(3.1) (1). $f(x) = x^2 + |x|$.

(a). $f(x)$ 显然在0点不连续可微.

(b,d). $f_1(x) = x^2$, $\frac{\partial f_1(x)}{\partial x} = 2x$, 无界.

$\therefore f(x)$ 是局部利普希兹, 不是全局 ~.

(c). 显然连续

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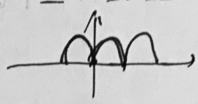
(3). $f(x) = \sin(x)\,\mathrm{sgn}(x)$.



a. $\lim_{x\to 0^+} f'(x) = 1$. $\lim_{x\to 0^-} f'(x) = -1$

$\therefore$ 在 $0.\pi \ldots$ 不连续可微.

(b.d). $|f'(x)| \le 1$, 一致有界

$\therefore f(x)$ 而为全局利普希兹.

(c). 显然连续.

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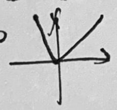
(5). $f(x) = -x + 2|x|$. $f(x) = \begin{cases} -3x & x<0 \\ x & x\ge 0 \end{cases}$



(a). $\lim_{x\to 0^+} f'(x) = 1$ $\lim_{x\to 0^-} f'(x) = -3$

$\therefore$ 在0点不连续可微.

(b.d). $|f'(x)| \le 3$.

$\therefore f(x)$ 为全局利普希兹.

(c). 显然连续

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(7). $f(x) = \begin{bmatrix} ax_1 + \tanh(bx_1) + \tanh(bx_2) \\ ax_2 + \tanh(bx_1) + \tanh(bx_2) \end{bmatrix}$

$$\left[\frac{\partial f(x)}{\partial x}\right] = \begin{bmatrix} a + b\cdot\frac{1}{\cosh^2(bx_1)} & b\cdot\frac{1}{\cosh^2(bx_2)} \\ b\cdot\frac{1}{\cosh^2(bx_1)} & a + b\cdot\frac{1}{\cosh^2(bx_2)} \end{bmatrix}$$

(a). $\therefore f(x)$ 在 R^2 是连续可微的.

(b.d). $\because \frac{1}{\cosh^2(u)} = 1 - \tanh^2(u)$, 一致有界

$\therefore |f'(x)|$ 一致有界

$\therefore f(x)$ 为全局利普希兹

(c). 显然连续.

(3.3.) 假设 $\begin{cases} \frac{|f_1(x) - f_1(y)|}{|x-y|} \le L_1 \\ \frac{|f_2(x) - f_2(y)|}{|x-y|} \le L_2. \end{cases}$

①. $f_3(x) = f_1(x) + f_2(x)$

$$\left|\frac{f_3(x) - f_3(y)}{|x-y|}\right| = \left|\frac{f_1(x) + f_2(x) - f_1(y) - f_2(y)}{|x-y|}\right|$$

$$= \frac{|f_1(x) - f_1(y) + f_2(x) - f_2(y)|}{|x-y|} \le (L_1 + L_2)$$

$\therefore f_3(x) = f_1(x) + f_2(x)$ 是局部利普希兹的.

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②. $f_4(x) = f_1(x)\cdot f_2(x)$

$$\frac{|f_4(x) - f_4(y)|}{|x-y|} = \frac{|f_1(x)\cdot f_2(x) - f_1(y)\cdot f_2(y)|}{|x-y|}$$

$$= \frac{|f_1(x) f_2(x) - f_1(x) f_2(y) + f_1(x) f_2(y) - f_1(y) f_2(y)|}{|x-y|}$$

$$\le \frac{|f_1(x)|\,|f_2(x) - f_2(y)| + |f_2(y)|\,|f_1(x) - f_1(y)|}{|x-y|}$$

$\le |f_1(x)| L_2 + |f_2(y)| L_1$.

$\because |f_1(x)|, |f_2(y)|$ 显然为有界数

$\therefore f_4(x) = f_1(x)\cdot f_2(x)$ 为局部利普希兹.

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③. $f_5(x) = f_2(f_1(x))$.

$$\frac{|f_5(x) - f_5(y)|}{|x-y|} = \frac{|f_2(f_1(x)) - f_2(f_1(y))|}{|x-y|}$$

$\because |f_2(x) - f_2(y)| \le L_2 |x-y|$

$\therefore$ 原式 $\le L_2 |f_1(x) - f_1(y)| \le L_1 L_2$.

$\therefore f_5(x) = f_2 \circ f_1$ 为局部利普希兹.

(3.6) (a). $x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,ds$.

$\therefore \|x(t)\| \le \|x_0\| + \int_{t_0}^{t} \|f(s, x(s))\|\,ds$.

$\because \|f(t,x)\| \le k_1 + k_2\|x\|$

$$\therefore \|x(t)\| \le \|x_0\| + \int_{t_0}^{t}(k_1 + k_2\|x(s)\|)\,ds$$

$$= \|x_0\| + k_1(t-t_0) + k_2\int_{t_0}^{t}\|x(s)\|\,ds$$

应用Gronwall-Bellman不等式：

有 $\|x(t)\| \le \|x_0\| + k_1(t-t_0) + \int_{t_0}^{t}[\|x_0\| + k_1(s'-t_0)] \cdot k_2 e^{k_2(t-s')}\,ds$

$$\therefore \|x(t)\| \le \|x_0\| \exp[k_2(t-t_0)] + \frac{k_1}{k_2}\{\exp[k_2(t-t_0)] - 1\}, \quad \forall t \ge t_0$$

3.14.

$$\dot{x}_1 = -\frac{1}{\tau}x_1 + \tanh(\lambda x_1) - \tanh(\lambda x_2)$$
$$\dot{x}_2 = -\frac{1}{\tau}x_2 + \tanh(\lambda x_1) + \tanh(\lambda x_2).$$

(b). $\dot{r} = (\sqrt{x_1^2+x_2^2})' = 2x_1\cdot\dot{x}_1\cdot\frac{1}{2}(x_1^2+x_2^2)^{-\frac{1}{2}} + 2x_2\cdot\dot{x}_2\cdot\frac{1}{2}(x_1^2+x_2^2)^{-\frac{1}{2}}$

$$= (x_1\dot{x}_1 + x_2\dot{x}_2)(x_1^2+x_2^2)^{-\frac{1}{2}}$$

$$= \frac{x_1\dot{x}_1 + x_2\dot{x}_2}{r}.$$

$$x_1\dot{x}_1 + x_2\dot{x}_2 = -\frac{1}{\tau}x_1^2 + x_1[\tanh(\lambda x_1) - \tanh(\lambda x_2)] - \frac{1}{\tau}x_2^2 + x_2[\tanh(\lambda x_1) + \tanh(\lambda x_2)]$$

$$\le -\frac{1}{\tau}r^2 + 2r(|\cos\theta| + |\sin\theta|)$$

$$\le -\frac{1}{\tau}r^2 + 2\sqrt{2}\,r$$

$$\therefore \dot{r} \le -\frac{1}{\tau}r + 2\sqrt{2}$$

(c). 由比较引理可得：

设 $\dot{u} = -\frac{1}{\tau}u + 2\sqrt{2}$, $r(0) = u(0) = \|x(0)\|_2$

$r(t) \le u(t)$.

$$u(t) = \exp\left(-\frac{t}{\tau}\right)\|x(0)\|_2 + \int_0^t \exp\left[\frac{-(t-\sigma)}{\tau}\right] \cdot 2\sqrt{2}\,d\sigma$$

$$= \exp\left(-\frac{t}{\tau}\right)\|x(0)\|_2 + 2\sqrt{2}\,\tau\left[1 - \exp\left(-\frac{t}{\tau}\right)\right].$$

$$\therefore \|x(t)\|_2 \le e^{-t/\tau}\|x(0)\|_2 + 2\sqrt{2}(1 - e^{-t/\tau}).$$