Journal of Engineering Management and Systems Engineering
https://www.acadlore.com/journals/JEMSE

A Parametric Similarity Measure for Spherical Fuzzy Sets and Its Applications in Medical Equipment Selection

Mehwish Sarfraz[1*], Dragan Pamucar[2*]

[1] Department of Mathematics, Riphah International University (Lahore Campus), 54000 Lahore, Pakistan
[2] Department of Operations Research and Statistics, Faculty of Organizational Sciences, University of Belgrade, 11000 Belgrade, Serbia

[*] Correspondence: Mehwish Sarfraz (mehwish.sarfraz.ms@gmail.com); Dragan Pamucar (dpamucar@gmail.com)

Received: 12-28-2023 **Revised:** 01-31-2024 **Accepted:** 02-10-2024

Citation: M. Sarfraz and D. Pamucar, "A parametric similarity measure for spherical fuzzy sets and its applications in medical equipment selection," *J. Eng. Manag. Syst. Eng.*, vol. 3, no. 1, pp. 38–52, 2024. https://doi.org/10.56578/jemse030104.

Abstract: The Spherical Fuzzy Set (SFS) framework extends the Picture Fuzzy Set (PFS) concept, offering enhanced precision in handling data characterized by conflict and uncertainty. Furthermore, similarity measures (SMs) are crucial for determining the extent of resemblance between pairs of fuzzy values. While existing SMs evaluate similarity by measuring the distance between values, they sometimes yield results that are illogical or unreasonable, due to certain properties and operational complexities. To address these anomalies, this paper introduces a parametric similarity measure based on three adjustable parameters $(\alpha_1, \alpha_2, \alpha_3)$, allowing decision-makers to fine-tune the measure to suit various decision-making styles. This paper also scrutinizes existing SMs from a mathematical standpoint and demonstrates the efficacy of the proposed SM through mathematical modeling. Finally, we apply the proposed SM to tackle Multi-Attribute Decision-Making (MADM) problems. Comparative analysis reveals that our proposed SM outperforms certain existing SMs in the context of SFS-based applications.

Keywords: Fuzzy set; Spherical fuzzy set; Similarity measure; Multi-attribute decision-making

1 Introduction

Information extraction and analysis from real-life problems are full of vagueness and uncertainties. Several attempts have been introduced to reduce this uncertainty. A famous way to reduce uncertainty was introduced by Zadeh by introducing the concept of the fuzzy set (FS) [1]. This concept is a generalization of the crisp set, describing the belongingness of an object with the help of the membership degree (MD). By generalizing the concept of the fuzzy set, Atanassov attempted to further reduce uncertainty by introducing the concept of intuitionistic fuzzy sets (IFS) [2], in which he described the belongingness of an object by both MD and non-membership degree (NMD). To achieve greater accuracy during information extraction, Atanassov [3] formalized the interval-valued IFS by considering MD and NMD as intervals from [0,1]. IFS has been used by researchers in various fields, such as pattern recognition [4], decision making [5], and medical diagnosis [6]. IFS had a limited range for assigning values to the MD $\varrho(\lambda)$ and NMD $\gamma'(\lambda)$ because the sum of the MD and NMD did not necessarily equal 1. The limitations of IFS were expanded by the ideas of the Pythagorean fuzzy set (PYFS) [7] and the q-rung orthopair fuzzy set (qROFS) [8], respectively.

The applications of IFS, PYFS, and qROFS have great potential in practical scenarios due to their ability to reduce vagueness in information extraction. However, in some instances, these tools could not extract information without some loss, as they account for only two degrees for the description of an element. To address this limitation and describe an object's belongingness with three degrees, Cuong [9] introduced the concept of picture fuzzy sets (PFS), which include an additional degree known as the abstinence degree (AD). The PFS has been used by many scholars for example, in the work [10]. But some time the concept of PFS failed when $0 \leq \varphi u \alpha\left(\varrho(\lambda), i(\lambda), \gamma'(\lambda)\right) \leq 1$ violated. For example, the values of the MD, AD and NMD are 0.8, 0.2 and 0.4 respectively. In this case the $\varphi u \alpha\left(\varrho(\lambda), i(\lambda), \gamma'(\lambda)\right) = 1.4 \nleq 1$. The PFS, while useful, had its limitations. To broaden the scope of PFS, Mahmood et al. [11] introduced the concept of Spherical Fuzzy Sets (SFS) and the Total Spherical Fuzzy Sets (TSFS). TSFS, being the latest framework, is designed to extract information with higher accuracy.

MADM is a compelling technique used to identify the best alternative from a set of options. The introduction of fuzzy theory has significantly transformed and enhanced MADM. Numerous scholars have refined the MADM process utilizing various approaches. Khan et al. [12] applied complex SFS to address MADM problems. Senapati et al. [13] adopted interval-valued Intuitionistic IFS for the same purpose. Jana et al. [14] utilized PyFS in their approach to MADM, while Senapati [15] employed PFS for MADM solutions. TSFS were used in work [12] to tackle MADM challenges. Mahmood and Ali [16] resolved MADM issues using complex single-valued neutrosophic sets (CSVNS), and Riaz and Farid [17] applied complex PFS for MADM problems. Khan et al. [18] again used complex SFS in the context of MADM. Riaz et al. [19] addressed MADM with bipolar Fuzzy Sets, while Garg [20] implemented IFS for MADM solutions. Ashraf et al. [21] utilized interval-valued PFS for MADM, and Garg [22] adopted PyFS for this purpose. Lastly, Riaz et al. [23] and Pamučar et al. [24] both used qROFS to solve MADM problems.

SM is the significant tool for evaluating the similarity between two fuzzy values (FVs). Numerous scholars have introduced various SMs, finding interesting applications in medical diagnosis, pattern recognition, and MADM. Boran and Akay [4] and Du and Hu [25] introduced SMs within the framework of IFS and discussed their application in pattern recognition. Donyatalab et al. [26] introduced an SM for qROFS, while Mohd and Abdullah [27] presented SMs for PyFS, discussing their intriguing applications. Wei [28] introduced Cosine Similarity Measures (CSMs) based on the cosine function and contingent similarity measures based on the contingent function for PFS and applied them to MADM. Wei and Geo [29] developed a Dice SM for PFS. Van Dinh et al. [30] introduced some SMs for PFS and applied them to MADM problems. Singh et al. [31] extended SMs by considering the refusal degree of PFS and applied them to clustering problems. Luo and Zhang [32] introduced SMs based on basic operations for PFS. In the work [33], the concept of SM was introduced for Spherical Fuzzy Sets (SFS) with applications to MADM. Zhao et al. [34] developed SMs for the SFS framework and applied them to pattern recognition and MADM. Shishavan et al. [35] and Khan et al. [36] introduced SMs for SFS, applying them to pattern recognition, while Mahmood et al. [37] applied them to medical diagnosis and pattern recognition.

From the SMs discussed above, we can draw some key points. All the SMs for IFS, PyFS, qROFS, and PFS are outdated because these frameworks have limited capacity to extract information from real-life scenarios. Consequently, decision-makers cannot find the best results due to uncertainty and information loss. Therefore, advanced SMs for SFS should be defined to assess the similarity between FVs with less uncertainty.

Some SMs fail to compute in certain scenarios. For instance, some cannot provide decision results due to division by zero problems. Thus, the major contribution of this study is to enhance the identification ability of SMs and overcome the defects of current SMs, necessitating the proposal of new SMs.

This paper is organized as follows: Section 2 discusses some basic concepts. Section 3 reviews existing SMs and discusses their limitations. Section 4 develops a new SM for SFS, which improves upon and generalizes existing SMs for SFS by using parameters. Section 5 presents the application of the proposed SMs to the MADM problem, and Section 6 summarizes the study.

2 Preliminaries

This section presents some basic concepts for understanding the article.

2.1 Definition [2]: On a set X a IFS is of the shape $I = \{(\lambda, (\varphi, \gamma')) : 0 \leq \text{sum}\,(\varphi(\lambda), \gamma'(\lambda)) \leq 1\}$. Further, $r(\lambda) = 1 - \text{sum}\,(\varphi, \gamma')$ represents the hesitancy degree of $\lambda \in X$ and the pair (φ, γ') is termed as an intuitionist FV (IFV).

2.2 Definition [7]: On a set X a PyFS is of the shape $I = \left\{(\lambda, (\varphi, \gamma')) : 0 \leq \text{sum}\left(\varphi^2(\lambda), \gamma'^2(\lambda)\right) \leq 1\right\}$. Further, $r(\lambda) = 1 - \text{sum}\left(\varphi^2(\lambda), \gamma'^2(\lambda)\right)$ represents the hesitancy degree of $\lambda \in X$ and the pair (φ, γ') is termed as a Pythagorean FV (PyFV).

2.3 Definition [9]: On a set X a PFS is of the shape $I = \{(\lambda, (\varphi, i, \gamma')) : 0 \leq \text{sum}\,(\varphi(\lambda), i(\lambda), \gamma'(\lambda)) \leq 1\}$. Further, $r(\lambda) = 1 - \text{sum}\,(\varphi(\lambda), i(\lambda), \gamma'(\lambda))$ represents the refusal degree of $\lambda \in X$ and the pair (φ, i, γ') is termed as a picture FV (PFV).

2.4 Definition [11]: For any universal set X a SFS is of the form $I = \{(\lambda, (\varphi, i, \gamma')) : \forall \lambda \in X\}$. Here φ, i, and γ' are mappings form $X \to [0, 1]$ denoting MD, AD, and ND respectively provided that $0 \leq \text{sum}\left(\varphi^2(\lambda), i^2(\lambda), \gamma'^2(\lambda)\right) \leq 1$ and $r(\lambda) = \sqrt{1 - \text{sum}\,(\varphi^2(\lambda), i^2(\lambda), \gamma'^2(\lambda))}$ is known as the RD of λ in I. The triplet (φ, i, γ') is considered as a spherical FV (SFV).

2.5 Definition [11]: Let $\vartheta = \left\{\left(\lambda, \varphi_\vartheta^2(\lambda), i_\vartheta^2(\lambda), \gamma'^2_\vartheta(\lambda)\right) \mid \lambda \in X\right\}$ and $\Phi = \left\{\left(\lambda, \varphi_\Phi^2(\lambda), i_\Phi^2(\lambda), \gamma'^2_\Phi(\lambda)\right) \mid \lambda \in X\right\}$ be any two be SFSs on universe λ, then

$\vartheta \subseteq \Phi$ If and only if $\varphi_\vartheta^2(\lambda) \leq \varphi_\Phi^2(\lambda), i_\vartheta^2(\lambda) \geq i_\Phi^2(\lambda), \gamma'^2_\vartheta(\lambda) \geq \gamma'^2_\Phi(\lambda)$. For $\lambda \in X$.

$\vartheta = \Phi$ If and only if $\vartheta \subseteq \Phi$ and $\Phi \subseteq \vartheta$ i.e. $\varphi_\vartheta^2(\lambda) = \varphi_\Phi^2(\lambda), i_\vartheta^2(\lambda) = i_\Phi^2(\lambda), \gamma'^2{}_\vartheta(\lambda) = \gamma'^2{}_\Phi(\lambda)$.

$\vartheta^c = \left\{\left(\lambda, \varphi_\vartheta^2(\lambda), i_\vartheta^2(\lambda), \gamma_\vartheta'^2(\lambda)\right) \mid \lambda \in X\right\}$.

2.6 Definition [34]: Let $\vartheta = \left\{\left(\lambda, \varphi_\vartheta^2(\lambda), i_\vartheta^2(\lambda), \gamma'^2_\vartheta(\lambda)\right) \mid \lambda \in X\right\}$ and $\Phi = \left\{\left(\lambda, \varphi_\Phi^2(\lambda), i_\Phi^2(\lambda), \gamma'^2_\Phi(\lambda)\right) \mid \lambda \in\right.$

$X\}$ be any two be SFSs on universe X, then the SM between ϑ and Φ is defined as $\varrho(\vartheta, \Phi)$, which satisfies the following axioms:

(ϱ_1) $0 \leq \varrho(\vartheta, \Phi) \leq 1$;

(ϱ_2) $\varrho(\vartheta, \Phi) = 1$ Iff $\vartheta = \Phi$;

(ϱ_3) $\varrho(\vartheta, \Phi) = \varrho(\Phi, \vartheta)$;

(ϱ_4) Let C be any SFS such that $\vartheta \subseteq \Phi \subseteq C$, then $\varrho(\vartheta, C) \leq \varrho(\vartheta, \Phi)$ and $\varrho(\vartheta, C) \leq \varrho(\Phi, C)$;

Now, we will review some existing similarity measures for SFSs in the following section.

Let $\vartheta = \{(\lambda_i, \varphi_\vartheta(\lambda_i), i_\vartheta(\lambda_i), \gamma'_\vartheta(\lambda_i)) \mid \lambda_i \in X\}$ and $\Phi = \{(\lambda_i, \varphi_\Phi(\lambda_i), i_\Phi(\lambda_i), \gamma'_\Phi(\lambda_i)) \mid \lambda_i \in X\}$ be any two be SFSs on $X = \{\lambda_1, \lambda_2, \ldots, \lambda_2\}, \rho_\vartheta(\lambda_i)$ and $\rho_\Phi(\lambda_i)$ be the refusal degrees of element λ_i belonging to SFSs A and B respectively, where $\rho_\vartheta(\lambda_i) = 1 - \varphi_\vartheta^2(\lambda_i), i_\vartheta^2(\lambda_i), \gamma'^2_\vartheta(\lambda_i)$ and $\rho_\Phi(\lambda_i) = 1 - \varphi_\Phi^2(\lambda_i), i_\Phi^2(\lambda_i), \gamma'^2{}_\Phi(\lambda_i)$. The existing similarity degrees between SFSs ϑ and Φ are reviewed as follows:
where, $i = 1, 2, 3 \ldots 2$.

The SMs for SFSs as defined by the work [38] are given as follows:

$$\varrho_1(\vartheta, \Phi) = 1 - \frac{1}{2\eta} \sum_{i=1}^{\eta} \Bigg(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| + \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| + \left|\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| + \left|\rho_\vartheta^2(\lambda_i) - \rho_\Phi^2(\lambda_i)\right| \Bigg) \tag{1}$$

$$\varrho_2(\vartheta, \Phi) = 1 - \frac{1}{2\eta} \sum_{i=1}^{\eta} \left| \left(\left(\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right) - \left(i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right) - \left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) \right) \right| \tag{2}$$

$$\varrho_3(\vartheta, \Phi) = \frac{1}{4\eta} \sum_{i=1}^{\eta} \begin{pmatrix} \left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| + \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| + \\ \left|\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| + \left|\rho_\vartheta^2(\lambda_i) - \rho_\Phi^2(\lambda_i)\right| \end{pmatrix} + \sum_{i=1}^{\eta} \begin{pmatrix} \mid \left(\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right) - \\ \left(i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right) - \left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) \mid \end{pmatrix} \tag{3}$$

$$\varrho_4(\vartheta, \Phi) = 1 - \frac{1}{\eta} \sum_{i=1}^{\eta} \left(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i) \vee \right| i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i) \mid \vee \left|\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| \right) \tag{4}$$

$$\varrho_5(\vartheta, \Phi) = 1 - \frac{1}{\eta} \sum_{i=1}^{\eta} \frac{1 - \left(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| \vee \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| \vee \left|\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| \right)}{1 + \left(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| \vee \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| \vee \left|\gamma'^2{}_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| \right)} \tag{5}$$

$$\varrho_6(\vartheta, \Phi) = \frac{\sum_{i=1}^{\eta} 1 - \left(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| \vee \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| \vee \left|\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| \right)}{\sum_{i=1}^{\eta} 1 + \left(\left|\varphi_\vartheta^2(\lambda_i) - \varphi_\Phi^2(\lambda_i)\right| \vee \left|i_\vartheta^2(\lambda_i) - i_\Phi^2(\lambda_i)\right| \vee \left|\gamma'^2(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right| \right)} \tag{6}$$

$$\varrho_7(\vartheta, \Phi) = \frac{1}{\eta} \sum_{i=1}^{\eta} \frac{\left(\varphi_\vartheta^2(\lambda_i) \wedge \varphi_\Phi^2(\lambda_i)\right) + \left(i_\vartheta^2(\lambda_i) \wedge i_\Phi^2(\lambda_i)\right) + \left(\gamma'^2_\vartheta(\lambda_i) \wedge \gamma'^2_\Phi(\lambda_i)\right)}{\left(\varphi_\vartheta^2(\lambda_i) \vee \varphi_\Phi^2(\lambda_i)\right) + \left(i_\vartheta^2(\lambda_i) \vee i_\Phi^2(\lambda_i)\right) + \left(\gamma'^2{}_\vartheta(\lambda_i) \vee \gamma'^2_\Phi(\lambda_i)\right)} \tag{7}$$

$$\varrho_8(\vartheta, \Phi) = \frac{\sum_{i=1}^{\eta} \left(\varphi_\vartheta^2(\lambda_i) \wedge \varphi_\Phi^2(\lambda_i)\right) + \left(i_\vartheta^2(\lambda_i) \wedge i_\Phi^2(\lambda_i)\right) + \left(\gamma'^2{}_\vartheta(\lambda_i) \wedge \gamma'^2{}_\Phi(\lambda_i)\right)}{\sum_{i=1}^{\eta} \left(\varphi_\vartheta^2(\lambda_i) \vee \varphi_\Phi^2(\lambda_i)\right) + \left(i_\vartheta^2(\lambda_i) \vee i_\Phi^2(\lambda_i)\right) + \left(\gamma'^2_\vartheta(\lambda_i) \vee \gamma'^2_\Phi(\lambda_i)\right)} \tag{8}$$

$$\varrho_9(\vartheta, \Phi) = \frac{1}{\eta} \sum_{i=1}^{\eta} \frac{\left(\varphi_\vartheta^2(\lambda_i) \wedge \varphi_\Phi^2(\lambda_i)\right) + \left(1 - i_\vartheta^2(\lambda_i)\right) \wedge \left(1 - i_\Phi^2(\lambda_i)\right) + \left(1 - \gamma'^2_\vartheta(\lambda_i)\right) \wedge \left(1 - \gamma'^2_\Phi(\lambda_i)\right)}{\left(\varphi_\vartheta^2(\lambda_i) \wedge \varphi_\Phi^2(\lambda_i)\right) + \left(1 - i_\vartheta^2(\lambda_i)\right) \vee \left(1 - i_\Phi^2(\lambda_i)\right) + \left(1 - \gamma'^2_\vartheta(\lambda_i)\right) \vee \left(1 - \gamma'_\Phi{}^2(\lambda_i)\right)} \tag{9}$$

$$\varrho_{10}(\vartheta,\Phi)=\frac{\sum_{i=1}^{\eta}\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\wedge\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)+\left(1-i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)\wedge\left(1-i_{\Phi}^{2}\left(\lambda_{i}\right)\right)+\left(1-\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)\wedge\left(1-\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)}{\sum_{i=1}^{\eta}\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\wedge\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)+\left(1-i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)\vee\left(1-i_{\Phi}^{2}\left(\lambda_{i}\right)\right)+\left(1-\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)\vee\left(1-\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)} \tag{10}$$

Ullah et al. [39] introduced the SMs for SFSs based on the cosine function, as provided below.

$$\varrho_{11}(\vartheta,\Phi)=\frac{1}{\eta}\sum_{i=1}^{\eta}\frac{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\cdot\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}{\sqrt{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}\sqrt{\left(\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}} \tag{11}$$

$$\varrho_{12}(\vartheta,\Phi)=\frac{1}{\eta}\sum_{i=1}^{\eta}\omega_{i}\frac{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\cdot\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}{\sqrt{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}\sqrt{\left(\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}} \tag{12}$$

$$\varrho_{13}(\vartheta,\Phi)=\frac{1}{\eta}\sum_{i=1}^{\eta}\frac{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\cdot\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}{\left(\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}\right)\cdot\left(\left(\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}\right)} \tag{13}$$

$$\varrho_{14}(\vartheta,\Phi)=\frac{1}{\eta}\sum_{i=1}^{\eta}\omega_{i}\frac{\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\cdot i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\cdot\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}}{\left(\left(\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\vartheta}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma^{\prime 2}{}_{\vartheta}\left(\lambda_{i}\right)\right)^{2}\right)\cdot\left(\left(\varphi_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(i_{\Phi}^{2}\left(\lambda_{i}\right)\right)^{2}+\left(\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)^{2}\right)} \tag{14}$$

3 An Analysis of Some Existing Spherical Fuzzy Similarity Measures

As a numerical tool for calculating the degree of similarity between objects, SMs have been utilized to solve problems in decision-making, clinical diagnosis, and pattern recognition. Although many SMs for SFSs have been proposed, they can yield unreasonable and counter-intuitive results in practical applications, bringing significant challenges to users. In this section, we comprehensively analyze some existing SMs from an arithmetic perspective, as presented in Table 1 below.

Table 1. A comprehensive analysis of some existing similarity measures for SFS

ϱ	Does Not Meet the Axiom ϱ_2	The Division by Zero Problem	Serious Information Loss
ϱ_1	Yes	No	No
ϱ_2	Yes	No	No
ϱ_3	Yes	No	No
ϱ_4	Yes	No	No
ϱ_5	No	Yes	No
ϱ_6	Yes	No	No
ϱ_7	Yes	No	No
ϱ_8	No	No	Yes
ϱ_9	Yes	No	No
ϱ_{10}	Yes	No	No
ϱ_{11}	Yes	No	No
ϱ_{12}	No	No	No
ϱ_{13}	Yes	No	No
ϱ_{14}	No	No	No
$\varrho_{\mathbf{m}}$	Yes	No	No

The axiom ϱ_2 is one of the most basic axioms of spherical SMs. By analyzing Table 1, we can easily find that the similarity measures $\varrho_5,\varrho_7,\varrho_8,\varrho_{12}$, and ϱ_{14} do not satisfy this axiom. The detailed discussion is as follows:

(1) Let $\boldsymbol{\vartheta}=\left\{\left(\lambda_{i},\varphi_{\vartheta}^{2}\left(\lambda_{i}\right),i_{\vartheta}^{2}\left(\lambda_{i}\right),\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\right)\mid\lambda_{i}\in X\right\}$ and $\Phi=\left\{\left(\lambda_{i},\varphi_{\Phi}^{2}\left(\lambda_{i}\right),i_{\Phi}^{2}\left(\lambda_{i}\right),\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)\right)\mid\lambda_{i}\in X\right\}$ be any two be SFSs on $X=\{\lambda_1,\lambda_2,\ldots,\lambda_2\}$. For the similarity measure ϱ_{12}, there are two cases in which ϱ_{12} does not satisfy the axiom (ϱ_2) $\varrho(\boldsymbol{\vartheta},\Phi)=1$ implies $\boldsymbol{\vartheta}=\Phi$ as shown below:

If $\varphi_{\vartheta}^{2}\left(\lambda_{i}\right)=i_{\vartheta}^{2}\left(\lambda_{i}\right)=\gamma_{\vartheta}^{\prime 2}\left(\lambda_{i}\right)\neq\varphi_{\Phi}^{2}\left(\lambda_{i}\right)=i_{\Phi}^{2}\left(\lambda_{i}\right)=\gamma_{\Phi}^{\prime 2}\left(\lambda_{i}\right)$

i.e. $\boldsymbol{\vartheta}\neq\boldsymbol{\Phi}$ based on Eq. (12), we have

$$\varrho_{12}(\boldsymbol{\vartheta},\Phi)=\frac{1}{2}\sum_{i=1}^{2}\omega_i\frac{\varphi_{\vartheta}^{2}(\lambda_i)\,\varphi_{\Phi}^{2}(\lambda_i)+i_{\vartheta}^{2}(\lambda_i)\,i_{\Phi}^{2}(\lambda_i)+\gamma'^{2}{}_{\vartheta}(\lambda_i)\,\gamma'^{2}{}_{\Phi}(\lambda_i)}{\sqrt{(\varphi_{\vartheta}^{2}(\lambda_i))^2+(i_{\vartheta}^{2}(\lambda_i))^2+(\gamma'^{2}{}_{\vartheta}(\lambda_i))^2}\sqrt{(\varphi_{\Phi}^{2}(\lambda_i))^2+(i_{\Phi}^{2}(\lambda_i))^2+(\gamma'^{2}{}_{\Phi}(\lambda_i))^2}}$$

$$=\omega_i\frac{3\varphi_{\vartheta}^{2}(\lambda_i)\,\varphi_{\Phi}^{2}(\lambda_i)}{\sqrt{3\varphi_{\vartheta}^{2}(\lambda_i)}\sqrt{3\varphi_{\Phi}^{2}(\lambda_i)}}=1$$

If $\varphi_{\vartheta}^{2}(\lambda_i)=2\varphi_{\Phi}^{2}(\lambda_i)\,,i_{\vartheta}^{2}(\lambda_i)=2i_{\Phi}^{2}(\lambda_i)\,a2\gamma\gamma'^{2}{}_{\vartheta}(\lambda_i)=2\gamma'^{2}_{\phi}(\lambda_i)$
i.e., $\boldsymbol{\vartheta}\neq\Phi$ based on Eq. (12), we have

$$\varrho_{12}(\boldsymbol{\vartheta},\Phi)=\frac{1}{2}\sum_{i=1}^{2}\omega_i\frac{\varphi_{\vartheta}^{2}(\lambda_i)\,\varphi_{\Phi}^{2}(\lambda_i)+i_{\vartheta}^{2}(\lambda_i)\,i_{\Phi}^{2}(\lambda_i)+\gamma'^{2}{}_{\vartheta}(\lambda_i)\,\gamma'^{2}_{\Phi}(\lambda_i)}{\sqrt{(\varphi_{\vartheta}^{2}(\lambda_i))^2+(i_{\vartheta}^{2}(\lambda_i))^2+\left(\gamma'^{2}{}_{\vartheta}(\lambda_i)\right)^2}\sqrt{(\varphi_{\Phi}^{2}(\lambda_i))^2+(i_{\Phi}^{2}(\lambda_i))^2+(\gamma'^{2}{}_{\Phi}(\lambda_i))^2}}$$

$$=\omega_i\frac{2\left(\varphi_{\vartheta}^{2}(\lambda_i)\right)^2+2\left(i_{\Phi}^{2}(\lambda_i)\right)^2+2\left(\gamma'^{2}_{\Phi}(\lambda_i)\right)^2}{\sqrt{4\left(\varphi_{\vartheta}^{2}(\lambda_i)\right)^2+4\left(i_{\vartheta}^{2}(\lambda_i)\right)^2+4\left(\gamma'^{2}_{\vartheta}(\lambda_i)\right)^2}\sqrt{(\varphi_{\Phi}^{2}(\lambda_i))^2+(i_{\Phi}^{2}(\lambda_i))^2+(\gamma'^{2}_{\Phi}(\lambda_i))^2}}=1$$

Obviously, in the above cases, the SMϱ_{12} is invalid.

(2) The similar SMs $\varrho_5,\varrho_7,\varrho_8,\varrho_{12}$, and ϱ_{14} do not satisfy the the axiom $\varrho(\boldsymbol{\vartheta},\Phi)=1$ implies $\boldsymbol{\vartheta}=\Phi$ and these SMs provide a counter-intuitive result for practical users in this case.

(3) For the SM $\varrho_3,\varrho_7,\varrho_8,\varrho_{11},\varrho_{12},\varrho_{13}$ and ϱ_{14}, when SFSs $A=\Phi=(\lambda,0.0,0.0,0.0)$ defined on $X=\{\lambda\}$, we have $\varrho_3(\boldsymbol{\vartheta},\Phi)=\varrho_7(\boldsymbol{\vartheta},\Phi)=\varrho_8(\boldsymbol{\vartheta},\Phi)=\varrho_{11}(\boldsymbol{\vartheta},\Phi)=\varrho_{12}(\boldsymbol{\vartheta},\Phi)=\varrho_{13}(\boldsymbol{\vartheta},\Phi)=\varrho_{14}(\boldsymbol{\vartheta},\Phi)=\frac{0}{0}$. In this case, these SM are invalid, they do not satisfy the axiom.

(4) The capacity of the SM to recognize the nearness of fuzzy not entirely settled by the articulation structure and the data contained in the articulation. The more data the SM focuses on the more grounded the identification ability. By analyzing Table 1 we find that the SM ϱ_1 only considers the difference of positive degree or neutral degree or negative degree or refusal degree between SFSs which brings a big amount of information losing. For example, let $\boldsymbol{\vartheta}=(0.1,0.2,0.1),\Phi=(0.6,0.2,0.1)$ be two SFSs. Since $|0.1-0.1|<|0.2-0.2|<|\,1-0.1-0.2-0.1|<|1-0.6-0.2-0.1|<|0.1-0.6\,|$, hence, the SM among A and B just considers the distinction of the positive degree between A and B by utilizing the SMϱ_1. For this situation, the SM will cause a great of data loss in viable application, so that it cannot provide more accurate results for practical users. In addition, in this situation, we also find that the SMs $\varrho_2,\varrho_4,\varrho_5,\varrho_6,\varrho_{p_3}$ have the same drawback.

3.1 A Parametric Similarity Measure Between Spherical Fuzzy Sets

Considering the reasons for the unsatisfactory results observed in the above analysis (Table 1), we propose an expanded parametric SM for SFSs in this section to address the limitations of existing SMs.

In this section, we introduce a parametric spherical fuzzy SM by developing a paired function. The analysis in Table 1 indicates that the SMs $\varrho_5,\varrho_7,\varrho_8,\varrho_{12}$, and ϱ_{14} have drawbacks. Consequently, we present the parametric SMs in Definition 8, which follows.

3.1.1 Definition 7: Let $\boldsymbol{\vartheta}=\{(\lambda_i,\varphi_{\vartheta}(\lambda_i),i_{\vartheta}(\lambda_i),\gamma'_{\vartheta}(\lambda_i))\mid\lambda_i\in X\}$ and $\Phi=\{((\lambda_i,\varphi_{\Phi}(\lambda_i),i_{\Phi}(\lambda_i),\gamma'_{\Phi}(\lambda_i)))\mid\lambda_i\in X\}$ be any two be SFSs on $X=\{\lambda_1,\lambda_2,\ldots,\lambda_2\}$ then the function $\varrho_\alpha:\mathrm{SFS}(\lambda)\times\mathrm{SFS}(\lambda)\to[0,1]$ defined by

$$\varrho_\alpha(\boldsymbol{\vartheta},\Phi)=1-\left[\frac{1}{3\eta}\sum_{i=1}^{\eta}\Delta_{1\vartheta\Phi}^{p}(\lambda_i)+\Delta_{2\vartheta\Phi}^{p}(\lambda_i)+\Delta_{3\vartheta\Phi}^{p}(\lambda_i)\right]^{\frac{1}{p}}\tag{15}$$

$\varrho_\alpha(\boldsymbol{\vartheta},\Phi)$ is a similarity measure between $\boldsymbol{\vartheta}$ and Φ, and $p=2$.
where,

$$\Delta_{1\vartheta\Phi}(\lambda_i)=\frac{1}{\alpha_1+1}\left|\alpha_1\left(\varphi_{\vartheta}^{2}(\lambda_i)-\varphi_{\Phi}^{2}(\lambda_i)\right)-\left(i_{\vartheta}^{2}(\lambda_i)-i_{\Phi}^{2}(\lambda_i)\right)-\left(\gamma'^{2}_{\vartheta}(\lambda_i)-\gamma'^{2}_{\Phi}(\lambda_i)\right)\right|\alpha_1\in[0,+\infty),$$

$$\Delta_{2\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_2+1}\left|\alpha_2\left(i_{\vartheta}^{2}(\lambda_i)-i_{\Phi}^{2}(\lambda_i)\right)-\left(\varphi_{\vartheta}^{2}(\lambda_i)-\varphi_{\Phi}^{2}(\lambda_i)\right)-\left(\gamma'^{2}_{\vartheta}(\lambda_i)-\gamma'^{2}_{\Phi}(\lambda_i)\right)\right|\alpha_2\in[0,+\infty),$$

$$\Delta_{3\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_3+1}\left|\alpha_3\left(\gamma'^{2}_{\vartheta}(\lambda_i)-\gamma'^{2}_{\Phi}(\lambda_i)\right)-\left(\varphi_{\vartheta}^{2}(\lambda_i)-\varphi_{\Phi}^{2}(\lambda_i)\right)-\left(i_{\vartheta}^{2}(\lambda_i)-i_{\Phi}^{2}(\lambda_i)\right)\right|\alpha_3\in[0,+\infty),$$

$\frac{1}{\alpha_1+1}+\frac{1}{2\alpha_2+1}+\frac{1}{2\alpha_2+1}\in(0,1]$, and p is any positive integer.

3.1.2 Theorem Let $\boldsymbol{\vartheta}=\left\{\left(\lambda,\varphi_{\vartheta}^{2}(\lambda),i_{\vartheta}^{2}(\lambda),\gamma'^{2}_{\boldsymbol{\vartheta}}(\lambda)\right)\mid\lambda\in X\right\}$ and $\Phi=\left\{\left(\lambda,\varphi_{\Phi}^{2}(\lambda),i_{\Phi}^{2}(\lambda),\gamma'^{2}_{\Phi}(\lambda)\right)\mid\lambda\in X\right\}$ be any two be SFSs on universe X, then the SM between $\boldsymbol{\vartheta}$ and Φ is defined as $\varrho_\alpha(\boldsymbol{\vartheta},\Phi)$, which satisfies the following axioms:

(ϱ_1) $0\le\varrho_\alpha(\boldsymbol{\vartheta},\Phi)\le 1$;

(ϱ_2) $\varrho_\alpha(\boldsymbol{\vartheta},\Phi)=1$ Iff $\boldsymbol{\vartheta}=\Phi$;

(ϱ_3) $\varrho_\alpha(\boldsymbol{\vartheta},\Phi)=\varrho_\alpha(\Phi,\boldsymbol{\vartheta})$;

(ϱ_4) Let C be any SFS such that $\boldsymbol{\vartheta}\subseteq\Phi\subseteq C$, then $\varrho_\alpha(\boldsymbol{\vartheta},C)\le\varrho_\alpha(\boldsymbol{\vartheta},\Phi)$ and $\varrho_\alpha(\boldsymbol{\vartheta},C)\le\varrho_\alpha(\Phi,C)$.

Proof: In order to prove that Eq. (15) is a SM, we only need to prove Eq. (15) satisfies axioms $(\varrho_1)-(\varrho_4)$ $\boldsymbol{\vartheta}=\{(\lambda_i,\varphi_\vartheta(\lambda_i),i_\vartheta(\lambda_i),\gamma'_\vartheta(\lambda_i))\mid\lambda_i\in X\}$ and $\Phi=\{((\lambda_i,\varphi_\Phi(\lambda_i),i_\Phi(\lambda_i),\gamma'_\Phi(\lambda_i)))\mid\lambda_i\in X\}$ and $C=\{((\lambda_i,\varphi_C(\lambda_i),i_C(\lambda_i),\gamma'_C(\lambda_i)))\mid\lambda_i\in X\}$ be any three SFSs on $X=\{\lambda_1,\lambda_2,\ldots,\lambda_2\}$.

(ϱ_1) We can write the following equations:

$$\Delta_{1\vartheta\Phi}(\lambda_i)=\frac{1}{\alpha_1+1}\left|\alpha_1\left(\varphi_\vartheta^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)\right)-\left(i_\vartheta^2(\lambda_i)-i_\Phi^2(\lambda_i)\right)-\left(\gamma'^2_\vartheta(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\right|\alpha_1\in[0,+\infty)$$
$$=\frac{1}{\alpha_1+1}\left|\left(\alpha_1\varphi_\vartheta^2(\lambda_i)-i_\vartheta^2(\lambda_i)-\gamma'^2_\vartheta(\lambda_i)\right)-\left(\alpha_1\varphi_\Phi^2(\lambda_i)-i_\Phi^2(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\right|$$

$$\Delta_{2\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_2+1}\left|\alpha_2\left(i_\vartheta^2(\lambda_i)-i_\Phi^2(\lambda_i)\right)-\left(\varphi_\vartheta^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)\right)+\left(\gamma'^2_\vartheta(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\right|\alpha_2\in[0,+\infty)$$
$$=\frac{1}{2\alpha_2+1}\left|\left(\alpha_2 i_\vartheta^2(\lambda_i)-\varphi_\vartheta^2(\lambda_i)-\gamma'^2_\vartheta(\lambda_i)\right)-\left(\alpha_2 i_\vartheta^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\right|$$

$$\Delta_{3\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_3+1}\left|\alpha_3\left(\gamma'^2_\vartheta(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)-\left(\varphi_\vartheta^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)\right)+\left(i_\vartheta^2(\lambda_i)-i_\Phi^2(\lambda_i)\right)\right|\alpha_3\in[0,+\infty)$$
$$=\frac{1}{2\alpha_3+1}\left|\left(\alpha_3\gamma'^2_\vartheta(\lambda_i)-\varphi_\vartheta^2(\lambda_i)-i_\vartheta^2(\lambda_i)\right)-\left(\alpha_3\gamma'^2_\vartheta(\lambda_i)-\varphi_\Phi^2(\lambda_i)-i_\Phi^2(\lambda_i)\right)\right|$$

By $\varphi_\vartheta^2(\lambda_i),i_\vartheta^2(\lambda_i),\gamma'^2{}_\vartheta(\lambda_i),\varphi_\Phi^2(\lambda_i),i_\Phi^2(\lambda_i),\gamma'^2{}_\Phi(\lambda_i)\in[0,1]$ and $\varphi_\vartheta^2(\lambda_i)+i_\vartheta^2(\lambda_i)+\gamma'^2{}_\vartheta(\lambda_i)\le 1\varphi_\Phi^2(\lambda_i)+i_\Phi^2(\lambda_i)+\gamma'^2_\Phi(\lambda_i)\le 1$

We have

$$-1\le\alpha_1\varphi_\vartheta^2(\lambda_i)-i_\vartheta^2(\lambda_i)-\gamma'^2{}_\vartheta(\lambda_i)\le\alpha_1$$
$$-\alpha_1\le-\left(\alpha_1\varphi_\Phi^2(\lambda_i)-i_\Phi^2(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\le 1$$
$$0\le\left|\left(\alpha_1\varphi_\vartheta^2(\lambda_i)-i_\vartheta^2(\lambda_i)-\gamma'^2_\vartheta(\lambda_i)\right)-\left(\alpha_1\varphi_\Phi^2(\lambda_i)-i_\Phi^2(\lambda_i)-\gamma'^2_\Phi(\lambda_i)\right)\right|\le\alpha_1+1$$

i.e, $0\le\Delta_{1\vartheta\Phi}(\lambda_i)\le 1$

Then

$$-1\le\alpha_2 i_\vartheta^2(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+\gamma'^2_\vartheta(\lambda_i)\le 1\vee\alpha_2$$
$$-(1\vee\alpha_2)\le-\left(\alpha_2 i_\Phi^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)+\gamma'^2_\Phi(\lambda_i)\right)\le 1$$

Similarly, we get the following inequalities:

$$-1\le\alpha_3\gamma'^2_\vartheta(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+i_\vartheta^2(\lambda_i)\le 1\vee\alpha_3$$
$$-(1\vee\alpha_3)\le-\left(\alpha_3\gamma'^2_\Phi(\lambda_i)-\varphi_\Phi^2(\lambda_i)+i_\Phi^2(\lambda_i)\right)\le 1$$

Then we obtain:

$$0\le\left|\left(\alpha_2 i_\vartheta^2(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+\gamma'^2{}_\vartheta(\lambda_i)\right)-\left(\alpha_2 i_\Phi^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)+\gamma'^2{}_\Phi(\lambda_i)\right)\right|\le 2\vee\alpha_2$$
$$0\le\left|\left(\alpha_3\gamma'^2{}_\vartheta(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+i_\vartheta^2(\lambda_i)\right)-\left(\alpha_3\gamma'^2{}_\Phi(\lambda_i)-\varphi_\Phi^2(\lambda_i)+i_\Phi^2(\lambda_i)\right)\right|\le 2\vee\alpha_3$$

It means that:

$$0\le\Delta_{2\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_2+1}\left|\left(\alpha_2 i_\vartheta^2(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+\gamma'^2{}_\vartheta(\lambda_i)\right)-\left(\alpha_2 i_\Phi^2(\lambda_i)-\varphi_\Phi^2(\lambda_i)+\gamma'^2_\Phi(\lambda_i)\right)\right|\le\frac{1}{\alpha_2+1}\vee\frac{1}{2}\le 1$$

$$0\le\Delta_{3\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_3+1}\left|\left(\alpha_3\gamma'^2_\vartheta(\lambda_i)-\varphi_\vartheta^2(\lambda_i)+i_\vartheta^2(\lambda_i)\right)-\left(\alpha_3\gamma'^2_\Phi(\lambda_i)-\varphi_\Phi^2(\lambda_i)+i_\Phi^2(\lambda_i)\right)\right|\le\frac{1}{\alpha_3+1}\vee\frac{1}{2}\le 1$$

Finally, we have:

$$0\le 1-\left[\frac{1}{32}\sum_{i=1}^{2}\Delta_{1\vartheta\Phi}^p(\lambda_i)+\Delta_{2\vartheta\Phi}^p(\lambda_i)+\Delta_{3\vartheta\Phi}^p(\lambda_i)\right]^{\frac{1}{p}}\le 1$$

Therefore,

$$(\varrho_1)\, 0 \leq \varrho_\alpha(\boldsymbol{\vartheta}, \Phi) \leq 1$$

(ϱ_2) If $\boldsymbol{\vartheta} = \Phi$ then $\varphi^2_\vartheta(\lambda_i) = \varphi^2_\Phi(\lambda_i), i^2_\vartheta(\lambda_i) = i^2_\Phi(\lambda_i)$ and $\gamma'^2_\vartheta(\lambda_i) = \gamma'^2_\Phi(\lambda_i)$. Therefore, $\Delta_{1\boldsymbol{\vartheta}\Phi}(\lambda_i) = 0, \Delta_{2\boldsymbol{\vartheta}\Phi}(\lambda_i) = 0a2\gamma, \Delta_{3\boldsymbol{\vartheta}\Phi}(\lambda_i) = 0$ i.e., $\varrho_\alpha(\boldsymbol{\vartheta}, \Phi) = 1$

If $\varrho_\alpha(\boldsymbol{\vartheta}, \Phi) = 1$ then

$$\Delta_{1\vartheta\Phi}(\lambda_i) = \frac{1}{\alpha_1+1}\left|\alpha_1\left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) - \left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right)\right| = 0$$

$$\Delta_{2\boldsymbol{\vartheta}\Phi}(\lambda_i) = \frac{1}{2\alpha_2+1}\left|\alpha_2\left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) + \left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right)\right| = 0$$

$$\Delta_{3\vartheta\Phi}(\lambda_i) = \frac{1}{2\alpha_3+1}\left|\alpha_3\left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) - \left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) + \left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right)\right| = 0$$

By the definition of absolute value, we have:

$$\alpha_1\left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) - \left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) = 0$$

$$\alpha_2\left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) + \left(\gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) = 0$$

$$\alpha_3\left(\gamma'^2{}_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right) - \left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) + \left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) = 0$$

i.e.,

$$\begin{pmatrix} \alpha_1 & -1 & -1 \\ -1 & \alpha_2 & 1 \\ -1 & 1 & \alpha_3 \end{pmatrix} \begin{pmatrix} \varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i) \\ i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i) \\ \gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since $\frac{1}{\alpha_1+1} + \frac{1}{2\alpha_2+1} + \frac{1}{2\alpha_3+1} \in (0, 1]$ then $2 \leq \alpha_1\alpha_2\alpha_3 - (\alpha_1 + \alpha_3 + \alpha_3)$.
By the definition of matrix determinant, we can get:

$$\begin{vmatrix} \alpha_1 & -1 & -1 \\ -1 & \alpha_2 & 1 \\ -1 & 1 & \alpha_3 \end{vmatrix} = \alpha_1\alpha_2\alpha_3 + 2 - (\alpha_1 + \alpha_3 + \alpha_3) \geq 4$$

Therefore, we have

$$\begin{pmatrix} \varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i) \\ i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i) \\ \gamma'^2_\vartheta(\lambda_i) - \gamma'^2_\Phi(\lambda_i) \end{pmatrix} = \begin{pmatrix} \alpha_1 & -1 & -1 \\ -1 & \alpha_2 & 1 \\ -1 & 1 & \alpha_3 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

It means that $\varphi^2_\vartheta(\lambda_i) = \varphi^2_\Phi(\lambda_i), i^2_\vartheta(\lambda_i) = i^2_\Phi(\lambda_i)$ and $\gamma'^2_\vartheta(\lambda_i) = \gamma'^2_\Phi(\lambda_i)$ then $\boldsymbol{\vartheta} = \Phi$ (ϱ_3). Based on the definition of absolute value, we can get the following equations:

$$\begin{aligned}
\Delta_{1\boldsymbol{\vartheta}\Phi}(\lambda_i) &= \frac{1}{\alpha_1+1}\left|\alpha_1\left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) - \left(i^2_\vartheta(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right)\right| \\
&= \frac{1}{\alpha_1+1}\left|(-1)\left[\alpha_1\left(\varphi^2_\Phi(\lambda_i) - \varphi^2_\vartheta(\lambda_i)\right) - \left(i^2_\Phi(\lambda_i) - i^2_\vartheta(\lambda_i)\right) - \left(\gamma'^2_\Phi(\lambda_i) - \gamma'^2_\vartheta(\lambda_i)\right)\right]\right| \\
&= \frac{1}{\alpha_1+1}\left|\alpha_1\left(\varphi^2_\Phi(\lambda_i) - \varphi^2_\vartheta(\lambda_i)\right) - \left(i^2_\Phi(\lambda_i) - i^2_\vartheta(\lambda_i)\right) - \left(\gamma'^2_\Phi(\lambda_i) - \gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i)\right)\right| \\
&= \Delta_{1\Phi\vartheta}(\lambda_i)
\end{aligned}$$

$$\begin{aligned}
\Delta_{2\boldsymbol{\vartheta}\Phi}(\lambda_i) &= \frac{1}{2\alpha_2+1}\left|\alpha_2\left(i^2_{\boldsymbol{\vartheta}}(\lambda_i) - i^2_\Phi(\lambda_i)\right) - \left(\varphi^2_\vartheta(\lambda_i) - \varphi^2_\Phi(\lambda_i)\right) + \left(\gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i) - \gamma'^2_\Phi(\lambda_i)\right)\right| \\
&= \frac{1}{2\alpha_2+1}\left|(-1)\left[\alpha_2\left(i^2_\Phi(\lambda_i) - i^2_\vartheta(\lambda_i)\right) - \left(\varphi^2_\Phi(\lambda_i) - \varphi^2_\vartheta(\lambda_i)\right) + \left(\gamma'^2_\Phi(\lambda_i) - \gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i)\right)\right]\right| \\
&= \frac{1}{2\alpha_2+1}\left|\alpha_2\left(i^2_\Phi(\lambda_i) - i^2_\vartheta(\lambda_i)\right) - \left(\varphi^2_\Phi(\lambda_i) - \varphi^2_\vartheta(\lambda_i)\right) + \left(\gamma'^2_\Phi(\lambda_i) - \gamma'^2_\vartheta(\lambda_i)\right)\right| \\
&= \Delta_{2\Phi\vartheta}(\lambda_i)
\end{aligned}$$

$$\Delta_{3\vartheta\Phi}(\lambda_i)=\frac{1}{2\alpha_3+1}\left|\alpha_3\left(\gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)-\left(\varphi^2_{\vartheta}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)\right)+\left(i^2_{\vartheta}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right)\right|$$
$$=\frac{1}{2\alpha_3+1}\left|(-1)\left[\mid\alpha_3\left(\gamma'^2_{\Phi}(\lambda_i)-\gamma'^2_{\boldsymbol{\vartheta}}(\lambda_i)\right)-\left(\varphi^2_{\Phi}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)\right)+\left(i^2_{\Phi}(\lambda_i)-i^2_{\vartheta}(\lambda_i)\right)\right]\right|$$
$$=\frac{1}{2\alpha_3+1}\left|\alpha_3\left(\gamma'^2_{\Phi}(\lambda_i)-\gamma'^2_{\vartheta}(\lambda_i)\right)-\left(\varphi^2_{\Phi}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)\right)+\left(i^2_{\Phi}(\lambda_i)-i^2_{\vartheta}(\lambda_i)\right)\right|$$
$$=\Delta_{3\Phi\vartheta}(\lambda_i)$$
$$\varrho_\alpha(\boldsymbol{\vartheta},\Phi)=\varrho_\alpha(\Phi,\boldsymbol{\vartheta})$$

(ϱ_4) Therefore $\boldsymbol{\vartheta}\subseteq\Phi\subseteq C$ then $\varphi^2_{\vartheta}(\lambda_i)\le\varphi^2_{\Phi}(\lambda_i)\le\varphi^2_C(\lambda_i), i^2_C(\lambda_i)\le i^2_{\Phi}(\lambda_i)\le i^2_{\vartheta}(\lambda_i)$, $\gamma'^2_C(\lambda_i)\le\gamma'^2_{\Phi}(\lambda_i)\le\gamma'^2_{\vartheta}(\lambda_i)$

Therefore, we can have

$$\alpha_1\left(\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)-\gamma'^2{}_{\vartheta}(\lambda_i)\right)\le\alpha_1\left(\varphi^2_{\Phi}(\lambda_i)-i^2_{\Phi}(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)\le\alpha_1\left(\varphi^2_C(\lambda_i)-i^2_C(\lambda_i)-\gamma'^2_C(\lambda_i)\right)$$
$$\alpha_2\left(i^2_C(\lambda_i)-\varphi^2_C(\lambda_i)-\gamma'^2_C(\lambda_i)\right)\le\alpha_2\left(i^2_{\Phi}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)\le\alpha_2\left(i^2_{\vartheta}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-\gamma'^2{}_{\vartheta}(\lambda_i)\right)$$
$$\alpha_3\left(\gamma'^2_C(\lambda_i)-\varphi^2_C(\lambda_i)-i^2_C(\lambda_i)\right)\le\alpha_3\left(\gamma'^2{}_{\Phi}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right)\le\alpha_3\left(\gamma'^2{}_{\boldsymbol{\vartheta}}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)\right)$$

By the property of inequality, we can obtain:

$$\left|\alpha_1\left(\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)-\gamma'^2{}_{\vartheta}(\lambda_i)\right)-\alpha_1\left(\varphi^2_{\phi}(\lambda_i)-i^2_{\phi}(\lambda_i)-\gamma'^2{}_{\phi}(\lambda_i)\right)\right|$$
$$\le\left|\alpha_1\left(\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)-\gamma'^2{}_{\vartheta}(\lambda_i)\right)-\alpha_1\left(\varphi^2_C(\lambda_i)-i^2_C(\lambda_i)-\gamma'^2_C(\lambda_i)\right)\right|$$
$$\left|\alpha_2\left(i^2_{\vartheta}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-\gamma'^2_{\vartheta}(\lambda_i)\right)-\alpha_2\left(i^2_{\Phi}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)\right|$$
$$\le\left|\alpha_2\left(i^2_{\vartheta}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-\gamma'^2_{\vartheta}(\lambda_i)\right)-\alpha_2\left(i^2_C(\lambda_i)-\varphi^2_C(\lambda_i)-\gamma'^2_C(\lambda_i)\right)\right|$$
$$\left|\alpha_3\left(\gamma'^2_{\vartheta}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)\right)-\alpha_3\left(\gamma'^2_{\Phi}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right)\right|$$
$$\le\left|\alpha_3\left(\gamma'^2_{\vartheta}(\lambda_i)-\varphi^2_{\vartheta}(\lambda_i)-i^2_{\vartheta}(\lambda_i)\right)-\alpha_3\left(\gamma'^2_C(\lambda_i)-\varphi^2_C(\lambda_i)-i^2_C(\lambda_i)\right)\right|$$
$$\Delta_{2\boldsymbol{\vartheta}\Phi}(\lambda_i)\le\Delta_{1\boldsymbol{\vartheta}C}(\lambda_i),\Delta_{2\boldsymbol{\vartheta}\Phi}(\lambda_i)\le\Delta_{2\boldsymbol{\vartheta}C}(\lambda_i),\Delta_{3\boldsymbol{\vartheta}\Phi}(\lambda_i)\le\Delta_{3\boldsymbol{\vartheta}C}(\lambda_i)$$

Therefore, we have

$$1-\left[\frac{1}{3\eta}\sum_{i=1}^{2}\Delta^p_{1\vartheta\boldsymbol{C}C}(\lambda_i)+\Delta^p_{2\vartheta C}(\lambda_i)+\Delta^p_{3\vartheta C}(\lambda_i)\right]^{\frac{1}{p}}$$
$$=1-\left[\frac{1}{3\eta}\sum_{i=1}^{2}\Delta^p_{1\vartheta\Phi}(\lambda_i)+\Delta^p_{2\vartheta\Phi}(\lambda_i)+\Delta^p_{3\vartheta\Phi}(\lambda_i)\right]^{\frac{1}{p}}$$

It means that $\varrho_\alpha(\boldsymbol{\vartheta},C)\le\varrho_\alpha(\boldsymbol{\vartheta},\Phi)$.

Similarity, we have $\varrho_\alpha(\boldsymbol{\vartheta},C)\le\varrho_\alpha(\Phi,C)$.

(1) When $\alpha_1=0,\alpha_2=\alpha_3=+\infty$, Eq. (15) can be written as:

$$\varrho_1(\boldsymbol{\vartheta},\Phi)=1-\left[\frac{1}{3\eta}\sum_{i=1}^{2}\left(\left|\left(i^2_{\vartheta}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right)\right.\right.\right.$$
$$\left.+\left(\gamma'_{\vartheta}{}^2(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)\right|^P+\frac{\left|i^2_{\vartheta}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right|^P}{2^P}$$
$$\left.\left.+\frac{\left|\left(\gamma'^2_{\vartheta}(\lambda_i)-\gamma'^2_{\Phi}(\lambda_i)\right)\right|^P}{2^P}\right)\right]^{\frac{1}{P}} \tag{16}$$

(2) When $\alpha_1=\alpha_2=+\infty\alpha_3=0$, Eq. (15) can be written as:

$$\varrho_2(\boldsymbol{\vartheta},\Phi)=1-\left[\frac{1}{3\eta}\sum_{i=1}^{2}\left(\left|\varphi^2_{\vartheta}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)\right|^P\right.\right.$$
$$\left.\left.+\frac{\left|i^2_{\vartheta}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right|^P}{2^P}+\frac{\left|\left(i^2_{\vartheta}(\lambda_i)-i^2_{\Phi}(\lambda_i)\right)-\left(\varphi^2_{\vartheta}(\lambda_i)-\varphi^2_{\Phi}(\lambda_i)\right)\right|^P}{2^P}\right)\right]^{\frac{1}{P}} \tag{17}$$

3.1.3 Theorem For any two SFSs

$$\boldsymbol{\vartheta} = \{(\lambda_i, \varphi_\vartheta(\lambda_i), i_\vartheta(\lambda_i), \gamma'_\vartheta(\lambda_i)) \mid \lambda_i \in X\},$$
$$\Phi = \{(\lambda_i, \varphi_\Phi(\lambda_i), i_\Phi(\lambda_i), \gamma'_\Phi(\lambda_i)) \mid \lambda_i \in X\},$$

on $X = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, $\omega_i \in [0,1]$ and $\sum_{i=1}^{n} \omega_i = 1$. The function ϱ_ω : SFS(λ) $\times$ SFS(λ) $\rightarrow [0,1]$ is defined by

$$\varrho_\omega(\boldsymbol{\vartheta}, \Phi) = 1 - \left[\frac{1}{3\eta}\sum_{i=1}^{2}\omega_i\left(\Delta^p_{1\vartheta\Phi}(\lambda_i) + \Delta^p_{2\vartheta\Phi}(\lambda_i) + \Delta^p_{3\vartheta\Phi}(\lambda_i)\right)\right]^{\frac{1}{p}}$$

$\varrho_\omega(\boldsymbol{\vartheta}, \Phi)$ is a weighted SM between $\boldsymbol{\vartheta}$ and Φ.

Proof: The proof is similar to Theorem 1 .

In the following, an example is added to clarify more.

3.1.4 Example Let $\boldsymbol{\vartheta} = (\lambda, 0.2, 0.4, 0.5)$, $\Phi = (\lambda, 0.4, 0.3, 0.4)$ and $C = (\lambda, 0.5, 0.0, 0.0)$ are three different SFVs on $X = \{\lambda\}$.$\boldsymbol{\vartheta}$ Is more similar to Φ than the C say $\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(\boldsymbol{\vartheta}, C)$. To prove the accuracy of this view for our proposed SM ϱ_α and the current ones to be specific $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6\varrho_7, \varrho_9, \varrho_{10}\varrho_{11}\varrho_{12}, \varrho_{13}, \varrho_{14}, \varrho_\alpha$. We can see the obtained values of the SMs in Table 2.

Table 2. The values of the SMs on SFVs ϑ, Φ and C

ϱ	$\varrho(\boldsymbol{\vartheta}, \Phi)$	$\varrho(\boldsymbol{\vartheta}, \boldsymbol{C})$	**Relation**
ϱ_1	0.902	0.811	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_2	0.923	0.847	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_3	0.3552	-0.1463	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_4	0.939	0.875	$\varrho(\boldsymbol{v}, \Phi) = \varrho(C, \Phi)$
ϱ_5	0.115	0.2222	$\varrho(\boldsymbol{\vartheta}, \Phi) < \varrho(C, \Phi)$
ϱ_6	0.8453	0.7273	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_7	0.3913	0.0255	$\varrho(\boldsymbol{v}, \Phi) > \varrho(C, \Phi)$
ϱ_8	0.832	0.712	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_9	0.9219	0.856	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_{10}	0.9219	0.856	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_{11}	0.077	0.153	$\varrho(\boldsymbol{\vartheta}, \Phi) < \varrho(C, \Phi)$
ϱ_{12}	0.0385	0.0765	$\varrho(\boldsymbol{\vartheta}, \Phi) < \varrho(C, \Phi)$
ϱ_{13}	0.6599	0.0765	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_{14}	0.2292	0.0196	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$
ϱ_m	0.4	0.0175	$\varrho(\boldsymbol{\vartheta}, \Phi) > \varrho(C, \Phi)$

4 Application of the Proposed Similarity Measures

In this section, we apply the proposed the SMs in MADM problems, which show the expected SM is sensible and in accordance with human cognition.

Let $X = \{\lambda_1, \lambda_2, \ldots \lambda_\alpha\}$ a set of attributes, the η alternatives $\boldsymbol{\vartheta}_i = (\boldsymbol{\vartheta}_{ij}) = \{(\lambda_j, \varphi_{\boldsymbol{\vartheta} i}(\lambda_j), i_{\boldsymbol{\vartheta} i}(\lambda_j), \gamma'_{\boldsymbol{\vartheta i}}(\lambda_j))\} \mid \lambda_j \in X$ Where $\lambda_j, \varphi^2_\vartheta(\lambda_i), i^2_\vartheta(\lambda_i), \gamma'^2_\vartheta(\lambda_i), \varphi^2_\Phi(\lambda_i), i^2_\Phi(\lambda_i), {\gamma'}^2_\Phi(\lambda_i) \in [0,1]$ and $\varphi^2_\vartheta(\lambda_i) + i^2_\vartheta(\lambda_i) + {\gamma'}^2{}_\vartheta(\lambda_i) \leq 1$, $\varphi^2_\vartheta(\lambda_i)$ is a positive degree which is use to alternative $\boldsymbol{\vartheta}_i$ satisfies the $\lambda_j(i = \{1, 2, \ldots 2\}, j = \{1, 2, \ldots \alpha\}).i^2_\vartheta(\lambda_i)$ a neutral degree which is use to alternative $\boldsymbol{\vartheta}_i$ does not satisfies the $\lambda_j \cdot \gamma'^2_\vartheta(\lambda_i)$, negative degree which is use to alternative $\boldsymbol{\vartheta}_i$ does not satisfies the λ_j. The decision making is used to choose best alternative steps are following.

Step 1. Standardize decision alternatives.

In this process multi attribute decision making can be divided in to type's amount type and interest type. The amount type can be changed into interest type by use the formula of decision-making process.

$$\boldsymbol{\vartheta}'_{ij} = \begin{cases} \boldsymbol{\vartheta}_{ij} \text{ for benefit attribute } \lambda_j \\ \boldsymbol{\vartheta}^c_{ij} \text{ for cost attribute } \lambda_j \end{cases}$$

$\boldsymbol{\vartheta}^c_{ij} = (\varphi_{\vartheta i}(\lambda_j), i_{\vartheta i}(\lambda_j), \gamma'_{\vartheta i}(\lambda_j)), i = \{1, 2, \ldots \eta\}, j = \{1, 2 \ldots \alpha\}$. The above formula based on the alternative $\boldsymbol{\vartheta}_i = \{\boldsymbol{\vartheta}'_{ij}\}$.

Step 2. The SM $\varrho = (\boldsymbol{\vartheta}_i, \boldsymbol{\vartheta})$ $(i = 1, 2, 3, \ldots, \eta)$ is calculated where, $\boldsymbol{\vartheta}$ = (0.2,0.4,0.5),(0.2,0.4,0.5),(0.2,0.4,0.5) is a standard provided by the decision maker in the form of the SFV. We find the similarity values with the help of the proposed SM.

Step 3. The maximum one is chosen in $\varrho = (\boldsymbol{\vartheta}_{i0}, \boldsymbol{\vartheta})$ from $\varrho = (\boldsymbol{\vartheta}_i, \boldsymbol{\vartheta})\; i = (i = 1, 2, 3, \ldots \eta)$ i.e $\varrho = (\boldsymbol{\vartheta}_{i0}, \boldsymbol{\vartheta}) = \max_{1 \leq i \leq 2} \{\varrho(\boldsymbol{\vartheta}_i, \boldsymbol{\vartheta})\}$. Then the maximum SMs alternative $\boldsymbol{\vartheta}_{i0}$ according to the principle of maximsum.

In the following example for the similarity measure $\varrho_\alpha, P = 3, \alpha_1 = \alpha_2 = \alpha_3 = 3$

3.1.5 Example There are three medical equipment $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2, \boldsymbol{\vartheta}_3$ with four different attributes $\lambda_1, \lambda_2, \lambda_3$ described the SFSs as shown in Table 3. The weight of $\lambda_j (1 \leq j \leq 3)$ are $(0.5, 0.3, 0.2)$.

Table 3. Three alternatives with three attributes

	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$
$\boldsymbol{\vartheta}_1$	$(0.15, 0.14, 0.12)$	$(0.13, 0.13, 0.33)$	$(0.33, 0.22, 0.16)$
$\boldsymbol{\vartheta}_2$	$(0.18, 0.13, 0.34)$	$(0.26, 0.26, 0.27)$	$(0.39, 0.1, 0.16)$
$\boldsymbol{\vartheta}_3$	$(0.12, 0.18, 0.37)$	$(0.25, 0.32, 0.21)$	$(0.38, 0.28, 0.35)$

In the following, Table 4 shows the values of the SMs of $\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta}_2$, and $\boldsymbol{\vartheta}_3$ with $\boldsymbol{\vartheta}$.

Table 4. Values of the similarity measures and decision results of the Example 2

	$\varrho(\boldsymbol{\vartheta}_1, \boldsymbol{\vartheta})$	$\varrho(\boldsymbol{\vartheta}_2, \boldsymbol{\vartheta})$	$\varrho(\boldsymbol{\vartheta}_3, \boldsymbol{\vartheta})$
ϱ_1	0.6544	0.6903	0.7537
ϱ_2	0.8293	0.8244	0.8642
ϱ_3	XXX	XXX	XXX
ϱ_4	0.7996	0.8185	0.8463
ϱ_5	0.3328	0.3059	0.2647
ϱ_6	0.6238	0.6474	0.6845
ϱ_7	0.2208	0.293	0.4125
ϱ_8	XXX	XXX	XXX
ϱ_9	0.8148	0.8222	0.849
ϱ_{10}	0.8312	0.8474	0.853
ϱ_{11}	0.8878	0.7858	0.8841
ϱ_{12}	0.3339	0.3191	0.3338
ϱ_{13}	0.2586	0.6521	0.8402
ϱ_{14}	0.1006	0.3041	0.375
ϱ_m	0.9086	0.9108	0.9253

Table 4 displays the values of the SMs for medical equipment based on their attributes. Now, we will determine the ranking of the alternatives using the values obtained from the SMs. The resulting decision rankings for the medical equipment are presented in Table 5.

Table 5. Ranking of medical equipment based on SM values

	The Best Alternative	**Doc**
ϱ_1	$\boldsymbol{\vartheta}_3$	0.1352
ϱ_2	$\boldsymbol{\vartheta}_3$	0.0398
ϱ_3	XXX	0.2706
ϱ_4	$\boldsymbol{\vartheta}_3$	0.0656
ϱ_5	$\boldsymbol{\vartheta}_3$	0.3059
ϱ_6	$\boldsymbol{\vartheta}_3$	0.0843
ϱ_7	$\boldsymbol{\vartheta}_3$	0.2639
ϱ_8	XXX	2.5184
ϱ_9	$\boldsymbol{\vartheta}_3$	0.0416
ϱ_{10}	$\boldsymbol{\vartheta}_3$	0.038
ϱ_{11}	$\boldsymbol{\vartheta}_1$	0.1057
ϱ_{12}	$\boldsymbol{\vartheta}_1$	0.0149
ϱ_{13}	$\boldsymbol{\vartheta}_3$	0.9751
ϱ_{14}	$\boldsymbol{\vartheta}_3$	0.4779
ϱ_m	$\boldsymbol{\vartheta}_3$	0.0189

It is cleared from Table 5, the alternative $\boldsymbol{\vartheta}_3$ is obtained by using the SMs $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_9, \varrho_{10}, \varrho_{11}, \varrho_{12}, \varrho_{13}, \varrho_{14}$ and ϱ_α. However, the alternative $\boldsymbol{\vartheta}_3$ is obtained the ϱ_8. The ranking of the medical equipment is geometrically represented by Figure 1 as follows.

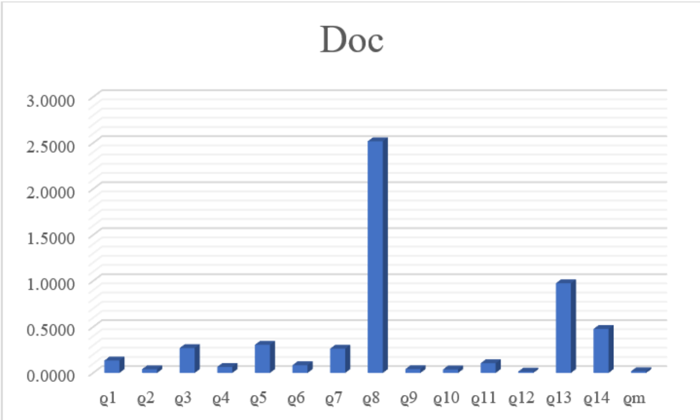

Figure 1. Ranking of the medical equipment obtained from the SMs in Table 5

It is cleared from Figure 1, the alternative $\boldsymbol{\vartheta}_3$ is obtained by using the SMs $\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_9, \varrho_{10}, \varrho_{11}, \varrho_{12}, \varrho_{13}, \varrho_{14}$ and ϱ_α. However, the alternative $\boldsymbol{\vartheta}_3$ is obtained by the ϱ_m.

3.1.6 Example In this example, a MADM problem related to selecting medical equipment is solved. When prioritizing clinical needs assessment, factors such as regulatory compliance, interoperability with current systems, user-friendly interfaces, dependable quality from reputable manufacturers, cost-effectiveness, scalability, patient comfort and safety, evidence-based decision-making, and extensive training and support are crucial. By considering these factors, healthcare facilities can make well-informed decisions that not only improve patient care but also ensure operational effectiveness and regulatory compliance. Based on some attribute i.e., Precision (λ_1), Versatility (λ_2) and Ease of maintenance (λ_3) the weight vector is $\omega = (0.2, 0.3, 0.5)$. Let there are six candidates to be assessed based on these attributes. Consider $\boldsymbol{\vartheta} = \{(0.11, 0.21, 0.32), (0.11, 0.21, 0.32), (0.11, 0.21, 0.32)\}$ the standard is set in the form of the SFV. The alternative which is more similar to $\boldsymbol{\vartheta}$ is considered as the best employee. After an initial assessment, the employees are assigned the SFVs with respect to the attributes provided in Table 6 in the following.

Table 6. Evaluation results of six faculty candidates in this example

	$\boldsymbol{x_1}$	$\boldsymbol{x_2}$	$\boldsymbol{x_3}$
$\boldsymbol{\vartheta_1}$	$(0.23, 0.33, 0.20)$	$(0.33, 0.13, 0.22)$	$(0.12, 0.32, 0.37)$
$\boldsymbol{\vartheta_2}$	$(0.10, 0.20, 0.24)$	$(0.02, 0.21, 0.10)$	$(0.22, 0.22, 0.22)$
$\boldsymbol{\vartheta_3}$	$(0.31, 0.31, 0.25)$	$(0.32, 0.24, 0.22)$	$(0.10, 0.22, 0.33)$
$\boldsymbol{\vartheta_4}$	$(0.13, 0.25, 0.23)$	$(0.23, 0.24, 0.22)$	$(0.32, 0.21, 0.1)$
$\boldsymbol{\vartheta_5}$	$(0.13, 0.22, 0.13)$	$(0.22, 0.13, 0.22)$	$(0.23, 0.23, 0.10)$
$\boldsymbol{\vartheta_6}$	$(0.17, 0.11, 0.14)$	$(0.12, 0.22, 0.21)$	$(0.32, 0.2, 0.12)$

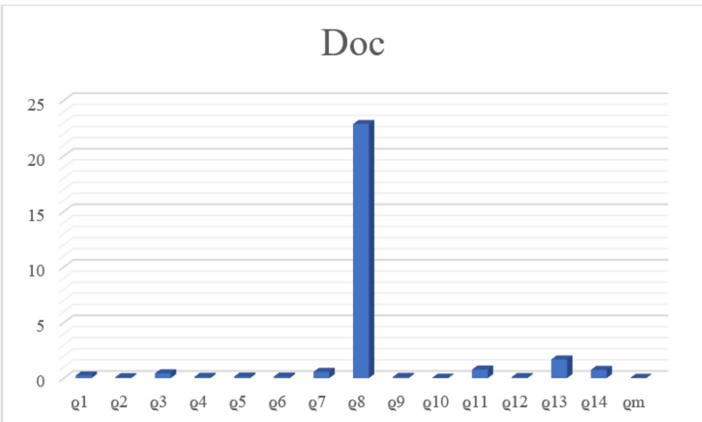

Figure 2. The ranking of the medical equipment obtained from the SMs in Table 8

Table 7. Similarity measures and decision results in this example

	$\varrho(\vartheta_1,\vartheta)$	$\varrho(\vartheta_2,\vartheta)$	$\varrho(\vartheta_3,\vartheta)$	$\varrho(\vartheta_4,\vartheta)$	$\varrho(\vartheta_5,\vartheta)$	$\varrho(\vartheta_6,\vartheta)$
ϱ_1	0.7326	0.6496	0.7206	0.6741	0.6468	0.6373
ϱ_2	0.8612	0.8466	0.8506	0.8322	0.8276	0.8205
ϱ_3	XXX	XXX	XXX	XXX	XXX	XXX
ϱ_4	0.8251	0.7887	0.8233	0.7871	0.7751	0.776
ϱ_5	0.2953	0.3485	0.2995	0.3506	0.3669	0.3658
ϱ_6	0.6559	0.612	0.6535	0.6102	0.5951	0.5981
ϱ_7	0.3916	0.2196	0.3497	0.2643	0.2116	0.1855
ϱ_8	XXX	XXX	XXX	XXX	XXX	XXX
ϱ_9	0.8482	0.8199	0.8359	0.8226	0.8183	0.8086
ϱ_{10}	0.8119	0.8151	0.8195	0.8365	0.8192	0.8221
ϱ_{11}	0.8996	0.8413	0.9386	0.7454	0.7405	0.6992
ϱ_{12}	0.3131	0.3277	0.321	0.3148	0.2943	0.276
ϱ_{13}	0.6767	0.417	0.7393	0.4639	0.2719	0.1951
ϱ_{14}	0.2896	0.1917	0.3291	0.2186	0.123	0.0838
ϱ_{m}	0.9125	0.9145	0.91	0.912	0.905	0.904

Table 8. Ranking of the medical equipment obtained by the proposed and existing SMs

	The Best Candidate	Doc
ϱ_1	ϑ_1	0.2372
ϱ_2	ϑ_1	0.0873
ϱ_3	XXX	0.4301
ϱ_4	ϑ_1	0.1211
ϱ_5	ϑ_1	0.14
ϱ_6	ϑ_1	0.1422
ϱ_7	ϑ_1	0.5597
ϱ_8	XXX	22.8696
ϱ_9	ϑ_1	0.1019
ϱ_{10}	ϑ_1	0.0371
ϱ_{11}	ϑ_3	0.767
ϱ_{12}	ϑ_3	0.0925
ϱ_{13}	ϑ_3	1.6719
ϱ_{14}	ϑ_3	0.7388
ϱ_{m}	ϑ_2	0.03

The similarity of each candidate with the standard is evaluated using both existing and proposed SMs. The results are tabulated in Table 7.

Table 7 displays the values of the SMs for the employees compared to the standard, using both the existing and proposed SMs based on their attributes. The ranking of the medical equipment based on these values is provided in Table 8.

It is cleared from Table 8, the candidate ϑ_1 is obtained by using the SMs $\varrho_1, \varrho_2, \varrho_4, \varrho_5, \varrho_6, \varrho_7, \varrho_9, \varrho_{10}$ the candidate ϑ_1 is obtained by using the SMs ϱ_α. Some SM not given the answer. The ranking is also geometrically represented in Figure 2.

5 Conclusions

In this study, new SMs are defined for SFS to evaluate the similarity between two SFVs. The newly defined SM for SFS generalizes the existing SMs by introducing parameters. The mathematical work and subsequent discussion demonstrate the viability and adaptability of the proposed SM. The limitations of the existing SMs for SFS have also been discussed. The following steps are discussed:

The proposed SM satisfies the axiom (S2), which ensures that the proposed SM avoids counterintuitive situations where $\vartheta = \Phi$ implies $\varrho(\vartheta, \Phi) = 1$, a condition not met by some existing SMs.

The proposed SMs are based on the parameters α_1, α_2 and α_3, giving decision-makers the flexibility to choose the values of these parameters independently. In this scenario, decision-makers can select appropriate values for the parameters $\alpha_1, \alpha_2, \alpha_3$ to obtain a sensible SM that aligns with the current leadership style and decision-making environment.

The proposed SM is capable of providing reliable and sensible decision-making results. It not only has a high level of credibility but can also address dynamic problems that current SMs cannot resolve, yielding reasonable decision outcomes. Therefore, the proposed SM is both practical and adaptable.

Author Contributions

Mehwish Sarfaraz: Conceptualization, writing, review, and editing. Dragan Pamucar: Validation, Supervision.

Data Availability

Not Available.

Conflicts of Interest

The authors declare that none of the work reported in this paper could have been influenced by any known competing financial interests or personal relationships.

References

[1] J. A. Goguen, "L. A. Zadeh. Fuzzy sets. Information and control, vol. 8 (1965), pp. 338–353. - L. A. Zadeh. Similarity relations and fuzzy orderings. information sciences, vol. 3 (1971), pp. 177–200." *J. Symb. Log.*, vol. 38, no. 4, pp. 656–657, 1973.

[2] K. T. Atanassov, "Intuitionistic fuzzy sets," *Fuzzy Sets Syst.*, vol. 20, no. 1, pp. 87–96, 1986.

[3] K. T. Atanassov, "Interval valued intuitionistic fuzzy sets," pp. 139–177, 1999. https://doi.org/10.1007/978-3-7908-1870-3_2

[4] F. E. Boran and D. Akay, "A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition," *Inf. Sci.*, vol. 255, pp. 45–57, 2014.

[5] H. Nguyen, "A new knowledge-based measure for intuitionistic fuzzy sets and its application in multiple attribute group decision making," *Expert Syst. Appl.*, vol. 42, no. 22, pp. 8766–8774, 2015.

[6] J. Mahanta and S. Panda, "A novel distance measure for intuitionistic fuzzy sets with diverse applications," *Int. J. Intell. Syst.*, vol. 36, no. 2, pp. 615–627, 2021. https://doi.org/10.1002/int.22312

[7] R. R. Yager, "Pythagorean fuzzy subsets," in *2013 Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada*, 2013, pp. 57–61. https://doi.org/10.1109/IFSA-NAFIPS.2013.6608375

[8] R. R. Yager, "Generalized orthopair fuzzy sets," *IEEE Trans. Fuzzy Syst.*, vol. 25, no. 5, pp. 1222–1230, 2016. https://doi.org/10.1109/TFUZZ.2016.2604005

[9] B. C. Cuong, "Picture fuzzy sets," *JCC*, vol. 30, no. 4, pp. 409–420, 2015.

[10] B. C. Cuong and V. H. Pham, "Some fuzzy logic operators for picture fuzzy sets," in *2015 Seventh International Conference on Knowledge and Systems Engineering (KSE), Ho Chi Minh City, Vietnam*, 2015, pp. 132–137.

[11] T. Mahmood, K. Ullah, Q. Khan, and N. Jan, "An approach toward decision-making and medical diagnosis problems using the concept of spherical fuzzy sets," *Neural Comput. Appl.*, vol. 31, no. 11, pp. 7041–7053, 2019. https://doi.org/10.1007/s00521-018-3521-2

[12] M. R. Khan, K. Ullah, D. Pamucar, and M. Bari, "Performance measure using a multi-attribute decision-making approach based on complex t-spherical fuzzy power aggregation operators," *J. Comput. Cogn. Eng.*, vol. 1, no. 3, pp. 138–146, 2022. https://doi.org/10.47852/bonviewJCCE696205514

[13] T. Senapati, G. Chen, R. Mesiar, and R. R. Yager, "Novel Aczel–Alsina operations-based interval-valued intuitionistic fuzzy aggregation operators and their applications in multiple attribute decision-making process," *Int. J. Intell. Syst.*, vol. 37, pp. 5059–5081, 2022. https://doi.org/10.1002/int.22751

[14] C. Jana, T. Senapati, and M. Pal, "Pythagorean fuzzy Dombi aggregation operators and its applications in multiple attribute decision-making," *Int. J. Intell. Syst.*, vol. 34, no. 9, pp. 2019–2038, 2019. https://doi.org/10.1002/int.22125

[15] T. Senapati, "Approaches to multi-attribute decision-making based on picture fuzzy Aczel–Alsina average aggregation operators," *Comput. Appl. Math.*, vol. 41, no. 1, pp. 1–19, 2022. https://doi.org/10.3934/math.2023847

[16] T. Mahmood and Z. Ali, "Prioritized muirhead mean aggregation operators under the complex single-valued neutrosophic settings and their application in multi-attribute decision-making," *J. Comput. Cogn. Eng.*, vol. 1, no. 2, pp. 56–73, 2021. https://doi.org/10.47852/bonviewJCCE2022010104

[17] M. Riaz and H. M. A. Farid, "Picture fuzzy aggregation approach with application to third-party logistic provider selection process," *Rep. Mech. Eng.*, vol. 3, no. 1, pp. 318–327, 2022. http://doi.org/10.31181/rme20023062022r

[18] M. R. Khan, K. Ullah, and Q. Khan, “Multi-attribute decision-making using Archimedean aggregation operator in T-spherical fuzzy environment,” *Rep. Mech. Eng.*, vol. 4, no. 1, 2023. https://doi.org/10.31181/rme20031012023k

[19] M. Riaz, H. Garg, H. A. F. Muhammad, and R. Chinram, “Multi-criteria decision making based on bipolar picture fuzzy operators and new distance measures,” *Comput. Model. Eng. Sci.*, vol. 127, no. 2, pp. 771–800, 2021. https://doi.org/10.32604/cmes.2021.014174

[20] H. Garg, “Intuitionistic fuzzy hamacher aggregation operators with entropy weight and their applications to multi-criteria decision-making problems,” *Iran. J. Sci. Technol. Trans. Electr. Eng.*, vol. 43, no. 3, pp. 597–613, 2019. https://doi.org/10.1007/s40998-018-0167-0

[21] A. Ashraf, K. Ullah, A. Hussain, and M. Bari, “Interval-valued picture fuzzy maclaurin symmetric mean operator with application in multiple attribute decision-making,” *Rep. Mech. Eng.*, vol. 3, no. 1, pp. 301–317, 2022. http://doi.org/10.31181/rme20020042022a

[22] H. Garg, “Confidence levels based Pythagorean fuzzy aggregation operators and its application to decision-making process,” *Comput. Math. Organ. Theory*, vol. 23, no. 4, pp. 546–571, 2017. https://doi.org/10.1007/s10588-017-9242-8

[23] M. Riaz, H. M. Athar Farid, H. Kalsoom, D. Pamučar, and Y. J. Chu, “A robust q-Rung orthopair fuzzy Einstein prioritized aggregation operators with application towards MCGDM,” *Symmetry*, vol. 12, no. 6, p. 1058, 2020. https://doi.org/10.3390/sym12061058

[24] D. Pamučar, A. E. Torkayesh, and S. Biswas, “Supplier selection in healthcare supply chain management during the COVID-19 pandemic: A novel fuzzy rough decision-making approach,” *Ann. Oper. Res.*, pp. 1–43, 2022. https://doi.org/10.1007/s10479-022-04529-2

[25] W. S. Du and B. Q. Hu, “Aggregation distance measure and its induced similarity measure between intuitionistic fuzzy sets,” *Pattern Recognit. Lett.*, vol. 60, pp. 65–71, 2015. https://doi.org/10.1016/j.patrec.2015.03.001

[26] Y. Donyatalab, E. Farrokhizadeh, and S. A. A. Shishavan, “Similarity measures of q-rung orthopair fuzzy sets based on square root cosine similarity function,” pp. 475–483, 2020. https://doi.org/10.1007/978-3-030-51156-2_55

[27] W. R. W. Mohd and L. Abdullah, “Similarity measures of pythagorean fuzzy sets based on combination of cosine similarity measure and euclidean distance measure,” *AIP Conf. Proc.*, p. 030017, 2018. http://doi.org/10.1063/1.5041661

[28] G. Wei, “Some cosine similarity measures for picture fuzzy sets and their applications to strategic decision making,” *Informatica*, vol. 28, no. 3, pp. 547–564, 2017. https://doi.org/10.15388/Informatica.2017.144

[29] G. Wei and H. Gao, “The generalized dice similarity measures for picture fuzzy sets and their applications,” *Informatica*, vol. 29, no. 1, pp. 107–124, 2018. https://doi.org/10.15388/Informatica.2018.160

[30] N. Van Dinh, N. X. Thao, and N. Xuan, “Some measures of picture fuzzy sets and their application in multi-attribute decision making,” *Int. J. Math. Sci. Comput.*, vol. 4, pp. 23–41, 2018. https://doi.org/10.5815/ijmsc.2018.03.03

[31] P. Singh, N. K. Mishra, M. Kumar, S. Saxena, and V. Singh, “Risk analysis of flood disaster based on similarity measures in picture fuzzy environment,” *Afr. Mat.*, vol. 29, pp. 1019–1038, 2018. https://doi.org/10.1007/s13370-018-0597-x

[32] M. Luo and Y. Zhang, “A new similarity measure between picture fuzzy sets and its application,” *Eng. Appl. Artif. Intell.*, vol. 96, p. 103956, 2020. https://doi.org/10.1016/j.engappai.2020.103956

[33] M. Rafiq, S. Ashraf, S. Abdullah, T. Mahmood, and S. Muhammad, “The cosine similarity measures of spherical fuzzy sets and their applications in decision making,” *J. Intell. Fuzzy Syst.*, vol. 36, no. 6, pp. 6059–6073, 2019. https://doi.org/10.3233/JIFS-181922

[34] R. R. Zhao, M. X. Luo, S. G. Li, and L. N. Ma, “A parametric similarity measure between picture fuzzy sets and its applications in multi-attribute decision-making,” *Iran. J. Fuzzy Syst.*, vol. 20, no. 1, pp. 87–102, 2023. https://doi.org/10.22111/ijfs.2023.7348

[35] S. A. S. Shishavan, F. K. Gündoğdu, E. Farrokhizadeh, Y. Donyatalab, and C. Kahraman, “Novel similarity measures in spherical fuzzy environment and their applications,” *Eng. Appl. Artif. Intell.*, vol. 94, p. 103837, 2020. https://doi.org/10.1016/j.engappai.2020.103837

[36] M. J. Khan, P. Kumam, W. Deebani, W. Kumam, and Z. Shah, “Distance and similarity measures for spherical fuzzy sets and their applications in selecting mega projects,” *Mathematics*, vol. 8, no. 4, p. 519, 2020. https://doi.org/10.3390/math8040519

[37] T. Mahmood, M. Ilyas, Z. Ali, and A. Gumaei, “Spherical fuzzy sets-based cosine similarity and information measures for pattern recognition and medical diagnosis,” *IEEE Access*, vol. 9, pp. 25 835–25 842, 2021. https://doi.org/10.1109/ACCESS.2021.3056427

[38] X. Shen, S. Sakhi, K. Ullah, M. N. Abid, and Y. Jin, “Information measures based on T-spherical fuzzy

sets and their applications in decision making and pattern recognition," *Axioms*, vol. 11, no. 7, 2022. https://doi.org/10.3390/axioms11070302

[39] K. Ullah, T. Mahmood, and N. Jan, "Similarity measures for T-spherical fuzzy sets with applications in pattern recognition," *Symmetry*, vol. 10, no. 6, 2018. https://doi.org/10.3390/sym10060193