8.3. $g_1(y,0)=0$, $g_2(y,0)=0$. $A_1=0$. 且满足定理8.1.

$\exists\delta>0$, $\forall\|y\|<\delta$, 存在 $z=h(y)=0$, 满足

$$0=A_2h(y)+g_2(y,h(y))-\frac{\partial h}{\partial y}(y)[A_1y+g_1(y,h(y))].$$

$\because g_1(y,0)=0$, $A_1=0$

$\therefore \dot{y}=0$

$\therefore$ 降阶系统原点稳定, Lyapunov $V(y)=y^Ty$ 满足推论8.1

即 $\frac{\partial V}{\partial y}[A_1y+g_1(y,h(y))]\le 0$.

$\therefore$ 整个系统原点稳定.

8.6. (1).

$$\dot{x}_1=ax_1^2-x_2^2,\quad a\neq 0$$
$$\dot{x}_2=-x_2+x_1^2+x_1x_2.\qquad A=\frac{\partial f}{\partial x}\Big|_{x=0}=\begin{bmatrix}0&0\\0&-1\end{bmatrix}.$$

令 $y=x_1$, $z=x_2$; $\dot{y}=ay^2-z^2$, $\dot{z}=-z+y^2+yz$.

$N(h(y))=h'(y)[ay^2-h^2(y)]+h(y)-y^2-yh(y)=0$, $h(0)=h'(0)=0$.

将 $h(y)=O(|y|^2)$ 代入降阶系统, $\dot{y}=ay^2+O(|y|^4)$, $a\neq 0$.

$\therefore$ 系统原点不稳定.

(4).

$$\dot{x}_1=x_1^2x_2$$
$$\dot{x}_2=-x_1^3-x_2\qquad A=\frac{\partial f}{\partial x}\Big|_{x=0}=\begin{bmatrix}0&0\\0&-1\end{bmatrix}.$$

令 $y=x_1$, $z=x_2$; $\dot{y}=y^2z$, $\dot{z}=-z-y^3$.

$N(h(y))=-h'(y)y^2h(y)-h(y)-y^3=0$, $h(0)=h'(0)=0$.

将 $h(y)=O(|y|^2)$ 代入, 无法得到; 将 $h(y)=h_2y^2+O(|y|^3)$ 代入, $h_2=0$.

将 $h(y)=h_3y^3+O(|y|^4)$ 代入, $h_3=-1$. 则 $\dot{y}=-y^5+O(|y|^6)$.

$\therefore$ 系统原点渐近稳定.

(6). $\dot{x}_1 = -x_1 + x_2^3(x_1 + x_2 - 1)$

$\dot{x}_2 = x_2^3(x_1 + x_2 - 1)$. $\quad A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$.

$\therefore$ 令 $y = x_2$，$z = x_1$；$\dot{y} = y^3(z + y - 1)$，$\dot{z} = -z - y^2 + y^3(z + y - 1)$.

$N(h(y)) = h'(y)[y^3(h(y)) + y - 1)] + h(y) - y^3(h(y) + y - 1) = 0$. $h(0) = h'(0) = 0$

$\phi(y) = 0$，$h(y) = O(|y|^3)$，降阶系统为 $\dot{y} = -y^3 + O(|y|^4)$.

$\therefore$ 系统原点渐近稳定

(8). $\dot{x}_1 = -2x_1 + 3x_2 + x_3 + x_3^2$.

$\dot{x}_2 = x_1 + x_1^2 + x_2$

$\dot{x}_3 = x_1^2$. $\quad A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

令 $y = x_3$，$z = \begin{bmatrix} x_1 + x_3 \\ x_2 - x_3 \end{bmatrix}$. $\dot{y} = (z_1 - y)^2$，$\dot{z}_1 = -2z_1 - 3z_2 + y^2 + (z_1 - y)^2$

$\dot{z}_2 = z_1 + z_2$.

$N_1(h(y)) = h_1'(y)[h_1(y) - y]^2 + 2h_1(y) + 3h_2(y) - y^2 - [h_1(y) - y]^2$

$N_2(h(y)) = h_2'(y)[h_1(y) - y]^2 - h_1(y) - h_2(y)$，$h(0) = h'(0) = 0$.

$\phi(y) = 0$，$N_1(0) = O(|y|^2)$，$N_2(0) = 0$.

$\dot{y} = y^2 + O(|y|)^3$，原点不稳定.

$\therefore$ 系统原点不稳定.

$\dot{x}_1 = a x_1 x_2 - x_1^3$

$\dot{x}_2 = -x_2 + b x_1 x_2 + c x_1^2$. $\quad A = \frac{\partial f}{\partial x}\Big|_{x=0} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.

令 $y = x_1$，$z = x_2$；$\dot{y} = ayz - y^3$，$\dot{z} = -z + byz + cy^2$.

$N(h(y)) = h'(y)[ay h(y) - y^3] + h(y) - by h(y) - cy^2 = 0$. $h(0) = h'(0) = 0$.

将 $h(y) = h_2 y^2 + O(|y|^3)$ 代入. $h_2 = c$，降阶系统为：

$\dot{y} = (ac - 1) y^3 + O(|y|^4)$.

①. $ac - 1 > 0$，原点不稳定.

②. $ac - 1 < 0$，原点渐近稳定.

③. $ac - 1 = 0$，令 $h(y) = cy^2 + h_3 y^3 + O(|y|^4)$. $h_3 = bc$.

降阶系统为：$\dot{y} = \underbrace{(ac-1)}_{0} y^3 + bay^4 + O(|y|^5) = by^4 + O(|y|^5)$.

3.1. $ac-1=0$. $b \neq 0$. 原点不稳定.

3.2 $ac-1=0$, $b=0$, $h(y) = cy^2$, $\dot{y} = 0 \Leftarrow$ 降阶系统, 原点稳定.

Lyapunov $V(y) = y^2$ 满足推论 8.1. 则整个系统原点稳定.

8.14. $\dot{x} = f(x)$. $V(x) = x_1^2 + x_2^2$. $D = \{x \in R^2 \mid |x_2| < 1, |x_1 - x_2| < 1\}$.

设 $[\partial V / \partial x] f(x)$ 在 D 上负定. 令 $c = \min_{x \in \partial D} V(x)$.

则 $\Omega_c = \{V(x) < c\}$ 为吸引区估计值.

$$\min_{x_2 = 1,\ 0 \le x_1 \le 2} \{x_1^2 + x_2^2\} = 1. \quad \min_{x_2 = -1,\ -2 \le x_1 \le 0} \{x_1^2 + x_2^2\} = 1.$$

$$\min_{x_1 - x_2 = 1,\ 0 \le x_1 \le x_2} \{x_1^2 + x_2^2\} = \min_{0 \le x_1 \le 2} \{x_1^2 + (x_1 - 1)^2\} = \frac{1}{2}.$$

$$\min_{x_1 - x_2 = -1,\ -2 \le x_1 \le 0} \{x_1^2 + x_2^2\} = \min_{-2 \le x_1 \le 0} \{x_1^2 + (x_1 + 1)^2\} = \frac{1}{2}.$$

$\therefore c = \frac{1}{2}$. 吸引区估计值为 $\{x \in R^2 \mid x_1^2 + x_2^2 < \frac{1}{2}\}$.

8.16. $\dot{x}_1 = x_2$.

$\dot{x}_2 = -x_2 - x_1 + x_1^3 = -x_2 - (x_1 - x_1^3)$.

令 $V(x) = \frac{1}{2} x_2^2 + \int_0^{x_1} (y - y^3)\,dy = \frac{1}{2} x_2^2 + \frac{1}{2} x_1^2 - \frac{1}{4} x_1^4$.

当 $|x_1| < 1$ 时, $V(x)$ 正定.

$\dot{V}(x) = (x_1 - x_1^3) x_2 + x_2 [-x_2 - (x_1 - x_1^3)] = -x_2^2$

$\dot{V}(x) = 0 \Rightarrow x_2(t) \equiv 0 \Rightarrow x_1(t) - x_1^3(t) \equiv 0 \Rightarrow x_1(t)$, $|x_1| < 1$.

$\therefore$ 原点渐近稳定, 吸引区估计值为 $\Omega_c = \{x \in R^2 \mid V(x) < c\}$.

$\Omega_c \subset D$ 对 $\forall c < \frac{1}{4}$ 满足, 故 $\Omega_c = \{x \in R^2 \mid V(x) < \frac{1}{4}\}$.

8.21. $\dot{x}_1 = x_2$.

$\dot{x}_2 = -\sin x_1 - g(t)x_2$.

$g(t)$ 连续可微, $\forall t \geq 0$, $0 < k_1 \leq g(t) \leq k_2$.

~~线~~ 将系统在 $x=0$ 线性化. $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 - g(t)x_2$.

令 $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, $\dot{V}(x) = x_1x_2 - x_1x_2 - g(t)x_2^2 \leq -k_1x_2^2$.

~~令~~ $A(t) = \begin{bmatrix} 0 & 1 \\ -1 & -g(t) \end{bmatrix}$, $\bar{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. $C(t) = [0 \quad \sqrt{g(t)}]$.

$g(t) \geq k_1$ $(\bar{A}, C(t))$ 为一致可观的,

$\because A(t) = \bar{A} - C^T(t)C(t)$, $C(t)$ 为一致有界的.

$\therefore (A(t), C(t))$ 为一致可观的. 即 $-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + C^T(t)C(t)$

, $P = \frac{1}{2}I$.

$\therefore$ 线性系统原点指数稳定.

$\therefore$ 据定理4.13, 非线性系统原点指数稳定.

8.22. $\dot{x}_1 = -x_1 - x_2 - \alpha(t)x_3$, $\dot{x}_2 = x_1$, $\dot{x}_3 = \alpha(t)x_1$. $\alpha(t) = \sin t + \sin 2t$.

$\dot{x} = \begin{bmatrix} -1 & -1 & -\alpha(t) \\ 1 & 0 & 0 \\ \alpha(t) & 0 & 0 \end{bmatrix} x = A(t)x$. 令 $V(x) = x^Tx$.

$\dot{V}(x) = x^T[A(t) + A^T(t)]x = -x_1^2 = -x^TC^TCx$. $C = [1, 0, 0]$.

令 $K(t) = [-1, 1, \alpha(t)]^T$, $A(t) - K(t)C = \begin{bmatrix} 0 & -1 & -\alpha(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$[A(t) - K(t)C]$ 的状态转移矩阵为 $\Phi(t,\tau) = \begin{bmatrix} 1 & -(t-\tau) & -\int_\tau^t \alpha(\sigma)d\sigma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$C\Phi(t,\tau) = [1, -(t-\tau), -\int_\tau^t \alpha(\sigma)d\sigma]$.

$\because \alpha(t) = \sin t + \sin 2t$.

$\therefore (A - KC, C)$ 为一致可观.

$\therefore$ 原点指数稳定