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A New Spectral Three-Term Conjugate Gradient Method for Unconstrained Optimization and Its Application

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Abstract: In this paper, we derive a new conjugate gradient (CG) direction with random parameters which are obtained by minimizing the deviation between search direction matrix and self-scaled memoryless Broyden-Fletcher-Goldfard-Shanno (BFGS) update. We propose a new spectral three-term CG algorithm and establish the global convergence of new method for uniformly convex functions and general nonlinear functions, respectively. Numerical experiments show that our method has nice numerical performance on nonconvex functions and supply chain problems.

Keywords: Random parameter; Spectral three-term conjugate gradient; Global convergence; Unconstrained optimization

1 Introduction

Many problems of relevance in compressing sensing [1], portfolio selection [2] and image restoration [3, 4] can be transformed into the following unconstrained optimization problem

$$\min_{x\in\mathbb{R}^n} f(x), \tag{1}$$

where,

$f:\ \mathbb{R}^n \to \mathbb{R}$ is continuous differentiable. The spectral conjugate gradient (SCG) method [5] is one of the most popular class of algorithm for solving problem Eq. (1), due to its simplicity, low memory requirement and strong convergence performance. Starting from an initial point $x_0 \in \mathbb{R}^n$, the iterative formula of the SCG method is given by

$$x_{k+1} = x_k + \alpha_k d_k,\ k \geq 0, \tag{2}$$

where, stepsize $\alpha_k > 0$ is generated by line search. The search direction d_k is defined by

$$d_0 = -g_0,\ d_k = -\theta_k g_k + \beta_{k-1} d_{k-1},\ k \geq 1, \tag{3}$$

where,

$g_k = \nabla f(x_k)$ is the gradient of $f(x)$ at current point x_k, θ_k is a spectral parameter and β_{k-1} is a scalar called the conjugate gradient (CG) parameter. There are many choices of θ_k and β_{k-1}, which may lead to different computational efficiency and convergence properties [5–10].

Babaie-Kafaki and Ghanbari [11] proposed a descent family of the Dai-Liao CG (DDL) method with the search direction

$$d_{k+1} = -\Big(I - \frac{s_k y_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k} + t_k \frac{s_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k}\Big) g_{k+1} \doteq -Q_{k+1} g_{k+1}, \tag{4}$$

where,

Q_{k+1} is called a search direction matrix. They obtained the relationship

$$d_{k+1}^{\mathrm{T}} g_{k+1} = -g_{k+1}^{\mathrm{T}} Q_{k+1} g_{k+1} = -g_{k+1}^{\mathrm{T}} A_{k+1} g_{k+1}, \tag{5}$$

where,

$$A_{k+1} \doteq \frac{Q_{k+1}^{\mathrm{T}} + Q_{k+1}}{2} = I - \frac{1}{2}\frac{s_k y_k^{\mathrm{T}} + y_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k} + t_k \frac{s_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k}. \tag{6}$$

By letting the smallest eigenvalue of A_{k+1} has a lower bound greater than 0, they determined parameter t_k. Then the search direction d_{k+1} was descent.

Livieris et al. [12] proposed a descent hybrid CG method with the search direction

$$d_{k+1} = -\Big(I - \lambda_k \frac{d_k g_{k+1}^{\mathrm{T}}}{d_k^{\mathrm{T}} y_k} - (1-\lambda_k)\frac{d_k y_k^{\mathrm{T}}}{d_k^{\mathrm{T}} y_k}\Big) g_{k+1} \doteq -P_{k+1} g_{k+1}. \tag{7}$$

They obtained the hybrization parameter λ_k by minimizing the distance between P_{k+1} and self-scaled memoryless Broyden-Fletcher-Goldfard-Shanno (BFGS) update in the Frobenius norm.

Recently, the spectral three-term CG methods have been paid more attention [13–17]. Chen and Yang [14] using subspace presented a three-term CG algorithm for large-scale unconstrained optimization. Faramarzi and Amini [15] proposed a spectral three-term Hestenes-Stiefel CG method. Al-Bayati and Abbas [16] gave a robust spectral three-term CG algorithm for solving unconstrained minimization problems. Eslahchi and Bojari [17] proposed a new sufficient descent spectral three-term CG class for large-scale optimization.

Motivated by the above works, in order to make better use of the properties of spectral parameter, we consider the search direction matrix

$$A_{k+1} = \theta_k I - \frac{1}{2}\frac{s_k y_k^{\mathrm{T}} + y_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k} + t_k \frac{s_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k} \tag{8}$$

to propose a new spectral three-term CG method, where scaling parameter θ_k and t_k are undetermined parameters. The corresponding search direction is

$$d_{k+1} = -A_{k+1} g_{k+1} = -\theta_k g_{k+1} + \beta_k s_k + \gamma_k y_k, \tag{9}$$

where,

$$\beta_k = \frac{1}{2}\frac{y_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k} - t_k \frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k} \text{ and } \gamma_k = \frac{1}{2}\frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}.$$

Based on the idea of the study [12], our main work is to give a new choice of parameters t_k and θ_k to propose a spectral three-term CG method. The contributions of this article are listed as follows:

◇ A random parameter which leads to our method more relaxed and elastic, is introduced to construct β_k and θ_k in the search direction.

◇ The search direction satisfies the sufficient descent condition. Under appropriate assumptions, we give global convergence of new method for general functions.

◇ The new method has good numerical performance for the objective function with sharp curvature change.

◇ The new method is applied to the supply chain problem, which shows that the new method is effective.

The motivation and algorithm are analyzed in the next section, we find parameter t_k including a random parameter at each iterate, and obtain parameters β_k and θ_k. Then we state a new spectral three-term CG method. In Section 3, global convergence of our method is proved for uniformly convex functions and general nonlinear functions. In Section 4, some numerical experiments and application results are reported. Conclusions are made in the last section.

2 Motivation and Algorithm

In this section, our main aim is to discuss how to choose t_k and propose a random spectral three-term CG method. The search direction is derived by minimizing the deviation between the search direction matrix and a quasi-Newton update, in conjunction with choices of random parameter. Consider model

$$\min \|D_{k+1}\|_F^2 \doteq \|A_{k+1} - B_{k+1}^{-1}\|_F^2, \tag{10}$$

where,

$\|\cdot\|_F$ denotes the Frobenius norm, A_{k+1} is determined by Eq. (8), B_{k+1}^{-1} is a self-scaled memoryless BFGS matrix

$$B_{k+1}^{-1} = \theta_k I - \theta_k \frac{s_k y_k^{\mathrm{T}} + y_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k} + \left(1 + \theta_k \frac{\|y_k\|^2}{s_k^{\mathrm{T}} y_k}\right)\frac{s_k s_k^{\mathrm{T}}}{s_k^{\mathrm{T}} y_k}, \tag{11}$$

and θ_k is a scaling parameter. From Eq. (8) and Eq. (11), we have

$$\begin{aligned}\|D_{k+1}\|_F^2 &= \operatorname{tr}(D_{k+1}^{\mathrm{T}}D_{k+1})\\ &= \frac{\|s_k\|^4}{(s_k^{\mathrm{T}}y_k)^2}t^2+2\left[(2\theta_k-1)\frac{\|s_k\|^2}{s_k^{\mathrm{T}}y_k}-\frac{\|s_k\|^4}{(s_k^{\mathrm{T}}y_k)^2}-\theta_k\frac{\|y_k\|^2\|s_k\|^4}{(s_k^{\mathrm{T}}y_k)^3}\right]t+\zeta,\end{aligned}$$

where,

ζ is a constant independent of t. This is a second-degree polynomial of variable t and the coefficient of t^2 is positive. Therefore, the minimum of problem Eq. (10) is

$$t_k=\arg\min\left\{\operatorname{tr}(D_{k+1}^{\mathrm{T}}D_{k+1})\right\}=1+\frac{\theta_k\chi_k}{\sqrt{p_k}}+(1-2\theta_k)\sqrt{p_k}\chi_k, \tag{12}$$

where,

$p_k=\cos^2\langle s_k,y_k\rangle$ and $\chi_k=\frac{\|y_k\|}{\|s_k\|}$. Instead of the mean value to p_k in the study [18], we set p_k is a random number in the interval $[\underline{m},\ \overline{m}]$, where $0<\underline{m}<\overline{m}<\frac{1}{2}$. There are many possible ways to choose θ_k, we prefer to use

$$\theta_k=\max\left\{\frac{1-\underline{m}}{2(1-\overline{m})},\frac{\|s_k\|^2}{s_k^{\mathrm{T}}y_k}\right\}\ \text{ or }\ \theta_k=\max\left\{\frac{1-\underline{m}}{2(1-\overline{m})},\frac{s_k^{\mathrm{T}}y_k}{\|y_k\|^2}\right\}. \tag{13}$$

Thus, t_k in Eq. (12) can be regarded as random parameters. Substitute Eq. (12) into Eq. (9), then we get the following new search direction

$$d_{k+1}=-\theta_kg_{k+1}+\left[\frac{1}{2}\frac{y_k^{\mathrm{T}}g_{k+1}}{s_k^{\mathrm{T}}y_k}-2\gamma_k\Big(1+\frac{\theta_k\chi_k}{\sqrt{p_k}}+(1-2\theta_k)\sqrt{p_k}\chi_k\Big)\right]s_k+\gamma_ky_k. \tag{14}$$

Now, we state a description of the random spectral three-term conjugate gradient algorithm (RSTTCG) as follows.

RSTTCG Algorithm

Step 0. Given $x_0\in\mathbb{R}^n$, $\varepsilon>0$, $0<\underline{m}<\overline{m}<\frac{1}{2}$ and $0<\rho<\sigma<1$. Let $f_0=f(x_0)$, $g_0=\nabla f(x_0)$, $d_0:=-g_0$ and $k:=0$.

Step 1. If $\|g_k\|\le\varepsilon$, stop and output x_k.

Step 2. Compute α_k satisfying the the strong Wolfe line search conditions

$$f(x_k+\alpha d_k)-f(x_k)\le\rho\alpha g_k^{\mathrm{T}}d_k, \tag{15}$$

$$|g_{k+1}^{\mathrm{T}}d_k|\le-\sigma g_k^{\mathrm{T}}d_k. \tag{16}$$

Step 3. Set $x_{k+1}=x_k+\alpha_kd_k$. Calculate f_{k+1}, g_{k+1}, s_k and y_k.

Step 4. Compute t_k by Eq. (12) and the search direction d_{k+1} by Eq. (13) and Eq. (14). Set $k:=k+1$ and go to Step 1.

The following lemma shows the sufficient descent property of search direction.

Lemma 2.1 Let the sequence $\{d_{k+1}\}$ be generated by RSTTCG algorithm. There exists a positive constant c satisfying

$$g_{k+1}^{\mathrm{T}}d_{k+1}\le-c\|g_{k+1}\|^2. \tag{17}$$

Proof By Eq. (16), we have $s_k^{\mathrm{T}}y_k=s_k^{\mathrm{T}}(g_{k+1}-g_k)\ge(\sigma-1)s_k^{\mathrm{T}}g_k>0$. Since $s_k^{\mathrm{T}}y_k=\|s_k\|\|y_k\|\cos\langle s_k,y_k\rangle$, then $\cos\langle s_k,y_k\rangle=\sqrt{p_k}>0$. Combined with Eq. (14), we get

$$\begin{aligned}g_{k+1}^{\mathrm{T}}d_{k+1} &= -\theta_k\|g_{k+1}\|^2+\frac{y_k^{\mathrm{T}}g_{k+1}g_{k+1}^{\mathrm{T}}s_ks_k^{\mathrm{T}}y_k}{(s_k^{\mathrm{T}}y_k)^2}\\ &\quad-\left[1+\frac{\theta_k\chi_k}{\sqrt{p_k}}+(1-2\theta_k)\sqrt{p_k}\chi_k\right]\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k}\\ &= -\theta_k\|g_{k+1}\|^2+\frac{y_k^{\mathrm{T}}g_{k+1}g_{k+1}^{\mathrm{T}}s_ks_k^{\mathrm{T}}y_k}{(s_k^{\mathrm{T}}y_k)^2}\\ &\quad-\left[1+\frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}}y_k}{\|s_k\|^2}+(1-2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}}y_k}\right]\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k}\\ &\le -\theta_k\|g_{k+1}\|^2-\left[1+\frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}}y_k}{\|s_k\|^2}+(1-2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}}y_k}\right]\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\frac{(g_{k+1}^{\mathrm{T}}s_k)^2\|y_k\|^2+(s_k^{\mathrm{T}}y_k)^2\|g_{k+1}\|^2}{(s_k^{\mathrm{T}}y_k)^2} \\
= \ & (\frac{1}{2}-\theta_k)\|g_{k+1}\|^2-\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k} \\
&-\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k}\left[\frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}}y_k}{\|s_k\|^2}+\Big((1-2\theta_k)p_k-\frac{1}{2}\Big)\frac{\|y_k\|^2}{s_k^{\mathrm{T}}y_k}\right] \\
\leq \ & (\frac{1}{2}-\theta_k)\|g_{k+1}\|^2-\frac{(s_k^{\mathrm{T}}g_{k+1})^2}{s_k^{\mathrm{T}}y_k}\left[1+\frac{\|y_k\|}{\sqrt{p_k}\|s_k\|}\Big(\theta_k+p_k-2\theta_kp_k-\frac{1}{2}\Big)\right] \\
\leq \ & (\frac{1}{2}-\theta_k)\|g_{k+1}\|^2\leq\left[\frac{1}{2}-\frac{1-\underline{m}}{2(1-\overline{m})}\right]\|g_{k+1}\|^2=-\frac{\overline{m}-\underline{m}}{2(1-\overline{m})}\|g_{k+1}\|^2.
\end{aligned}
$$

The second of above inequality is from the fact $a^{\mathrm{T}}b\leq\frac{1}{2}(\|a\|^2+\|b\|^2)$, in which $a=g_{k+1}^{\mathrm{T}}s_ky_k$ and $b=s_k^{\mathrm{T}}y_kg_{k+1}$. Combining Eq. (13), let $c=\frac{\overline{m}-\underline{m}}{2(1-\overline{m})}$, the proof is completed. □

3 Convergence Analysis

In this section, to prove the global convergence of RSTTCG algorithm, we give the following assumptions.

Assumption (i). The level set $\Omega=\{x\in\mathbb{R}^n:\ f(x)\leq f(x_0)\}$ is bounded, namely, there exists $\delta>0$ satisfying $\|x\|\leq\delta,\ \ \forall x\in\Omega$.

Assumption (ii). The gradient of function f is Lipschitz continuous in some neighborhood $\mathbb{N}$ of Ω, namely, there exists $L>0$ satisfying

$$\|g(x)-g(y)\|\leq L\|x-y\|,\ \ \forall x,y\in\mathbb{N}. \tag{18}$$

Based on the above assumptions, we know

$$\|s_k\|=\|x_{k+1}-x_k\|\leq\|x_{k+1}\|+\|x_k\|\leq 2\delta. \tag{19}$$

Besides, we can easily see that $g(x)$ is bounded, namely, there exists a positive constant L_1 such that $\|g(x)\|\leq L_1,\ \ \forall x\in\Omega$.

Lemma 3.1 Let the sequence $\{d_k\}$ be generated by RSTTCG algorithm. If Assumption (ii) holds, then

$$\alpha_k\geq\frac{(1-\sigma)|g_k^{\mathrm{T}}d_k|}{L\|d_k\|^2}. \tag{20}$$

Proof Subtracting $g_k^{\mathrm{T}}d_k$ from both sides of the left inequality of Eq. (16) and using the Lipschitz condition, we get

$$(\sigma-1)g_k^{\mathrm{T}}d_k\leq(g_{k+1}-g_k)^{\mathrm{T}}d_k=y_k^{\mathrm{T}}d_k\leq\|y_k\|\|d_k\|\leq\alpha_kL\|d_k\|^2.$$

Since $0<\sigma<1$ and d_k is a descent direction, then Eq. (20) holds. The proof is completed. □

Lemma 3.2 Let the sequence $\{d_k\}$ be generated by RSTTCG algorithm. If Assumptions (i)-(ii) hold, we have the following Zoutendijk condition

$$\sum_{k=0}^{\infty}\frac{(g_k^{\mathrm{T}}d_k)^2}{\|d_k\|^2}<+\infty. \tag{21}$$

Proof From the first inequality Eq. (15) of the strong Wolfe conditions, Assumption (ii) and Lemma 3.1, we have

$$f_k-f_{k+1}\geq-\rho\alpha_kg_k^{\mathrm{T}}d_k\geq-\rho\frac{(1-\sigma)(g_k^{\mathrm{T}}d_k)^2}{L\|d_k\|^2}.$$

From Assumption (i), we know $f(x)$ is bounded from below, then Eq. (21) is obtained. The proof is completed.□

Theorem 3.1 Suppose that Assumptions (i) and (ii) hold. The sequence $\{x_k\}$ is generated by RSTTCG algorithm. If f is a uniformly convex function on Ω, namely, there exists $\mu>0$ such that

$$(\nabla f(x)-\nabla f(y))^{\mathrm{T}}(x-y)\geq\mu\|x-y\|^2,\ \ \forall x,y\in\mathbb{N}, \tag{22}$$

Then, we have

$$\lim_{k\to\infty}\|g_k\|=0.$$

Proof From the Lipschitz condition Eq. (18), we have

$$\|y_k\| = \|g_{k+1} - g_k\| \le L\|s_k\|. \tag{23}$$

It follows Eq. (22) and the Cauchy inequality that

$$\mu\|s_k\|^2 \le y_k^{\mathrm{T}} s_k \le \|y_k\|\|s_k\|, \tag{24}$$

i.e.,

$$\mu\|s_k\| \le \|y_k\|. \tag{25}$$

Then, from Eqs. (23)–(25), we get

$$\frac{1}{L} = \frac{\|s_k\|}{L\|s_k\|} \le \frac{\|s_k\|^2}{\|y_k\|\|s_k\|} \le \frac{\|s_k\|^2}{s_k^{\mathrm{T}} y_k} \le \frac{\|s_k\|^2}{\mu\|s_k\|^2} = \frac{1}{\mu}, \tag{26}$$

$$\frac{\mu}{L^2} = \frac{\mu\|s_k\|^2}{L^2\|s_k\|^2} \le \frac{\mu\|s_k\|^2}{\|y_k\|^2} \le \frac{s_k^{\mathrm{T}} y_k}{\|y_k\|^2} \le \frac{\|y_k\|\|s_k\|}{\|y_k\|} \le \frac{\|s_k\|}{\|y_k\|} \le \frac{\|s_k\|}{\mu\|s_k\|} \le \frac{1}{\mu}. \tag{27}$$

Let $\theta_{\max} = \max\{\frac{1-\underline{m}}{2(1-\overline{m})}, \frac{1}{\mu}\}$, we have $\theta_k \le \theta_{\max}$. From Eq. (26) and Eq. (27), we obtain

$$t_k = 1 + \frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}} y_k}{\|s_k\|^2} + (1 - 2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}} y_k} \le 1 + \frac{L\theta_{\max}}{\underline{m}} + \frac{\overline{m}L^2}{\mu}. \tag{28}$$

Therefore, from the Cauchy inequality, the triangle inequality, Eq. (14), Eq. (23), Eq. (24) and Eq. (28), we have

$$\begin{aligned}
\|d_{k+1}\| &= \|-\theta_k g_{k+1} + \beta_k s_k + \gamma_k y_k\| \\
&\le \theta_k\|g_{k+1}\| + \frac{1}{2}\left|\frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}\right|\|y_k\| + \frac{1}{2}\left|\frac{y_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}\right|\|s_k\| \\
&\quad + |t_k|\left|\frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}\right|\|s_k\| \\
&\le \theta_{\max}\|g_{k+1}\| + \frac{1}{2}\frac{\|g_{k+1}\|L}{\mu} + \frac{1}{2}\frac{\|g_{k+1}\|L}{\mu} \\
&\quad + \left(1 + \frac{L\theta_{\max}}{\underline{m}} + \frac{\overline{m}L^2}{\mu}\right)\frac{\|g_{k+1}\|}{\mu} \\
&= \left[\theta_{\max} + \frac{1}{\mu}\big(1 + L + \frac{L\theta_{\max}}{\underline{m}} + \frac{\overline{m}L^2}{\mu}\big)\right]\|g_{k+1}\| \doteq M\|g_{k+1}\|.
\end{aligned} \tag{29}$$

From Lemma 2.1 and Eq. (29), we have $\frac{(g_{k+1}^{\mathrm{T}} d_{k+1})^2}{\|d_{k+1}\|^2} \ge \frac{c^2\|g_{k+1}\|^2}{M^2}$. Combined with Lemma 3.2, we get

$$\sum_{k=0}^{\infty}\|g_k\|^2 < \infty.$$

The proof is completed. □

For the general nonlinear functions, we can establish a weaker convergence result

$$\liminf_{k\to\infty}\|g_k\| = 0. \tag{30}$$

Lemma 3.3 Suppose that Assumptions (i) and (ii) hold. Let the sequence $\{x_k\}$ be generated by RSTTCG algorithm, then we have $d_k \ne 0$ and

$$\sum_{k=0}^{\infty}\|u_{k+1} - u_k\|^2 < \infty, \tag{31}$$

whenever $\inf\{\|g_k\| : k \ge 0\} > 0$, in which $u_k = \frac{d_k}{\|d_k\|}$.

Proof Define $\tau = \inf\{\|g_k\| : k \geq 0\}$, we know $\|g_k\| \geq \tau > 0$. From the sufficient descent condition Eq. (17), we have $d_k \neq 0$ for each k, so u_k is well defined. To prove global convergence, we define $\beta_k^+ = \max\{\beta_k^{'}, 0\}$, where $\beta_k^{'} = \frac{1}{2}\frac{y_k^{\mathrm{T}} g_{k+1}}{d_k^{\mathrm{T}} y_k} - \left[1 + \frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}} y_k}{\|s_k\|^2} + (1-2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}} y_k}\right]\frac{s_k^{\mathrm{T}} g_{k+1}}{d_k^{\mathrm{T}} y_k}$. From Eq. (14), we have

$$\frac{d_{k+1}}{\|d_{k+1}\|} = \frac{-\theta_k g_{k+1}}{\|d_{k+1}\|} + \beta_k^+ \frac{d_k}{\|d_{k+1}\|} + \gamma_k \frac{y_k}{\|d_{k+1}\|} = \frac{-\theta_k g_{k+1} + \gamma_k y_k}{\|d_{k+1}\|} + \beta_k^+ \frac{\|d_k\|}{\|d_{k+1}\|}\frac{d_k}{\|d_k\|},$$

namely,

$$u_{k+1} = \omega_k + \xi_k u_k,$$

where,

$$\omega_k = \frac{-\theta_k g_{k+1} + \gamma_k y_k}{\|d_{k+1}\|}, \xi_k = \beta_k^+ \frac{\|d_k\|}{\|d_{k+1}\|} \geq 0.$$

By using the condition $\|u_{k+1}\| = \|u_k\| = 1$, we have $\|\omega_k\| = \|u_{k+1} - \xi_k u_k\| = \|\xi_k u_{k+1} - u_k\|$. Since $\xi_k \geq 0$, it follows that

$$\|u_{k+1} - u_k\| \leq \|(1+\xi_k)u_{k+1} - (1+\xi_k)u_k\| \leq \|u_{k+1} - \xi_k u_k\| + \|\xi_k u_{k+1} - u_k\| = 2\|\omega_k\|.$$

From Eq. (16), we have

$$\left|\frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}\right| = \left|\frac{d_k^{\mathrm{T}} g_{k+1}}{d_k^{\mathrm{T}} y_k}\right| \leq \frac{\sigma}{1-\sigma}, \|y_k\| \leq \|g_{k+1}\| + \frac{\|g_k\|}{\|g_{k+1}\|}\|g_{k+1}\| \leq 1 + \frac{L_1}{\tau}\|g_{k+1}\|. \tag{32}$$

By the definition of ω_k, γ_k and Eq. (32), we get

$$\begin{aligned}\|\omega_k\| = \frac{\|-g_{k+1} + \gamma_k y_k\|}{\|d_{k+1}\|} &\leq \frac{\|g_{k+1}\| + \frac{1}{2}\left|\frac{s_k^{\mathrm{T}} g_{k+1}}{s_k^{\mathrm{T}} y_k}\right| \cdot \|y_k\|}{\|d_{k+1}\|} \\ &\leq \left[1 + \frac{\sigma}{2(1-\sigma)}(1 + \frac{L_1}{\tau})\right]\frac{\|g_{k+1}\|}{\|d_{k+1}\|}.\end{aligned}$$

If $\|g_{k+1}\| > \tau$, from Lemma 2.1 and Lemma 3.2, we have

$$\sum_{k=0}^{\infty}\frac{c^2\tau^2\|g_{k+1}\|^2}{\|d_{k+1}\|^2} \leq \sum_{k=0}^{\infty}\frac{c^2\|g_{k+1}\|^4}{\|d_{k+1}\|^2} \leq \sum_{k=0}^{\infty}\frac{(g_{k+1}^{\mathrm{T}} d_{k+1})^2}{\|d_{k+1}\|^2} < +\infty,$$

therefore, Eq. (31) holds. □

Property(*) Consider a method of the form Eq. (2) and Eq. (14), and suppose

$$0 < \tau \leq \|g_k\| \leq \overline{\tau}, \ \ k \geq 0. \tag{33}$$

We call that a method has Property(*) if there exist constants $b > 1$ and $\lambda > 0$ such that $|\beta_k^{'}| < b$ and $\|s_k\| \leq \lambda \ \Rightarrow \ |\beta_k^{'}| \leq \frac{1}{2b}$.

Lemma 3.4 Suppose that Assumptions (i)-(ii) hold. Let the sequence $\{d_k\}$ be generated by RSTTCG algorithm, then RSTTCG algorithm has Property (*).

Proof From Eq. (15) and Eq. (16), we have

$$d_k^{\mathrm{T}} y_k \geq (\sigma - 1)g_k^{\mathrm{T}} d_k \geq c(1-\sigma)\|g_k\|^2. \tag{34}$$

By using Assumption (i), Eqs. (32)–(34), we obtain

$$\begin{aligned}|\beta_k^{'}| &= \left|\frac{1}{2}\frac{y_k^{\mathrm{T}} g_{k+1}}{d_k^{\mathrm{T}} y_k} - \left(1 + \frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}} y_k}{\|s_k\|^2} + (1-2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}} y_k}\right)\frac{s_k^{\mathrm{T}} g_{k+1}}{d_k^{\mathrm{T}} y_k}\right| \\ &\leq \frac{1}{2}\frac{\|y_k\|\|g_{k+1}\|}{c(1-\sigma)\|g_k\|^2} + \left(1 + \frac{\theta_k}{p_k}\frac{s_k^{\mathrm{T}} y_k}{\|s_k\|^2} + (1-2\theta_k)p_k\frac{\|y_k\|^2}{s_k^{\mathrm{T}} y_k}\right)\frac{\|s_k\|\|g_{k+1}\|}{c(1-\sigma)\|g_k\|^2} \\ &\leq \frac{1}{2}\frac{\|g_{k+1} - g_k\|\|g_{k+1}\|}{c(1-\sigma)\|g_k\|^2} + \left(1 + \frac{L\theta_{\max}}{\underline{m}} + \frac{\overline{m}L^2}{\mu}\right)\frac{\|s_k\|\|g_{k+1}\|}{c(1-\sigma)\|g_k\|^2} \\ &\leq \frac{\overline{\tau}^2}{c(1-\sigma)\tau^2} + \left(1 + \frac{L\theta_{\max}}{\underline{m}} + \frac{\overline{m}L^2}{\mu}\right)\frac{2\delta\overline{\tau}}{c(1-\sigma)\tau^2} := b.\end{aligned} \tag{35}$$

Define

$$\lambda := \frac{c^2(1-\sigma)^2\tau^4}{2\overline{\tau}^2\left[\overline{\tau}+(1+\frac{L\theta_{\max}}{\underline{m}}+\frac{\overline{m}L^2}{\mu})L_1\right]\left(1+\frac{L}{2}+\frac{L\theta_{\max}}{\underline{m}}+\frac{\overline{m}L^2}{\mu}\right)}, \tag{36}$$

if $\|s_k\| \leq \lambda$, from Eq. (35) and Eq. (36), we obtain

$$\begin{aligned}
|\beta_k^{'}| &\leq \frac{1}{2}\frac{L\|s_k\|\|g_{k+1}\|}{c(1-\sigma)\tau^2} + \left(1+\frac{L\theta_{\max}}{\underline{m}}+\frac{\overline{m}L^2}{\mu}\right)\frac{\|s_k\|\|g_{k+1}\|}{c(1-\sigma)\tau^2} \\
&\leq \left[\frac{1}{2}\frac{L\overline{\tau}}{c(1-\sigma)\tau^2} + \left(1+\frac{L\theta_{\max}}{\underline{m}}+\frac{\overline{m}L^2}{\mu}\right)\frac{\overline{\tau}}{c(1-\sigma)\tau^2}\right]\|s_k\| \\
&\leq \left[\frac{1}{2}\frac{L\overline{\tau}}{c(1-\sigma)\tau^2} + \left(1+\frac{L\theta_{\max}}{\underline{m}}+\frac{\overline{m}L^2}{\mu}\right)\frac{\overline{\tau}}{c(1-\sigma)\tau^2}\right]\lambda = \frac{1}{2b}. \qquad \square
\end{aligned}$$

We will show that if the gradient sequence is bounded away from zero, then a fraction of the steps cannot be too small in next lemma. Let $\mathbb{N}$ be the set of positive integers, $K^{\lambda} := \{i \in \mathbb{N} : i \geq 1, \|s_i\| > \lambda\}$, for $\lambda > 0$, namely, the set of integers corresponding to steps greater than λ. Here, we need to discuss groups of $\triangle$ consecutive iterates and let $K^{\lambda}_{k,\Delta} := \{i \in \mathbb{N} : k \leq i \leq k+\Delta-1, \|s_i\| > \lambda\}$, where $|K^{\lambda}_{k,\Delta}|$ denotes the number of elements of $K^{\lambda}_{k,\Delta}$.

Lemma 3.5 Suppose that Assumptions (i)-(ii) hold. Let the sequences $\{x_k\}$ and $\{d_k\}$ be generated by RSTTCG algorithm. When Eq. (33) holds, there exists $\lambda > 0$ such that

$$\left|K^{\lambda}_{k,\Delta}\right| > \frac{\Delta}{2}, \text{ for } \Delta \in \mathbb{N},$$

where,

$k \geq k_0$, in which k_0 is any index.

Proof Suppose on the contrary that there exists $\lambda > 0$ such that $\left|K^{\lambda}_{k,\Delta}\right| \leq \dfrac{\Delta}{2}$ for $\Delta \in \mathbb{N}$ and for any $k \geq k_0$. By Eq. (32), we have

$$\|\gamma_k y_k\| = \frac{1}{2}\left|\frac{s_k^{\mathrm{T}}g_{k+1}}{s_k^{\mathrm{T}}y_k}\right|\|y_k\| \leq \frac{\sigma}{4(1-\sigma)}\left(1+\frac{L_1}{\tau}\right)\|g_{k+1}\| \doteq L_2\|g_{k+1}\|.$$

According to the definition of Eq. (14), we have

$$\begin{aligned}
\|d_{k+1}\|^2 &\leq \left(\beta_k^{'}\|d_k\| + \|-g_{k+1}+\gamma_k y_k\|\right)^2 \leq 2\beta_k^{'2}\|d_k\|^2 + 2\|-g_{k+1}+\gamma_k y_k\|^2 \\
&\leq 2\beta_k^{'2}\|d_k\|^2 + 2(2\|g_{k+1}\|^2 + 2\|\gamma_k y_k\|^2) \leq 2\beta_k^{'2}\|d_k\|^2 + 4\left(1+L_2^2\right)\|g_{k+1}\|^2,
\end{aligned}$$

the above inequalities are established based on $2ab \leq a^2+b^2$ for any scalars a and b, so $(a+b)^2 \leq 2a^2+2b^2$. By induction, we have

$$\|d_l\|^2 \leq c_1\left(1+2\beta_{l-1}^{'2}+2\beta_{l-1}^{'2}2\beta_{l-2}^{'2}+\cdots+2\beta_{l-1}^{'2}2a\beta_{l-2}^{'2}\cdots2\beta_{k_0}^{'2}\right), \tag{37}$$

for any given index $l \geq k_0+1$, where c_1 depends on $\|d_{k_0-1}\|$, not depends on l. Next, we consider $2\beta_{l-1}^{'2}2\beta_{l-2}^{'2}\cdots2\beta_k^{'2}$, where $k_0 \leq k \leq l-1$. We divide the $2(l-k)$ factors of Eq. (37) into groups of each 2Δ elements, namely, if $\Lambda := (l-k)/\Delta$, then Eq. (37) can be divided into Λ or $\Lambda+1$ groups

$$(2\beta_{l_1}^{'2}\cdots2a\beta_{k_1}^{'2}),\cdots,(2\beta_{l_\Lambda}^{'2}\cdots2\beta_{k_\Lambda}^{'2}), \tag{38}$$

and a possible group

$$(2\beta_{l_{\Lambda+1}}^{'2}\cdots2\beta_k^{'2}), \tag{39}$$

where,

$l_i = l-1-(i-1)\Delta$ for $i = 1,2,\cdots,\Lambda+1$, and $k_i = l_{i+1}+1$ for $i = 1,2,\cdots,\Lambda$. It is clear that $k_i \geq k_0$ for $i = 1,2,\cdots,\Lambda$, from assumption condition, we get $q_i := \left|K^{\lambda}_{k_i,\Delta}\right| \leq \frac{\Delta}{2}$. Thus, there are q_i indices j such that $\|s_j\| > \lambda$ and $(\Delta-q_i)$ indices j such that $\|s_j\| \leq \lambda$ on $[k_i, k_i+\Delta-1]$.

From Eq. (35), we have $b > \frac{\overline{\tau}^2}{c(1-\sigma)\tau^2} > 1$, i.e., $2b^2 > 1$. In conjunction with $2q_i - \Delta \leq 0$, we have $2\beta_{l_i}^{'2}\cdots2a\beta_{k_i}^{'2} \leq 2^{\Delta}b^{2q_i}(\frac{1}{2b})^{2(\Delta-q_i)} = (2b^2)^{2q_i-\Delta} \leq 1$. So every item in Eq. (38) is less than or equal to 1, and so is their product. In Eq. (39), we have $2\beta_{l_{\Lambda+1}}^{'2}\cdots2\beta_k^{'2} \leq (2b^2)^{\Delta}$. Then, we get

$$\|d_l\|^2 \leq c_2(l-k_0+2),$$

where,

$c_2 > 0$ and independent of l. Furthermore, $\sum\limits_{k\geq 0} \frac{1}{\|d_k\|^2} = \infty$. But from sufficient condition Eq. (17), Zoutendijk condition Eq. (21) and Eq. (33), we have

$$c^2\tau^4 \sum_{k\geq 0} \frac{1}{\|d_k\|^2} \leq c^2 \sum_{k\geq 0} \frac{\|g_k\|^4}{\|d_k\|^2} \leq \sum_{k\geq 0} \frac{(g_k^{\mathrm{T}} d_k)^2}{\|d_k\|^2} < \infty.$$

It leads to a contradiction. The proof is completed. □

Theorem 3.2 Suppose that Assumptions (i)-(ii) hold. Let the sequence $\{x_k\}$ be generated by RSTTCG algorithm, then Eq. (30) holds.

Proof Suppose on the contrary that we can get a contradiction similarly to Theorem 4.3 in the study [19]. □

4 Numerical Results

In this section, the numerical performance of RSTTCG algorithm will be listed. All experiments were done on a PC with CPU 2.40 GHz and 2.00 GB RAM using Matlab R2015b.

Table 1. List of the test functions, dimensions, and initial points

P.	Functions	Dim.	P.	Functions	Dim.
1	Freudenstein and Roth	2	41	Chebyquad	1000
2	Powell badly scaled	2	42	Chebyquad	5000
3	Brown badly scaled	2	43	Chebyquad	10000
4	Beale	2	44	Chebyquad	50000
5	Helical valley	3	45	Broyden banded	1000
6	Wood	4	46	Broyden banded	5000
7	Biggs EXP6	6	47	Broyden banded	10000
8	Extended Rosenbrock	1000	48	Broyden banded	50000
9	Extended Rosenbrock	5000	49	Generalized Rosebrock	1000
10	Extended Rosenbrock	10000	50	Generalized Rosebrock	5000
11	Extended Rosenbrock	50000	51	Generalized Rosebrock	10000
12	Extended Powell singular	1000	52	Generalized Rosebrock	50000
13	Extended Powell singular	5000	53	Boundary value	1000
14	Extended Powell singular	10000	54	Boundary value	5000
15	Extended Powell singular	50000	55	Boundary value	10000
16	Penalty function I	1000	56	Boundary value	50000
17	Penalty function I	5000	57	Integral equation	1000
18	Penalty function I	10000	58	Integral equation	5000
19	Penalty function II	1000	59	Integral equation	10000
20	Penalty function II	5000	60	Integral equation	50000
21	Penalty function II	10000	61	Broyden tridiagonal	1000
22	Gaussian	3	62	Broyden tridiagonal	5000
23	Gaussian	3	63	Broyden tridiagonal	10000
24	Box	3	64	Broyden tridiagonal	50000
25	Box	3	65	Separable cubic	1000
26	Variable dimension	1000	66	Separable cubic	5000
27	Variable dimension	5000	67	Separable cubic	10000
28	Variable dimension	10000	68	Separable cubic	50000
29	Variable dimension	50000	69	Nearly separable	1000
30	Watson	1000	70	Nearly separable	5000
31	Watson	5000	71	Nearly separable	10000
32	Watson	10000	72	Nearly separable	50000
33	Watson	50000	73	Yang tridiagonal	1000
34	Brown and Dennis	4	74	Yang tridiagonal	5000
35	Brown and Dennis	4	75	Yang tridiagonal	10000
36	Trigonometric	500	76	Allgower	1000
37	Trigonometric	1000	77	Allgower	5000
38	Trigonometric	5000	78	Allgower	10000
39	Trigonometric	10000	79	Schittkowski 302	5000
40	Trigonometric	50000	80	Schittkowski 302	10000

4.1 Normal Unconstrained Optimization Problems

Two classes of test problems are selected here which are listed in Table 1. One class is drawn from the CUTEr library [20], and the other class come from Andrei [13]. A total of twenty-eight test functions with eighty problems from different dimensions are considered.

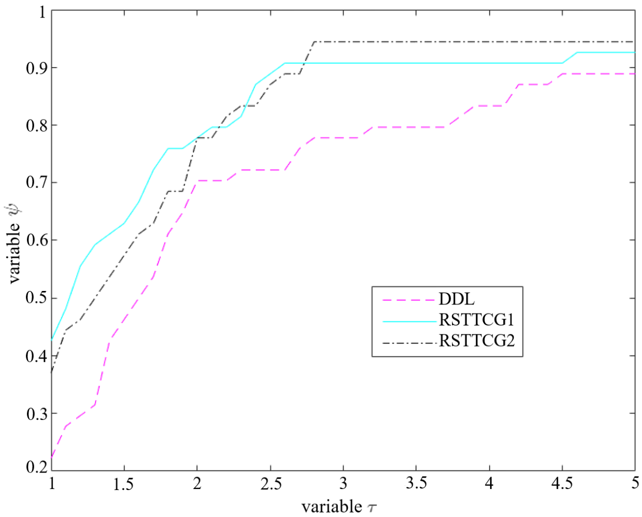

Figure 1. The number of iterations

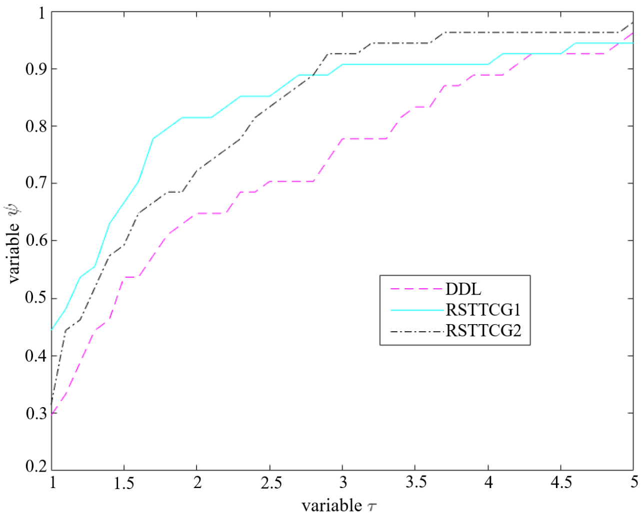

Figure 2. The number of function evaluations

We compare RSTTCG algorithm against DDL method [11] which possess better numerical performance. When

$$\theta_k = \max\left\{\frac{1-\underline{m}}{2(1-\overline{m})}, \frac{\|s_k\|^2}{s_k^{\mathrm{T}} y_k}\right\} \text{ and } \theta_k = \max\left\{\frac{1-\underline{m}}{2(1-\overline{m})}, \frac{s_k^{\mathrm{T}} y_k}{\|y_k\|^2}\right\}$$

are chosen, RSTTCG Algorithm are denoted by "RSTTCG1" and "RSTTCG2", respectively. All test methods are terminated when satisfies condition

$$\|g_k\| \leq \varepsilon \ \text{ or } \text{ the number of iterations exceeds 1000.} \tag{40}$$

We set parameter as $\varepsilon = 10^{-5}$, $\rho = 0.1$, $\sigma = 0.6, \underline{m} = 0.05, \overline{m} = 0.45$. And we set $p = 0.8$ and $q = 0.1$ in DDL algorithm.

The performance profile of four algorithms, included number of iterations, function evaluations, gradient evaluations and CPU time, was analyzed using the profiles of Dolan and Moré [21]. In a performance profile plot, the horizontal axis gives the percentage (τ) of the test problems for which a method is the fastest (efficiency), while the vertical side gives the percentage (ψ) of the test problems that are successfully solved by each of the methods.

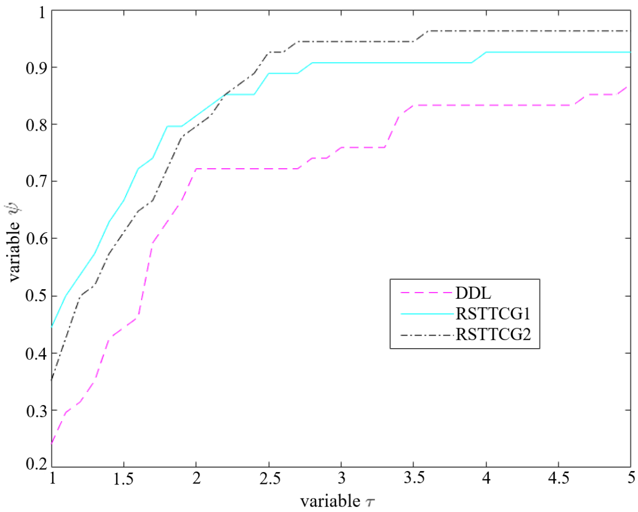

Figure 3. The number of gradient evaluations

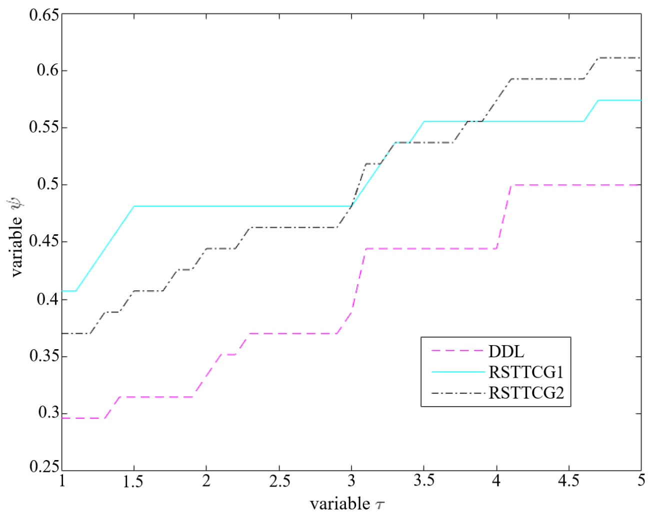

Figure 4. CPU time

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Figures 1-4 plot the performance profiles for the number of iterations, the number of function evaluations, the number of gradient evaluations, and the CPU time, respectively. As can be observed in Figures 1-4, the curves corresponding to RSTTCG1 stays others curves representing RSTTCG2 and DDL methods. This indicates that RSTTCG1 outperforms RSTTCG2 and DDL in all aspects. Furthermore, when $\tau < 2$, the performance of RSTTCG1 is slightly better than RSTTCG2. Whereas, when $\tau \geq 2$, the performance of RSTTCG2 is slightly better than RSTTCG1. We deem that RSTTCG1 may be more competitive than RSTTCG2.

4.2 Fresh Agricultural Products Supply Chain Optimization Problems

In this section, we use RSTTCG algorithm to study the profit-maximization pricing strategy of supply chain of fresh agricultural products led by suppliers under centralized decision making. Consider a two-level fresh produce supply chain in a single-cycle production-sales mode composed of a single fresh produce supplier and a single retailer. The fresh produce supplier, as the leader of the Stackeberg game, supplies both ordinary fresh produce (ofp) and green fresh produce (gfp) of the same variety to the retailer as the follower. Fresh agricultural products supply chain (FPSC) system the overall decision-making structure is shown in Figure 5.

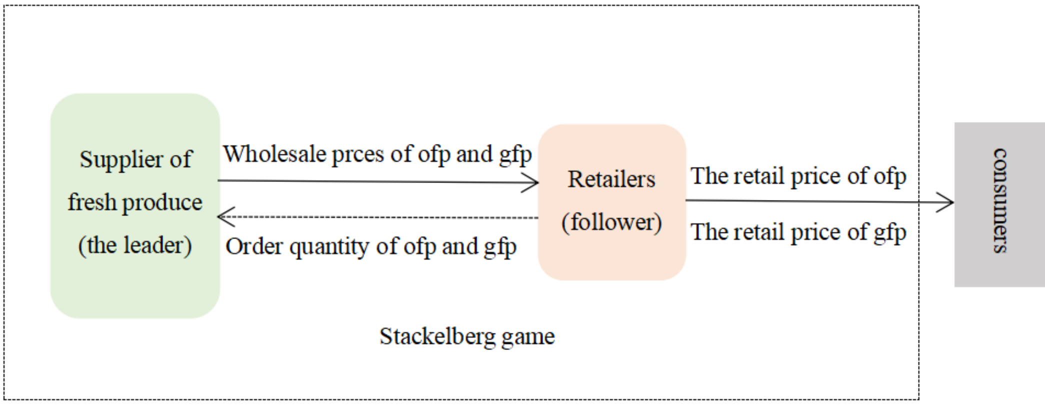

Figure 5. FPSC operation flow chart

Table 2. The optimal solution corresponding to different initial points by RSTTCG1

Initial Point	k	p_1^*	p_2^*	q_1^*	q_2^*	π^*
(1;1)	7	45.0265	43.7651	21.2879	26.4818	1.9449×10^3
(10;10)	6	44.9792	43.7150	21.3107	26.5163	1.9449×10^3
(30;30)	8	44.9970	43.7427	21.3177	26.4825	1.9449×10^3
(50;50)	6	45.0255	43.7949	21.3427	26.4100	1.9449×10^3
(100;100)	7	44.9776	43.7113	21.3079	26.5222	1.9449×10^3
(1000;1000)	6	44.9809	43.7130	21.3033	26.5238	1.9449×10^6

Table 3. The optimal solution corresponding to different initial points by RSTTCG2

Initial Point	k	p_1^{**}	p_2^{**}	q_1^{**}	q_2^{**}	π^{**}
(1;1)	7	44.9459	43.6127	21.2085	26.6981	1.9448×10^3
(10;10)	5	44.9491	43.7202	21.3907	26.4509	1.9449×10^3
(30;30)	6	45.0035	43.7488	21.3132	26.4796	1.9449×10^3
(50;50)	6	45.0234	43.7905	21.3400	26.4165	1.9449×10^3
(100;100)	6	45.0273	43.7657	21.2870	26.4818	1.9449×10^3
(1000;1000)	5	45.0032	43.7530	21.3213	26.4692	1.9449×10^6

Because these two kinds of fresh products are substitutable, there is competition in the demand market, based on the demand function theory of substitute price competition, the demand function of two fresh agricultural products is assumed as follows

$$q_i = a - b\frac{p_i}{\theta} + r\frac{p_j}{\theta}, i = 1, 2, j = 3 - i, \tag{41}$$

where,

q_1, q_2 represent the market demand of gfp and ofp, respectively, a represents the total potential market capacity of fresh agricultural products, p_1, p_2 represent the retail price of gfp and ofp, respectively, b is the price sensitivity coefficient, r is the competitive substitution coefficient of the two products, and satisfy $b > r > 0$, $\theta(0 \leq \theta \leq 1)$ is the freshness of fresh produce when it arrives at the retailer's store. In centralized decision-making, we regard suppliers and retailers as subjects with identical interests, and both sides cooperate to maximize FPSC profits.

Now, under the establishment of centralized decision, the profit function of FPSC is as follows

$$\max_{p_1,p_2} \pi^c = (p_1 - \frac{c_1}{1-\beta})(a - b\frac{p_1}{\theta} + r\frac{p_2}{\theta}) + (p_2 - \frac{c_2}{1-\beta})(a - b\frac{p_2}{\theta} + r\frac{p_1}{\theta}), \tag{42}$$

where,

β $(0 < \beta < 1)$ represents the quantity loss of fresh produce when it reaches the retailer's store. c_1, c_2 represents the unit production cost of gfp and ofp, respectively. Obviously, $p_1 > p_1 > 0$ and $c_1 > c_2 > 0$. We transform Eq. (42) into the following optimization problem

$$\min_{p_1,p_2} \pi^c = -(p_1 - \frac{c_1}{1-\beta})(a - b\frac{p_1}{\theta} + r\frac{p_2}{\theta}) - (p_2 - \frac{c_2}{1-\beta})(a - b\frac{p_2}{\theta} + r\frac{p_1}{\theta}). \tag{43}$$

With reference to the setting of the parameters in the relevant literature [22], we set: $a = 50$, $b = 2$, $c_1 = 4$, $c_2 = 2$, $r = 1.5$, $\beta = 0.2$, $\theta = 0.85$. These values satisfy the theoretical proof in reference [22] and can guarantee that the optimal value has practical significance. We choose different initial points and use RSTTCG algorithm to solve the optimization problem Eq. (43), the results are shown in Table 2 and Table 3. In Table 2 and Table 3, k represents the number of iterations.

As can be seen from Table 2 and Table 3, with certain parameters, RSTTCG algorithm can be used to solve the optimization problem, so as to obtain the optimal pricing strategy with maximum profit in the supply chain led by suppliers under centralized decision making. In addition, the global convergence and effectiveness of RSTTCG algorithm are verified according to different initial values and the number of iterations.

5 Conclusion

In this paper, based on the random technique and a new search direction, a class of spectral three-term spectral three-term CG methods with random parameters are proposed. The random parameters are introduced to simplify the derived parameter. This is achieved by minimizing distance between the symmetric matrix A_{k+1} and memoryless BFGS matrix in Frobenius norm. Global convergence of new algorithm is proved for uniformly convex functions and general nonlinear functions. Some classical test problems are selected for numerical experiments and compared with other methods to verify the effectiveness of proposed algorithm. Numerical experiments show that our methods have nice numerical performance, more relaxed and elastic.

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Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflict of Interest

The authors declare no conflict of interest.

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