Dequantization

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November 25, 2015

1 Question

Dequantization, in a broad sense, should be some kind of adjoint or inverse to quantization. How can we describe it mathematically? As a starting point: Is there an inverse the the Borel-Weil construction?

2 Motivation and History

The question of dequantization is easily motivated for anyone who has dealt with the topic of quantization, for example somebody coming from physics. So first a brief note on

Quantization

The word comes from physics, where it means to reconstruct from a **classical physical system** the underlying **quantum mechanical system** from which it arises. Now let's translate into math:

Classical system $\rightarrow$ symplectic manifold (M, ω) with line bundle and connection $(\mathcal{L}, \nabla)$ such that the curvature satisfies $\nabla^2 = \omega$.

Quantum system $\rightarrow$ Hilbert space $\mathcal{H}$ with an algebra $\mathfrak{A}$ of self-adjoint operators acting in it.

Thus quantization is a map(functor?) between the spaces of all possible classical/quantum data

$$\mathcal{Q} : \{(M, \omega, \mathcal{L}, \nabla)\} \rightarrow \{(\mathcal{H}, \mathfrak{A})\}$$

Quantization is notoriously delicate, and not really known to be functorial in any good sense. It comes in many flavors (geometric, deformation, path integral, canonical, and in field theory BRST, BV, etc). Today let's focus on geometric quantization. In this case one is required to make an extra choice of 'polarization' which is meant to restrict the vector space in some sense. Very roughly the idea is as follows: Look at some suitable space of global sections (usually L^2) $\Gamma(M, \mathcal{L})$. The polarization you've chosen gives you a collection of differential operators whose kernels cut out a subspace $\mathcal{H} \subset \Gamma(M, \mathcal{L})$ which you declare to be your quantum Hilbert space. Then the algebra $\mathfrak{A}$ is the algebra $C^\infty(M)$ acting as certain vector fields (differential operators) defined by the symplectic form and the connection. (I've skipped giving details to save time and get to the algebraic geometry, and also not to bore you) So let's restrict our attention even further to the special case when $M = G \cdot \lambda$ is coadjoint orbit of a compact Lie group G (a.k.a. the orbit method) with $\lambda \in \mathfrak{g}^*$ a dominant integral weight. In this case the Borel-Weil-Bott theorem tells us that the resulting quantum Hilbert space carries an irreducible unitary representation of G.

The history of quantization is long. Briefly, it can be said to have originated with Dirac's canonical quantization (Poisson brackets$\rightarrow$commutators + quantum conditions), in 1925. Geometric quantization and its representation-theoretic incarnation, the orbit method, were pioneered by the likes of Kostant and Kirillov in the 1970's.

As remarked, dequantization is a natural phenomenon to consider mathematically. (It is also natural in the literal, physical sense, in that our Universe is governed by quantum mechanics, yet we experience "classical approximations" day to day). Dequantization is present in physics with the work of Wenzel, Kramers, and Brillioun (WKB semiclassical approximation) and also Feynman (one-loop Feynman diagrams). Some ideas also present in mathematics, in the work of Tian (1990) on almost-isometric embeddings of projective algebraic varieties. And in the world of representation theory, dequantization was kicked off by the 1981 paper of Beilinson and Bernstein. Their work was recently generalized by David Nadler and David Ben-Zvi. Another approach (which extends to loop groups of compact groups) is given in the series of papers by Freed, Hopkins and Teleman (2007-2011). A paper is on the horizon which generalizes this method to non-compact real semisimple Lie groups.

3 Some Existing Approaches

3.1 D-modules

If you've just learned about sheaves of modules over an affine scheme $X =$Spec$R$ in algebraic geometry class, then the Borel-Weil construction of $\mathbb{C}G$-modules will strike you as being vaguely similar to the construction of R-modules in the sense of taking global sections of some sheaf. The actual details are very different of course but nevertheless it leads one to ask the question: Is there an inverse in the Lie Group context similar to the one for R-mod?

Let's recall the R-mod "dequantization." One begins with an arbitrary R-module V. Then the "classical object" assigned to V is the sheaf $\mathcal{V} := V \otimes_R \mathcal{O}_X$. The "quantization" procedure, which in this case is an inverse on the nose, is just given by $V = \Gamma(X, \mathcal{V})$.

Let G be a compact (semisimple) Lie group. Let $G \cdot \lambda \cong G/T$ be the flag variety ($\lambda \in \mathfrak{g}^*$ dominant integral) Let $V_\lambda = \Gamma(G/T, \mathcal{L}_\lambda)$ given by the Borel-Weil construction. Inspired by the R-mod dequantization above, we see that we must take $X = G/T$). Let $R_\lambda := \Gamma(G/T, \mathcal{O}_{G/T})$. Unfortunatley, when we try to write

$$V_\lambda \otimes_{R_\lambda} \mathcal{O}_{G/T}$$

we run into a serious problem: R_λ acts commutatively on $\mathcal{O}_{G/T}$ but not on V_λ!

So to complete our task in this way, we must find a kind of non-commutative analog of a sheaf of modules on the flag variety.

We can let ourselves be guided further by the Borel-Weil example. As mentioned briefly above, the global functions $f \in R_\lambda = \Gamma(G/T, \mathcal{O}_{G/T})$ act in the representation $V_\lambda \subset \Gamma(G/T, \mathcal{L}_\lambda)$ as differential operators. Thus we should try to "dequantize" by tensoring against the sheaf of differential operators.

Enter Beilinson and Bernstein. In their 1981 paper they expounded the theory of localization of D-modules. The action of G on G/T can be differentiated to an action of $\mathfrak{g}$ by differential operators. Let $\mathcal{D}_{G/T}$ be the sheaf of differential operators, defined on local affine opens as $\mathbb{C}\langle x_1, x_2, ..., x_n, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\rangle$ with commutation relations trivial except $[\frac{\partial}{\partial x_i}, x_i] = 1$. They showed that the map $\mathcal{U}\mathfrak{g}_\mathbb{C} \to \mathcal{D}_{G/T}(G/T)$ extending the differentiated G-action is surjective. The sheaf $\mathcal{D}_{G/T}$ can be 'twisted' to act on sections of a line bundle $\mathcal{L}$ by putting $\mathcal{D}^{\mathcal{L}}_{G/T} := \mathcal{L} \otimes \mathcal{D}_{G/T} \otimes \mathcal{L}^{-1}$. Then they defined a localization map Δ similar to the one above for R-mod. Without going into detail, let's see the formula to appreciate its similarity to the one above:

$$\Delta(V_\lambda) := V_\lambda \otimes_{\mathcal{U}\mathfrak{g}_\mathbb{C}} \mathcal{D}^{\mathcal{L}_\lambda}_{G/T}$$

Of course there is a lot of fine-print to go through here, some of which I do not completely understand. There are also a couple of 'flaws'. For example, if one interprets the resulting D-module as a vector bundle with connection, you get (I think) a rank $|W|$ (Weyl group) bundle, instead of a line. Then there is the problem of needing to know the infinitesimal character to properly twist the differential operators. This is cured in the Nadler Ben-Zvi paper by passing to a torus bundle $G_\mathbb{C}/N \to G/T$ and identifying certain Weyl invariants in the category of monodromic D-modules there.

4 Some Other Approaches

4.1 Embedding the Orbit As usual, let V_λ be a unitary irrep of G. We can try to concoct another recipe for dequantization using a technique discussed in class this semester: projective embeddings via line bundles. Choose a maximal torus T and system of positive roots Δ^+ to identify the highest weight line $L_\lambda \subset V_\lambda$. Consider

$[L_\lambda] \in \mathbb{P}(V_\lambda)$ and note that G acts naturally on $\mathbb{P}(V_\lambda)$ via the linear action on V_λ. Consider the orbit $\mathfrak{O} := G \cdot [L_\lambda] \subset \mathbb{P}(V_\lambda)$. A straightforward calculation will show that $\mathfrak{O} \cong G/T$. (E.g. if one prefers $\mathbb{C}$ over $\mathbb{R}$, then one can identify the annihilator of $[L_\lambda]$, $\mathfrak{b} \subset \mathfrak{g}_\mathbb{C}$ as $\mathrm{Lie}(T)_\mathbb{C} \oplus \mathfrak{n}$, pass to the complex group $G_\mathbb{C}$ and exponentiate $B :=\exp\mathfrak{b}$, to get $\mathfrak{O} \cong G_\mathbb{C}/B$ and then use $G_\mathbb{C}/B \cong G/T$). Thus we have an embedding

$$\iota : G/T \hookrightarrow \mathbb{P}(V_\lambda)$$

If we pretend this is an embedding coming from a very ample line bundle $\mathcal{L} \to G/T$ then we can reconstruct $\mathcal{L}$ as $\mathcal{L} = \iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$. What about the symplectic form and the connection? Note that this line bundle also has a natural hermitian metric, adopted from the hermitian metric on V_λ. Also note that $V_\lambda \cong \Gamma(G/T, \iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1))^\vee$ via a G-equivariant isomorphism taking $[L_\lambda]$ to $[\mathrm{ev}_{eT}]$, the evaluation functional

$$\mathrm{ev}_{eT} : \Gamma\big(G/T, \iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)\big) \to \big(\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)\big)_{eT}$$

$$\xi \mapsto \xi(eT)$$

at the identity coset, which is only defined up to a scalar due to ambiguity in choice of local trivialization of the line bundle. Thus we may write

$$\iota : G/T \hookrightarrow \mathbb{P}(V_\lambda)$$

$$gT \mapsto [\mathrm{ev}_{gT}] = [\xi \mapsto \xi(gT)]$$

There is a natural symplectic structure on $\mathbb{P}(V_\lambda) \cong \mathbb{CP}^N$ that we can pull back to G/T. It is the Fubini-Study metric, which is most elegantly described in homogeneous coordinates $\{z_j\}_{j=1}^N$ as (unfortunately it uses complex analysis)

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\overline{\partial}\log\sum |z_j|^2$$

Now in a local trivialization $\mathbb{T}$ of $\iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$, the hermitian metric is represented by some function $h = ||\mathbb{T}||^2$. We can write a basis of global sections $\{\xi_j\}_{j=1}^N$ as $\xi_j = f_j\mathbb{T}$ for some functions f_j. Then the pullback symplectic form is

$$\iota_*\omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\overline{\partial}\log\sum |f_j|^2$$

$$= \frac{i}{2\pi}\partial\overline{\partial}\log\sum h|f_j|^2 - \frac{i}{2\pi}\partial\overline{\partial}\log h$$

$$= -\frac{i}{2\pi}\partial\overline{\partial}\log||\mathbb{T}||^2 + \frac{i}{2\pi}\partial\overline{\partial}\log\sum ||\xi_j||^2$$

Those who know a bit of topology will recognize the first term as a representative for the first Chern class $c_1(\iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1))$. Now, because the representation V_λ is *unitary* and G acts on G/T transitively, $||\xi_j||^2$ is a constant function, so the second term vanishes!

I'll add (without detail, since I've already strayed too far afield) that there is also a natural connection, obtained from realizing $\iota^*\mathcal{O}_{\mathbb{P}(V_\lambda)}(1)$ as a subbundle of the trivial $V_\lambda \times G/T$. This connection, together with the line bundle and the symplectic form above, combine to a compatible set of classical data.

To compare, informally but intuitively, to Beilinson-Bernstein localization, what we've done here is place over $\mathfrak{b} \in G_\mathbb{C}/B$ the $\mathfrak{b}$-highest weight line in V_λ which we can do because we chose a system of positive roots at the beginning. BB localization doesn't make this choice, so you get a Weyl group's worth of *extremal* weight lines, (much like if one did Lie algebra $\mathfrak{n}$ homology), which can, however, still be told apart in $\mathcal{D}_{G_\mathbb{C}/B}$-mod by their curvature, corresponding to Weyl group translates of λ.

4.2 Spectral Covers

Let's try to use one last technique we learned in class this semester: spectral covers. The set up I need is: Let V be a finite dimensional vector space and T and endomorphism. Consider the ring $\mathbb{C}[t]$ generated by T, acting on V. Then V becomes an $\mathcal{O}_{\mathbb{A}^1}$ module and we can localize to $\mathcal{V} = V \otimes_{\mathbb{C}[t]} \mathcal{O}_{\mathbb{A}^1}$. We saw in class that this sheaf (assuming the Jordan blocks are all unique) is concentrated at the Jordan values, over which sit the corresponding subspaces of V. In other words, $V = \Gamma(\mathrm{Supp}\mathcal{V}, \mathcal{O}_{\mathrm{Supp}\mathcal{V}})$. Now let's parametrize. Suppose $V \to X$ is a vector bundle over some variety and $T \in \mathrm{End}(V)$. That is, $T(x) \in \mathrm{End}(V_x)$. Then we can 'localize fiberwise' by $\mathcal{V} := V_x \otimes_{\mathbb{C}[t_x]} \mathcal{O}_{\mathbb{A}^1 \times \{x\}} \in \mathrm{Coh}(X \times \mathbb{A}^1)$. With the same assumtion on $T(x)$, one can rebuild the vector bundle $V \to X$ as $f_*\mathcal{O}_{\mathrm{Supp}\mathcal{V}}$ for $f : \mathrm{Supp}\mathcal{V} \to X$ the restriction of the projection $X \times \mathbb{A}^1 \to X$

Now let $X = \mathfrak{g}^*$, $V = V_\lambda \times \mathfrak{g}^*$ and T be as follows (The construction will seem ad hoc, bear with me, it's secretly a Dirac operator, I'll mention at the end): Let $\mathfrak{h}$ be the centralizer of λ in $\mathfrak{g}$, it is a Cartan subalgebra. Define

$$T : \mathfrak{h}^* \to \mathrm{End}(V_\lambda)$$

$$\xi \mapsto -|\lambda|^2 + 2i\xi - |\xi|^2$$

Here the norms are taken in the (positive-definite signed) Killing form. Note that here $\mathfrak{g}$ means the real Lie algebra of the compact group G. This operator is diagonalizable for every ξ, and has kernel iff $\xi = \lambda$. The kernel is 1-dimensional, it is the lowest weight line (of weight λ) with respect to the Weyl chamber for which λ is dominant. Extend this T to all of $\mathfrak{g}^*$ by requiring G-equivariance. Then T is still diagonalizable and has kernel only on the coadjoint orbit of λ. You can guess what's coming: we just need to assemble these kernel lines to a sheaf on $\mathfrak{g}^*$, which will be concentrated over a single coadjoint orbit.

We can use the language of spectral covers: Localize $V_\lambda \times \mathfrak{g}^* \to \mathfrak{g}^*$ using T as above to get $\mathcal{V} \in$Coh$(\mathfrak{g}^* \times \mathbb{A}^1)$. Take the scheme-theoretic intersection S:=Supp$\mathcal{V} \cap \mathfrak{g}^* \times \{0\}$ with the zero-section (to single out the kernel Jordan block) and let $f : S \to \mathfrak{g}^*$ be the restriction of the projection $\mathfrak{g}^* \times \mathbb{A}^1 \to \mathfrak{g}^*$ as above. Consider the sheaf

$$f_* \mathcal{O}_S \in \mathrm{Coh}(\mathfrak{g}^*)$$

We know it corresponds to a 'line bundle' concentrated along $G \cdot \lambda$. We can take the sheaf and pull back to G/T (T=Stab(λ) via the map $gT \mapsto g \cdot \lambda$. We have a symplectic form (Kostant-Kirillov) given by $\omega(\mu, \nu) = \langle \lambda, [\mu, \nu] \rangle$, and a connection coming from the trivial bundle as in the previous section. In fact, all these data agree with those from the previous section. That is, this is an inverse to the Borel-Weill construction. This is a theorem of Freed-Hopkins-Teleman in disguise.

In reality, the operator $T(\xi)$ I defined above is a left-invariant cubic Dirac operator coupled to L^2 spinors on G (which decomposed by Peter-Weyl). It defines a family of Fredholm operators over $\mathfrak{g}^*$, which, when restricted to each irreducible summand V_λ in $L^2(G)$, represent a class in equivariant K-theory (there is also some subtle twisting going on due to the central extension coming from the spin structure) whose Dirac index is exactly V_λ. So this construction cuts out the symplectic manifold and the line-bundle-with-connection starting with just the representation! An even better way to say it is that the kernel of the family of cubic Dirac operators defines a matrix factorization which is an 'Atiya-Bott-Shapiro' resolution of the sheaf constructed above via spectral covers.

5 Conclusion

So we've seen a couple different ways to recast the idea of dequantization in terms of constructions that appear in an algebraic geometry class. It was interesting for me to see how some things I've been thinking about recently look when translated into these languanges. A topic for further research would be to try and generalize the spectral cover construction to infinite dimensional representations, which would be necessary for non-compact groups. The difficulty would arise in the fact that there may be interesting parts of the spectrum of an endomorphism which are not eigenvalues. So one would have to somehow work with cokernels or something. A topic of further investigation for myself would be to understand the details of localization, in particular the Weyl multiplicity and perhaps a geometric realization thereof (there is a vaguely related multiplicity that appears in the Freed-Hopkins-Teleman method when you try to apply it to principal series representations).