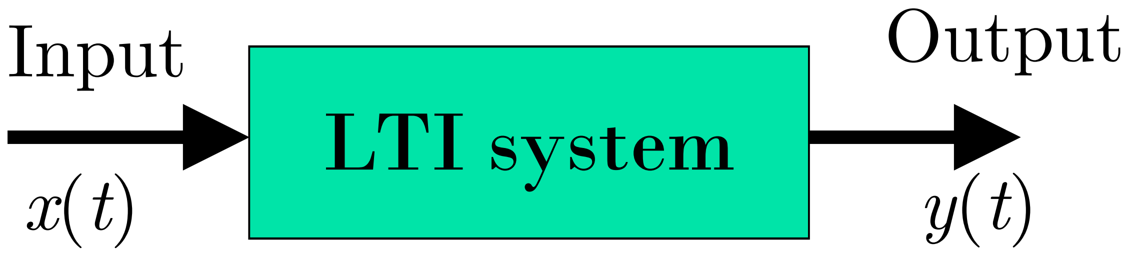
2-2 Transfer function and impulse-response function

2. Transfer function



Definition: The transfer function of an **LTI system** is defined as the ratio of the Laplace transform of the output variable to the Laplace transform of the input variable when **all initial conditions are zero**.

$$G(s) = \frac{Y(s)}{X(s)}$$

Consider a linear time-invariant system described by the following differential equation:

$$a_0 \frac{d^n}{dt^n} y + a_1 \frac{d^{n-1}}{dt^{n-1}} y + \cdots + a_{n-1} \frac{dy}{dt} + a_n y$$

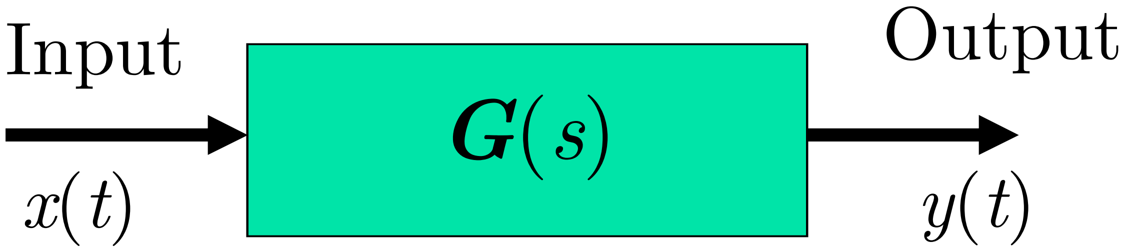
$$= b_0 \frac{d^m}{dt^m} x + b_1 \frac{d^{m-1}}{dt^{m-1}} x + \cdots + b_{m-1} \frac{dx}{dt} + b_m x, \quad n \geq m$$

By definition, the transfer function is

$$\frac{Y(s)}{X(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} := G(s)$$

The advantage of transfer function: It represents system dynamics by algebraic equations and clearly shows the input-output relationship:

$$Y(s) = G(s)X(s)$$

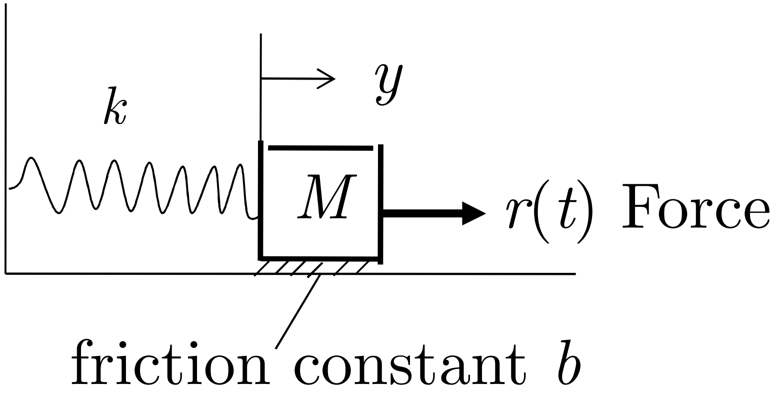


Example. Given a system described by the following differential equation

$$\frac{d^2}{dt^2}y + 2\frac{d}{dt}y + 3y = u(t)$$

Find its transfer function.

Example. Spring-mass-damper system:



Let the input be the force $r(t)$ and the output be the displacement $y(t)$ of the mass. Find its transfer function.

Solution: The system differential equation is

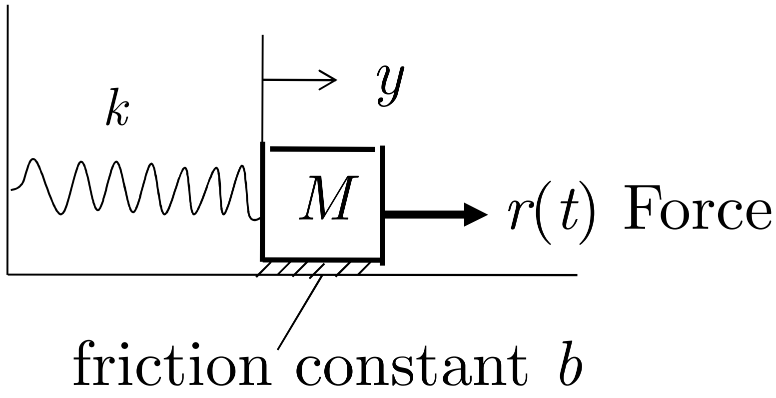
$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

from which we obtain its transfer function

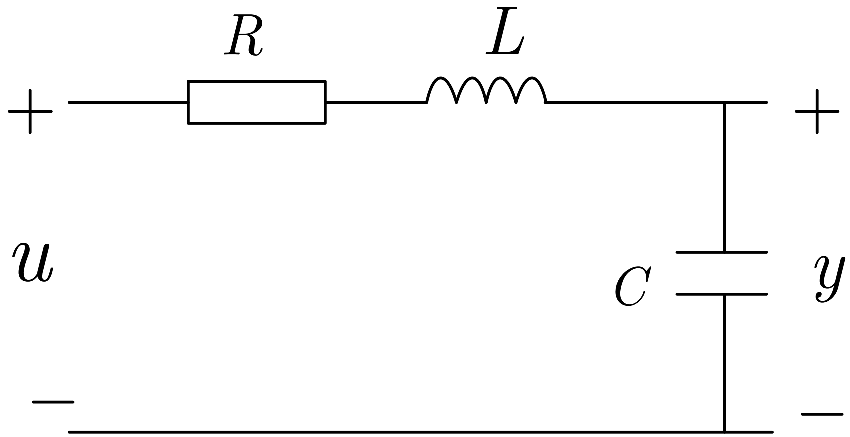
$$\frac{Y(s)}{R(s)}=\frac{1}{Ms^2+bs+k}$$

Comments on transfer function:

- is limited to LTI systems.
- is an operator to relate the output variable to the input variable of a differential equation.
- is a property of a system itself, independent of the magnitude and nature of the input (or driving function).
- does not provide any information concerning the physical structure of the system. That is, the transfer functions of many physically different systems can be identical.



k
y
M
$r(t)$ Force
friction constant b



R
L
$+$
u
$-$
C
y
$+$
$-$

3. Convolution integral

From

$$Y(s) = G(s)X(s)$$

and by using the convolution theorem, we have

$$\begin{aligned} y(t) &= \int_0^t g(\tau)x(t-\tau)d\tau \\ &= \int_0^t g(t-\tau)x(\tau)d\tau \end{aligned}$$

where both $g(t)$ and $x(t)$ are 0 for $t<0$.

Example. Given

$$Y(s) = G(s)R(s) = \frac{1}{s+1}R(s)$$

If R(s)=$1/s$, find $y(t)$.

Solution: By definition of convolution integral,

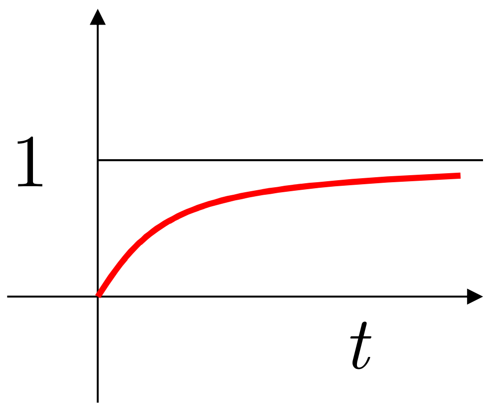
$$g(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t},\ t \geq 0$$

$$r(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1(t)$$

Hence,

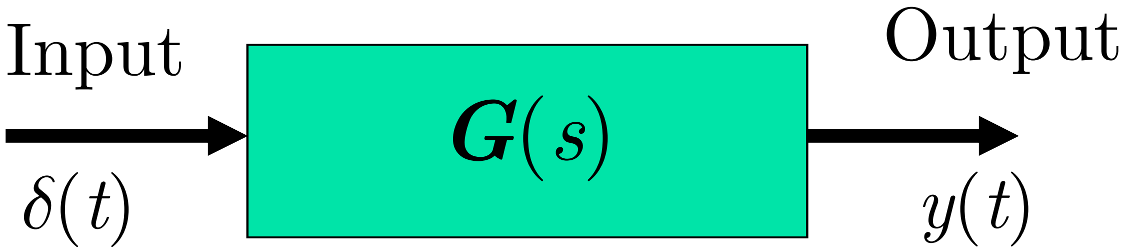
$$y(t) = \int_0^t g(t-\tau)r(\tau)d\tau = \int_0^t e^{-(t-\tau)} \cdot 1 d\tau$$

$$= e^{-t}\int_0^t e^{\tau}d\tau = 1 - e^{-t},\ t \geq 0$$



4. Impulse response function

Consider the output (response) of an LTI system to a **unit-impulse input** when the initial conditions are zero:



Hence,

$$Y(s) = G(s)R(s) = G(s)\mathcal{L}(\delta(t))$$
$$= G(s)\cdot 1 = G(s)$$

and

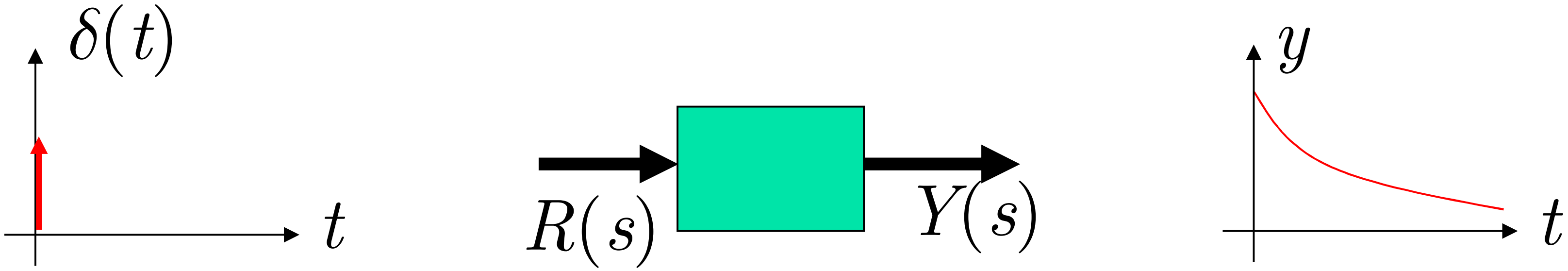
$$y(t) = \mathcal{L}^{-1}[G(s)] := g(t)$$

where $g(t)$ is called **impulse response function**.

An impulse response function $g(t)$ is the inverse Laplace transform of the system's transfer function!

Example. Given

$$Y(s) = G(s)R(s)$$



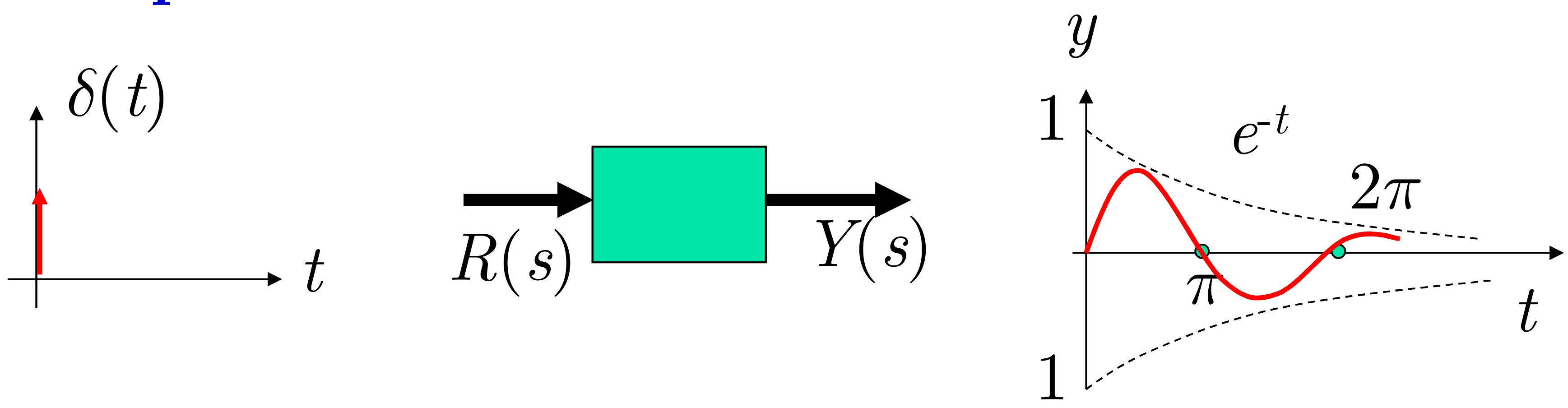
If

$$y(t) = e^{-t}, t \geq 0$$

determine the system's transfer function.

It is hence possible to obtain complete information about the dynamic characteristics of the system by exciting it with an impulse input and measuring the response (In practice, a pulse input with a very short duration can be considered an impulse).

Example. Let



Assume that the system is LTI. Determine its transfer function.

Example. Let

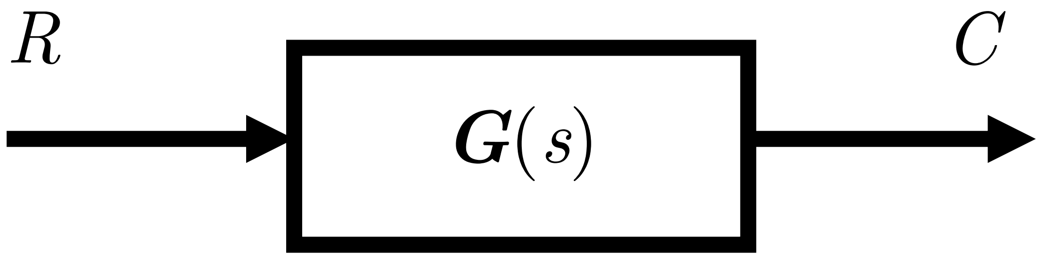
$$Y(s) = G(s)R(s) = \frac{1}{s^3 + 3s^2 + 3s + 1} R(s)$$

Determine its impulse response function.

2-3 Automatic control systems

1. Block Diagrams

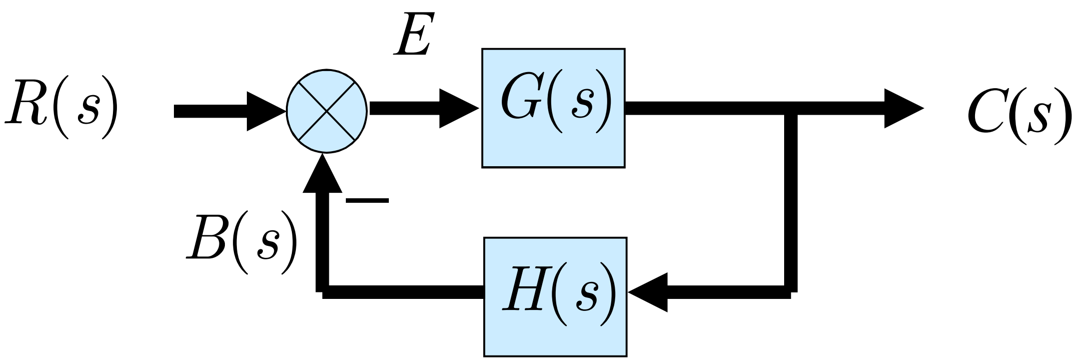
Definition: A block diagram of a system is a pictorial representation of the functions performed by each component and of the flow of signals.



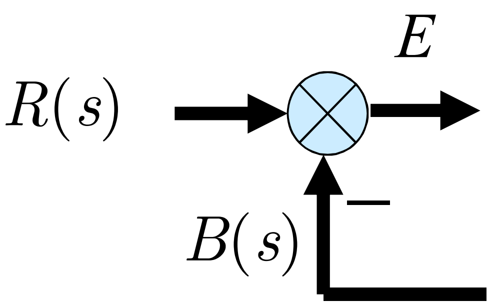
$$\frac{C(s)}{R(s)} = G(s) \qquad \text{or} \qquad C(s) = G(s)R(s)$$

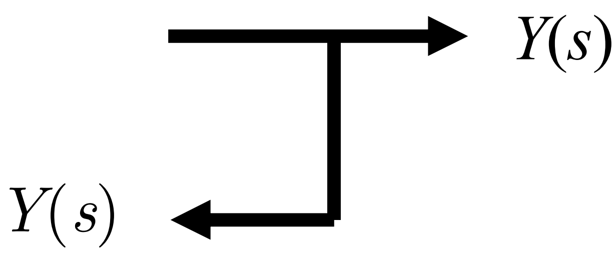
2. Block Diagram of a closed-loop system

A real physical system includes more than one components. The following is a typical **feedback system** represented by block diagram:

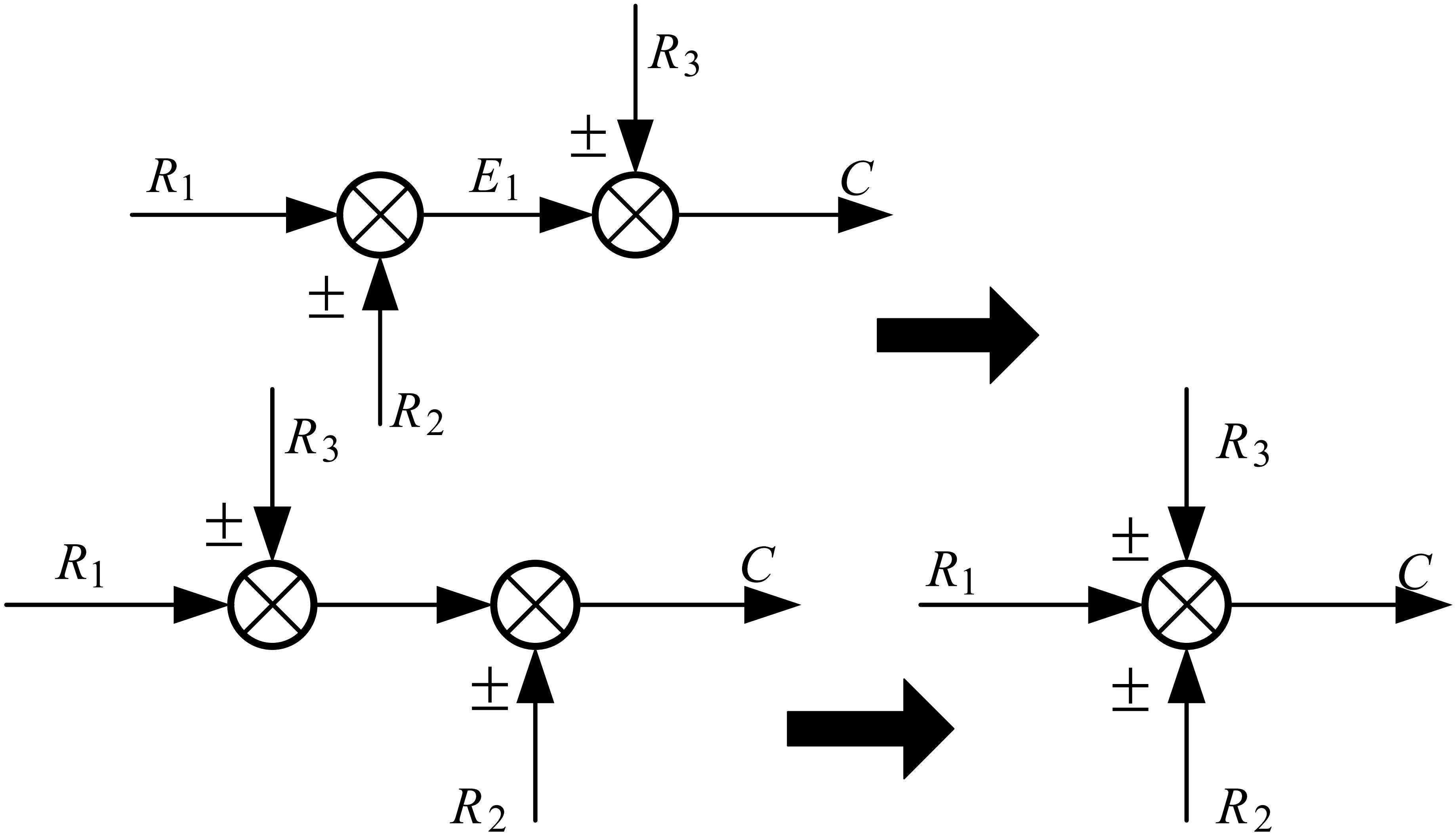


where the summing point $E(s)=R(s)-B(s)$ and the branch point are shown below.





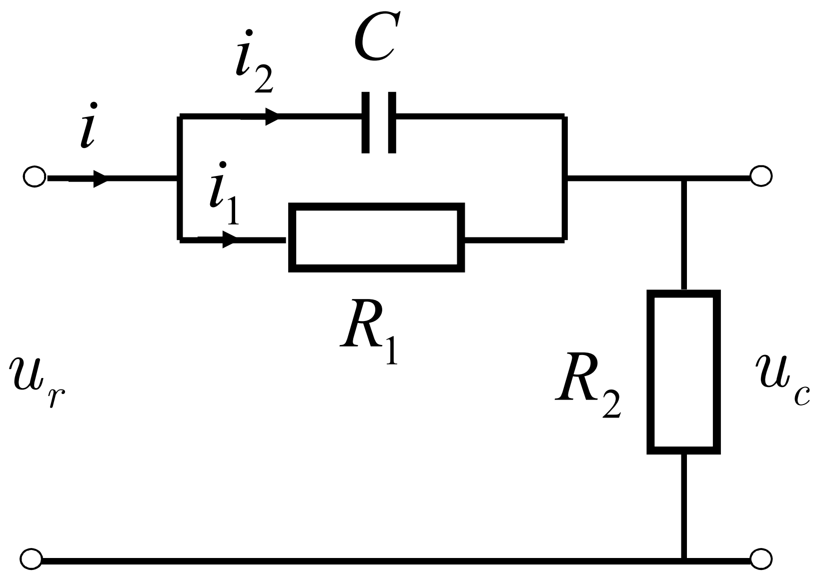
The property of summing point:



Example. A network system is shown below, where u_c is the output and u_r is the input. Draw its block diagram.

Solution:

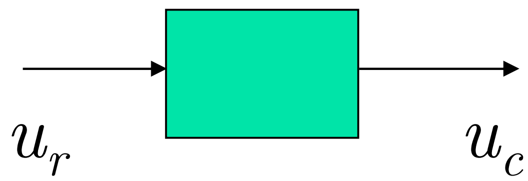
Step 1. Write the input and output relationship of each device:



$$u_{R1} = R_1 i_1 = (u_r - u_c)$$

$$i_2 = C\frac{du_{R1}}{dt} = CR_1\frac{di_1}{dt}$$

$$u_c = R_2(i_1 + i_2)$$



Step 2. Taking the Laplace transform of both sides of the above equations yields:

$$R_1 I_1(s) = (U_r(s) - U_c(s))$$

$$I_2(s) = R_1 C s I_1(s)$$

$$U_c(s) = R_2(I_1(s) + I_2(s))$$

Step 3. Rearrange each above equation so that its left-hand side is the output variable and the right-hand side is the transfer function multiplied by input signals:

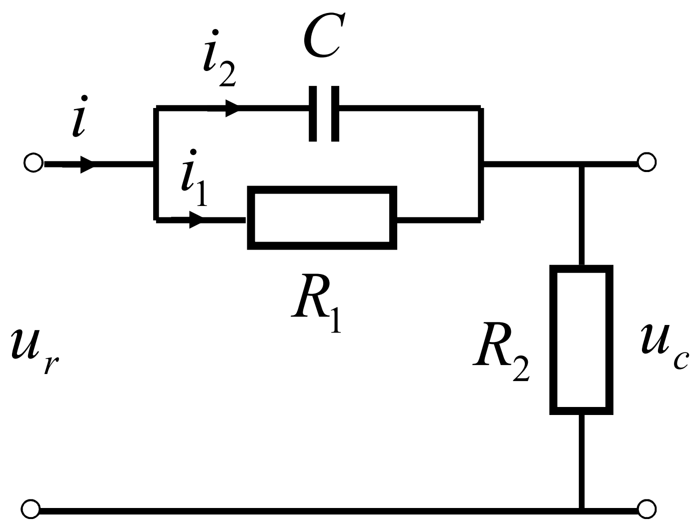
$$I_1(s) = \frac{1}{R_1}[U_r(s) - U_c(s)] \quad (1)$$

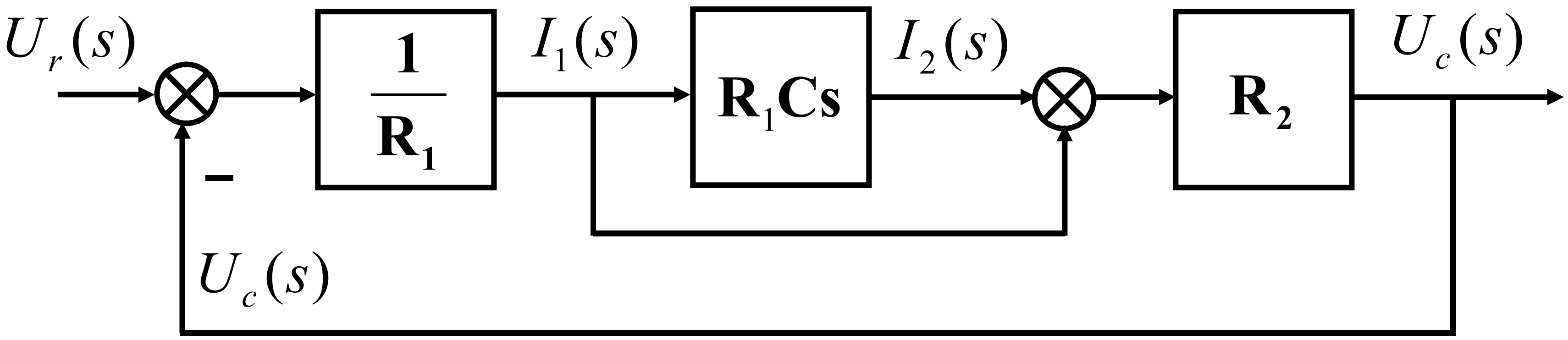
$$I_2(s) = R_1 C s I_1(s) \quad (2)$$

$$U_c(s) = R_2[I_1(s) + I_2(s)] \quad (3)$$

Step 4. Based on (1)-(3), draw the block diagram:

$$
\begin{cases}
I_1(s) = \dfrac{1}{R_1}[U_r(s) - U_c(s)] & (1) \\
I_2(s) = R_1 C s I_1(s) & (2) \\
U_c(s) = R_2[I_1(s) + I_2(s)] & (3)
\end{cases}
$$





Example. A system is described by the following equations:

$$x_1 = r - c$$

$$x_2 = \tau \dot{x}_1 + K_1 x_1$$

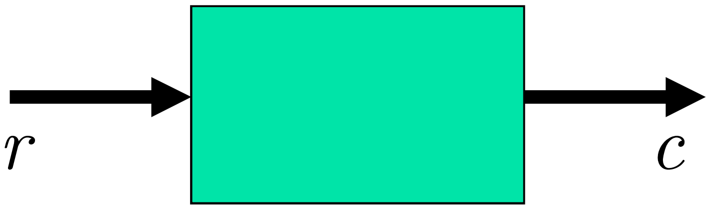
$$x_3 = K_2 x_2$$

$$x_4 = x_3 - x_5 - K_5 c$$

$$\dot{x}_5 = K_3 x_4$$

$$K_4 x_5 = T\dot{c} + c$$

Draw its block diagram, where τ, K_i and T are positive constants, the input and output signals are r and c, respectively, and $x_1 \sim x_5$ are intermediate variables. .



Example. A system is described by the following equations:

$$x_1 = r - c + n_1$$

$$x_2 = K_1 x_1$$

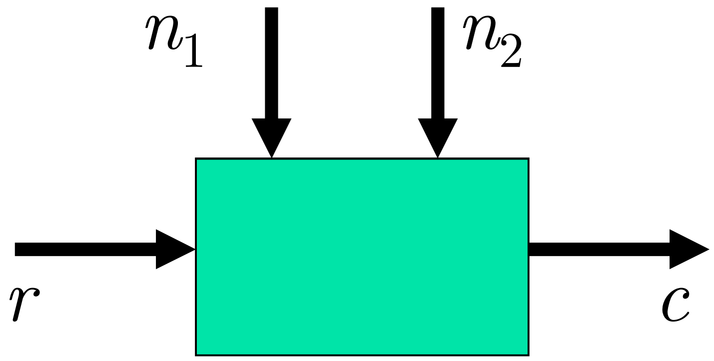
$$x_3 = x_2 - x_5$$

$$T\dot{x}_4 = x_3$$

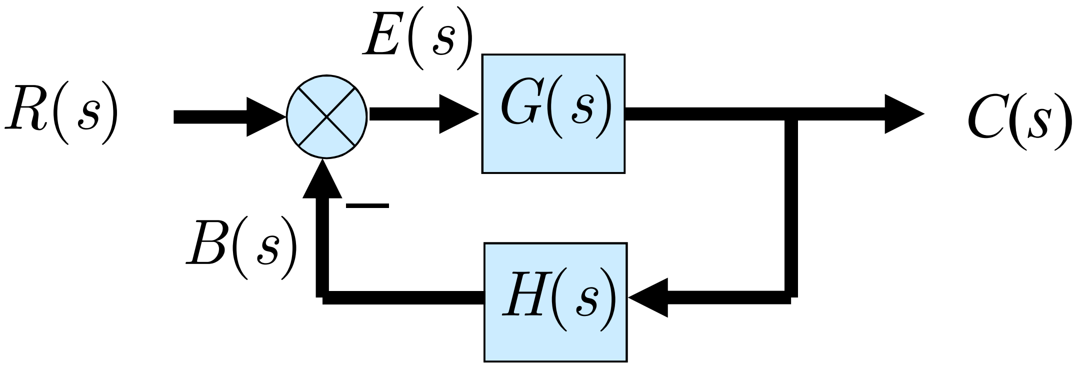
$$x_5 = x_4 - K_2 n_2$$

$$K_0 x_5 = \ddot{c} + \dot{c}$$

Draw its block diagram, where K_i and T are positive constants, the input and output signals are r and c, respectively, n_1, n_2 are disturbances, and $x_1 \sim x_5$ are intermediate variables.



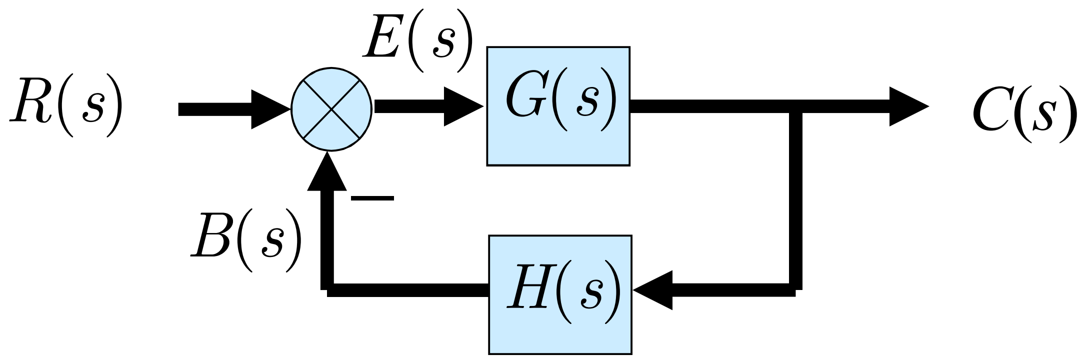
3. Open-loop transfer function and feedforward transfer function



Two important concepts:

Open-loop transfer function: The ratio of the feedback signal $B(s)$ to the actuating error signal $E(s)$ is called the **open-loop transfer function**. That is,

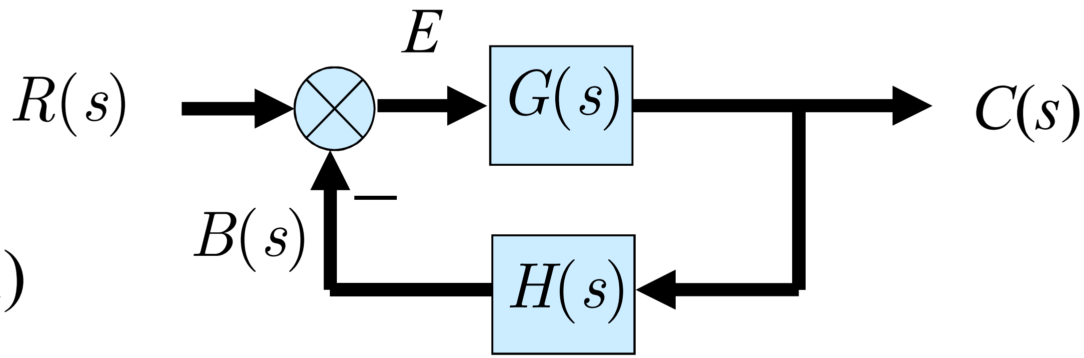
$$\frac{B(s)}{E(s)} = G(s)H(s)$$



Feedforward transfer function: The ratio of the output $C(s)$ to the actuating error signal $E(s)$ is called the **feedforward transfer function**. That is

$$\frac{C(s)}{E(s)} = G(s)$$

4. Closed-loop transfer function



$$C(s) = G(s)E(s) \quad (1)$$

$$E(s) = R(s) - B(s) = R(s) - H(s)C(s) \quad (2)$$

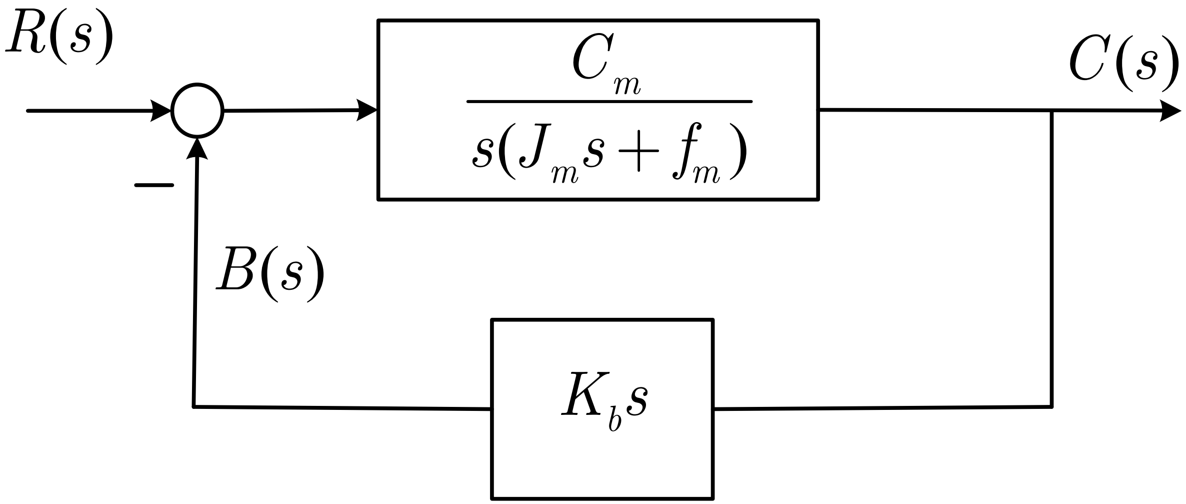
Substituting (2) into (1) yields

$$C(s) = G(s)R(s) - G(s)H(s)C(s) \quad (3)$$

from which we obtain the closed-loop transfer function as

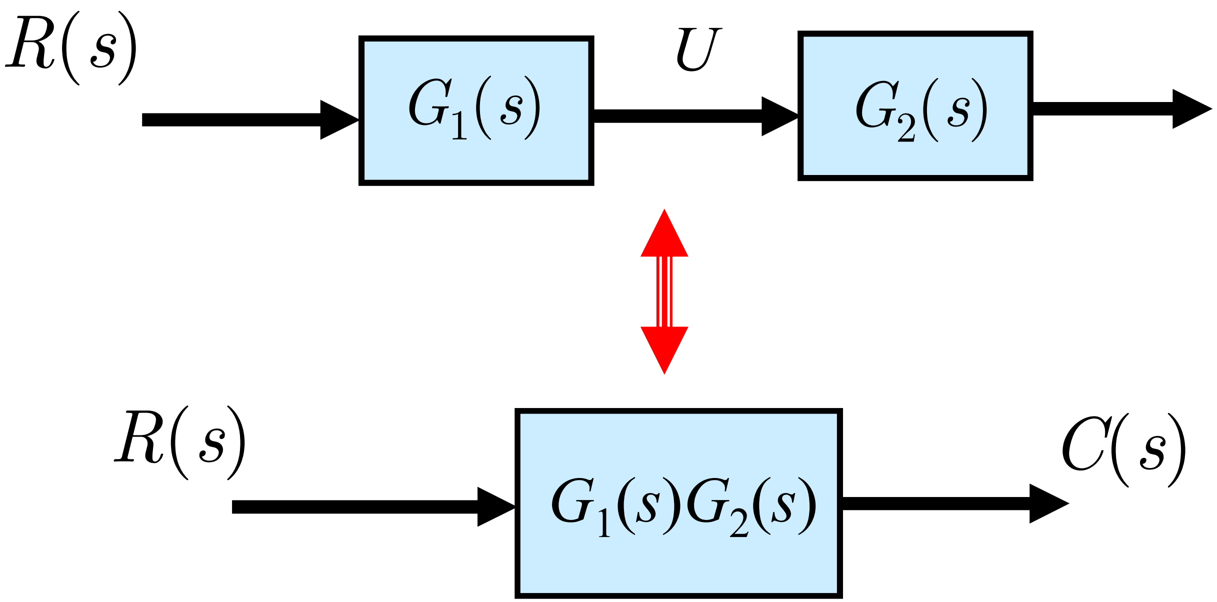
$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Example. A block diagram of a system is shown below. Determine its closed-loop transfer function $C(s)/R(s)$.



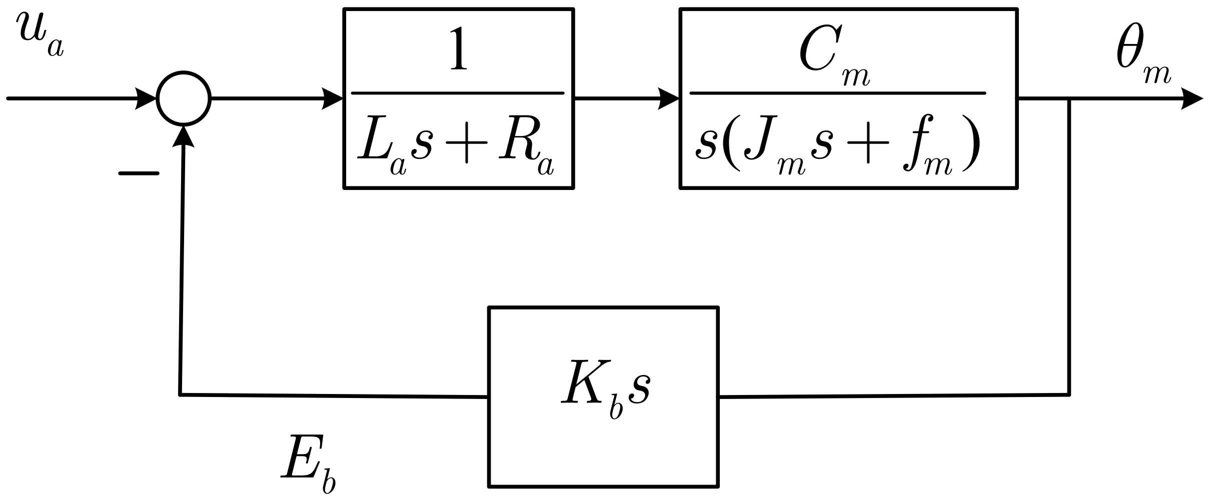
5. Obtaining cascaded, parallel, and feedback transfer functions

a) Cascaded system

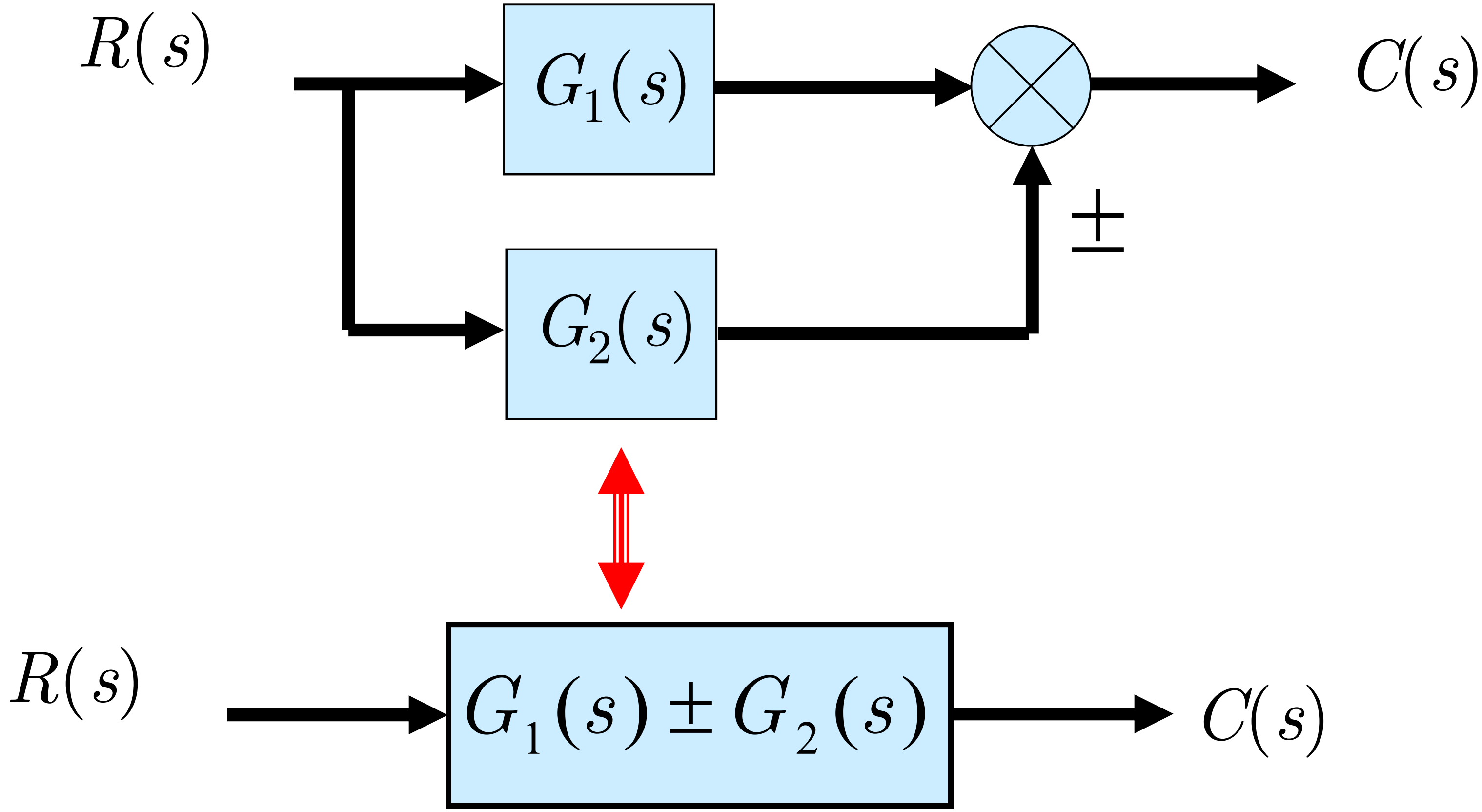


$$C(s) = G_2(s)U(s) = G_2(s)G_1(s)R(s)$$

Example. A block diagram of DC motor is shown below. Determine its open-loop transfer function and feedforward transfer function.

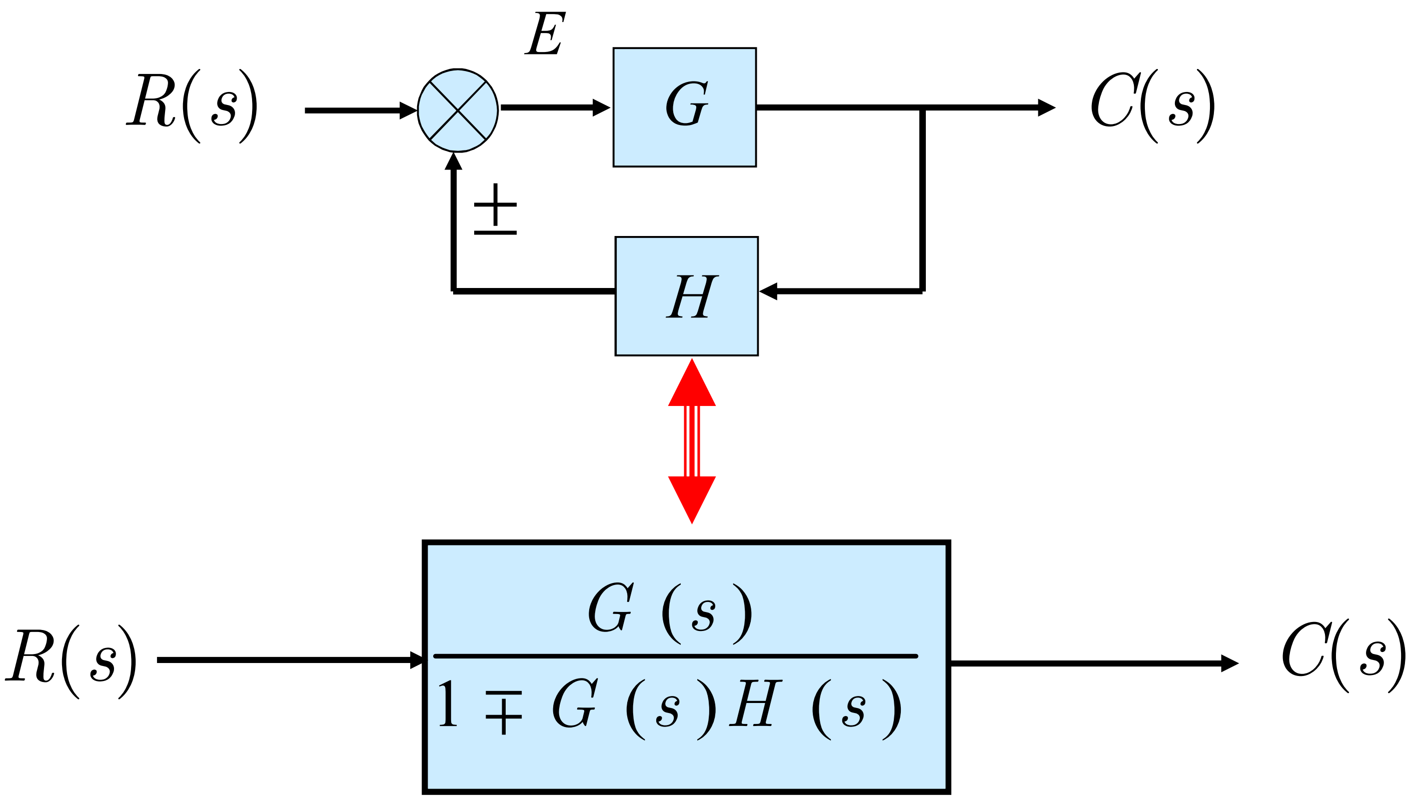


b) Parallel Blocks



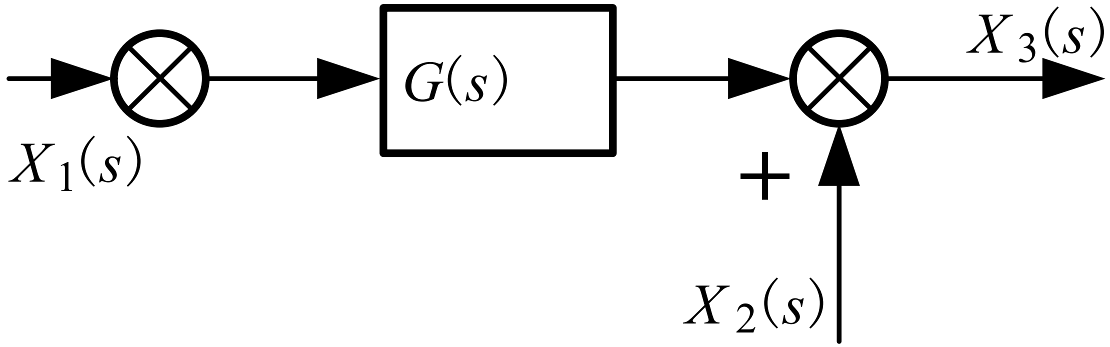
$$C(s) = G_1(s)R(s) \pm G_2(s)R(s) = (G_1(s) \pm G_2(s))R(s)$$

c) Feedback loop



$$C(s) = \frac{G(s)}{1 \mp G(s)H(s)} R(s)$$

Diagram simplification: Moving a summing point ahead of a block:



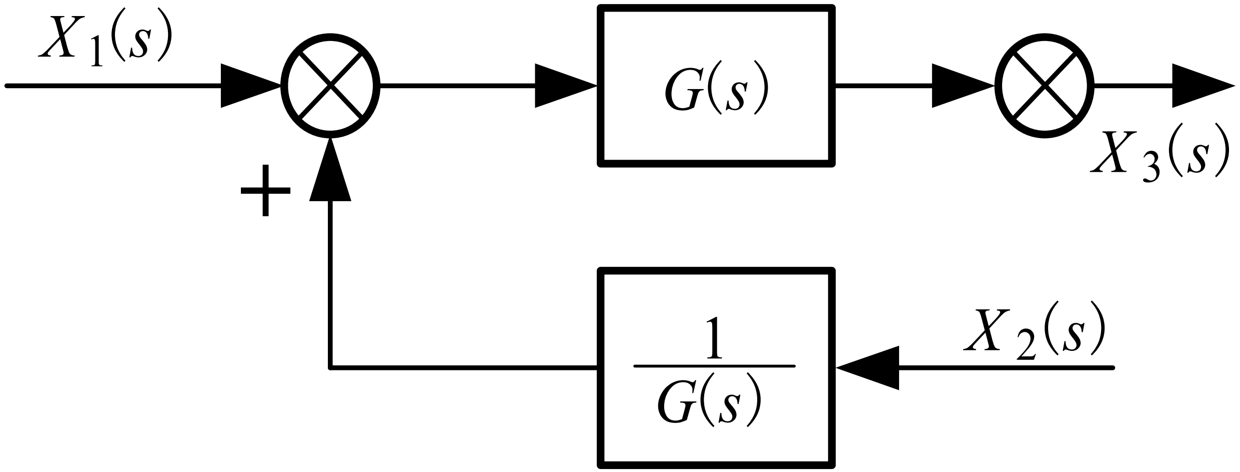
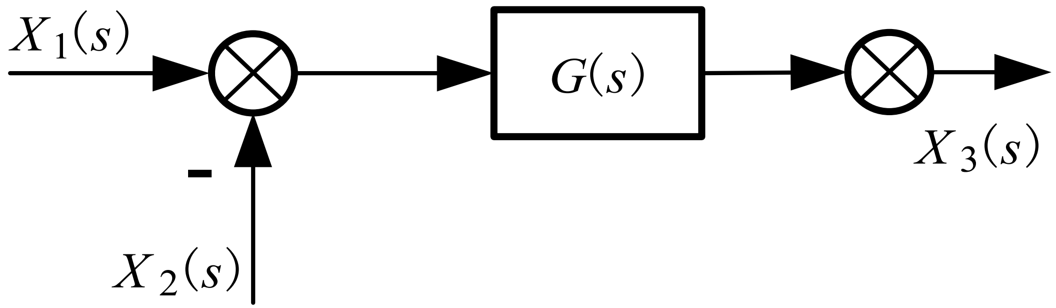
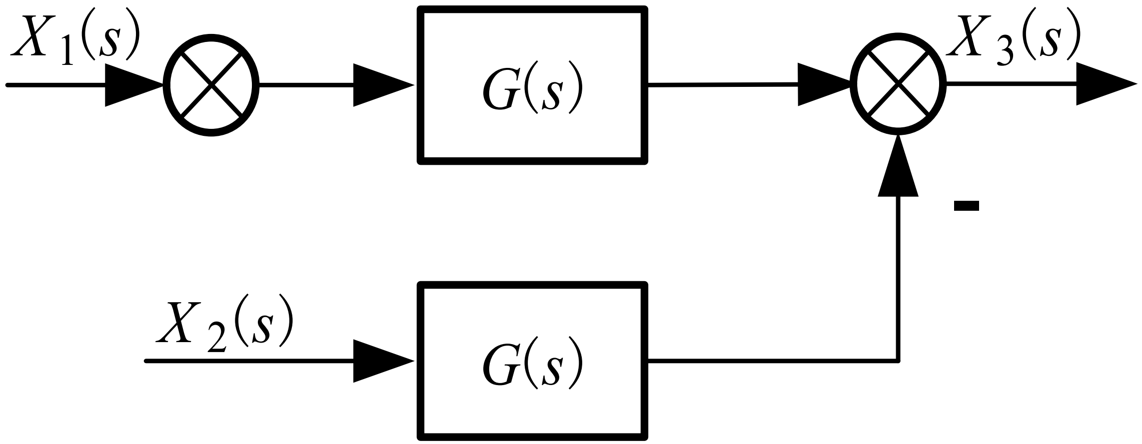
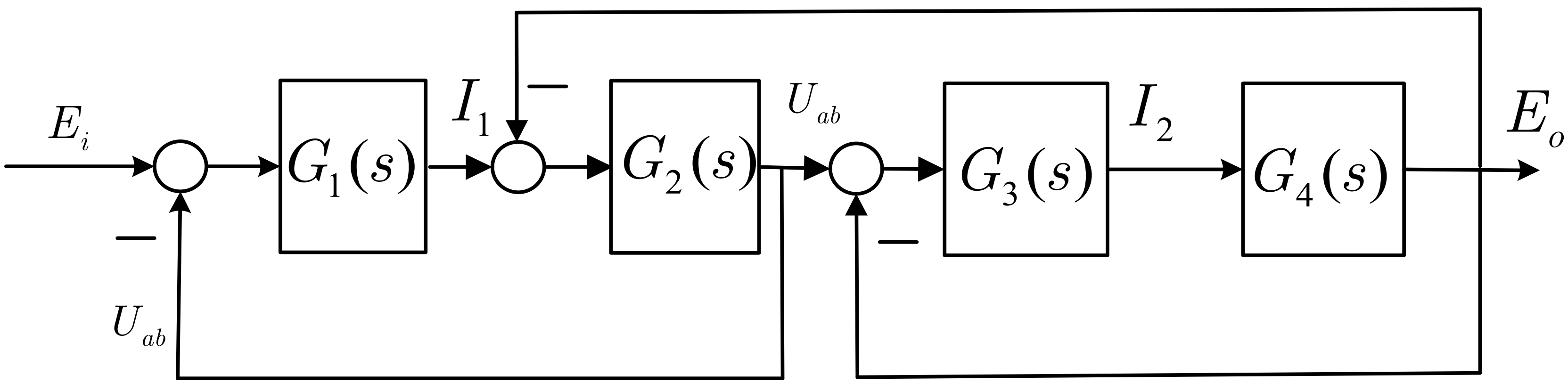


Diagram simplification: Moving a summing point behind a block:



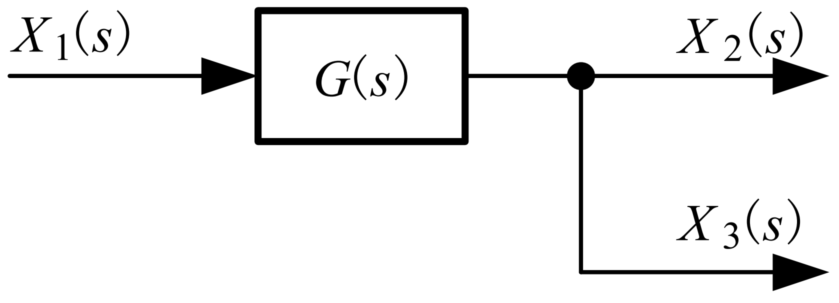


Example. Simplify the following block diagram. Then find the closed-loop transfer function $E_o(s)/E_i(s)$.



If there exists a branch point between two summing points, do not move the summing point.

Diagram simplification: Moving a branch point ahead of a block



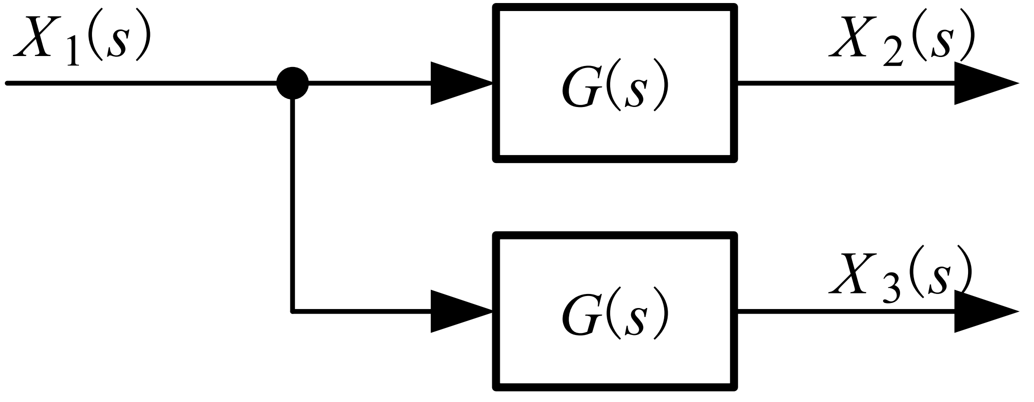
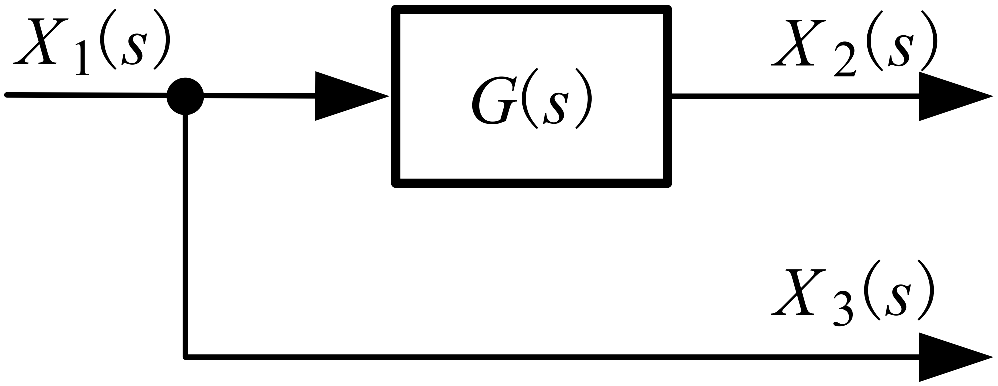
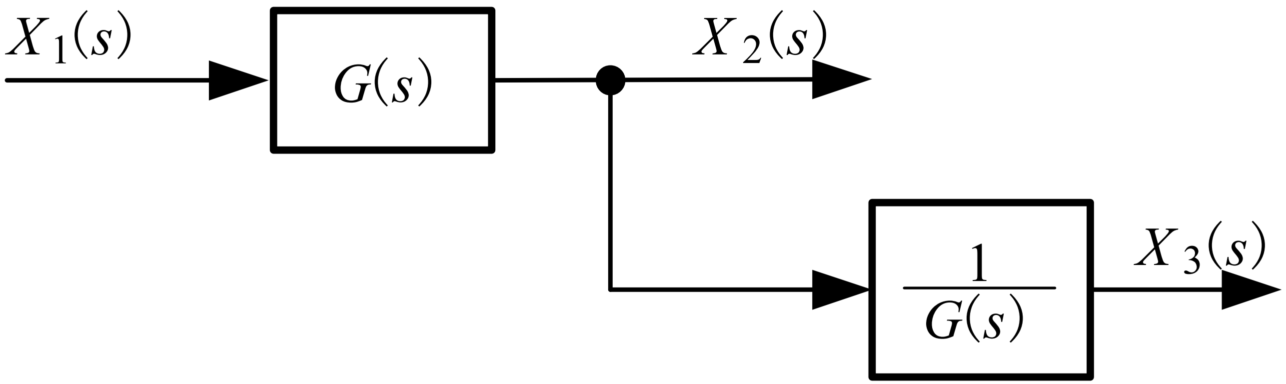


Diagram simplification: Moving a branch point behind of a block





Example. Simplify the following block diagram. Then obtain the closed-loop transfer function $E_o(s)/E_i(s)$.

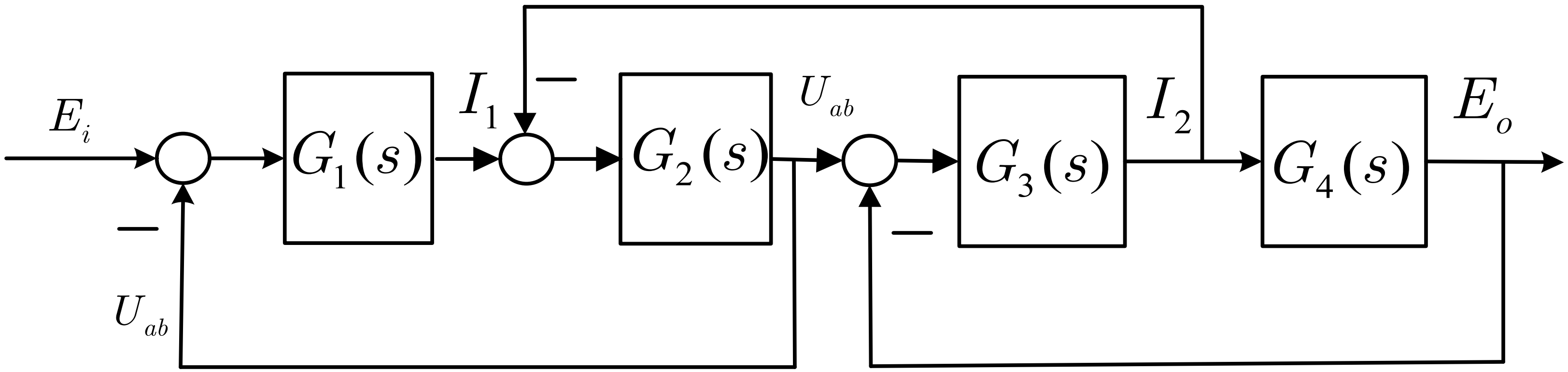
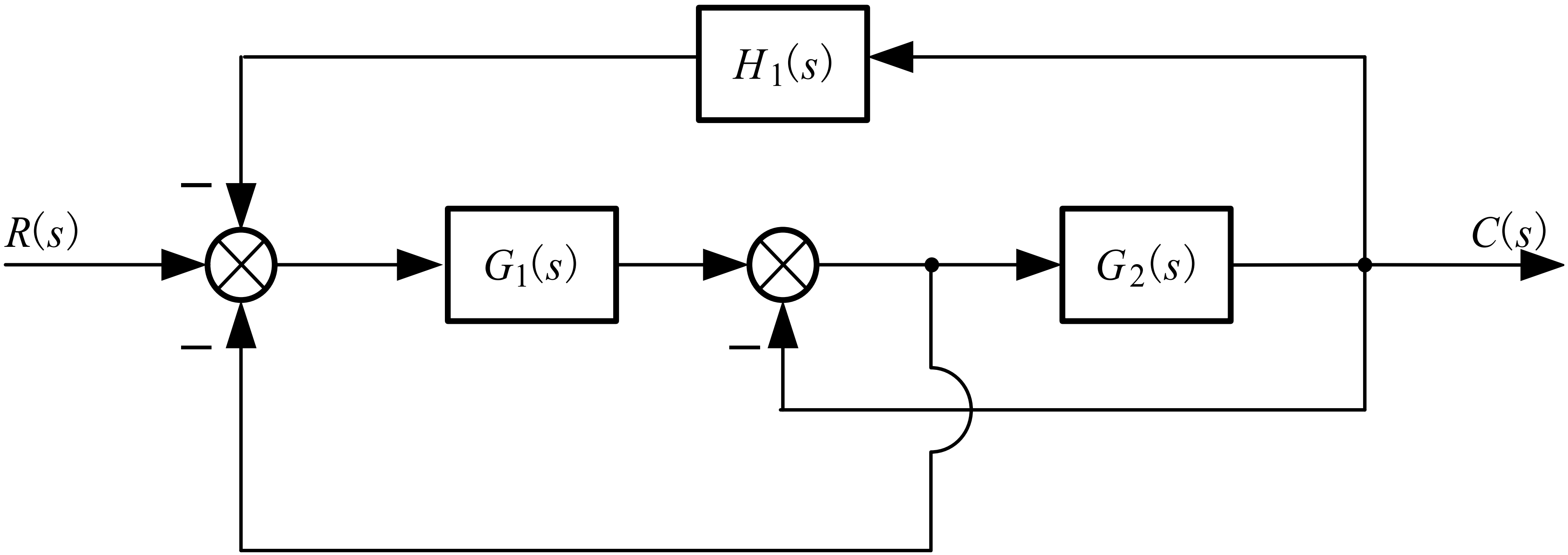


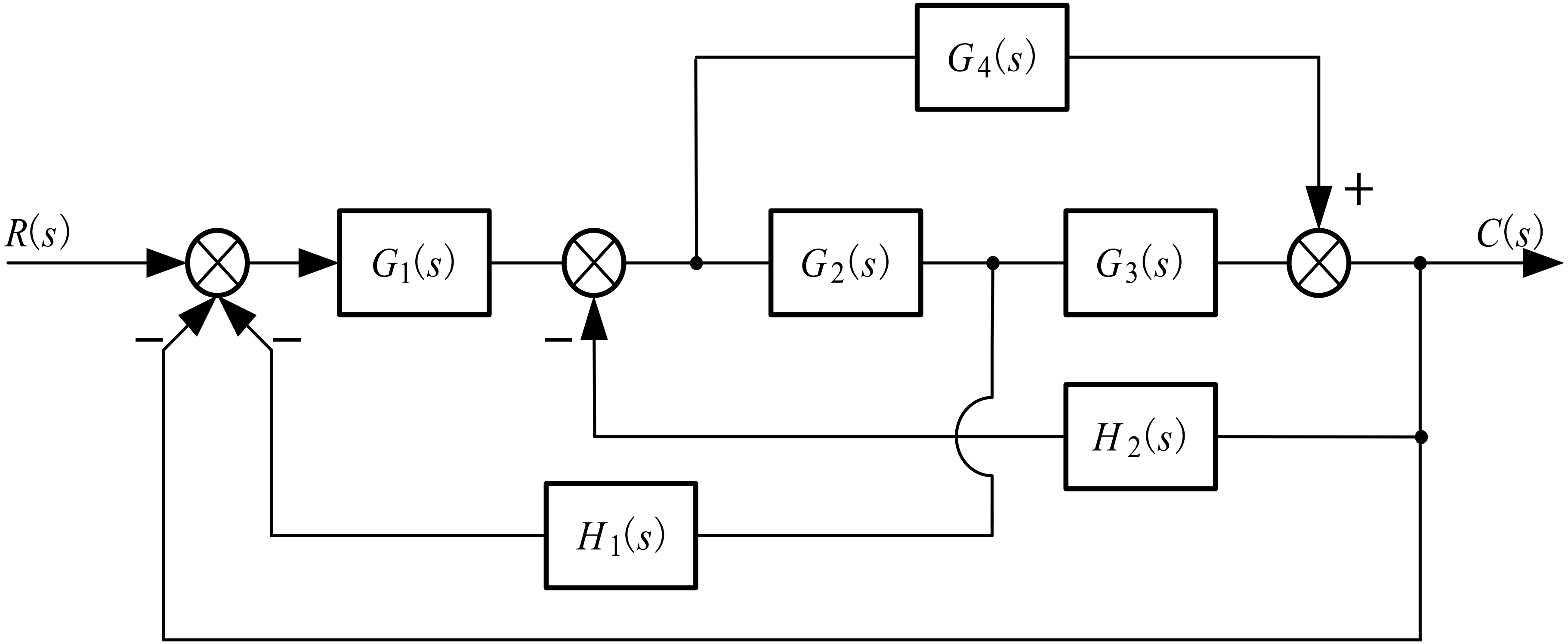
Diagram simplification: Examples

Example. The block diagram of a system is shown below. Find $C(s)/R(s)$.



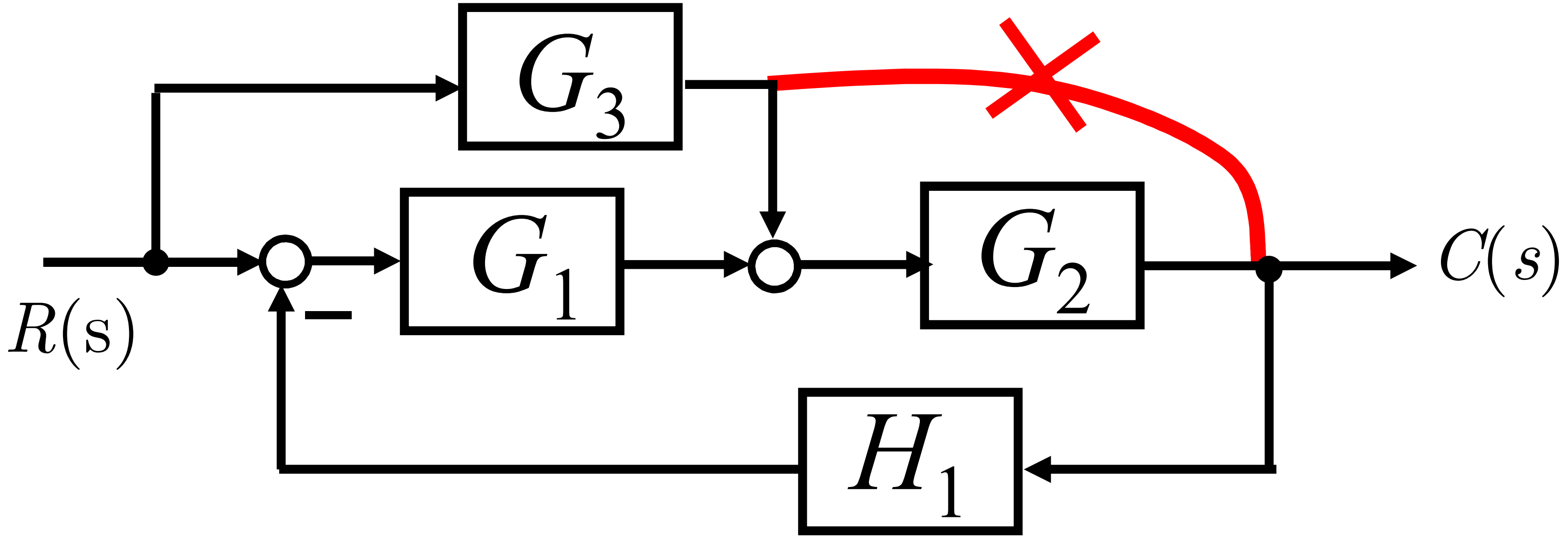
Note that only the movement between two summing points (two branch points) is valid.

Example. The block diagram of a system is shown below. Find $C(s)/R(s)$.

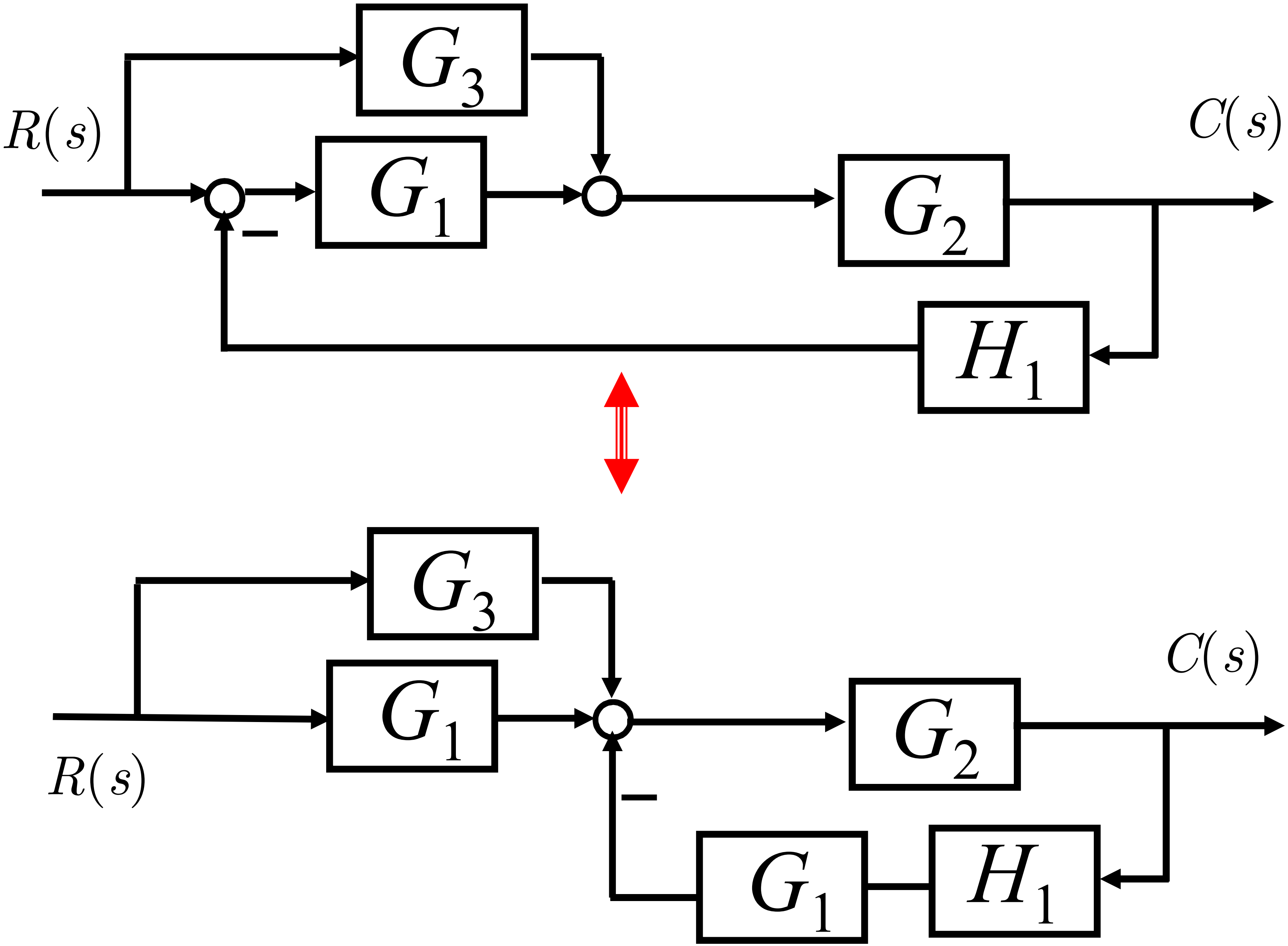


If there exists a branch point (summing point) between two summing points (two branch points), do not move them.

Example. The block diagram of a given system is shown below. Obtain the transfer function that relates the output $C(s)$ in function of the input R(s), i.e., $C(s)/R(s)$.

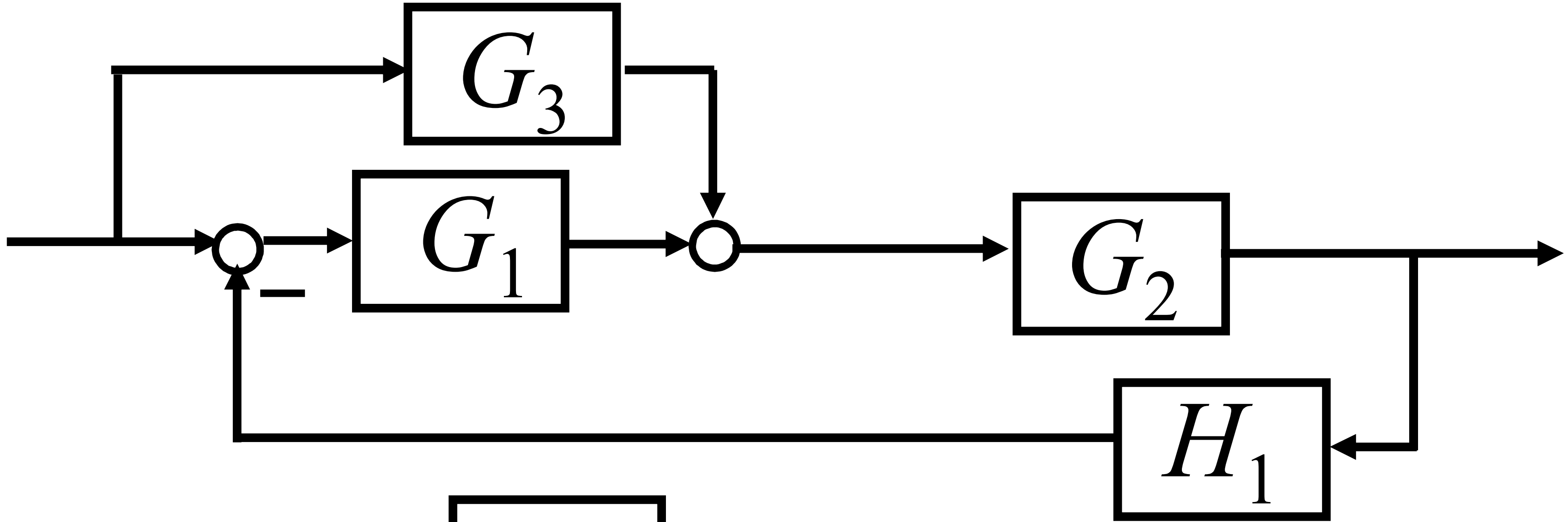


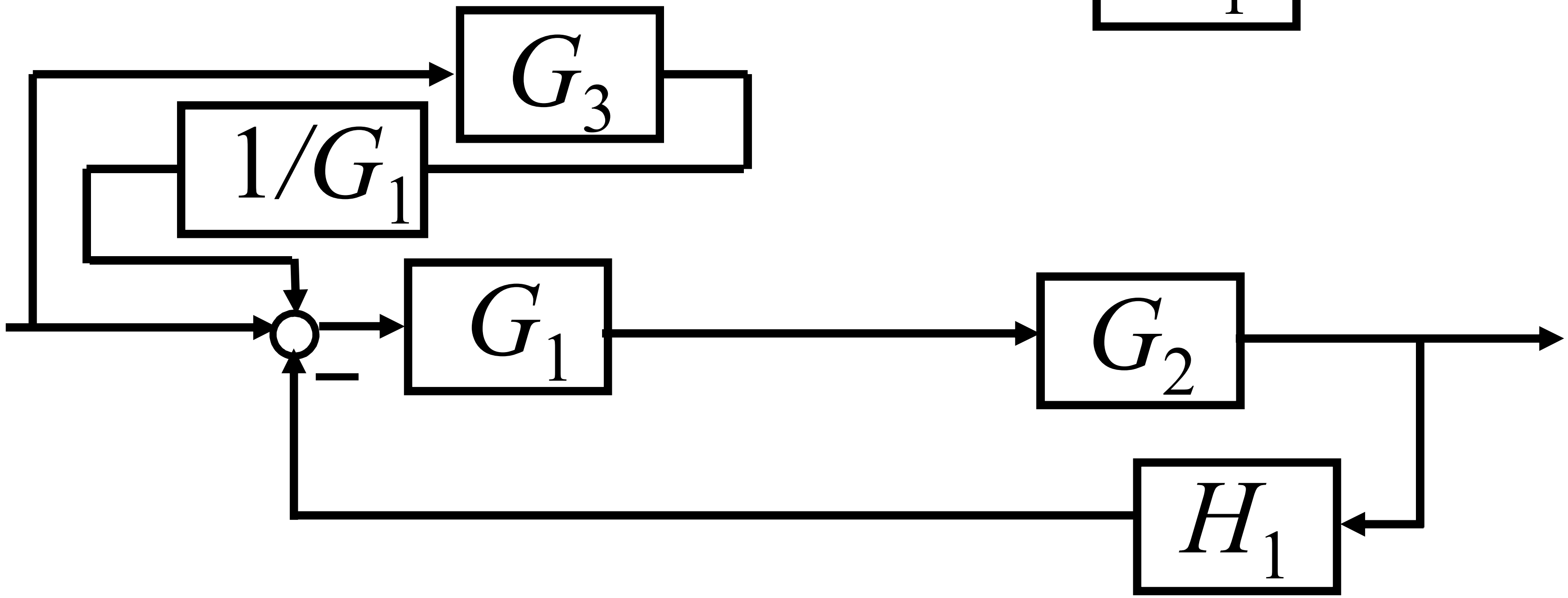
Note that only the movement between two summing points (two branch points) is valid.



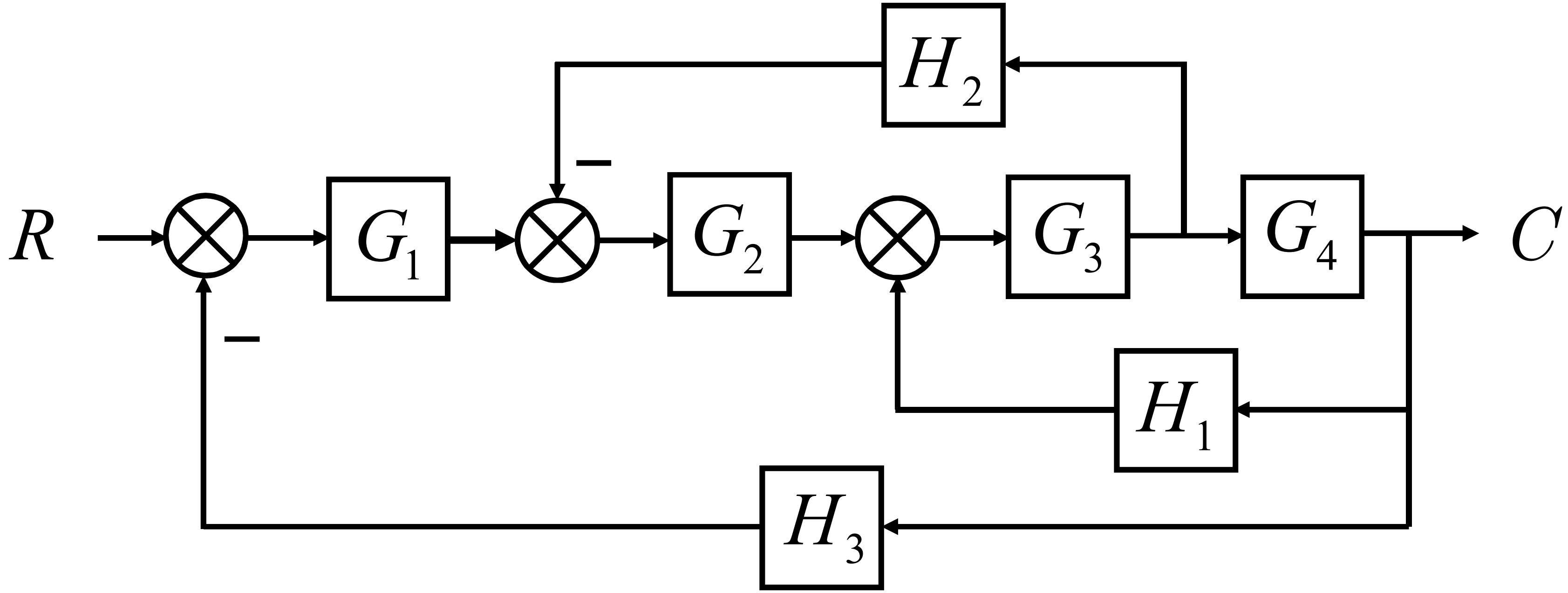
$R(s)$
G_3
G_1
G_2
H_1
$C(s)$
$R(s)$
G_3
G_1
G_2
G_1
H_1
$C(s)$

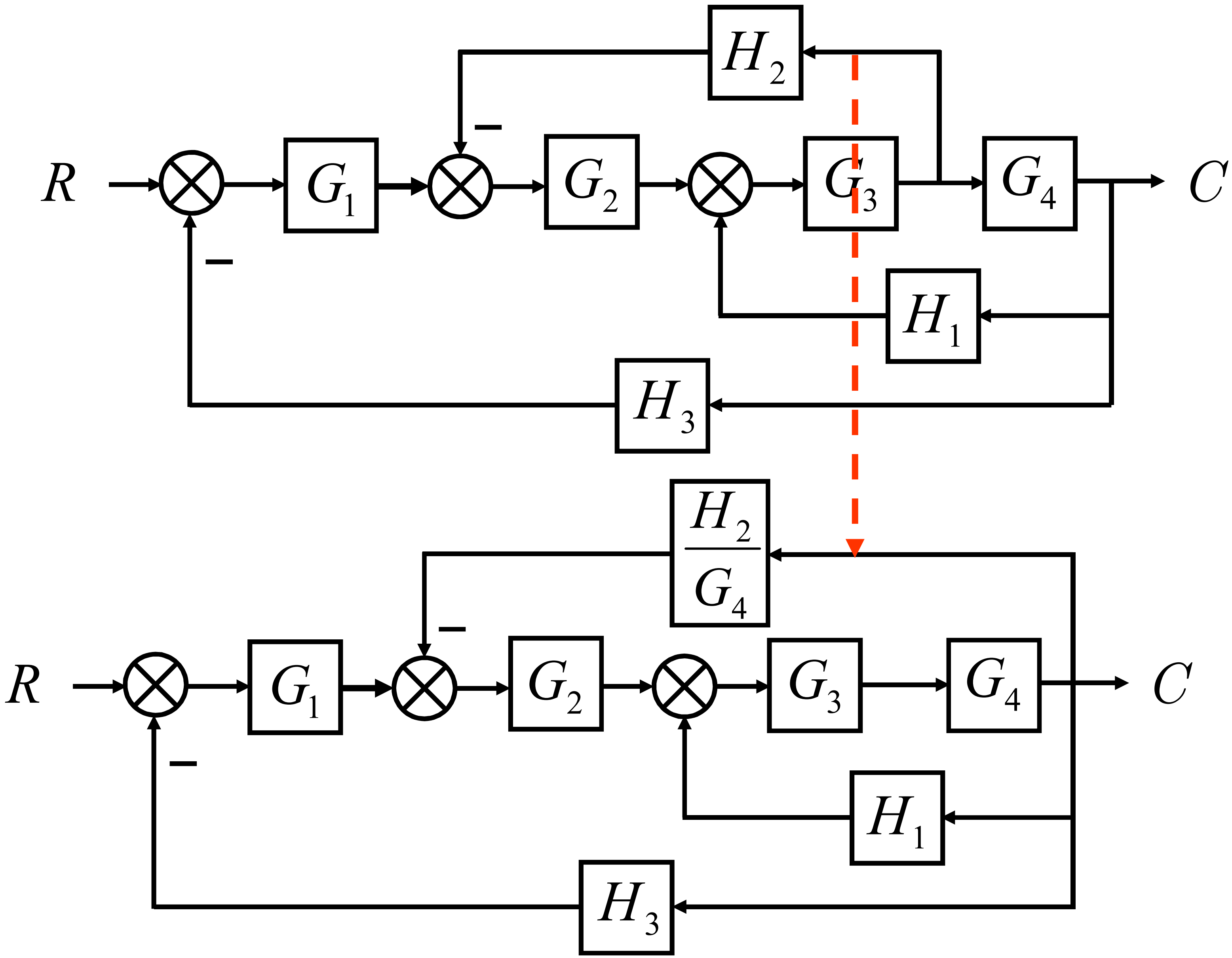
An alternative way to simplify the diagram is



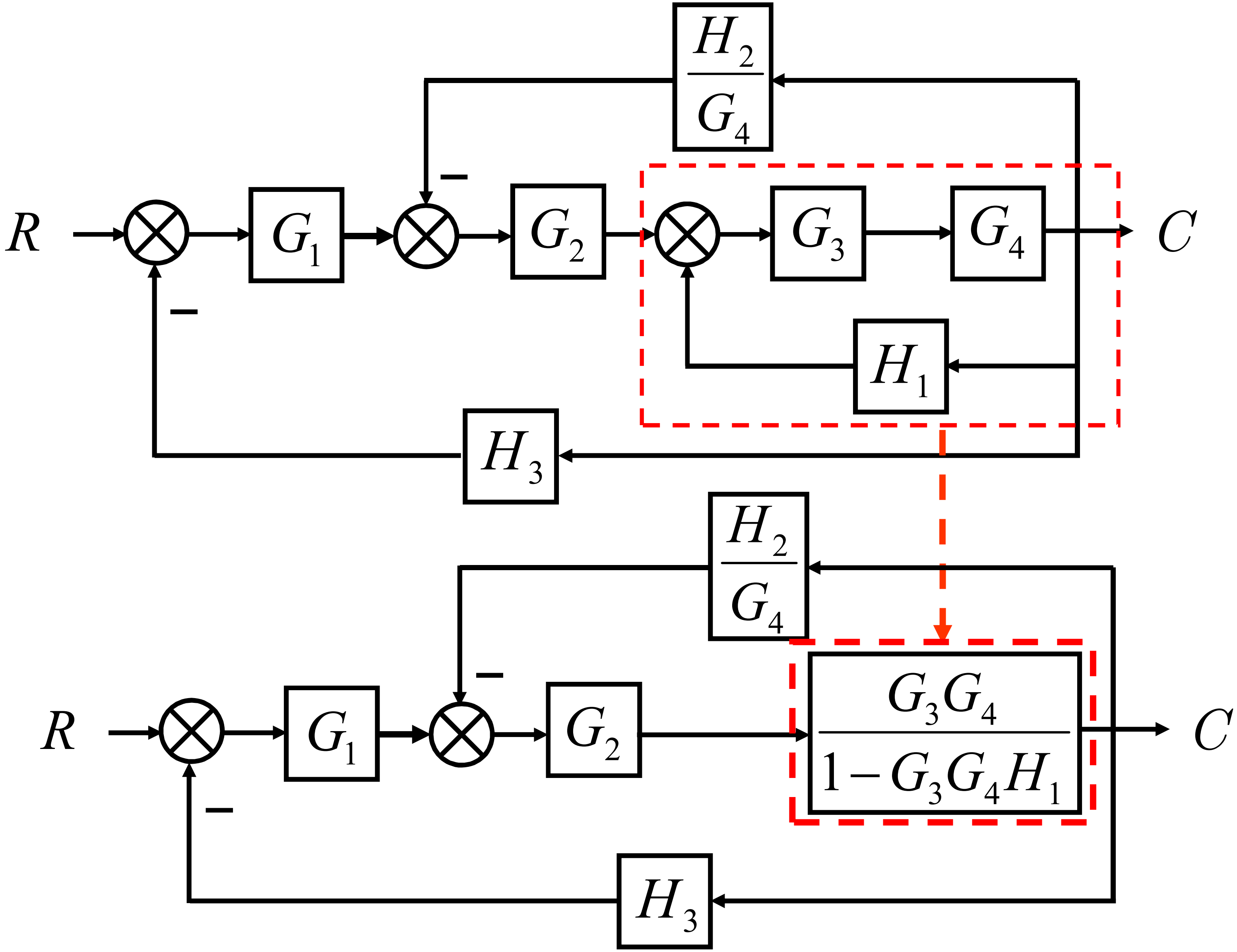


Example. The block diagram of a given system is shown below. Find $C(s)/R(s)$.

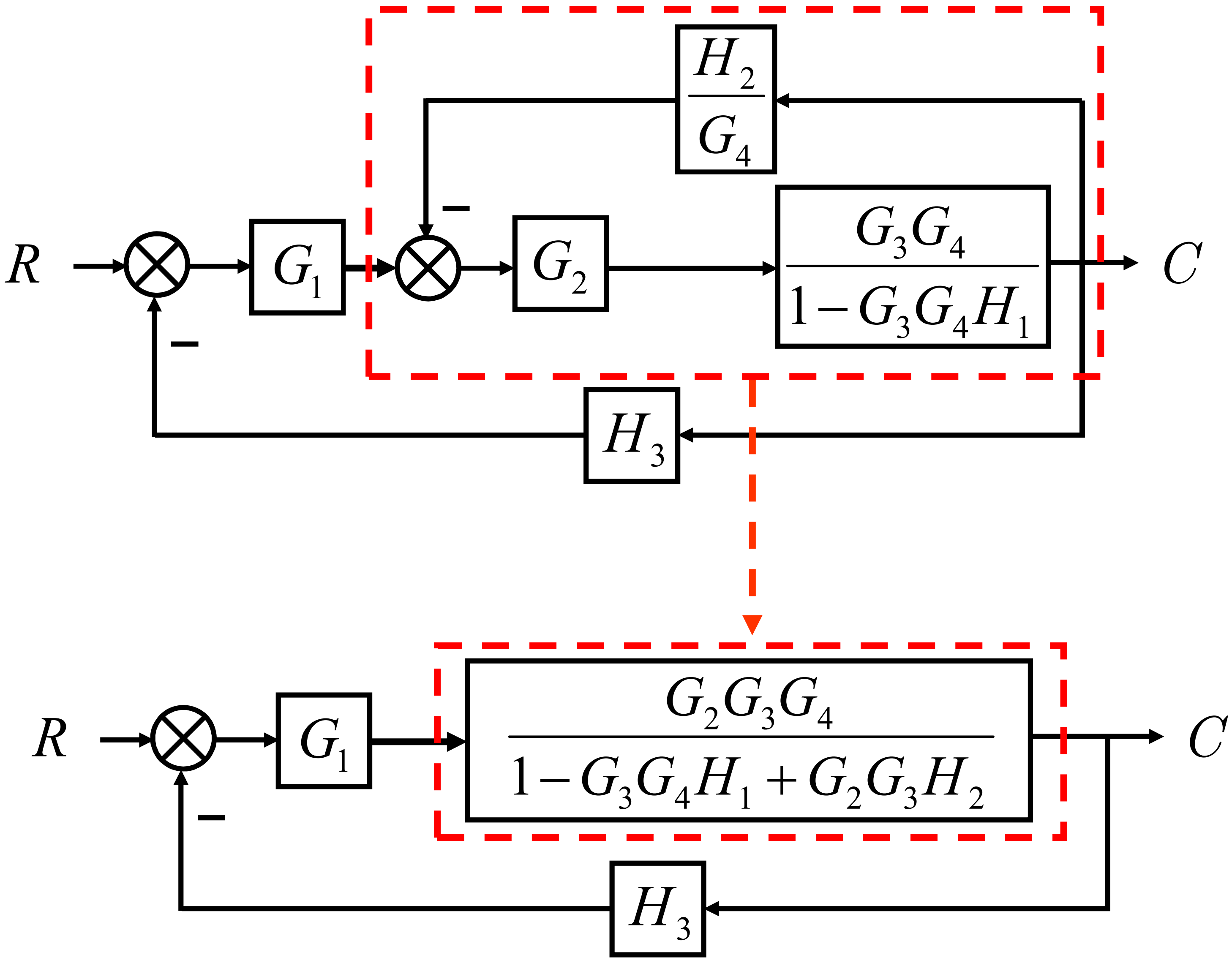




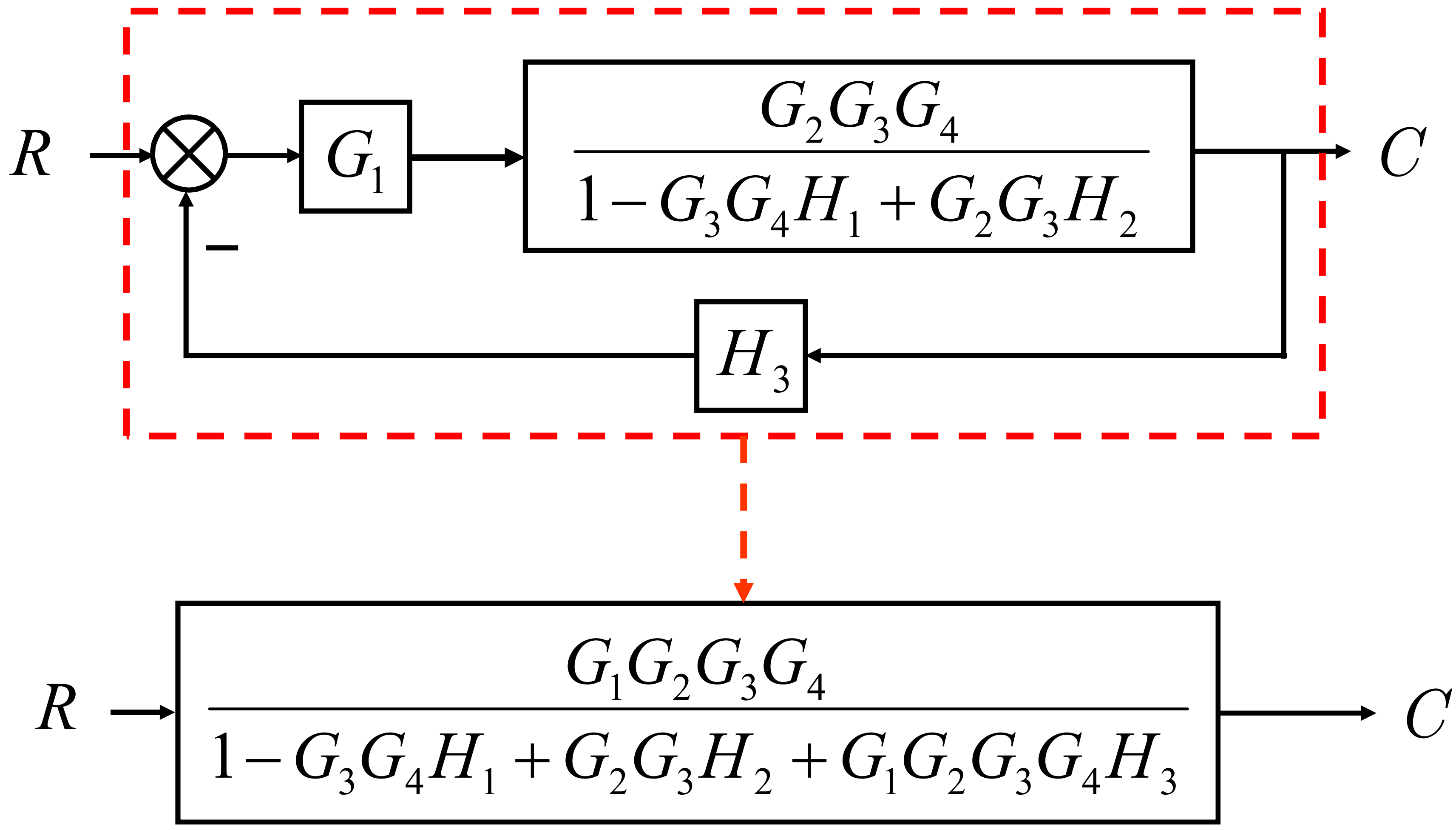
R
G1
G2
G3
G4
C
H2
H1
H3
R
G1
G2
G3
G4
C
H2/G4
H1
H3



R
G1
G2
G3
G4
C
H2/G4
H1
H3
R
G1
G2
G3G4/(1−G3G4H1)
C
H2/G4
H3

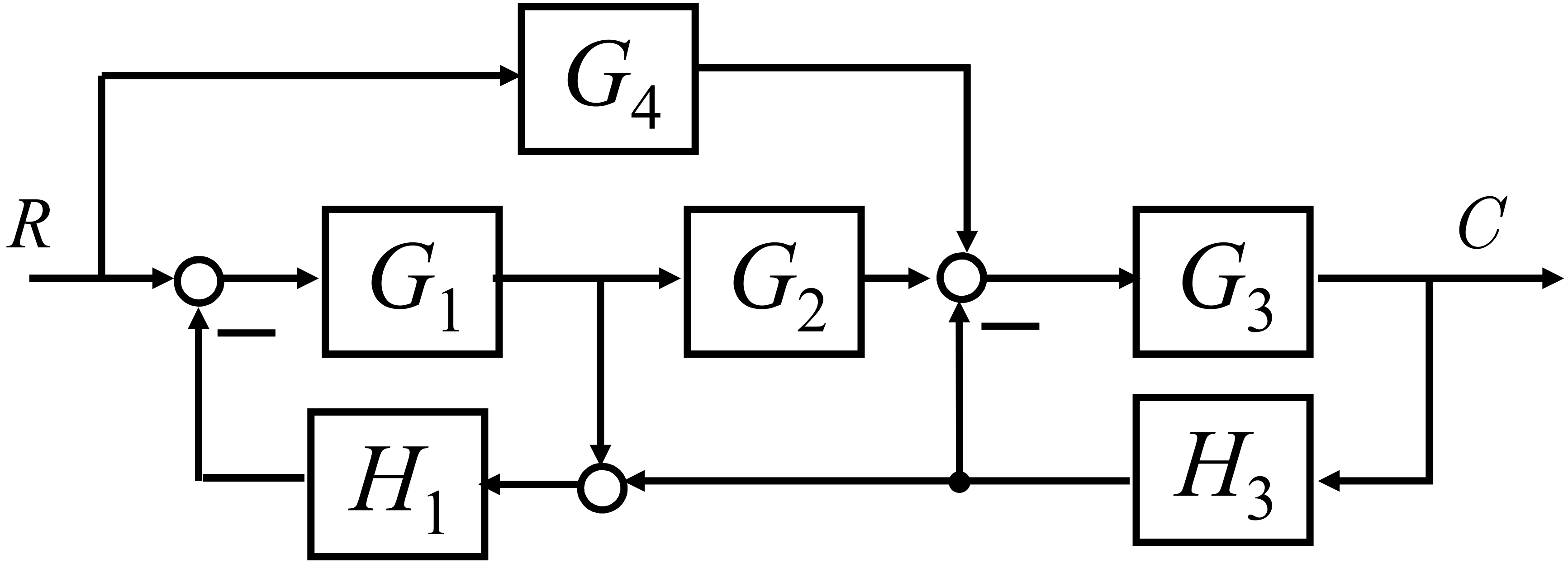


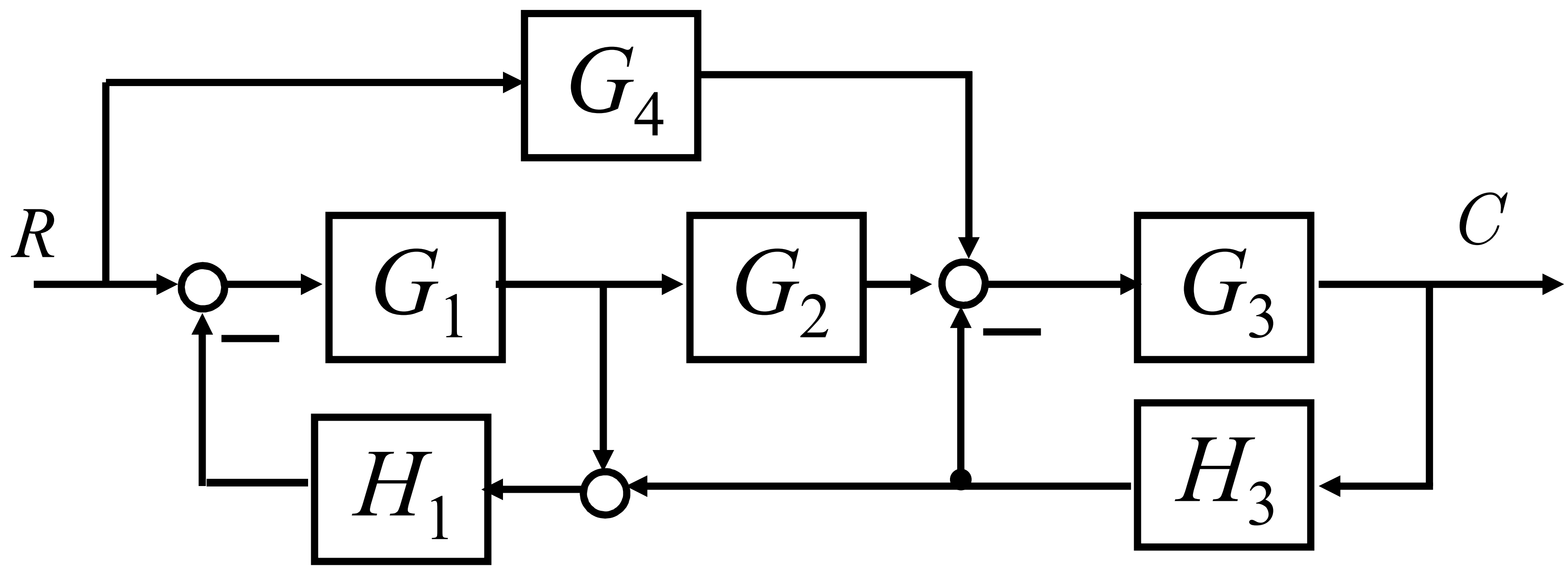
R
G1
G2
H2/G4
G3G4/(1−G3G4H1)
H3
C
R
G1
G2G3G4/(1−G3G4H1+G2G3H2)
H3
C

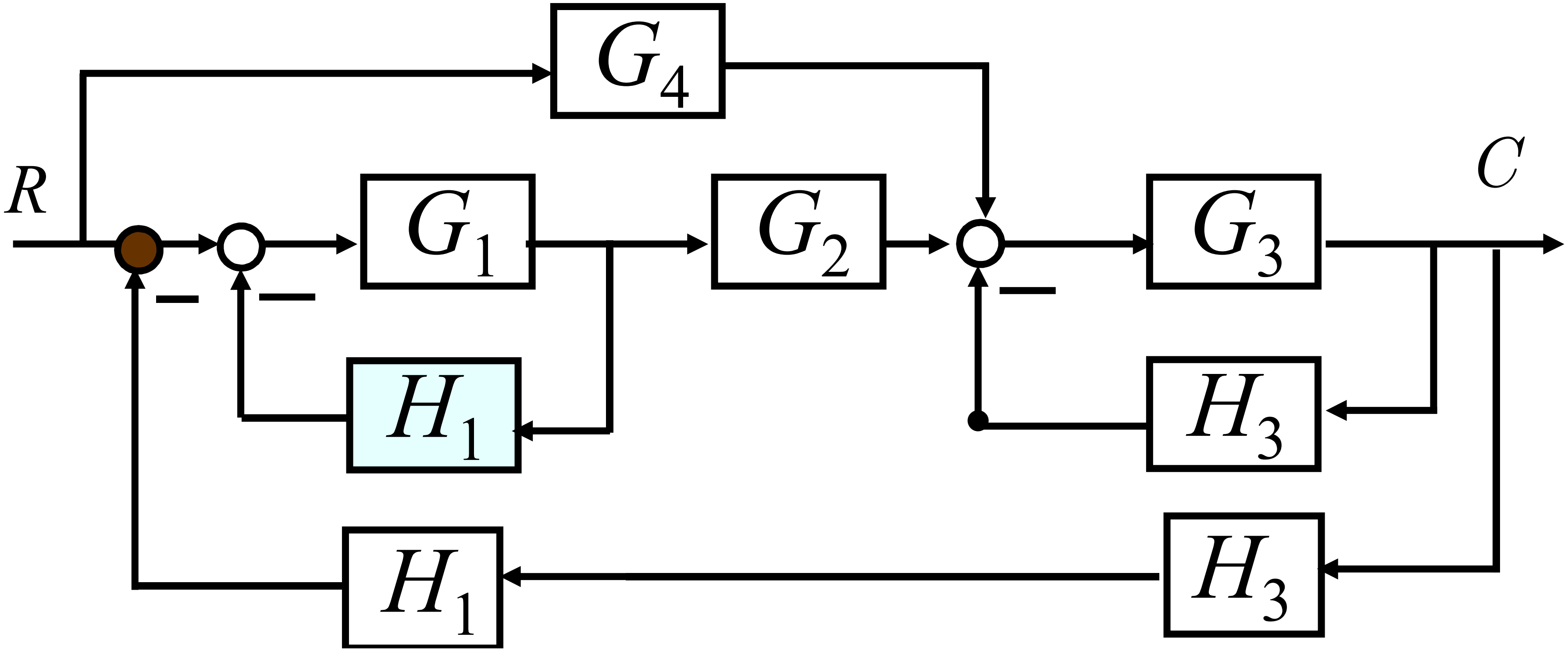


R
G_1
$\dfrac{G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2}$
C
$-$
H_3
R
$\dfrac{G_1G_2G_3G_4}{1-G_3G_4H_1+G_2G_3H_2+G_1G_2G_3G_4H_3}$
C

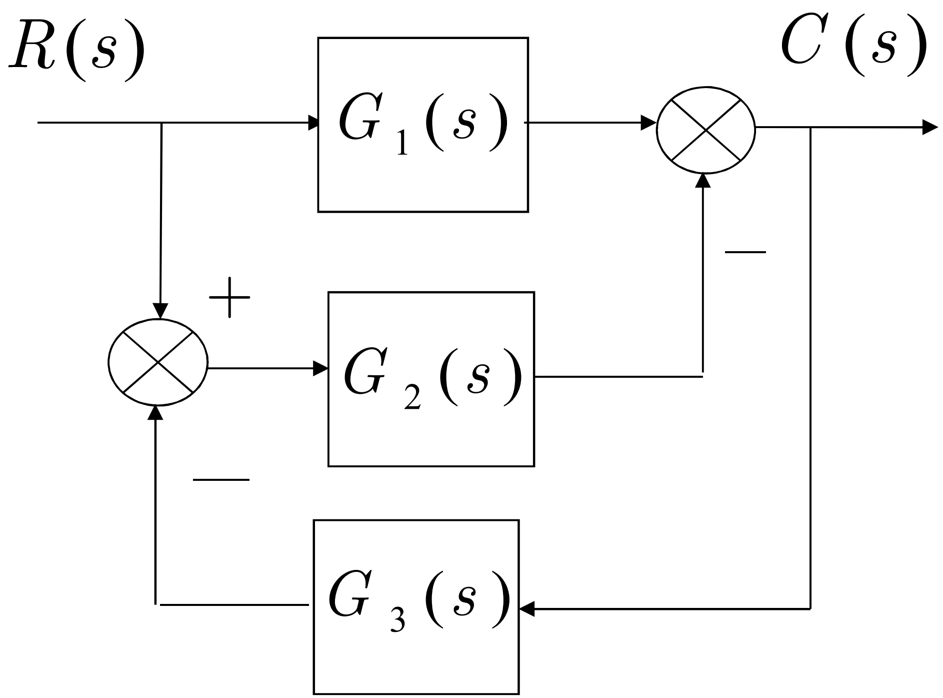
Example. **Block reduction: Redrawing a block diagram.**
Consider the following diagram. With redrawing the diagram, the simplification can be proceeded.




R
C
G4
G1
G2
G3
H1
H3
—
—


R
C
G4
G1
G2
G3
H1
H3
H1
H3
—
—
—

Example. The block diagram of a system is shown below. Find $C(s)/R(s)$.



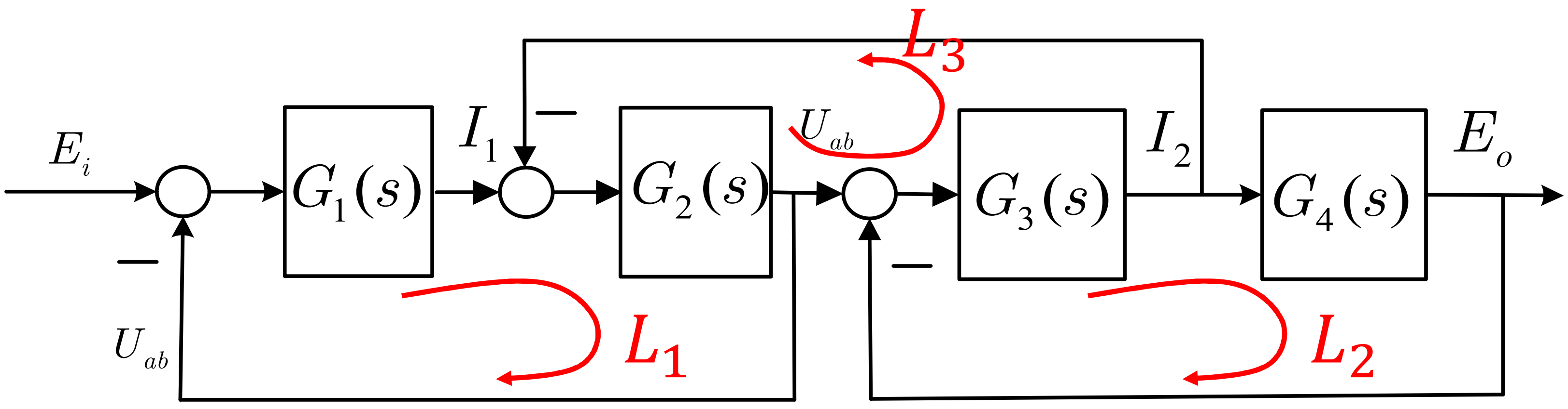
5. Mason' Formula (Supplement material)

$$\frac{C(s)}{R(s)} = \frac{1}{\Delta}\sum_{k=1}^{N} P_k \Delta_k$$

- Δ= the characteristic polynomial of the system $=1-\sum L_i+\sum L_i L_j -\sum L_i L_j L_k+\ldots;$
- L_i= transfer function of the ith loop, where a **loop** is a path that originates from and terminates at the same point and along which no device is encountered more than once.
- $L_i L_j$= product of transfer functions of two non-touching loops, where nontouching loops refer to loops sharing no common device.

- $L_iL_jL_k$= product of transfer functions of three non-touching loops;
- $L_iL_jL_kL_l.....;$
- N=total number of forward paths (from $R(s)$ to $C(s)$ without visiting a point more than once);
- P_k=transfer function of kth forward path;
- Δ_k= the cofactor of the kth forward path, which is determined with the loops **touching** the kth forward path removed from Δ.

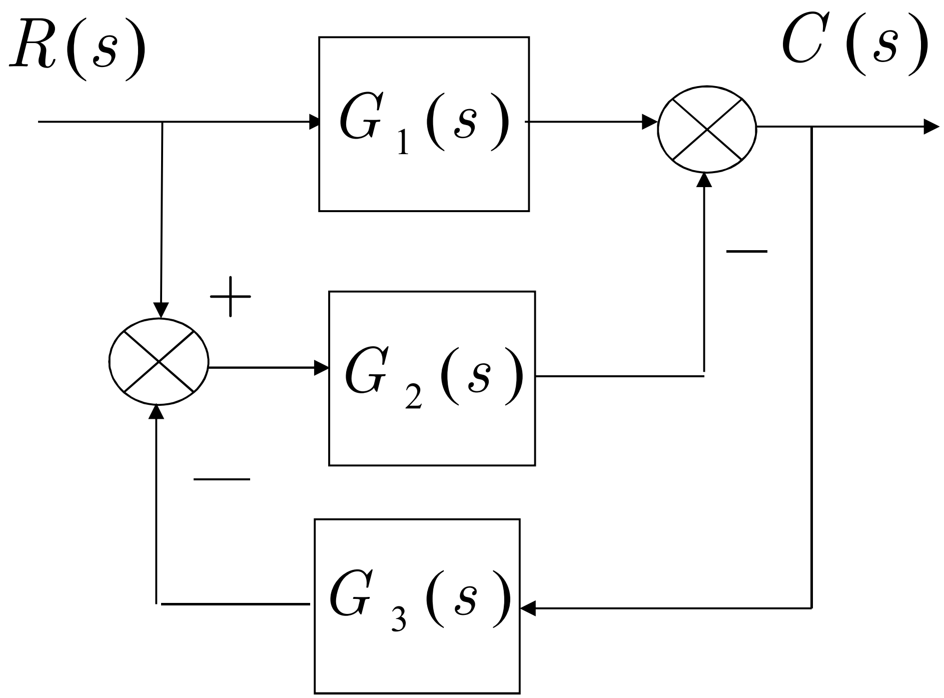
Example. Determine the characteristic polynomial of the following block diagram:



$$L_1 = -G_1G_2 \qquad L_2 = -G_3G_4 \qquad L_3 = -G_2G_3$$

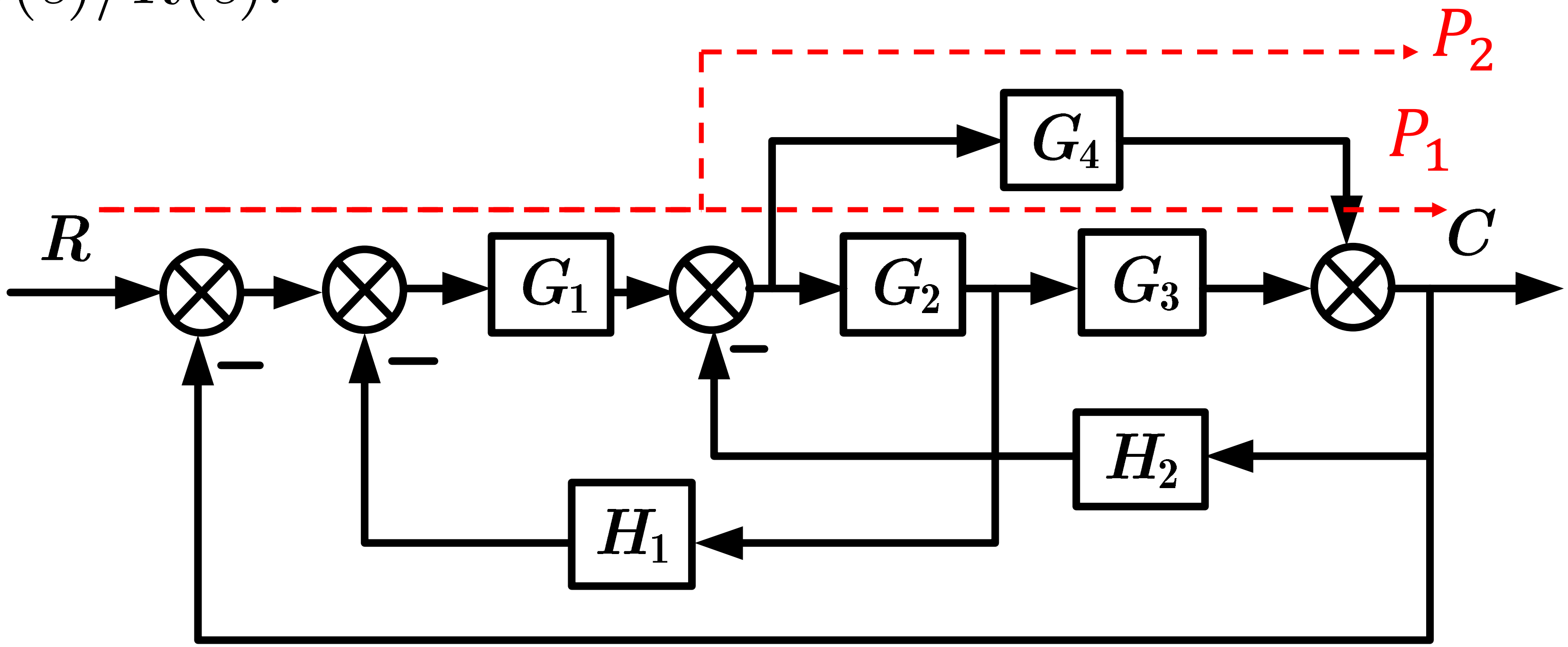
$$\begin{aligned}\Delta &= 1 - L_1 - L_2 - L_3 + L_1L_2 \\ &= 1 + G_1G_2 + G_3G_4 + G_2G_3 + G_1G_2G_3G_4\end{aligned}$$

Example. The block diagram of a system is shown below. Find $C(s)/R(s)$.



Solution: We have two methods for obtaining the transfer function.

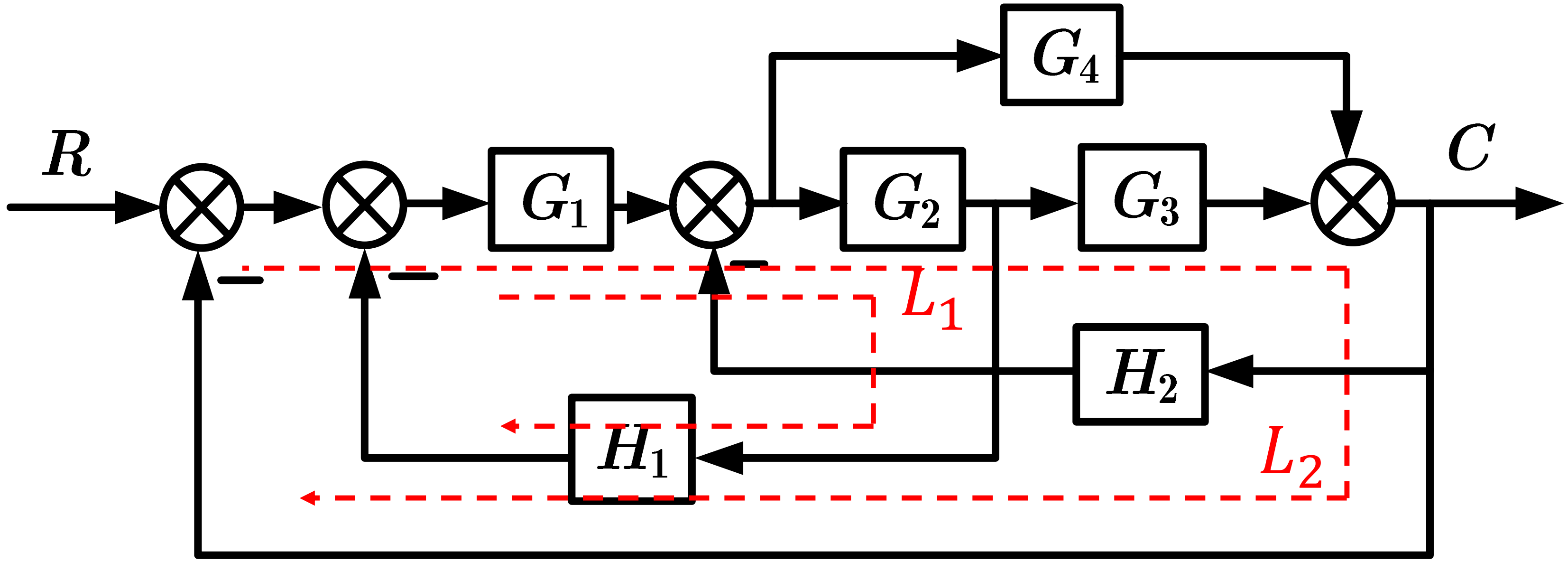
Example. The block diagram of a given system is shown below. Determine the transfer function $C(s)/R(s)$.



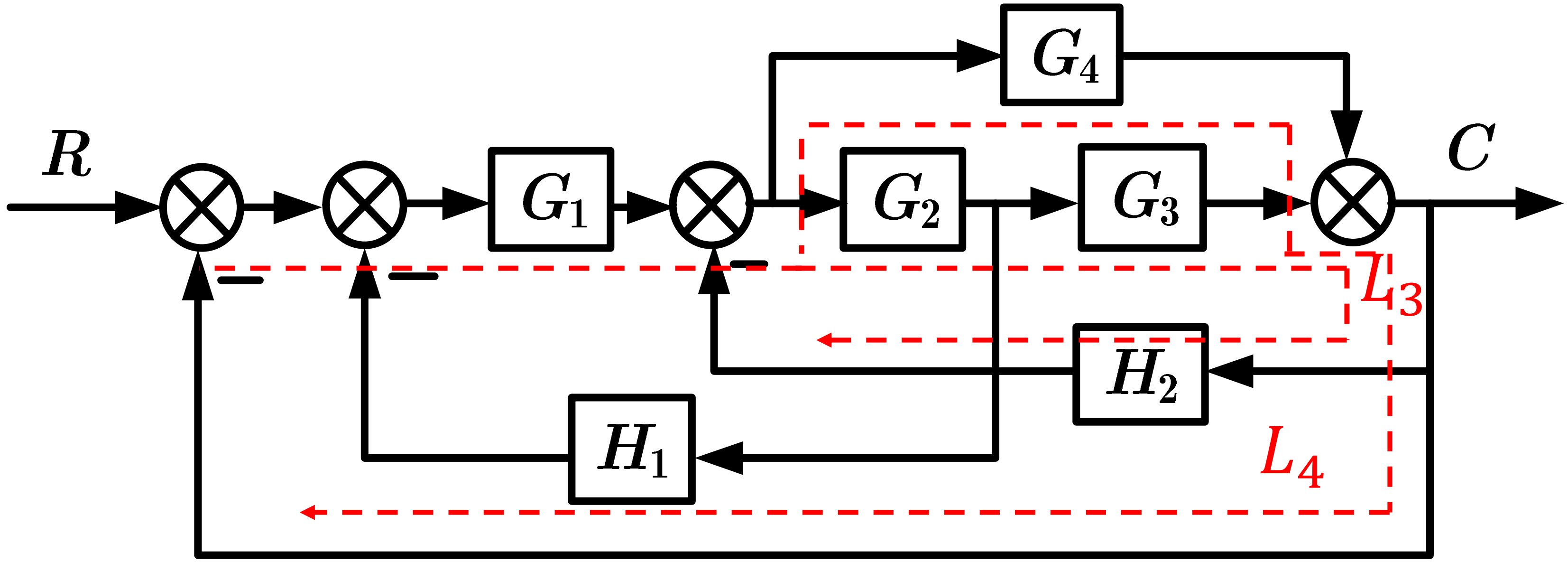
Solution:

Two forward paths: $\begin{cases} P_1 = G_1 G_2 G_3 \\ P_2 = G_1 G_4 \end{cases}$

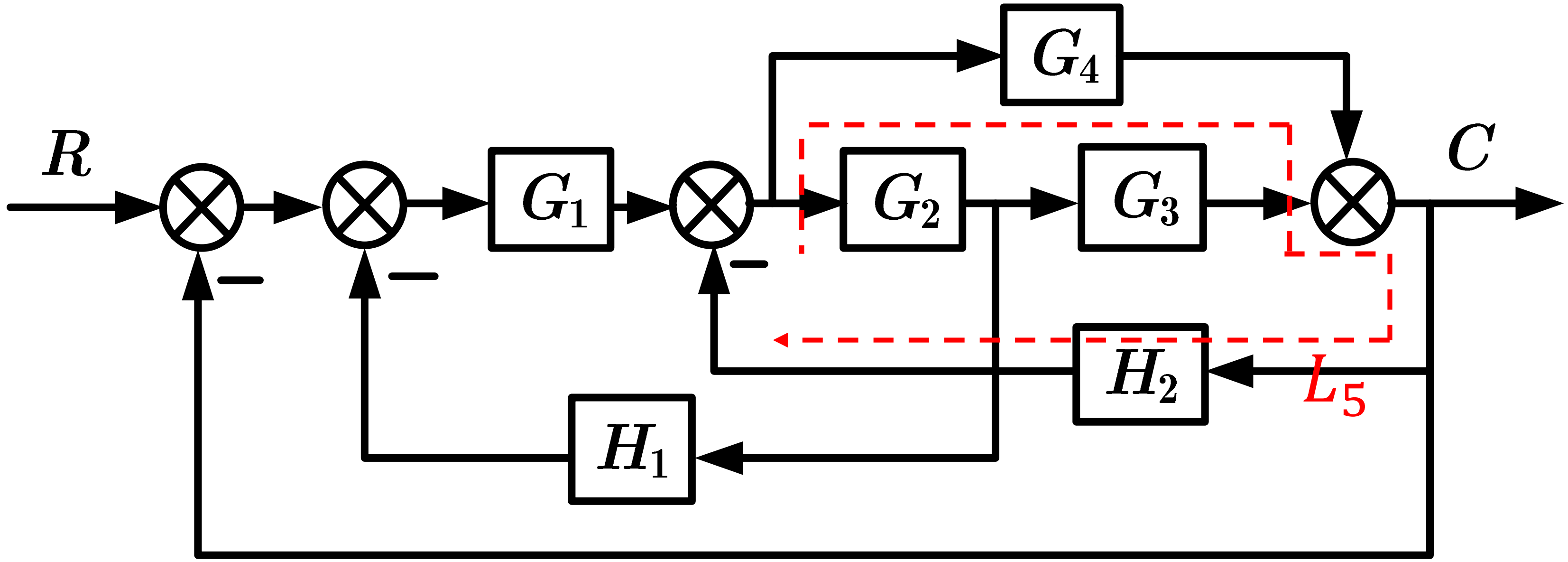
Example. The block diagram of a given system is shown below. Determine the transfer function $C(s)/R(s)$.



Example. The block diagram of a given system is shown below. Determine the transfer function $C(s)/R(s)$.



Example. The block diagram of a given system is shown below. Determine the transfer function $C(s)/R(s)$.



Five individual loops:

$$\begin{cases} L_1 = -G_1 G_2 H_1 \\ L_2 = -G_1 G_2 G_3 \\ L_3 = -G_2 G_3 H_2 \\ L_4 = -G_1 G_4 \\ L_5 = -G_4 H_2 \end{cases}$$

No nontouching loops, therefore,

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4 + L_5)$$

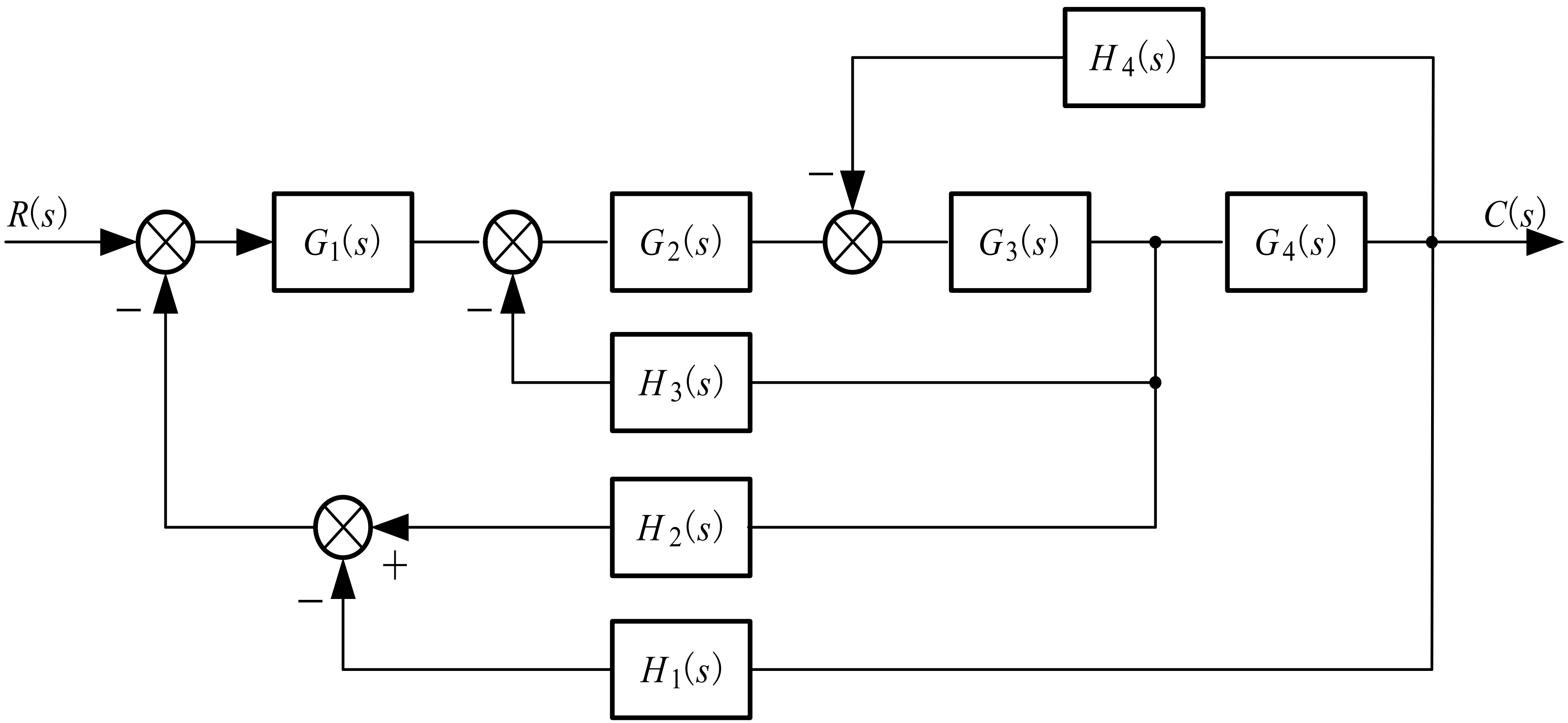
The cofactors are: $\Delta_1 = 1 \quad \Delta_2 = 1$

Consequently,

$$\frac{C}{R} = \frac{\sum_{k=1}^{n} P_k \Delta_k}{\Delta} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

$$= \frac{G_1 G_2 G_3 + G_1 G_4}{1 + G_1 G_2 H_1 + G_1 G_2 G_3 + G_2 G_3 H_2 + G_1 G_4 + G_4 H_2}$$

Example. For the following block diagram, find $C(s)/R(s)$.



Solution: By using Mason's formula,

Only one forward path: $P_1 = G_1 G_2 G_3 G_4$

Four individual loops:

No nontouching loops, therefore,

$$\begin{cases} L_1 = -G_2 G_3 H_3 \\ L_2 = -G_1 G_2 G_3 H_2 \\ L_3 = G_1 G_2 G_3 G_4 H_1 \\ L_4 = -G_3 G_4 H_4 \end{cases}$$

$$\Delta = 1 - (L_1 + L_2 + L_3 + L_4)$$

The cofactor is: $\Delta_1 = 1$

Consequently,

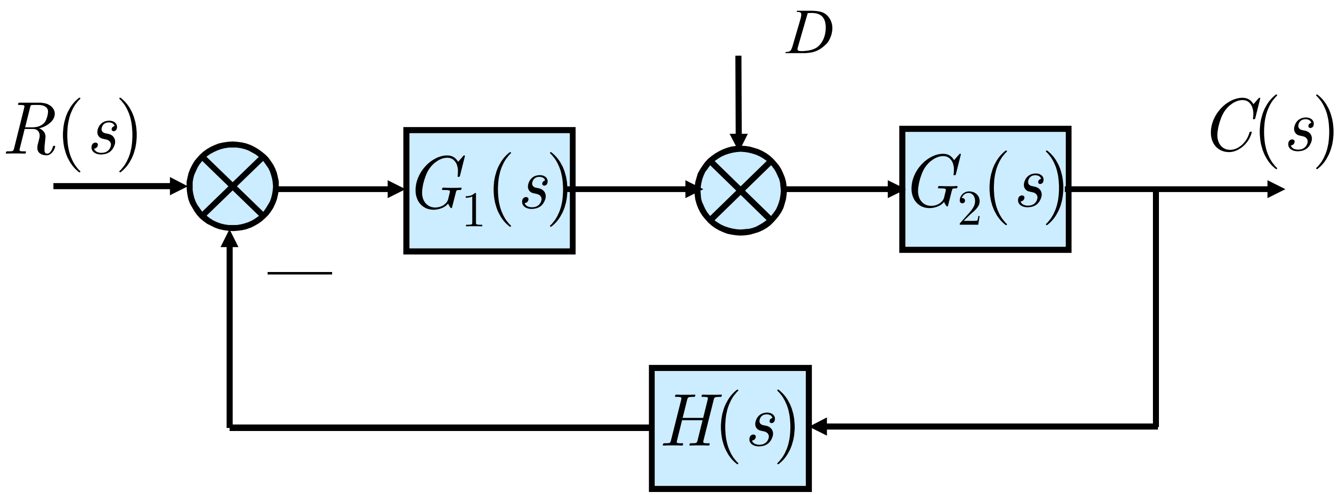
$$\frac{C}{R} = \frac{\sum_{k=1}^{n} P_k \Delta_k}{\Delta}$$

$$= \frac{G_1 G_2 G_3 G_4}{1 + G_2 G_3 H_3 + G_1 G_2 G_3 H_2 + G_3 G_4 H_4 - G_1 G_2 G_3 G_4 H_1}$$

6. Closed-loop systems subjected to a disturbance

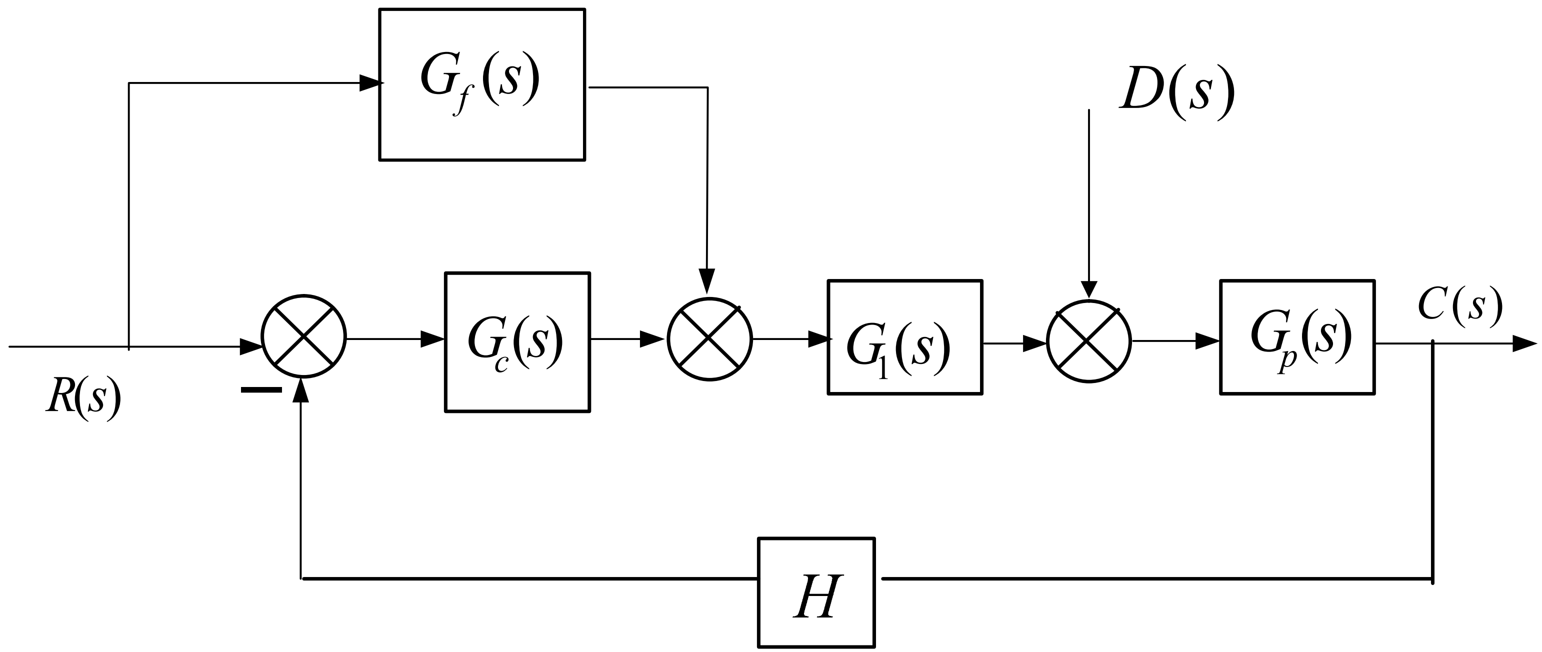
By **principle of superposition**, let, respectively, $R(s)=0$ and $D(s)=0$ and calculate the corresponding outputs $C_D(s)$ and $C_R(s)$. Then,

$$C = C_D(s) + C_R(s)$$



$$C(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)} D(s) + \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)} R(s)$$

Example. The block diagram of a given system is shown below. Find $C(s)/R(s)$ and $C(s)/D(s)$.



Summary of Chapter 2

In this chapter, we mainly studied the following issues:

1. Laplace transformation and the related theorems.

2. Transfer function:

$$\frac{C(s)}{R(s)} = G(s) := \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

from which we know that

- $C(s) = G(s)R(s)$;
- In time domain, the above equation is a convolution integral:

$$c(t) = \int_0^t g(t-\tau)r(\tau)d\tau$$

where $g(t) = \mathcal{L}^{-1}[\mathrm{G(s)}]$.

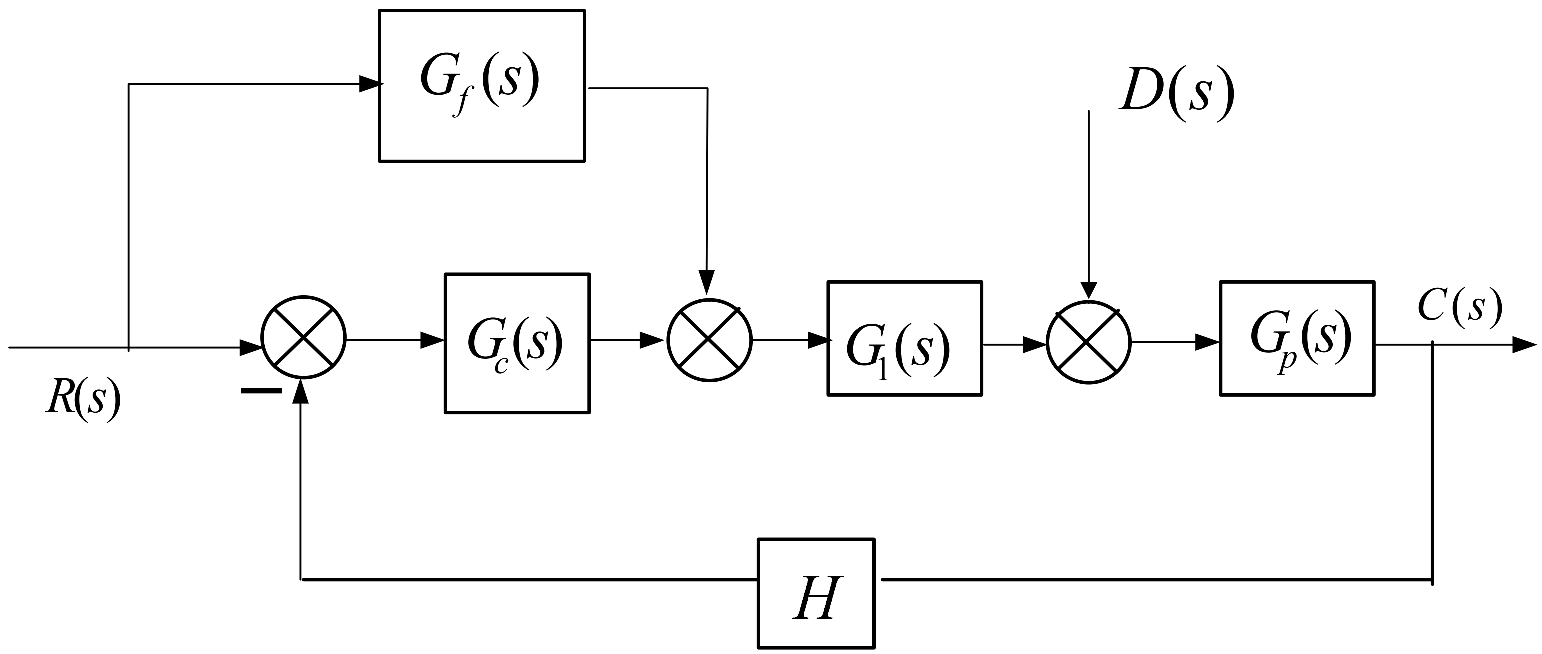
- In particular, if $r(t) = \delta(t)$,

$$c(t) = \mathcal{L}^{-1}[\mathrm{G(s)}],$$

that is, the impulse response is the inverse Laplace transform of its transfer function.

3. Block diagram:

Note that only one block is less meaning. However, with a block diagram, the interrelationship between components and the signal flows can be revealed pictorially compared with the mathematical expression of a set of equations; for instance:



4. Block diagram simplification (reduction):

- feedback;
- cascade;
- parallel;
- moving between two summing (branch) points;
- redrawing the block diagram for some special cases.

Mason's Formula is an alternative option for reducing a block diagram.

By using diagram simplification techniques, one can finally obtain, no matter how complex a system may be,

$$C_r(s) = G(s)R(s);$$
$$C_d(\mathrm{s}) = \Phi(\mathrm{s})D(s),$$

based on which, system analysis can be proceeded.