The Kostant Dirac Operator on Discrete Series Representations

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1 Intro

In this note I will prove the two propositions below. Let G be a real semisimple Lie Group with a Cartan decomposition of its Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Let K be the maximal compact subgroup ($\mathrm{Lie}(K) = \mathfrak{k}$) and suppose rank$G$=rank$K$ so that G has discrete series representations. Throughout the first section $\mu \in \mathfrak{k}$ will be a regular elliptic element, so that $\mathfrak{h} := Z_{\mathfrak{k}}(\mu)$ is a Cartan subalgebra with $\mathfrak{h} \subseteq \mathfrak{k} \subseteq \mathfrak{g}$. We choose the system of positive roots $\Delta^+ = \Delta_c^+ \cup \Delta_n^+$ for which μ is antidominant. Let $\tilde{\rho} = \rho_c - \rho_n$. Let $\mathcal{H}_\Lambda$ be the discrete series rep with Blattner parameter (lowest K-type) $-\Lambda$ and $\mathcal{S}$ the spinor representation on $\mathfrak{g}^*$ (defined explicitly later). Let $L_{-\Lambda}$ be the lowest weight line of K_Λ and $S_{-\tilde{\rho}}$ the lowest weight line of $S(\mathfrak{g}^*)$.For a basis $\{e^\nu\}$ of $\mathfrak{g}^*$ with dual basis $\{e_\nu\}$ of $\mathfrak{g}$ let $\gamma^v = \gamma(e^\nu)$ denote Clifford multiplication by e^ν, σ_ν the action of e_v on $S(\mathfrak{g}^*)$ by pullback via the coadjoint representation, and let R_ν denote the action of e_ν on $\mathcal{H}_\Lambda$ (or rather the dense subspace of K-finite vectors). Recall (see FHT I,II, III) the definiton of the coupled Dirac operator: for $\mu = \mu_\nu e^\nu \in \mathfrak{g}^*$

$$\not{D}_\mu := \gamma^\nu (iR_\nu + \frac{i}{3}\sigma_\nu + \mu_\nu)$$

Note that the map $\mu \mapsto \not{D}_\mu$ is G-equivariant.

Proposition 1: The operator $\not{D}_\mu^2 \in \mathrm{End}(\mathcal{H}_\Lambda \otimes S(\mathfrak{g}^*))$ is invertible for all $\mu \in \mathfrak{g}^*$ except on the coadjoint orbit $\mathcal{O}_{-\Lambda-\tilde{\rho}} = \mathrm{Ad}^\# G \cdot (-\Lambda - \tilde{\rho})$. For μ in this orbit,

$$\ker \not{D}_\mu^2 = L_{-\Lambda} \otimes S_{-\tilde{\rho}} \otimes S(\mathfrak{h}^*)$$

Away from this subspace the spectrum of $\not{D}_\mu^2$ is bounded a finite distance away from 0 for all $\mu \in \mathfrak{g}^*$.

Proposition 2: There is a concomitant family of Fredholm operators over $\mathfrak{g}^*$, also denoted by $\not{D}^2_\mu$, acting in a Hilbert space $\mathcal{H} := \mathcal{H}_\Lambda \otimes \left|S(\mathfrak{g}^*)\right|$ whose spectra are identical to the operators in Proposition 1 for every μ. (Here $\left|S(\mathfrak{g}^*)\right|$ denotes the definite inner product space obtained by flipping the sign of the metric on the indefinite subspace of $S(\mathfrak{g}^*)$)

2 A Formula for the Dirac Operator

I'll show a preliminary result: that the negative non-compact roots elements kill $L_{-\Lambda}$. Let $v \in L_{-\Lambda}$. Take $X_{-\beta} \in \mathfrak{g}_{-\beta}$ a root element for some $\beta \in \Delta_n^+$.Then $X_{-\beta} \cdot v = 0$ or $X_{-\beta} \cdot v \in L_{-\Lambda-\beta}$. In the latter case, by applying some (possibly empty) combination of compact roots Δ_c we may conclude that $X_{-\beta} \cdot v$ is contained in a K type of highest weight

$$-\Lambda - \beta + \sum_{\alpha \in \Delta_c^+} n_\alpha \alpha; (n_\alpha \geq 0)$$

But by Theorem 9.20 Knapp (restated in terms of lowest weights, using the longest element of the Weyl group W_K) the only possible K-types in such a setting are of the form

$$-\Lambda' = -\Lambda + \sum_{\nu \in \Delta^+ \cup \Delta_c^-} n_\nu \nu; (n_\nu \geq 0)$$

Hence it must be that $X_{-\beta} \cdot v = 0$, as desired, because the above formulas disagree on the sign of the non-compact simple root in β.

Spin Module

Let $\mathfrak{p} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha^{\mathbb{C}}$. Consider the spin module

$$S(\mathfrak{g}^*) \cong S(\mathfrak{h}^*) \otimes \bigwedge \mathfrak{p}_c \otimes \bigwedge \overline{\mathfrak{p}}_n \otimes \det(\mathfrak{p}_c^*)^{1/2} \otimes \det(\mathfrak{p}_n)^{1/2}$$

Let's recall two standard facts about Clifford algebras: firstly, an orthogonal decomposition of an inner product space $V = V_1 \oplus V_2$ leads to a decomposition $C(V) \cong C(V_1) \otimes C(V_2)$; secondly, the dual of $C(\mathbb{R})$ with the standard inner product on $\mathbb{R}$ is isomorphic to $C(\mathbb{R})$ with the negative definite one. (E.g. Lawson-Michelson pg 21) Thus $\gamma(\xi)$ for $\xi \in \mathfrak{s}^*$ acts contragrediently compared to $\gamma(\mathfrak{k}^*)$. Hence, the spin module for a non-compact Lie algebra will be a direct sum of copies of a representation of lowest (resp. highest) weight $-\tilde{\rho}$ (resp. $\tilde{\rho}$) where $-\tilde{\rho} = \rho_c - \rho_n$. Here we see why $\rho_n - \rho_c$ appears in the Blattner parameter above. Note that Clifford multiplication by the root basis vector e_ν (for ν an positive root index), denoted γ^ν, kills the lowest weight space $S_{-\tilde{\rho}}$.

Taking into account the two facts outlined above we may steal the calculation of Prop 1.9 of [FHT II] to find that on $L_{-\Lambda} \otimes S_{-\tilde{\rho}}$ we have

$$\gamma^{\alpha_c} = R_{-\alpha_c} = \sigma_{-\alpha_c} = 0$$

for $\alpha_c \in \Delta_c$ and

$$\gamma^{\alpha_n} = R_{-\alpha_n} = \sigma_{-\alpha_n} = 0$$

for $\alpha_n \in \Delta_n$.

Hence on $L_{-\Lambda} \otimes S_{-\tilde{\rho}}$ we have $\not{D}_\mu = \gamma(\Lambda + \tilde{\rho} + \mu)$, so $\not{D}_\mu^2 = -|\Lambda + \tilde{\rho} + \mu|^2$, meaning that $\not{D}_\mu^2$ has kernel on this weight space only when $\mu = -\Lambda - \tilde{\rho}$.

In FHT II the authors show that $\not{D}_{-\Lambda-\tilde{\rho}}^2$ has no kernel elsewhere in the finite dimensional representation $V \otimes S(\mathfrak{g}^*)$, and that $\not{D}_\mu^2$ is invertible for other μ by showing that the Dirac Laplacian $\not{D}_0^2$ (Recall from FHT that $\not{D}_\mu^2 = \not{D}_0^2 + 2iT(\mu) - |\mu|^2$) commutes with both the Lie algebra and the Clifford action which together generate the representation. From this it follows that $\not{D}_\mu^2$ is non-positive definite with a maximum on the lowest weight space, whose behavior we investigated above.

In our current case we are not so lucky. Indeed, our discrete series representations are infinite dimensional and there is in general no lowest weight space, only a lowest K-type. If it was the case that our operator looked like $\not{D}_\mu^2 = \not{D}_0^2 + 2iT(\mu) - |\mu|^2$ with $\not{D}_0^2$ a constant (recall that $T(\mu) = \mu_*^\nu(R_\nu + \sigma_\nu)$ denotes the total Lie algebra action of $\mu \mapsto \mu_* \in \mathfrak{g}$ on $\mathcal{H} \otimes \mathcal{S}$), we would be out of luck because the operator $2iT(\mu) - |\mu|^2$ need not have a maximum. This implies that there exists a weight space in some large enough K-type where the Dirac operator for, say, $\mu = -\Lambda - \tilde{\rho}$, has a positive eigenvalue. Scaling $\mu \mapsto \epsilon\mu$ down to zero inside $\mathfrak{h}^*$, we see that the eigenvalue on this weight space will eventually become very close to $\not{D}_0^2 = -|\Lambda + \tilde{\rho}|^2$ which is negative. It does so continuously; hence at some point this eigenvalue vanishes and we get a extra kernel line on a different elliptic orbit.

The relevant formulas from FTH II are the following:

$$[\not{D}_\mu, \pi(\xi_*)] = \gamma\Big([\mu_*, \xi_*]^*\Big)$$

$$[\not{D}_\mu, \gamma(\xi)] = -2i\pi(\xi_*)$$

Here $\eta \mapsto \eta_*$ is the map $\mathfrak{g}^* \to \mathfrak{g}$ given by the metric, and $\eta \mapsto \eta^*$ is its inverse.

Iterating the commutators gives, $\forall \xi \in \mathfrak{g}^*$:

$$[\not{D}_\mu^2, \pi(\xi_*)] = [2T(\xi_*) - |\mu|^2, \pi(\xi_*)]$$

$$[\not{D}_\mu^2, \gamma(\xi)] = [2T(\xi_*) - |\mu|^2, \gamma(\xi)]$$

Hence

$$[\not{D}_0^2, \pi(\xi_*)] = [\not{D}_0^2, \pi(\xi_*)] = 0$$

In these calculations $\not{D}_0^2 = g^{ij}(R_iR_j + \frac{1}{3}\sigma_i\sigma_j)$ where g^{ij} is the metric tensor in coordinates. If we choose an orthogonal basis compatible with the weight basis of the adjoint representation then we can write

$$\not{D}_0^2 = \left(\not{D}_0^2\right)_c + \left(\not{D}_0^2\right)_n$$

$$\left(\not{D}_0^2\right)_c = \sum_{\nu \in \mathfrak{k}} g^{\nu\nu}\left(R_\nu^2 + \frac{1}{3}\sigma_\nu^2\right)$$

$$\left(\not{D}_0^2\right)_n = \sum_{\nu \in \mathfrak{s}} g^{\nu\nu}\left(R_\nu^2 + \frac{1}{3}\sigma_\nu^2\right)$$

Our calculations show that the $\not{D}_0^2$ commutes with $\pi(\xi_*)$ and $\gamma(\xi)$ (i.e. it is in the center of the algebra $G \rtimes \mathrm{Cliff}(\mathfrak{g}^*)$. These operators generate $\mathcal{H} \otimes \mathcal{S}$ from $L_{-\Lambda-\tilde{\rho}}$, and so $\not{D}_0^2$ a constant. This constant is easily computed to be $-|\Lambda + \tilde{\rho}|^2$. From our considerations above we find that the coupled Dirac operator

$$\not{D}_0^2 + 2iT(\mu) - |\mu|^2$$

$$= \left(\not{D}_0^2\right)_c + \left(\not{D}_0^2\right)_n + 2iT(\mu) - |\mu|^2$$

is not the one we want, since it'll have kernel all over the place. Instead we must use the Dirac Laplacian

$$\not{D}_\mu^2 := \left(\not{D}_0^2\right)_c - \left(\not{D}_0^2\right)_n + 2iT(\mu) - |\mu|^2$$

This can be seen as the Dirac Laplacian for the metric $\tilde{g}_{ij} := \left(g_{ij}\right)_{\mathfrak{k}} - \left(g_{ij}\right)_{\mathfrak{s}}$, made positive definite by flipping the sign on $\mathfrak{s}$.

So let us determine the behavior of $\left(\not{D}_0^2\right)_c - \left(\not{D}_0^2\right)_n$. But this is simple: $\left(\not{D}_0^2\right)_c$ is just the corresponding Dirac Laplacian for the maximal compact subgroup; hence it is constant on each K-type in $\mathcal{H} \otimes \mathcal{S}$. Using Knapp's Theorem from above and taking into account the spinors, any K-type has highest weight

$$\tau = \Lambda + \tilde{\rho} + \sum_{\beta \in \Delta_n^+} n_\beta \beta + \sum_{\nu \in \Delta^+} c_\nu \nu$$

where $n_\beta \in \mathbb{Z}_{\geq 0}$ and $c_\nu \in \{0, 1\}$. The value of $\left(\not{D}_0^2\right)_c$ on such a K-type is $-|\tau|^2$. Then we can compute

$$\left(\not{D}_0^2\right)_n = \not{D}_0^2 - \left(\not{D}_0^2\right)_c$$

$$= -|\Lambda + \tilde{\rho}|^2 + |\tau|^2$$

$$= \Big|\sum_{\beta\in\Delta_n^+} n_\beta \beta\Big|^2 - 2\langle w_0\Big(\sum_{\beta\in\Delta_n^+} n_\beta\beta\Big), \Lambda + \tilde{\rho}\rangle$$

where w_0 is the longest element in the Weyl group of K.

Hence we may finally write our desired coupled Dirac operator $\not{D}_\mu^2$ over the regular elliptic element μ, diagonalized on the weight basis of K-finite vectors in $\mathcal{H} \otimes \mathcal{S}$ as follows: In a weight space W_Θ of weight

$$\Theta = -\Lambda - \tilde{\rho} + w_0\Big(\sum_{\beta\in\Delta_n^+} n_\beta\beta\Big) + \sum_{\alpha\in\Delta_c^+} m_\alpha \alpha$$

(We've absorbed the c_ν in to the m_α and n_β) we have :

$$\not{D}_\mu^2\Big|_{W_\Theta} =$$

$$-\Big|\Lambda + \tilde{\rho} + \sum_{\beta\in\Delta_n^+} n_\beta\beta \ + \mu\Big|^2 + 2\langle\mu, \sum_{\alpha\in\Delta_c^+} m_\alpha\alpha\rangle - \Big|\sum_{\beta\in\Delta_n^+} n_\beta\beta\Big|^2 + 2\langle w_0\Big(\sum_{\beta\in\Delta_n^+} n_\beta\beta\Big), \Lambda + \tilde{\rho}\rangle$$

In particular all of these terms are nonpositive at best (remember μ is antidominant) and so the expression is 0 if and only if $\Theta = \mu = -\Lambda - \tilde{\rho}$, as desired.

3 Bounding the Spectrum Away from 0

Recall that we've defined $\tilde{\rho} = \rho_c - \rho_n$.

In this section I'll show that the Dirac operator is invertible away from $G\cdot(-\Lambda-\tilde{\rho})$. I will give a lower bound for the square norm of the eigenvalues of the Dirac operator away from the subspace $L_{-\Lambda-\tilde{\rho}} \otimes S(\mathfrak{h}^*)$, where as before $L_{-\Lambda-\tilde{\rho}} = L_{-\Lambda} \otimes S_{-\tilde{\rho}}$.

My notation will be a bit lax, in that I will not distinguish between $\mu \in \mathfrak{g}^*$ and its image in $\mathfrak{g}$ under the isomorphism coming from the invariant metric. I proceed case by case:

Case 1: Let μ be elliptic. Since $\mu \mapsto \not{D}_\mu$ is G equivariant we may conjugate μ and thus by standard results assume $\mu \in \mathfrak{k}$. In fact, we may conjugate so that $\mu \in \mathfrak{h} := Z_{\mathfrak{k}}(-\Lambda - \tilde{\rho})$. Our computations in part A, in particular the long formula at the end, show that the eigenvalue with smallest square norm occurs on $L_{-\Lambda-\tilde{\rho}} \otimes S(\mathfrak{h}^*)$. Its value there is $-|\Lambda+\tilde{\rho}+\mu|^2$. Because the metric is positive definite on $\mathfrak{k}$, this is only 0 on $G\cdot(-\Lambda-\tilde{\rho})$. Furthermore, our computation shows that away from this subspace the spectrum of $\not{D}_\mu^2$ is bounded away from 0 in square norm by the minimum (in the completely-negative orthant) of the strictly positive real quadratic polynomial in μ: (not all m_α are 0)

$$p(\mu) = |\Lambda + \tilde{\rho} + \mu|^2 - 2\langle \mu, \sum_{\alpha \in \Delta_c^+} m_\alpha \alpha \rangle$$

We are restricted to the negative orthant by the condition that our positive root system is chosen to make μ antidominant.

Case 2: Now suppose that μ is hyperbolic and hence, after possible conjugation, in $\mathfrak{s}$. We still have

$$\not{D}_\mu^2 = \left(\not{D}_0^2\right)_c - \left(\not{D}_0^2\right)_n + 2iT(\mu) - |\mu|^2$$

Because the invariant form is negative definite on $\mathfrak{s}$ and because $T(\mu)$ has only real eigenvalues on K-finite vectors we see that $\not{D}_\mu^2$ will have kernel only when $T(\mu) = 0$ and μ is scaled properly to cancel the contribution from the first two tierms. We show that $T(\mu)$ has no kernel. Since μ is hyperbolic and in $\mathfrak{g}_\mathbb{R}$ we can write it as a sum of non-compact root elements X_α in the following way:

$$\mu = \sum_{j \in \Delta_n^+} \mu^{\alpha_j} X_{\alpha_j} + \overline{\mu^{\alpha_j}} X_{-\alpha_j}$$

Now choose a weight basis $\{e_\tau\}_{\tau \in \mathfrak{h}_\mathbb{C}^*}$ of $\mathcal{H}_\Lambda \otimes \mathcal{S}$. Using these facts we find that if $T(\mu)$ has a kernel then: (Einstein summation for τ and α_j ranging over the weight basis and Δ_n^+, respectively)

$$\begin{aligned} 0 = T(\mu)v &= v^\tau T(\mu) e_\tau \\ &= v^\tau \Big(\mu_{\alpha_j} T(X_{\alpha_j}) + \overline{\mu_{\alpha_j}} T(X_{-\alpha_j}) \Big) e_\tau \\ &= v^\tau \mu_{\alpha_j} e_{\tau + \alpha_j} + v^\tau \overline{\mu_{\alpha_j}} e_{\tau - \alpha_j} \end{aligned}$$

Because of elementary properties of the spin rep $\mathcal{S}$ (See, for example, FHTIII Prop. 1.6) it is clear that if the weight τ appears in $\mathcal{H}_\Lambda \otimes \mathcal{S}$, then so does either $\tau + \alpha$ or $\tau - \alpha$, for any $\alpha \in \Delta^+$. Hence we must conclude that $\mu_{\alpha_j} = 0$ for every α_j, hence $\mu = 0$. Hence the spectrum is bounded away from 0 by $|\Lambda + \tilde{\rho}|^2$.

Case 3: Let μ be nilpotent. But then $|\mu|^2 = 0$ and $T(\mu)$ is nilpotent on K-finite vectors, so the spectrum of $\not{D}_\mu^2$ is contained in the spectrum of $\left(\not{D}_0^2\right)_c - \left(\not{D}_0^2\right)_n$, which is bounded away from 0 by $|\Lambda + \tilde{\rho}|^2$.

Case 4: Let $\mu = \mu_e + \mu_n$ be mixed elliptic/nilpotent. Standard results say that we can assume the two term commute. Then we have

$$\not{D}_\mu^2 = \not{D}_{\mu_e}^2 - 2iT(\mu_n)$$

The two terms in the last line commute and the second is nilpotent, so our only hope for a kernel would be if $\mu_e = -\Lambda - \tilde{\rho}$ and $T(\mu_n) = 0$ on $L_{-\Lambda-\tilde{\rho}}$. But then

$$\begin{aligned} 0 &= [T(\mathfrak{h}), T(\mu_n)]L_{-\Lambda-\tilde{\rho}} \\ &= T(\mathrm{ad}(\mathfrak{h})\mu_n)L_{-\Lambda-\tilde{\rho}} \\ &= T(\sum_{j\in\Delta} \alpha_j(\mathfrak{h})\mu_n^{\alpha_j} X_{\alpha_j})L_{-\Lambda-\tilde{\rho}} \\ &= \sum_{j\in\Delta} \alpha_j(\mathfrak{h})\mu_n^{\alpha_j} T(X_{\alpha_j})L_{-\Lambda-\tilde{\rho}} \end{aligned}$$

but because of what we know above about how the root elements X_α act on the lowest weight space, we get

$$\begin{aligned} 0 &= \sum_{j\in\Delta_c^+\cup\Delta_n^-} \alpha_j(\mathfrak{h})\mu_n^{\alpha_j} T(X_{\alpha_j})L_{-\Lambda-\tilde{\rho}} \\ &= \sum_{j\in\Delta_c^+\cup\Delta_n^-} \alpha_j(\mathfrak{h})\mu_n^{\alpha_j} L_{-\Lambda-\tilde{\rho}+\alpha_j} \end{aligned}$$

Hence, since (as mentioned above) the weights $-\Lambda - \tilde{\rho} + \alpha_j$ all appear in $\mathcal{H}_\Lambda \otimes \mathcal{S}$ we must conclude that $\mu_n^{\alpha_j} = 0$ for $j \in \Delta_c^+ \cup \Delta_n^-$ and as before since $\mu_n \in \mathfrak{g}_\mathbb{R}$, $\mu_n^{-\alpha_j} = \overline{\mu_n^{\alpha_j}} = 0$, implying that $\mu_n = 0$. Thus we reduce to Case 1.

Case 5: Let $\mu = \mu_e + \mu_h$ be mixed elliptic/hyperbolic, the two terms commuting. Then we have

$$= \not{D}_{\mu_e}^2 - 2iT(\mu_h) - |\mu_h|^2$$

Since we can simultaneously diagonalize these three terms, we see that there is kernel only if $\mu_h = 0$ and so we reduce to Case 1.

Case 6: After all that warm up, let's take a general μ. Then after conjugation, $\mu = \mu_e + \mu_h + \mu_n$ with $\mu_e \in \mathfrak{k}$, $\mu_h \in \mathfrak{s}$, and all three terms commuting. So we have:

$$\not{D}_\mu^2 = \not{D}_{\mu_e}^2 - 2iT(\mu_h) - 2iT(\mu_n) - |\mu_h|^2$$

Again, the hyperbolic element prevents the existence of a kernel, so we reduce to Case 4.

Hence, for all $\mu \in \mathfrak{g}^*$, the spectrum of the Dirac operator restricted to the orthogonal complement of the subspace $L_{-\Lambda-\tilde{\rho}} \otimes S(\mathfrak{h}^*)$ is contained in the complement of an open ball of radius equal to the global minimum of the polynomial $p(\mu)$ from Case 1. Proposition 1 follows.

4 The Fredholm Family

Now we construct the associated Fredholm family.

The vector space $\mathcal{H}_\Lambda \otimes S(\mathfrak{g}^*)$ is a Krein space since it inherits the indefinitness of the metric on the spin module, which itself originates from the indefinitness of the Killing form on $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Choosing as we have done, an orthogonal basis $\{e_\nu\}$ of $\mathfrak{g} \cong \mathfrak{g}^*$ compatible with the root decomposition gives an orthogonal basis of $S(\mathfrak{g}^*)$ which respects the orthogonal decomposition $S(\mathfrak{g}^*) \cong S_+ \oplus S_-$ into positive and negative definite subspaces. The vectors which span the negative definite subspace are precisely those k-vectors $e_{\alpha_1} \wedge ... \wedge e_{\alpha_j} \otimes e_{-\beta_1} \wedge ... \wedge e_{-\beta_{2k+1}}$ with an odd number of basis vectors from $\overline{\mathfrak{p}}_n$. Flipping the sign of the metric tensor on each of these basis elements gives a (finite dimensional) Hilbert space denoted by $\left|S(\mathfrak{g}^*)\right|$. We then define the Hilbert space $\mathcal{H}$ to be the Hilbert space tensor product with the discrete series representation: $\mathcal{H} := \mathcal{H}_\Lambda \otimes \left|S(\mathfrak{g}^*)\right|$. Let $\mathcal{K}$ denote the old Krein space $\mathcal{K} := \mathcal{H}_\Lambda \otimes S(\mathfrak{g}^*)$. The operators $\not{D}_\mu^2$ were self adjoint in the indefinite inner product on $\mathcal{K}$. Let $\{e_\tau\}$ be the weight basis of $\mathcal{H}_\Lambda$. Then the (closed) subspace $\mathfrak{S}$ spanned by the vectors $e_\tau \otimes e_{\alpha_1} \wedge ... \wedge e_{\alpha_j} \otimes e_{-\beta_1} \wedge ... \wedge e_{-\beta_{2k+1}}$ is a maximal nonpositive definite subspace $\mathfrak{S} \subset \mathcal{K}$. But we know that $\not{D}_\mu^2 \mathfrak{S} \subseteq \mathfrak{S}$. In fact we showed above that all the vectors in the spanning set defining $\mathfrak{S}$ are eigenvectors of $\not{D}_\mu^2$. Hence by standard results in Krein space operator theory [Nakagama88] these operators are actually also self adjoint for the Hilbertian positive definite inner product obtained as above by flipping signs, and their spectral theory is the same.