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Enhanced Industrial Control System of Decision-Making Using Spherical Hesitant Fuzzy Soft Yager Aggregation Information

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Abstract: In the realm of emergency response, where time and information constraints are paramount, and scenarios often involve high levels of toxicity and uncertainty, the effective management of industrial control systems (ICS) is critical. This study introduces novel methodologies for enhancing decision-making processes in emergency situations, specifically focusing on ICS-security. Central to this research is the employment of spherical hesitant fuzzy soft set (SHFSS), a concept that thrives in the presence of ambiguity and incomplete information. The research adopts and extends the parametric families of t-norms and t-conorms, as introduced by Yager, to analyze these sets. This approach is instrumental in addressing multi-attribute decision-making (MADM) problems within the ICS-security domain. To this end, four distinct aggregation operators (AOs) are proposed: spherical hesitant fuzzy soft yager weighted averaging aggregation, spherical hesitant fuzzy soft yager ordered weighted averaging aggregation, spherical hesitant fuzzy soft yager weighted geometric aggregation, and spherical hesitant fuzzy soft yager ordered weighted geometric aggregation. These operators are tailored to harness the operational benefits of Yager's parametric families, thereby offering a robust framework for dealing with decision-making problems under uncertainty. Further, an algorithm specifically designed for MADM is presented, which integrates these AOs. The efficacy and precision of the proposed methodology are demonstrated through a numerical example, applied in the context of an ICS security supplier. This example serves as a testament to the superiority of the approach in handling complex decision-making scenarios inherent in ICS-security management.

Keywords: Spherical hesitant fuzzy soft set (SHFSS), Multi-attribute decision making (MADM), Decision making problems (DMPs)

1 Introduction

Decision-making processes are universally recognized as pivotal in selecting optimal options under specific criteria. Daily, individuals are tasked with navigating through a plethora of choices to identify those that best align with their requirements. Recent research in emergency decision-making, especially under conditions of data uncertainty, has yielded significant insights. These advancements have been applied across various domains, including economics, management, and machine learning, as evidenced in the studies [1–3]. Historically, numerical methods were deemed adequate for presenting essential information regarding alternatives in a clear and concise manner. However, the acquisition of comprehensive data, essential for decision-makers, necessitates a focused approach on the ultimate objective. The handling of empirical data mandates acknowledgment of its inherent imprecision and uncertainty. It is well-established that decision-making is an effective method for enhancing the selection of the most suitable option amidst numerous decision-making challenges [4, 5]. Concentrating on the primary goal is imperative to gather the necessary and comprehensive information required by decision-makers. Acknowledging the fundamental imprecision and uncertainty of empirical data is crucial in this context.

The foundational concept of fuzzy set (FS) theory was initially introduced by Zadeh et al. [6], marking a significant advancement in decision-making (DM) processes, particularly in scenarios characterized by ambiguity. FS theory provides a robust framework for addressing situations involving inaccurate and uncertain information. Subsequently, numerous researchers [7–10] have contributed to the evolution of FS theory, developing extensions that enhance its applicability in DM under unclear conditions. One notable development in this field was the establishment of intuitionistic fuzzy sets (IFS) by Atanassov [11]. This was achieved by integrating the concept of non-membership grade (NMG) with traditional fuzzy sets, offering a more nuanced representation of uncertainty. IFS is characterized by both membership grade (MG) and NMG, with their sum not exceeding 1. This extension of FS theory has become a critical tool in addressing ambiguity in complex real-world DM challenges. The significant interest in IFS theory has spurred numerous researchers to explore its potential further. These explorations have involved the application of novel combinations of operators, information measures, score, and accuracy functions [12–14], across diverse contexts.

IFSs have proven effective in handling partial and ambiguous data, yet they encounter limitations when faced with inconsistent data in practical applications. For instance, in a voting system, human opinions often include diverse responses such as "vote for", "neutral voting", and "vote against". Addressing this gap, Cuong introduced the concept of neutral membership into IFS, leading to the development of picture fuzzy sets (PFS). Cuong and Kreinovich [15]'s seminal work on PFS, an extension of IFS, incorporated the notion of neutrality. In PFS theory, it is stipulated that the combined positive, neutral, and negative membership degrees should not exceed one. Further elaborating on this concept, Cuong and Pham [16] established the foundational logical operations for PFS. Recognizing the challenges posed by uncertainty in complex real-world DM scenarios, particularly within a picture fuzzy context, Ashraf et al. [17] identified limitations in existing operational laws and proposed advanced AOs. Subsequent developments by Ashraf et al. [18] included a cleaner production evaluation method employing PFS. Moreover, Ashraf et al. [19] introduced algebraic norm-based AOs within the realm of cubic image FS, demonstrating their utility in DM processes.

Ashraf et al. [20, 21] have significantly expanded the FS theory by introducing the concept of spherical fuzzy sets (SFS), an extension that further relaxes the traditional constraints of FS. This new formulation modifies the condition to $0 \leqq (\vartheta(a))^2 + (\xi(a))^2 + (\partial(a))^2 \leqq 1$, providing a more comprehensive framework for capturing membership. The introduction of SFS allows for a broader expression of support in decision-making scenarios, enhancing the ability to articulate ambiguous information more effectively. This attribute makes spherical FS particularly adept at managing uncertain information. The implementation of SFS has shown promising results, especially in fields that heavily rely on decision-making processes [22]. The significance of AOs in decision-making problems cannot be overstated, and there has been a concerted effort by researchers to develop AOs specifically for SFS. Notable developments include spherical AOs based on algebraic norms [23, 24]. Building on this concept, Khan et al. [25] further extended these principles to spherical hesitant fuzzy sets, broadening the applicability and utility of SFS in complex decision-making environments.

Despite the advances in FS theory and its extensions, challenges have emerged in accurately characterizing uncertainty in data within practical applications. Addressing this shortfall, Molodtsov [26] introduced the concept of soft sets (SSs). SSs were perceived as a significant improvement in dealing with uncertainty, devoid of the limitations that constrained earlier methodologies. The development of hybrid models, which amalgamate various concepts of uncertainty, has proven to be more effective than standalone models. This has been corroborated by numerous researchers [27, 28], who have extended traditional SSs to enhance their efficacy. One persistent challenge within FS theory is the difficulty in determining the optimal membership degree for assessment data. To offer a more versatile numerical representation of membership levels, Torra [29] developed hesitant fuzzy sets (HFS). HFS has emerged as a common method for managing ambiguity in DMPs. Building on this concept, the most widely recognized version of hesitant fuzzy soft sets (HFSS) was introduced by Babitha and John [30]. This development represents a significant stride in the field, effectively combining the strengths of both hesitant and soft set theories to better address the complexities of real-world data uncertainty.

Ashraf et al. [31] have made notable contributions by introducing the concept of SHFSS, elaborating on its properties, and defining its score and accuracy functions. These developments have highlighted the importance of AOs in the execution of decision-making processes, particularly those based on SFS. The focus has been on various norms under SFS to facilitate the basic operations that define the union, intersection, product, and sum of SFSs. In this context, Yager operations emerge as viable alternatives to traditional algebraic norms, offering smoother estimations. However, the application of Yager operations on fuzzy sets for aggregating fuzzy numbers remains underexplored in existing literature. Addressing this gap, Akram and Shahzadi [32] developed q-rung orthopair fuzzy Yager AOs. Additionally, Akram et al. [33] have extended this work to the realm of complex Pythagorean FSs. Another significant application is demonstrated by the study [34], which utilized norm data from Fermatean FS and Yager operations to optimize COVID-19 testing facilities. In the broader scope of decision-making methods, similar approaches like the VIKOR method [35] and the TOPSIS method [36] have been implemented, especially in determining standard distances for the EDAS method. Further related work can be found in the studies [37–42], indicating a growing interest and continued development in this area.

The review of existing studies reveals a diverse array of AOs that have been proposed and applied in various real-world scenarios. While these AOs are adept at addressing a multitude of real-world challenges, there are instances where they yield results that are either irrational or counterintuitive. Particularly in emergency decision-making processes (EmDMPs), these limitations of the extensions of fuzzy sets become more pronounced. Furthermore, the formulation of new rules, which may seem overly complex or lack clear applicability, adds to the challenges in this domain. Notably, the development of generalized AOs for SHFSS remains a crucial area of interest and research.

In response to these challenges, this study introduces innovative spherical hesitant fuzzy soft Yager operational rules-based AOs. These are specifically designed to effectively manage uncertainty in practical DM scenarios. Moreover, this paper presents a unique approach to multi-criteria decision making (MCDM) issues, utilizing weighted averaging, weighted geometric, and their ordered operators within the spherical hesitant fuzzy soft framework. The primary objective is to offer a novel strategy for MADM challenges, integrating the TODIM approach with SHFSS. To aid in the comprehension of this concept, Figure 1 provides a graphical representation of the geometrical layout of SHFSS.

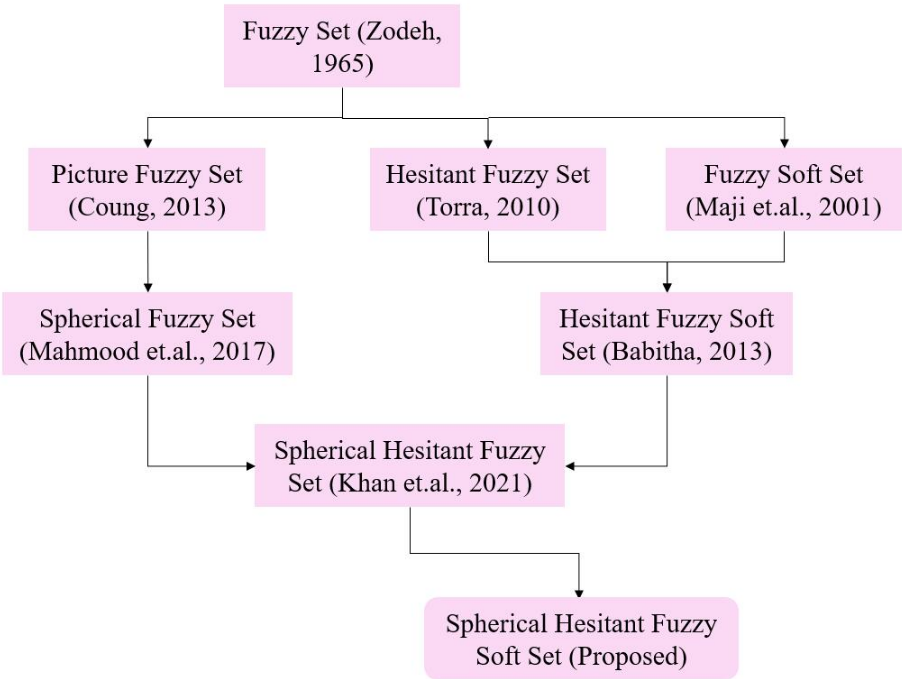

Figure 1. Geometric representation of SHFSS

This paper delineates its unique contributions as follows: Section 2 introduces the fundamental concepts of SHFSS. In Section 3, the core principles of SHFSS are detailed, including Yager operations within the SHFSS framework, alongside discussions on score and accuracy functions. The innovative spherical hesitant fuzzy soft Yager AOs are presented in Section 4. This section also includes a verification of their properties, offering a novel methodology for the accurate assessment of spherical hesitant fuzzy numbers. Section 5 introduces a unique approach for addressing ambiguity in DM processes, focusing on optimal selection based on a set of attributes. A case study related to the selection of an ICS security supplier is included to exemplify the MADM approach. Section 6 delves into the TODIM approach, demonstrating its application in tackling MADM challenges under the framework of spherical hesitant fuzzy soft Yager weighted averaging operators (SHFSYWAOs). Finally, Section 6 concludes the paper, highlighting the superior performance of the methodologies developed within this study. It also acknowledges the limitations of the current work and suggests potential directions for future research in this field.

1.1 List of Abbreviations

The basic abbreviation that are used in our manuscript is given as:

Abbrevations	Explanation
FSs	Fuzzy Sets
SSs	Soft Sets
FSSs	Fuzzy Soft Sets
HFSs	Hesitant Fuzzy Sets
HFSSs	Hesitant Fuzzy Soft Sets
PFSs	Picture Fuzzy Sets
PHFSs	Picture Hesitant Fuzzy Sets
PHFSSs	Picture Hesitant Fuzzy Soft Sets
SFSs	Spherical Fuzzy Sets
SHFSs	Spherical Hesitant Fuzzy Sets
SHFSS	Spherical Hesitant Fuzzy Soft Sets
MG	Membership Grade
NMG	Non-Membership Grade
nMG	neutral Membership Grade
MAGDM	Multiple Attribute Group Decision-Making
DMPs	Decision-Making Problems
EDM	Emergency Decision-Making
AO's	Aggregation Operators
ICS	Industrial Control System
SHFSYWA	Spherical hesitant fuzzy soft yager weighted average
SHFSYOWA	Spherical hesitant fuzzy soft yager ordered weighted average
SHFSYWGA	Spherical hesitant fuzzy soft yager weighted geometric aggregation
SHFSYOWG	Spherical hesitant fuzzy soft yager ordered weighted geometric aggregation

2 Preliminaries

Definition 2.1. [6] Over the universal set R, in which $\vartheta(a) : R \to [0, 1]$ indicate membership of fuzzy set (FS) defined as:

$$\widetilde{G} = \{(z, \vartheta(z)) : z \in R\}$$

Definition 2.2. [27] If parametric set $M \subseteq T$. A pair (S, M) on R, is known as fuzzy soft set (FSS), with the mapping $S : M \to$Æ, denoted as:

$$S_{\rho j}(z_i) = \{(z_i, \vartheta_j(z_i)) : z_i \in R\}$$

where, Æ is the set of fuzzy soft sets on R.

Definition 2.3. [30] Let $\hat{h}(z)$ is the set of hesitant fuzzy sets; with the mapping $\hat{H}$. So, a pair $(\hat{H}, M)$ is refer to as hesitant fuzzy soft set $(HFSS)$ expressed as:

$$\hat{H} : M \to \hat{h}(z)$$

Definition 2.4. [15] A PFS over R is defined as:

$$\widetilde{K} = \{(z, (\vartheta(z), \xi(z), \partial(z))) : z \in R\}$$

where, $\vartheta(z) : R \to [0, 1]$ is the MG, $\xi(z) : R \to [0, 1]$ is the nMG and $\partial(z) : R \to [0, 1]$ is NMG in the condition; $0 \leq \vartheta(z), \xi(z), \partial(z) \leq 1$.

Definition 2.5. [29] A SFS over R, is denoted as:

$$\c{S} = \{(z, (\vartheta(z), \xi(z), \partial(z))) : z \in R\}$$

where, $\vartheta(z) : R \to [0, 1]$ is the MG, $\xi(z) : R \to [0, 1]$ is the nMG and $\partial(r) : R \to [0, 1]$ is NMG with $0 \leq (\vartheta(z))^2 + (\xi(z))^2 + (\partial(z))^2 \leq 1$.

Definition 2.6. [25] A $\check{R}$ is SHFS over R, when implemented to R yields a subsets of $[0, 1]$, such as:

$$\check{R} = \{(z, (\vartheta(z), \xi(z), \partial(z))) | z \in R\}$$

where, $\vartheta(z) = \{\phi | \phi \in \vartheta(z)\}, \xi(z) = \{\chi | \chi \in \xi(z)\}, \partial(z) = \{\psi | \psi \in \partial(z)\}$ be a set of finite elements in $[0, 1]$, i.e., MG, nMG, NMG of the elements, with the condition $0 \leq (\phi^+)^2 + (\chi^+)^2 + (\psi^+)^2 \leq 1$, where $\phi^+ = \cup_{\phi \in \vartheta(Z)} \max\{\phi\}, \chi^+ = \cup_{\chi \in \xi(Z)} \max\{\chi\}$ and $\psi^+ = \cup_{\psi \in \partial(Z)} \max\{\psi\}$. And $\check{Z} = \{\vartheta, \xi, \partial\}$ represents the SHFE.

3 Operating Laws under Spherical Hesitant Fuzzy Soft Yager Environment

In this section, we define some of the, Yager weighted arithmetic aggregation operators under SHFS environment.

Definition 3.1. [31] Over the universal set R a duo (S, T) is refereed to as SHFSS, with $M \subseteq T, m \in M$ and the mapping S is; $S : M \to \beth$ defined by:

$$S_{\rho ij}(m) = \{(z_i, (\vartheta_j(z_i), \xi_j(z_i), \partial_j(z_i)) : z_i \in R\}$$

where, $\beth$ on R is the collection of SHFSS. With $\vartheta_j(Z_i) = \{\phi | \phi \in \vartheta_j(z_i)\}, \xi_j(z_i) = \{\chi | \chi \in \xi_j(z_i)\}, \partial_j(z_i) = \{\psi | \psi \in \partial_j(z_i)\}$ is a collection of three hesitant sets of several values in $[0, 1]$, and $0 \leq (\vartheta_j(z_i))^2 + (\xi_j(z_i))^2 + (\partial_j(z_i))^2 \leq 1$ is the MG, nMG and NMG, over $0 \leq (\phi^+)^2 + (\chi^+)^2 + (\psi^+)^2 \leq 1$, where $(\phi^+) = \cup_{\phi_y \in \vartheta_j(z_i)} \max\{\phi_y\}$, $(\chi^+) = \cup_{\chi_y \in \xi_j(z_i)} \max\{\chi_y\}$ and $(\psi^+) = \cup_{\psi_y \in \partial_j(z_i)} \max\{\psi_y\}$. And $S_{\rho ij} = \{\vartheta_j(z_i), \xi_j(z_i), \partial_j(z_i)\}$ represents the SHFSE.

Definition 3.2. Let $S_{\rho rs} = \{\vartheta_s(z_r), \xi_s(z_r), \partial_s(z_r)\}, S_{\rho rt} = \{\vartheta_t(z_r), \xi_t(z_r), \partial_t(z_r)\}$ be two SHFSNs and $\kappa > 0, \digamma > 0$. Then, Yager t-norm and t-conorm operations of SHFSSNs are defined by:

$$(1)\ \kappa S_{\rho rs} = \bigcup_{\phi_s \in \vartheta_s, \chi_s \in \xi_s, \psi_s \in \partial_s} \left\{ \begin{array}{c} \sqrt{\min(1, (\kappa(\phi_s)^{2\digamma})^{1/\digamma})}, \sqrt{1 - \min(1, (\kappa(1 - \chi_s^2)^{\digamma})^{1/\digamma})}, \\ \sqrt{1 - \min(1, (\kappa(1 - \psi_s^2)^{\digamma})^{1/\digamma})} \end{array} \right\}.$$

$$(2)\ S_{\rho rs}^{\kappa} = \bigcup_{\phi_s \in \vartheta_s, \chi_s \in \xi_s, \psi_s \in \partial_s} \left\{ \begin{array}{c} \sqrt{1 - \min(1, (\kappa(1 - \phi_s^2)^{\digamma})^{1/\digamma})}, \sqrt{1 - \min(1, (\kappa(1 - (\chi_s^2)^{\digamma})^{1/\digamma}}, \\ \sqrt{\min(1, (\kappa(\psi_s)^{2\digamma})^{1/\digamma})} \end{array} \right\}.$$

$$(3)\ S_{\rho rs} \oplus S_{\rho rt} = \bigcup_{\phi_s \in \vartheta_s, \chi_s \in \xi_s, \psi_s \in \partial_s, \phi_t \in \vartheta_t, \chi_t \in \xi_t, \psi_t \in \partial_t} \left\{ \begin{array}{c} \sqrt{(\min(1, ((\phi_s)^{2\digamma} + (\phi_t)^{2\digamma})^{1/\digamma})}, \\ \sqrt{1 - \min(1, ((1 - \chi_s^2)^{\digamma} + (1 - \chi_t^2)^{\digamma})^{1/\digamma})}, \\ \sqrt{1 - \min(1, ((1 - \psi_s^2)^{\digamma} + (1 - \psi_t^2)^{\digamma})^{1/\digamma})} \end{array} \right\}.$$

$$(4)\ S_{\rho rs} \otimes S_{\rho rt} = \bigcup_{\phi_s \in \vartheta_s, \chi_s \in \xi_s, \psi_s \in \partial_s, \phi_t \in \vartheta_t, \chi_t \in \xi_t, \psi_t \in \partial_t} \left\{ \begin{array}{c} \sqrt{1 - \min(1, ((1 - \phi_s^2)^{\digamma} + (1 - \phi_t^2)^{\digamma})^{1/\digamma})}, \\ \sqrt{1 - \min(1, ((1 - \chi_s^2)^{\digamma} + (1 - \chi_t^2)^{\digamma})^{1/\digamma})}, \\ \sqrt{\min(1, ((\psi_s)^{2\digamma} + (\psi_t)^{2\digamma})^{1/\digamma})}. \end{array} \right\}.$$

Definition 3.3. [31] Let $\jmath = \{\vartheta_j(z_i), \xi_j(z_i), \partial_j(z_i)\}$ be a SHFSE, such as $\vartheta_j(z_i) = \{\phi | \phi \in \vartheta_j(z_i)\}, \xi_j(z_i) = \{\chi | \chi \in \xi_j(z_i)\}$, $\partial_j(z_i) = \{\psi | \psi \in \partial_j(z_i)\}$ and the numbers of elements in $\vartheta_j, \xi_j, \partial_j$ are m, n, o respectively. Thus, the score function is defined as:

$$sc(\jmath) = \frac{\left\{1 + \frac{1}{m} \sum_{j=1}^{m} \phi_j - \frac{1}{n} \sum_{j=1}^{n} \chi_j - \frac{1}{o} \sum_{j=1}^{o} \psi_j \right\}}{2}, sc(\jmath) \in [0, 1].$$

and the accuracy function is:

$$ac(\jmath) = \left\{ \frac{1}{m} \sum_{j=1}^{m} \phi_j - \frac{1}{o} \sum_{j=1}^{o} \psi_j \right\}, ac(\jmath) \in [0, 1].$$

4 Aggregation Operators Based on Yager's Norms

Yager operators hold a pivotal role in the realm of SFSS, primarily due to their capacity to effectively aggregate measures of hesitancy. These operators are instrumental in facilitating decision-making in contexts laden with uncertainty. They also play a crucial role in representing complex systems and ensuring compatibility with established fuzzy set operations. In this section, we introduce a series of AOs for SHFSSs, grounded in the Yager t-norm and t-conorm.

4.1 Yager Weighted Averaging Aggregation Operators

Definition 4.1. Suppose $\Gamma(j_r) = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)}), (r = 1 \to p)$ represent a collection of SHFSNs (Γ, J) having weights $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^\intercal$ for $\Gamma(j_r)$ parameters (attributes). Here $\acute{w}_r \in [0, 1]$ with $\Sigma_{r=1}^{p} \acute{w}_r = 1$ and $\acute{w}_r \geq 0$.

The SHFSYWA operator is a mapping denoted as SHFSWA: $\beth^p \to \beth$, where ($\beth$ be the class of all SHFSNs) such that SHFSYWA $\Gamma(j_r) = (\phi_{\Gamma(j_r)}, \chi_{\Gamma(j_r)}, \psi_{\Gamma(j_r)}) \forall (r = 1 \to p)$.

$$SHFSYWA(\Gamma(j_1), \Gamma(j_2), ..., \Gamma(j_p)) = \oplus_{r=1}^{p} \acute{w}_r \Gamma(j_r).$$

Theorem 4.1. Let $\Gamma(j_r) = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)}), (r = 1 \to p)$, being a SHFSNs, then the aggregated data provided by the SHFSYWA operator is also considered as SHFSNs, and is expressed as:

$$SHFSWA(\Gamma(j_1), \Gamma(j_2), \Gamma(j_3), ..., \Gamma(j_p)) = \oplus_{r=1}^{p} \acute{w}_r \Gamma(j_r)$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\ldots,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\ldots,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)},\ldots,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.}} \left\{ \begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \end{array} \right\}.$$

where, $r = 1, \rightarrow, p$, if $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^\intercal$ denote the WV of $\acute{w}_r$ parameters with condition $\acute{w}_r \in [0,1]$ with $\Sigma_{r=1}^{p}\acute{w}_r = 1$ and $\acute{w}_r \geq 0$.

Proof. The use of mathematical induction on p is required to back up this statement. For $p = 2$, we get:

$$SHFSWA(\Gamma(j_1),\Gamma(j_2)) = \oplus_{k=1}^{2}\acute{w}_r\Gamma(j_r) = \acute{w}_1\Gamma(j_1) \oplus \acute{w}_2\Gamma(j_2).$$

By using the operational law, we have:

$$\acute{w}_1\Gamma(j_1) = \bigcup_{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)}.} \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \end{array} \right\}$$

$$\acute{w}_2\Gamma(j_2) = \bigcup_{\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.} \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})}. \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1) \oplus \acute{w}_2\Gamma(j_2) = \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{ \begin{array}{c} \left(\begin{array}{c} \sqrt{\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \end{array} \right) \oplus \\ \left(\begin{array}{c} \sqrt{\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})}. \end{array} \right) \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1) \oplus \acute{w}_2\Gamma(j_2)$$
$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{ \begin{array}{c} \sqrt{\begin{array}{c} \min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F}) + \min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F}) - \\ \min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F}).\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F}) \end{array}}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}. \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})} \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1) \oplus \acute{w}_2\Gamma(j_2)$$
$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{ \begin{array}{c} \sqrt{1-\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F}).\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}. \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})}. \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1)\oplus\acute{w}_2\Gamma(j_2)=\bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}}\left\{\begin{array}{c}\sqrt{\min(1,(\Pi_{r=1}^{2}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{2}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{2}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})}\end{array}\right\}$$

This means that the conclusions hold for $p=2$. Assuming that the outcomes are valid for p= $z+1$, we get the following form by combining the above two assumptions.

$$SHFSWA(\Gamma(j_1),\Gamma(j_2),...,\Gamma(j_z),\Gamma(j_{z+1}))=\oplus_{r=1}^{z}\acute{w}_r\Gamma(j_r)+\acute{w}_{z+1}\Gamma(j_{z+1})$$

$$=\bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_z)}\in\vartheta_{\Gamma(j_z)},\phi_{\Gamma(j_{z+1})}\in\vartheta_{\Gamma(j_{z+1})},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},,...,\chi_{\Gamma(j_z)}\in\xi_{\Gamma(j_z)},\chi_{\Gamma(j_{z+1})}\in\xi_{\Gamma(j_{z+1})},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_z)}\in\partial_{\Gamma(j_z)},\psi_{\Gamma(j_{z+1})}\in\partial_{\Gamma(j_{z+1})}.}}\left\{\begin{array}{c}\left\{\begin{array}{c}\sqrt{\min(1,(\Pi_{r=1}^{z}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})}\end{array}\right\}\oplus\\ \sqrt{\min(1,(\acute{w}_{z+1}(\phi_{\Gamma(j_{z+1})})^{2F})^{1/F})},\\ \sqrt{1-\min(1,(\acute{w}_{z+1}(1-(\chi_{\Gamma(j_{z+1})})^2)^F)^{1/F})},\\ \sqrt{1-\min(1,(\acute{w}_{z+1}(1-(\psi_{\Gamma(j_{z+1})})^2)^F)^{1/F})}\end{array}\right\}$$

$$=\bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_z)}\in\vartheta_{\Gamma(j_z)},\phi_{\Gamma(j_{z+1})}\in\vartheta_{\Gamma(j_{z+1})},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},...,\chi_{\Gamma(j_z)}\in\xi_{\Gamma(j_z)},\chi_{\Gamma(j_{z+1})}\in\xi_{\Gamma(j_{z+1})},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_z)}\in\partial_{\Gamma(j_z)},\psi_{\Gamma(j_{z+1})}\in\partial_{\Gamma(j_{z+1})}.}}\left\{\begin{array}{c}\sqrt{\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})}\end{array}\right\}$$

The equation above makes it clear that the aggregated values are also SHFSNs. For this reason, the result holds true for any r.

$$SHFSWA(\Gamma(j_1),...,\Gamma(j_p))=\bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},...,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.}}\left\{\begin{array}{c}\sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})},\\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})}\end{array}\right\}$$

which presents the proof.

4.2 Yager Ordered Weighted Averaging Operator

The YOWA operator is of significant importance in the context of SHFSSs. It provides decision-makers with a mechanism to aggregate hesitant information, taking into account their preferences and the relative importance of each element. The flexibility and robustness of YOWA, coupled with its compatibility with existing fuzzy aggregation methods, render it an indispensable tool in the management of uncertainty and vagueness in decision-making scenarios. The integration of YOWA enhances the applicability and effectiveness of SHFSSs across a broad spectrum of domains.

Definition 4.2. Suppose $\Gamma(j_r)=(\vartheta_{\Gamma(j_r)},\xi_{\Gamma(j_r)},\partial_{\Gamma(j_r)}),(r=1\to p)$ be collection of SHFSNs (Γ,J) with $\acute{w}=(\acute{w}_1,\acute{w}_2,\acute{w}_3,..,\acute{w}_p)^{\intercal}$ WV for $\Gamma(j_r)$ parameters, where $\acute{w}_r\in[0,1]$ with $\Sigma_{r=1}^{p}\acute{w}_r=1$ and $\acute{w}_r\geq 0$. SHFSOWA operator is the mapping defined as $SHFSOWA:\daleth^p\to\daleth$, where ($\daleth$ be the class of all SHFSNs) such that SHFSOWA $\Gamma(j_{\sigma r})=(\vartheta_{\Gamma(j_{\sigma r})},\xi_{\Gamma(j_{\sigma r})},\partial_{\Gamma(j_{\sigma r})}),(r=1\to p)$, where $(\sigma_1,\sigma_2,...,\sigma_p)$ is the variation of $m_{\sigma(r-1)}\geq m_{\sigma(r)}\forall$ $(r=2\to p)$.

$$SHFSOWA(\Gamma(j_{\sigma 1}),\Gamma(j_{\sigma 2}),\Gamma(j_{\sigma 3}),....,\Gamma(j_{\sigma p}))=\oplus_{r=1}^{p}\acute{w}_r\Gamma(j_{\sigma r}).$$

Theorem 4.2. Let $\Gamma(j_{\sigma r})=(\vartheta_{\Gamma(j_{\sigma r})},\xi_{\Gamma(j_{\sigma r})},\partial_{\Gamma(j_{\sigma r})}),(r=1\to p)$, being a SHFSNs, then the aggregated data provided by the SHFSYWA operator is also considered as SHFSNs, and is expressed as:

$$SHFSOYWA(\Gamma(j_{\sigma 1}),\Gamma(j_{\sigma 2}),\Gamma(j_{\sigma 3}),....,\Gamma(j_{\sigma p}))=\oplus_{r=1}^{p}\acute{w}_r\Gamma(j_{\sigma r})$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_{\sigma 1})}\in\vartheta_{\Gamma(j_{\sigma 1})},\phi_{\Gamma(j_{\sigma 2})}\in\vartheta_{\Gamma(j_{\sigma 2})},\ldots,\phi_{\Gamma(j_{\sigma p})}\in\vartheta_{\Gamma(j_{\sigma p})},\\ \chi_{\Gamma(j_{\sigma 1})}\in\xi_{\Gamma(j_{\sigma 1})},\chi_{\Gamma(j_{\sigma 2})}\in\xi_{\Gamma(j_{\sigma 2})},\ldots,\chi_{\Gamma(j_{\sigma p})}\in\xi_{\Gamma(j_{\sigma p})},\\ \psi_{\Gamma(j_{\sigma 1})}\in\partial_{\Gamma(j_{\sigma 1})},\psi_{\Gamma(j_{\sigma 2})}\in\partial_{\Gamma(j_{\sigma 2})},\ldots,\psi_{\Gamma(j_{\sigma p})}\in\partial_{\Gamma(j_{\sigma p})}.}} \left\{ \begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\phi_{\Gamma(j_{\sigma r})})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_{\sigma r})})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\psi_{\Gamma(j_{\sigma r})})^2)^F)^{1/F})} \end{array} \right\}.$$

where, $r = 1, \rightarrow, p$, if $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^{\intercal}$ denote the weight vector (WV) of $\acute{w}_r$ parameters with condition $\acute{w}_r \in [0, 1]$ with $\Sigma_{r=1}^{p}\acute{w}_r = 1$ and $\acute{w}_r \geq 0$. Where $(\sigma_1, \sigma_2, ..., \sigma_p)$ is the permutation of $m_{\sigma(r-1)} \geq m_{\sigma(r)} \forall r = 2 \rightarrow p$.

Proof. The proof follows the preceding Theorem 4.1.

4.3 Yager Weighted Geometric Aggregation Operator

Definition 4.3. Suppose $\Gamma(j_r) = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)}), (r = 1 \rightarrow p)$ be collection of SHFSNs (Γ, J) having $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^{\intercal}$ is WV of $\Gamma(j_r)$ parameters, where $\acute{w}_r \in [0, 1]$ with $\Sigma_{r=1}^{p}\acute{w}_r = 1$ and $\acute{w}_r \geq 0$, then SHFSWGA operator is the mapping defined as $SHFSWGA : \daleth^p \rightarrow \daleth$, where ($\daleth$ is the family of all SHFSNs) such that SHFSWGA $\Gamma(j_r) = (\phi_{\Gamma(j_r)}, \chi_{\Gamma(j_r)}, \psi_{\Gamma(j_r)}), (r = 1 \rightarrow p)$.

$$SHFSWGA(\Gamma(j_1), \Gamma(j_2), \Gamma(j_3), ..., \Gamma(j_p)) = \prod_{r=1}^{p}(\Gamma(j_r))^{\acute{w}_r}.$$

Theorem 4.3. Let $\Gamma(j_r) = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)}), (r = 1 \rightarrow p)$, be an SHFSNs, the aggregated data by SHFSWGA operator is also an SHFSNs, and given by:

$$SHFSWGA(\Gamma(j_1), \Gamma(j_2), \Gamma(j_3), ..., \Gamma(j_p)) = \prod_{r=1}^{p}(\Gamma(j_r))^{\acute{w}_r}$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\ldots,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\ldots,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)},\ldots,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.}} \left\{ \begin{array}{c} \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\phi_{\Gamma(j_r)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})}, \\ \sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\psi_{\Gamma(j_r)})^{2F})^{1/F})} \end{array} \right\}.$$

where, $r = 1 \rightarrow p$, if $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^{\intercal}$ denote the WV of $\acute{w}_r$ parameters with condition $\acute{w}_r \in [0, 1]$ with $\Sigma_{r=1}^{p}\acute{w}_r = 1$ and $\acute{w}_r \geq 0$.

Proof. The use of mathematical induction on p is required to back up this statement. For $p = 2$, we get:

$$SHFSWGA(\Gamma(j_1), \Gamma(j_2)) = \oplus_{k=1}^{2}(\Gamma(j_r))^{\acute{w}_r} = \Gamma(j_1)^{\acute{w}_1} \oplus \Gamma(j_2)^{\acute{w}_2}.$$

By using the operational law, we have:

$$\acute{w}_1\Gamma(j_1) = \bigcup_{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)}.} \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \end{array} \right\}$$

$$\acute{w}_2\Gamma(j_2) = \bigcup_{\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.} \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})}. \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1) \oplus \acute{w}_2\Gamma(j_2) = \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{ \begin{array}{c} \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F})}. \end{array} \right\} \oplus \\ \left\{ \begin{array}{c} \sqrt{\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F})}, \\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})}. \end{array} \right\} \end{array} \right\}$$

$$\acute{w}_1\Gamma(j_1)\oplus\acute{w}_2\Gamma(j_2)$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{\begin{array}{c} \sqrt{\begin{array}{c}\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F})+\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})-\\ \min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F}).\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F})\end{array}},\\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F}}.\\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F}},\\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F}}.\\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F})} \end{array}\right\}$$

$$\acute{w}_1\Gamma(j_1)\oplus\acute{w}_2\Gamma(j_2)$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{\begin{array}{c} \sqrt{1-\min(1,(\acute{w}_1(\phi_{\Gamma(j_1)})^{2F})^{1/F}).\min(1,(\acute{w}_2(\phi_{\Gamma(j_2)})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\acute{w}_1(1-(\chi_{\Gamma(j_1)})^2)^F)^{1/F}}.\\ \sqrt{1-\min(1,(\acute{w}_2(1-(\chi_{\Gamma(j_2)})^2)^F)^{1/F}},\\ \sqrt{1-\min(1,(\acute{w}_1(1-(\psi_{\Gamma(j_1)})^2)^F)^{1/F}}.\\ \sqrt{1-\min(1,(\acute{w}_2(1-(\psi_{\Gamma(j_2)})^2)^F)^{1/F}}. \end{array}\right.$$

$$\acute{w}_1\Gamma(j_1)\oplus\acute{w}_2\Gamma(j_2) = \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)}.}} \left\{\begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{2}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{2}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{2}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \end{array}\right\}$$

This means that the conclusions hold for $p = 2$. Assuming that the outcomes are valid for p= $z + 1$, we get the following form by combining the above two assumptions.

$$SHFSWA(\Gamma(j_1),\Gamma(j_2),...,\Gamma(j_z),\Gamma(j_{z+1})) = \oplus_{r=1}^{z}\acute{w}_r\Gamma(j_r)+\acute{w}_{z+1}\Gamma(j_{z+1})$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_z)}\in\vartheta_{\Gamma(j_z)},\phi_{\Gamma(j_{z+1})}\in\vartheta_{\Gamma(j_{z+1})},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},...,\chi_{\Gamma(j_z)}\in\xi_{\Gamma(j_z)},\chi_{\Gamma(j_{z+1})}\in\xi_{\Gamma(j_{z+1})},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_z)}\in\partial_{\Gamma(j_z)},\psi_{\Gamma(j_{z+1})}\in\partial_{\Gamma(j_{z+1})}.}} \left\{\begin{array}{c} \left\{\begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{z}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \end{array}\right\}\oplus\\ \sqrt{\min(1,(\acute{w}_{z+1}(\phi_{\Gamma(j_{z+1})})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\acute{w}_{z+1}(1-(\chi_{\Gamma(j_{z+1})})^2)^F)^{1/F}}),\\ \sqrt{1-\min(1,(\acute{w}_{z+1}(1-(\psi_{\Gamma(j_{z+1})})^2)^F)^{1/F})} \end{array}\right\}$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_z)}\in\vartheta_{\Gamma(j_z)},\phi_{\Gamma(j_{z+1})}\in\vartheta_{\Gamma(j_{z+1})},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},...,\chi_{\Gamma(j_z)}\in\xi_{\Gamma(j_z)},\chi_{\Gamma(j_{z+1})}\in\xi_{\Gamma(j_{z+1})},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_z)}\in\partial_{\Gamma(j_z)},\psi_{\Gamma(j_{z+1})}\in\partial_{\Gamma(j_{z+1})}.}} \left\{\begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{z+1}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \end{array}\right\}$$

The equation above makes it clear that the aggregated values are also SHFSNs. For this reason, the result holds true for any r.

$$SHFSWA(\Gamma(j_1),...,\Gamma(j_p)) = \bigcup_{\substack{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},...,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},\\ \chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},...,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},\\ \psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},...,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.}} \left\{\begin{array}{c} \sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F}}),\\ \sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \end{array}\right\}.$$

which presents the proof.

4.4 Yager Ordered Weighted Geometric Aggregation Operator

Definition 4.4. Suppose $\Gamma(j_r) = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)}), (r = 1 \to p)$ be collection of SHFSNs (Γ, J) with WV $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{r}_3, .., \acute{w}_p)^\intercal$ of $\Gamma(j_r)$ parameters, where $\acute{w}_r \in [0,1]$ with $\Sigma_{r=1}^{p} \acute{w}_r = 1$ and $\acute{r}_r \geq 0$. SHFSOWGA operator is the mapping defined as $SHFSOWGA : \daleth^p \to \daleth$, where ($\daleth$ be the class of SHFSNs) such that SHFSYOWGA $\Gamma(j_{\sigma r}) = (\vartheta_{\Gamma(j_{\sigma r})}, \xi_{\Gamma(j_{\sigma r})}, \partial_{\Gamma(j_{\sigma r})}), (r = 1 \to p)$, where $(\sigma_1, \sigma_2, ..., \sigma_p)$ is the variation such that $m_{\sigma(r-1)} \geq m_{\sigma(r)} \forall\, r = 2 \to p$.

$$SHFSYOWGA(\Gamma(j_{\sigma 1}), \Gamma(j_{\sigma 2}), \Gamma(j_{\sigma 3}), ..., \Gamma(j_{\sigma p})) = \prod_{r=1}^{p} (\Gamma(j_{\sigma r}))^{\acute{w}_r}.$$

Theorem 4.4. Let $\Gamma(j_{\sigma r}) = (\vartheta_{\Gamma(j_{\sigma r})}, \xi_{\Gamma(j_{\sigma r})}, \partial_{\Gamma(j_{\sigma r})}), (r = 1 \to p)$, be an SHFSNs, the aggregated data of SHFSOWGA operator is also an SHFSNs, presented as:

$$SHFSYOWGA(\Gamma(j_{\sigma 1}), \Gamma(j_{\sigma 2}), \Gamma(j_{\sigma 3}), ..., \Gamma(j_{\sigma p})) = \prod_{r=1}^{p} (\Gamma(j_{\sigma r}))^{\acute{w}_r}$$

$$= \bigcup_{\substack{\phi_{\Gamma(j_{\sigma 1})} \in \vartheta_{\Gamma(j_{\sigma 1})}, \phi_{\Gamma(j_{\sigma 2})} \in \vartheta_{\Gamma(j_{\sigma 2})}, ..., \phi_{\Gamma(j_{\sigma p})} \in \vartheta_{\Gamma(j_{\sigma p})}, \\ \chi_{\Gamma(j_{\sigma 1})} \in \xi_{\Gamma(j_{\sigma 1})}, \chi_{\Gamma(j_{\sigma 2})} \in \xi_{\Gamma(j_{\sigma 2})}, ..., \chi_{\Gamma(j_{\sigma p})} \in \xi_{\Gamma(j_{\sigma p})}, \\ \psi_{\Gamma(j_{\sigma 1})} \in \partial_{\Gamma(j_{\sigma 1})}, \psi_{\Gamma(j_{\sigma 2})} \in \partial_{\Gamma(j_{\sigma 2})}, ..., \psi_{\Gamma(j_{\sigma p})} \in \partial_{\Gamma(j_{\sigma p})}.}} \left\{ \begin{array}{c} \sqrt{1 - \min(1, (\Pi_{r=1}^{p} \acute{w}_r (1 - (\phi_{\Gamma(j_{\sigma r})})^2)^F)^{1/F})}, \\ \sqrt{1 - \min(1, (\Pi_{r=1}^{p} \acute{w}_r (1 - (\chi_{\Gamma(j_{\sigma r})})^2)^F)^{1/F})}, \\ \sqrt{\min(1, (\Pi_{r=1}^{p} \acute{w}_r (\psi_{\Gamma(j_{\sigma r})})^{2F})^{1/F})} \end{array} \right\}.$$

where, $r = 1 \to p$, if $\acute{w} = \{\acute{w}_1, \acute{w}_2, \ldots, \acute{w}_p\}$ denote the weight vector (WV) of $\acute{w}_r$ parameters with condition $\acute{w}_r \in [0,1]$ with $\Sigma_{r=1}^{p} \acute{w}_r = 1$ and $\acute{w}_r \geq 0$, where $(\sigma_1, \sigma_2, ..., \sigma_p)$ is the permutation $(1 \to p)$ such that $m_{\sigma(r-1)} \geq m_{\sigma(r)}$ for all $r = 2 \to p$.

Proof. The proof follows the preceding Theorem 4.3.

5 Algorithm for MCDM Problems under Picture SHFSSs

In the realm of complex and uncertain real-life problems, decision-makers often face hesitancy in expressing their opinions, primarily due to limited knowledge or experience. SHFSSs have proven to be more adaptable in capturing the judgement processes in MADM scenarios. SHFSS enables decision-makers to select a balanced subset of attributes based on their intuition, leveraging a parameterized structure that encompasses their confidence levels as well as hesitancy. This paper delves into a MCDM technique that utilizes Yager Weighted Averaging (YWA), Yager Ordered Weighted Averaging (YOWA), Yager Weighted Geometric Averaging (YWGA), and Yager Ordered Weighted Geometric Averaging (YOWGA) AOs. This technique is specifically tailored for resolving MCDM problems in the context of SHFSS. The focus is on the application and analysis of these AOs in enhancing the decision-making process under the framework of SHFSS.

Consider an alternative set $¥ = \{y_1, y_2, y_3, ..., y_t\}$ and a set of attributes $\mp = \{z_1, z_2, z_3, ..., z_p\}$ with weight vectors $\acute{w} = (\acute{w}_1, \acute{w}_2, \acute{w}_3, .., \acute{w}_p)^\intercal$, where $\acute{w}_r \in [0,1]$ with $\Sigma_{r=1}^{p} \acute{w}_r = 1$ and $\acute{w}_r \geq 0$. We must take the following steps in order to make a decision:

Step 1: Organize the experts' evaluations for each alternative across the specified parameters, thereby constructing the decision matrix, denoted as I.

Step 2: Rank the alternatives in matrix I by applying a score function, aligning each alternative with its corresponding parameters.

Step 3: Perform aggregation on the SHFS decision matrix for each parameter, utilizing the designated Yager operators.

Step 4: Compute the score for each aggregated alternative using the prescribed formula, which integrates the outcomes of the previous steps.

$$sc(\jmath) = \frac{\left(1 + \frac{1}{m} \sum\limits_{j=1}^{m} \phi_j - \frac{1}{n} \sum\limits_{j=1}^{n} \chi_j - \frac{1}{o} \sum\limits_{j=1}^{o} \psi_j \right)}{2}, sc(\jmath) \in [0,1].$$

Step 5: Rate the effectiveness of each alternative's $¥ = \{y_1, y_2, y_3, ..., y_t\}$ consequences in descending order and select the most effective one.

The proposed model's flowchart is depicted in Figure 2.

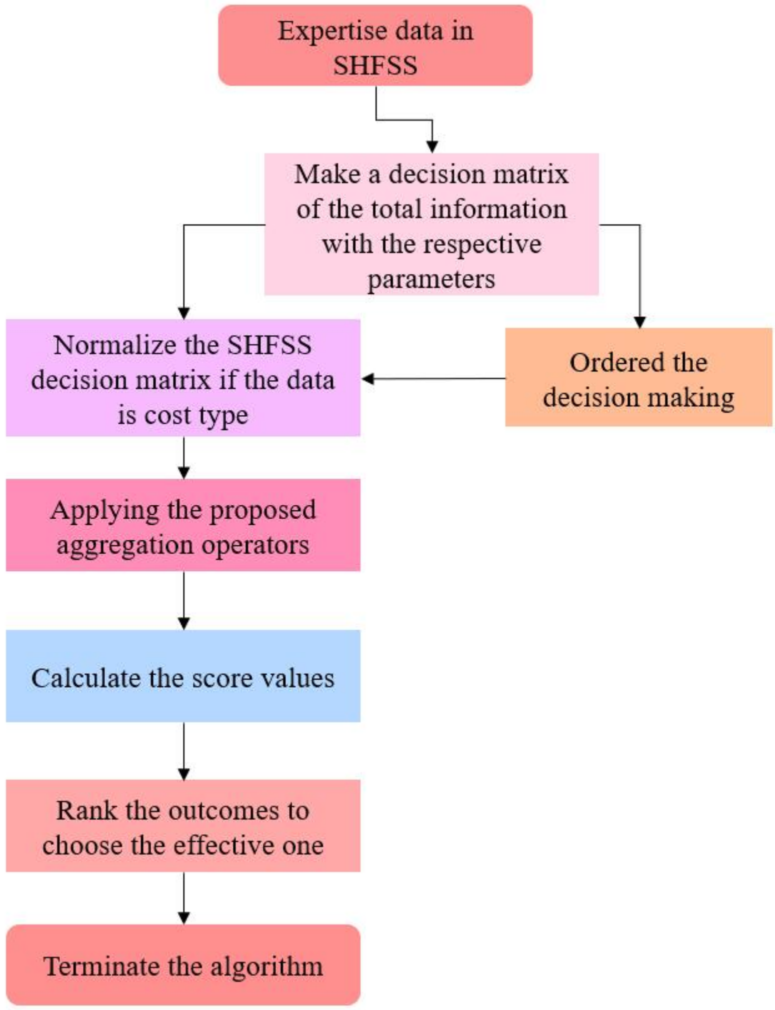

Figure 2. Flowchart of the algorithm

Numerical Illustration

In this section, a detailed numerical example is presented to demonstrate the practical application and effectiveness of the proposed approach to MADM.

Case Study

The case study focuses on ICS security suppliers. These entities are crucial in safeguarding critical infrastructure across various sectors such as energy, water, transportation, and manufacturing, against cyber threats. ICS security suppliers offer an array of protective measures for ICS networks and equipment. Their services encompass the provision of firewalls, intrusion detection and prevention systems, secure remote access solutions, and security information and event management systems. Beyond hardware and software solutions, these suppliers also extend consulting and training services to aid organizations in identifying and mitigating cyber risks to their ICS systems. They engage closely with clients, conducting thorough security needs assessments, crafting tailored solutions, and overseeing the implementation and ongoing maintenance of these security frameworks.

The significance of ICS security suppliers has escalated markedly in recent times, a trend primarily attributed to the increasing prevalence of cyber-attacks targeting critical infrastructure. The expansion of connected devices and the burgeoning Internet of Things (IoT) ecosystem have heightened the vulnerability of CIS to a range of cyber threats. This shift underscores the necessity for organizations to engage with credible and experienced ICS security suppliers as a fundamental component of their cybersecurity strategy. ICS security suppliers play a critical role in safeguarding vital infrastructure systems. Their expertise is instrumental in mitigating cyber threats, ensuring compliance with relevant regulations, and fortifying the overall cybersecurity posture of organizations.

These suppliers bring specialized knowledge and experience that are indispensable for the identification, analysis, and neutralization of potential cyber risks. Collaborating with a proficient ICS security supplier is not merely a protective measure; it is a strategic move that contributes to the safety, reliability, and productivity of an organization's operations. By leveraging the expertise of these suppliers, organizations can significantly enhance their resilience against cyber threats, thereby securing their operational continuity and safeguarding critical assets.

Implementing security measures for ICS through a qualified supplier equips organizations with the essential expertise, resources, and tools to fortify their critical infrastructure against cyber threats. The process of selecting an ICS security supplier is pivotal, as the integrity of an organization's ICS is crucial to maintaining the safety, reliability, and productivity of its operations. In the context of choosing an ICS security supplier, decision-makers are presented with a range of alternatives, denoted as $¥ = \{y_1, y_2, y_3, ..., y_t\}$. Each of these alternatives possesses distinct attributes, represented as $Ŧ= \{z_1, z_2, z_3, ..., z_p\}$, which are integral to determining the effectiveness and suitability of the security solutions they offer.

(1) z_1: Reputation and Experience.

(2) z_2: Technology and Innovation.

(3) z_3: Customer Support and Service.

1. Reputation and Experience:

When choosing an ICS security supplier, the reputation and experience of the supplier emerge as critical attributes. The reputation can be gauged through customer reviews, case studies, and industry accolades. To assess experience, one should consider their history in providing ICS security solutions, including their expertise in securing ICS systems, familiarity with industrial protocols and architectures, and insights into ICS-specific threats and vulnerabilities. A supplier renowned for its reputation and armed with extensive experience is more likely to possess a profound understanding of ICS security needs and the capability to deliver effective solutions.

2. Technology and Innovation:

The technological prowess and innovation capacity of the supplier stand as vital considerations. It is essential that the supplier offers state-of-the-art security technologies tailored for ICS environments, along with innovative solutions to counteract evolving threats. The supplier's competency in integrating their solutions with other security technologies and platforms is also a key factor.

3. Customer Support and Service:

Customer support and service play a significant role in the selection of an ICS security supplier. Optimal suppliers should boast a dedicated customer support team, available round-the-clock to resolve any emergent issues. Comprehensive training and ongoing support to ensure effective utilization of the security solutions are imperative. Suppliers must also be proactive in delivering timely updates and patches to address new threats. Transparent and responsive communication with customers, particularly regarding security incidents or vulnerabilities affecting their systems, is crucial for maintaining trust and efficacy.

To access an ICS security supplier (shown in Figure 3), we have three important attributes for the decision-makers, and have been displayed the overall decision matrix $I = \Gamma(j_r)_{t\times p} = (\vartheta_{\Gamma(j_r)}, \xi_{\Gamma(j_r)}, \partial_{\Gamma(j_r)})_{3\times 3}$ in Table 1. Let $\acute{w} = (0.4, 0.35, 0.25)^{\intercal}$ be the attribute WV's for SHFSWA, SHFSYOWA, SHFSYWG and SHFSYOWG operators, In this problem, normalization is not required due to benefit type attributes.

Figure 3. ICS security supplier

Table 1. Spherical hesitant fuzzy soft decision matrix

	z_1	z_2	z_3
y_1	$(0.4, 0.1)(0.2, 0.1)(0.4)$	$(0.2)(0.3, 0.15)(0.1, 0.5)$	$(0.4, 0.3)(0.2)(0.1, 0.4)$
y_2	$(0.5, 0.2)(0.1)(0.3, 0.1)$	$(0.6)(0.2, 0.1)(0.2)$	$(0.4, 0.1)(0.3, 0.2)(0.3, 0.2)$
y_3	$(0.4, 0.2)(0.1, 0.4)(0.2, 0.1)$	$(0.15, 0.3)(0.2, 0.3)(0.1, 0.4)$	$(0.6)(0.2)(0.2)$

5.1 Using SHFSYWA Operator

By using step 3 of the algorithm aggregate the SHFS decision matrix represented in Table 1 for each alternative. Such as:

For membership grade:

$$a_{tp} = \bigcup_{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\ldots,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},} \left\{ \sqrt{\min(1, (\Pi_{r=1}^{p}\acute{w}_r(\phi_{\Gamma(j_r)})^{2F})^{1/F})}, \right\}$$

$$a_{11} = \sqrt{\min(1, (((0.4^{(2*1)}) * 0.4) + ((0.2^{(2*1)}) * 0.35) + ((0.4^{(2*1)}) * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.343511281$$

For neutral grade:

$$a_{tp} = \bigcup_{\chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\ldots,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},} \left\{\sqrt{1-\min(1, (\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})}, \right\}$$

$$a_{11} = \sqrt{1-\min(1, (((1-0.2^{(2)})^1 * 0.4) + ((1-0.3^{(2)})^1 * 0.35) + ((1-0.2^{(2)})^1 * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.239791576$$

For non-membership grade:

$$a_{tp} = \bigcup_{\psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)},\ldots,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.} \left\{ \sqrt{1-\min(1, (\Pi_{r=1}^{p}\acute{w}_r(1-(\psi_{\Gamma(j_r)})^2)^F)^{1/F})} \right\}$$

$$a_{11} = \sqrt{1-\min(1, (((1-0.4^{(2)})^1 * 0.4) + ((1-0.1^{(2)})^1 * 0.35) + ((1-0.1^{(2)})^1 * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.264575131$$

The overall aggregated table for SHFSYWA Operator represented in Tables 2–4.

Table 2. The values obtained by SHFSYWA operator for y_1

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.343511281	0.239791576	0.264575131
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.31701735	0.184051623	0.327871926
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.240831892	0.21330729	0.392428337
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.201246118	0.147901995	0.43760713

Table 3. The values obtained by SHFSYWA operator for y_2

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.515751878	0.201246118	0.26925824
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.478016736	0.132287566	0.244948974
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.380131556	0.167332005	0.201246118
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.42661458	0.173205081	0.167332005

Table 4. The values obtained by SHFSYWA operator for y_3

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.402336923	0.167332005	0.17175564
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.430697109	0.21330729	0.286356421
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.337453701	0.296647939	0.132287566
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.370809924	0.324807635	0.264575131

Now, by using step 4 of the algorithm calculate the score values, represented in Table 5 and rank all the alternatives accordingly.

Table 5. The score values obtained by SHFSYWA operator

Alternatives	Score Values
y_1	0.361883954
y_2	0.53045733
y_3	0.460528504
Ranking Order: $y_2 > y_3 > y_1$	

5.2 Using SHFSYWG Operator

By using step 3 of the algorithm aggregate the SHFS decision matrix represented in Table 1 for each alternative. Such as:

For membership grade:

$$a_{tp} = \bigcup_{\phi_{\Gamma(j_1)}\in\vartheta_{\Gamma(j_1)},\phi_{\Gamma(j_2)}\in\vartheta_{\Gamma(j_2)},\ldots,\phi_{\Gamma(j_p)}\in\vartheta_{\Gamma(j_p)},} \left\{ \sqrt{1-\min(1,(\Pi_{r=1}^{p}(1-\acute{w}_r(\phi_{\Gamma(j_r)})^2)^F)^{1/F})}, \right\}$$

$$a_{11} = \sqrt{1-\min(1,(((1-0.4^{(2)})^1 * 0.4) + ((1-0.2^{(2)})^1 * 0.35) + ((1-0.4^{(2)})^1 * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.343511281$$

For neutral grade:

$$a_{tp} = \bigcup_{\chi_{\Gamma(j_1)}\in\xi_{\Gamma(j_1)},\chi_{\Gamma(j_2)}\in\xi_{\Gamma(j_2)},\ldots,\chi_{\Gamma(j_p)}\in\xi_{\Gamma(j_p)},} \left\{\sqrt{1-\min(1,(\Pi_{r=1}^{p}\acute{w}_r(1-(\chi_{\Gamma(j_r)})^2)^F)^{1/F})},\right\}$$

$$a_{11} = \sqrt{1-\min(1,(((1-0.2^{(2)})^1 * 0.4) + ((1-0.3^{(2)})^1 * 0.35) + ((1-0.2^{(2)})^1 * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.239791576$$

For non-membership grade:

$$a_{tp} = \bigcup_{\psi_{\Gamma(j_1)}\in\partial_{\Gamma(j_1)},\psi_{\Gamma(j_2)}\in\partial_{\Gamma(j_2)},\ldots,\psi_{\Gamma(j_p)}\in\partial_{\Gamma(j_p)}.} \left\{ \sqrt{\min(1,(\Pi_{r=1}^{p}\acute{w}_r(\psi_{\Gamma(j_r)})^{2F})^{1/F})} \right\}$$

$$a_{11} = \sqrt{\min(1, (((0.4^{(2*1)}) * 0.4) + ((0.1^{(2*1)}) * 0.35) + ((0.1^{(2*1)}) * 0.25))^{(1/1)}))}$$

$$a_{11} = 0.264575131$$

The overall aggregated table for SHFSYWG Operator represented in Tables 6–8.

Now, by using step 4 of the algorithm calculate the score values, represented in Table 9 and rank all the alternatives accordingly.

Table 6. The values obtained by SHFSYWG operator for y_1

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^\intercal$	0.343511281	0.239791576	0.264575131
$\{0.4, 0.35, 0.25\}^\intercal$	0.31701735	0.184051623	0.327871926
$\{0.4, 0.35, 0.25\}^\intercal$	0.240831892	0.21330729	0.392428337
$\{0.4, 0.35, 0.25\}^\intercal$	0.201246118	0.147901995	0.43760713

Table 7. The values obtained by SHFSYWG operator for y_2

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^\intercal$	0.515751878	0.201246118	0.26925824
$\{0.4, 0.35, 0.25\}^\intercal$	0.478016736	0.132287566	0.244948974
$\{0.4, 0.35, 0.25\}^\intercal$	0.380131556	0.167332005	0.201246118
$\{0.4, 0.35, 0.25\}^\intercal$	0.42661458	0.173205081	0.167332005

Table 8. The values obtained by SHFSYWG operator for y_3

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^\intercal$	0.402336923	0.167332005	0.17175564
$\{0.4, 0.35, 0.25\}^\intercal$	0.430697109	0.21330729	0.286356421
$\{0.4, 0.35, 0.25\}^\intercal$	0.337453701	0.296647939	0.132287566
$\{0.4, 0.35, 0.25\}^\intercal$	0.370809924	0.324807635	0.264575131

Table 9. The score values obtained by SHFSYWG operator

Alternatives	Score Values
y_1	0.361883954
y_2	0.53045733
y_3	0.460528504

Ranking Order: $y_2 > y_3 > y_1$

Now, switch the decision matrix I into the ordered matrix, presented in Table 10. Let $\acute{w} = (0.4, 0.35, 0.25)^\intercal$ be the attribute WV's and same as weighted averaging and geometric operators, we have done for their ordered operators such as: SHFSYOWA and SHFSYOWGA operators, sequentially.

Table 10. Ordered spherical hesitant fuzzy soft decision matrix

	z_1	z_2	z_3
y_1	$(0.4, 0.3)(0.2)(0.1, 0.4)$	$(0.2)(0.3, 0.15)(0.1, 0.5)$	$(0.4, 0.3)(0.2)(0.1, 0.4)$
y_2	$(0.6)(0.2, 0.1)(0.2)$	$(0.6)(0.2, 0.1)(0.2)$	$(0.4, 0.1)(0.3, 0.2)(0.3, 0.2)$
y_3	$(0.6)(0.2)(0.2)$	$(0.15, 0.3)(0.2, 0.3)(0.1, 0.4)$	$(0.6)(0.2)(0.2)$

5.3 Using SHFSYOWA Operator

By using step 3 of the algorithm aggregate the SHFS ordered decision matrix represented in Table 10 for each alternative. The overall aggregated table for SHFSYOWA Operator represented in Tables 11–13.

Now, by using step 4 of the algorithm calculate the score values, represented in Table 14 and rank all the alternatives accordingly.

Table 11. The values obtained by SHFSYOWA operator for y_1

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.360555128	0.229128785	0.25
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.278388218	0.158508675	0.427200187
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.319374388	0.188745861	0.35
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.222485955	0.204939015	0.35

Table 12. The values obtained by SHFSYOWA operator for y_2

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.521056619	0.204939015	0.264575131
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.400624512	0.17175564	0.239791576
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.483735465	0.173205081	0.204939015
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.444971909	0.132287566	0.17175564

Table 13. The values obtained by SHFSYOWA operator for y_3

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.453458929	0.17175564	0.180277564
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.471699057	0.204939015	0.264575131
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.404505871	0.286356421	0.14832397
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.424852916	0.307408523	0.243926218

Table 14. The score values obtained by SHFSYOWA operator

Alternatives	Score Values
y_1	0.377785146
y_2	0.53589248
y_3	0.493369286

Ranking Order: $y_2 > y_3 > y_1$

5.4 Using SHFSYOWG Operator

By using step 3 of the algorithm aggregate the SHFS decision matrix represented in Table 10 for each alternative. The overall aggregated table for SHFSYOWG Operator represented in Tables 15–17.

Now, by using step 4 of the algorithm calculate the score values, represented in Table 18 and rank all the alternatives accordingly.

Table 19 provides a comparison of the several explained operators.

Table 15. The values obtained by SHFSYOWG operator for y_1

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.343511281	0.239791576	0.264575131
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.31701735	0.184051623	0.327871926
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.240831892	0.21330729	0.392428337
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.201246118	0.147901995	0.43760713

Table 16. The values obtained by SHFSYOWG operator for y_2

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.515751878	0.201246118	0.26925824
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.478016736	0.132287566	0.244948974
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.380131556	0.167332005	0.201246118
$\{0.4, 0.35, 0.25\}^{\mathsf{T}}$	0.42661458	0.173205081	0.167332005

Table 17. The values obtained by SHFSYOWG operator for y_3

WV's	ϑ	ξ	∂
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.402336923	0.167332005	0.17175564
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.430697109	0.21330729	0.286356421
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.337453701	0.296647939	0.132287566
$\{0.4, 0.35, 0.25\}^{\intercal}$	0.370809924	0.324807635	0.264575131

Table 18. The score values obtained by SHFSYOWG operator

Alternatives	Score Values
y_1	0.361883954
y_2	0.53045733
y_3	0.460528504

Ranking Order: $y_2 > y_3 > y_1$

Table 19. Overall evaluation of the given operators

Proposed Operators	Scoring Order
SHFSYWA	$y_2 > y_3 > y_1$
SHFSYWG	$y_2 > y_3 > y_1$
SHFSYOWA	$y_2 > y_3 > y_1$
SHFSYOWGA	$y_2 > y_3 > y_1$

The alternatives ranking is depicted graphically in Figure 4.

So, the decision makers evaluate potential suppliers to ensure that they select the best one for their organization.

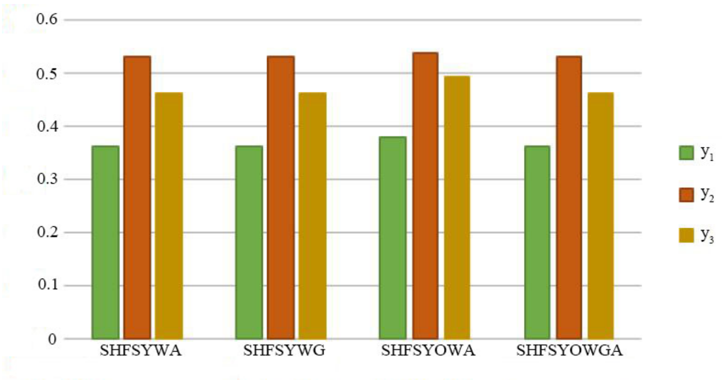

Figure 4. Alternatives ranking

6 Conclusion

The integration of a spherical structure into HFSS culminates in the creation of SHFSS, significantly enhancing the capability to model complex systems characterized by multiple layers of uncertainty. This research contributes to the state-of-the-art in statistical modeling, offering a refined approach to effective management through the use of fuzzy systems in decision-making contexts. To meet our objectives, a hybrid framework comprising AOs, specifically the spherical hesitant fuzzy soft Yager operators, has been developed. These operators are tailored to aggregate spherical hesitant fuzzy soft data efficiently.

Further, we have introduced an algorithm designed for tackling Single-Objective Hierarchical Fuzzy Sets MADM problems. A numerical example was presented to validate the effectiveness and practicality of our approach, demonstrating the application of the AOs. This example confirms the robustness, accuracy, and utility of our proposed model, particularly in comparison to existing models.

This paper advances the field of decision-making by proposing the development of a spherical hesitant fuzzy soft decision matrix. The findings of this study indicate that the proposed methodology is not only convenient but also well-aligned with other established selection procedures. It is our hope that this revised conceptual framework will be instrumental in overcoming various challenges associated with uncertainty, thereby yielding more reliable and impactful results.

Future work:

In forthcoming research endeavors, the focus will be placed on the development of novel operators to further refine the effectiveness and accuracy of deliberation processes within SHFSS. The ambit of this theory will extend to encompass not only complex SHFSSs but also the realm of Aczel-Alsina AOs. A key challenge in SHFSS group decision-making lies in eliciting individual perspectives in a manner that is both consistent over time and reliable.

A notable concern in the use of SHFSS is the subjectivity involved in assigning hesitation degrees and determining the spherical nature of the set. These processes, often influenced by personal preferences, have the potential to introduce bias and inconsistency in decision-making outcomes.

To surmount these challenges, there is a need for further research and the standardization of methodologies for collective decision-making utilizing SHFSS. This effort may involve the establishment of new standards, akin to those proposed by Zhao, M., Wei, G., Guo, Y., and Chen, X., in 2021. Their work, Decision Making in Interval-Valued Bipolar Fuzzy Multiple Attribute Groups Using the CPT-TODIM Approach with an Application to the Choice of a Security Service Provider for Industrial Control Systems, exemplifies the type of innovative approaches required to address these challenges.

Data Availability

The data used to support the research findings are available from the corresponding author upon request.

Conflicts of Interest

The authors declare no conflict of interest.

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