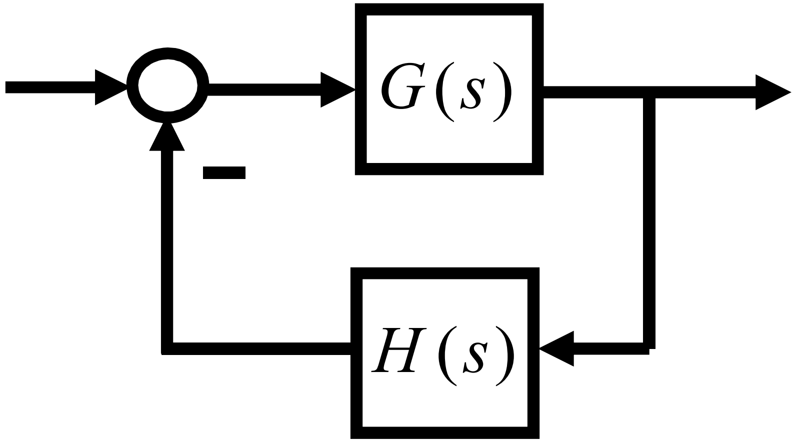
Chapter 5

Time-Domain Analysis of Control Systems (3)

5-5 Routh's Stability Criterion

Consider the following typical closed-loop system:



$$\frac{C(s)}{R(s)}=\frac{G(s)}{1+G(s)H(s)}$$

which can be written as

$$\frac{C(s)}{R(s)}=\frac{b_0 s^m+b_1 s^{m-1}+\cdots+b_m}{s^n+a_1 s^{n-1}+\cdots+a_n}=\frac{N(s)}{D(s)}$$

where a_i's and b_i's are constants and $m \le n$.

Routh's stability criterion enables us to determine **the number of closed-loop poles that lie in the right-half plane** without having to factor the denominator polynomial.

1. Routh's stability criterion

The procedure in *Routh's stability criterion* is as follows:

1). Write the polynomial in s in the following form:

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

where we assume that $a_n \neq 0$; that is, any zero root has been removed.

2) Ascertain that all the coefficients are positive (or negative). Otherwise, the system is unstable. If we are interested in only the stability, there is no need to follow the procedure further.

3) If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern (called *Routh Array*):

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

The evaluation of b_i's is continue until the remaining ones are all zero.

The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating c_i's, d_i's and so on:

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Routh Array:

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\
s^0 & g_1 = a_n & & & &
\end{array}$$

$$b_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

This process is continued until the nth row has been completed:

Routh Array:

$$
\begin{array}{c|c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
s^2 & e_1 & e_2 & & & \cdots \\
s^1 & f_1 & & & & \\
s^0 & g_1 = a_n & & & &
\end{array}
$$

Routh's stability criterion

1. The system is stable **if and only if** (1) all the coefficients of $D(s)$ are positive and (2) all the terms in the first column of the array have positive signs.
2. The number of roots of the D(s) with **positive** real parts is equal to the number of **changes in sign of the coefficients of the first column** of the array.

Example. Consider the following polynomial

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Determine the number of unstable roots.

Example. The characteristic polynomial of a second-order system is

$$D(s) = a_0 s^2 + a_1 s + a_2$$

The Routh array is written as

$$\begin{array}{lll} s^2 & a_0 & a_2 \\ s^1 & a_1 & \\ s^0 & a_2 & \end{array}$$

Therefore, the requirement for a stable second-order system is simply that all the coefficients be positive or all the coefficients be negative.

Example. The characteristic polynomial of a third-order system is

$$D(s) = a_0 s^3 + a_1 s^2 + a_2 s + a_3$$

The Routh array is

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & \\ s^0 & c_1 & \end{array}$$

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad c_1 = \frac{b_1 a_3}{b_1} = a_3$$

For third-order system to be stable, it is necessary and sufficient that the coefficients be positive and

$$a_1 a_2 - a_0 a_3 > 0$$

2. Special Cases

If one of the following cases occurs, a suitable modification of the array calculation procedure must be made.

(1) (**Case 1**) If a first column term in any row is zero, but the remaining terms are not zero or there is no remaining term;

(2) (**Case 2**) All the coefficients in any derived row are zero.

Modified Routh's Stability Criterion: Case 1:

In this case, the zero is replaced by a very small positive number ε and the rest of the array is evaluated then.

Example. $D(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11
s^4	2	4	10
s^3	$0 \approx \varepsilon$	6	
s^2	c_1	10	
s	d_1		
s^0	10		

where $\varepsilon \to 0$

$$c_1 = \frac{4\varepsilon - 12}{\varepsilon} < 0$$

$$d_1 = \frac{6c_1 - 10\varepsilon}{c_1} \to 6 > 0$$

There are two changes in sign, that is, two poles lie in the right-half s-plane.

Example.

$$D(s) = s^3 + 2s^2 + s + 2$$

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \varepsilon & \\ s^0 & 2 & \end{array}$$

Questions: Is the system stable? Is the system unstable?

In this case, with $\varepsilon > 0$, the signs of the first column are the same, which indicates that there is a pair of imaginary roots. The system is marginally stable.

Modified Routh's Stability Criterion: Case 2

All the coefficients in any derived row are zero. For example,

$$D(s) = s^6 + s^5 - 2s^4 - 3s^3 - 7s^2 - 4s - 4 = 0$$

$$\begin{array}{lrrrr}
s^6 & 1 & -2 & -7 & -4 \\
s^5 & 1 & -3 & -4 & \\
s^4 & 1 & -3 & -4 & \\
s^3 & 0 & 0 & &
\end{array}$$

To solve the problem, from s^4 row we obtain an auxiliary polynomial:

$$P(s) = s^4 - 3s^2 - 4$$

which indicates that there are two roots on the $j\omega$ axis and the system is in the marginal stability case. Then, consider

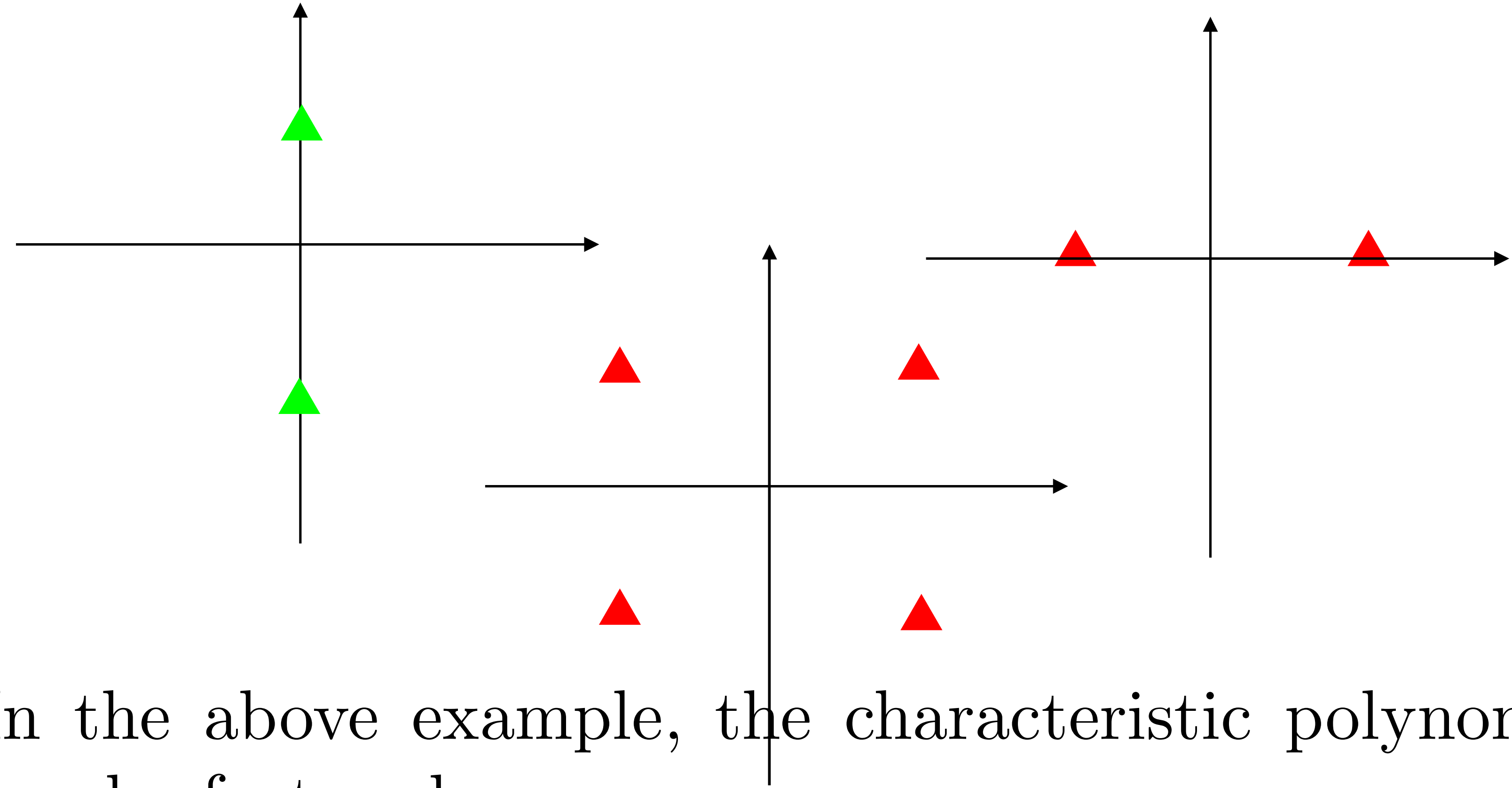
$$dP(s)/ds = 4s^3 - 6s$$

Let the term in s^3 row be replaced by $4s^3-6s$. The array becomes

s^6	1	−2	−7	−4
s^5	1	−3	−4	
s^4	1	−3	−4	
s^3	4	−6		
s^2	−1.5	−4		
s^1	−16.7	0		
s^0	−4			

There is one change in sign of the first column of the array. Hence, only one pole lies in the right-half s-plane.

In general, the terms of a row are all zero indicate that there are roots of equal magnitude lying radially opposite in the s-plane.



In the above example, the characteristic polynomial can be factored as:

$$(s+2)(s-2)(s+j)(s-j)(s+1+j\sqrt{3})(s+1-j\sqrt{3})/4$$

Example. Consider the following equation:

$$D(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

$$\begin{array}{cccc} s^5 & 1 & 24 & -25 \\ s^4 & 2 & 48 & -50 \\ s^3 & 0 & 0 & \end{array}$$

The auxiliary polynomial is then formed from the coefficients of the second row:

$$P(s) = 2s^4 + 48s^2 - 50$$

Hence,

$$dP(s)/ds = 8s^3 + 96s$$

Let the term in s^3 row be replaced by $4s^3+96s$. The array becomes

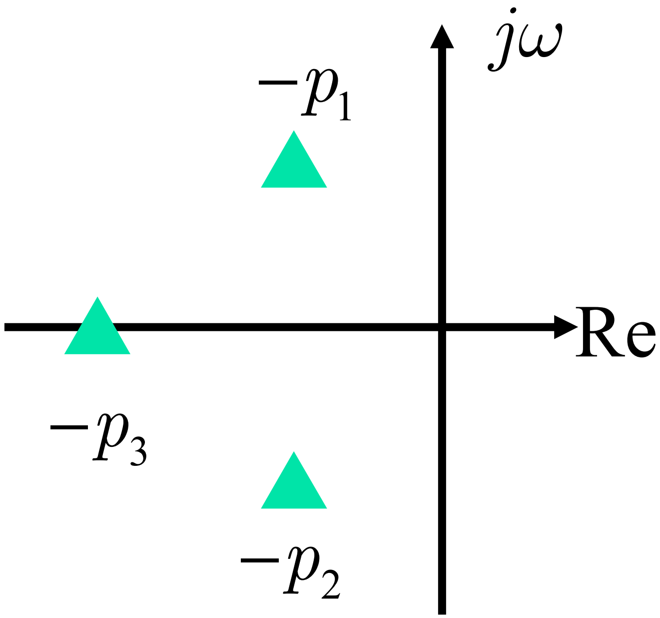
s^5	1	24	–25
s^4	2	48	–50
s^3	8	96	
s^2	24	–50	
s	112.7		
s^0	–50		

The system has one pole in the right-half s-plane and two poles on the $j\omega$ axis. In fact,

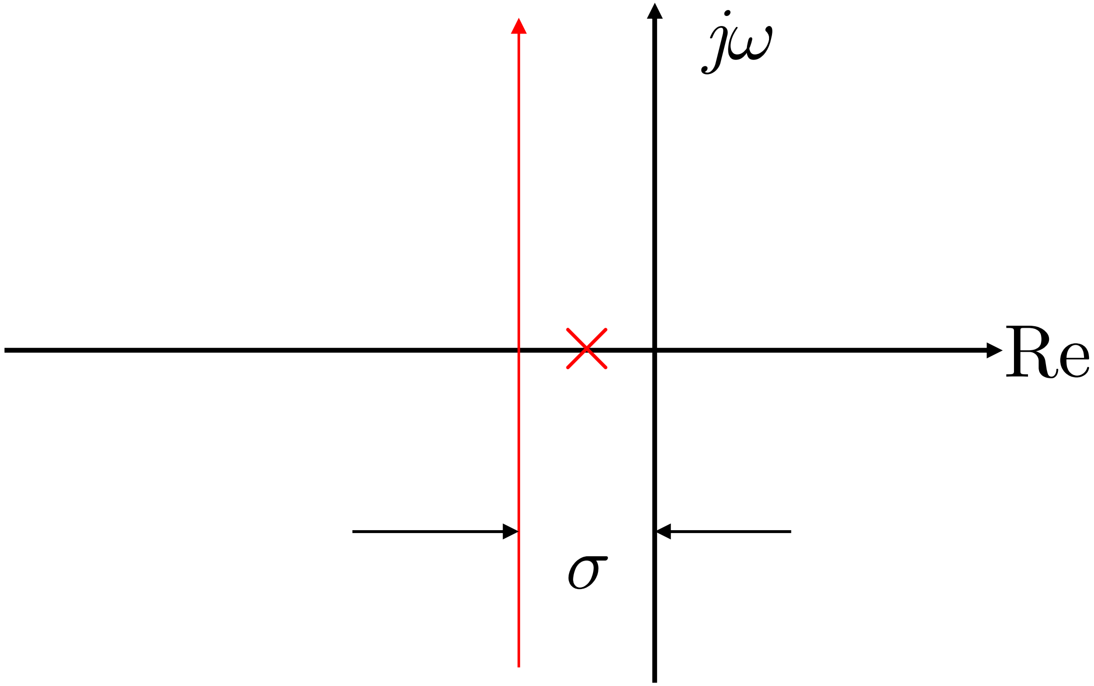
$$D(s) = (s+1)(s-1)(s+j5)(s-j5)(s+2)$$

3.Relative Stability Analysis

The relative stability of a system can be defined as the property that is measured by the relative real part of each root or pair of roots.



The root $-p_3$ is relatively more stable than the roots $-p_1$ and $-p_2$.

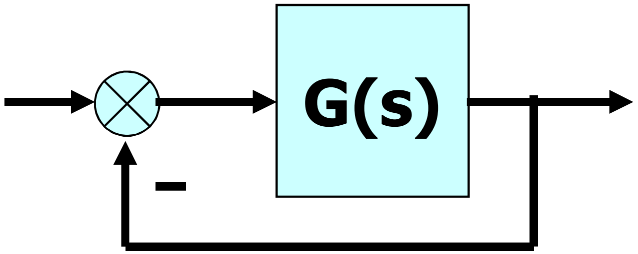


One useful approach for examining relative stability is to shift the s-plane axis as shown above. Then

$$s = \hat{s} - \sigma \Rightarrow \hat{s} = s + \sigma$$

where $\sigma > 0$ implies that to make all the poles lie in the left-half $\hat{s}$-plane, more restrictions must be made.

Example (Axis shift). Consider a unity-feedback system:



$$G(s) = \frac{K}{s(0.1s+1)(0.25s+1)}$$

Determine the range of K for which the system is stable.

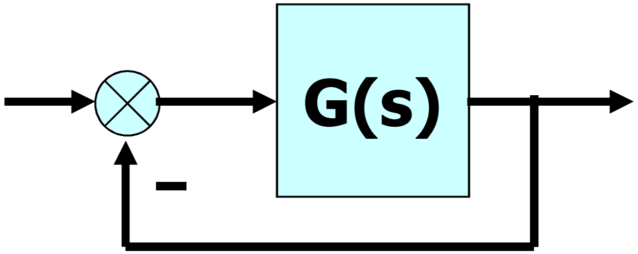
Solution: The closed-loop characteristic equation is

$$0.025s^3 + 0.35s^2 + s + K = 0$$

1) $K > 0$

2) $0.35 - 0.025K > 0 \quad \Rightarrow \quad K < 14$

The range of K for system to be stable is $\quad 0 < K < 14$



$$G(s)=\frac{K}{s(0.1s+1)(0.25s+1)}$$

Now, if it is required that **the real part of all roots be less than -1**, what is the range of K for the system to be stable?

The closed-loop characteristic equation is

$$0.025s^3+0.35s^2+s+K=0$$

Let $s=\hat{s}-1$, then

$$\hat{s}^3+11\hat{s}^2+15\hat{s}+(40K-27)=0$$

$$a_i>0,\quad K>0.675$$

$$11\times 15-(40K-27)>0$$

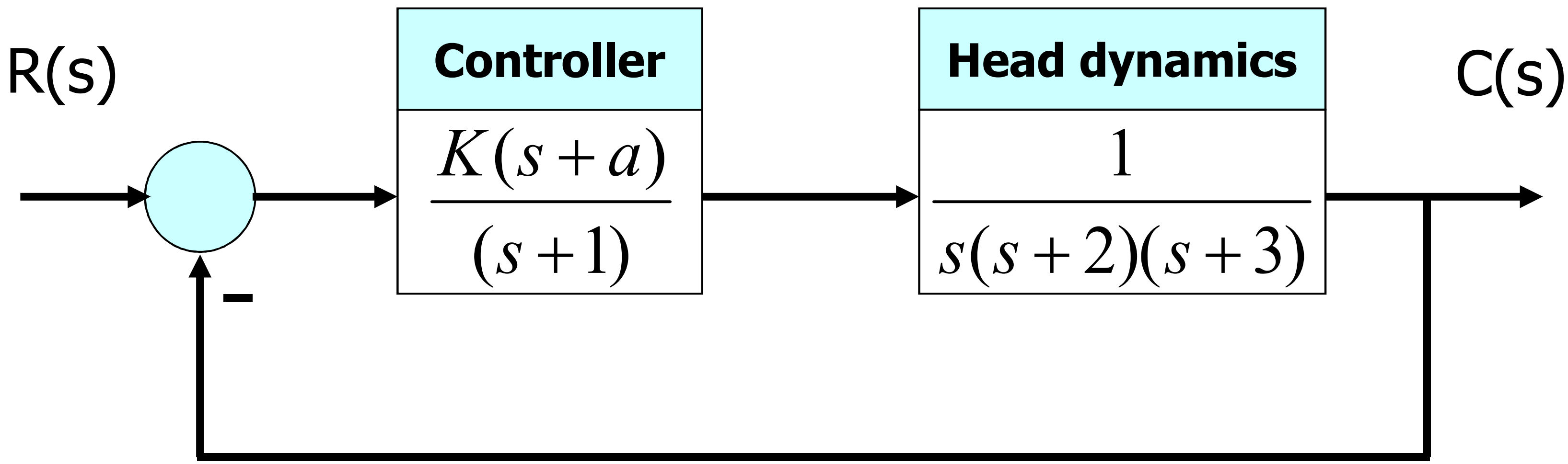
$$K<4.8$$

The range of K is

$$0.675<K<4.8$$

4. Application of Routh's stability criterion to control system analysis

Example. Welding head position control



The close-loop characteristic equation is

$$s^4 + 6s^3 + 11s^2 + (K+6)s + Ka = 0$$

where it is assumed that $a>0$ and $K>0$.

$$s^4 + 6s^3 + 11s^2 + (K+6)s + Ka = 0$$

Routh array:

s^4	1	11	Ka
s^3	6	$(K+6)$	
s^2	b_1	Ka	
s^1	c_1	Ka	
s^0	Ka		

$$b_1 = \frac{60 - K}{6}$$

$$c_1 = \frac{b_1(K+6) - 6Ka}{b_1}$$

By Routh's stability criterion,

$$b_1 > 0 \Rightarrow K < 60$$

$$c_1 > 0 \Rightarrow (60 - K)(K+6) - 36Ka > 0$$

If, for instance, K=40, then c_1>0 when a<0.639.