequently, the true optimization is computed using V for the closed interval approximation at $\alpha = \frac{1}{2}$.

An approach is delineated whereby approximations of closed intervals, derived from PFNs, are subjected to optimization. This procedure is depicted in Table 6. Table 7 shows the calculation of $\mathrm{Re}\left[\tilde{c}_{i_P}\right]$ and $\left\| \mathrm{Re}\left[\tilde{c}_{i_P}\right] - \mathrm{Re}\left[\widetilde{D}_P\right]\right\|$, with $\left[\widetilde{D}_P\right] = [35, 50]$.

Therefore, $\hat{V} = \sum_{i=1}^{10} \left\| R\left(\tilde{c}_{i_P}\right) - R\left(\widetilde{D}_P\right)\right\| = 199$ (days).

This process can be extended to other types of scheduling problems. In addition, it is noted that hybrid numbers can be used in some particular cases.

Table 6. Closed interval approximation of processing time for the 10-job scheduling problem

J_i	$\tilde{t}_{i_P}$
J_1	$[19, 21]$
J_2	$[17, 20]$
J_3	$[13, 18]$
J_4	$[9, 12]$
J_5	$[8, 11]$
J_6	$[7, 10]$
J_7	$[5, 9]$
J_8	$[4, 8]$
J_9	$[3, 7]$
J_{10}	$[2, 6]$
Total	$[87, 122]$

Table 7. Closed interval approximation of the optimal schedule

J_i	$\tilde{t}_{i_P}$	$\text{Re}\left[\tilde{c}_{i_P}\right]$	$\left|\text{Re}\left[\tilde{c}_{i_P}\right] - \text{Re}\left[\widetilde{D}_P\right]\right|$
J_2	$[17, 20]$	$[17, 20]$	26.5
J_4	$[9, 12]$	$[26, 32]$	16
J_6	$[7, 10]$	$[33, 42]$	7.5
J_8	$[4, 8]$	$[37, 50]$	1.5
J_{10}	$[2, 6]$	$[39, 56]$	2.5
J_9	$[3, 7]$	$[42, 63]$	7.5
J_7	$[5, 9]$	$[47, 72]$	14.5
J_5	$[8, 11]$	$[55, 83]$	24
J_3	$[13, 18]$	$[68, 101]$	39.5
J_1	$[19, 21]$	$[87, 122]$	59.5
Sum	—	—	199

6 Conclusions and Future Works

In the conducted study, a heuristic approach was employed to optimize the aggregate costs associated with earliness and tardiness in a scheduling problem for n jobs on a SMS, wherein processing time was characterized by PFNs. The integration of a common due date was a pivotal aspect of the research, aimed at addressing the complexities of scheduling within uncertain operational environments. This approach involved only n iterations and was computationally expensive for large problems. In addition, it has been assumed that the unit earliness and tardiness costs are constant for all jobs. The advantages of this approach are that it can be applied to multiple machines, development in supply chain management, the Internet, and e-commerce. In the future, this study might be extended to other fuzzy-like structures, i.e., neutrosophic set, interval-valued fuzzy set, spherical fuzzy set, pythagorean fuzzy set, etc. In addition, new fuzzy systems can be considered, such as interval type-2, interval type-3, possibility interval-valued intuitionistic fuzzy set, possibility neutrosophic set, possibility interval-valued neutrosophic set, possibility interval-valued fuzzy set, possibility fuzzy expert set, etc., with applications in decision-making.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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