Automatic Control

Transient requirements

Introduction

- Transient requirements of a feedback system, are defined considering the controlled output response y(t) when the reference signal r(t) is a step function.
- In this context, the step reference introduces a sudden change in the desired behavior of the controlled output causing critical solicitations during the transient phase.
- Transient performance should be expressed in terms of
  - accuracy
  - trigger off and extinction quickness
- Suitable transient performance indices can be defined if we refer to prototype behaviors (e.g. a 2nd order prototype system, see AC_L07).

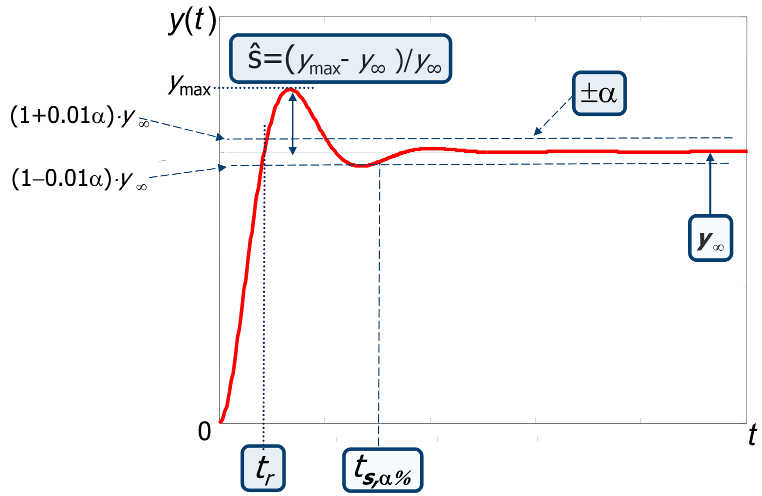
Introduction

- In particular, it can be assumed that the desired transient performance of a feedback control system can be described by the one of a suitable 2nd order prototype system of the form

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

- In this case, the following indices (defined in AC_L07) are used to define the transient performance of a feedback control system
  - Maximum overshoot $\hat{S}$ → accuracy.
  - Rise time t_r → trigger off quickness.
  - Settling time $t_{s,\alpha\%}$ → extinction quickness.

Step response parameters of 2nd order system



Step response of prototype 2nd order control system

- For a prototype 2nd order control system of the form

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

the transient response indices $\hat{s}$, t_r e $t_{s,\alpha\%}$ can be expressed as functions of parameters ζ and ω_n (see also AC_L07).

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = f_{\hat{s}}(\zeta)$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right) = f_{t_r}(\zeta, \omega_n)$$

$$t_{s,\alpha\%} = \frac{1}{\omega_n \zeta}\ln(\alpha/100)^{-1} = f_{t_s}(\zeta, \omega_n, \alpha)$$

Transient time response requirements

- Transient performance requirements are introduced as inequalities of the form

$$\hat{s} \le \bar{\hat{s}} \quad t_r \le \bar{t}_r \quad t_{s,\alpha\%} \le \bar{t}_{s,\alpha\%}$$

- Example

$$\hat{s} \le 10\% \quad t_r \le 0.5\, s \quad t_{s,1\%} \le 1.5\, s$$

Frequency response of a control system

- Since the transient controller $C_T(s)$ will be designed using $L(j\omega)$, we need to link the time domain requirements of the transient response to the most significant parameters of the following relevant frequency responses (see AC_L12).

  - *L(jω)* of the **loop function** *L(s)* → **crossover frequency ω_c**
  - *T(jω)* of the **complementary sensitivity function** *T(s)* → **resonant peak T_p (bandwidth ω_B)**
  - *S(jω)* of the **sensitivity function** → **resonant peak S_p**

Frequency response of prototype 2nd order system

- For a prototype 2nd order feedback control system described by:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

the corresponding loop function can be computed as:

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}} \quad \underset{L(s)=\frac{T(s)}{1-T(s)}}{\Rightarrow} \quad L(s) = \frac{\omega_n/(2\zeta)}{s\left(1 + \frac{s}{2\zeta\omega_n}\right)}$$

- In this case, the crossover frequency ω_c of $L(j\omega)$ can be expressed as a function of parameters ζ and ω_n as:

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2} = f_{\omega_c}(\zeta, \omega_n)$$

Frequency response of prototype 2nd order system

- The resonant peak T_p, and the bandwidth ω_B of the complemenatry sensitivity function

$$T(s) = \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

can be expressed as functions of parameters ζ and ω_n

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = f_{T_p}(\zeta)$$

$$\omega_B = \omega_n \sqrt{1 - 2\zeta^2 + \sqrt{2 - 4\zeta^2 + 4\zeta^4}} = f_{\omega_B}(\zeta, \omega_n)$$

Frequency response of prototype 2nd order system

- The resonant peak S_p of the sensitivity function

$$S(s) = 1 - T(s) = 1 - \frac{1}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}} = \frac{s\left(\frac{2\zeta}{\omega_n} + \frac{s}{\omega_n^2}\right)}{1 + \frac{2\zeta}{\omega_n} s + \frac{s^2}{\omega_n^2}}$$

is given by the following function of parameter ζ

$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2\text{-}1} = f_{S_p}(\zeta)$$

Frequency response and time response relations

- In order to define suitable relations between frequency response and time response parameters, we consider the expressions of the relevant indices obtained for the 2nd order prototype model.

Time response

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right)$$

$$t_{s,\alpha\%} = \frac{1}{\omega_n\zeta}\ln(\alpha/100)^{-1}$$

Frequency response

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2\text{-}1}$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

Requirements translation

- The objective is to translate time domain requirements into frequency domain requirements to be exploited during the controller design.

- In particular, we would like to translate requirements on $\hat{s}$, t_r and $t_{s,\alpha\%}$ into requirements on T_p, S_p and ω_c.

- In order to show the requirements translation procedure, a practical example will be considered.

Requirements translation: example

- Consider the following time response requirements

$$\hat{s} \leq 10\%$$
$$t_r \leq 0.5\ s$$
$$t_{s,1\%} \leq 1.5\ s$$

Requirements translation: example

- First, we consider the maximum overshoot requirement

$$\hat{s} \leq 10\%$$

- Recalling that

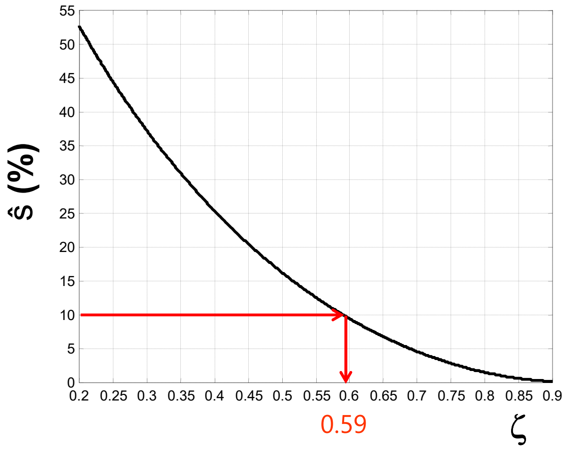
$$\hat{s} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}}$$

- It is possible to obtain a requirement on the minimum damping coefficient

$$\zeta \geq \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}} \underset{\hat{s}=0.1}{\overset{\uparrow}{=}} 0.59$$

Graphical procedure

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$



Requirements translation: example

- Using the obtained minimum damping coefficient

$$\zeta \geq 0.59$$

and recalling that

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1}$$

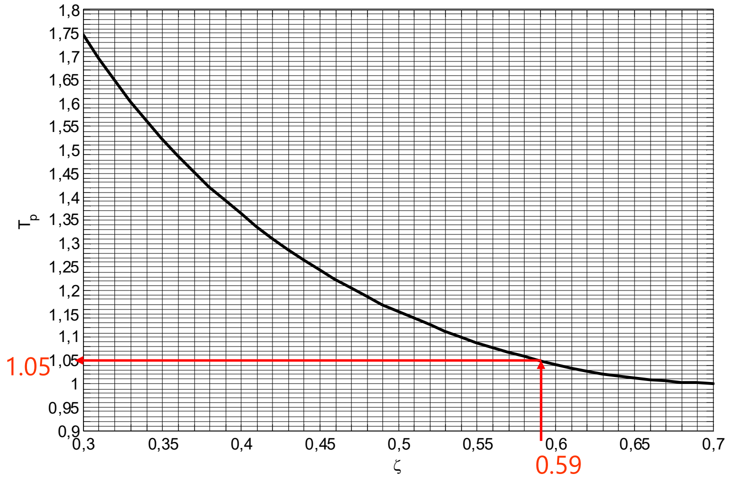
- Requirements on the resonant peaks of the complementary sensitivity (T_p) and sensitivity (S_p) functions can be derived.

$$T_p \leq \frac{1}{2\zeta\sqrt{1-\zeta^2}} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 1.0496 = 0.42\,\text{dB}$$

$$S_p \leq \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 1.3622 = 2.68\,\text{dB}$$

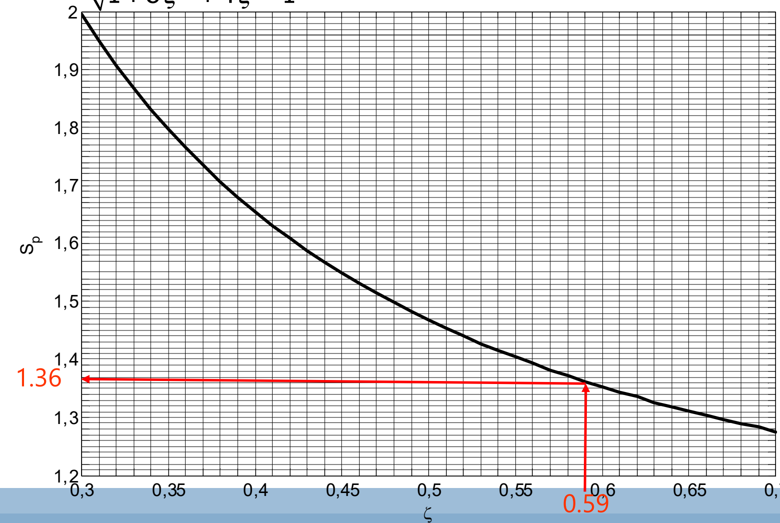
Graphical procedure

$$T_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$



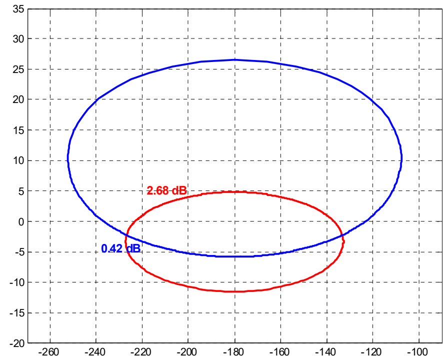
Graphical procedure

$$S_p = \frac{2\zeta\sqrt{2+4\zeta^2+2\sqrt{1+8\zeta^2}}}{\sqrt{1+8\zeta^2}+4\zeta^2-1}$$



Requirements on T_p and S_p: a remark

- The values of the resonant peaks T_p and S_p of $|T(j\omega)|$ and $|S(j\omega)|$ respectively obtained via the requirement on $\hat{S}$ can be reported on the Nichols plane using the corresponding constant magnitude loci.
- Such loci can be interpreted as constraints to be satisfied by the course of $L(j\omega)$ (Nichols plot).



Requirements translation: example

- Let us now consider the rise time requirement

$$t_r \leq 0.5\ s$$

- Recalling that

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right)$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$$

- we obtain

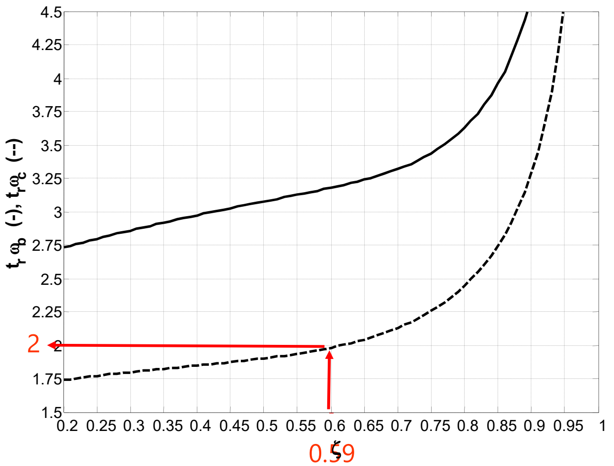
$$t_r \cdot \omega_c = \frac{1}{\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right)\cdot\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2} \underset{\substack{\uparrow \\ \zeta=0.59}}{=} 1.9708$$

- Therefore

$$\omega_c \geq \frac{1.9708}{t_r} \underset{\substack{\uparrow \\ t_r=0.5}}{=} 3.9417\ \text{rad/s}$$

Graphical procedure

$t_r \cdot \omega_c$



Requirements translation: example

- A similar procedure can be used for the settling time requirement

$$t_{s,1\%} \leq 1.5\ s$$

- Recalling that

$$t_{s,\alpha\%} = \frac{\ln(\alpha/100)^{-1}}{\omega_n\zeta} \underset{\alpha=1}{\overset{\uparrow}{=}} \frac{4.6052}{\omega_n\zeta}$$

$$\omega_c = \omega_n\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2}$$

- we obtain

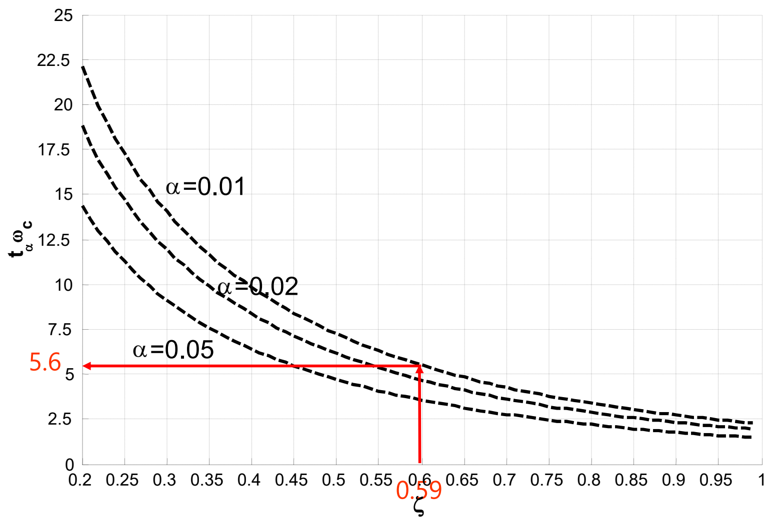
$$t_{s,\alpha\%}\cdot\omega_c = \frac{4.6052}{\zeta}\cdot\sqrt{\sqrt{1+4\zeta^4}-2\zeta^2} \underset{\zeta=0.59}{\overset{\uparrow}{=}} 5.6409$$

- then

$$\omega_c \geq \frac{5.6409}{t_{s,1\%}} \underset{t_{s,1\%}=1.5}{\overset{\uparrow}{=}} 3.7606 \text{ rad/s}$$

Graphical procedure

$t_{s,1\%} \cdot \omega_c$



Requirements translation: example

- The translation of the rise time and the settling time requirements leads to two different values of the crossover frequency.
- The crossover frequency value to be employed for the design is chosen as

$$\omega_{c,des} \geq \max(\underset{t_r}{\overset{\uparrow}{\omega_c}}, \underset{t_{s,1\%}}{\overset{\uparrow}{\omega_c}}) = (\underset{t_r}{\overset{\uparrow}{3.94}}, \underset{t_{s,1\%}}{\overset{\uparrow}{3.76}}) = 3.94 \Rightarrow \omega_{c,des} = 4 \text{ rad/s}$$

Time domain requirements translation: a final remark

- The requirements on $\omega_{c,des}$, T_p and S_p have been obtained through the assumption that T(s) is exactly described by a 2nd order prototype model.
- However, such assumption though reasonable is not, in general, satisfied.
- Therefore, satisfaction of requirements on $\omega_{c,des}$, T_p and S_p at the end of the design procedure, does not guarantee, in general, satisfaction of requirements on $\hat{S}$, t_r and $t_{s,\alpha\%}$.
- Thus, in order to check fulfilment of such time domain requirements, simulation of the control feedback system has to be performed.