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Stock Portfolio Optimization Using Pythagorean Fuzzy Numbers

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Abstract: Linear programming problems (LPP) have been widely used to address real-world problems, including the stock portfolio problem. In this study, an approach is proposed that incorporates Pythagorean fuzzy numbers (PFN) in the rate of risked return, portfolio risk amount, and expected return rate. The problem is transformed into a deterministic form using the scoring function, and a solution algorithm is being developed to provide portfolio investment choices. One of the key features of this study is the investor's ability to choose risk coefficients to increase expected returns and set their circumstances while determining their strategies. The optimum return rate is identified using the TORA program. An example is provided to demonstrate the efficiency and reliability of the method.

Keywords: Investment; Stock Portfolio; Score Function; Pythagorean Fuzzy Numbers; Optimization problem; TORA

1 Introduction

Portfolio optimization is an important problem related to asset allocation [1]. Its primary objective is to minimize the investment risk by dividing it into various assets that are expected to vary independently [2]. Financial assets such as securities, bonds, currencies, commodities, and cash equivalents are included in a portfolio [3]. Additionally, a portfolio may include non-publicly traded securities such as private investments, real estate, and the arts. Skrinjaric and Sego [4] used the Grey Relational Analysis approach to examine how a sample of stocks performed under different conditions.

Researchers have shown great interest in the fuzzy set (FS) theory developed by Zadeh [5] to address practical problems, such as economic risk management [6], which enables us to clarify and manage ambiguity in decision-support systems. Fuzzy values or restrictions may also be used to consider the fuzzy nature of financial market activity and the inaccurate information in asset reports. Fuzzy numbers (FN), the phenomenon of R fuzzy subsets, may be used to represent fuzzy numerical data. Dubois and Prade [7] expanded algebraic operations on real numbers to include FNs using the concept of fuzzification.

The process of selecting the best portfolio from a variety of prospective portfolios is known as portfolio selection (PS), which emphasizes how to invest in the best way to increase profits and reduce risk [8]. Since the precise return of any security cannot be predicted, Markowitz [9] developed the concepts of optimal portfolios and proposed the mean-variance models. Since all securities returns are linear constants, the PS problem is often an LPP. Several investigations on PS have been conducted in recent years, including [10–15]. Several academics have examined stock price evaluation. In reference [16], Lindberg modified the n-stock Black-Choles model by adding new drift rate parameterizations and solving Markowitz's continuous time PS within this framework. In reference [17], Khalifa and Kumar studied the stock portfolio in neutrosophic settings.

The first Pythagorean fuzzy sets (PFS), known as "intuitionistic fuzzy sets of the second kind," were established by Atanassov in 1999 [18]. In 2013, Yager introduced the first practical PFS implementations for decision-making [19]. Yang and Hussain [20] proposed new estimates of fuzzy entropy values for Pythagorean fuzzy sets based on the probabilistic type, distance, and Pythagorean index. Zulqarnain et al. [21] presented the Pythagorean fuzzy soft Einstein-ordered weighted geometric operator (PFSEOWG) and used it to solve a grouped decision-making problem.

The structure of this study is as shown in Figure 1:

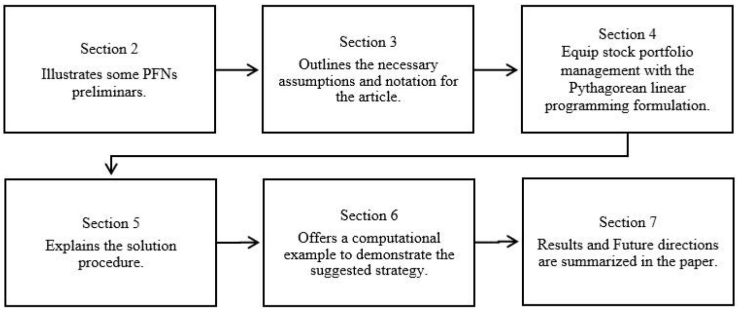

Figure 1. The paper structure

2 Preliminaries

Core concepts and terms from the fuzzy-like literature are reviewed in this section.

Definition 1 [5]. Fuzzy set $\tilde{z}$ determined on the real numbers set R will be an FN if its membership function $\mu_{\tilde{z}}(y) : R \to [0, 1]$ contains the listed features:

1. $\mu_{\tilde{z}}(y)$ is an upper semi-continuous membership function;
2. $\tilde{z}$ is a convex fuzzy set, i.e., $\mu_{\tilde{z}}(ay + (1-a)t) \geq \min\{\mu_{\tilde{z}}(y), \mu_{\tilde{z}}(t)\}\, \forall y, t \in R; 0 \leq a \leq 1$;
3. $\tilde{z}$ is normal, i.e., $\exists z_0 \in R$ s.t $\mu_{\tilde{z}}(z_0) = 1$;
4. The support of $\tilde{z} : \operatorname{Supp}(\tilde{z}) = \{y \in R : \mu_{\tilde{z}}(y) > 0\}$;
5. The closure: $\operatorname{Cl}(\operatorname{Supp}(\tilde{z}))$ is a compact set.

Definition 2 [21]. Let Z be a fixed set; a Pythagorean fuzzy set is defined as

$A = \{\langle y, (\alpha_A(y), \beta_A(y))\rangle : y \in Z\}$, where, $\alpha_p(y) : Z \to [0, 1], \beta_p(y) : Z \to [0, 1]$ are the degree of membership and non-membership functions, respectively and $(\alpha_A(y))^2 + (\beta_A(y))^2 \leq 1$.

Definition 3 [22]. Let $\tilde{p}^A = \left(\alpha_i^A, \beta_i^A\right)$ and $\tilde{q}^A = \left(\alpha_j^A, \beta_j^A\right)$ be two $PFNs$. Their arithmetic operations are:

(i) $\tilde{p}^A(+)\tilde{q}^A = \left(\sqrt{\left(\alpha_i^A\right)^2 + \left(\alpha_j^A\right)^2 - \left(\alpha_i^A\right)^2 \cdot \left(\alpha_j^A\right)^2}, \beta_i^A \cdot \beta_j^A\right)$,

(ii) $\tilde{p}^A(\times)\tilde{q}^A = \left(\alpha_i^A \cdot \alpha_j^A, \sqrt{\left(\beta_i^A\right)^2 + \left(\beta_j^A\right)^2 - \left(\beta_i^A\right)^2 \cdot \left(\beta_j^A\right)^2}\right)$,

(iii) $\omega \cdot \tilde{p}^A = \left(\sqrt{1 - \left(1 - \alpha_i^A\right)^\omega}, \left(\beta_i^A\right)^\omega\right), \omega > 0$.

Definition 4 [21, 22]. Let $\tilde{p}^A = \left(\alpha_i^A, \beta_i^A\right)$ and $\tilde{q}^A = \left(\alpha_j^A, \beta_j^A\right)$ be two PFNs,

(i) Score function: $S\left(\tilde{p}^A\right) = \left(\alpha_i^A\right)^2 - \left(\beta_i^A\right)^2$.

(ii) Accuracy function: $A\left(\tilde{p}^A\right) = \left(\alpha_i^A\right)^2 + \left(\beta_i^A\right)^2$.

Definition 5 [21, 22]. Let $\tilde{p}^A$, and $\tilde{q}^A$ be any two $PFNs$, then

(i) $\tilde{p}^A > \tilde{q}^A$ if and only if $S\left(\tilde{p}^A\right) > S\left(\tilde{q}^A\right)$,

(ii) $\tilde{p}^A < \tilde{q}^A$ if and only if $S\left(\tilde{p}^A\right) < S\left(\tilde{q}^A\right)$,

(iii) $S\left(\tilde{p}^A\right) = S\left(\tilde{q}^A\right)$, and $A\left(\tilde{p}^A\right) < A\left(\tilde{q}^A\right)$ then $\tilde{p}^A < \tilde{q}^A$,

(iv) $S\left(\tilde{p}^A\right) = S\left(\tilde{q}^A\right)$, and $A\left(\tilde{p}^A\right) > A\left(\tilde{b}^P\right)$ then $\tilde{p}^A > \tilde{q}^A$,

(v) $S\left(\tilde{p}^A\right) = S\left(\tilde{q}^A\right)$, and $A\left(\tilde{p}^A\right) = A\left(\tilde{b}^P\right)$ then $\tilde{p}^A = \tilde{q}^A$.

3 Assumptions and Notations

3.1 Assumptions

As the investing climate is extremely sensitive, slight changes might impact portfolio selection. To make problem formulation clearer, we assumed that:

1) The predicted return rate and loss risk rate are employed to analyze the securities;
2) Securities can be separated and have uncertain values;
3) Transactions are not required to be paid for;
4) Avoiding risk and dissatisfaction should be the top priority for investors;
5) The bank's interest rate is fixed during the investment duration;
6) Risk security consists of n distinct components;
7) It is not allowed to participate in short selling.

Notations:

c_0 : Bank interest rate;

c_n : Expected rates of return, $n = 1 : k$;
B_{nm} : Risked rates of return, $n = 1 : k, m = 1 : j$;
y_0 : The total investment percentage made throughout the investment period;
y_n : The percentage of investments financed in secondary securities, $n = 1 : k$;
T: Expected total return rate;
r: Investment portfolio risk factor;
V: All security risks maximum value.

3.2 Formulating the Problem

Assume the stock issue raised by Yin [23]. The predicted return of the portfolio is expressed as $T = \sum_{n=0}^{k} c_n y_n$. Investors seek to maximize investment interest while minimizing risk in risky securities. The portfolio risk coefficient r refers to the market risk. It is the ratio between stock portfolio risk and the market risk average value. The maximum value of all securities risks, denoted

$$V = \max\left(B_1 y_1, B_2 y_2, \ldots, B_k y_k\right)$$

.

The following classical linear programming model may be formulated:

$$\max T = \sum_{n=0}^{k} c_n y_n$$
$$\text{s.t.} \begin{cases} By \leq r \\ \sum_{n=0}^{k} y_n = 1 \\ y_n \geq 0, n = 1 : k \end{cases} \tag{1}$$

To be more generic and adaptable, we describe c_n, r_n and B_n as PFNs. So, we construct the following model:

$$\max \tilde{T}^A = c_0 y_0 + \sum_{n=1}^{k} \tilde{c}_n{}^A y_n$$
$$\text{s.t.} \begin{cases} \tilde{B}^A y \leq \tilde{r}^A \\ \sum_{n=0}^{k} y_n = 1 \\ y_m \geq 0, \quad n = 1 : k, \quad m = 1 : j \end{cases} \tag{2}$$

4 Solution Procedure for PFNs

In this section, the solution procedure of LPP involves $PFNs$ in expected return rates, risk loss rates, and risk coefficients is presented: Set, $\tilde{B}^A = \left(\tilde{B}^A_{nm}\right)_{k\times j}$, $\tilde{r}^A = \left(\tilde{r}^A_1, \tilde{r}^A_2, \ldots, \tilde{r}^A_j\right)^T, \tilde{c}^A = \left(\tilde{c}^A_1, \tilde{c}^A_2, \ldots, \tilde{c}^A_n\right)$ and $Y = (y_1, y_2, \ldots, y_k)^T$. Then, we have,

$$\max \tilde{T}^A = c_0 y_0 + \sum_{n=1}^{k} \tilde{c}_n{}^A y_n$$
$$\text{s.t.} \begin{cases} \sum_{m=1}^{j} \tilde{B}^A_{nm} y_m \leq \tilde{r}^A_n \\ \sum_{m=0}^{j} y_m = 1 \\ y_m \geq 0, \quad m = 1 : j; n = 1 : k \end{cases} \tag{3}$$

Depending on the score function, model (3) can be transformed to model (1), which is simple and solvable:

$$\max T = c_0 y_0 + \sum_{n=1}^{k} S\left(\tilde{c}_n{}^A\right) y_n$$
$$\text{s.t.} \begin{cases} \sum_{m=1}^{j} S\left(\tilde{B}^A_{nm}\right) y_m \leq S\left(\tilde{r}^A_n\right) \\ \sum_{m=0}^{j} y_m = 1 \\ y_m \geq 0, \quad m = 1 : j \end{cases} \tag{4}$$

5 Numerical Example

Five equities are available for selection, with the first stock being a bank savings portfolio that earns an annual rate of return of $c_0 = 6\%$. Tables 1, 2, and 3 present the data for the remaining four stocks.

Table 1. Risked loss rate %

$\widetilde{\mathbf{B}}^{\mathbf{A}}_{\mathbf{nm}}$	**Risk loss rate**
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{11}$	$[0.5, 0.2]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{12}$	$[0.7, 0.3]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{13}$	$[0.6, 0.1]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{14}$	$[0.3, 0.1]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{21}$	$[0.5, 0.2]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{22}$	$[0.4, 0.2]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{23}$	$[0.7, 0.4]$
$\widetilde{\mathrm{B}}^{\mathrm{A}}_{24}$	$[0.6, 0.3]$

Table 2. Risk coefficient %

$\tilde{\mathbf{r}}^{\mathrm{A}}_{\mathbf{n}}$	**Risk coefficient rate**
$\mathrm{r}^{\mathrm{A}}_{1}$	$[0.4, 0.1]$
$\mathrm{r}^{\mathrm{A}}_{2}$	$[0.5, 0.3]$

Table 3. Risk coefficient %

Stocks	$\tilde{\mathbf{c}}^{\mathbf{A}}$
St_1	$[0.8, 0.3]$
St_2	$[0.7, 0.2]$
St_3	$[0.9, 0.4]$
St_4	$[0.6, 0.1]$

The problem can be formulated using the following model:

$$\max \widetilde{T}^A = c_0 y_0 + \sum_{m=1}^{4} S\left(\tilde{c}_n^A\right) x_m$$
$$\text{s.t.} \begin{cases} S\left(\tilde{B}_{11}^A\right) y_1 + S\left(\tilde{B}_{12}^A\right) y_2 + S\left(\tilde{B}_{13}^A\right) y_3 + S\left(\tilde{B}_{14}^A\right) y_4 \leq S\left(\tilde{r}_1^A\right) \\ S\left(\tilde{B}_{21}^A\right) y_1 + S\left(\tilde{B}_{22}^A\right) y_2 + S\left(\tilde{B}_{23}^A\right) y_3 + S\left(\tilde{B}_{24}^A\right) y_4 \leq S\left(\tilde{r}_2^A\right) \\ y_0 + y_1 + y_2 + y_3 + y_4 = 1 \\ y_m \geq 0, \quad 0 \leq m \leq 4 \end{cases} \tag{5}$$

Applying the score function yields the following result:

$$\max \widetilde{T}^A = 0.06y_0 + 0.55y_1 + 0.45y_2 + 0.65y_3 + 0.35y_4$$
$$\text{s.t.} \begin{cases} 0.21y_1 + 0.40y_2 + 0.35y_3 + 0.08y_4 \leq 0.15 \\ 0.21y_1 + 0.12y_2 + 0.45y_3 + 0.27y_4 \leq 0.16 \\ y_0 + y_1 + y_2 + y_3 + y_4 = 1 \\ y_m \geq 0, \quad 0 \leq m \leq 4 \end{cases} \tag{6}$$

The optimal solution is:

$$y_0 = 0.25, \quad y_1 = 0.0, y_2 = 0.28, y_3 = 0.0.0, y_4 = 0.47$$

The optimal value $\widetilde{T}^A = 31\%$.

The findings show that 25% of total money stored in a bank at 6% interest, 28% of total capital invested in St security and 47% of total capital can be invested in S_4 security is the best investment, under the information provided. Based on the assumption that risk coefficients $\tilde{r}_1^A$ and $\tilde{r}_2^A$ exist, this approach yields a maximum predicted return of 31%.

6 Conclusion Remarks and Future Work

This study presents a formulation of the stock portfolio problem that incorporates Pythagorean fuzzy numbers (PFNs) in the risked return rate, portfolio risk amount, and expected return rate. The model is transformed into a crisp version using the score function, and a solution strategy is being developed to provide portfolio investment decisions for both securities and savings investors. The proposed research has two important features: the ability to select risk coefficients according to individual circumstances, and the flexibility to adopt strategies based on these factors. The TORA software is used to determine the best return rate. The proposed method is illustrated through a numerical example of Pythagorean fuzzy stock portfolios. In the future, various fuzzy-like structures such as interval-valued fuzzy sets [24], neutrosophic sets [25], and spherical fuzzy sets [26] can be explored, with further discussion and recommendations.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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