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Modeling the Analysis for the Exploitation of Fertilizers and Pesticides on Rice Production in Bangladesh

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Abstract: Rice is a dietary staple and vital economic crop for the majority of Bangladesh's population. To sustain yields amidst growing demand, the widespread application of chemical fertilizers and pesticides has led to concerns over soil health and long-term productivity. This study presents a novel mathematical model, comprising a system of non-linear ordinary differential equations (ODEs), to analyze the effects of diminishing soil fertility due to excessive fertilizer use. The model was investigated analytically and numerically, examining equilibrium points, stability, and the interactions between soil nutrients, plant nutrients, and rice yield. Numerical solutions were obtained using the Runge-Kutta method. Findings indicate that while the initial application of chemical fertilizers results in an increase in yield, prolonged usage ultimately depletes soil organic matters, causing a decline in long-term productivity. The improper use of organic fertilizers exacerbates soil salinity, further hindering rice cultivation. Additionally, rising global temperatures encourage pest proliferation, necessitating higher pesticide usage that adversely affects human health and the environment. The study underscores that optimal fertilizer application, combined with sustainable practices such as straw residue incorporation and land relaxation, improves soil fertility and ensures long-term productivity, addressing food security concerns. Optimal fertilizer application strategies are recommended to sustain rice yields and minimize adverse environmental impacts. The model's insights are crucial for policymakers and farmers in optimizing fertilizer and pesticide use to secure long-term rice productivity in Bangladesh while mitigating the risk of soil degradation.

Keywords: Land relaxation; Long-term productivity; Organic matters; Soil salinity; Straw residue

1. Introduction

Rice serves as a basic food source for more than 50% of the global population, which steadily rises as the world's population expands. Meanwhile, owing to limited agricultural land, farmers have been using more chemical fertilizers and pesticides to produce more rice in a shorter period of time on this restricted arable land, declining the farmland's fertility. As a result, paddy production is jeopardized day after day.

In Bangladesh, the agriculture sector contributes to 19.41% of the total Gross Domestic Product (GDP) and employs 47.5% of the total labor force (Nahar & Hamid, 2016). The country has a population of approximately 164.7 million (World Bank Group, 2022), despite being one of the world's smallest nations. The number rises year after year, similar to the world population, as shown in subgraphs (a) and (b) of Figure 1. Food demand expands in parallel with population growth. If the present trend of population growth continues, it is projected that by 2050, an additional 60%-70% increase in food production would be required to adequately feed the global population (Le Mouël & Forslund, 2017). Although rice and wheat are the most prominent food sources, wheat production increases are heavily sensitive to temperature which is difficult to attain in Bangladesh (Hasan et al., 2019). Therefore, there is no other alternative than to boost paddy yield numerous times in agriculture. Although organic fertilizer increases soil fertility, chemical fertilizer aids in rice production. Therefore, chemical fertilizer is more efficient. Every year, the amount of chemical fertilizer applied to agricultural land grows a bit more, and the land is utilized numerous times without respite. As a result, the land's own or permanent fertility, as shown in subgraph (c) of Figure 1, is dwindling along with its organic matters (Rahman & Zhang, 2018). In most parts of Bangladesh, the amount of organic matters in the soil is less than 2%, whereas it is recommended to have 3-8% or more organic matters for plant growth (Banglapedia, 2021; Kremser & Schnug, 2002; Kumar et al., 2019; LoveToKnow, 2024; Macrotrends, 2023; Srivastav, 2020; Vasquez et al., 2024). As it loses its potential, the land continues to deteriorate and becomes increasingly salty.

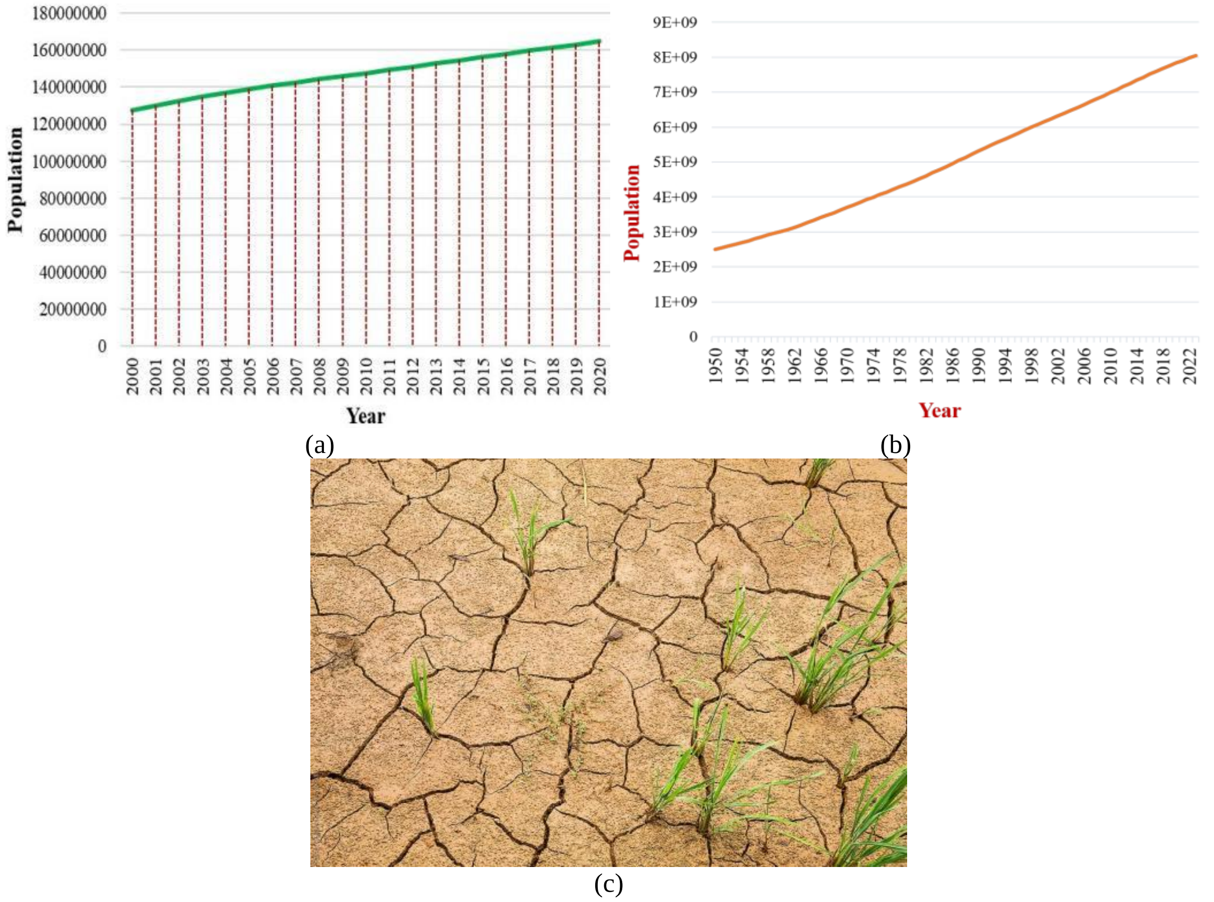


Figure 1. (a) The global population continues to rise at an alarming rate (Macrotrends, 2023); (b) The people living in Bangladesh are always on the rise (World Bank Group, 2022); (c) Dramatic drop in soil fertility poses a hazard to rice production (LoveToKnow, 2024)

This use of chemical fertilizers in agriculture will permanently deplete the fertility of the paddy-growing land, making it impossible to harvest the crop in the future, even with increased use of chemical fertilizers. Hence, a solution should be found to these issues to ensure that Bangladesh will no longer have food shortages. Even though chemical fertilizers are largely responsible for rising food cultivation these days, their excessive use is posing serious problems for both current and future generations, such as soil, water, and air pollution, land degradation, soil depletion, and increased emissions of greenhouse gases (Kumar et al., 2019a). Moreover, nutrient runoff from fertilizer usage causes water bodies to become eutrophic, which in turn causes oxygen deprivation and toxic algal blooms, which are harmful to aquatic ecosystems (Kremser & Schnug, 2002). Furthermore, fertilizers often include nitrogen compounds that have the potential to contaminate drinking water sources, leak into groundwater, and pose health concerns to people (Srivastav, 2020; Vasquez et al., 2024).

Again, about 25% of Bangladesh's cropland is located on the country's coasts, which are about 2.5 million hectares in size. Local varieties, adapted to poor water management, occupy about 60% of the paddy-cropped area, leading to waterlogging and salinity (Kumar et al., 2019b). Salinity surveys and monitoring data show that over 1.02 million hectares (almost 70%) of Bangladesh's arable land is afflicted by various levels of soil salinity, and Khulna, Bagerhat, Satkhira, and Patuakhali have the highest salt levels, affecting rice plants (Nahar & Hamid, 2016). Recently, a report stated that the farmers in the present Boro paddy field in the Khulna region had lost their way owing to a pest infestation. The number of attacks on the Mazra pest has never been so significant. The rice plant's spikes damaged their grains and became white (Sangbad, 2022). In the meantime, to increase production, a wide range of chemical pesticides were used to get rid of pests, unfortunately reducing the nutrients of crops (Kundu & Mallick, 2024) and causing various health diseases for humans, including birth defects, autism, amyotrophic lateral sclerosis, asthma, bronchitis, respiratory disorders, organ diseases, and system failures (Mostafalou & Abdollahi, 2013).

In agriculture, accurate prediction is the most essential aim for economic benefit through producing crops and maintaining soil fertility. It is a worry that if various kinds of prediction need to be evaluated experimentally or with fieldwork, one must work on each new project to receive the results, which is expensive and time-consuming. Instead, if we can define the agricultural issue as a model using mathematical modelling and fit it using data, we can use different model parameters to produce more exact predictions. This can be done quickly and cheaply, allowing us to meet agricultural economic goals.

Inspired by this approach, as there has been no previous work for illustrating this problem with dynamical system, this study proposes a mathematical model implementing ordinary differential equations to determine soil fertility and develop long-term rice production plans to overcome the life-threatening situation and analyzed it theoretically and numerically with graphs, using MATLAB software, to predict the situation for the next ten years. Using this model, it has been mathematically identified that the application of these two types of fertilizers at an appropriate level is needed to improve both productivity and soil fertility. At the same time, straw residues and land relaxation as a treatment of this problem might be helpful to increase the yield by enhancing the soil organic matters as well as soil fertility.

The rest of the paper is organized as follows: Section 2 illustrates the literature review where Methodology is described in Section 3. Section 4 demonstrates a mathematical modeling approach theoretically and numerically to analyze the threats to rice production along with statistical significance for the model's variables. The dynamical behavior of the system to find some solution strategy theoretically and numerical illustrations are given in Section 5. After that, Section 6 declares the limitation of this study. Finally, Section 7 presents the conclusions and discusses their implications while Section 8 highlights the future plan of this work.

2. Literature Review

Rice is not only the major source of nutrition in Bengali culture but also serves as the focal point of all aspects of Bengali life, including the political system, economics, and cultural practices. Despite the transformation of many other aspects of Bengali life over time, rice continues to shine in its unique way. As a side effect of the imbalanced use of fertilizer and pesticides recently, it is a terrible reality that rice production is now in danger. In the meantime, global warming causes a rise in the soil's salinity, which is adverse to the efforts to cultivate rice.

Several studies have been conducted to address this problem. Basak et al. (2015) conducted an experiment to compare the effects of organic and inorganic fertilizers on rice yield. It was found that organic fertilizers led to less rice production for a shorter period with less impact on the environment than inorganic fertilizers and increased soil organic matters. But the use of organic and chemical fertilizers to increase production and soil fertility was not mentioned. On this track, after evaluating the efficacy of chemical fertilizer across four different treatments in a field experiment in Dhaka, Mahmud et al. (2016) mixed organic and chemical fertilizers as the best option. Overuse of fertilizers resulted in a long-term decline in rice yield and soil fertility. However, pesticides were not used, though pests increased as the temperature became more appropriate for their breeding due to global warming. Rahaman et al. (2018) used numerous chemical pesticides to boost rice production, despite their negative consequences on human health. It was found that natural pesticides might benefit farmers by reducing insect populations to produce safe food. However, the effect of chemical residues after using pesticides and fertilizers was not analyzed. Agaba et al. (2020) demonstrated a mathematical model to analyze the effect of agro-chemical residues on rice production. With the aid of analytical and numerical solutions, the findings of the study corroborated the idea that agro-chemical residues may have harmful effects on human health if they are absorbed via farm products like rice grains and others. However, no study has explored the impact of increasing soil salinity on rice production in this region, given the rising sea level resulting from global warming. Besides, in three districts of Patuakhali, Khulna, and Cox's Bazar, Jakariya et al. (2021) used a log-log model and panel data analysis to examine rice production and the relationship between input variables, such as seed, urea, pesticide, and so on. It focused on fertilizer mismanagement and irrigation water scarcity in Kutubdia and introduced measures like a mobile-based online data collection system. According to the findings of the study, researchers are significantly concerned that the rising salinity levels can lead to a decline in rice production and that changes in agricultural practices and improvements in technical efficiency cannot be enough to make up for the lost productivity (Sarker et al., 2022).

By analyzing all related studies, it can be found that mathematical modeling has not been used to predict the impact of joint use of fertilizers and pesticides by considering soil salinity. This helps decrease the amount of time and money spent on doing fresh field trials using a different strategy. The level of accuracy required for mathematical findings is relatively high. Thus, this study rigorously assessed the impact of fertilizers and pesticides while investigating potential mitigation strategies, such as the application of straw residues and land relaxation, to counteract issues arising from the excessive use of agro-chemicals. These strategies were examined using a mathematical model based on a system of ODEs.

3. Methodology

3.1 Modeling and Its Analytical Solution

The newly proposed mathematical model was proposed in this study with the help of a non-linear dynamical system of five variables using ODEs. A Jacobian matrix was used to convert a nonlinear system to a linear one because the behaviour of the non-linear system is almost the same near the equilibrium point like linearity. To find the equilibrium point, some non-linear algebraic equations were solved with the help of MATLAB using the ‘solve’ command. The stability of the model at the equilibrium point was examined by eigenvalue of the co-efficient matrix. In addition, the Routh-Hurwith criterion was used to find stability when it was difficult to find the co-efficient of eigenvalues. Hypothetically, equilibrium points changing the behaviour of the model were observed through the characteristic equations.

3.2 Data Collection

An experiment was conducted in Shovna village, Dumria upazila, Khulna district, Bangladesh, on 2.5 kathas of agricultural land to collect the value of the parameter. This area's soil contains salinity. Rice was cultivated using a combination of organic and chemical fertilizers. Table 1 shows the framework for this cultivation system. Figure 2 illustrates the application of chemical fertilizers (urea, TSP, and potash) and organic fertilizers to the land during the experiment. Then the proposed model was fitted through data collection from this field experiment and the study of some published research works (Al Mamun et al., 2021; Hossain, 2022; Islam et al., 2021; Nasiruddin & Roy, 2012).

Table 1. Activity timeline for the field experiment

Date	Activities
06.02.23	Soil collection
07.02.23	Plantation
01.03.23	Soil collection
02.03.23	Drying land
10.03.23	Fertilizer Apply half of prescribed level
18.03.23	Soil collection
02.04.23	Fertilizer Apply full of prescribed level
17.04.23	Fertilizer Apply full of prescribed level
20.04.23	Soil collection
02.05.23	Fertilizer Apply full of prescribed level
05.05.23	Soil collection
19.05.23	Cutting of yield
20.05.23	Soil collection
05.06.23	Soil collection

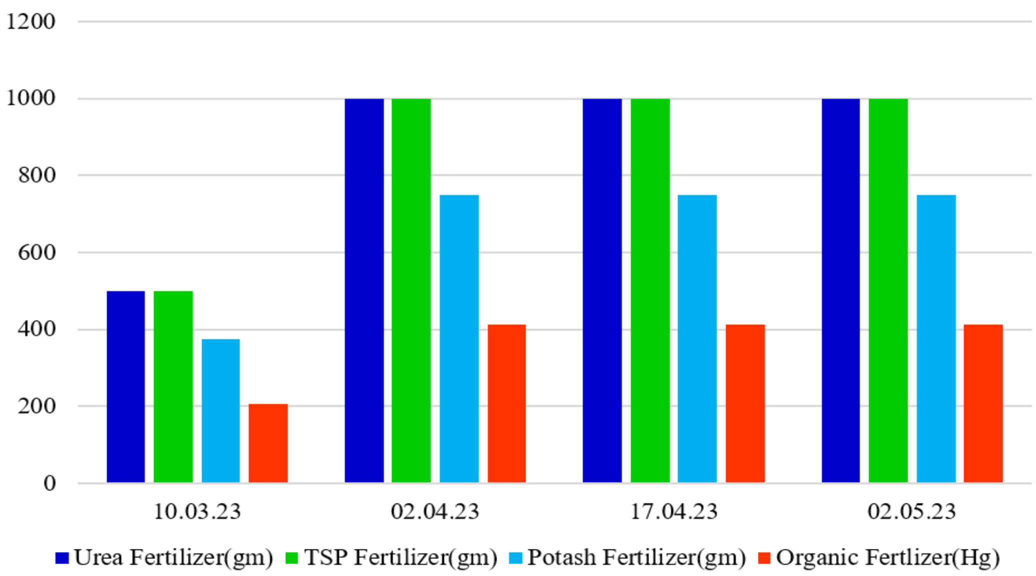


Figure 2. Fertilizer application scenario at the time of the field experiment

3.3 Method for Numerical Simulation

Numerical simulations were conducted utilizing MATLAB software, with the Runge-Kutta fourth-order algorithm being employed to forecast and model various incident scenarios based on the data collected. Concurrently, multiple regression analysis was applied to identify significant relationships among the state variables. Microsoft Excel was utilized to execute the regression analysis and visually represent the outcomes.

Therefore, Figure 3 depicts all of the processes involved in this study.

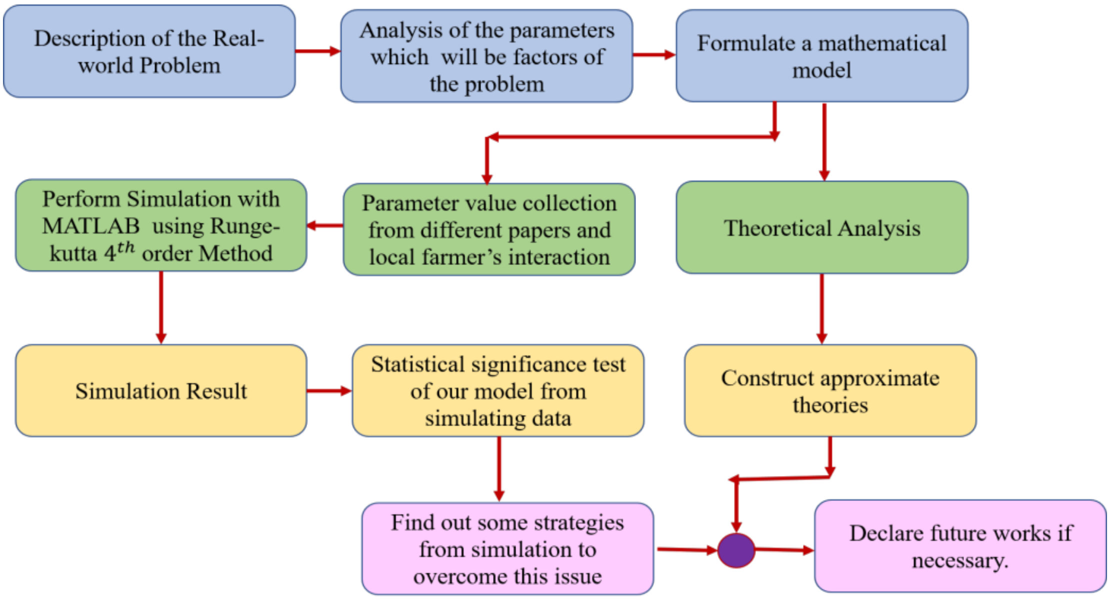


Figure 3. Methodology of the research steps

4. Threats to Rice Production: A Mathematical Modeling Approach

4.1 Model Framework

To explore the threats to the imbalanced use of fertilizers and pesticides on rice production in Bangladesh, the newly proposed model consists of five most concerning variables, such as yield (weight of rice grain and straw), soil organic matters, plant nutrients, soil salinity and pest, which are denoted by $Y(t), O_M(t), P_N(t), S(t)$ and $P(t)$, respectively.

Soil organic matters and plant nutrients both contribute to enhanced rice yields. Conversely, if the quantity of organic matters in the soil and the number of plant nutrients available to the plants are below the standard level, it hampers the increased rate of yield. Tragically, most farmable land in Bangladesh now contains less than the optimum amount of organic matters and plant nutrients. Moreover, pest attacks cause the harvested rice grains to become white and destroy the grain, drastically lowering the production of a field. Taking this into account, the first equation can be expressed as follows:

$$\frac{dY}{dt} = \mu\left(O_M - O_{M_s}\right)Y + \psi\left(P_N - P_{N_s}\right)Y - \delta PY \tag{1}$$

where, μ and ψ are the yield-increasing rates caused by soil organic matters and plant nutrients, respectively, with standard quantities O_{M_s} and P_{N_s}. δ represents the rate at which yield falls because of the bug.

Organic matters in the soil are major contributors to the fertile nature. A higher concentration of organic matters in the soil is one of the benefits of using organic fertilizers. Besides, if rice straw is not picked up and instead left on the land, it eventually breaks down into organic matters. Conversely, constant use of chemical fertilizers can reduce organic matters in the soil by changing its pH, increasing pests, acidifying it, killing beneficial organisms, and thinning the soil crust. The quantity of organic matters in the soil diminishes as plant nutrients are extracted from it, and plants use organic substances to grow yield. Additionally, organic matters are negatively affected by soil salinity. All of this can be expressed in the following equation:

$$\frac{dO_M}{dt} = \xi O_M + \phi O_M - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M \tag{2}$$

where, ξ and ϕ are the rates at which organic fertilizers and straw residues contribute to an increase in organic matters, while β, γ, μ_1, and τ_1 are the rates at which chemical fertilizers, soil salinity, yield, and the conversion of organic matters into plant nutrients contribute to a reduction in soil organic matters.

Furthermore, chemical fertilizers may improve plant nutrition when organic matters are present in the soil. Unlike chemical fertilizers, which may quickly boost plant nutrients, organic fertilizers have a more gradual effect and can be used for a longer period of time without diminishing yields. In light of these considerations, the following equation can be established:

$$\frac{dP_N}{dt} = \varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \psi_1 P_N Y + \tau_1 O_M \tag{3}$$

where, ε, ε_1 and τ_1 are the rates of increase achieved through chemical fertilizers, organic fertilizers, and the conversion of organic matters into plant nutrients, respectively. In addition, ε_2 and ψ_1 are decrease rates of plant nutrients due to overwater flow and yield, respectively.

One of the biggest challenges for rice production is soil salinity, especially in coastal areas. Global warming leads to increasing sea levels. Low-lying coastal regions have been progressively flooding with seawater, polluting the soil. In addition, waterlogging renders plants shallow-rooted, which promotes salinization as salts rise due to capillary action, ultimately making the ground unsuitable for farming. Soil salinity is also raised by the excessive application of organic fertilizers. However, soil salinity may decrease using the leaching approach. The following equation can be generated using the whole set of information:

$$\frac{dS}{dt} = e_1 + \xi\sigma S + \sigma_1 S - \sigma_2 S \tag{4}$$

where, e_1 and $\xi\sigma$ are the salinity increase rates due to sea-level rising and excessive use of organic fertilizers. σ_1 and σ_2 are the increase and decrease rates of soil salinity due to the waterlogging and leaching approaches, respectively.

However, pests are a well-known barrier to increasing paddy production or yield. The number of pests is rising each day as the temperature becomes more appropriate for their breeding due to global warming. As a result, it's high time to minimize it. Chemical and organic pesticide applications can reduce the number of hazardous pests. The last equation can be expressed as follows:

$$\frac{dP}{dt} = \alpha P - \omega P - \kappa P \tag{5}$$

where, α represents the Malthusian natural birth rate of pests, ω and κ denote the mortality rates of pests when treated with organic and chemical pesticides, respectively.

Therefore, the diagram of Figure 4 represents the model's structure. In a nutshell, our proposed model consists of the following equations:

$$f_1(Y, O_M, P_N, S, P) = \frac{dY}{dt} = \mu\left(O_M - O_{M_s}\right)Y + \psi\left(P_N - P_{N_s}\right)Y - \delta PY \tag{6}$$

$$f_2(Y, O_M, P_N, S, P) = \frac{dO_M}{dt} = \xi O_M + \phi O_M - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M \tag{7}$$

$$f_3(Y, O_M, P_N, S, P) = \frac{dP_N}{dt} = \varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \psi_1 P_N Y + \tau_1 O_M \tag{8}$$

$$f_4(Y, O_M, P_N, S, P) = \frac{dS}{dt} = e_1 + \xi\sigma S + \sigma_1 S - \sigma_2 S \tag{9}$$

$$f_5(Y, O_M, P_N, S, P) = \frac{dP}{dt} = \alpha P - \omega P - \kappa P \tag{10}$$

with the initial conditions of $Y(0) \geq 0, O_M(0) \geq 0, P_N(0) \geq 0, S(0) \geq 0$ and $P(0) \geq 0$.

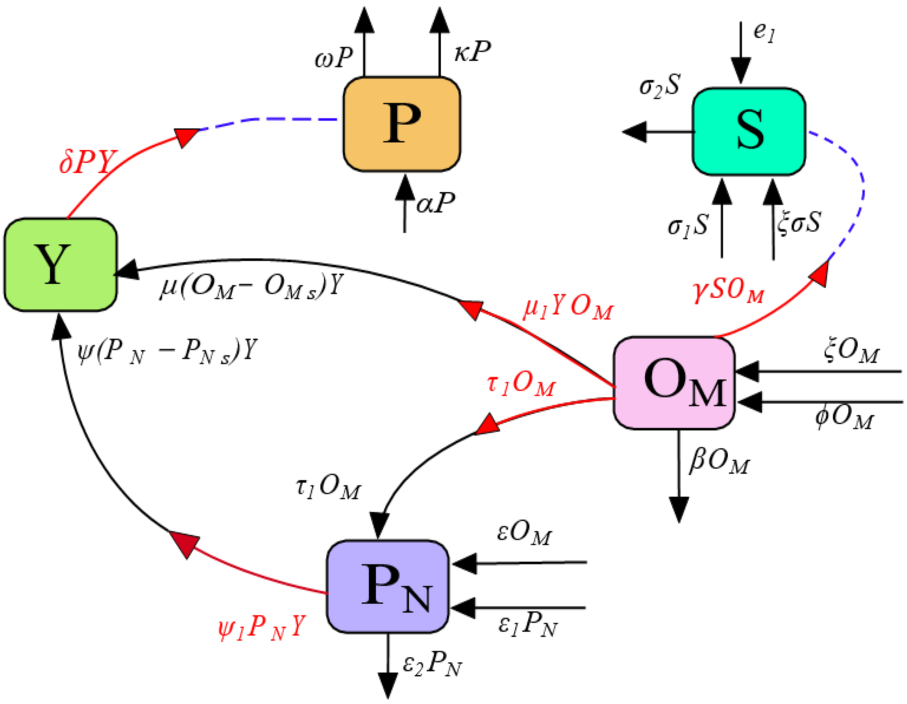


Figure 4. Flow diagram of the proposed model

4.2 Positivity and Boundedness of the Model

Lemma 1. Considering $Y(0) \geq 0$, $O_M(0) \geq 0$, $P_N(0) \geq 0$, $S(0) \geq 0$ and $P(0) \geq 0$, $Y(t)$, $O_M(t)$, $P_N(t)$, $S(t)$ and $P(t)$ are always positive for all $t \in [0, T]$ in R_5^+ where $T > 0$.
Proof. Refer to Appendix A for details.
Lemma 2. The functions f_1, f_2, f_3, f_4 and f_5 are bounded within a closed interval or domain [a,b].
Proof. Refer to Appendix A for details.
Lemma 3. All the solution trajectories of the model $(Y(t), O_M(t), P_N(t), S(t), P(t))$ are constrained inside certain boundaries and attract all solutions originating inside the positive interior.
Proof. Refer to Appendix A for details.
Theorem 4. Existence and uniqueness of the model solution. Let D be a domain defined in such a way that Lipschitz conditions are satisfied. Then, for all non-negative initial conditions, the solutions of the system exist and are unique at the same time for all time $T \geq 0$ in the domain D.
Proof. Refer to Appendix B for details.

4.3 Equilibrium Analysis

By solving the following equations below, the equilibrium points can be obtained:

$$\mu(O_M - O_{M_s})Y + \psi(P_N - P_{N_s})Y - \delta PY = 0 \tag{11}$$

$$\xi O_M + \phi O_M - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M = 0 \tag{12}$$

$$\varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \psi_1 P_N Y + \tau_1 O_M = 0 \tag{13}$$

$$e_1 + \xi\sigma S + \sigma_1 S - \sigma_2 S = 0 \tag{14}$$

$$\alpha P - \omega P - \kappa P = 0 \tag{15}$$

By solving the Eqs. (11)-(15), the following two non-negative equilibrium points E_1 and E_2 can be obtained:

$$E_1 = \left(0, 0, 0, \frac{e_1}{\sigma_2 - \sigma\xi - \sigma_1}, 0\right)$$
$$E_2 = \left(\frac{\varepsilon_1 - \varepsilon_2}{\psi_1}, 0, \frac{P_{N_s}\psi + O_{M_s}\mu}{\psi}, \frac{e_1}{\sigma_2 - \xi\sigma - \sigma_1}, 0\right) \tag{16}$$

They are positive under the following conditions:

$$\sigma_2 > \xi\sigma + \sigma_1, \varepsilon_1 > \varepsilon_2 \tag{17}$$

4.4 Future Status of Soil Organic Matters

Soil fertility can be enhanced by increasing the organic matter content if $\xi + \phi > \beta + \gamma S^* + \mu_1 Y^* + \tau_1$. Conversely, if $\xi + \phi < \beta + \gamma S^* + \mu_1 Y^* + \tau_1$, organic matters in the soil can diminish progressively, eventually leading to a significant decline in soil fertility and thus reducing its capacity to sustain rice production.

Refer to Appendix C for details.

4.5 Stability Analysis of the Model

Theorem 5. Local stability theorem. The dynamical system is unstable at equilibrium point E_1.

Proof. Refer to Appendix C for details.

Theorem 6. Local stability theorem. Under the conditions $\xi + \phi > \beta + \tau_1$, $\varepsilon_1 > \varepsilon_2, \sigma_2 > \sigma o + \sigma_1$ and $e_1\gamma > (\xi + \phi - \beta - \tau_1)(\sigma_2 - \xi\sigma - \sigma_1)$, the equilibrium point E_2 is asymptotically stable if $\alpha < \kappa + \omega$ and is unstable if $\alpha > \kappa + \omega$.

Proof. Refer to Appendix C for details.

4.6 Characteristics of State Equilibrium Values with Respect to $\boldsymbol{\beta}$

Refer to Appendix D for details.

4.7 Convergence in the Quantity of Plant Nutrients

Constant yield and organic matters lead to a convergence in the quantity of plant nutrients. Refer to Appendix E for details.

4.8 Convergence of the Amount of Organic Matters in the Soil

When yield and soil salinity are held constant, this leads to a convergence of the amount of organic matters in the soil. Refer to Appendix F for details.

4.9 Nature of Yield Trends

When the soil organic matters and plant nutrients do not decrease, but the pest population declines, then the nature of yield shows a rising trends. Refer to Appendix G for details.

4.10 Sensitivity Analysis

The sensitivity index quantifies the level to which a certain parameter is important in this model that captures the impact of increasing or reducing organic matters on soil fertility. A procedure that is similar to the one described was used to compute the sensitivity index parameter for the future status of the soil organic matters (R_0).

As a function of the differentiable parameter ν, the normalized forward sensitivity index of the variable M is defined as follows:

$$\mathcal{O}_{\nu}^{M} = \frac{\partial M}{\partial \nu} \times \frac{\nu}{M} \tag{18}$$

The sensitivity indices are expected to vary from -1 to 1. Specifically,

$$
\begin{aligned}
\mho_{\xi}^{R_0} &= \frac{\partial R_0}{\partial \xi} \times \frac{\xi}{R_0} = 0.0279 \\
\mho_{\phi}^{R_0} &= \frac{\partial R_0}{\partial \phi} \times \frac{\phi}{R_0} = 0.3507 \\
\mho_{\beta}^{R_0} &= \frac{\partial R_0}{\partial \beta} \times \frac{\beta}{R_0} = -0.0410 \\
\mho_{e_1}^{R_0} &= \frac{\partial R_0}{\partial e_1} \times \frac{e_1}{R_0} = -0.0671 \\
\mho_{\tau_1}^{R_0} &= \frac{\partial R_0}{\partial \tau_1} \times \frac{\tau_1}{R_0} = -0.0410 \\
\mho_{\epsilon_2}^{R_0} &= \frac{\partial R_0}{\partial \epsilon_2} \times \frac{\epsilon_2}{R_0} = -5.1134 \times 10^{-4} \\
\mho_{\psi}^{R_0} &= \frac{\partial R_0}{\partial \psi} \times \frac{\psi}{R_0} = -6.0729 \times 10^{-10} \\
\mho_{\sigma}^{R_0} &= \frac{\partial R_0}{\partial \sigma} \times \frac{\sigma}{R_0} = -0.96809
\end{aligned} \tag{19}
$$

A number of parameters affect the sensitivity index of the system. Table 2 displays the sensitivity index values for each parameter, which are based on the values in Table 3. The parameters ξ and ϕ are positive, according to the results of this analysis. In other words, when these two parameters rise or fall, R_0 does as well. Conversely, the parameters β, e_1, τ_1, e_2, ψ, and σ have negative values for their sensitivity indices. This indicates that R_0 can be lowered (or raised) if any of these factors increase (or decrease).

Table 2. Sensitivity indices of R_0 with parameters

Parameter	Sensitivity Index	Parameter	Sensitivity Index
ξ	0.0279	ϕ	0.3507
β	-0.0410	e_1	-0.0671
τ_1	-0.0410	ε_2	-5.1134×10^{-4}
ψ	-6.0729×10^{-10}	σ	-0.96809

Table 3. Description of parameters with values

Notation	Descriptions of the Parameter	Unit	Value
μ	Increase rate of yield due to organic matters	H/T	0.0003
O_{M_s}	Standard value of organic matters	T/H	80
ψ	Increase rate of yield due to plant nutrients	H/T	0.32
P_{N_s}	Standard value of plant nutrients	T/H	0.25
δ	Decrease rate of yield due to pest attraction	H/L	0.12
ξ	Growth rate of organic matters due to organic fertilizers		0.45
ϕ	Escalating rate of organic matters due to straw residues		0.15
β	Decrease rate of organic matters for chemical fertilizers		0.45
γ	Decrease rate of organic matters for salinity	m/ds	0.0167
μ_1	Decrease rate of organic matters for yield	H/T	5.26×10^{-3}
τ_1	Conversion rate of organic matters into plant nutrients		1.44×10^{-2}
ε_1	Growth rate of plant nutrients for organic fertilizers		6.35×10^{-5}
ε_2	Decrease rate of plant nutrients due to overwater flow		6.3×10^{-5}
ε	Increase rate of plant nutrients for chemical fertilizers		1.35×10^{-3}
ψ_1	Decrease rate of plant nutrients for yield	H/T	5.24×10^{-2}
e_1	Increase rate for salinity per year due to rising sea levels	ds/m	0.5
$\xi\sigma$	Increase rate of soil salinity due to excessive organic fertilizer usage		0.0198
σ_1	Increase rate of salinity for waterlogging		0.03
σ_2	Decrease rate of salinity by applying the leaching method		0.06
α	Natural growth rate of pests		2.5
ω	Death rate of pests due to the use of organic pesticides		1.00
κ	Death rate of pests due to the use of chemical pesticides		1.00

4.11 Numerical Simulations: Model Visualization

Numerical analysis of the proposed model was conducted using MATLAB (R2018b). Numerical simulations using Runge-Kutta fourth-order approaches graphically showed the dependence of state variables on model parameters and phase portraits of the relevant variables. The parameter values for the system of non-linear differential equations were taken from Table 3, which were collected in the ways mentioned in Subsection 3.2. Based on this information, nowadays, most of the land in this area produces 24.7 Ton/Hectare (T/H) yield, while soil salinity is 6-17 Decisiemens/Metre (ds/m). However, the land's organic matters are less than 5%, and plant nutrients can be up to 0.78 T/H by using fertilizers in Bangladesh. Therefore, to analyze the proposed model, the following initial values of variables were considered: $Y(0) = 24.7$ T/H, $OM(0) = 80$ T/H, $PN(0) = 0.25$T/H, $S(0) = 9$ ds/m, and $P(0) = 0.001$ Lac/Hectare (L/H).

4.11.1 Parametric dynamic variation of state variables

Figure 5 shows the changes in the state variables with respect to ξ, where ξ represents the rate at which organic fertilizer boosts soil organic matters. Up to the first year and five months, the alteration in ξ does not affect the amount of yield, as shown in subgraph (a) of Figure 5. The yield then gradually rises, but it dramatically rises after three years. Then the peak becomes larger, reaching 34.22 T/H at the beginning of the fifth year, with an increase in the quantity of ξ, due to the increase in the amount of ξ. After attaining the yield peaks, the yield consistently drops as the organic fertilizer application rate ξ drops. This trend begins after the yield peaks are reached.

As seen in subgraph (b) of Figure 5, organic matters decline little by little, with a lesser value of ξ up to nearly one year and five months. After this period of time, the yield is 22.01 T/H, 29.64 T/H, and 17.37 T/H in the seventh year when ξ is 0.47, 0.45, and 0.43, respectively, indicating that organic matters begin to fall significantly more proportionally to the value of ξ. Interestingly, the decrease in organic matters, which has eased to a rate of ξ, slows again after the ninth month.

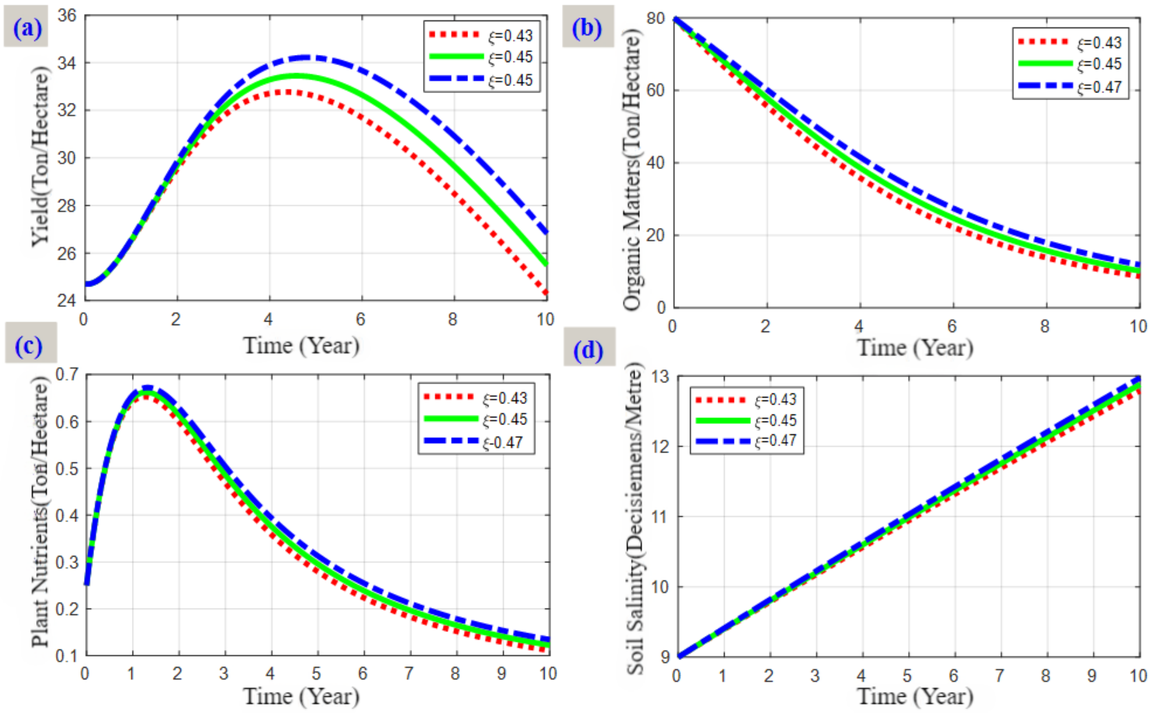


Figure 5. (a) An increased application of organic fertilizers results in enhanced yields; (b) An increased application of organic fertilizers correlates with a lesser reduction in soil organic matter content; (c) Plant nutrients continue to increase with the aid of more organic fertilizers as soil fertility rises; (d) Increased use of organic fertilizers gradually enhances soil salinity, but excessive application can lead to dangerously high levels

It can be observed that there are no shifts in the values of plant nutrients for the variation of ξ in subgraph (c) of Figure 5. However, as plant nutrients approach the second year, they rise more rapidly for higher ξ and reach a maximal value of nearly 0.67 T/H. Once plant nutrients have passed their peak value, they begin to decline again at a slower rate, with the parameter's higher value reflecting a continual increase in its decaying trend, leading to a final value that is far lower than the initial value. Subgraphs (d) of Figure 5 demonstrates that though ξ has no impact on plant nutrients until nearly two years, but progressively more ξ creates more salinity in the soil, resulting in soil salinity reaching 13 ds/m for the higher ξ in the tenth year. Chemical fertilizers reduce organic matters while simultaneously increasing the tendency of organic matters to be converted into plant nutrients.

Therefore, the rate at which organic matters tend to transform into plant nutrients directly correlates with the impact of chemical fertilizers on organic matters.

Again, providing the plant with the necessary nutrients helps to improve productivity. It is believed that if the rate β at which organic matters are reduced for chemical fertilizer rises, this would instantly result in an increase in the rate ψ at which plant nutrients cause a rise in yield. Figure 6 shows how the state variables change as a function of β and ψ, where β is the decrease rate of organic matters due to chemical fertilizers while ψ represents the rate at which yield enhances for plant nutrients. As shown in subgraph (a) of Figure 6, the variation in β and ψ has no impact on the amount of yield from the day of initialization to the fourth month. However, once a period of time equal to four months has elapsed, the yield varies significantly depending on the pace of beta growth. However, it can be observed that when $\beta = 0.50$ and $\psi = 0.421$ are the highest among them, the quantity of yield that is most favorable is 33.97 T/H after three years and eight months. However, the maximum amount of yield that may be achieved decreases slowly when the values of β and ψ are lower. After reaching its highest point, the yield starts to gradually fall. Surprisingly, the values of β and ψ have a negative relationship with the dropping tendency of yield.

According to the data in subgraph (b) of Figure 6, there has been a change in the variation of the β and ψ values since the very first day, but this shift is quite gradual. On the other hand, this change process starts to speed up in the second year. However, from the first day to the next ten years, the amount of organic matters falls as β and ψ increase. It is really fascinating to note that the dynamical pattern of subgraph (c) of Figure 6 is almost identical to that of subgraph (a) of Figure 6, despite the fact that the numerical value is different in each case. This graph shows that there is no difference in beta levels between the beginning of the study and the end of that period after five months. After five months, however, there is a discernible upward trend in plant nutrients proportional to the values of β and ψ. If the value of β is high, then the peak is also greater, which can be observed after an interval of one year. The greatest quantity of plant nutrients is 0.66, 0.64, and 0.63 T/H with $\beta = 0.50$, 0.45, and 0.40 and $\psi = 0.421$, 0.32, and 0.24 after one year and two months, three months, and four months, respectively. Then the amount of plant nutrients available begins to decrease even more, as shown by the ever-increasing values of β and ψ, and the rate at which these plant nutrients diminish continues to steadily accelerate over time.

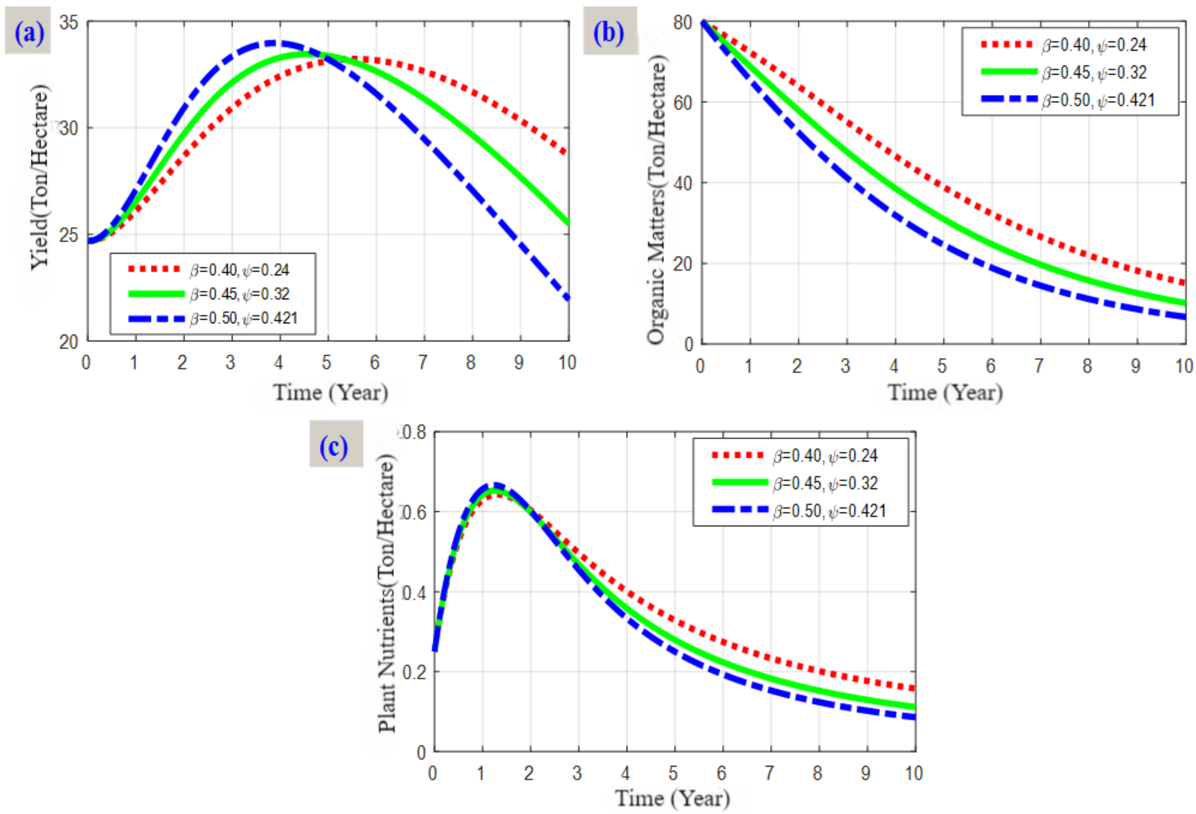


Figure 6. (a) More chemical fertilizers initially lead to a higher yield, but this advantage quickly fades and the production actually drops more as time goes on; (b) With an increase in the use of chemical fertilizers, soil organic matter depletion accelerates; (c) After initially improving plant nutrition, more chemical fertilizers often diminish it more due to the reduction of organic matters

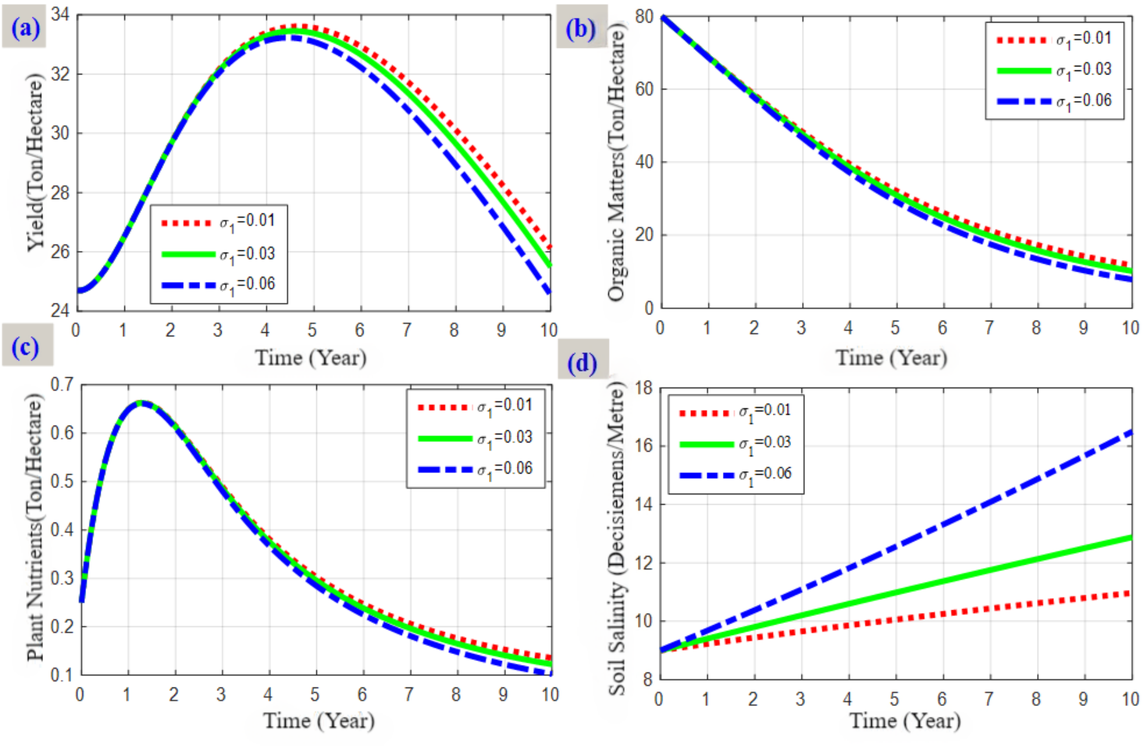


Figure 7. (a) The higher the level of waterlogging, the lower the amount of yield that can be produced; (b) Because of waterlogging, the amount of organic matters in the soil is going down; (c) Increased waterlogging speeds up the process of depleting plant nutrients; (d) Soil salinity rises relentlessly in relation to the rate of waterlogging

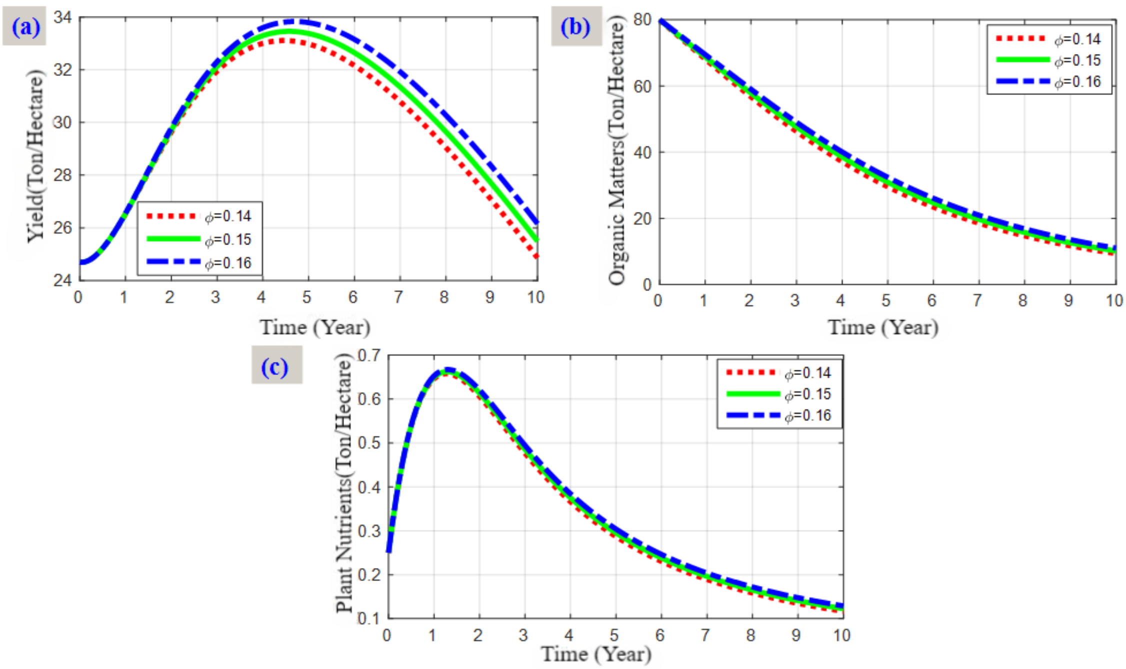


Figure 8. (a) More straw residues have the potential for growth, resulting in increased yield; (b) An increased rate of straw residue gradually improves the depletion of soil organic matters; (c) Straw residues at a higher rate steadily enhance plant nutrition

Figure 7 shows clearer dynamical changes of state variables with parameter σ_1, where σ_1 reflects the rate at which waterlogging enhances the soil salinity. It can be observed from subgraph (a) of Figure 7 that the amount of yield stays constant for up to two years and six months, despite fluctuations in σ_1. Following that, the yield starts to grow dramatically in response to a minor fall in beta. After four years, the peak is reached which is high or low depending on the increasing value of σ_1. Nevertheless, the height of the top is inversely linked to σ_1. After reaching the maximum yield level, the rate of yield drop accelerates when the value of σ_1 is greater which is guaranteed by the scenario that, at the tenth-year mark, the amount of yields is 26.12, 25.49 and 24.55 T/H with a rising value of σ_1 = 0.01, 0.03, and 0.06, respectively. Subgraph (b) of Figure 7 illustrates that waterlogging cannot exhibit its influence on soil organic materials for at least one year and two months. Unfortunately, beyond that point, the soil organic matters start to decline at a rate of σ_1, and this rate of decline steadily rises as waterlogging continues to have a negative impact on the soil carbon over time which is a source of tremendous sadness for the soil health. In response to a change in σ_1, plant nutrients remain unaltered for nearly the first year and two months. But after plant nutrients pass their pick point, the amount of plant nutrients either drops less or more according to the less or more reductive value of σ_1 shown in subgraph (c) of Figure 7. It can be seen from subgraph (d) of Figure 7 that the soil salinity is very sensitive to the value of σ_1. When the value of σ_1 increases, the soil salinity increases to such a great extent that at the tenth year, the soil salinity is 10.97, 16.5 and 12.88, respectively, when σ_1 = 0.01, 0.03 and 0.06, respectively.

The dynamical fluctuation of ϕ can be seen in Figure 8. ϕ represents the rate at which organic materials increase as a result of straw residues. The structure of this figure is quite similar to that of Figure 5, which shows that straw residue plays a significant role in its capacity to function as organic fertilizers. Therefore, when the value of ϕ is large, there is an immediate rise in the soil organic matters as well as the plant nutrients. This ensures that the plant receives the optimal nutrients it needs to produce a higher yield.

According to Figure 9, the growth rate of the pest α poses a danger to rice production currently. This threat may be seen in the figure. According to this number, there is no significant relationship between the rate of pest growth and crop yield over the first one and a half years. Regrettably, after that, the yield is drastically affected by the varied values of the α parameter. When α is low, the apex yield is high. However, when α is high, the yield drops off more quickly after six years.

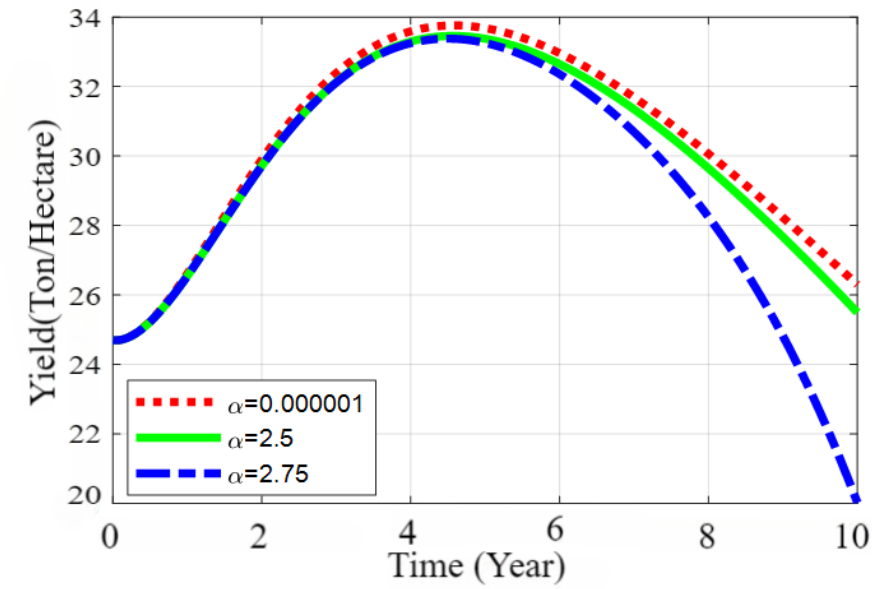


Figure 9. The burgeoning birth rate of pests reduces crop yields

4.11.2 Graphical visualizations of relationships between state variables

In this part, phase portraits obtained from the model simulations were used to explain its qualitative behavior. Figure 10 depicts the non-linear behaviour of the model.

The relationship between organic matters and yields is shown in subgraph (a) of Figure 10. The organic matters are 80 T/H at the beginning, but the yield is 24.7 T/H. Then the organic materials start to drop as a means of making up for the rise in yield. However, as yield hits its peak at four years and four months, yield begins to decline while organic matters continue to fall. It can be seen from subgraph (b) of Figure 10 that when the number of pests in an area increases, so does the rate at which pesticides are applied, resulting in an increase in yield. But after gradually reaching the point of maximum production, yield starts to fall along with growing pest populations because pest populations at this time cannot be controlled with the same rate of pesticide applications, leading to a decrease in yield along with rising pest populations.

Subgraph (c) of Figure 10 reveals that when organic matters are at the highest of 80 T/H, plant nutrients are at 0.25 T/H. Then plant nutrients steadily grow until they reach 0.66 T/H, but the amount of organic matters gradually declines. However, if the value of organic matters is less than 65.71 T/H, the amount of plant nutrients begins to decline. The application of the same amount of chemical and organic fertilizers, therefore, results in a reduction in the amount of plant nutrients and organic matters. Subgraph (d) of Figure 10 clearly demonstrates that the initial yield is 24.7 T/H for plant nutrients at 0.25 T/H. Increasing the supply of plant nutrients leads to a maximum yield of 28.52 T/H. Then plant nutrients begin to decline while yield continues to rise. This trend continues until plant nutrients are 0.3248 T/H, since this is the limit of the optimal range for plant nutrients. But when nutrient levels in the soil are below that range, yield naturally begins to decline.

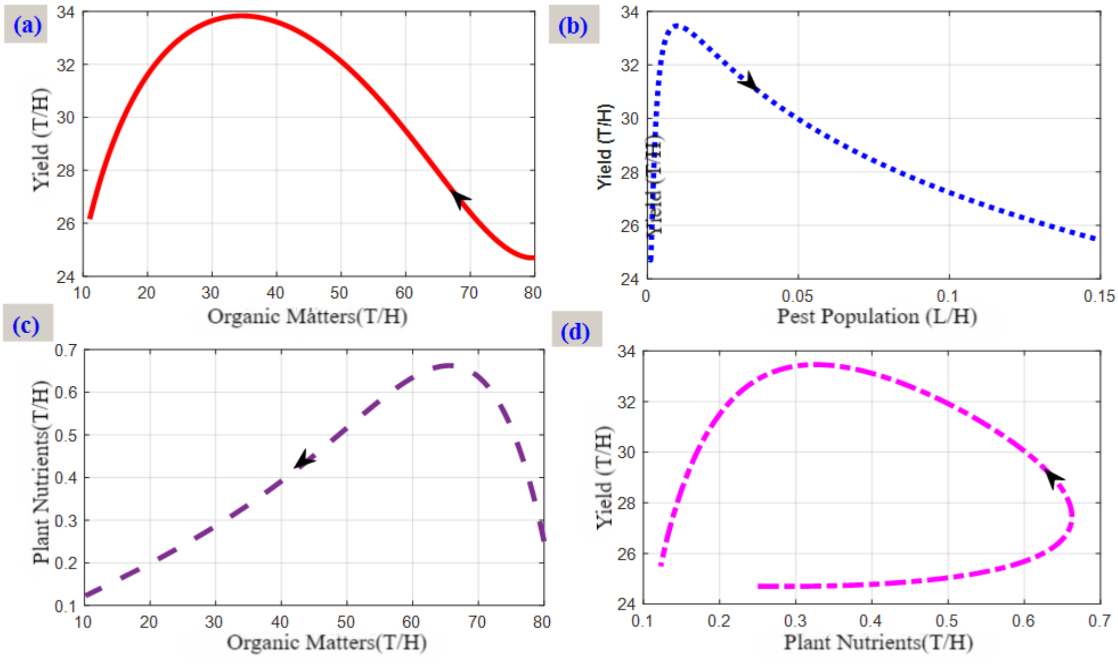


Figure 10. (a) The reduction of organic matters first enhances yield largely due to the use of chemical fertilizers, but yield falls over time along with the decline of organic matters; (b) Yield improves with a relatively small uptick in pest population, but when pest population grows significantly, yield drops precipitously; (c) Plant nutrients increase when organic matters fall as a result of the effects of chemical fertilizers, but the plant nutrients actually decline in the end; (d) Increased plant nutrients improve yield within the standard value of plant nutrients; otherwise, production tends to decline

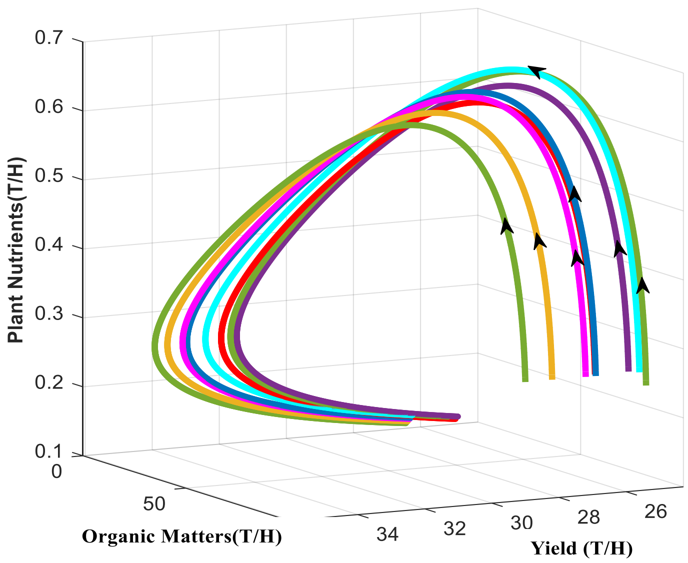


Figure 11. State variables converge with different initial values

State variables, such as yield, organic matters, and plant nutrients, converge over a wide range of initial values, as exemplified by Figure 11. These factors instill confidence in the proposed model's boundedness and the reliability of the study's analysis. As a logical consequence, it may be concluded that the newly proposed model is biologically valid.

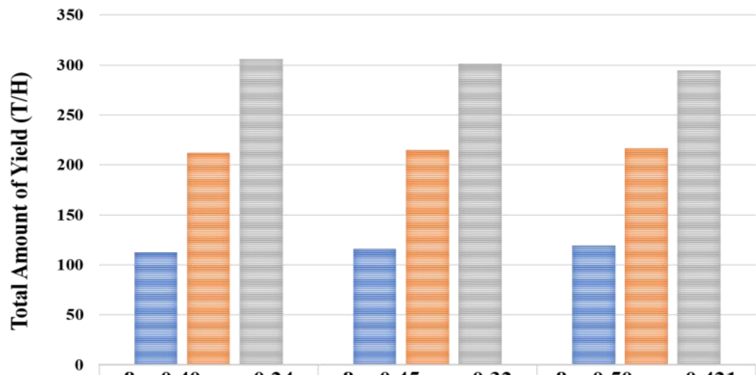
4.12 Results and Discussion

Bangladesh is predominantly an agriculture-based country with rice as the major food crop. Furthermore, farming heavily depends on healthy soil. To put it simply, poor soil quality results in subpar yields. It took just ten kilograms of fertilizers to grow five mounds of paddy two years ago, but now it takes between 14 and 15 kilograms to get the same result. Soil fertility has been declining due to the widespread use of chemical fertilizers (Muhammad, 2022). Meanwhile, production is declining. Research in the field of agriculture warns that if the current trend continues, the agricultural system might eventually collapse, causing a serious food shortage in the near future.

By analyzing the proposed model for this food security problem, some results can be observed virtually in the form of graphs. Generally, yield is affected by plant nutrients and soil organic matters.

Figure 5 suggests that if farmers use more organic fertilizers, it leads to a slow increase in yield and plant nutrients. Commonly used organic fertilizers include composted animal manure, compost, sewage sludge, food processing wastes, and municipal biosolids. They improve soil health and release nutrients to soils gradually. Therefore, their increased use decreases the amount of organic matters in the soil. This finding aligns with the result from the field experiment conducted by Basak et al. (2015). At the same time, increasing the use of organic fertilizers results in a slight rise in soil salinity. Therefore, using them in excess might be detrimental to the soil because the paddy plant cannot tolerate the stress necessary to generate yield if the soil salt rises beyond 20 ds/m.

If farmers continue to apply an ever-increasing rate of chemical fertilizers constantly, the initial rise in plant nutrients can be higher on one side, while the decline in organic matters on the other side. As a direct consequence of this, the amount of organic matters can diminish to an increased proportion over time. Figure 12 helps to illustrate this point in further detail, as shown in Figure 6.



	β = 0.40, ψ = 0.24	β = 0.45, ψ = 0.32	β = 0.50, ψ = 0.421
Total Yield in 4 Year (T/H)	112.83	115.94	119.3012
Total Yield in 7 Year (T/H)	212.48	215.19	216.787
Total Yield in 10 Year (T/H)	305.924	301.64	294.688

Figure 12. Use of more chemical fertilizers leads to a higher total yield up to the seventh year, with the opposite situation occurring in a decade

Total yields in four years are 112.83 T/H when β = 0.40, meaning the use of fewer chemical fertilizers, and 119.3012 T/H when β is larger, meaning the use of more chemical fertilizers. The difference in yield between these two results is 6.4712 T/H. But surprisingly, the gap between the two reduces day by day over time. This difference is just 4.307 T/H after seven years. Interestingly, a decade has revealed the other side of the coin. When fewer chemical fertilizers are applied, the total yield is higher. This shows that the use of more chemical fertilizers has resulted in greater production in recent years by increasing plant nutrients. However, this can ultimately result in a lower yield in the years to come as plant nutrients cannot possibly rise due to the lack of organic matters, as shown in subgraph (b) of Figure 6, which is similar to the situations observed by Haque & Biswas (2020).

Figure 7 clearly shows that waterlogging poses a significant problem for rice cultivation. An increase in soil salinity disrupts soil health, reducing the amount of organic matters in the soil. This can reduce the amount of plant nutrients. Waterlogging reduces production because of the loss of organic matters and plant nutrients.

If farmers keep the straw residue after harvesting on the cultivable land, it can boost soil fertility by functioning organic fertilizers, as seen in Figure 8. Although the pest population does not affect yield much at the beginning, the growth rate of pests can affect yield slowly but surely. Initially, pest control through a specific percentage of pesticide application may enhance production. However, as the pest population increases, maintaining the same quantity of pesticide becomes insufficient to protect the crop, leading to a decline in production. More pesticides can be required this time in order to maintain pest control, resulting in a rise in the demand for the safety food referred to in Figure 9.

Finally, as shown in Figure 10, in order to produce yield, soil organic matters and plant nutrients are reduced every time as the paddy plant needs them for growth. Reducing organic matters may boost production as long as plant nutrients are at their ideal levels. Chemical fertilizers have the ability to improve plant nutrients initially when organic matters in the soil decrease, but those fertilizers inevitably lose this ability when there are relatively few organic matters in the soil over time.

In a nutshell, in order to boost yield and maintain a high level of rice production, attention should be paid to organic matters, thereby helping to utilize more organic fertilizers while employing fewer chemical fertilizers.

4.13 Statistical Significance of the Model

In this section, efforts were made to determine the relationships between yield and all other state variables, such as organic matters, plant nutrients, soil salinity, and pests. To accomplish this, multiple regression was conducted, treating organic matters, plant nutrients, soil salinity, and pests as independent variables and yield as the dependent variable based on a t-test with the help of Microsoft Excel 2016.

Table 4. Results of statistical significance tests

Regression Statistics		Variable	Coefficients	P-value
Multiple R Value	0.9975	Organic Matter	-0.839721	0.00
R Square	0.9950	Plant Nutrients	-0.321064	0.000252
Error	0.1947	Salinity	-16.17364	0.00
Observations	1153	Pest	40.143806	0.00

Table 4 shows the statistical significance test results. After analyzing 1153 simulation data, the R-square value is 0.9950, showing that there is a strong (99%) relationship between yield and all other state variables. Statistically, taken as a set, the predictor, i.e., organic matters, plant nutrients, salinity, and pest, accounts for 99% of the variance in yield, and the standard error is only 0.1947, which is considered as a good level for a 95% confidence test.

Furthermore, this can be highlighted by the fact that none of the p-values are greater than 0.05. Therefore, it can be concluded that yield is strongly affected by factors, such as salinity, organic materials, plant nutrients, and pests.

5. Dynamism of Organic Matters for Consistent Yields

5.1 Model Formulation

Most of the farmers in Bangladesh want to get the same amount of yield. There are a number of reasons why farmers in many countries, including Bangladesh, may want to achieve consistent crop yields (Tbsnews, 2021), like economic stability, food security, market demand, government policies, personal goals and so on. They use more and more chemical fertilizers to satisfy their demands. But chemical fertilizers reduce soil organic matters day by day (Muhammad, 2022). Generally speaking, if farmers wish to maintain an increasing number of crops on a consistent basis, it is vital for them to realize the dynamic changes in organic matters, as this allows them to readily comprehend the fertility conditions of the soil. In conjunction with this, if farmers wish to maintain a lower level of yield, this study can help them by constructing a model, showing the time-dependent changes of the organic matters in the soil along with the previous one.

To formulate the model, four variables were considered, such as soil organic matters (O_M), plant nutrients (P_N) and soil salinity (S) with a constant yield. Then the model equations are as follows:

$$\frac{dO_M}{dt} = \xi O_M + \phi O_M - \beta O_M - \gamma O_M S - \nu_1 O_M - \tau_1 O_M \tag{20}$$

$$\frac{dP_N}{dt} = \varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \nu_2 P_N + \tau_1 O_M \tag{21}$$

$$\frac{dS}{dt} = e_1 + \xi \sigma S + \sigma_1 S - \sigma_2 S \tag{22}$$

In initial conditions of $O_M(0)$, $P_N(0)$, $S(0) \geq 0$, where v_1 and v_2 reflect the declining rate of organic matters and plant nutrients owing to the constant yield. However, all the other parameters have the same identities, as shown in Eq. (6).

5.2 Numerical Simulations

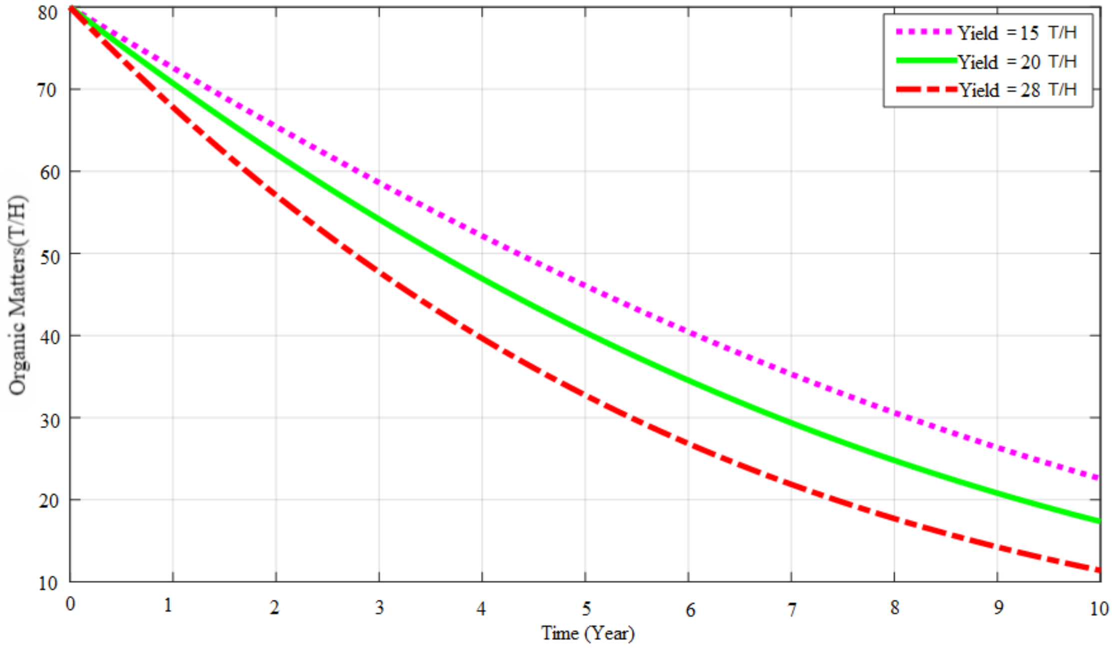


Figure 13. Higher consistent yields mean more degradation of organic matters

Figure 13 illustrates the dynamical shifts in organic matters as a result of maintaining a constant amount of yield year after year. Within a time frame of up to one year, this variation is not significantly linked with the shifting value of yield (Y). The change, though, starts to become more noticeable over time. When the value of yield increases, the link with organic matters weakens and becomes unfavorable. When there is a larger, more consistent yield, there is also a greater reduction in the amount of organic matters.

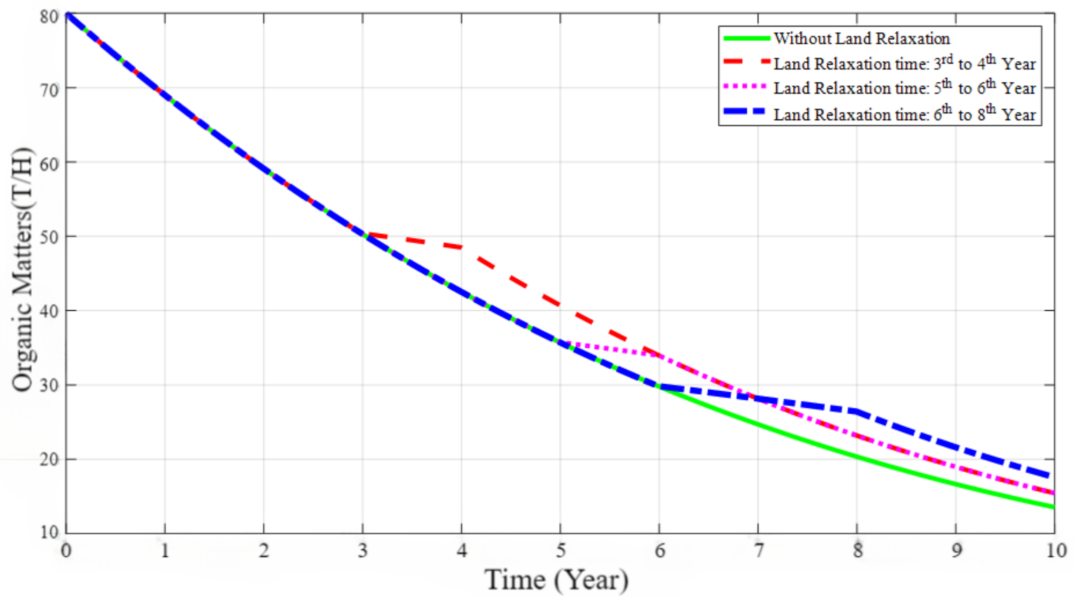


Figure 14. The reduction in soil organic matters is less dramatic when land is allowed to rest for longer periods of time

A plant makes greater use of organic matters throughout its development and other processes in order to increase its yield. Figure 14 presents the difference in organic matter improvement between various time periods with and without land relaxation. This figure shows how continual rice cultivation in the same area reduces soil organic matters. Without land relaxation, organic matters reduce continuously. But when the land is relaxed between the third and fourth years, the soil organic matters increase at that time. The land returns to rice cultivation after the fourth year. Consequently, organic matters begin to decrease. This same incident occurs for land relaxation between the fifth and sixth years. During the sixth and eighth years of a land relaxation phase, organic matters begin to rise in these two years. Then land is used again, causing the decline of organic matters. All of this suggests that land relaxation may be a fantastic strategy for making the land fruitful by boosting soil organic matters over long time intervals.

5.3 Results and Discussion

Organic matters are not only a storehouse of nutrients but also contributes to other factors that regulate fertility in moderation. It is a painful fact that 80% of the land is deficient in organic matters. Farmers constantly apply chemical fertilizers and pesticides in an uncontrolled manner in the hope of higher crop yields. According to Table 5, the organic matters are 52.12, 35.27 and 22.57 T/H at the fourth, seventh and tenth years, provided farmers maintain a production of 15 T/H. But keeping a higher yield, such as 28 T/H, results in organic matters of 39.65, 21.85, and 11.39 T/H after the fourth, seventh and tenth years, respectively.

Table 5. Amount of organic matters in different years due to yield

Yield	Organic Matters (T/H) in		
	4^{th} year	7^{th} year	10^{th} year
Y=15 T/H	52.12	35.27	22.57
Y=20 T/H	46.92	29.33	17.35
Y=28 T/H	39.65	21.85	11.39

This provides more evidence that plants need to extract a greater quantity of nutrients from the soil, thereby enabling farmers to maintain the same level of output. As a direct result of this, farmers use more chemical fertilizers to boost the levels of plant nutrients. By increasing the quantity of plant nutrients, a consistent output may be maintained. Nevertheless, increasing the amount of chemical fertilizers results in a reduction in the amount of organic matters. Therefore, applying more chemical fertilizers in the future won't yield more output. Consequently, farmers have to expand the usage of organic fertilizers while keeping track of the state of the organic matters in the soil.

Land relaxation can be a useful process to increase the organic matters in the soil for long-term cultivation. Table 6 shows that if the land is used continuously to produce rice without any break, the soil organic matters at the tenth year is 13.5483 T/H. Even though it seems implausible, organic matter levels can increase after a year of no cultivation because plants do not take any nutrients from the soil. In this scenario, the soil organic matters can reach 17.5803 T/H if the land relaxation time is extended by two years between the sixth and eighth years. Therefore, farmers may simply boost soil fertility if they leave a portion of their arable land uncultivated from rice production from time to time. At this time of relaxation, the fertility of the land can be enhanced if it is cultivated with green manure or through regular animal grazing. Jaha Delwar Jahan, a graduate of Chittagong University, has been successful in trying to prevent the land from using chemical fertilizers through this process (Desh Rupantor, 2019).

Table 6. Organic matter status for land relaxation

Land Relaxation Time Period	Organic Matters in Soil at 10th Year
No Relaxation	13.5483 T/H
Between 3th to 4th Year	15.4419 T/H
Between 5th to 6th Year	15.4419 T/H
Between 6th to 8th Year	17.5803 T/H

6. Limitations of the Study

The findings of this study ensure that this proposed model can be a new addition to solve the life-threatening rice cultivation problem. But there may be some limitations. It is assumed that rice farming on the land is continuous. Thus, the land where rice and other crops grow cannot be a good fit for this model. To predict the impact of fertilizers and pesticides, it is supposed that farmers apply the same quantity of fertilizers and pesticides to the soil every time which is not probable for all cultivable land.

7. Conclusions

Despite being one of the smallest nations in the world, Bangladesh has 164.6 million residents (World Bank Group, 2022). Due to the decrease in arable land due to the increasing population in fractures, 0.86 to 1.16% of the nation's basic food, rice, has been lost. As arable land decreases, feeding an expanding population is increasingly the primary concern. And more chemical fertilizers are used each time to make up for this productivity deficit. In other words, the soil's efficiency for production decreases and its resources are continually drained. It presents a life-threatening danger to rice production in the future. To analyze this issue, a mathematical model was proposed for analysis both theoretically and numerically.

In the theoretical analysis, positivity and the boundedness theorem help to ensure that the proposed model has a unique solution. In addition, this model is biologically valid. Basic reproduction numbers show the conditions under which soil fertility improves or declines. The equilibrium point is unstable and stable because of some conditions related to real-life situations. Then one parameter was taken for the characteristics test, which obtained an R-square value of 0.9950 and a p-value of less than 0.05, ensuring that the model's variables are significantly related.

Then numerical simulations were conducted. It can be observed that organic fertilizers increase soil organic matters and plant nutrients, but their excessive use increases soil salinity. On the other side, soil salinity is changed by waterlogging, leading to a reduction in organic matters. One of the interesting findings is that more application of chemical fertilizers increases yields for some years, but it is also a great threat to long-term productivity because it decreases production when soil organic matters are reduced due to the harmful effects of chemical fertilizers. Another important finding is that the growth of the pest population can be an obstacle in the future if it is not under control. Soil organic matters are decreased to maintain the consistency of rice production because of the use of chemical fertilizers, which makes us thoughtful. To find out the process of this solution, it has been found that keeping straw residue on the cultivable land after harvesting and keeping a portion of that land rested can be useful to increase soil fertility. Ultimately, based on the prediction of the proposed model, it is realistic to state that Bangladesh, which has a large population, is incapable of having its rice production decreased. In order to preserve the organic materials in the soil, it is now absolutely necessary to apply fertilizers on land in an optimum amount and to increase the amount of organic fertilizers applied. The sooner farmers become aware of this, the better they will be able to maintain food security in the future.

8. Future Work

Following the interpretation of the model, it can be observed that there is a pressing need to control the use of both chemical and organic fertilizers if the quality of the rice harvest in Bangladesh can be maintained. Meanwhile, controlling the use of both chemical and natural pesticides is becoming increasingly significant in the pursuit of producing safe food. As a result, as part of the ongoing study, the concept of optimal control could be applied to the proposed model, thereby optimizing the amount of rice and safe food produced.

Author Contributions

Conceptualization, P.K. and U.K.M.; methodology, P.K. and U.K.M.; software, P.K. and U.K.M.; validation, U.K.M.; formal analysis, P.K.; data collection, P.K.; writing—original draft, P.K.; writing—review and editing, P.K. and U.K.M.; supervision, U.K.M. All authors have read and agreed to the published version of the manuscript.

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Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Appendix

Appendix A

Proof for Lemma 1:

If all parameters of the system and all initial values are positive, $Y(t)$, $O_M(t)$, $P_N(t)$, $S(t)$ and $P(t)$ are positive for all $t \in [0,T]$ in R_5^+ should be proved.

Eq. (4) can be written as follows (Athanassov, 1985):

$$\frac{dS}{dt} = e + \sigma o S + \sigma_1 S - \sigma_2 S \tag{23}$$

$$\frac{dS}{dt} \geq \sigma o S + \sigma_1 S - \sigma_2 S \tag{24}$$

$$\frac{dS}{S} \geq (\sigma o + \sigma_1 - \sigma_2) dt \tag{25}$$

Integrating both sides leads to the following equation:

$$\ln S - \ln S_0 \geq (\xi\sigma + \sigma_1 - \sigma_2)t \tag{26}$$

Then this result can be expressed as follows:

$$\ln \frac{S}{S_0} \geq (\sigma o + \sigma_1 - \sigma_2)t \tag{27}$$

$$\frac{S}{S_0} \geq e^{(\xi\sigma + \sigma_1 - \sigma_2)t} \tag{28}$$

$$\therefore S(t) \geq e^{(\xi\sigma + \sigma_1 - \sigma_2)t} > 0 \tag{29}$$

Again Eq. (5) can be written as follows:

$$\frac{dP}{dt} = \alpha P - (\omega + \kappa)P \tag{30}$$

Integrating both sides leads to the following equation:

$$\therefore P(t) = P_0 e^{(\alpha - \omega - \kappa)t} > 0 \tag{31}$$

Since it has been found that $S > 0$ and γ is a positive parameter, therefore, $\gamma S = \gamma_1 > 0$.
Then Eq. (2) leads to the following equations:

$$\frac{dO_M}{dt} = \xi O_M + \phi O_M - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M \tag{32}$$

$$\frac{dO_M}{O_M} = (\xi + \phi - \beta - \gamma_1 - \tau_1 - \mu Y)dt \tag{33}$$

Integrating both sides lead to the following equations:

$$\ln O_M = (\xi + \phi - \beta - \gamma_1 - \tau_1)t - \mu \int Y dt \tag{34}$$

$$\therefore O_M(t) = e^{(\xi + \phi - \beta - \gamma_1 - \tau_1)t - \mu \int Y dt} > 0 \tag{35}$$

Again, it can be observed that O_M is positive. Therefore, the value of $(\varepsilon + \tau)O_M$ is also positive. Consequently, it can be removed as a positive term, leading to the following equation:

$$\therefore P_N(t) \geq e^{(\varepsilon_1 - \varepsilon_2)t - \psi_1 \int Y dt} > 0 \tag{36}$$

Now, the following equation can be obtained after similar calculation from Eq. (1):

$$Y = e^{\mu \int (O_M - O_{Ms})dt + \psi \int (P_N - P_{Ns})dt - delta \int P dt} > 0 \tag{37}$$

These all complete the proof.

Proof for Lemma 2:

A function is said to be continuous on a closed interval $[a, b]$ when it is defined at every point on that interval $[a, b]$ and undergoes no interruptions, jumps, or breaks.

It can be clearly observed from Eq. (6) that all the functions f_1, f_2, f_3, f_4 and f_5 are not undefined for any point. Therefore, they must be defined for every point in an interval $[a, b]$ because the prediction in this study was performed using this model for ten years. The conditions of $a = 0$ and $b = 10$ are required for the model's equation. Therefore, the functions f_1, f_2, f_3, f_4 and f_5 are continuous within a closed interval or domain $[a, b]$.

According to the boundedness theorem mentioned by Fu & Yu (2020), a continuous function on a closed interval must be bounded on that interval. Therefore, the functions considered in this study are also bounded in the interval $[a, b]$.

This completes the proof.

Proof for Lemma 3:

First, Eqs. (2) and (4) were added, implying that:

$$\dot{O}_M + \dot{S} = e_1 + (\xi\sigma + \sigma_1 - \sigma_2) + (\xi + \phi - \beta - \tau_1)O_M - \gamma O_M S - \mu O_M Y \tag{38}$$

$$\frac{d(O_M + S)}{dt} \leq e_1 - k(S + O_M) \tag{39}$$

$$\frac{d(O_M + S)}{dt} - k(S + O_M) \leq e_1 \tag{40}$$

where, $k = min\{(\xi\sigma + \sigma_1 - \sigma_2), (\xi + \phi - \beta - \tau_1)\}$. Then after integrating with the help of integrating factor, the following equation can be obtained:

$$\lim_{t\to\infty} Sup\{O_M + S\} \le \frac{e_1}{k} = M_0 \tag{41}$$

Again,

$$\frac{dP}{dt} = (\alpha - \kappa - \omega)P \tag{42}$$

$$P = e^{-\kappa+\omega-\alpha t} \tag{43}$$

The above results demonstrate that when $t \to \infty$, P is bounded.
Then the following equations can be found from Eq. (3):

$$\dot{P}_N \le -k_1 P_N + dM_0 \tag{44}$$

$$\dot{P}_N + k_1 P_N \le dM_0 \tag{45}$$

where, $k_1 = min\{\epsilon_1, \epsilon_2\}$.
Then $\underset{t\to\infty}{limSup}\{P_N\} \le \frac{dM_0}{k_1}$ can be obtained using the comparison theorem.
Furthermore, Eq. (1) leads to the following equations:

$$Y = e^{\mu\int (O_M - O_{Ms})dt + \psi\int (P_N - P_{Ns})dt - delta\int Pdt} \tag{46}$$

$$Y = e^{\mu\int O_M dt + \psi\int P_N dt - \int Pdt - (\mu O_{Ms} + \psi P_{Ns})t} \tag{47}$$

$$\therefore Y = e^{\mu\beta_m + \psi\beta_n - \delta\beta_{mn} - (\mu O_{Ms} + \psi P_{Ns})t} \tag{48}$$

Therefore, $Y(t)$ is also bounded.
This completes the proof.

Appendix B

Proof for Theorem 4:
The following is continuous and bounded in the domain D where $n = 5$.

$$\frac{\partial f_i}{\partial x_j}, \qquad i, j = 1,2,3,..,n \tag{49}$$

Eq. (6) leads to the following equations:

$$\frac{\partial f_1}{\partial Y} = \mu(O_M - O_{Ms}) - \psi(P_N - P_{Ns}) - \delta P \tag{50}$$

$$\therefore \left|\frac{\partial f_1}{\partial Y}\right| = \left|\mu(O_M - O_{Ms}) - \psi(P_N - P_{Ns}) - \delta P\right| < \infty \tag{51}$$

$$\frac{\partial f_1}{\partial O_M} = \mu Y, \left|\frac{\partial f_1}{\partial O_M}\right| = |\mu Y| < \infty \tag{52}$$

$$\frac{\partial f_1}{\partial P_N} = \psi Y, \left|\frac{\partial f_1}{\partial P_N}\right| = |\psi Y| < \infty \tag{53}$$

$$\frac{\partial f_1}{\partial S} = 0, \left|\frac{\partial f_1}{\partial P_N}\right| < \infty \tag{54}$$

$$\frac{\partial f_1}{\partial P} = -\delta Y, \left|\frac{\partial f_1}{\partial P_N}\right| = \left|-\delta Y\right| < \infty \tag{55}$$

Again,

$$\frac{\partial f_2}{\partial Y} = -\mu_1 O_M, \left|\frac{\partial f_2}{\partial Y}\right| = \left|-\mu_1 O_M\right| < \infty \tag{56}$$

$$\frac{\partial f_2}{\partial O_M} = \xi - \phi - \beta - \tau_1 - \gamma S - \mu_1 Y \tag{57}$$

$$\therefore \left|\frac{\partial f_2}{\partial O_M}\right| = \left|\xi - \phi - \beta - \tau_1 - \gamma S - \mu_1 Y\right| < \infty \tag{58}$$

$$\frac{\partial f_2}{\partial P_N} = 0, \left|\frac{\partial f_2}{\partial P_N}\right| < \infty \tag{59}$$

$$\frac{\partial f_2}{\partial S} = -\gamma O_M, \left|\frac{\partial f_2}{\partial S}\right| = \left|-\gamma O_M\right| < \infty \tag{60}$$

$$\frac{\partial f_2}{\partial P} = 0, \left|\frac{\partial f_1}{\partial P}\right| < \infty \tag{61}$$

In this same manner, the following can be easily observed:

$$\left|\frac{\partial f_3}{\partial Y}\right|, \left|\frac{\partial f_3}{\partial O_M}\right|, \left|\frac{\partial f_3}{\partial P_N}\right|, \left|\frac{\partial f_3}{\partial S}\right|, \left|\frac{\partial f_3}{\partial P}\right| < \infty \tag{62}$$

$$\left|\frac{\partial f_4}{\partial Y}\right|, \left|\frac{\partial f_4}{\partial O_M}\right|, \left|\frac{\partial f_4}{\partial P_N}\right|, \left|\frac{\partial f_4}{\partial S}\right|, \left|\frac{\partial f_4}{\partial P}\right| < \infty \tag{63}$$

$$\left|\frac{\partial f_5}{\partial Y}\right|, \left|\frac{\partial f_5}{\partial O_M}\right|, \left|\frac{\partial f_5}{\partial P_N}\right|, \left|\frac{\partial f_5}{\partial S}\right|, \left|\frac{\partial f_5}{\partial P}\right| < \infty \tag{64}$$

Hence, all the partial derivatives are continuous and bounded, satisfying the Lipschitz condition. According to the theorem discussed by Sowole et al. (2019), there exists a unique solution of system (1)-(5) in the region *D*.

This verifies the theorem.

Appendix C

Details for Section 4.4:

Using the next generation matrix method mentioned in study (Diekmann et al., 1990), the future status of soil organic matters R_0 can be found. To examine the dynamical behavior of soil organic matters associated with soil fertility, this state variable is associated for the discussion.

$$\frac{dO_M}{dt} = \xi O_M + \phi O_N - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M \tag{65}$$

Differentiating it with respect to O_M, the following equation can be obtained:

$$\frac{d}{dO_M}\left(\frac{dO_M}{dt}\right) = (\xi + \phi) - (\beta + \gamma S + \mu_1 Y + \tau_1) \tag{66}$$

Therefore, at the equilibrium point $(Y^*, O_M^*, P_N^*, S^*, P^*)$, two matrices M and D which represent increase and decrease of soil organic matters are $M = (\xi + \phi)$ and $D = (\beta + \gamma S^* + \mu_1 Y^* + \tau_1)$.

Then the future status of soil organic matters R_0 is as follows:

$$R_0 = \frac{M}{D} = \frac{\xi + \phi}{\beta + \gamma S^* + \mu_1 Y^* + \tau_1} \tag{67}$$

Proof for Theorem 5:

To proof theorem 5, the following equations were first considered:

$$f_1(Y, O_M, P_N, S, P) = \mu(O_M - O_{Ms})Y + \psi(P_N - P_{Ns})Y - \delta PY = 0 \tag{68}$$

$$f_2(Y, O_M, P_N, S, P) = \xi O_M + \phi O_N - \beta O_M - \gamma O_M S - \mu_1 O_M Y - \tau_1 O_M \tag{69}$$

$$f_3(Y, O_M, P_N, S, P) = \varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \psi_1 P_N Y + \tau_1 O_M \tag{70}$$

$$f_4(Y, O_M, P_N, S, P) = e_1 + \xi\sigma S + \sigma_1 S - \sigma_2 S \tag{71}$$

$$f_5(Y, O_M, P_N, S, P) = \alpha P - \omega P - \kappa P \tag{72}$$

For the Eqs. (68)-(72), the Jacobian matrix is as follows:

$$J = \frac{\partial(f_1, f_2, f_3, f_4, f_5)}{\partial(Y, O_M, P_N, S, P)} = \begin{pmatrix} \frac{\partial f_1}{\partial Y} & \frac{\partial f_1}{\partial O_M} & \frac{\partial f_1}{\partial P_N} & \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial Y} & \frac{\partial f_2}{\partial O_M} & \frac{\partial f_2}{\partial P_N} & \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial P} \\ \frac{\partial f_3}{\partial Y} & \frac{\partial f_3}{\partial O_M} & \frac{\partial f_3}{\partial P_N} & \frac{\partial f_3}{\partial S} & \frac{\partial f_3}{\partial P} \\ \frac{\partial f_4}{\partial Y} & \frac{\partial f_4}{\partial O_M} & \frac{\partial f_4}{\partial P_N} & \frac{\partial f_4}{\partial S} & \frac{\partial f_4}{\partial P} \\ \frac{\partial f_5}{\partial Y} & \frac{\partial f_5}{\partial O_M} & \frac{\partial f_5}{\partial P_N} & \frac{\partial f_5}{\partial S} & \frac{\partial f_5}{\partial P} \end{pmatrix} \tag{73}$$

$$\Rightarrow J = \begin{pmatrix} \psi(P_N - P_{Ns}) - \delta P + \mu O_M - O_{Ms} & \mu Y & \psi Y & 0 & -\delta Y \\ \mu_1 O_M & c - \gamma S - \mu_1 Y & 0 & -\gamma O_M & 0 \\ -\psi_1 P_N & v & d - \psi_1 Y & 0 & 0 \\ 0 & 0 & 0 & -u & 0 \\ 0 & 0 & 0 & 0 & \alpha - \omega - \kappa \end{pmatrix} \tag{74}$$

where, $c = \xi + \phi - \beta - \tau_1$, $d = \varepsilon_1 - \varepsilon_2, v = \tau_1 + \varepsilon$ and $u = \sigma_2 - \xi\sigma - \sigma_1$.

At the equilibrium point $E = \left(0,0,0,\frac{e_1}{u},0\right)$, the Eq. (74) becomes

$$J_{|E_1} = \begin{pmatrix} -(\psi P_{N_s} + \mu O_{Ms}) & 0 & 0 & 0 & -\delta \\ 0 & c - \frac{e_1\gamma}{u} & 0 & 0\gamma & 0 \\ 0 & v & d & 0 & 0 \\ 0 & 0 & 0 & -u & 0 \\ 0 & 0 & 0 & 0 & \alpha - \omega - \kappa \end{pmatrix} \tag{75}$$

Therefore, the characteristic equation is as follows:

$$\left| J_{|E_1} - \lambda I \right| = 0 \tag{76}$$

$$(-\lambda + \alpha - \omega - \kappa) \begin{vmatrix} -(\psi P_{N_s} + \mu O_{Ms}) - \lambda & 0 & 0 & 0 \\ 0 & c - \frac{e_1\gamma}{u} - \lambda & 0 & 0 \\ 0 & v & d - \lambda & 0 \\ 0 & 0 & 0 & -u - \lambda \end{vmatrix} = 0 \tag{77}$$

$$\left(-\lambda+\alpha-\omega-\kappa\right)\left(-\lambda-u\right)\left(\lambda-d\right)\left(-\psi P_{N_s}-\mu O_{Ms}-\lambda\right)\left(c-\frac{e_1\gamma}{u}-\lambda\right)=0 \tag{78}$$

It can be clearly observed that there are five different values of eigenvalue as follows:

$$\lambda_1=\alpha-\omega-\kappa \tag{79}$$

$$\lambda_2=-(\sigma_2-\xi\sigma-\sigma_1) \tag{80}$$

$$\lambda_3=\epsilon_1-\epsilon_2 \tag{81}$$

$$\lambda_4=\omega+\kappa-\alpha \tag{82}$$

$$\lambda_5=\xi+\phi-\beta-\tau_1-\frac{e_1\gamma}{\sigma_2-\sigma o-\sigma_1} \tag{83}$$

It has been found that the value of λ_3 is not negative because $\epsilon_1>\epsilon_2$ and λ_3 is assumed to be positive. As all the eigenvalues (λ) are not negative, the proposed model or dynamical system is unstable at the equilibrium point E_1. This finishes the proof.

Proof for Theorem 6:

At the following equilibrium point

$$E_2=\left(\frac{\varepsilon_1-\varepsilon_2}{\psi_1},0,\frac{P_{Ns}\psi+O_{Ms}\mu}{\psi},\frac{e_1}{\sigma_2-\xi\sigma-\sigma_1},0\right) \tag{84}$$

Eq. (75) can be expressed as follows:

$$J_{|E_2}=\begin{pmatrix} -\left(\psi\left(P_{N_s}-\frac{r}{\psi}\right)+\mu O_{Ms}\right) & \frac{d\psi}{\psi_1} & \psi & 0 & \frac{-\delta d}{\psi_1} \\ 0 & c-\gamma-\frac{e_1\gamma}{u}-\frac{d\mu_1}{\psi_1} & 0 & 0 & 0 \\ \frac{-\psi_1 r}{\psi} & v & 0 & 0 & 0 \\ 0 & 0 & 0 & -u & 0 \\ 0 & 0 & 0 & 0 & \alpha-\omega-\kappa \end{pmatrix} \tag{85}$$

where, $r=\psi P_{Ns}+\mu O_{Ms}$.

Therefore, the characteristic equation is as follows:

$$\begin{vmatrix} -q_1-\mu O_{Ms}-\lambda & \frac{d\mu}{\psi_1} & \frac{d\psi}{\psi_1} & 0 & \frac{-\delta d}{\psi_1} \\ 0 & c-\gamma-q_2-\lambda & 0 & 0 & 0 \\ \frac{-\psi_1 r}{\psi} & v & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -u-\lambda & 0 \\ 0 & 0 & 0 & 0 & \alpha-\omega-\kappa-\lambda \end{vmatrix}=0 \tag{86}$$

where, $q_1=\psi\left(P_{N_s}-\frac{r}{\psi}\right), q_2=\gamma+\frac{e_1\gamma}{u}+\frac{d\mu_1}{\psi_1}$. Then the following equations can be obtained:

$$\lambda_1=\alpha-\omega-\kappa \tag{87}$$

$$\lambda_2=-(\sigma_2-\sigma o-\sigma_1) \tag{88}$$

Both of them are negative if $\alpha<\omega+\kappa$. Another three values for λ can be found from the following equation:

$$
\begin{vmatrix}
-\psi\left(P_{N_s}-\dfrac{r}{\psi}\right)-\mu O_{Ms}-\lambda & \dfrac{d\mu}{\psi_1} & \dfrac{d\psi}{\psi} \\
0 & c-\gamma-\dfrac{e_1\gamma}{u}-\dfrac{d\mu_1}{\psi_1}-\lambda & 0 \\
\dfrac{-\psi_1 r}{\psi} & v & -\lambda
\end{vmatrix}=0 \tag{89}
$$

$$
\begin{aligned}
&\lambda^3+\frac{e_1\gamma\psi_1+d\mu_1u-c\psi_1u}{\psi_1u}\lambda^2+\frac{P_{Ns}d\psi_1\psi u+O_{Ms}d\mu\psi_1u}{\psi_1u}\lambda+\frac{1}{\psi_1u}(P_{Ns}d^2\mu_1\psi u+O_{Ms}d^2\mu\mu_1u \\
&+P_{Ns}de_1\gamma\psi_1\psi+O_{Ms}de_1\gamma\mu\psi_1-P_{Ns}cd\psi_1\psi u-O_{Ms}cd\mu\psi_1u)=0
\end{aligned} \tag{90}
$$

Comparing with the following standard equation

$$\lambda^3+A_1\lambda^2+A_2\lambda+A_3=0 \tag{91}$$

The following equations can be obtained:

$$A_1=\frac{e_1\gamma\psi_1+d\mu_1u-c\psi_1u}{\psi_1u} \tag{92}$$

$$A_2=\frac{P_{Ns}d\psi_1\psi u+O_{Ms}d\mu\psi_1u}{\psi_1u} \tag{93}$$

$$A_3=\frac{1}{\psi_1u}(P_{Ns}d^2\mu_1\psi u+O_{Ms}d^2\mu\mu_1u+P_{Ns}de_1\gamma\psi_1\psi+O_{Ms}de_1\gamma\mu\psi_1-P_{Ns}cd\psi_1\psi u-O_{Ms}cd\mu\psi_1u) \tag{94}$$

The following equation can be obtained:

$$A_1A_2-A_3=\frac{1}{\psi u}\left(d^2\mu_1uP_{Ns}\psi_1\psi+P_{Ns}d^2\mu_1\psi_1u\right) \tag{95}$$

However, it has been found that A_1, A_3 are positive if and only if $e_1\gamma > (\xi+\phi-\beta-\tau_1)(\sigma_2-\xi\sigma-\sigma_1)$. As a result, all the elements of the first column for Table 7 are the same sign when $R_0 > 1$ with $e_1\gamma > (\xi+\phi-\beta-\tau_1)(\sigma_2-\xi\sigma-\sigma_1)$. Therefore, according to the Routh-Hurwitz criterion described in the study by DeJesus & Kaufman (1987), all the eigenvalues for Eq. (89) are negative.

Table 7. The Routh-Hurwitz stability criterion table

λ^3	1	A_2
λ^2	A_1	A_3
λ^1	$\dfrac{A_1A_2-A_3}{A_1}$	0
λ^0	A_3	

As a result, the equilibrium point E_2 is asymptotically stable provided the conditions $\xi+\phi > \beta+\tau_1$, $\varepsilon_1 > \varepsilon_2, \sigma_2 > \xi\sigma+\sigma_1, \alpha < \kappa+\omega$ and $e_1\gamma > (\xi+\phi-\beta-\tau_1)(\sigma_2-\xi\sigma-\sigma_1)$ are satisfied, otherwise it is unstable. In contrast, in everyday lives, pest populations increase dramatically as warmer temperatures create an ideal breeding environment for pests. That signifies $\alpha > \kappa+\omega$. As a result, the proposed model is inherently unstable, which is reflected in the numerical result obtained.

Appendix D

This study discusses the characterization of the equilibrium values of yield, soil organic matters, plant nutrients, soil salinity and pests based on the study of Misra & Singh (2011). The following two functions of P_N^*, O_M^* and β can be obtained from Eq. (6):

$$g(P_N^*, O_M^*, \beta) = (\varepsilon_1 - \varepsilon_2)P_N^* + (\tau_1 + \varepsilon)O_M^* \tag{96}$$

$$h(P_N^*, O_M^*, \beta) = O_M^*\left[\frac{e_1\gamma}{\xi\sigma + \sigma_1 - \sigma_2}\right] + (\xi + \phi - \beta - \tau_1)O_M^* \tag{97}$$

$$\therefore \frac{dP_N^*}{d\beta} = \frac{\begin{vmatrix} \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial O_M^*} & \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial \beta} \\ \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial O_M^*} & \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial \beta} \end{vmatrix}}{\begin{vmatrix} \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial P_N^*} & \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial O_M^*} \\ \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial P_N^*} & \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial O_M^*} \end{vmatrix}} \tag{98}$$

$$\Rightarrow \frac{dP_N^*}{d\beta} = -\frac{(\tau_1 + \varepsilon)O_M^*}{(\varepsilon_1 - \varepsilon_2)\left[-\frac{e_1\gamma}{u} + \xi - \beta - \tau_1 + \phi\right]} \tag{99}$$

where, $u = \sigma_2 - \xi\sigma - \sigma_1$.

It is clear that the numerator is always negative. Under the condition of $e_1\gamma > (\xi + \phi)(\sigma_2 - \xi\sigma - \sigma_1)$, it can be observed that the denominator is negative if $\epsilon_1 > \epsilon_2$ and, therefore, $\frac{dP_N^*}{d\beta} > 0$.

But when the denominator is positive if $\epsilon_1 < \epsilon_2$, consequently $\frac{dP_N^*}{d\beta} < 0$.

Again,

$$\therefore \frac{dO_M^*}{d\beta} = \frac{\begin{vmatrix} \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial \beta} & \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial P_N^*} \\ \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial \beta} & \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial P_N^*} \end{vmatrix}}{\begin{vmatrix} \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial P_N^*} & \frac{\partial g(P_N^*, O_M^*, \beta)}{\partial O_M^*} \\ \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial P_N^*} & \frac{\partial h(P_N^*, O_M^*, \beta)}{\partial O_M^*} \end{vmatrix}} \tag{100}$$

$$= \frac{O_M^*}{\left[-\frac{e_1\gamma}{u} + \xi - \beta - \tau_1 + \phi\right]} \tag{101}$$

Applying the previous conditions, it can be observed that the denominator is always negative, leading to $\frac{dO_M^*}{d\beta} < 0$.

The following equation can be obtained from Eq. (4):

$$0 + \xi\sigma\frac{dS^*}{d\beta} + \sigma_1\frac{dS^*}{d\beta} - \sigma_2\frac{dS^*}{d\beta} = 0 \tag{102}$$

$$\frac{dS^*}{d\beta}\left(\xi\sigma + \sigma_1 - \sigma_1\right) = 0 \therefore \frac{dS^*}{d\beta} = 0$$

Similarly, $\frac{dP^*}{d\beta} = 0$.

Finally, it can be observed from Eq. (1) that if P_N^* is in standard amount or $\frac{dP_N^*}{dt} > 0, \frac{dY^*}{dt} > 0$ and if $\frac{dP_N^*}{dt} < 0, \frac{dY^*}{dt} < 0$.

Appendix E

When farmers want to get same amount of yield, they also need to keep soil organic matters constant. Therefore, when the quantity of yield and organic matters is constant, $Y = Y_c$ and $O_M = O_{Mc}$ can be used to determine the convergence of plant nutrients.

According to study of Misra & Singh (2011), the following equations can be obtained from Eq. (3):

$$\frac{dP_N}{dt} = \varepsilon O_M + \varepsilon_1 P_N - \varepsilon_2 P_N - \psi_1 P_N Y + \tau_1 O_M \tag{103}$$

$$\dot{P}_N + (\varepsilon_2 - \varepsilon_1 + \psi_1 Y_c) P_N = (\varepsilon + \tau_1) O_{M_c} \tag{104}$$

The following integrating factor can be used to find the solutions:

$$I.F. = e^{\int (\varepsilon_2 + \psi_1 Y_c - \varepsilon_1) dt} = e^{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)t} \tag{105}$$

Using this integrating factor (*I.F.*), the following equations can be obtained after integrating for the initial condition of $P_N(0) = P_{N_0}$:

$$\therefore P_N e^{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)t} = \frac{(\varepsilon + \tau_1) e^{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)t}}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} + C_1 \tag{106}$$

$$P_N = \frac{(\varepsilon + \tau_1)}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} + C_1 e^{-(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)t} \tag{107}$$

Thus, the following relation can be obtained from the above:

$$C_1 = P_{N_0} - \frac{(\varepsilon + \tau_1)}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} \tag{108}$$

Taking these values leads to:

$$\Rightarrow P_N = \frac{(\varepsilon + \tau_1)}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} + \left\{ P_{N_0} - \frac{(\varepsilon + \tau_1)}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} \right\} e^{-(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)t} \tag{109}$$

Taking $\lim_{x \to \infty}$ for both sides, then

$$\lim_{t \to \infty} P_N(t) = \frac{(\varepsilon + \tau_1)}{(\varepsilon_2 + \psi_1 Y_c - \varepsilon_1)} \tag{110}$$

The sequence of plant nutrients $\{P_N(t)\}$ converses to $\frac{(\varepsilon+\tau_1)}{(\varepsilon_2+\psi_1 Y_c-\varepsilon_1)}$.

Appendix F

In order to figure out the convergence of soil organic materials, the amount of soil salinity and yield can be utilized as constants. Therefore, for the purposes of the following analysis, it is assumed that $S = S_c$ and $Y = Y_c$.

According to the study of Misra & Singh (2011), the following equations can be obtained from Eq. (2):

$$\frac{dO_M}{dt} = \xi O_M + \phi O_M - \beta O_M - \gamma O_M S_c - \mu_1 O_M Y_c - \tau_1 O_M \tag{111}$$

$$\dot{O}_M + K O_M = 0 \tag{112}$$

where, $K = (\beta + \tau_1 + \gamma S_c + \mu_1 Y_c - \xi - \phi)$. Then,

$$I.F. = e^{\int Ktdt} = e^{Kt} \quad (113)$$

The following equations can be obtained after integrating:

$$e^{Kt} O_M = C \quad (114)$$

$$\therefore O_M = O_{M_0} e^{-Kt} \quad (115)$$

Therefore, if $K > 0$ or $\beta + \tau_1 + \gamma S_c + \mu_1 Y_c > \xi + \phi$, then the sequence of organic matters converges. Otherwise, it does not converge.

Appendix G

The dynamics of yield can be examined in light of monotonically increasing soil organic matters, plant nutrients and pest elimination. This signifies that organic matters and plant nutrients continue to rise, and those current levels must be larger than those previous ones.

Taking Eq. (1), both sides with respect to time *t* can be differentiated:

$$\frac{dY}{dt} = \mu(O_M - O_{Ms})Y + \psi(P_N - P_{Ns})Y - \delta PY \quad (116)$$

$$\frac{d^2Y}{dt^2} > \mu O_M \frac{dY}{dt} + \mu Y \frac{O_M}{dt} - \mu O_{Ms} \frac{dY}{dt} + \psi Y \frac{dP_N}{dt} + \psi P_N \frac{dY}{dt} - \psi P_{Ns} \frac{dY}{dt} - \delta P \frac{dY}{dt} \quad (117)$$

Because of the fact that Pest population is decreasing, so $\frac{dP}{dt}$, must be negative and $-\delta Y \frac{dP}{dt}$ would be a positive term.

Again, for the critical points or equilibrium point,

$$\frac{dY}{dt} = \frac{dO_M}{dt} = \frac{dP_N}{dt} = 0 \quad (118)$$

Then it can be found that at the equilibrium point

$$\frac{d^2Y}{dt^2} > 0 \quad (119)$$

The result of this shows that the yield is the minimum at the critical points.

This implies that the levels of organic matters in the soil and nutrients available to the plants endlessly rise to ensure that each harvest produces a higher yield than the previous one.