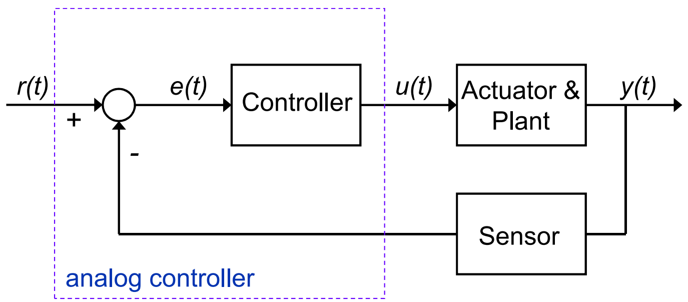
Automatic Control

Introduction to digital control

- **Motivations**
- **Structure of digital control systems**
- **Discrete time signals and systems**

Motivations for digital control

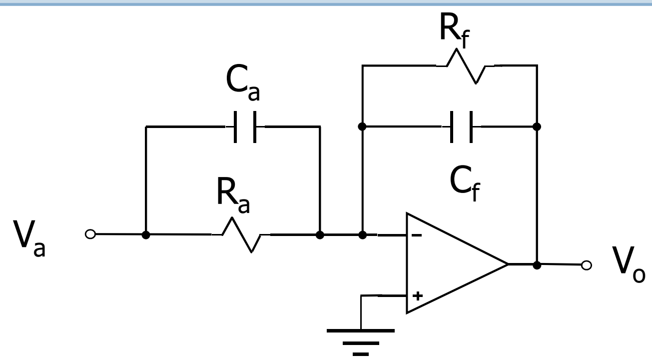
The structure of an analog feedback control system



In an analog feedback control system:

- relevant signals are analog (i.e. continuous in time and amplitude)
- the controller is typically realized through an active electronic filter

Lead and lag networks implementation using ideal OA*



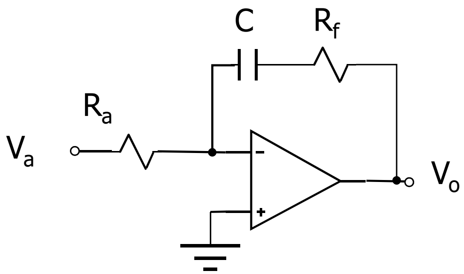
$$V_o(s) = -\frac{R_f}{R_a} \cdot \frac{1 + R_a C_a s}{1 + R_f C_f s} V_a(s)$$

Lead if: $R_a C_a > R_f C_f$

Lag if: $R_a C_a < R_f C_f$

* see C. Greco, M. Indri, Controlli Automatici, Politecnico di Torino - CELM (2007)

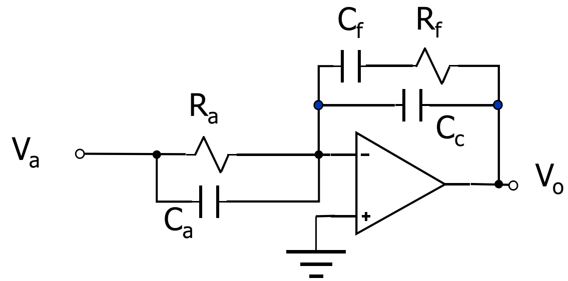
PI network implementation using ideal OA*



$$V_o(s) = -\left(\frac{R_f}{R_a} + \frac{1}{R_aCs}\right)V_a(s) = -\frac{1}{R_aC}\cdot\frac{1+R_fCs}{s}V_a(s)$$

* see C. Greco, M. Indri, Controlli Automatici, Politecnico di Torino - CELM (2007)

PID network implementation using ideal OA*



$$V_o(s) = -\frac{(1+R_fC_fs)(1+R_aC_as)}{R_aC_fs\cdot(1+C_c/C_f+R_fC_cs)}V_a(s)$$

$$V_o(s) = -\left(\underbrace{\frac{C_a}{C_f}+\frac{R_f}{R_a}}_{P}+\underbrace{\frac{1}{R_aC_fs}}_{I}+\underbrace{R_fC_as}_{D}\right)\cdot\frac{1}{(1+C_c/C_f+R_fC_cs)}V_a(s)$$

* see C. Greco, M. Indri, Controlli Automatici, Politecnico di Torino - CELM (2007)

Analog controllers drawbacks and solutions

Problems related to analog controllers

- Components degradation due to aging
- Parameters uncertainty and variability as a function of working conditions
- Actual circuits show nonlinear behaviors
- Very expensive in case of re-tuning and/or re-design
- Coupling with EM disturbance

Possible solutions

- Improve robustness in the design (→ more conservative)
- Realize controllers improving accuracy and "stability" of the components (→ high costs)
- **Design and realize digital controllers**

The structure of a digital feedback control system

- At present, most control systems use digital computers for controller implementation
- Digital computers allow one to calculate the control input through SW algorithms rather than using suitable electronic filters. This gives relevant advantages:
  - Flexibility in making modifications to the controller after the hardware design is fixed
  - Hardware and software design can proceed almost independently, saving a large amount of time
  - Logic and nonlinear operations can be easily included in the controller
  - Rapid prototyping

The structure of a digital feedback control system

In digital feedback control systems, the analog controller is replaced by a digital computer (μ-processor)

The digital computer receives and operates on digital signals (i.e. discrete in both time and amplitude)

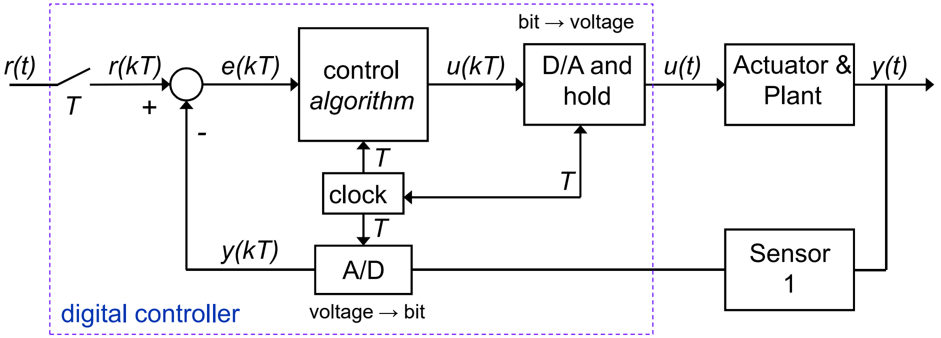
The analog measured signals are converted by means of analog-to-digital converters (A/D)

The digital controller output signal is converted to an analog signal to be provided to the plant by a digital-to-analog converter (D/A)

The structure of a digital feedback control system

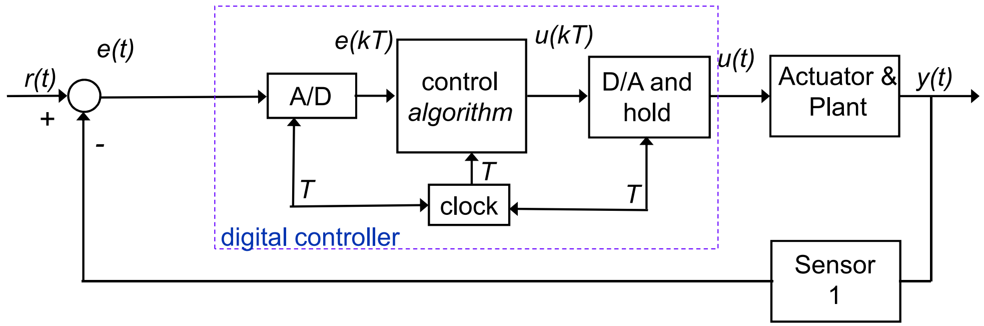
In a digital feedback control system:
- the digital controller is interfaced to the analog system made up by actuator-plant connection
- both analog and digital signals are present

$T \rightarrow$ sample time (s) $\rightarrow$ $x(kT)$ $\rightarrow$ sampled signal



The structure of a digital feedback control system

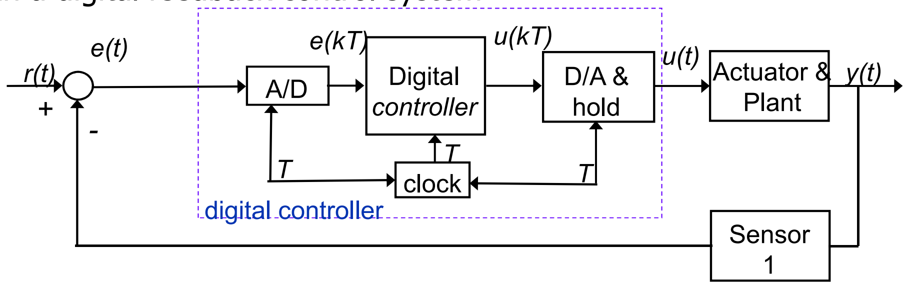
Sampled data feedback control system with error sampling



Discrete time signals and systems

Discrete time signals

In a digital feedback control system



the **digital controller**, at each sampling time kT, computes the control input $u(kT)$ using the sampled value of the tracking error $e(kT)$

Since both $u(kT)$ and $e(kT)$ are **discrete-time signals**

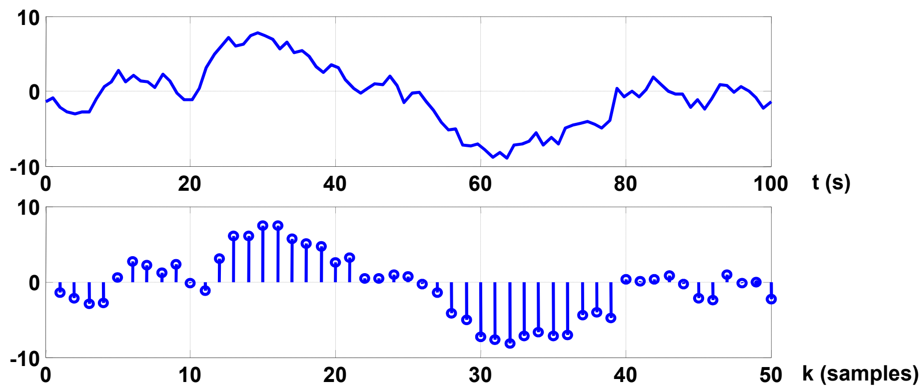
→ the digital controller is a **discrete-time dynamic system**

In the following, the simplified notation $f(k)$ will be used instead of $f(kT)$

Discrete-time signals: sequences

A **discrete-time signal** is made up by a sequence of real numbers

Therefore, the signal is not defined between two sampling instants.



Discrete-time signals: examples

- Unit impulse sequence:

$$\delta(k) = \begin{cases} 0 & k \neq 0 \\ 1 & k = 0 \end{cases}$$

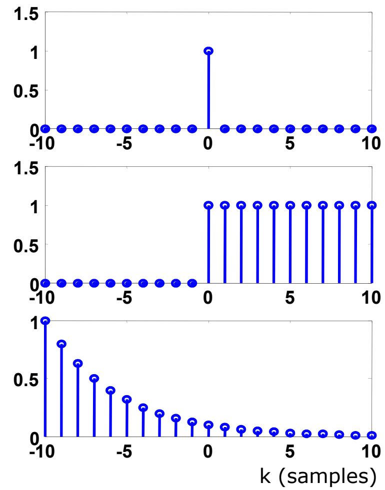
- Unit step sequence:

$$\varepsilon(k) = \begin{cases} 0 & k < 0 \\ 1 & k \geq 0 \end{cases}$$

- "Geometric" sequence:

$$e^{aTk} \underset{\uparrow}{=} \alpha^k, |\alpha| < 1$$

$$e^{aT} = \alpha$$



Introduction

Discrete time dynamical systems are used to describe:

- Phenomena whose events are defined (observed) in discrete time instants only (e.g. once a year, once a day, once a second, …) like, e.g., social end economic studies
- The sampling of continuous time signals

A simple example: bank account

Let $x(k)$ be the capital stored in the account at the generic year k

Lat $u(k)$ be the net paid up at the generic year k

Let $\eta > 0$ be the simple year interest

The capital increment at year $k+1$ is given by: $x(k+1) = (1+\eta)x(k) + u(k)$

Discrete time dynamical systems

A finite dimensional, discrete time ($k \in \mathbb{Z}^+$), dynamical system can be described through a state space representation made up by:

- a system of nonlinear finite difference equations
- a static output equation

$$\begin{cases} x(k+1) = f(k, x(k), u(k)) \rightarrow \textbf{state equation} \\ y(k) = g(k, x(k), u(k)) \rightarrow \textbf{output equation} \end{cases}$$

$$x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}^p, y(k) \in \mathbb{R}^q$$

Discrete time linear dynamic systems

If both $f(\cdot)$ and $g(\cdot)$ are linear functions in both the arguments $x(k)$ and $u(k)$, the system becomes **linear**

$$x(k+1) = A(k)x(k) + B(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$
$$A(k) \in \mathbb{R}^{n,n} \quad B(k) \in \mathbb{R}^{n,p} \quad C(k) \in \mathbb{R}^{q,n} \quad D(k) \in \mathbb{R}^{q,p}$$

Moreover, if matrices $A(\cdot)$, $B(\cdot)$, $C(\cdot)$ and $D(\cdot)$ do not depend on time, the system is **linear time invariant**

$$x(k+1) = Ax(k) + Bu(k)$$
$$y(k) = Cx(k) + Du(k)$$
$$A \in \mathbb{R}^{n,n} \quad B \in \mathbb{R}^{n,p} \quad C \in \mathbb{R}^{q,n} \quad D \in \mathbb{R}^{q,p}$$

Solution of discrete time LTI dynamic systems

Consider the state space description:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \quad \text{given } u(k), x(0)$$

$$\downarrow$$

$$\text{compute } x(k), y(k), \forall k \geq 0$$

The solution can be compute using the time domain iterative relations:

$$x(k) = \underbrace{A^k x(0)}_{x_{zi}(k)} + \underbrace{\sum_{i=0}^{k-1} A^{k-i-1} Bu(i)}_{x_{zs}(k)} = x_{zi}(k) + x_{zs}(k)$$

$$y(k) = \underbrace{CA^k x(0)}_{y_{zi}(k)} + \underbrace{C\sum_{i=0}^{k-1} A^{k-i-1} Bu(i) + Du(k)}_{y_{zs}(k)} = y_{zi}(k) + y_{zs}(k)$$

The $\mathcal{Z}$-transform

Analysis and solution of discrete time dynamical systems can be effectively performed using the **$\mathcal{Z}$-transform** which, for a generic discrete time sequence $f(kT)$, is defined as:

$$F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}$$

The $\mathcal{Z}$-transform is the discrete-time signals counterpart of the Laplace transform for analog signals

An exhaustive development of the $\mathcal{Z}$-transform theory (e.g. existence, convergence, uniqueness,… properties) is outside the scopes of an Automatic Control course

Thus, basic properties of the $\mathcal{Z}$-transform only will be introduced

$\mathcal{Z}$-transform properties

	Theorem	Name
1.	$z\{af(t)\} = aF(z)$	Linearity theorem
2.	$z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$	Linearity theorem
3.	$z\{e^{-aT} f(t)\} = F(e^{aT} z)$	Complex differentiation
4.	$z\{f(t-nT)\} = z^{-n}F(z)$	Real translation
5.	$z\{tf(t)\} = -Tz\dfrac{dF(z)}{dz}$	Complex differentiation
6.	$f(0) = \lim\limits_{z\to\infty} F(z)$	Initial value theorem
7.	$f(\infty) = \lim\limits_{z\to 1}(1-z^{-1})F(z)$	Final value theorem

Note: kT may be substituted for t in the table.

Some $\mathcal{Z}$-transform pairs

$f(k)$	$F(z)$
a^k	$\dfrac{z}{z-a}$
$\dbinom{k}{\ell} a^{k-\ell}, \ell > 0$	$\dfrac{z}{(z-a)^{\ell+1}}$
$\sin(\vartheta k), \vartheta \in \mathbb{R}$	$\dfrac{z\sin(\vartheta)}{z^2 - 2\cos(\vartheta)z + 1}$
$\cos(\vartheta k), \vartheta \in \mathbb{R}$	$\dfrac{z(z-\cos(\vartheta))}{z^2 - 2\cos(\vartheta)z + 1}$
$A^k, A \in \mathbb{R}^{n,n}$	$z(zI-A)^{-1}$

$\mathcal{Z}$-domain solution of discrete time LTI systems

Considering the $\mathcal{Z}$-transform of the state space description:

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{cases} \xrightarrow{\mathcal{Z}} \begin{cases} zX(z) - zx(0) = AX(z) + BU(z) \\ Y(z) = CX(z) + DU(z) \end{cases}$$

we obtain:

$$X(z) = \underbrace{z(zI-A)^{-1}x(0)}_{X_{zi}(z)} + \underbrace{(zI-A)^{-1}BU(z)}_{X_{zs}(z)} = X_{zi}(z) + X_{zs}(z)$$

$$Y(z) = \underbrace{Cz(zI-A)^{-1}x(0)}_{Y_{zi}(z)} + \underbrace{[\overbrace{C(zI-A)^{-1}B + D}^{H(z)}]U(z)}_{Y_{zs}(z)} = Y_{zi}(z) + Y_{zs}(z)$$

$$H(z) = C(zI-A)^{-1}B + D \rightarrow \textbf{transfer function}$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

All the $\mathcal{Z}$-transform expressions of the solution of discrete time LTI system are made up by real rational functions

The inverse $\mathcal{Z}$-transform can thus be performed by suitably using the PFE procedure:

$$F(z) = \frac{z}{(z-0.5)(z-0.4)} = \frac{5}{z-0.5} - \frac{4}{z-0.4}$$

$$\mathcal{Z}^{-1}\left\{\frac{5}{z-0.5}\right\} = ??$$

$$\mathcal{Z}^{-1}\left\{\frac{-4}{z-0.4}\right\} = ??$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

In order to apply the PFE procedure to compute the inverse $\mathcal{Z}$-transform, a preliminary step is needed:

$$\tilde{F}(z) = \frac{F(z)}{z}$$

$$F(z) = \frac{z}{(z-0.5)(z-0.4)}$$

$$\tilde{F}(z) = \frac{F(z)}{z} = \frac{1}{(z-0.5)(z-0.4)} = \frac{10}{z-0.5} - \frac{10}{z-0.4}$$

$$F(z) = z \cdot \tilde{F}(z) = \frac{10z}{z-0.5} - \frac{10z}{z-0.4}$$

$$f(k) = \mathcal{Z}^{-1}\{F(z)\} = \mathcal{Z}^{-1}\left\{\frac{10z}{z-0.5} - \frac{10z}{z-0.4}\right\} = \left(10 \cdot 0.5^k - 10 \cdot 0.4^k\right)\varepsilon(k)$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

Inverse $\mathcal{Z}$-transform, in the presence of a couple of complex conjugate roots

Compute the PFE as

$$F(z) = \frac{Rz}{z-\lambda} + \frac{R^* z}{z-\lambda^*}, \lambda = \sigma + j\omega = \nu e^{j\theta}$$

then:

$$f(k) = \mathcal{Z}^{-1}\left\{\frac{Rz}{z-\lambda} + \frac{R^* z}{z-\lambda^*}\right\} = 2\,|R|\,\nu^k \cos\left(\theta k + \angle R\right)$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

Example:

$$x(k+1) = \begin{bmatrix} 3 & 0 \\ -3.5 & -0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(k)$$

Compute $x_{zi}(k)$ and $x(k)$ when:

$$u(k) = 2\varepsilon(k), x(0) = \begin{bmatrix} 1 & -2 \end{bmatrix}^T$$

$$X(z) = z\left(zI - A\right)^{-1} x(0) + \left(zI - A\right)^{-1} BU(z)$$

$$(zI - A)^{-1} = \left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ -3.5 & -0.5 \end{bmatrix}\right]^{-1} = \begin{bmatrix} \frac{1}{z-3} & 0 \\ \frac{-3.5}{(z-3)(z+0.5)} & \frac{1}{z+0.5} \end{bmatrix}$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

$$X(z) = z\left(zI - A\right)^{-1} x(0) + \left(zI - A\right)^{-1} BU(z)$$

$$X_{zi}(z) = (zI - A)^{-1}x(0) = z\begin{bmatrix} \dfrac{1}{z-3} & 0 \\ \dfrac{-3.5}{(z-3)(z+0.5)} & \dfrac{1}{z+0.5} \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = z\begin{bmatrix} \dfrac{1}{z-3} \\ \dfrac{-2z+2.5}{(z-3)(z+0.5)} \end{bmatrix}$$

$$x_{zi}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 3^k \\ -3^k - (-0.5)^k \end{bmatrix}\varepsilon(k)$$

$\mathcal{Z}$-domain solution of discrete time LTI systems

$$X_{zs}(z) = (zI - A)^{-1}BU(z) = \begin{bmatrix} \dfrac{1}{z-3} & 0 \\ \dfrac{-3.5}{(z-3)(z+0.5)} & \dfrac{1}{z+0.5} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\frac{2z}{z-1} = z\begin{bmatrix} \dfrac{2}{(z-3)(z-1)} \\ \dfrac{4z-19}{(z-3)(z+0.5)(z-1)} \end{bmatrix}$$

$$x_{zs}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 3^k - 1 \\ -3^k - 4\cdot(-0.5)^k + 5 \end{bmatrix}\varepsilon(k)$$

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} 2\cdot 3^k - 1 \\ -2\cdot 3^k - 5\cdot(-0.5)^k + 5 \end{bmatrix}\varepsilon(k)$$

Discrete-time LTI systems transfer function

The transfer function of a discrete time LTI system is expressed as a real rational function of the form:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_m z^m + b_{m-1}z^{m-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0}, m \le n$$

Let's divide both numerator and denominator by z^n:

$$H(z) = \frac{b_m z^{m-n} + b_{m-1}z^{m-n-1} + \ldots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0 z^{-n}}$$

The $\mathcal{Z}$-transform of $Y(z)$ is given by

$$Y(z) = -a_{n-1}z^{-1}Y(z) - a_{n-2}z^{-2}Y(z) - \ldots - a_0 z^{-n}Y(z) + \\ + b_m z^{m-n}U(z) + b_{m-1}z^{m-1-n}U(z) + \ldots + b_0 z^{-n}U(z)$$

Discrete-time LTI systems transfer function

$$Y(z) = -a_{n-1}z^{-1}Y(z) - a_{n-2}z^{-2}Y(z) - \ldots - a_0 z^{-n}Y(z) + \\ + b_m z^{m-n}U(z) + b_{m-1}z^{m-1-n}U(z) + \ldots + b_0 z^{-n}U(z)$$

Recalling that

$$\mathcal{Z}^{-1}\{z^{-\ell}F(z)\} = f(k-\ell)$$

we can express the current output $y(k)$ by means of the finite difference equation

$$y(k) = -a_{n-1}y(k-1) - a_{n-2}y(k-2) - \ldots - a_0 y(k-n) + \\ + b_m u(k-n+m) + b_{m-1}u(k-n+m-1) + \ldots + b_0 u(k-n)$$

The current output $y(k)$ of a discrete time LTI system can be recursively computed through the values of the input and output signals at the previous sampling instants.

Example

Consider a discrete-time LTI system described by a difference equation:

$$y(k) = -a_1 y(k-1) - a_0 y(k-2) + b_1 u(k) + b_0 u(k-1)$$

Applying the $\mathcal{Z}$-transform:

$$Y(z) = \left(-a_1 z^{-1} - a_0 z^{-2}\right) Y(z) + \left(b_1 + b_0 z^{-1}\right) U(z)$$

we obtain the system transfer function:

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_1 + b_0 z^{-1}}{1 + a_1 z^{-1} + a_0 z^{-2}}$$

Discrete time transfer functions

Definition of a discrete-time transfer function with MatLab:

$$H(z) = \frac{N(z)}{D(z)} = \frac{1}{z^2 - 1.7\,z + 0.72} = \frac{1}{(z-0.8)(z-0.9)}$$

```
>> T=1
>> z=tf('z',T)
>> H=1/(z^2 - 1.7*z + 0.72)
```


Natural modes and modal analysis of discrete time LTI systems

Natural modes of discrete time LTI systems

The natural modes of the LTI system

$$x(k+1) = Ax(k) + Bu(k)$$

associated with the i^{th} distinct eigenvalue λ_i ($i = 1,\ldots,r$) with minimal polynomial multiplicity μ'_i, are given by the following functions $m_{ij}(k)$ ($i = 1,\ldots,r$, $j = 1,\ldots,\mu'_i$)

$$m_{i,0}(k) = \lambda_i^k, m_{i,1}(k) = k\lambda_i^{k-1}, \ldots, m_{i,\mu'_i}(k) = \binom{k}{\mu'_i - 1} \lambda_i^{k-\mu'_i+1}$$

$$\binom{k}{\mu'_i - 1} \lambda_i^{k-\mu'_i+1} = \frac{k(k-1)\cdots(k-\mu'-2)}{(\mu_i'-1)!} \lambda_i^{k-\mu'_i+1}$$

Natural modes of discrete time LTI systems

The natural modes of the LTI system

$$x(k+1) = Ax(k) + Bu(k)$$

associated with a couple of complex conjugate eigenvalues of the form $\lambda = \sigma_0 \pm j\omega_0 = \nu e^{\pm j\theta}$ having minimal polynomialmultipl icity μ', are given by the following functions $m_j(k)$ ($j = 1,\ldots,\mu'$)

$$m_0(k) = \begin{cases} \nu^k \cos(\theta k) \\ \nu^k \sin(\theta k) \end{cases}, m_1(k) = \begin{cases} k\nu^{k-1} \cos(\theta(k-1)) \\ k\nu^{k-1} \sin(\theta(k-1)) \end{cases}, \ldots$$

$$\ldots, m_{\mu'}(k) = \begin{cases} \begin{pmatrix} k \\ \mu'-1 \end{pmatrix} \nu^{k-\mu'+1} \cos(\theta(k-\mu'+1)) \\ \begin{pmatrix} k \\ \mu'-1 \end{pmatrix} \nu^{k-\mu'+1} \sin(\theta(k-\mu'+1)) \end{cases}$$

Modal analysis of discrete time LTI systems

The natural mode λ^k, associated with the eigenvalue $\lambda \in \mathbb{R}$ having unitary minimal polynomial multiplicity is:

- **Geometrically convergent** if $|\lambda| < 1$ (Example: 0.5^k, $(-0.5)^k$)
- **Bounded** if $|\lambda| = 1$ (Example: $1^k = 1$, $(-1)^k$)
- **Geometrically divergent** if $|\lambda| > 1$ (Example 2^k, $(-2)^k$)

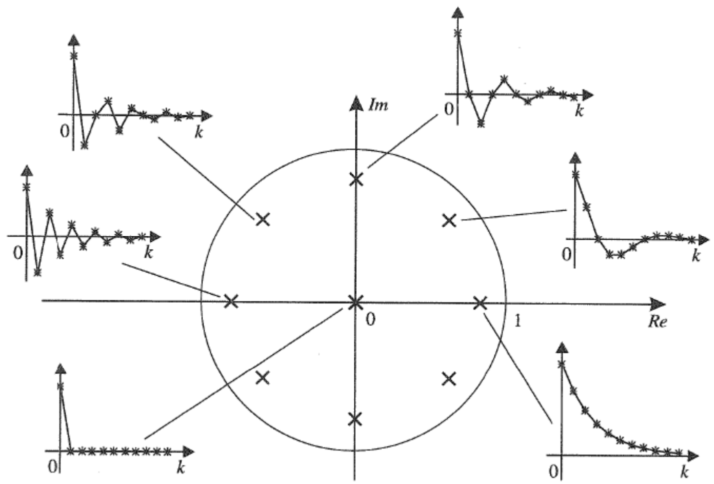
Note that, if $\mathbb{R}e(\lambda) < 0$, the corresponding mode gives rise to a samples sequence (**alternate mode**) whose sign changes at every sample time

Modal analysis of discrete time LTI systems

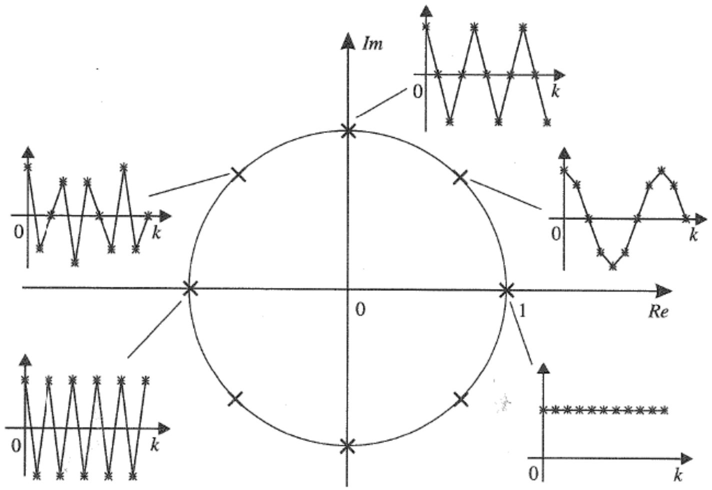
The natural modes of the form $\nu^k\cos(\theta k)$, $\nu^k\sin(\theta k)$, associated with the eigenvalue $\lambda = \sigma \pm j\omega = \nu e^{\pm j\theta} \in \mathbb{C}$ having unitary minimal polynomial multiplicity are:

- **Geometrically convergent** if $|\lambda| < \nu < 1$ (Example $0.5^k\sin(k)$)
- **Bounded (oscillating)** if $|\lambda| = \nu = 1$, $\text{Arg}(\lambda) = \theta \neq 0$ (Example $\sin(5k)$)
- **Geometrically divergent** if $|\lambda| = \nu > 1$ (Example $1.5^k\sin(k)$)

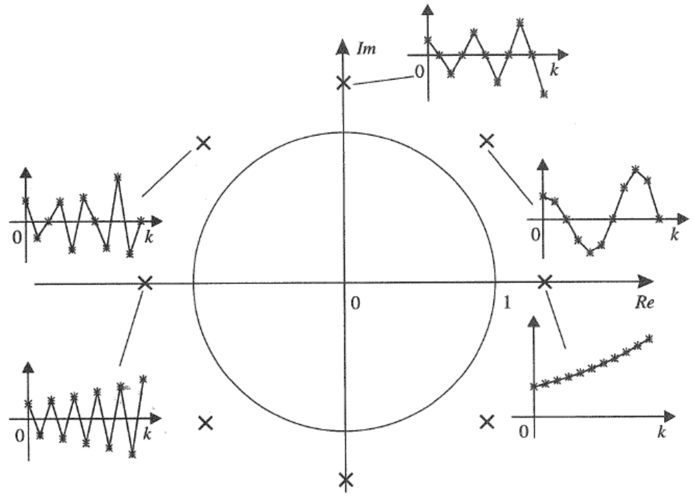
Modal analysis of discrete time LTI systems



Modal analysis of discrete time LTI systems



Modal analysis of discrete time LTI systems



Modal analysis of discrete time LTI systems

The μ' natural modes of the form $k(k-1)..(k-\mu'-2)\lambda^{k-\mu'+1}$, …, $k\lambda^{k-1}$, associated with the eigenvalue $\lambda \in \mathbb{R}$ having unitary minimal polynomial multiplicity μ' are:

- **Geometrically convergent** if $|\lambda| < 1$ (Example: $k \cdot 0.5^{k-1}$, $k \cdot (-0.5)^{k-1}$)
- **Polynomially divergent** if $|\lambda| = 1$ (Example: $k\,1^{k-1} = k$)
- **Geometrically divergent** if $|\lambda| > 1$ (Examples: $k \cdot 1.5^{k-1}$, $k \cdot (-1.5)^{k-1}$)

Note that, if $\mathbb{R}\mathrm{e}(\lambda) < 0$, the corresponding mode gives rise to a samples sequence (**alternate mode**) whose sign changes at every sample time

Modal analysis of discrete time LTI systems

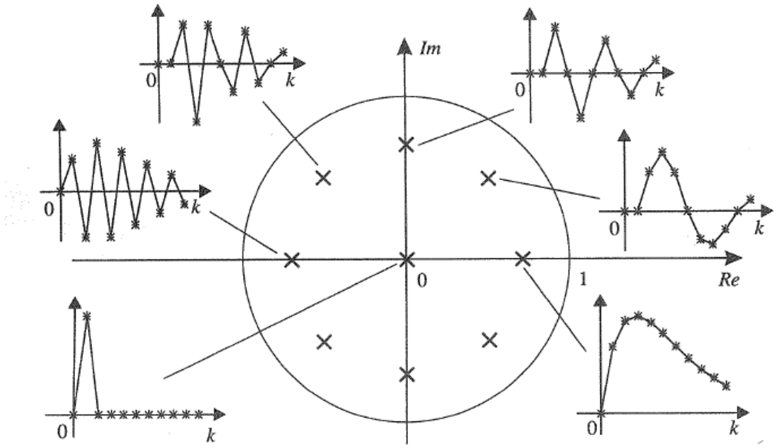
The natural modes of the form

$$\binom{k}{\mu'-1}\nu^{k-\mu'+1}\cos(\theta(k-\mu'+1)),\binom{k}{\mu'-1}\nu^{k-\mu'+1}\sin(\theta(k-\mu'+1))$$

associated with the eigenvalue $\lambda = \sigma \pm j\omega = \nu e^{\pm j\theta} \in \mathbb{C}$ having minimal polynomial multiplicity μ' are:

- **Geometrically convergent** if $|\lambda| = \nu < 1$ (Example $k\,0.5^{k-1}\sin(k-1)$)
- **Polynomially divergent** if $|\lambda| = \nu = 1$, $\mathrm{Arg}(\lambda) = \theta \neq 0$ (Example: $k\sin(5(k-1))$)
- **Geometrically divergent** if $|\lambda| = \nu > 1$ (Example: $k\,1.5^{k-1}\sin(k-1)1.5^{k}\sin(k)$)

Modal analysis of discrete time LTI systems



Modal analysis: synthetic resume

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A then

- The natural mode associated with eigenvalue λ_i is **bounded** if:

  $|\lambda_i(A)| = 1$ and $\mu'(\lambda_i(A)) = 1$

- The natural mode associated with eigenvalue λ_i is **convergent** if:

  $|\lambda_i(A)| < 1$

- The natural mode associated with eigenvalue λ_i is **divergent** if:

  $|\lambda_i(A)| > 1$ OR $|\lambda_i(A)| = 1$ and $\mu'(\lambda_j(A)) > 1$

Stability of discrete time dynamical systems

Internal stability of discrete time LTI systems

Denote with $\lambda_i(A), i = 1, \ldots, n$ the i^{th} eigenvalue of matrix A then

Result (Internal stability of discrete time LTI systems)

- A discrete time LTI system is **internally stable** if and only if:

  $|\lambda_i(A)| \leq 1,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) = 1$ for all the eigenvalues such that $|\lambda_i(A)| = 1$ ($\mu'(\cdot)$ is the minimal polynomial multiplicity)

- A discrete time LTI system is **asymptotically stable** if and only if:

  $|\lambda_i(A)| < 1,\ i = 1, \ldots, n$

- A discrete time LTI system is **unstable** if and only if:

  $\exists i: |\lambda_i(A)| > 1$ OR $|\lambda_i(A)| \leq 1,\ i = 1, \ldots, n$ and $\mu'(\lambda_j(A)) > 1$ for some j such that $|\lambda_j(A)| = 1$ ($\mu'(\cdot)$ is the minimal polynomial multiplicity)

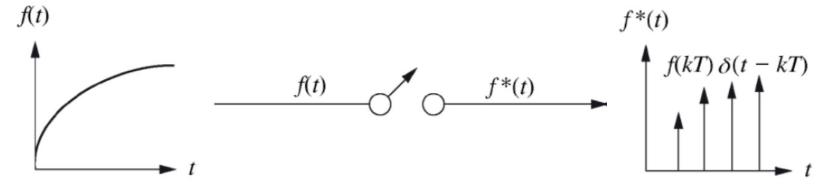
BIBO stability of discrete time LTI systems

Result (BIBO stability of LTI system)
A discrete-time LTI system is **BIBO stable** if and only if all the poles of its transfer function $H(z)$ lie strictly inside the unit circle:

$$|p_i| < 1,\ i=1,\dots,n\ ,\ \text{where } p_i \text{ are the poles of } H(z)$$

Relationship between $\mathcal{L}$-transform and $\mathcal{Z}$-transform

Consider a signal $f(t)$ ideally sampled with uniform sampling period T



the sampled signal $f^*(t)$ can be represented as a continuous time signal as:

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

Relationship between $\mathcal{L}$-transform and $\mathcal{Z}$-transform

$$f^*(t) = \sum_{k=0}^{\infty} f(kT)\delta(t-kT)$$

the Laplace of $f^*(t)$ is given by:

$$F^*(s) = \sum_{k=0}^{\infty} f(kT)e^{-kTs} = \sum_{k=0}^{\infty} f(kT)(e^{Ts})^{-k}$$

Let $z = e^{Ts}$, then:

$$F^*(s) = F(z)\big|_{z=e^{Ts}}$$

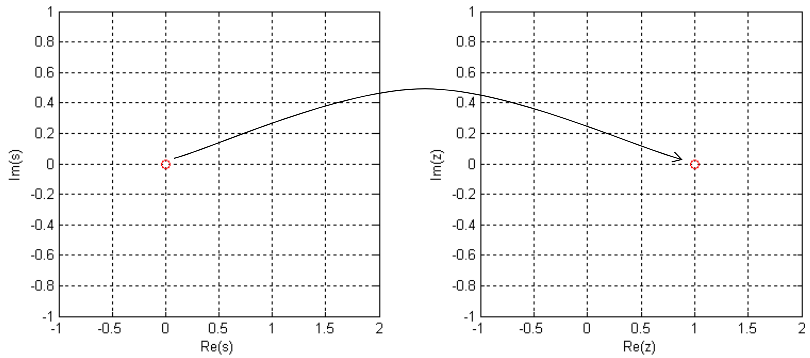
moreover: $z = e^{Ts} \Rightarrow s = 1/T \log(z)$

$$F^*(s)\big|_{s=1/T\log(z)} = \sum_{k=0}^{\infty} f(kT)(e^{Ts})^{-k} = \sum_{k=0}^{\infty} f(kT)z^{-k} = F(z)$$

The mapping $z = e^{Ts}$ is referred to as the **sampling transformation**

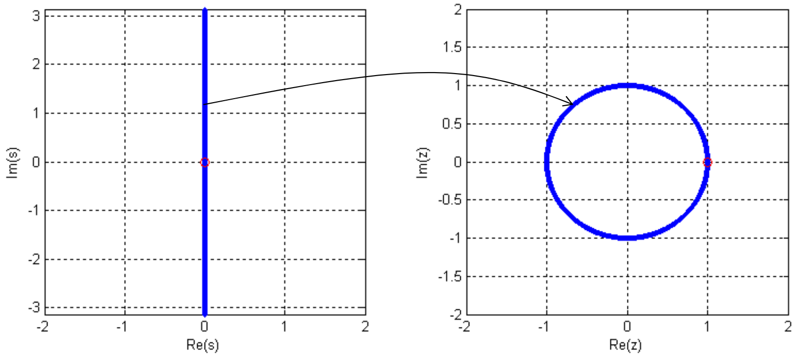
$\mathcal{Z}$-transform: mapping s-plane → z-plane

Axes origin: $\quad s = 0 \quad \rightarrow \quad z = e^{sT} = 1$



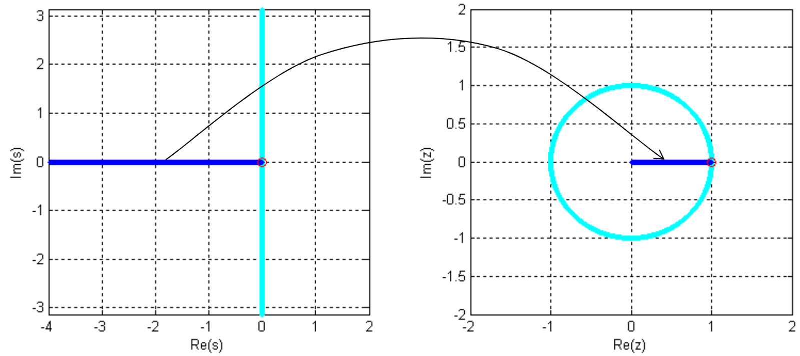
$\mathcal{Z}$-transform: mapping s-plane → z-plane

Imaginary axis: $s = j\omega \quad \rightarrow \quad z = e^{j\omega T} = \cos(\omega T) + j\sin(\omega T)$



$\mathcal{Z}$-transform: mapping s-plane → z-plane

Negative real axis: $s = \sigma < 0 \quad \rightarrow \quad 0 < z = e^{\sigma T} < 1$



$\mathcal{Z}$-transform: mapping s-plane → z-plane

Left half-plane:

$s = \sigma + j\omega,\ \sigma < 0 \quad \rightarrow \quad z = re^{j\omega T} = r\cos(\omega T) + jr\sin(\omega T),\ r = e^{\sigma T} < 1$