# State space representation of dynamical systems

## State and output equation

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) & x \in \mathbb{R}^n, u \in \mathbb{R}^p, \\ y(t) = g(x(t), u(t)) & y \in \mathbb{R}^q \end{cases}$$

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & A \in \mathbb{R}^{n,n}, B \in \mathbb{R}^{n,p}, \\ y(t) = Cx(t) + Du(t) & C \in \mathbb{R}^{q,n}, D \in \mathbb{R}^{q,p} \end{cases}$$

## Solution of LTI systems

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= C\left[e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau\right] + Du(t) \end{aligned}$$

$$\begin{aligned} X(s) &= (sI-A)^{-1}x(0) + (sI-A)^{-1}BU(s) \\ Y(s) &= C(sI-A)^{-1}x(0) + \left[C(sI-A)^{-1}B + D\right]U(s) \end{aligned}$$

$$H(s) = \left[C(sI-A)^{-1}B + D\right]$$

transfer matrix

## Steady state analysis

**Step input** $u(t) = A_u\varepsilon(t)$

$$y_{ss}(t) = A_y\varepsilon(t)$$

$$A_y = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sH(s)U(s)$$

**Sinusoidal input** $u(t) = A_u \sin \omega t$

$$y_{ss}(t) = A_y \sin(\omega t + \phi)$$

$$A_y = A_y(j\omega) = A_u|H(j\omega)| \quad \phi = \phi(j\omega) = \angle H(j\omega)$$

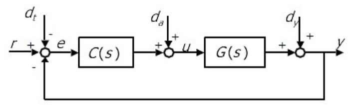
## Linearization of nonlinear systems in a neighborhood of an equilibrium point $(\bar{x}, \bar{u})$

$$\delta x(t) = x(t) - \bar{x}, \delta u(t) = u(t) - \bar{u}, \delta y(t) = y(t) - \bar{y}$$

$$\begin{cases} \dot{\delta x}(t) = A\delta x(t) + B\delta u(t) \\ \delta y(t) = C\delta x(t) + D\delta u(t) \end{cases} \quad \delta x(0) = x(0) - \bar{x}$$

$$A = \left.\frac{\partial f(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \quad B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$

$$C = \left.\frac{\partial g(x,u)}{\partial x}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}} \quad D = \left.\frac{\partial g(x,u)}{\partial u}\right|_{\substack{x=\bar{x}\\ u=\bar{u}}}$$




# Feedback control system

| | |
|---|---|
| $L(s)$ | Loop function |
| $S(s) = \frac{1}{1+L(s)}$ | Sensitivity function |
| $T(s) = \frac{L(s)}{1+L(s)}$ | Complementary sensitivity fn |
| $R(s) = \frac{C(s)}{1+L(s)}$ | Control sensitivity fn |
| $Q(s) = \frac{G(s)}{1+L(s)}$ | Actuator disturbance sensitivity fn |
| $S_p$ | $S(s)$ resonant peak |
| $T_p$ | $T(s)$ resonant peak |

## Stability analysis

**Nyquist stability criterion**: $Z = P + N$ where
$Z = \#$ poles of $T(s)$
$P = \#$ poles of $L(s)$ inside the Nyquist contour
$N =$ number of encirclement of $L(s)$ around $(-1, 0)$
$N > 0$ if clockwise
$N < 0$ if counterclockwise

## Steady state analysis
**(polynomial input)**

| | h → g ↓ | 0 $r(t) = \rho\epsilon(t)$ | 1 $r(t) = \rho t\epsilon(t)$ | 2 $r(t) = \rho\frac{t^2}{2}\epsilon(t)$ |
|---|---|---|---|---|
| $\lvert e_r^\infty \rvert$ | 0 | $\left\lvert\frac{\rho}{1+K_0}\right\rvert$ | $\infty$ | $\infty$ |
| | 1 | 0 | $\left\lvert\frac{\rho}{K_1}\right\rvert$ | $\infty$ |
| | 2 | 0 | 0 | $\left\lvert\frac{\rho}{K_2}\right\rvert$ |

$$K_g = \lim_{s\to 0} s^g L(s)$$

| | h → g ↓ | 0 $d_y(t) = \delta_y\epsilon(t)$ | 1 $d_y(t) = \delta_y t\epsilon(t)$ | 2 $d_y(t) = \delta_y\frac{t^2}{2}\epsilon(t)$ |
|---|---|---|---|---|
| $\lvert y_{dy}^\infty \rvert$ | 0 | $\left\lvert\frac{\delta_y}{1+K_0}\right\rvert$ | $\infty$ | $\infty$ |
| | 1 | 0 | $\left\lvert\frac{\delta_y}{K_1}\right\rvert$ | $\infty$ |
| | 2 | 0 | 0 | $\left\lvert\frac{\delta_y}{K_2}\right\rvert$ |

$$K_g = \lim_{s\to 0} s^g L(s)$$

| | $h \rightarrow$ / $g_c \downarrow$ | 0 $d_a(t) = \delta_a \epsilon(t)$ | 1 $d_a(t) = \delta_a t \epsilon(t)$ | 2 $d_a(t) = \delta_a \frac{t^2}{2} \epsilon(t)$ |
|---|---|---|---|---|
| $\lvert y^{\infty}_{d_a}\rvert$ | 0 | $\left\lvert\frac{\delta_a}{K_0}\right\rvert$ | $\infty$ | $\infty$ |
| | 1 | 0 | $\left\lvert\frac{\delta_a}{K_1}\right\rvert$ | $\infty$ |
| | 2 | 0 | 0 | $\left\lvert\frac{\delta_a}{K_2}\right\rvert$ |

$$K_0 = \begin{cases} K_C & \text{if } G(s) \text{ has poles in } 0 \\ \frac{1+K_C K_G}{K_C} & \text{if } G(s) \text{ has not poles in } 0 \end{cases}$$

$$K_{g_c} = \lim_{s \to 0} s_c^g C(s), \quad g_c \geq 1$$

# Feedback systems design

## Prototype 2$^{\text{nd}}$ order model

$$T(s) = K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

## Step output response ($u(t) = \bar{u}\varepsilon(t)$)

$$y_\infty = K\bar{u}$$

$$\hat{S} = \exp^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad \zeta = \frac{|\ln(\hat{S})|}{\sqrt{\pi^2 + \ln^2(\hat{S})}}$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}(\pi - \arccos\zeta)$$

$$t_{s,\alpha\%} = \frac{1}{\omega_n\zeta}\ln\left(\frac{\alpha}{100}\right)^{-1}$$

$$\hat{t} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

## Frequency response

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_B = \omega_n\sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}}$$

$$S_p = \frac{2\zeta\sqrt{2 + 4\zeta^2 + 2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2} + 4\zeta^2 - 1}$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

## Lead network / Lag network

$$C_D(s) = \frac{1 + \frac{s}{\omega_D}}{1 + \frac{s}{m_D\omega_D}} \qquad m_D > 1$$

$$C_I(s) = \frac{1 + \frac{s}{m_I\omega_I}}{1 + \frac{s}{\omega_I}} \qquad m_I > 1$$

## Negative real zero

$$C_z(s) = 1 + \frac{s}{\omega_z}$$

## PID standard controllers

### PI

$$C(s) = \frac{K_c(1 + \frac{s}{\omega_z})}{s} \qquad C(s) = K_p\left(1 + \frac{1}{T_i s}\right)$$

### PID

$$C(s) = \frac{K_c(1 + \frac{s}{\omega_{z1}})(1 + \frac{s}{\omega_{z2}})}{s} \qquad C(s) = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

### PD

$$C(s) = K_c\left(1 + \frac{s}{\omega_z}\right) \qquad C(s) = K_p(1 + T_d s)$$

## Sinusoidal disturbances

$$d(t) = \delta \sin\omega t \qquad \omega \in \left[\omega^L, \omega^H\right]$$

$$y_{ss}(t) = \delta|W_{dy}(j\omega)|\sin\left(\omega t + \angle W_{dy}(j\omega)\right)$$

$$|y_d^{\infty}| = \max|y_{perm}(t)| = \delta \cdot \max|W_{dy}(j\omega)|$$

$$\omega \in [\omega^L, \omega^H]$$

## Digital Control Design

$$G_{\text{ZOH}}(s) = \frac{1 - e^{Ts}}{s} \simeq \frac{1}{1 + sT/2}$$

### Butterworth filter design

$$\omega_f = \left(\omega_h\frac{\gamma^2}{1-\gamma^2}\right)^{\frac{1}{2n}} \simeq \omega_h\gamma^{\frac{1}{n}}$$