Chapter 6

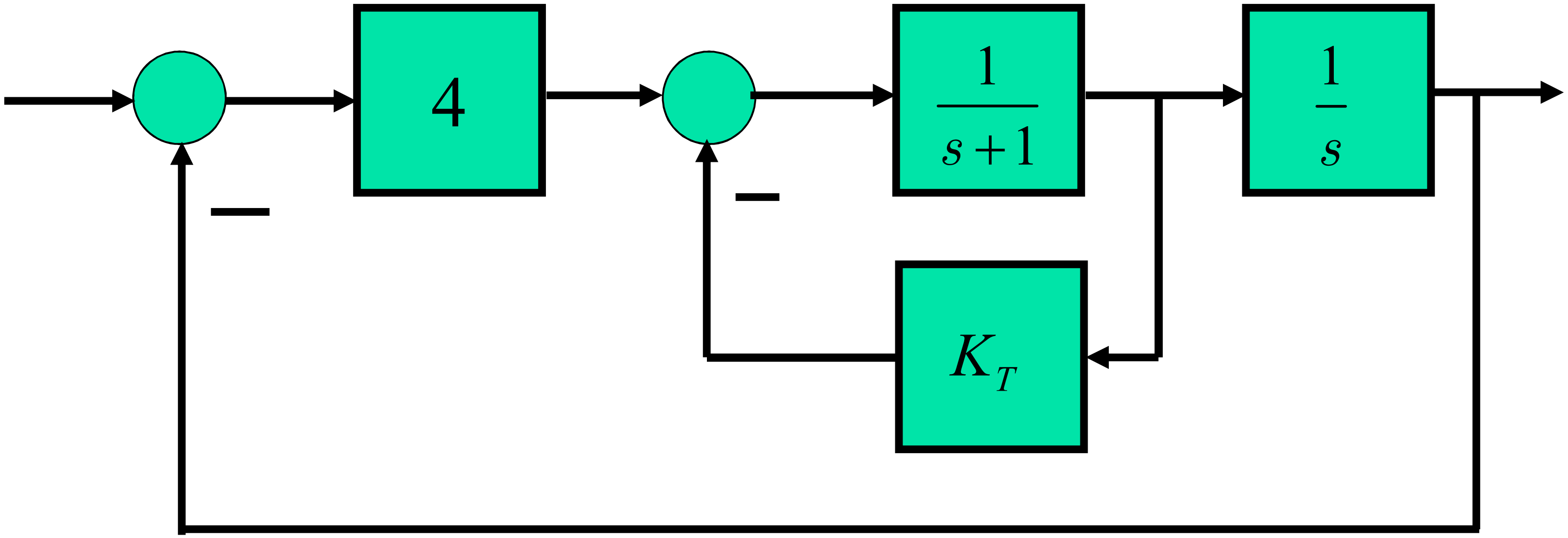
Root-Locus Analysis (2)

6-4 Extension of The Root Locus

Thus far we only considered the root loci as open loop gain K varies from zero to infinity.

In this section, we shall show that the technique introduced in Section 6-3 can be readily extended to the root loci as any system parameter varies and even the system is of positive feedback form.

1. Loci versus other parameters



The characteristic equation of the system is

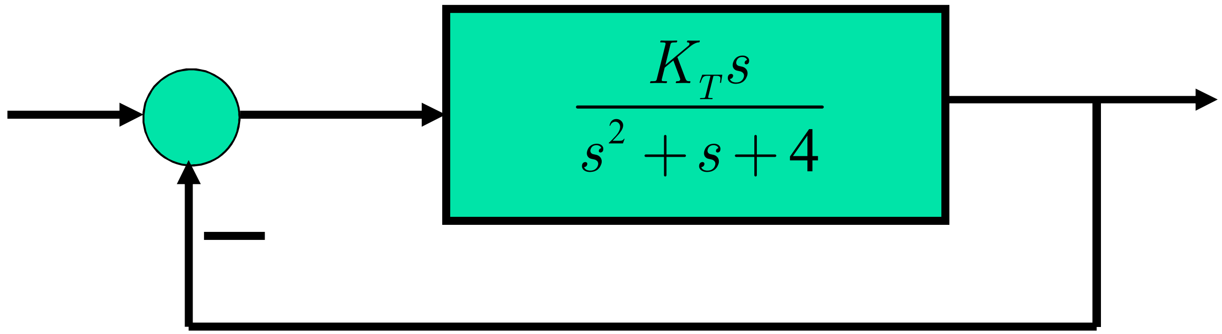
$$1+\frac{4}{s(s+1+K_T)}=0$$

which is not in the standard $1+KG(s)=0$ form.

Splitting the term with respect to K_T in the close-loop characteristic equation yields

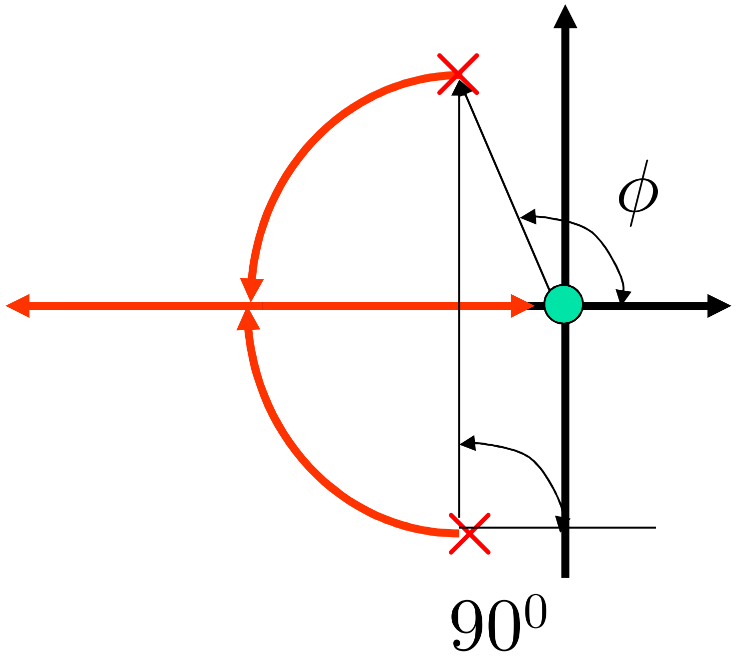
$$s^2 + s + 4 + K_T s = 0$$

from which, we can form a **new** open-loop transfer function as



The new system has **the same** closed-loop characteristic polynomial and therefore, has the same root loci as the original one.

Thus, we have an open-loop zero at $s=0$, and poles at $-1/2+j3.75$ and $-1/2-j3.75$. Following the rules introduced above, the root loci are shown below:



where the angle of departure is

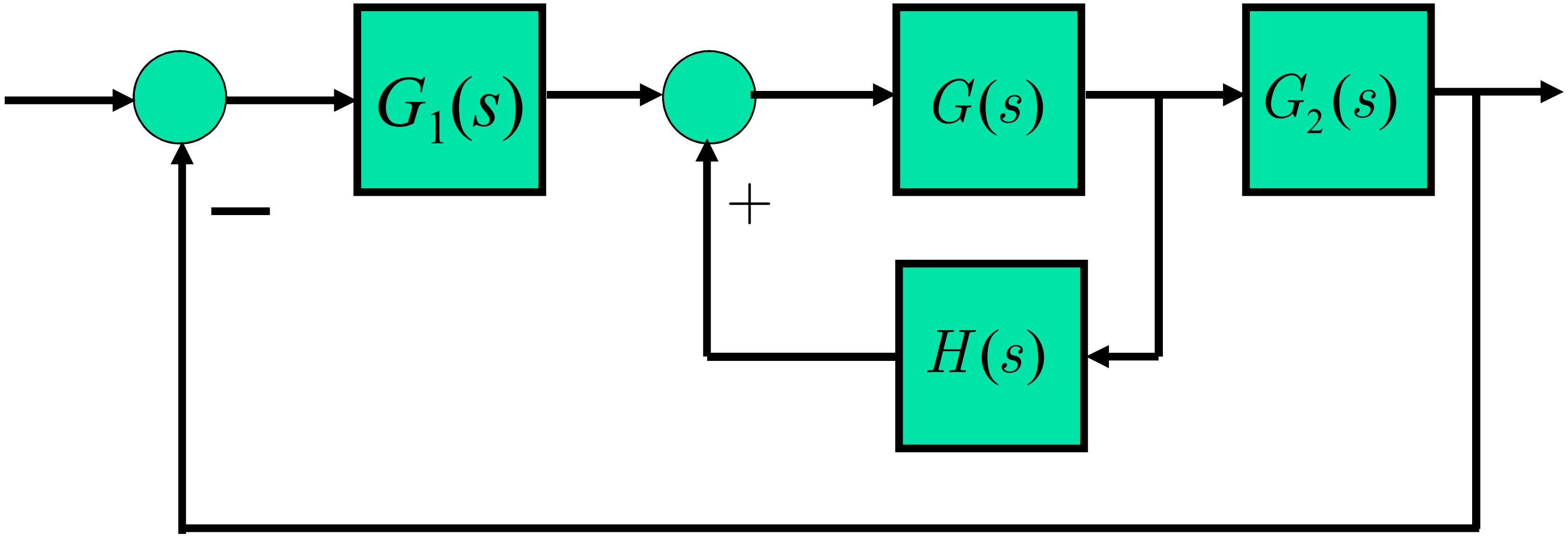
$$\theta_{p_1} \approx 180^0 - 90^0 + 96^0 = 186^0$$

The breakaway point is evaluated from

$$\frac{2d+1}{d^2+d+4} = \frac{1}{d}$$

$$\Rightarrow d_1 = -2, d_2 = 2$$

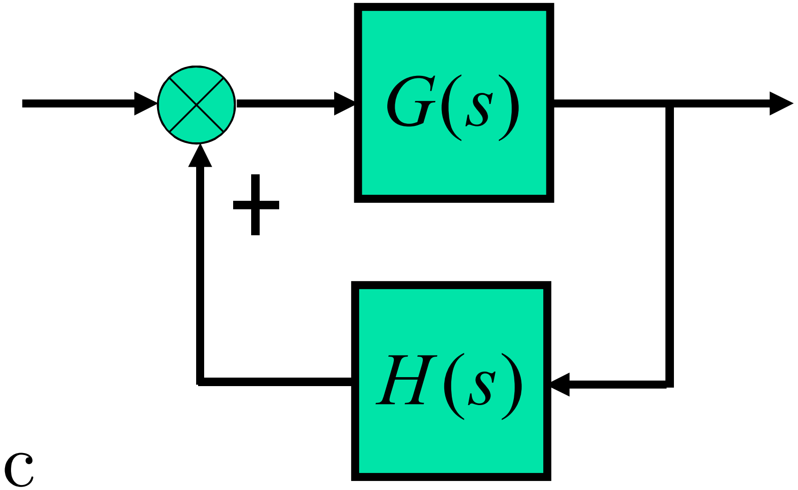
2. Root loci for positive-feedback systems: Zero-Degree Root Loci



In a complex control system, there may be a positive feedback inner loop.

The open loop transfer function

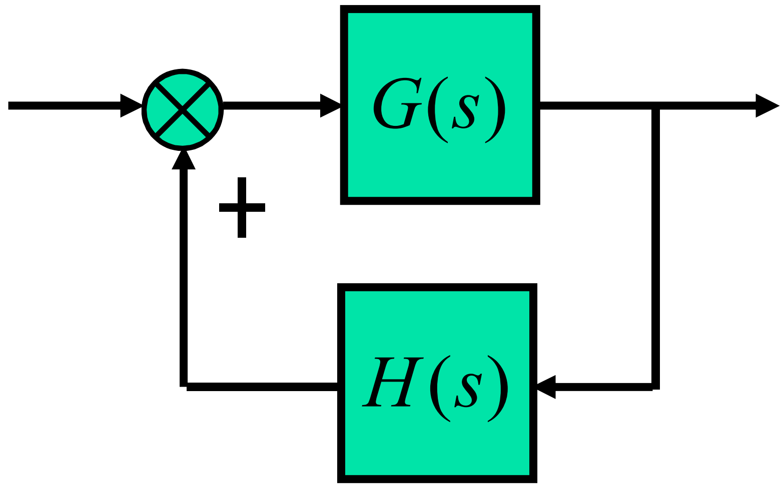
$$G(s)H(s) = K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$



The closed-loop characteristic equation is $1 - G(s)H(s) = 0$, or

$$1 - K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0 \quad \Rightarrow \quad K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 1$$

$$K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 1$$



The magnitude and angle requirements for the zero-degree root loci are

$$K^* \frac{\prod_{i=1}^{m}|s+z_i|}{\prod_{j=1}^{n}|s+p_j|} = 1$$

$$\boxed{\angle G(s)H(s) = \sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = 360^0 k, k = 0, \pm 1, \cdots}$$

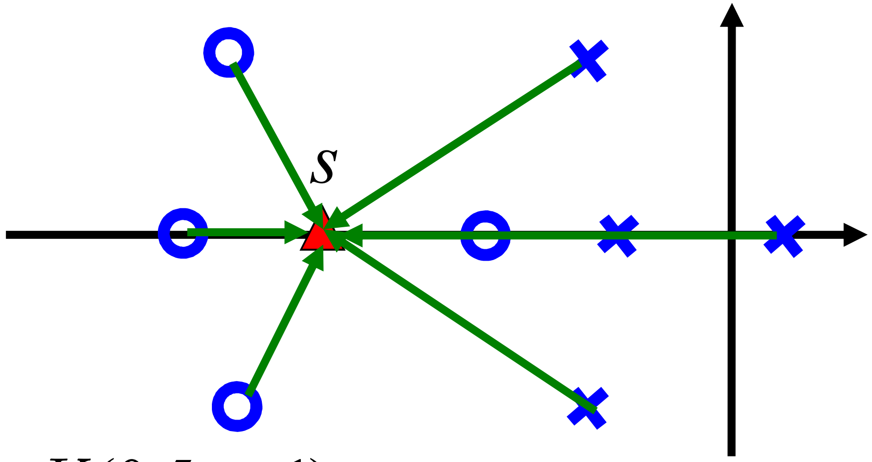
Therefore, the rules associated with the **new angle condition** must be modified and the magnitude condition remains unaltered.

- The number of separate loci is equal to the order of the characteristic equation.

- The loci are symmetrical about the real axis.

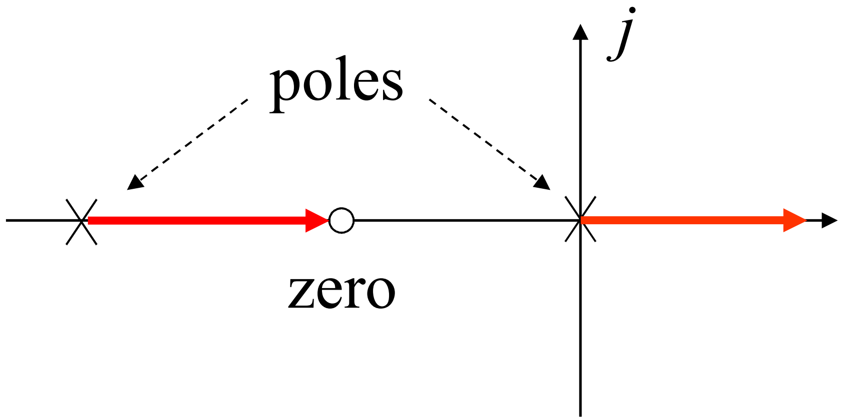
Rule 1. The root locus branches start from open-loop poles and end at open-loop zeros or zeros at infinity.

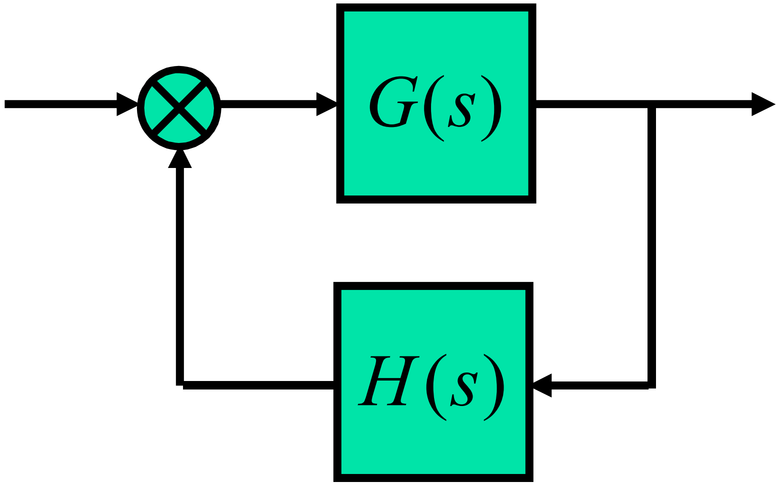
Rule 2. If the total number of real poles and real zeros to the right of a test point on the real axis is even, then the test point lies on a root locus (Modified).



Example.

$$G(s)H(s)=\frac{K(0.5s+1)}{s(0.25s+1)}$$





Rule 3. Asymptotes of root loci: The loci proceed to the zeros at infinity along Asymptotes (Modified).

The linear asymptotes intersection point on the real axis given by (remains unchanged)

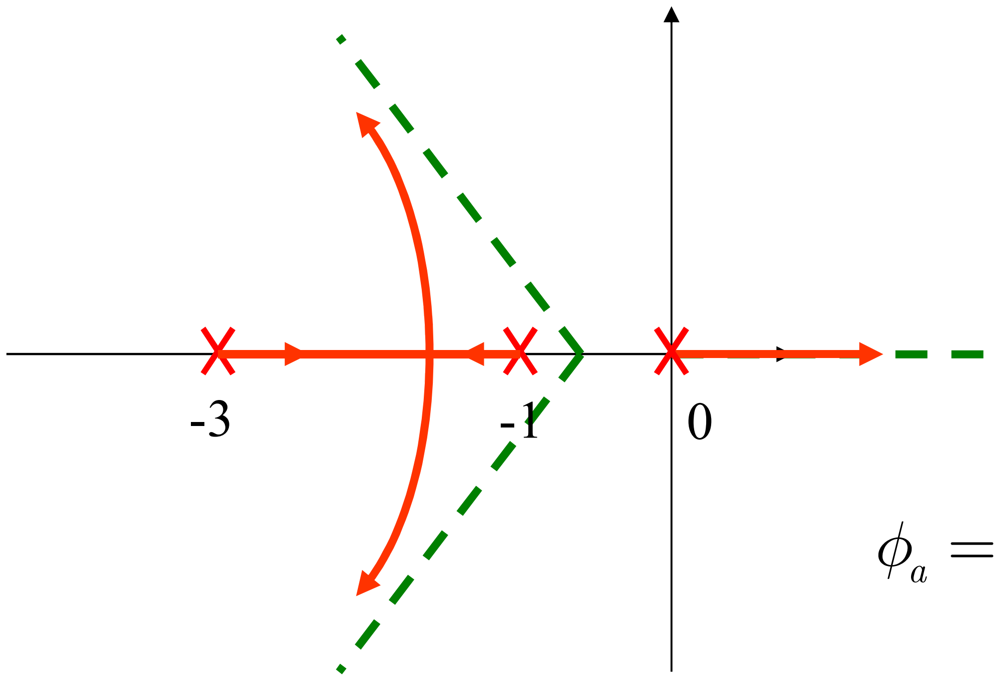
$$\boxed{\sigma_a = \frac{\sum_{j=1}^{n} -p_j - \sum_{i=1}^{m} -z_i}{n-m}}$$

The angle of the asymptotes with respect to the real axis is **modified as**

$$\boxed{\phi_a = 360^0 \times \frac{k}{n-m} \qquad (k = 0, \pm 1, \cdots)}$$

Example. The open-loop transfer function of a single loop positive feedback control system is

$$G(s) = \frac{K^*}{s(s+1)(s+3)} \qquad \left(1 - G(s) = 1 - \frac{K^*}{s(s+1)(s+3)}\right)$$



By rule 2, root loci exist on $[0, +\infty)$ and $[-3, -1]$. By rule 3,

$$\sigma_a = \frac{-1-3}{3} = -\frac{4}{3}$$

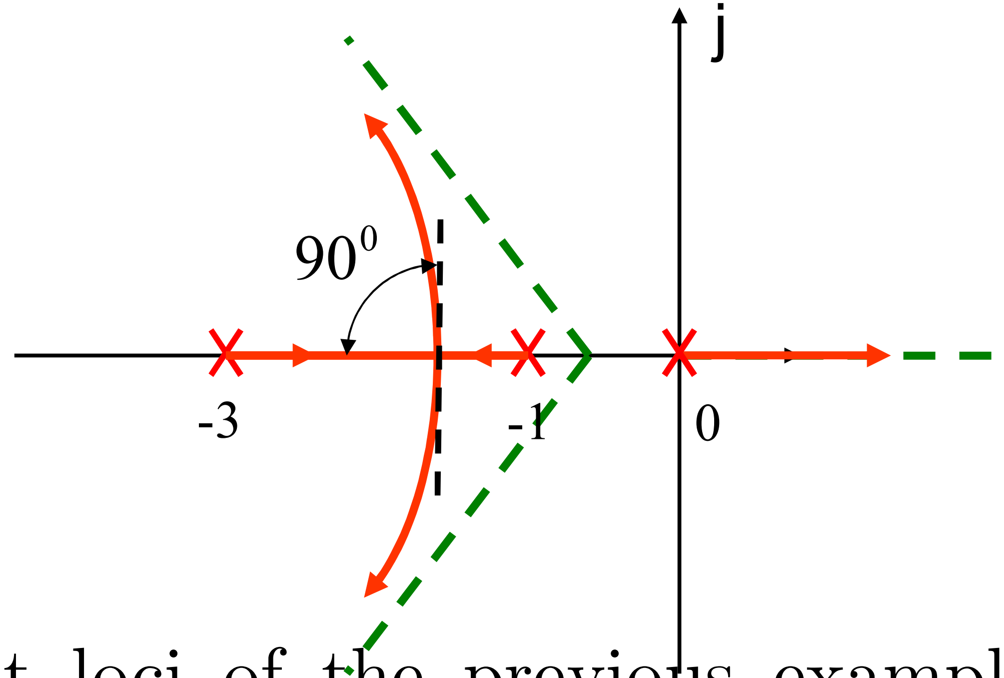
$$\phi_a = 360^0 \times \frac{k}{3} = \begin{cases} 0^0 & (k=0) \\ 120^0 & (k=1) \\ -120^0 & (k=-1) \end{cases}$$

Rule 4. Breakaway (break in) point on the root loci.

1) The breakaway point d can be computed by solving

$$\boxed{\sum_{j=1}^{m} \frac{1}{d+z_j} = \sum_{i=1}^{n} \frac{1}{d+p_i}}$$

2) The angle of breakaway is $180^0/k$, where k is the number of poles intersecting at the breakaway point.



The root loci of the previous example, where the angle of breakaway is $180^0/2=90^0$.

Example. Consider a positive feedback system with its open-loop transfer function as

$$G(s)=\frac{K^*(s+2)}{(s+3)(s^2+2s+2)}$$

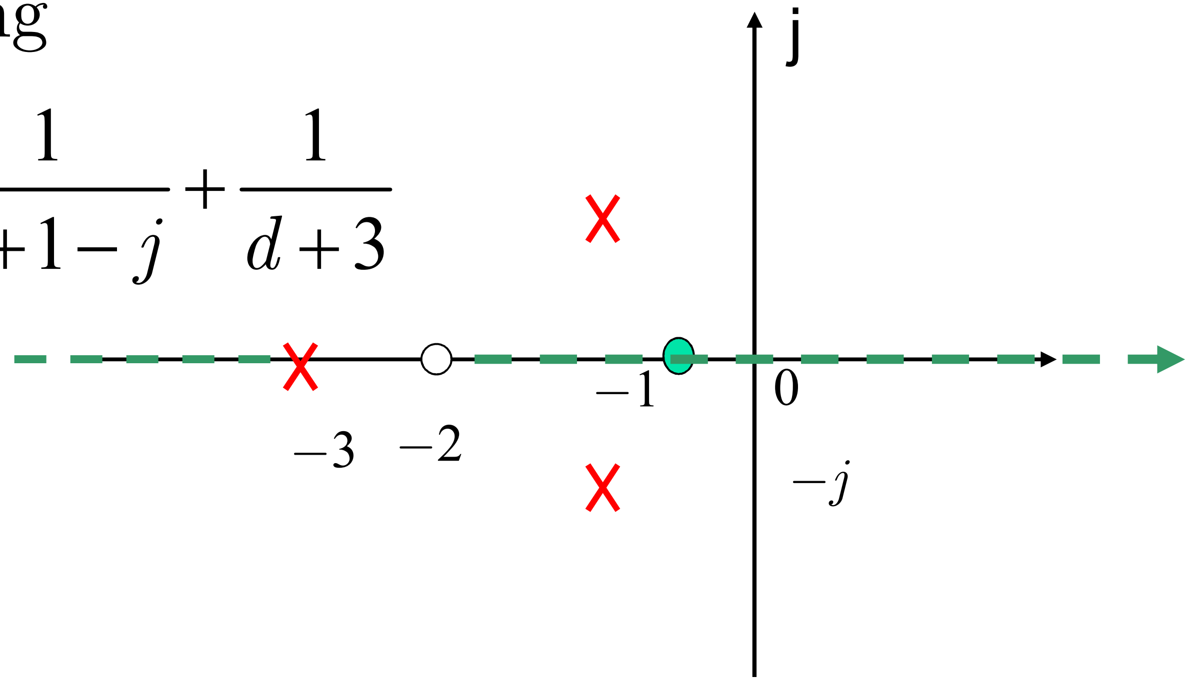
Sketch its root loci.

- By rule 2, root loci exist on $[-2, +\infty)$ and $(-\infty, -3]$.
- By rule 3, we obtain the asymptotes:

$$\sigma_a = \frac{(-3-2)-(-2)}{2} = -\frac{3}{2} \quad \phi_a = 360^0 \times \frac{k}{2} = \begin{cases} 0^0 & (k=0) \\ 180^0 & (k=1) \end{cases}$$

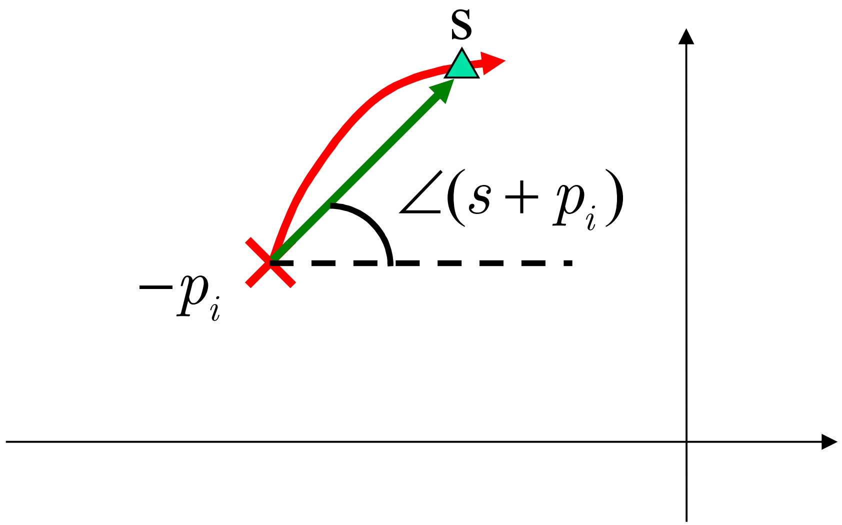
- By rule 4, d $(=-0.8)$ is obtained by solving

$$\frac{1}{d+2} = \frac{1}{d+1+j} + \frac{1}{d+1-j} + \frac{1}{d+3}$$



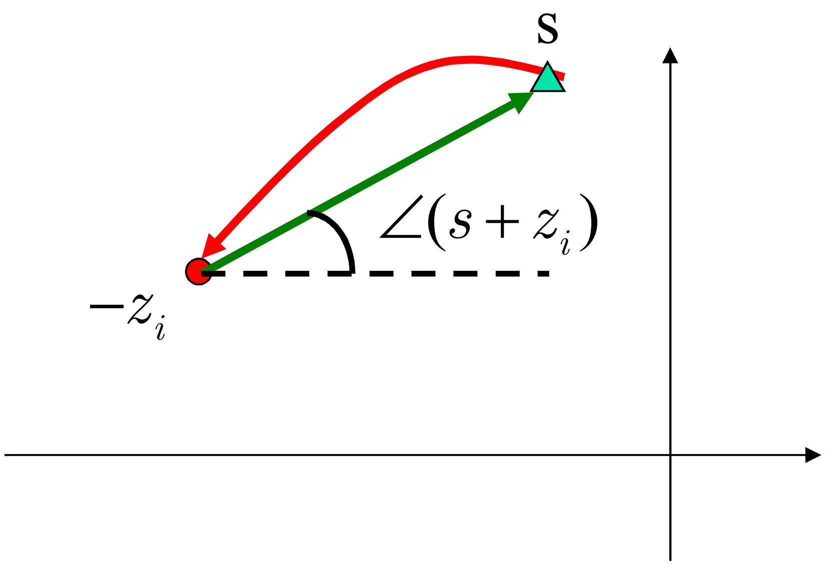
Rule 5. The angle of arrival and the angle of departure (Modified)

1). Angle of departure



$$\theta_{p_i} = 360^0 \times k + \sum_{j=1}^{m} \angle(-p_i + z_j) - \sum_{\substack{j=1 \\ j \neq k}}^{n} \angle(-p_i + p_j)$$

2). Angle of arrival



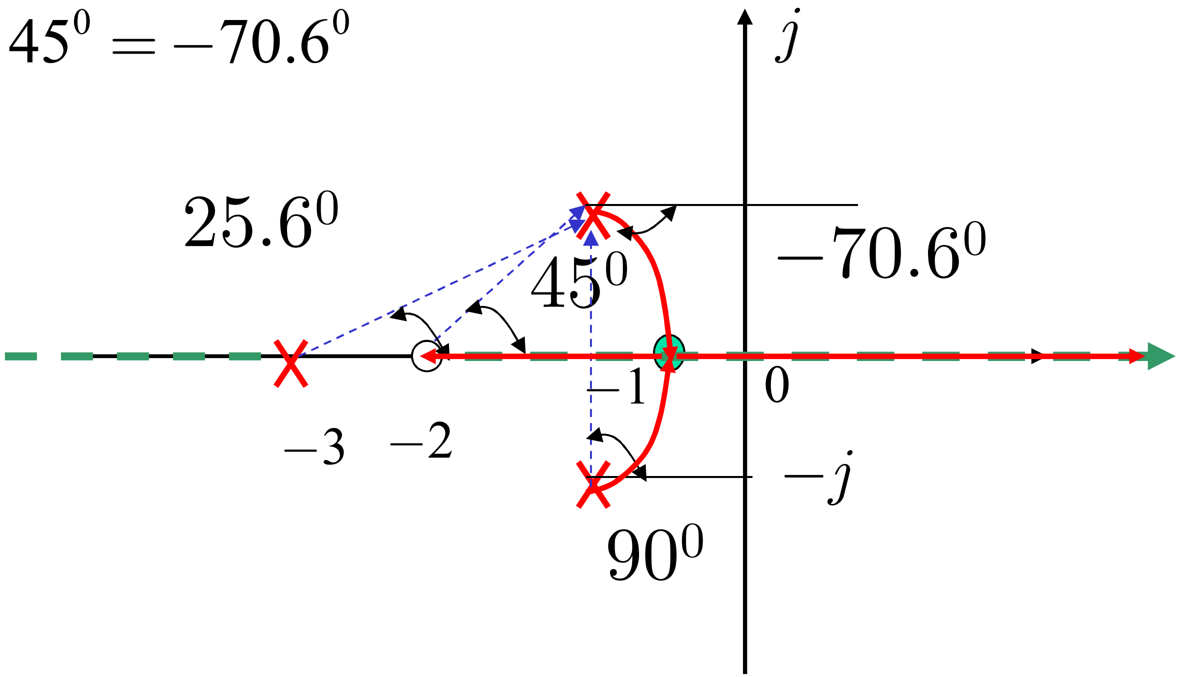
$$\phi_{z_i} = 360^0 \times k - \sum_{\substack{j=1 \\ j \neq i}}^{m} \angle(-z_i + z_j) + \sum_{j=1}^{n} \angle(-z_i + p_j)$$

Example. Consider a positive feedback system with its open-loop transfer function as

$$G(s) = \frac{K^*(s+2)}{(s+3)(s^2+2s+2)}$$

Sketch its root loci.

$$\theta_{p_1} = -90^0 - 25.6^0 + 45^0 = -70.6^0$$



Rule 6. Intersection of the root loci with the imaginary axis.

Substituting $s = j\omega$ into the characteristic equation yields

$$s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \Big|_{s=j\omega}$$

$$= \left[\prod_{j=1}^{n} (s + p_j) - K^* \prod_{i=1}^{m} (s + z_i) \right]_{s=j\omega} = 0$$

Then by splitting its real and imaginary parts one can solve the intersection points and the corresponding open-loop gain.

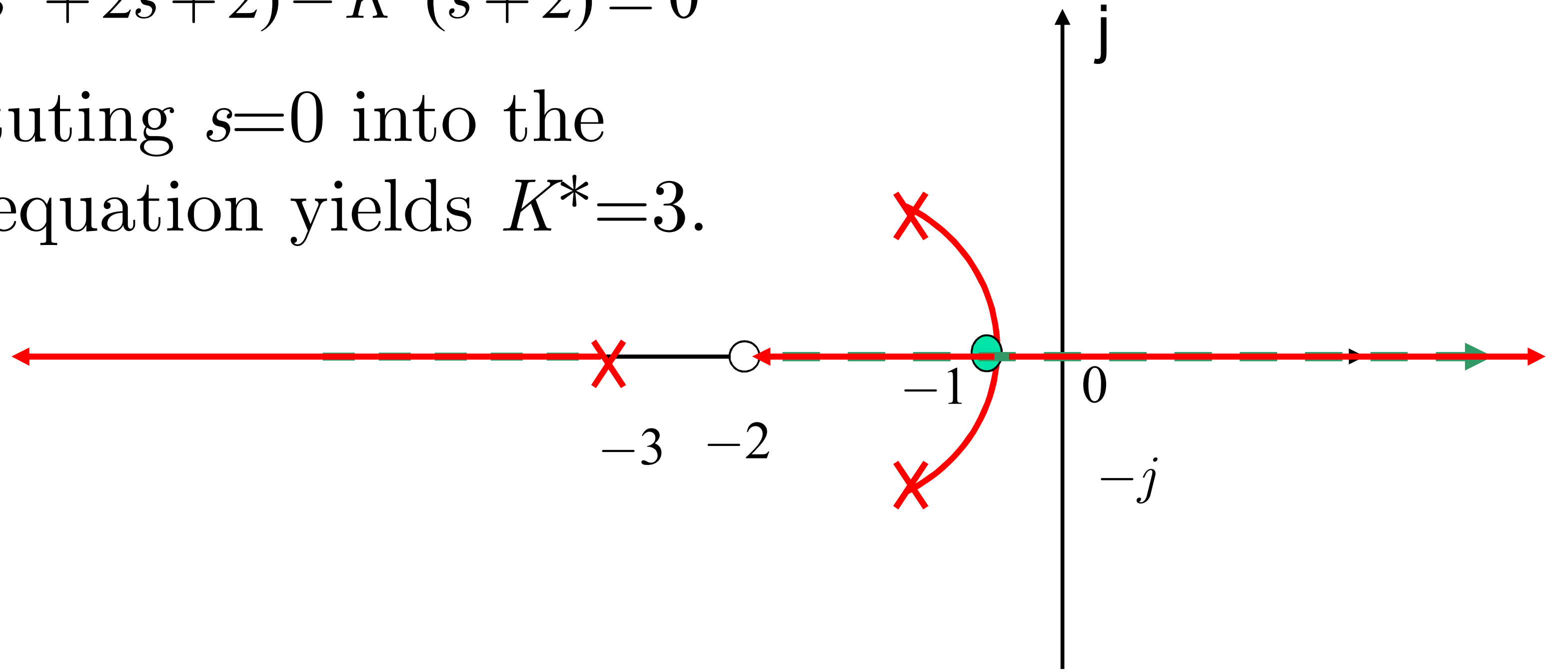
Example. Consider a positive feedback system with its open-loop transfer function as

$$G(s) = \frac{K^*(s+2)}{(s+3)(s^2+2s+2)}$$

- The angle of departure: $\theta_{p_i} \approx -70.6^0$
- Intersection point with imaginary axis:

$$(s+3)(s^2+2s+2) - K^*(s+2) = 0$$

Substituting s=0 into the above equation yields K*=3.



Rule 7. The sum of closed loop poles

If $n \geq m+2$, then **the sum of poles remains unchanged** as K^* varies form zero to infinity.

$$1 - K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

$$\prod_{j=1}^{n}(s+p_j) - K^* \prod_{i=1}^{m}(s+z_i) = \prod_{i=1}^{n}(s+s_i)$$

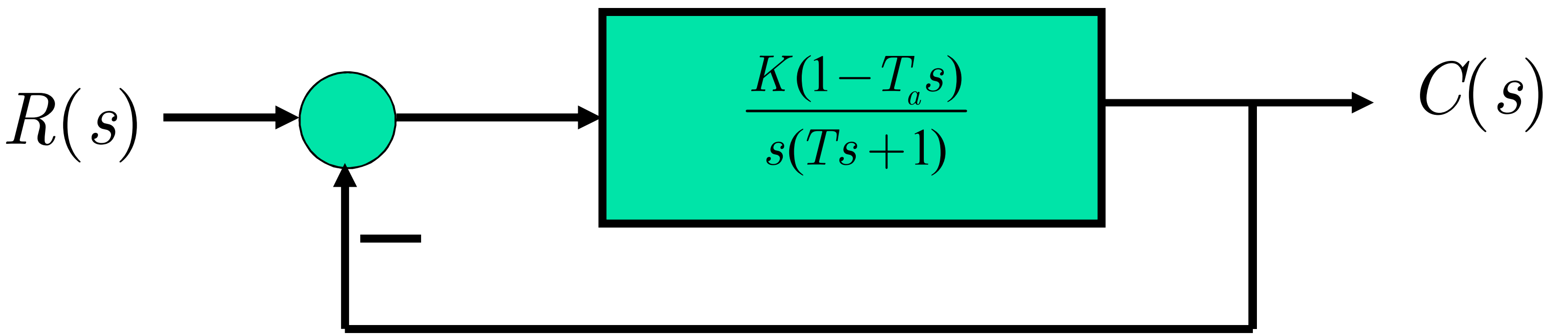
$$= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n = 0$$

$$a_1 = (p_1 + p_2 + \cdots + p_n) = (s_1 + s_2 + \cdots + s_n) = \text{const}$$

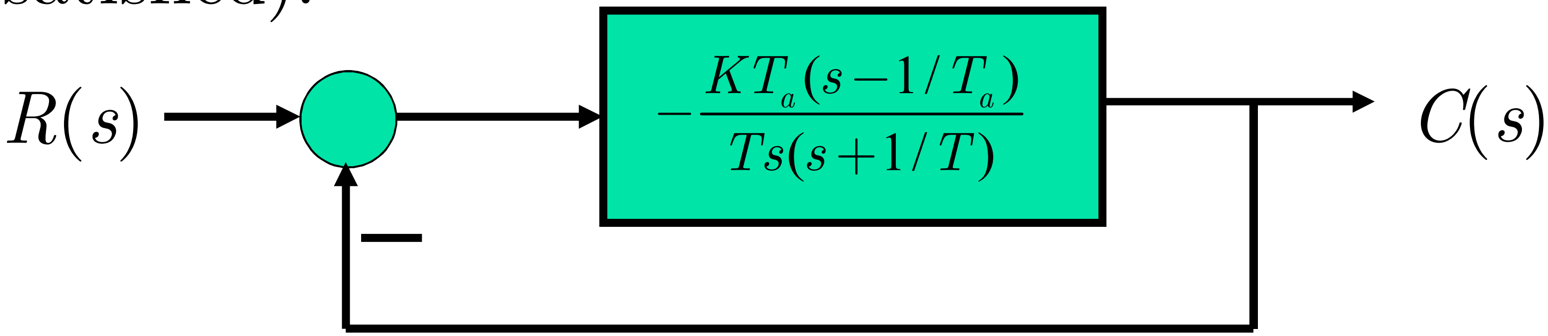
3. Nonminimum-phase systems

If all the poles and zeros of a system lie in the left half s plane, then the system is called **minimum phase**. If a system has at least one pole or zero in the right-half s plane, then the system is called **nonminimum phase**.

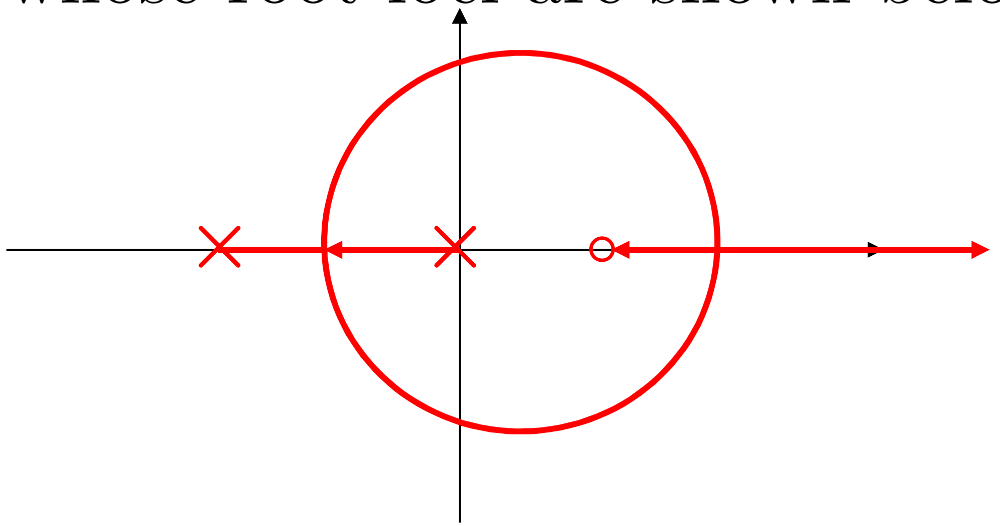
Consider the system shown below:

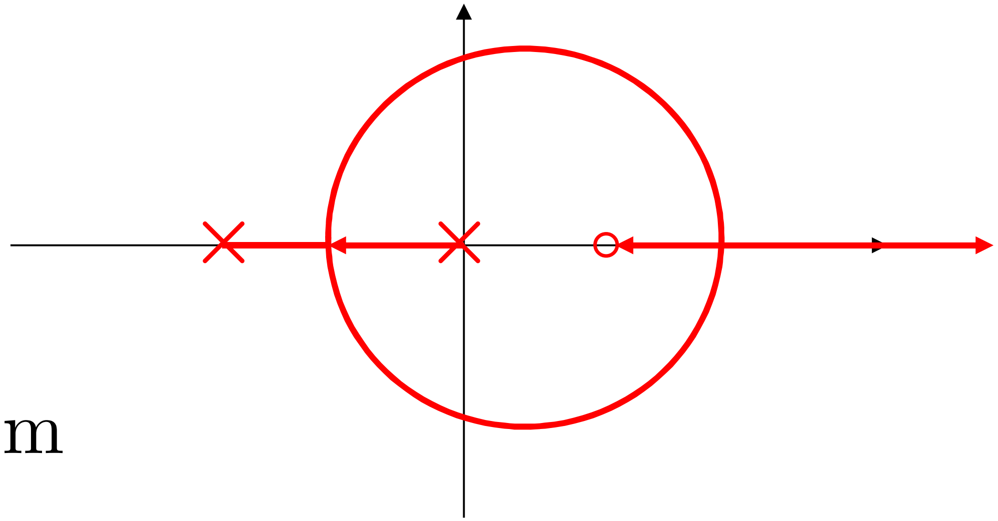


which is a nonminimum phase system and is not written in a conventional form. Write it in the conventional form (therefore, the angle condition is satisfied):



The system is equivalent to a positive feedback system, whose root loci are shown below,





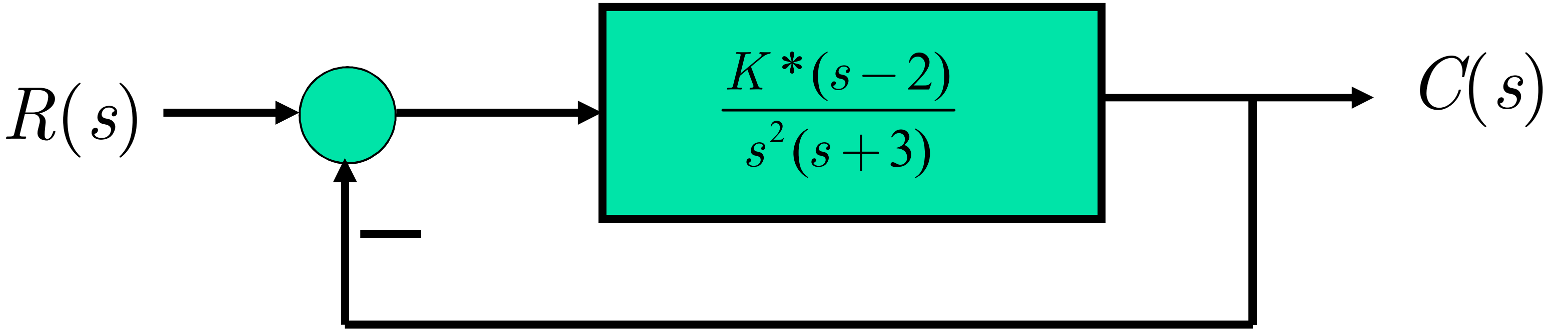
where from

$$\left(s(s+\frac{1}{T}) - K^{*}(s-\frac{1}{T_{a}})\right)_{s=j\omega} = 0$$

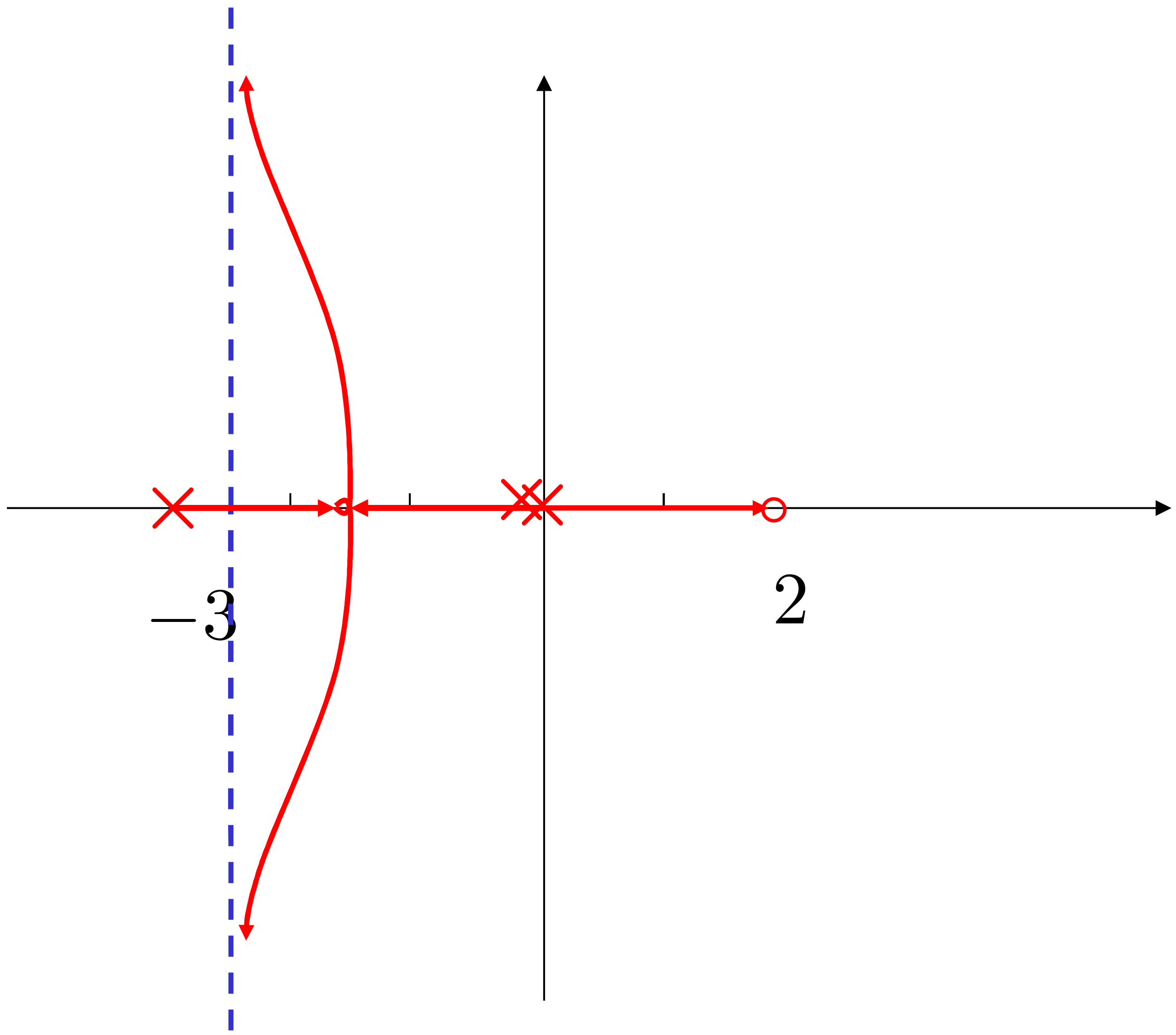
it can be computed that the intersection points of the imaginary axis are

$$\omega = \pm\frac{1}{\sqrt{T_{a}T}}, \quad K = \frac{1}{T_{a}}$$

Note that it does not mean that the root loci of all nonminimum phase systems need to be plotted by using positive feedback rules. For example, consider the following system:



which is nonminimum phase but in a **conventional form**. Therefore, the rules for plotting root loci of negative feedback systems should be applied.



-3
2