1) Proof: Let $P = \{p^* \in W \mid \|f-p^*\| \le \|f-g\| \ \forall g \in W\}$

Then $\|f-p^*\| = \|f-q^*\| = M$ for $\forall p^*, q^* \in P$ $M \in \mathbb{R}^+$

Show $\|\theta(f-p^*) + (1-\theta)(f-q^*)\| \le M$ $0<\theta<1$

$\|\theta(f-p^*) + (1-\theta)(f-q^*)\| \le M \iff \|\theta(f-p^*)+(1-\theta)(f-q^*)\| \le \theta\|f-p\| + (1-\theta)\|f-q^*\| \le M$

$\iff \theta M + (1-\theta)M \le M$

$M = M$

$\|\theta(f-p) + (1-\theta)(f-q)\| \le \theta\|f-p\| + (1-\theta)\|f-q\| = \theta M + (1-\theta)M = M$

By definition $c^* = \theta p^* + (1-\theta)q^* \in P$.

Therefore $\|f-c^*\| = M$ and P is convex.

2) Claim: The best uniform approximation is $p_n^* = 0$

Proof: Let $p_n^* = 0$, then $e_n = f - 0 = \sin(2x)$

Thus the error equioscillates, $\pm\|e_n\|_\infty = \pm 1$, for $n+2$ points on $[0, 2\pi]$. Thus, the error has 4 extrema and by Chebyshev's Equioscillation Thm, p_n^* is the best uniform approx.

3) $f \in C[a,b]$ Find the best uniform approx. of f by a constant

Claim: The best uniform approx by a constant is $\frac{\max(f)+\min(f)}{2}$

Proof: Assume $M = \frac{\max(f)+\min(f)}{2}$ is not the best uniform approx using a constant

Then $\exists c \in \mathbb{R}$ s.t. $c > M \vee c < M$. $\|e_M\| \overset{!!}{=} \frac{\max(f)+\min(f)}{2}$

Case 1: $c > M$

Then $\|f-c\|_\infty < \|f-M\|_\infty$

$\iff |c - \min(f)| < \|f-M\|_\infty$

Since $c > M$, we form a contradiction and M is a better approx.

Case 2: $c < M$

Then $\|f-c\|_\infty < \|f-M\|_\infty$

$\iff |c - \max(f)| < \|f-M\|_\infty$

Since $c < M$, we form a contradiction and M is a better approx.

Therefore, $\frac{\max(f)+\min(f)}{2}$ is the best uniform approximation.

4) $V=\mathbb{R}^3$ $\|\cdot\|_\infty$ $W=\text{span}\{(0,1,0),(0,0,1)\}$ $f=(3,6,4)$

Prove best approx to f is not unique

Claim: $\|e_n\|_\infty=3$ is the minimium of $\|e_n\|_\infty$

Proof: Assume $\exists$ $\|e_n\|_\infty<3$

Then there exists $c\in W$ s.t. $\|f-c\|_\infty<3$

$\Leftrightarrow$ $\max\{(3-0x),(6-1y),(4-1z)\}$

$\Leftrightarrow$ $\max\{3,(6-1y),(4-1z)\}$

By definition max function cannot be reduced below 3

therefore $\|e_n\|_\infty \not< 3$ and we have found a contradiction

Claim: $(0,6,4)$ and $(0,5,3)$ are both best approx to f under $\|\cdot\|_\infty$

Proof: As shown the best approx. is $\|e_n\|_\infty=3$

Then $\|f-(0,6,4)\|_\infty=3$ & $\|f-(0,5,3)\|_\infty=3$

$\Leftrightarrow$ $\max\{|3-0|,|6-6|,|4-4|\}=3$ & $\max\{|3-0|,|6-5|,|4-3|\}=3$

$\Leftrightarrow$ $\max\{3,0,0\}=3$ & $\max\{3,1,1\}=3$

Both $(0,6,4)$ and $(0,5,3)$ are best approx of f under $\|\cdot\|_\infty$

therefore the best approx of f is not unique.

5) $p\in P_n$ prove unique rep $p(x)=a_0+a_1T_1(x)+\dots+a_nT_n(x)$ T_j Chebyshev polynomial degree j

Proof: By induction

Base case:

$p_0=c_0$ for any $c_0\in\mathbb{R}$ Set $a_0=c_0$ then $T_0(x)a_0=c_0=p_0$

Take $p_1(x)=c_0+c_1x$ for any $c_0,c_1\in\mathbb{R}$ Since $T_1(x)=1\cdot x$ take $a_1=c_1$

Thus, $a_1T_1(x)+a_0T_0(x)=c_1x+c_0=p_1(x)$

We have shown existence of unique $a_0=c_0$ and $a_1=c_1$ s.t. $p(x)=a_0+a_1T(x)$

Induction: Assume $p_n(x)$ can be uniquely represented by

$$p_n(x)=\sum_{i=0}^{n}a_iT_i(x)$$

Take $p_{n+1}=p_n+c_{n+1}x^{n+1}$

$$T_{n+1}(x)=2^{n-1}x^{n+1}+\sum_{i=0}^{n}b_ix^i$$

Take $a_{n+1}=\frac{c_{n+1}}{2^{n-1}}$

By our induction hypothesis define $\hat{p}_n(x)=\sum_{i=0}^{n}(c-b_i)x^i$ then

we have $\hat{p}_n(x)=\sum_{i=0}^{n}\hat{a}_iT_i(x)$

Then $a_{n+1}T_{n+1}(x)+\sum_{i=0}^{n}\hat{a}_iT_i(x)=\sum_{i=0}^{n+1}a_iT_i(x)=\sum_{i=0}^{n+1}c_iT_i(x)$

Thus, we have found a_n to be uniquely determined by c_n, therefore $p\in P_n$ has a unique representation

$p_n(x)=\sum_{i=0}^{n}a_iT_i(x)$