Automatic Control

State space representation of dynamical systems

Solution of dynamical systems

State space representation of dynamical systems

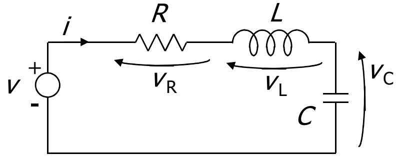
Informal definition of system

System literary means **composition**

From Greek, a system is a "whole compounded of several parts or members"

A system is a set of interacting or interdependent entities forming a set of relationships.

Example: RLC circuit



$$\begin{cases} v_R(t) &= R\,i(t) \\ v_L(t) &= L\,di(t)/dt \\ i(t) &= C\,dv_C(t)/dt \\ v(t) &= v_R(t)+v_L(t)+v_C(t) \end{cases},\ t \in \mathbb{R}^+$$

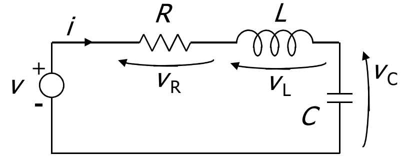
Informal definition of system

The class of considered systems is assumed to have some **inputs** and **outputs**

The inputs *u(t)* are independent **causes** (**excitations**) applied to the system.

The outputs *y(t)* are the measurable **effects** (**responses**) *we are interested in* produced by the inputs application.

Example: RLC circuit



$$u(t) = v(t)$$
$$y(t) = v_C(t)$$

System block diagram representation and solution

A system can be described through the following **block diagram** representation:

Solution of a system

given

- the time course of u(t)
- a mathematical model of the system

compute the time course of y(t) → **system response**

Examples of typical system inputs

$f(t)$	Description
$\delta(t)$	Dirac's delta
$\varepsilon(t)$	Unitary amplitude step
$\dfrac{t^n}{n!}$	Monomial of degree n

$f(t)$	Description
e^{at}	Exponential function
$\dfrac{t^n e^{at}}{n!}$	Polynomially modulated exponential
$\sin(\omega_o t)$	Harmonic signals
$\cos(\omega_o t)$	

Mathematical description: static system

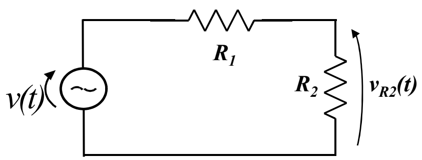
In a **static system** the input – output relationship is a static function:

$$y(t) = h(u(t)), \quad \forall t$$

i.e. the value $y(t)$ depends on the value $u(t)$ only

Example: voltage partition

- $v(t) = u(t)$ input (generator voltage)
- $v_{R2}(t) = y(t)$ output (voltage on the load R_2)



$$\underbrace{v_{R2}(t)}_{y(t)} = \frac{R_2}{R_1 + R_2} \underbrace{v(t)}_{u(t)} = h(u(t))$$

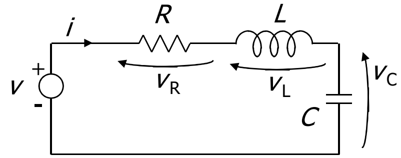
Mathematical description: dynamical system (1/3)

In a **dynamical system** the input – output relationship is dynamical:

$$y(t) = h\left(u([0,t]), \ldots\right), \quad \forall t$$

i.e. the value $y(t)$ does not depend on the value $u(t)$ only but also on its past values up to time t and on the initial condition of the system

Example: RLC circuit



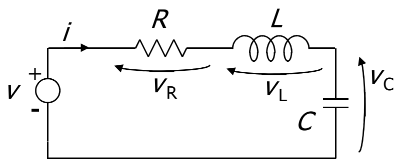
$$u(t) = v(t)$$
$$y(t) = v_C(t)$$

$$\begin{cases} v_R(t) & = R\,i(t) \\ v_L(t) & = L\,di(t)/dt \\ i(t) & = C\,dv_C(t)/dt \\ v(t) & = v_R(t) + v_L(t) + v_C(t) \end{cases}, t \in \mathbb{R}^+$$

Mathematical description: dynamical system (2/3)

In fact, the behavior of such systems, is described through a system of ordinary differential equations

Example: RLC circuit



$$\begin{cases} v_R(t) & = R\,i(t) \\ v_L(t) & = L\,di(t)/dt \\ i(t) & = C\,dv_C(t)/dt \\ v(t) & = v_R(t)+v_L(t)+v_C(t) \end{cases}, t \in \mathbb{R}^+$$

$$\begin{aligned} v(t) &= v_R(t)+v_L(t)+v_C(t) = \\ &= R\,i(t)+L\,di(t)/dt+v_C(t) \\ i(t) &= C\,dv_C(t)/dt \end{aligned} \rightarrow \begin{cases} \dfrac{di(t)}{dt} = \dfrac{1}{L}\left[-R\,i(t)-v_C(t)+v(t)\right] \\ \dfrac{dv_C(t)}{dt} = \dfrac{1}{C}\,i(t) \end{cases}$$

Mathematical description: dynamical system (3/3)

In order to compute the time behavior of the system output $v_C(t)$, the following system of ordinary differential equations has to be solved:

$$\begin{cases} \dfrac{di(t)}{dt} = \dfrac{1}{L}\left[-R\,i(t)-v_C(t)+v(t)\right] \\ \dfrac{dv_C(t)}{dt} = \dfrac{1}{C}\,i(t) \end{cases}$$

- the unknowns are i(t) and $v_C(t)$
- the needed data are
  - the input time course v(t)
  - the initial conditions i(0), $v_C(0)$

Once the solution has been computed, we get, in particular, the time course of $v_C(t)$ (i.e. the system output)

Mathematical description: state equations

The RLC circuit example, allows us to introduce the general formalism actually employed for the study of dynamical systems.

Let us consider again the system of ordinary differential equations of the RLC circuit

$$\begin{cases} \dfrac{di(t)}{dt} = \dfrac{1}{L}\left[-R\,i(t)-v_C(t)+v(t)\right] \\ \dfrac{dv_C(t)}{dt} = \dfrac{1}{C}\,i(t) \end{cases}$$

and perform the following substitutions:

$$u(t)=v(t), x(t)=\begin{bmatrix} i(t) \\ v_C(t) \end{bmatrix}=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, y(t)=v_C(t)$$

Mathematical description: state equations

$$\begin{cases} \dfrac{di(t)}{dt} = \dfrac{1}{L}\left[-R\,i(t)-v_C(t)+v(t)\right] \\ \dfrac{dv_C(t)}{dt} = \dfrac{1}{C}\,i(t) \end{cases}$$

$$u(t)=v(t), x(t)=\begin{bmatrix} i(t) \\ v_C(t) \end{bmatrix}=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, y(t)=v_C(t)=x_2(t)$$

The differential equations of the RLC circuit can be rewritten as:

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}\left[-R\,x_1(t)-x_2(t)+u(t)\right] \\ \dot{x}_2(t) = \dfrac{1}{C}\,x_1(t) \end{cases}$$
$$y(t)=x_2(t)$$

notation $\rightarrow \dfrac{dx(t)}{dt}=\dot{x}(t)$

Mathematical description: state equations

Notation

$u(t) \in \mathbb{R} \to$ input vector

$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^2 \to$ state vector

$y(t) \in \mathbb{R} \to$ output vector

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}\left[-R\,x_1(t) - x_2(t) + u(t)\right] \\ \dot{x}_2(t) = \dfrac{1}{C}x_1(t) \end{cases} \to \text{state}^{*} \text{ equation}$$

$$y(t) = x_2(t) \qquad \to \text{output equation}$$

* In electric circuits the state is usually chosen as voltage across capacitors and current through inductors

Mathematical description: state equations

Now we derive the general form for the state equations of dynamical systems.

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}\left[-R\,x_1(t) - x_2(t) + u(t)\right] = f_1(x(t),u(t)) \\ \dot{x}_2(t) = \dfrac{1}{C}x_1(t) = f_2(x(t),u(t)) \end{cases}$$

$$y(t) = x_2(t) = g(x(t),u(t))$$

$$\begin{cases} \dot{x}_1(t) = f_1(x(t),u(t)) \\ \dot{x}_2(t) = f_2(x(t),u(t)) \end{cases}$$

$$y(t) = g(x(t),u(t))$$

$$f(x(t),u(t)) = \begin{bmatrix} f_1(x(t),u(t)) \\ f_2(x(t),u(t)) \end{bmatrix} \to \dot{x}(t) = f(x(t),u(t))$$

Mathematical description: state equations

Thus, the general form for the state equations of a dynamical system is

$$\left.\begin{array}{l} \dot{x}(t) = f(x(t),u(t)) \to \text{state equation} \\ y(t) = g(x(t),u(t)) \to \text{output equation} \end{array}\right\} \textbf{state space representation}$$

As a matter of fact, functions $f(\cdot)$ and $g(\cdot)$ may depend explicitely on the time variable t, i.e. $f(t,x(t),u(t))$, $g(t,x(t),u(t))$, in such a case the system is said **time-variant**

Anyway, in this course, we will consider the case of **time-invariant** systems described by the above state space representation

Mathematical description: state equations

$$\dot{x}(t) = f(x(t),u(t))$$
$$y(t) = g(x(t),u(t))$$

If $g(\cdot)$ does not depend explicitly on $u(t)$ the system is said **(strictly) proper**

$$y(t) = g(x(t))$$

$$y(t) = Cx(t)$$

Mathematical description: state equations

In general

$$\left\{\begin{array}{l}\dot{x}_1(t)=f_1(x_1(t),\dots,x_n(t),u_1(t),\dots,u_p(t))\\ \dot{x}_2(t)=f_2(x_1(t),\dots,x_n(t),u_1(t),\dots,u_p(t))\\ \quad\vdots\\ \dot{x}_n(t)=f_n(x_1(t),\dots,x_n(t),u_1(t),\dots,u_p(t))\end{array}\right. \rightarrow \underbrace{\dot{x}(t)=f(x(t),u(t))}_{\textbf{state equation}}$$

$$\rightarrow \underbrace{x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\\ \vdots\\ x_n(t)\end{bmatrix}\in\mathbb{R}^n}_{\textbf{state vector}}, \underbrace{u(t)=\begin{bmatrix}u_1(t)\\u_2(t)\\ \vdots\\ u_p(t)\end{bmatrix}\in\mathbb{R}^p}_{\textbf{input vector}}, f(\cdot)=\begin{bmatrix}f_1(\cdot)\\f_2(\cdot)\\ \vdots\\ f_n(\cdot)\end{bmatrix}:\mathbb{R}^{n+p}\rightarrow\mathbb{R}^n$$

Mathematical description: state equations

...

$$\left\{\begin{array}{l}y_1(t)=g_1(x(t),u(t))\\ y_2(t)=g_2(x(t),u(t))\\ \quad\vdots\\ y_q(t)=g_q(x(t),u(t))\end{array}\right. \rightarrow \underbrace{y(t)=\begin{bmatrix}y_1(t)\\y_2(t)\\ \vdots\\ y_q(t)\end{bmatrix}\in\mathbb{R}^q}_{\textbf{output vector}}, g(\cdot)=\begin{bmatrix}g_1(\cdot)\\g_2(\cdot)\\ \vdots\\ g_q(\cdot)\end{bmatrix}:\mathbb{R}^{n+p}\rightarrow\mathbb{R}^q$$

$$\underbrace{y(t)=g(x(t),u(t))}_{\textbf{output equation}}$$

Mathematical description: state equations

Summing up

$$\dot{x}(t)=f(x(t),u(t))$$
$$y(t)=g(x(t),u(t))$$
$$x(t)\in\mathbb{R}^n, u(t)\in\mathbb{R}^p, y(t)\in\mathbb{R}^q$$

n is referred to as "system dimension" (system order)
→ $n<\infty$ → finite dimensional system

p is the input dimension → $p = 1$ Single Input system
$p > 1$ Multi Input system

q is the output dimension → $q = 1$ Single Output system
$q > 1$ Multi Output system

Continuous time linear dynamical systems

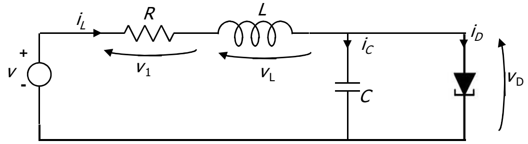
$$\dot{x}(t)=f(x(t),u(t))$$
$$y(t)=g(x(t),u(t))$$

If both $f(\cdot)$ and $g(\cdot)$ are linear functions in <u>both</u> the arguments $x(t)$ and $u(t)$, the system is said **linear time invariant** (LTI)

$$\dot{x}(t)=Ax(t)+Bu(t)$$
$$y(t)=Cx(t)+Du(t)$$
$$A\in\mathbb{R}^{n,n}\quad B\in\mathbb{R}^{n,p}\quad C\in\mathbb{R}^{q,n}\quad D\in\mathbb{R}^{q,p}$$

Example: a nonlinear circuit

In the nonlinear circuit below

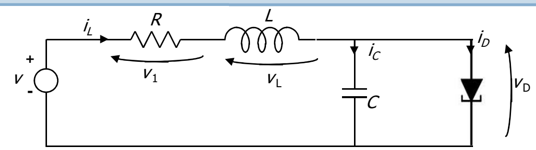


the voltage $v(t)$ and the current $i_L(t)$ are the input and the output variables respectively. The characteristic of the nonlinear element (tunnel diode) is described by the following static relation:

$$i_D(t) = \alpha_1 v_D(t) + \alpha_2 v_D^2(t) + \alpha_3 v_D^3(t) + \alpha_4 v_D^4(t) + \alpha_5 v_D^5(t) = h(v_D(t))$$

Due to the presence of a nonlinear component the state space representation of this system is nonlinear. In fact, …

Example: a nonlinear circuit



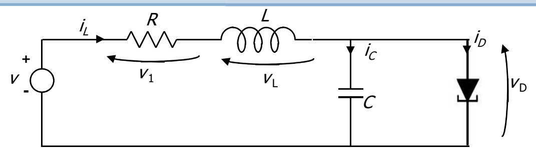
1) $v_L(t) = L\, di_L(t)/dt$
2) $i_C(t) = C\, dv_D(t)/dt$
3) $v_R(t) = R\, i_L(t)$
4) $v(t) = v_R(t) + v_L(t) + v_D(t)$
5) $i_L(t) = i_C(t) + i_D(t)$

6) $i_D(t) = \alpha_1 v_D(t) + \alpha_2 v_D^2(t) + \alpha_3 v_D^3(t) + \alpha_4 v_D^4(t) + \alpha_5 v_D^5(t) = h(v_D(t))$

$$\frac{di_L(t)}{dt} = \frac{1}{L}\left(v(t) - v_R(t) - v_D(t)\right) = \frac{1}{L}\left(v(t) - Ri_L(t) - v_D(t)\right)$$

$$\frac{dv_D(t)}{dt} = \frac{1}{C}\left(i_L(t) - i_D(t)\right) = \frac{1}{C}\left[i_L(t) - \left(\alpha_1 v_D(t) + \alpha_2 v_D^2(t) + \alpha_3 v_D^3(t) + \alpha_4 v_D^4(t) + \alpha_5 v_D^5(t)\right)\right]$$

Example: a nonlinear circuit



$$\frac{di_L(t)}{dt} = \frac{1}{L}\left(v(t) - Ri_L(t) - v_D(t)\right)$$

$$\frac{dv_D(t)}{dt} = \frac{1}{C}\left[i_L(t) - \left(\alpha_1 v_D(t) + \alpha_2 v_D^2(t) + \alpha_3 v_D^3(t) + \alpha_4 v_D^4(t) + \alpha_5 v_D^5(t)\right)\right]$$

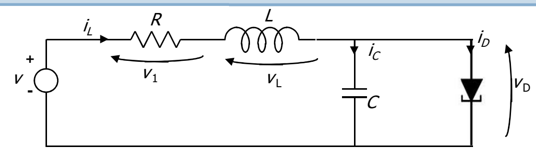
$$x(t) = \begin{bmatrix} i_L(t) \\ v_D(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},$$

$$u(t) = v(t),\ y(t) = i_L(t)$$

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}\left(u(t) - Rx_1(t) - x_2(t)\right) \\ \dot{x}_2(t) = \dfrac{1}{C}\left[x_1(t) - \left(\alpha_1 x_2(t) + \alpha_2 x_2^2(t) + \alpha_3 x_2^3(t) + \alpha_4 x_2^4(t) + \alpha_5 x_2^5(t)\right)\right] \end{cases}$$

$$y(t) = x_1(t)$$

Example: a nonlinear circuit

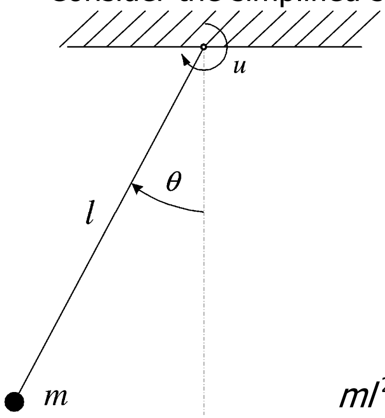


$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{L}\left(u(t) - Rx_1(t) - x_2(t)\right) = f_1(x(t), u(t)) \\ \dot{x}_2(t) = \dfrac{1}{C}\left[x_1(t) - \left(\alpha_1 x_2(t) + \alpha_2 x_2^2(t) + \alpha_3 x_2^3(t) + \alpha_4 x_2^4(t) + \alpha_5 x_2^5(t)\right)\right] = f_2(x(t), u(t)) \end{cases}$$

$$y(t) = x_1(t) = g(x(t), u(t))$$

Example: the single link manipulator

Consider the simplified scheme of a single link 2DOF manipulator:



θ = angular position (variable of interest)

m = mass

l = link length

β = hinge friction coefficient

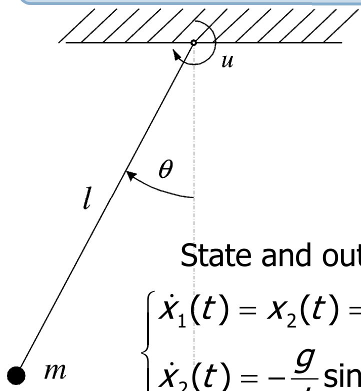
u = applied torque at the hinge

g = gravity acceleration

$$ml^2\ddot{\theta}(t) = -mgl\sin(\theta(t)) - \beta\dot{\theta}(t) + u(t)$$

$$\rightarrow \ddot{\theta}(t) = -\frac{g}{l}\sin(\theta(t)) - \frac{\beta}{ml^2}\dot{\theta}(t) + \frac{1}{ml^2}u(t)$$

Example: the single link manipulator



State vector* $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$

State and output equations

$$\begin{cases} \dot{x}_1(t) = x_2(t) = f_1(x(t),u(t)) \\ \dot{x}_2(t) = -\frac{g}{l}\sin(x_1(t)) - \frac{\beta}{ml^2}x_2(t) + \frac{1}{ml^2}u(t) = f_2(x(t),u(t)) \end{cases}$$

$$y(t) = x_1(t) = g(x(t),u(t))$$

* In mechanical systems, the state is usually chosen as the positions and speeds of inertial elements

Continuous time linear dynamical systems

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Example: the RLC circuit

$$\begin{cases} \dot{x}_1(t) = \frac{1}{L}[-R\,x_1(t) - x_2(t) + u(t)] \\ \dot{x}_2(t) = \frac{1}{C}x_1(t) \end{cases}$$

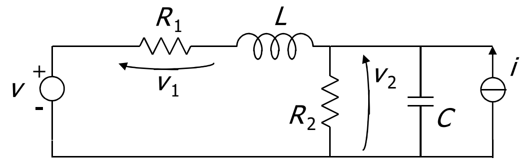
$$y(t) = x_2(t)$$

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$$

$$C = [0 \;\; 1], D = 0$$

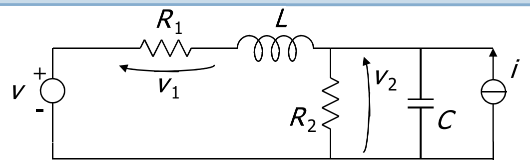
Example: a linear circuit

In the circuit below



the voltage $v(t)$ and the current $i(t)$ are the inputs while the voltages $v_1(t)$ and $v_2(t)$ are the outputs variables. Derive the state space representation.

Example: a linear circuit



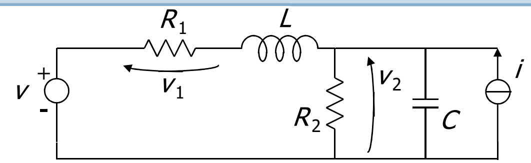
1) $v_L(t) = L\, di_L(t)/dt$ 3) $v(t) = v_1(t) + v_L(t) + v_2(t)$

2) $i_C(t) = C\, dv_2(t)/dt$ 4) $i_L(t) + i(t) = i_2(t) + i_C(t)$

$$x(t) = \begin{bmatrix} i_L(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

1) & 3) →
$$\begin{aligned} \dot{x}_1 &= di_L/dt = v_L/L = (v - v_1 - v_2)/L = \\ &= (u_1 - v_1 - x_2)/L = (u_1 - R_1 i_L - x_2)/L = \\ &= -\frac{R_1}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u_1 = f_1(t,x,u) \end{aligned}$$

Example: a linear circuit



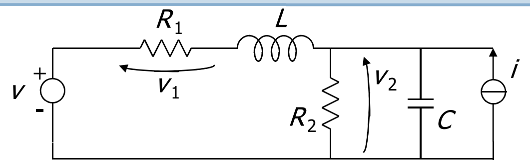
1) $v_L(t) = L\, di_L(t)/dt$ 3) $v(t) = v_1(t) + v_L(t) + v_2(t)$

2) $i_C(t) = C\, dv_2(t)/dt$ 4) $i_L(t) + i(t) = i_2(t) + i_C(t)$

$$x(t) = \begin{bmatrix} i_L(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

2) & 4) →
$$\begin{aligned} \dot{x}_2 &= dv_2/dt = i_C/C = (i_L + i - i_2)/C = \\ &= (x_1 + u_2 - i_2)/C = (x_1 + u_2 - v_2/R_2)/C = \\ &= \frac{1}{C} x_1 - \frac{1}{R_2 C} x_2 + \frac{1}{C} u_2 = f_2(t,x,u) \end{aligned}$$

Example: a linear circuit



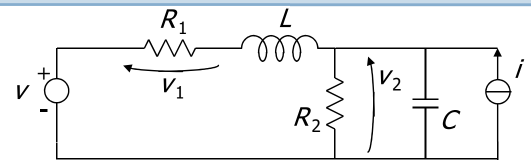
1) $v_L(t) = L\, di_L(t)/dt$ 3) $v(t) = v_1(t) + v_L(t) + v_2(t)$

2) $i_C(t) = C\, dv_2(t)/dt$ 4) $i_L(t) + i(t) = i_2(t) + i_C(t)$

$$x(t) = \begin{bmatrix} i_L(t) \\ v_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \qquad u(t) = \begin{bmatrix} v(t) \\ i(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$y_1 = v_1 = R_1 i_L = R_1 x_1 = g_1(t,x,u)$$
$$y_2 = v_2 = x_2 = g_2(t,x,u)$$
$$y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Example: a linear circuit



State space representation:

$$\begin{cases} \dot{x}_1 = -\frac{R_1}{L} x_1 - \frac{1}{L} x_2 + \frac{1}{L} u_1 \\ \dot{x}_2 = \frac{1}{C} x_1 - \frac{1}{R_2 C} x_2 + \frac{1}{C} u_2 \end{cases} \qquad \begin{aligned} \dot{x}(t) &= A x(t) + B u(t) \\ y(t) &= C x(t) + D u(t) \end{aligned}$$

$$\begin{cases} y_1 = R_1 x_1 \\ y_2 = x_2 \end{cases}$$

$$A = \begin{bmatrix} -R_1/L & -1/L \\ 1/C & -1/R_2 C \end{bmatrix}, B = \begin{bmatrix} 1/L & 0 \\ 0 & 1/C \end{bmatrix}, C = \begin{bmatrix} R_1 & 0 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

State space representation

- The MatLab statement **ss** allows us to introduce a dynamical system as an object to be employed in related computations
- Example

$$\dot{x}(t) = \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t)$$

- Introduce the system matrices A, B, C (and D)

```
>> A=[-3 2;-2 -3]; B=[1;0]; C=[0 1]; D=0;
```


State space representation

- Issue the **ss** statement

```
>> sys=ss(A,B,C,D)
a =
        x1   x2
   x1   -3    2
   x2   -2   -3
b =
        u1
   x1    1
   x2    0

c =
        x1   x2
   y1    0    1
d =
        u1
   y1    0
```


State space representation

- The object **sys** stores in a compact format all the matrices A, B, C, D
- In order to access the matrices you can do as follows

```
>> A=sys.A
A =
    -3     2
    -2    -3
>> B=sys.B
B =
     1
     0
```


Solution of LTI continuous time systems: problem setup

Linear Time Invariant (LTI) state space representation

In this course we will be mainly concerned with the case of **linear-time-invariant** (LTI) systems:

$$\dot{x}(t) = A\,x(t) + B\,u(t)$$ State equation

$A \in \mathbb{R}^{n\times n}$: state matrix
$B \in \mathbb{R}^{n\times p}$: input matrix

$$y(t) = C\,x(t) + D\,u(t)$$ Output equation

$C \in \mathbb{R}^{q\times n}$: output matrix
$D \in \mathbb{R}^{q\times p}$: input-output matrix

State response

The solution of the state equation is obtained through the Lagrange equation:

$$x(t) = \underbrace{e^{At}x(0)}_{x_{zi}(t)} + \underbrace{\int_{0_-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}_{x_{zs}(t)} = x_{zi}(t) + x_{zs}(t)$$

This result is easily verified by direct substitution in the state equation.

The solution of LTI systems can be split into two independent contributions

$X_{zi}(t)$: **zero-input response**

$X_{zs}(t)$: **zero-state response**

Output response

The output response can be obtained from the output equation

$$y(t) = C\,x(t) + D\,u(t)$$

$$y(t) = Ce^{At}x(0) + C\int_{0_-}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) = y_{zi}(t) + y_{zs}(t)$$

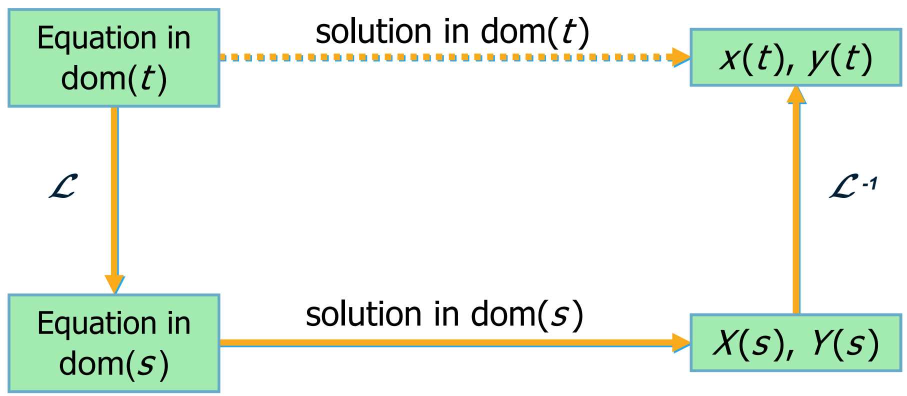
$y_{zi}(t)$: **zero-input output response**

$y_{zs}(t)$: **zero-state output response**

In order to simplify the computation of the time responses (i.e. avoid to solve differential equations) Laplace transform can be suitably exploited

Sketch of the solution procedure

Computation of $x(t)$ and $y(t)$ through the Laplace transform is obtained according to the following scheme:



Solution by Laplace trasform

In order to simplify the computation of the time responses (i.e. avoid to solve differential equations), Laplace transform can be suitably exploited.

Transformation of the state and the output equations and use of the real differentiation theorem give

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases}$$

$$\downarrow \mathcal{L}$$

$$\begin{cases} sX(s) - x(0) = AX(s) + B\,U(s) \\ Y(s) = C\,X(s) + D\,U(s) \end{cases}$$

Solution by Laplace trasform

The transformed state response is given by

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}BU(s) = X_{zi}(s) + X_{zs}(s)$$

$$X_{zi}(s) = (sI - A)^{-1}x(0) = H_{x,zi}(s)x(0)$$

$$X_{zs}(s) = (sI - A)^{-1}BU(s) = H_x(s)U(s)$$

While the transformed output response is given by

$$Y(s) = C(sI - A)^{-1}x(0) + \left[C(sI - A)^{-1}B + D\right]U(s) = Y_{zi}(s) + Y_{zs}(s)$$

$$Y_{zi}(s) = C(sI - A)^{-1}x(0) = H_{zi}(s)x(0)$$

$$Y_{zs}(s) = \left[C(sI - A)^{-1}B + D\right]U(s) = H(s)U(s)$$

Some unilateral ($t \geq 0$) Laplace transform pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$\varepsilon(t)$	$\dfrac{1}{s}$
$\dfrac{t^n}{n!}$	$\dfrac{1}{s^{n+1}}$

$f(t)$	$F(s)$
e^{at}	$\dfrac{1}{s-a}$
$\dfrac{t^n e^{at}}{n!}$	$\dfrac{1}{(s-a)^{n+1}}$
$\sin(\omega_o t)$	$\dfrac{\omega_o}{s^2 + \omega_0^2}$
$\cos(\omega_o t)$	$\dfrac{s}{s^2 + \omega_0^2}$

$f(t)$	$F(s)$
e^{At}	$(sI - A)^{-1}$

Solution of LTI continuous time systems: explicit computation

Solution procedure

The solution is obtained exploiting the following procedure:

1. Compute $X(s)$, $Y(s)$ in the Laplace variable s domain
2. Derive the Heaviside's partial fraction expansion (PFE) of $X(s)$ and $Y(s)$
3. Compute the coefficient (residues) of the PFE
4. Obtain $x(t)$ and $y(t)$ via Laplace inverse transform of $X(s)$ and $Y(s)$

Computation example 1

Consider the following LTI system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

compute the state response $x(t)$ when $u(t) = 2\varepsilon(t)$ and $x(0) = [2 \;\; 2]^T$

The solution in Laplace domain can be computed as

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

Computation example 1

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, U(s) = \frac{2}{s}$$

$$(sI - A)^{-1} = \frac{1}{\det(sI - A)} Adj(sI - A) = \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} =$$

$$= \frac{1}{\underbrace{s^2 + 3s + 2}_{\det(sI-A)}} \underbrace{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}_{Adj(sI-A)} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

Computation example 1

$$X(s) = \underbrace{(sI - A)^{-1} x(0)}_{X_{zi}(s)} + \underbrace{(sI - A)^{-1} BU(s)}_{X_{zs}(s)}$$

$$X_{zi}(s) = (sI - A)^{-1} x(0) = \underbrace{\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{x(0)} = \begin{bmatrix} \frac{2s+8}{(s+1)(s+2)} \\ \frac{2s-4}{(s+1)(s+2)} \end{bmatrix}$$

$$X_{zs}(s) = (sI - A)^{-1} BU(s) = \underbrace{\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\frac{2}{s}}_{U(s)} = \begin{bmatrix} \frac{2(s+3)}{s(s+1)(s+2)} \\ \frac{-4}{s(s+1)(s+2)} \end{bmatrix}$$

Computation example 1

$$X(s)=\underbrace{(sI-A)^{-1}x(0)}_{X_{zi}(s)}+\underbrace{(sI-A)^{-1}BU(s)}_{X_{zs}(s)}$$

$$X(s)=X_{zi}(s)+X_{zs}(s)=\underbrace{\begin{bmatrix}\dfrac{2s+8}{(s+1)(s+2)}\\[2mm]\dfrac{2s-4}{(s+1)(s+2)}\end{bmatrix}}_{X_{zi}(s)}+\underbrace{\begin{bmatrix}\dfrac{2(s+3)}{s(s+1)(s+2)}\\[2mm]\dfrac{-4}{s(s+1)(s+2)}\end{bmatrix}}_{X_{zs}(s)}=\begin{bmatrix}\dfrac{2s^2+10s+6}{s(s+1)(s+2)}\\[2mm]\dfrac{2s^2-4s-4}{s(s+1)(s+2)}\end{bmatrix}$$

$$X(s)=\begin{bmatrix}X_1(s)\\X_2(s)\end{bmatrix}=\begin{bmatrix}\dfrac{2s^2+10s+6}{s(s+1)(s+2)}\\[2mm]\dfrac{2s^2-4s-4}{s(s+1)(s+2)}\end{bmatrix}=\begin{bmatrix}\dfrac{R_1^{(1)}}{s}+\dfrac{R_2^{(1)}}{s+1}+\dfrac{R_3^{(1)}}{s+2}\\[2mm]\dfrac{R_1^{(2)}}{s}+\dfrac{R_2^{(2)}}{s+1}+\dfrac{R_3^{(2)}}{s+2}\end{bmatrix}=\begin{bmatrix}\dfrac{3}{s}+\dfrac{2}{s+1}-\dfrac{3}{s+2}\\[2mm]-\dfrac{2}{s}-\dfrac{2}{s+1}+\dfrac{6}{s+2}\end{bmatrix}$$

Computation example 1

$$X(s)=\begin{bmatrix}X_1(s)\\X_2(s)\end{bmatrix}=\begin{bmatrix}\dfrac{3}{s}+\dfrac{2}{s+1}-\dfrac{3}{s+2}\\[2mm]-\dfrac{2}{s}-\dfrac{2}{s+1}+\dfrac{6}{s+2}\end{bmatrix}$$

$$Re^{at}\varepsilon(t)=\mathcal{L}^{-1}\left\{\frac{R}{s-a}\right\}$$

$$x(t)=\begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix}=\begin{bmatrix}3+2e^{-t}-3e^{-2t}\\-2-2e^{-t}+6e^{-2t}\end{bmatrix}\varepsilon(t)$$

Computation example 1: MatLab procedure

- Define the Laplace variabile **`s`** using **`tf`** statement

```
>> s=tf('s')
Transfer function:
s
```

- Define the system input and initial condition

```
>> U=2/s, x0=[2;2]
Transfer function:
2
-
s
x0 =
     2
     2
```


Computation example 1: MatLab procedure

- Introduce the system matrices A and B

```
>> A=[0 1;-2 -3], B=[1;0]
A =
     0     1
    -2    -3
B =
     1
     0
```

Computation example 1: MatLab procedure

- Compute $X(s)=(sI-A)^{-1}x(0)+(sI-A)^{-1}BU(s)=(sI-A)^{-1}\left[x(0)+BU(s)\right]$

  use statements **minreal** and **zpk**, in order to simplify and highlight denominator roots respectively

```
>> X=zpk(minreal(inv(s*eye(2)-A)*(B*U+x0),1e-3))

 Zero/pole/gain from input to output...
      2 (s+4.303) (s+0.6972)
 #1:  ----------------------
          s (s+2) (s+1)

      2 (s-2.732) (s+0.7321)
 #2:  ----------------------
          s (s+2) (s+1)
```


Computation example 1: MatLab procedure

- For each of the two components of X(s) , compute the PFE using the statements **tfdata** and **residue**

```
>> [num_X1,den_X1]=tfdata(X(1),'v')
num_X1 =
        0    2.0000   10.0000    6.0000
den_X1 =
   1.0000    3.0000    2.0000         0
>> [r1,p1]=residue(num_X1,den_X1)
r1 =
  -3.0000
   2.0000
   3.0000
p1 =
  -2.0000
  -1.0000
        0
```

$$\rightarrow X_1(s)=\frac{3}{s}+\frac{2}{s+1}-\frac{3}{s+2} \qquad \rightarrow x_1(t)=\left(3+2e^{-t}-3e^{-2t}\right)\varepsilon(t)$$

Computation example 1: MatLab procedure

- For each of the two components of X(s) , compute the PFE using the statements **tfdata** and **residue**

```
>> [num_X2,den_X2]=tfdata(X(2),'v')
num_X2 =
        0    2.0000   -4.0000   -4.0000
den_X2 =
   1.0000    3.0000    2.0000         0
>> [r2,p2]=residue(num_X2,den_X2)
r2 =
   6.0000
  -2.0000
  -2.0000
p2 =
  -2.0000
  -1.0000
        0
```

$$\rightarrow X_2(s)=-\frac{2}{s}-\frac{2}{s+1}+\frac{6}{s+2} \qquad \rightarrow x_2(t)=\left(-2-2e^{-t}+6e^{-2t}\right)\varepsilon(t)$$

Computation example 2

Consider the following LTI system:

$$\dot{x}(t)=\begin{bmatrix}-3 & 2\\ -2 & -3\end{bmatrix}x(t)+\begin{bmatrix}1\\ 0\end{bmatrix}u(t)$$
$$y(t)=\begin{bmatrix}0 & 1\end{bmatrix}x(t)$$

compute the output response $y(t)$ when $u(t) = \varepsilon(t)$ and $x(0) = [1 \;\; 1]^T$

The solution in Laplace domain can be computed as

$$Y(s)=\underbrace{C(sI-A)^{-1}x(0)}_{Y_{zi}(s)}+\underbrace{\left[C(sI-A)^{-1}B+D\right]U(s)}_{Y_{zs}(s)}$$

Computation example 2

$$Y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{Y_{zi}(s)} + \underbrace{\left[C(sI-A)^{-1}B+D\right]U(s)}_{Y_{zs}(s)}$$

$$A = \begin{bmatrix} -3 & 2 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = [0], x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, U(s) = \frac{1}{s}$$

$$(sI-A)^{-1} = \begin{bmatrix} s+3 & -2 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{\underbrace{s^2+6s+13}_{\det(sI-A)}} \underbrace{\begin{bmatrix} s+3 & 2 \\ -2 & s+3 \end{bmatrix}}_{Adj(sI-A)} =$$

$$= \begin{bmatrix} \dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13} \\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13} \end{bmatrix}$$

Computation example 2

$$Y(s) = \underbrace{C(sI-A)^{-1}x(0)}_{Y_{zi}(s)} + \underbrace{\left[C(sI-A)^{-1}B+D\right]U(s)}_{Y_{zs}(s)}$$

$$Y_{zi}(s) = C(sI-A)^{-1}x(0) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13} \\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x(0)} = \frac{s+1}{s^2+6s+13}$$

$$Y_{zs}(s) = \left[C(sI-A)^{-1}B+D\right]U(s) \underset{D=0}{=} \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \underbrace{\begin{bmatrix} \dfrac{s+3}{s^2+6s+13} & \dfrac{2}{s^2+6s+13} \\ \dfrac{-2}{s^2+6s+13} & \dfrac{s+3}{s^2+6s+13} \end{bmatrix}}_{(sI-A)^{-1}} \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} \underbrace{\frac{1}{s}}_{U(s)} = \frac{-2}{s^3+6s^2+13s}$$

$$Y(s) = Y_{zi}(s) + Y_{zs}(s) = \frac{s^2+s-2}{s^3+6s^2+13s} = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)}$$

Computation example 2

$$Y(s) = Y_{zi}(s) + Y_{zs}(s) = \frac{s^2+s-2}{s^3+6s^2+13s} = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)}$$

Note that in the denominator of Y(s) there is a factor of the form

$(s - \sigma_0 - j\omega_0)\ (s - \sigma_0 + j\omega_0)$ leading to two complex conjugate roots with $\sigma_0 = -3$ and $\omega_0 = 2$

The PFE is:

$$Y(s) = \frac{s^2+s-2}{s(s+3-2j)(s+3+2j)} = \frac{R_1}{s+3-2j} + \frac{R_1^*}{s+3+2j} + \frac{R_2}{s} =$$

$$= \frac{0.57..+0.38..j}{s+3-2j} + \frac{0.57..-0.38..j}{s+3+2j} - \frac{0.15..}{s}$$

Please also note that R_1 is the residue associated with the complex root having positive imaginary part (this is important for the inverse transform procedure)

Computation example 2

A reminder on the inverse Laplace transform in the presence of complex conjugate roots.

The inverse Laplace transform of the following PFE (considered as a whole):

$$\frac{R}{s-\sigma_0-j\omega_0} + \frac{R^*}{s-\sigma_0+j\omega_0}$$

is given by:

$$2|R|e^{\sigma_0 t}\cos(\omega_0 t + \arg(R))\varepsilon(t)$$

$$|R| = \sqrt{\mathrm{Re}^2(R)+\mathrm{Im}^2(R)}, \arg(R) = \arctan\frac{\mathrm{Im}(R)}{\mathrm{Re}(R)}$$

Computation example 2

$$Y(s)=\frac{R_1}{s+3-2j}+\frac{R_1^*}{s+3+2j}+\frac{R_2}{s}=\frac{0.57..+0.38..j}{s+3-2j}+\frac{0.57..-0.38..j}{s+3+2j}-\frac{0.15..}{s}$$

$$\sigma_0=-3,\omega_0=2$$

$$R_1=0.57..+0.38..j$$

$$|R_1|=\sqrt{(0.57..)^2+(0.38..)^2}=0.69..$$

$$\arg(R_1)=\arctan\left(\frac{0.38..}{0.57..}\right)=0.58..\text{rad}$$

$$R_2=0.15..$$

$$y(t)=\left(2|R_1|e^{\sigma_0 t}\cos(\omega_0 t+\arg(R_1))+R_2\right)\varepsilon(t)$$

$$y(t)=\left(\underbrace{1.38}_{2|R_1|}e^{\overset{\sigma_0}{\downarrow}{-3}\,t}\cos(\overset{\omega_0}{\downarrow}{2}\,t+\underbrace{0.58}_{\arg(R_1)})\underbrace{-0.15}_{R_2}\right)\varepsilon(t)$$

Computation example 2: MatLab procedure

- Define the Laplace variabile **s** using **tf** statement

```
>> s=tf('s')
Transfer function:
s
```

- Define the system input and initial condition

```
>> U=1/s, x0=[1;1]
Transfer function:
1
-
s
x0 =
      1
      1
```


Computation example 2: MatLab procedure

- Introduce the system matrices A, B and C

```
>> A=[-3 2;-2 -3], B=[1;0], C=[0 1]
A =
    -3     2
    -2    -3
B =
     1
     0
C =
     0     1
```


Computation example 2: MatLab procedure

- Compute $Y(s)=C(sI-A)^{-1}x(0)+C(sI-A)^{-1}BU(s)=C(sI-A)^{-1}[x(0)+BU(s)]$ use statements **minreal** and **zpk**, in order to simplify and highlights denominator roots respectively

```
>> Y=zpk(minreal(C*inv(s*eye(2)-A)*(B*U+x0),1e-3))

Zero/pole/gain:
   (s+2) (s-1)
------------------
s (s^2  + 6s + 13)
```

Computation example 2: MatLab procedure

- For Y(s) , compute the PFE using the statements **tfdata** and **residue**

```
>> [num_Y,den_Y]=tfdata(Y,'v')
num_Y =
         0    1.0000    1.0000   -2.0000
den_Y =
    1.0000    6.0000   13.0000
>> [r,p]=residue(num_Y, den_Y)
r =
   0.5769 + 0.3846i
   0.5769 - 0.3846i
  -0.1538
p =
  -3.0000 + 2.0000i
  -3.0000 - 2.0000i
        0
```

$$\to Y(s) = \frac{0.57..+0.38..j}{s+3-2j} + \frac{0.57..-0.38..j}{s+3+2j} - \frac{0.15..}{s}$$

Computation example 2: MatLab procedure

- Compute magnitude and phase of the residue corresponding to the complex root with positive imaginary part

```
>> M=abs(r(1)), 2*M
M =
    0.6934
ans =
    1.3868
>> phi=angle(r(1))
phi =
    0.5880
```

$$\to y(t) = \left(1.38e^{-3t}\cos(2t+0.58) - 0.15\right)\varepsilon(t)$$