Real Dirac Quantiazation

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Motivation from Complex Polarizations

For (M, ω, J) a spin^c-manifold one gets the Dirac quantum Hilbert space as follows. Let $\mathcal{L}_\omega$ be the line-bundle with curvature ω and consider the Dirac operator $\bar{\partial\!\!\!/} : \Gamma(\mathcal{S}) \to \Gamma(\mathcal{S})$ where $\mathcal{S}$ is the canonical spinor rep and the spin^c-structure we've chosen has determinant a bundle-equivalent to $\mathcal{L}_\omega^{\otimes 2}$. Then

$$\mathcal{H}_{\text{quantum}} := \text{ind}\bar{\partial\!\!\!/} = [\bar{\partial\!\!\!/}, L^2(M)] \in KK(M, \text{pt})$$

This quantization procedure generalizes the usual recipe for geometric quantization with a *complex* polarization. Of course when there is a Hamiltonian action on M by a compact Lie group G we get a representation $\mathcal{H}_q \in K_G(\text{pt})$.

Where did the L^2 come from? One the one hand it came from physics where quantum states are also wave functions $|\psi\rangle$ whose square norm $|\psi|^2 = \langle\psi|\psi\rangle$ represents probability density and so $\int_M |\psi|^2$ had better be finite so that it may be normalized to be 1. On the other hand it came through math from the definitions of K-theory and from the "need" for a Hilbert space; that $L^p(M)$ is a Hilbert space iff $p = 2$ and integration is somehow fundamental.

What happens when M is not compact? One finds that $\ker\bar{\partial\!\!\!/}$ is no longer quite in $L^2(M)$, and the set up must be tweaked a bit.

What follows is a geometric approach that may not work itself but certainly contains elements that a proper solution must have.

Generalized Spectral Covers

The idea is simple: Develop a method of forming spectral covers over $\mathfrak{g}^*$ corresponding to the operator $D\!\!\!/_\mu$ acting fiber-wise on the trivial bundle $V \times \mathfrak{g}^*$ for some irreducible tempered representation V. Intersect the spectral curve (which should support some properly constructed sheaf carrying information about the spectral measure) with the zero section and push down to $\mathfrak{g}^*$. The resulting sheaf should be precisely the one which is resolved by the matrix factorization constructed in the FHT way. The novelty would be that this presumably gives a nice geometric picture of the Weyl multiplicity on hyperbolic orbits.

There are several obvious issues. The representations are infinite-dimensional and only in the discrete series case is 0 actually an eigenvalue. For principal series representations 0 will only be in the continuous part of the spectrum of $\not{D}_\mu$. This means that the usual scheme-theoretic construction of the spectral cover which sees only eigenvalues (without multiplcity) will not suffice. Which is to be expected since it's really a construction mean for Coh(X). What one needs is a method of detecting the type of a point in the spectrum and tailoring the construction of a sheaf over the spectral support or perhaps a the spectral support itself appropriately. For example one could imagine a construction that produces a sheaf supported on the spectrum of $\not{D}_\mu$ inside $\mathbb{A}^1$ such that over an eigenvalue one puts the corresponding kernel and over a continuous point (where $\not{D}_\mu - \lambda$ is injective but not surjective) one puts (the dual of) the cokernel. Let us approach this steep mountain ascent in the way that steep mountain ascents are usually approached-by switchbacks. Therefore:

A Homological Angle

A principal series representation V has almost by definition a realization as the the vector space of global sections of a sheaf on some space. Specifically

$$V = \Gamma(G/MAN, \mathcal{O}(\sigma, \nu))$$

for some element $\nu \in i\mathfrak{a}^*$ and some $\sigma \in \hat{M}$ which will henceforth be suppressed out of laziness. Also let $X := G/MAN$. This realization of V allows one attempt to localize the Dirac operator $\not{D}_\mu$ to a (twisted) differential operator on the sheaf $\mathcal{O}(\nu) \otimes \mathcal{S}$ ($\mathcal{S}$ = spinors) since $\not{D}_\mu$ acts on global sections $\Gamma(X, \mathcal{O}(\nu) \otimes \mathcal{S}) = V \otimes \mathcal{S}$. This localization will surely work since $\not{D}_\mu = \not{D}_0 + \gamma(\mu)$ and the two terms on the left are easily interpreted differential operators on X. Let $\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X\not{D}_\mu$ be the $\mathcal{D}_X$-module corresponding to $\not{D}_\mu \in \mathcal{D}_X(X)$. At this point one applies the general theory of $\mathcal{D}$-modules and forms the solution complex

$$\mathcal{R}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}(\nu) \otimes \mathcal{S})$$