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(1). $\dot{x}_1 = -x_1 + 2x_1^3 + x_2 \quad \dot{x}_2 = -x_1 - x_2$.

$$\begin{cases} 0 = -x_1 + 2x_1^3 + x_2 \\ 0 = -x_1 - x_2 \end{cases}$$

∴ $x_2 = -x_1$, $2x_1^3 - 2x_1 = 0$, $x_1^3 - x_1 = 0$

∴ $x_1(x_1^2 - 1) = 0$.

解得 $x_1 = 0$ 或 1 或 -1.

∴ 平衡点为 $(0,0)$ 或 $(1,-1)$ 或 $(-1,1)$

$\frac{\partial f}{\partial x} = \begin{bmatrix} -1+6x_1^2 & 1 \\ -1 & -1 \end{bmatrix}$.

①. $(0,0)$, $\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $\lambda_{1,2} = -1 \pm j$

$\alpha < 0$. 平衡点为稳定焦点.

② $(1,-1)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} 5 & 1 \\ -1 & -1 \end{bmatrix}$

$(\lambda - 5)(\lambda + 1) - 1 = 0$. $\lambda_{1,2} = 2 \pm \sqrt{10}$.

∴ 平衡点为鞍点

③ 同理, 与②相同.

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(2). $\dot{x}_1 = x_1 + x_1 x_2 \quad \dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$

$$\therefore \begin{cases} (x_2+1)x_1 = 0 \\ -x_2 + x_2^2 + x_1 x_2 - x_1^3 = 0 \end{cases}$$

若 $x_1 = 0$, $x_2 = 0$ 或 1.

$x_2 = -1$, $x_1 = 1$

∴ 平衡点为 $(0,0)$ $(0,1)$ $(-1,1)$.

①. $(0,0)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} 1+x_2 & x_1 \\ x_2 - 3x_1^2 & -1+2x_2+x_1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_{1,2} = \pm 1$

∴ 为鞍点

②. $(0,1)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$(\lambda - 2)(\lambda - 1) - 0 = 0$

∴ $\lambda_{1,2} = 2, 1$; 为非稳定结点

③. $(-1,1)$ $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$, $\lambda_{1,2} = -1 \pm j\sqrt{3}$; 为稳定焦点.

(3). $\dot{x}_1 = [1 - x_1 - 2h(x)]x_1$, $\dot{x}_2 = [2 - h(x)]x_2$

$h(x) = \frac{x_2}{1+x_1}$, $\begin{cases} (1-x_1)x_1 - \frac{2x_1x_2}{1+x_1} = 0 \\ 2 - \frac{x_2}{1+x_1} = 0 \end{cases}$

若 $x_2 = 0$. $x_1 = 0$ 或 1

$x_2 = 2(1+x_1)$, $x_1 = 0$ 或 -3.

∴ 平衡点为 $(0,0)$, $(1,0)$, $(0,2)$, $(-3,-4)$.

$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 - 2x_1 - \frac{2x_2}{(1+x_1)^2} & -2\frac{x_1}{1+x_1} \\ \frac{x_2^2}{(1+x_1)^2} & 2 - \frac{2x_2}{1+x_1} \end{bmatrix}$

①. $(0,0)$ $\frac{\partial f}{\partial x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. $\lambda_{1,2} = 1, 2$.

∴ 为非稳定结点

②. $(1,0)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ $\lambda_{1,2} = -1, 2$.

∴ 为鞍点

③. $(0,2)$ $\frac{\partial f}{\partial x} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}$ $\lambda_{1,2} = -3, -2$

∴ 为稳定结点

④. $(-3,-4)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} 9 & -3 \\ 4 & -2 \end{bmatrix}$ $\lambda_{1,2} = 7.72, -0.772$

∴ 为鞍点.

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(4). $\dot{x}_1 = x_2 \quad \dot{x}_2 = -x_1 + x_2(1 - x_1^2 + 0.1x_1^4)$.

$\begin{cases} x_2 = 0 \\ -x_1 = 0 \end{cases}$ ∴ 平衡点为 $(0,0)$

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2x_1x_2 + 0.4x_1^3x_2 & 1 - x_1^2 + 0.1x_1^4 \end{bmatrix}$

∴ $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$, $\lambda_{1,2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$.

∴ 为非稳定焦点

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(5). $\begin{cases} \dot{x}_1 = (x_1 - x_2)(1 - x_1^2 - x_2^2) = 0 \\ \dot{x}_2 = (x_1 + x_2)(1 - x_1^2 - x_2^2) = 0. \end{cases}$

若 $x_1^2 + x_2^2 = 1$, 则成立. 即满足为平衡.

若 $x_1^2 + x_2^2 \neq 1$, 则 $x_1 = x_2 = 0$

∴ 对于孤立点 $(0,0)$,

$\frac{\partial f}{\partial x} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. $\lambda_{1,2} = 1 \pm j$

∴ 为非稳定焦点.

(6). $\dot{x}_1 = -x_1^3 + x_2$ $\dot{x}_2 = x_1 - x_2^3$

$$\begin{cases} -x_1^3 + x_2 = 0 \\ x_1 - x_2^3 = 0 \end{cases} \Rightarrow \quad x_2 = x_1^3$$
$x_1 = 0$ 或 $x_1^8 = 1$. $x_1 = \pm 1$.

∴ 平衡点为 $(0,0)$, $(1,1)$, $(-1,-1)$.

①. $\frac{\partial f}{\partial x} = \begin{bmatrix} -3x_1^2 & 1 \\ 1 & -3x_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Big|_{(0,0)}$

$\lambda_{1,2} = \pm 1$. 为鞍点.

②. $\frac{\partial f}{\partial x} = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \Big|_{(1,1)或(-1,-1)}$

$\lambda_{1,2} = -2, -4$. 为稳定结点

2.5 $\dot{x}_1 = -x_1 - \frac{x_2}{\ln\sqrt{x_1^2 + x_2^2}}$

$\dot{x}_2 = -x_2 + \frac{x_1}{\ln\sqrt{x_1^2 + x_2^2}}$

(a).
$$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 + \underbrace{\frac{x_1x_2}{(x_1^2+x_2^2)(\ln\sqrt{x_1^2+x_2^2})^2}}_{A} & \underbrace{-\frac{1}{\ln\sqrt{x_1^2+x_2^2}}}_{B} + x_2^2 A \\ B - x_1^2 A & -1 - x_1x_2A \end{bmatrix}$$

在 $(0,0)$, $\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. $\lambda_{1,2} = -1, -1$

∴ 为稳定结点

(b). $r = \sqrt{x_1^2 + x_2^2}$ $\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$.

$\dot{r} = -r$, ∴ $r(t) = r_0 e^{-t}$.

∴ 当 $0 < r_0 < 1$ 时, $\dot{\theta} = \frac{1}{\ln r} = \frac{1}{\ln r_0 - t}$

∴ $\theta(t) = \theta_0 - \ln(|\ln r_0| + t) + \ln(|\ln r_0|)$

∴ 当 $0 < r_0 < 1$ 时, $r(t)$ 与 $\theta(t)$ 严格递减

$\lim_{t \to \infty} r(t) = 0$. $\lim_{t \to \infty} \theta(t) = -\infty$.

∴ 为顺时针趋近原点, 与稳定焦点类似.

(c). $f_1(x_1, x_2)$ 与 $f_2(x_1, x_2)$ 在平衡点 $(0,0)$

邻域内有连续一阶偏导数.

2.8 $\dot{x}_1 = x_2$. $\dot{x}_2 = -x_1 + \frac{1}{16}x_1^5 - x_2$.

$$\begin{cases} x_2 = 0 \\ x_1(16 - x_1^4) = 0. \end{cases}$$

∴ 平衡点为 $(0,0)$, $(2,0)$, $(-2,0)$

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 + \frac{5}{16}x_1^4 & -1 \end{bmatrix}$

①. $(0,0)$ $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ $\lambda_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

∴ 为稳定焦点.

②. $(2,0)$. $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 4 & -1 \end{bmatrix}$ $\lambda_{1,2} = \frac{-1 \pm \sqrt{17}}{2}$

∴ 为鞍点

③. $(-2,0)$ 与②同理.

2.17 (1). $\ddot{y} + y = \varepsilon \dot{y}(1 - y^2 - \dot{y}^2)$

若 $\varepsilon > 0$, 设 $x_1 = y$. $x_2 = \dot{y}$. $V(x) = x_1^2 + x_2^2$

$\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)$.

$f(x) \cdot \nabla V(x) = 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) = 2\varepsilon x_2^2(1 - V)$.

∴ 当 $V(x) \geq 1$ 时 $f(x) \cdot \nabla V(x) \leq 0$.

原点邻域内线性化为 $\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$

∴ 原点平衡点为非稳定焦点或结点.

∴ 有周期轨道.

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(2). $\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2)$

令 $V(x) = x_1^2 + x_2^2$

$f(x) \cdot \nabla V(x) = 2x_2^2(2 - 3x_1^2 - 2x_2^2)$

$= 4x_2^2(1 - x_1^2 - x_2^2) - 2x_1^2x_2^2$

$\leq 4x_2^2(1 - x_1^2 - x_2^2)$.

∴ 当 $x_1^2 + x_2^2 \geq 1$, $f(x) \cdot \nabla V(x) \leq 0$.

∴ 原点邻域内为线性化 $\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$. $\lambda = 1, 1$

∴ 为非稳定结点.

∴ 有周期轨道.

2.20

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(2). $\dot{x}_1 = -x_1 + x_1^3 + x_1 x_2^2$ $\qquad$ $\dot{x}_2 = -x_2 + x_2^3 + x_1^2 x_2$

$$\begin{cases} x_1(-1+x_1^2+x_2^2)=0 \\ x_2(-1+x_1^2+x_2^2)=0. \end{cases}$$

∴系统有孤立平衡点 $(0,0)$，且单位圆 $x_1^2+x_2^2=1$ 为平衡点.

对于 $(0,0)$，平衡点为稳定结点。

令 $x_1 = r\cos\theta$. $x_2 = r\sin\theta$. $\dot{r} = r(1-r^2)$. 当 $r<1$ 时，当 $t\to\infty$ 时，趋近于原点；当 $r>1$ 时，当 $t\to\infty$ 时，向无穷处发散.

∴系统无极限环.

(3). $\dot{x}_1 = 1 - x_1 x_2^2$ $\qquad$ $\dot{x}_2 = x_1$.

$$\begin{cases} x_1 x_2^2 - 1 = 0 \\ x_1 = 0. \end{cases}$$ $\qquad -1=0$，即无实根.

∴无平衡点

∴显然无极限环