Chapter 6

Root-Locus Analysis (1)

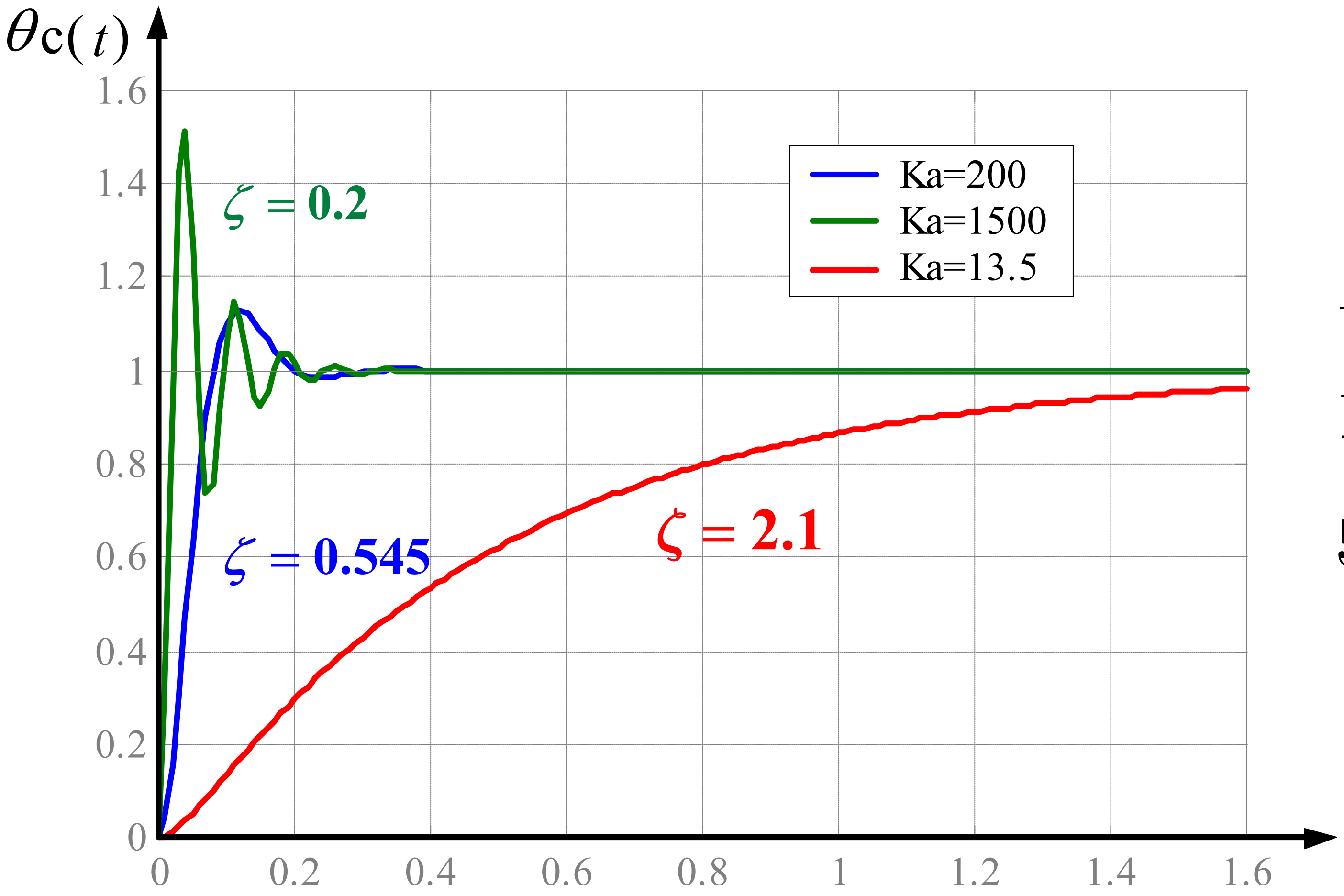
6-1 Introduction

It is well known that the transient response of a feedback system is closely related to the locations of the closed-loop poles.

From the design viewpoint, for some systems, simple gain adjustment may move the closed-loop poles to desired locations. For example, consider a unity-feedback (servo) system with open-loop transfer function

$$G(s) = \frac{5K_{\mathrm{A}}}{s(s+34.5)}$$

Let the input signal be $1(t)$. Then, with open-loop gain taking the values K_A=13.5, 200 and 1500, respectively, the system exhibits different responses:

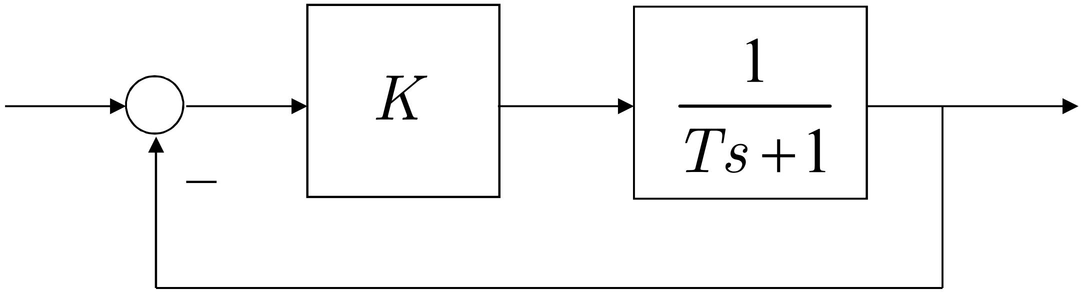


M_p: 0,

13%,

52.7%.

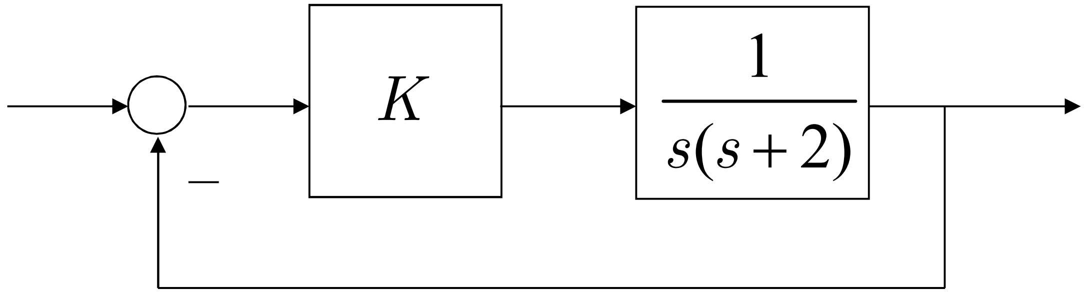
Example. Determine the path of closed-loop pole when K varies from 0 to $+\infty$ continuously.



Definition: The root locus is the path of a root of **closed-loop** characteristic equation traced out in the s-plane as a system parameter varies from 0 to $+\infty$.

Remark: The root locus analysis aims to investigate the **closed-loop** stability and the system controller design through **the open-loop transfer function** with variation of a certain system parameter, commonly the open-loop gain.

Example. Determine the closed-loop root loci when K varies from 0 to $+\infty$.



The **closed-loop** characteristic equation is:

$$s^2 + 2s + K = 0 \Rightarrow s_{1,2} = -1 \pm \sqrt{1-K}$$

1) $K = 0 \quad s_1 = 0 \quad s_2 = -2$

2) $K = 1 \quad s_1 = s_2 = -1$

3) $K = 2 \quad s_{1,2} = -1 \pm j1$

4) $K \to \infty \quad s_{1,2} = -1 \pm j\infty$

$$s_{1,2} = -1 \pm \sqrt{1-K}$$

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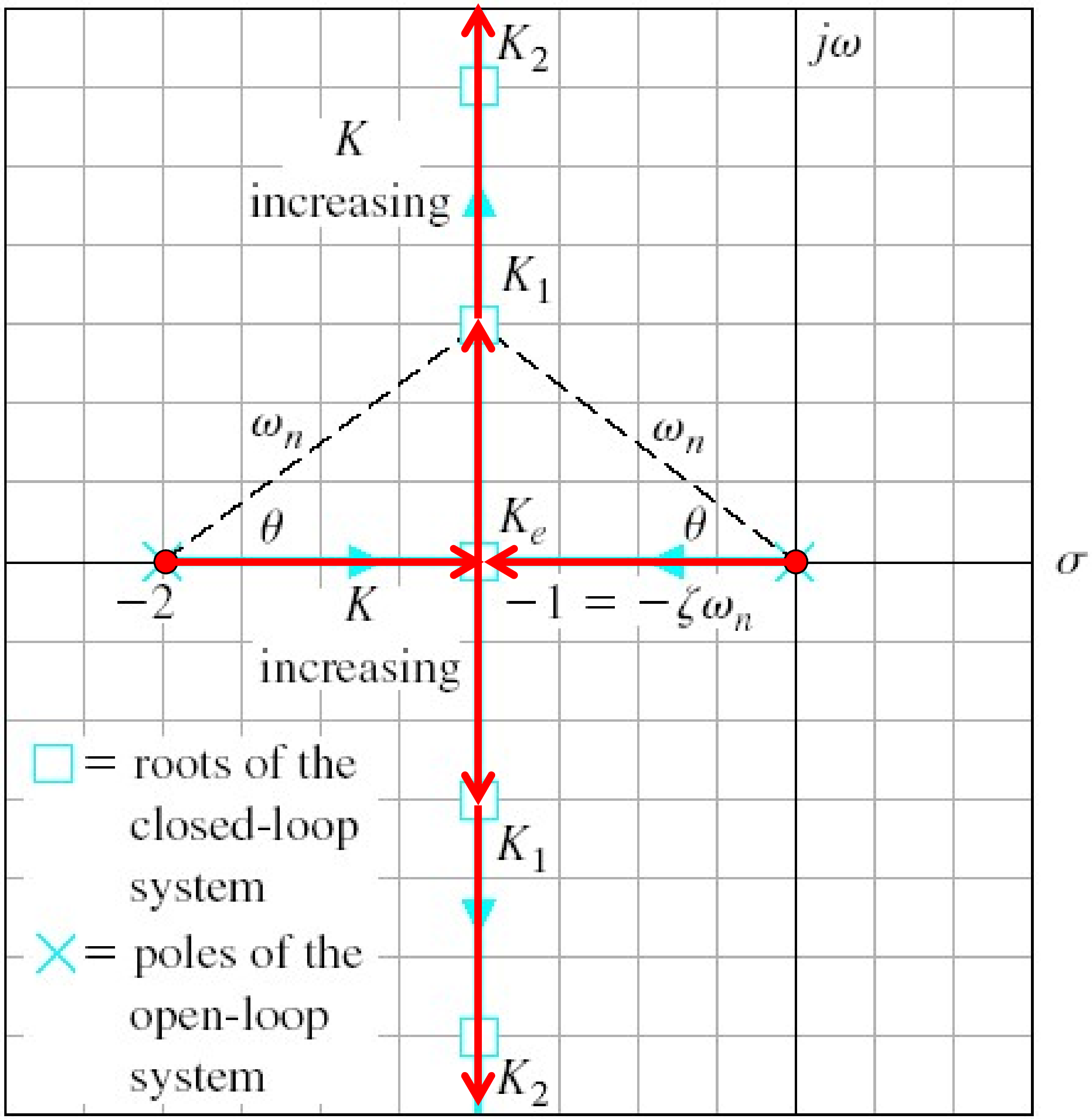
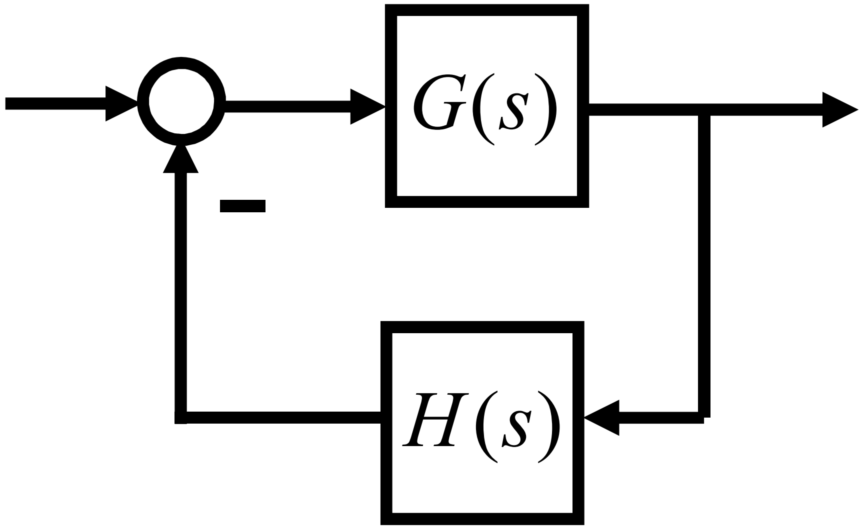


6-2 Root Locus Plots



Angle and Magnitude Conditions

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

The characteristic equation is defined as

$$1 + G(s)H(s) = 0$$

or

$$G(s)H(s) = -1$$

which can be split into two equations:

Angle Condition:

$$\boxed{\angle G(s)H(s) = 180^0(2k+1), \quad (k = 0, \pm 1, \pm 2, \ldots)}$$

Magnitude Condition:

$$\boxed{\left|G(s)H(s)\right| = 1}$$

More precisely, we can write the characteristic equation as

$$G(s)H(s) = -1 \Rightarrow \underbrace{\frac{K^*(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}}_{GH} = -1$$

Then the root loci are the loci of the **closed-loop poles** as the gain K^* varies from 0 to $+\infty$.

It is easy to see that **the magnitude condition is always satisfied by a suitable** $K^* \geq 0$. Thus, the key is to find all those points that satisfy the angle condition:

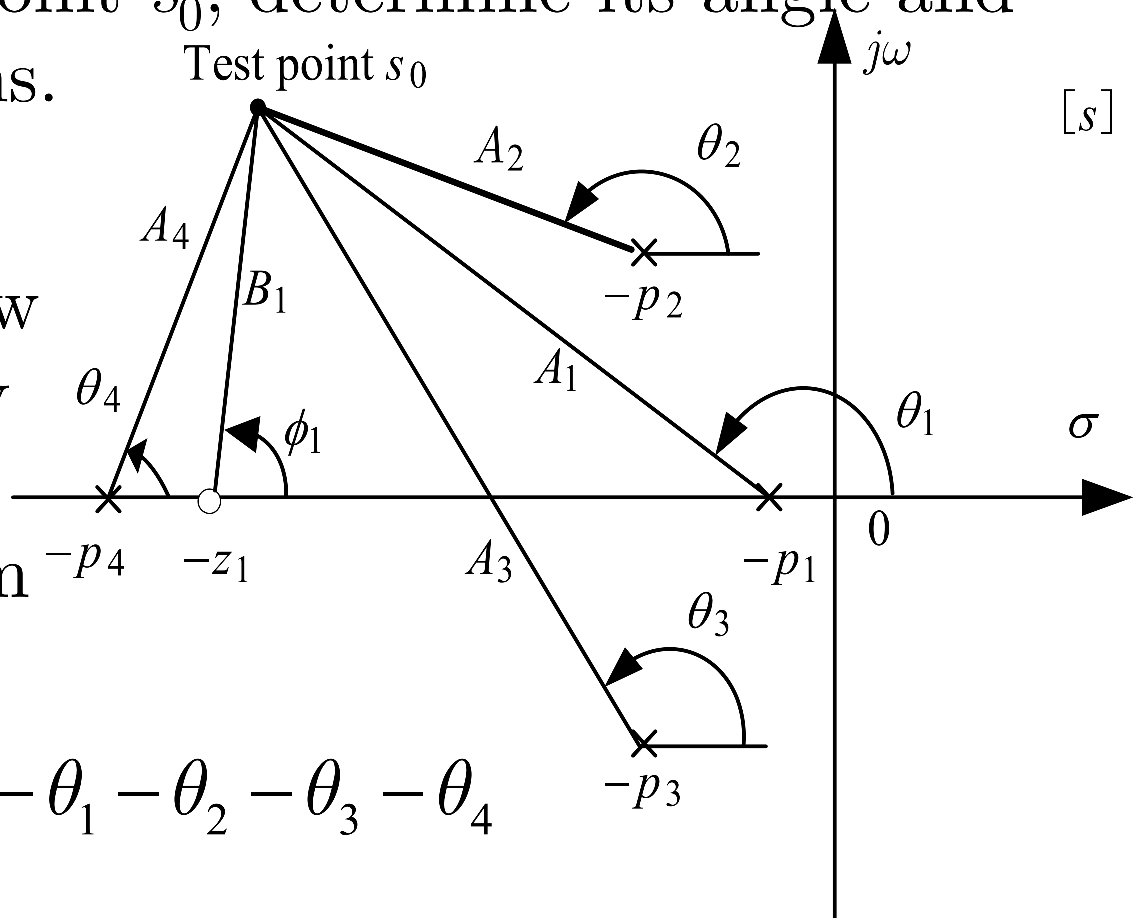
$$\angle G(s)H(s) = 180^0(2k+1)$$

Example. Consider the following open-loop plant:

$$G(s)H(s) = \frac{K^*(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

$$= \left|G(s)H(s)\right| \angle G(s)H(s)$$

For any given test point s_0, determine its angle and magnitude conditions.



Solution: For the given s_0, we can draw vectors in s-plane by the location of the poles and zeros, from which we have

$$\angle G(s)H(s)\Big|_{s=s_0} = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$$

If $\angle GH(s_0)=180^0$, s_0 is a point of the root loci; otherwise, it is not a point of the root loci. Meanwhile,

$$\left|G(s)H(s)\right\|_{s=s_0} = \frac{K^* B_1}{A_1 A_2 A_3 A_4}$$

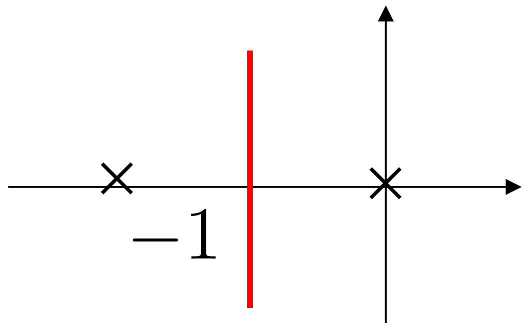
where A_i and B_1 are the magnitudes of the complex quantities s_0+p_i and s_0+z_1, respectively. Let

$$\left|G(s)H(s)\right\|_{s=s_0} = \frac{K^* B_1}{A_1 A_2 A_3 A_4} = 1$$

from which we can determine the value of K^* so that the magnitude condition is satisfied.

Example. Consider the following open-loop plant:

$$G(s)H(s) = \frac{K^*}{s(s+2)}$$

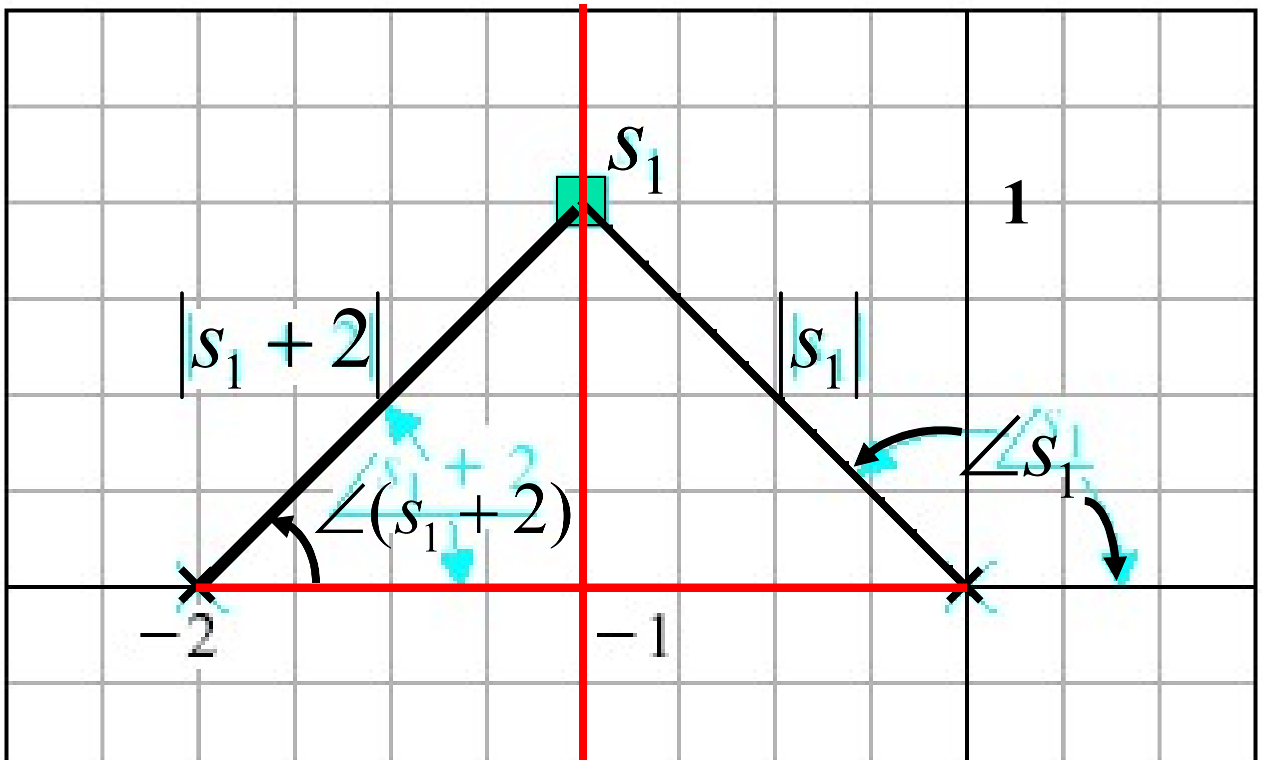


Show that any point on the -1 line belongs to the root locus. Moreover, determine K^* when $s_1 = -1 \pm j$.

Solution: Because any point on the -1 line always satisfies the angle condition. Then, let

$$\left|G(s)H(s)\right|\Big|_{s=s_1} = \left|\frac{K^*}{s(s+2)}\right|_{s=s_1} = 1$$

it follows that $K^* = 2$.



$$-\angle s_1 - \angle(s_1 + 2) = -135^0 - 45^0 = -180^0$$

$$K^* = |s_1||s_1 + 2| = \sqrt{2} \cdot \sqrt{2} = 2$$

6-3 Root Loci Construction Rules

Note that for root loci, the following facts are true:

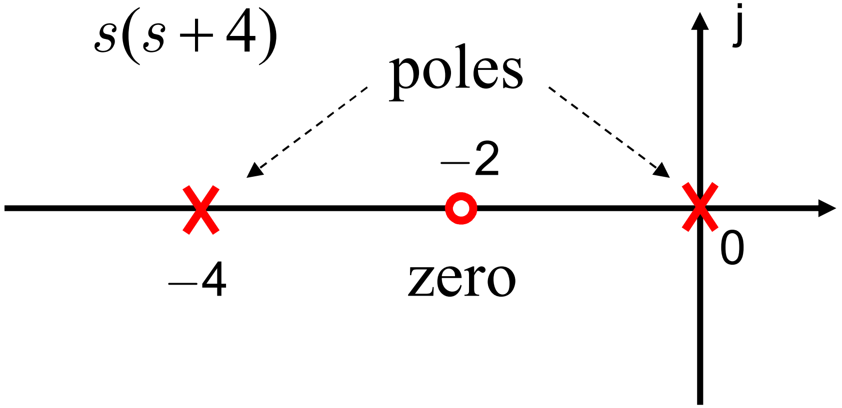
- The number of root locus branches is equal to the order of the characteristic equation.
- The loci are symmetrical about the real axis.

The root loci are symmetrical about the real axis since the roots of $1+G(s)H(s)=0$ must either be real or appear as complex conjugates. Therefore, we only need to construct the upper half of the root loci and draw the mirror image of the upper half in the lower-half s-plane.

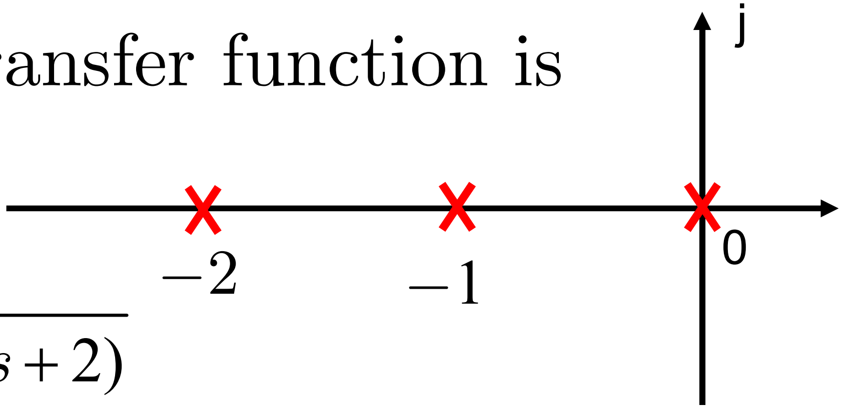
Rule 1. The root locus branches start from open-loop poles and end at open-loop zeros or zeros at infinity (see Appendix for the proof).

Example. Second-order system

$$G(s)H(s)=\frac{K(0.5s+1)}{s(0.25s+1)}=\frac{K^*(s+2)}{s(s+4)}$$



Example. The open-loop transfer function is

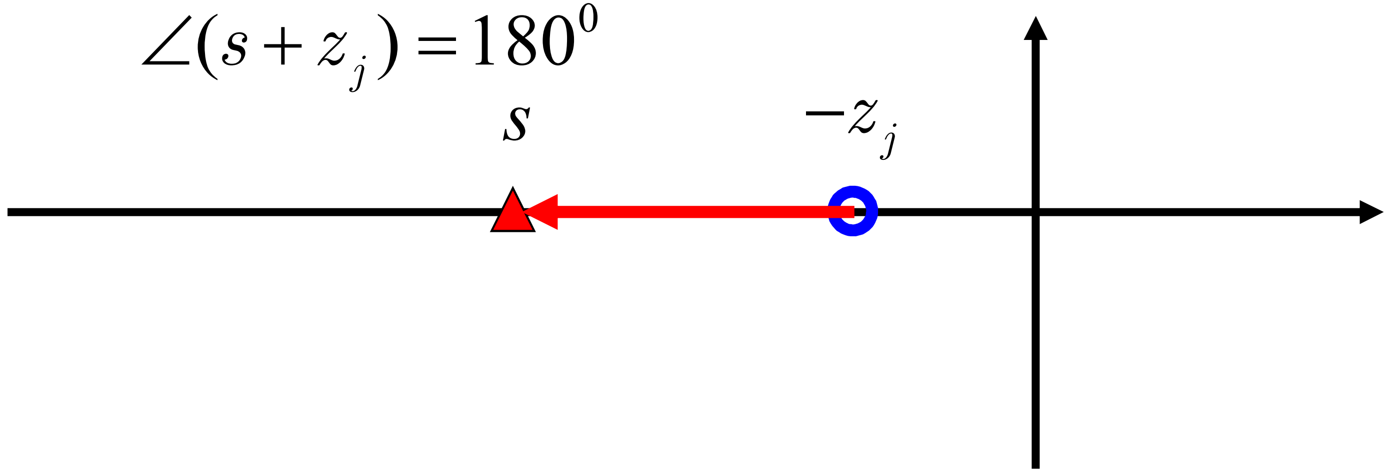


$$G(s)=\frac{K}{s(s+1)(0.5s+1)}=\frac{K^*}{s(s+1)(s+2)}$$

Rule 2. Root Loci on the Real Axis: If the total number of real poles and real zeros to the right of a test point on the real axis is odd, then the test point lies on a root locus.

Let s be a test point on the real axis as shown below. Since zero $-z_j$ of $G(s)H(s)$ lies to the right of s, it follows that

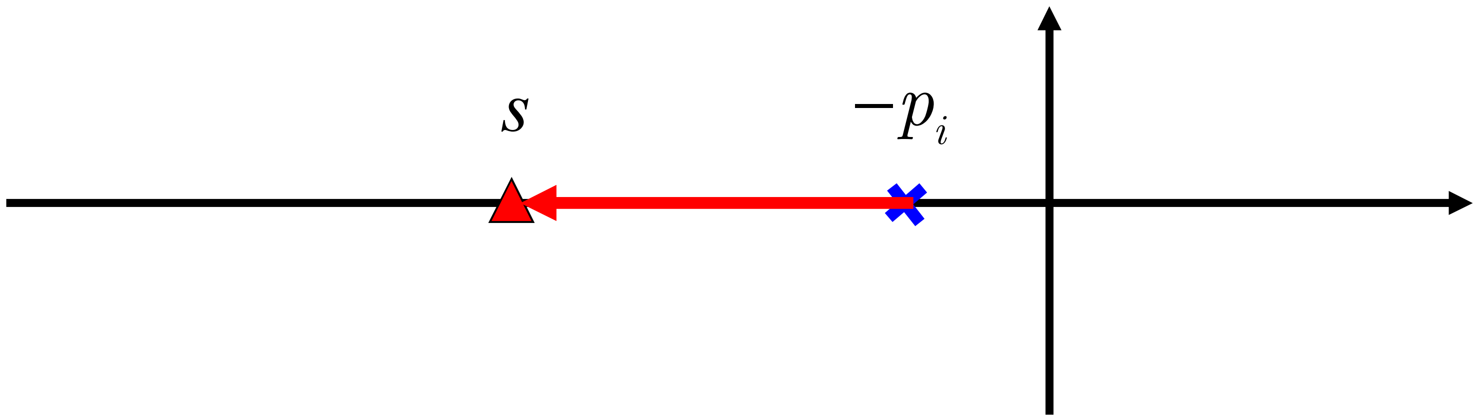
$$\angle(s+z_j)=180^0$$



Therefore, root locus exists on $(-\infty, -z_j]$.

Let s be a test point on the real axis as shown below. Since the pole $-p_i$ of $G(s)H(s)$ lies to the right of s, it follows that

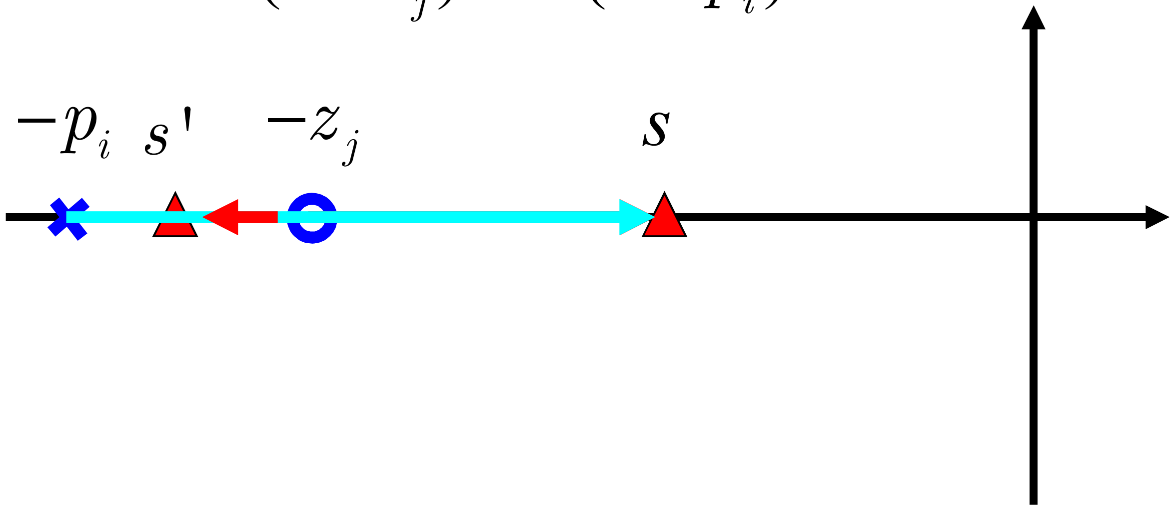
$$\angle(s+p_i)=180^0$$



Therefore, root locus exists on $(-\infty, -p_i]$.

Whereas, let s be a test point as shown below. Since there is no real pole or real zero to the right of s (the pole $-p_{\mathrm{i}}$ and zero $-z_j$ lies to the left of s), it follows that

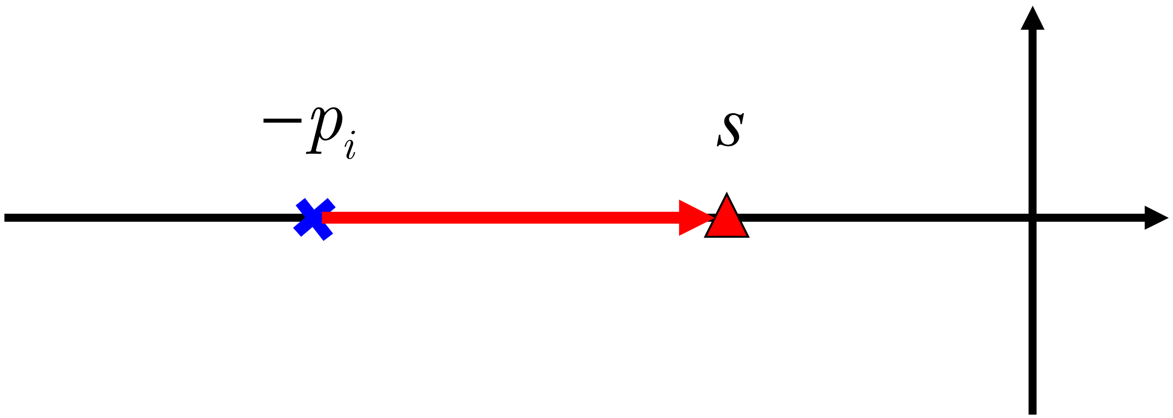
$$\angle(s+z_j)-\angle(s+p_i)=0^0$$



Therefore, no root locus exists on $[-z_j\ ,\ +\infty)$. However, by the rule, root locus exists on $[-p_i\ ,\ -z_j]$ since for the test point s', angle condition holds.

Whereas, let s be a test point as shown below. Since there is no real pole or real zero to the right of s (the pole $-p_i$ lies to the left of s, it follows that

$$\angle(s + p_i) = 0$$

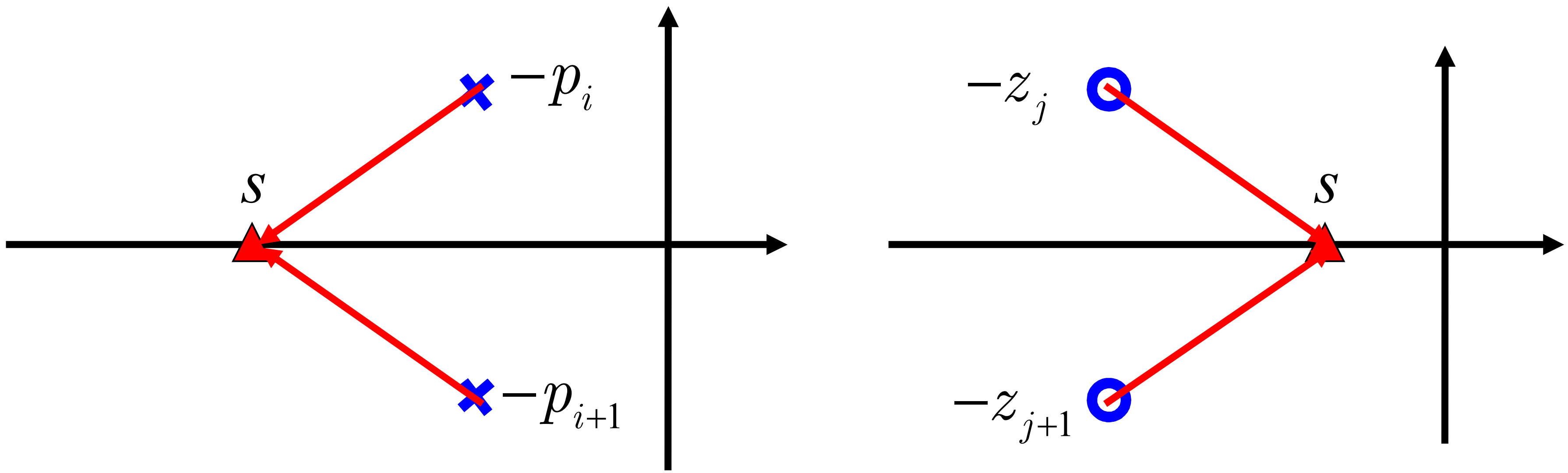


Therefore, no root locus exists on $[-p_i\ ,\ +\infty)$.

Now, consider the case that $-p_i$ ($-z_i$) and $-p_{i+1}$ ($-z_{i+1}$) are a pair of complex poles (or zeros) of $G(s)H(s)$. It is clear that

$$\angle(s+p_i)+\angle(s+p_{i+1})=0^0$$

$$\angle(s+z_i)+\angle(s+z_{i+1})=0^0$$



In general, for a **real** test point s, let

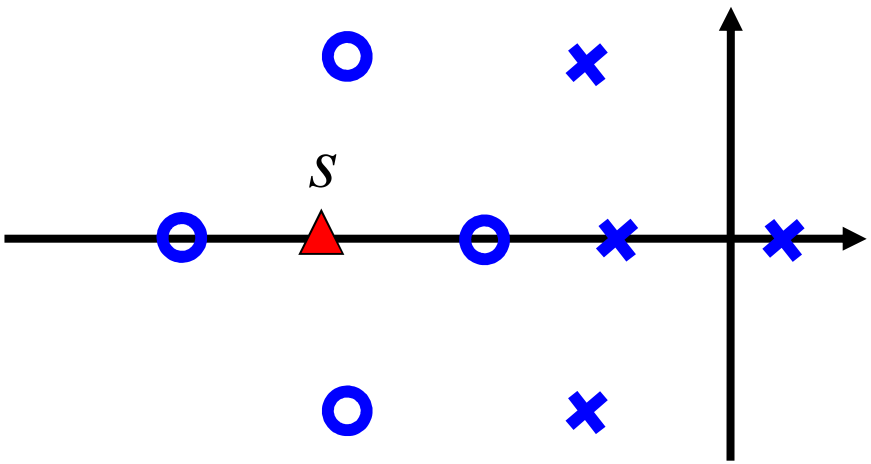
m_1: the number of **real** zeros to the right of test point s,

n_1: the number of **real** poles to the right of test point s.

If m_1+n_1 is odd, then

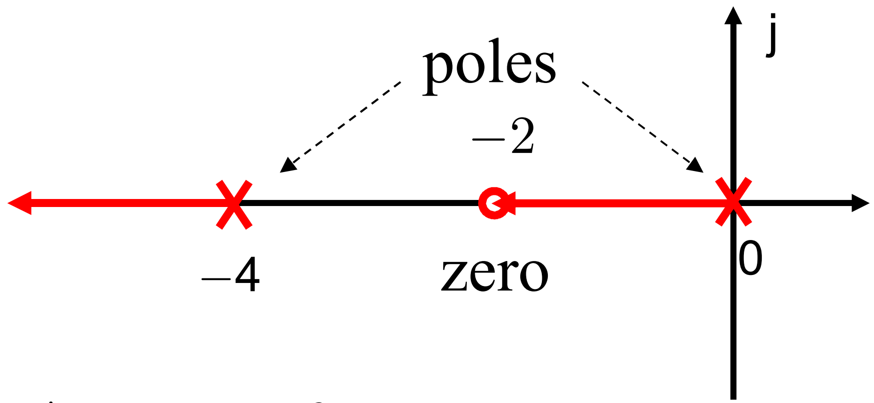
$$\sum_{j=1}^{m} \angle(s - z_j) - \sum_{i=1}^{n} \angle(s - p_i) = 180^0 \times (2k + 1)$$

That is, s is on a root locus.



Example. Second-order system:

$$G(s)H(s) = \frac{K(0.5s+1)}{s(0.25s+1)} = \frac{K^*(s+2)}{s(s+4)}$$



Example. A unity-feedback system with open-loop transfer function is

$$G(s) = \frac{K^*}{s(s+1)(s+2)}$$

Rule 3. Asymptotes of root loci: The loci proceed to the zeros at infinity along asymptotes.

These linear asymptotes intersection point on the real axis is given by

$$\boxed{\sigma_a = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m} = \frac{\sum \text{poles of } GH - \sum \text{zeros of } GH}{n-m}}$$

The angle of the asymptotes with respect to the real axis is

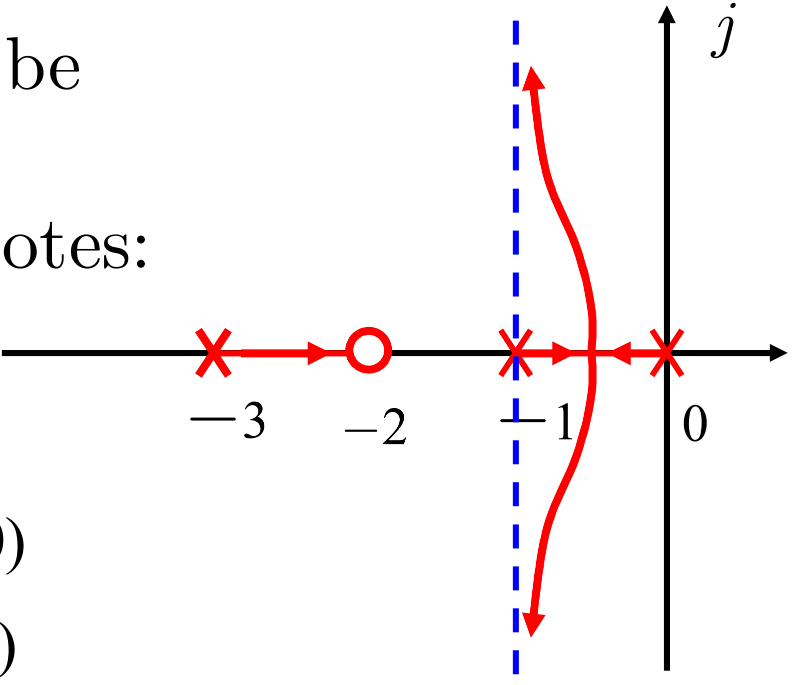
$$\varphi_a = 180^0 \times \frac{(2k+1)}{n-m} \qquad (k = 0, \pm 1, \cdots, \pm n-m-1)$$

Example. An open-loop transfer function of a unity-feedback system is

$$G(s)=\frac{K^*(s+2)}{s(s+1)(s+3)}$$

Sketch the root locus plot.

- By rule 2, root loci exist on $[-1, 0]$ and $[-3, -2]$. By rule 1, the root locus from -3 to -2 can be determined.
- By rule 3, for the asymptotes:



$$\sigma_a=\frac{(-1-3)-(-2)}{3-1}=-1$$

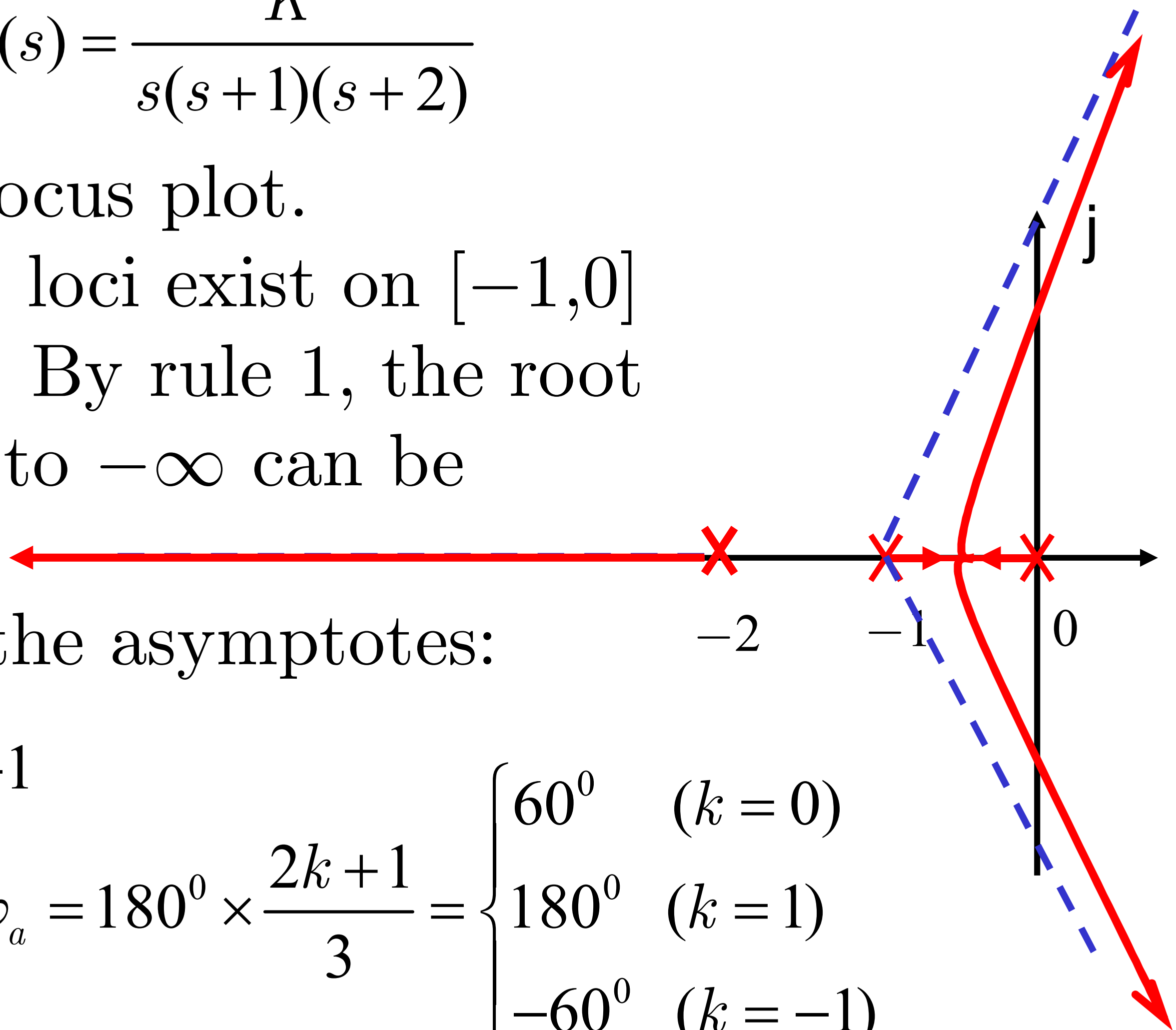
$$\varphi_a=180^0\times\frac{2k+1}{3-1}=\begin{cases}90^0 & (k=0)\\ 270^0 & (k=1)\end{cases}$$

Example. A unity-feedback system with open-loop transfer function is

$$G(s)=\frac{K^*}{s(s+1)(s+2)}$$

Sketch the root locus plot.

- By rule 2, root loci exist on $[-1,0]$ and $(-\infty, -2]$. By rule 1, the root locus from -2 to $-\infty$ can be determined.



- By rule 3, for the asymptotes:

$$\sigma_a=\frac{(-1-2)-0}{3}=-1$$

$$\varphi_a=180^0\times\frac{2k+1}{3}=\begin{cases}60^0 & (k=0)\\ 180^0 & (k=1)\\ -60^0 & (k=-1)\end{cases}$$

Rule 4. Breakaway (break in) point on the root loci.

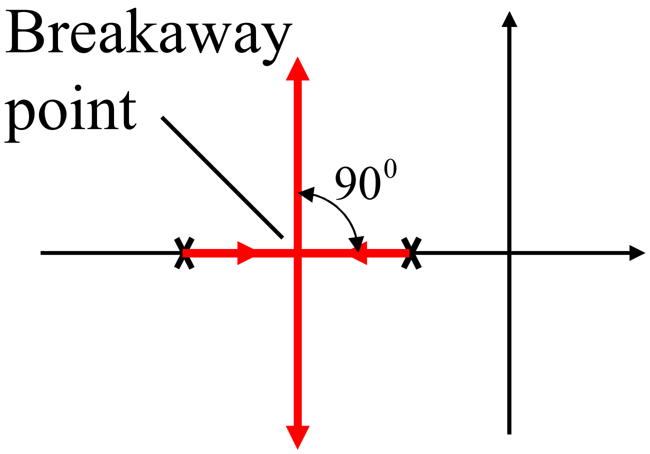
1) The breakaway (break in) point d can be computed by solving (see the Appendix for a proof)

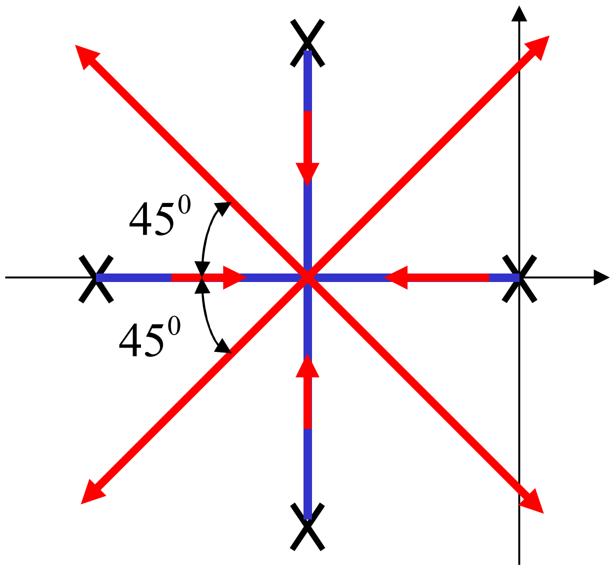
$$\boxed{\sum_{j=1}^{m} \frac{1}{d+z_j} = \sum_{i=1}^{n} \frac{1}{d+p_i}}$$

• Note that breakaway (break in) point on the root loci corresponds to multiple roots of the equation $1+G(s)H(s)=0$.

• The condition given above is **only necessary but not sufficient**.

2) The angle of breakaway is $180^0/k$, where k is the number of poles intersecting at the breakaway point.

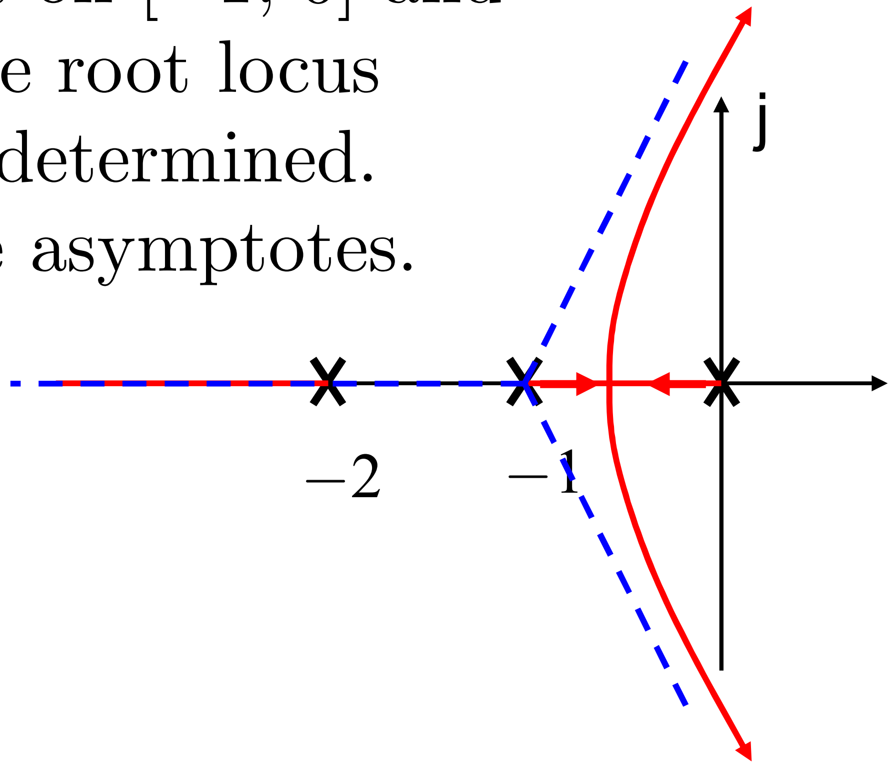




Example. Again, consider the open-loop transfer function

$$G(s) = \frac{K^*}{s(s+1)(s+2)}$$

- By rule 2, root loci exist on $[-1, 0]$ and $(-\infty, -2]$. By rule 1, the root locus from -2 to $-\infty$ can be determined.
- By rule 3, we obtain the asymptotes.
- By rule 4,



$$\frac{1}{d} + \frac{1}{d+1} + \frac{1}{d+2} = 0$$

$$3d^2 + 6d + 2 = 0$$

$$d_1 = -0.42, \quad \cancel{d_2 = -1.58}$$

Example. Consider the open-loop transfer function

$$G(s) = \frac{K^*}{s(s+4)(s^2+4s+20)}$$

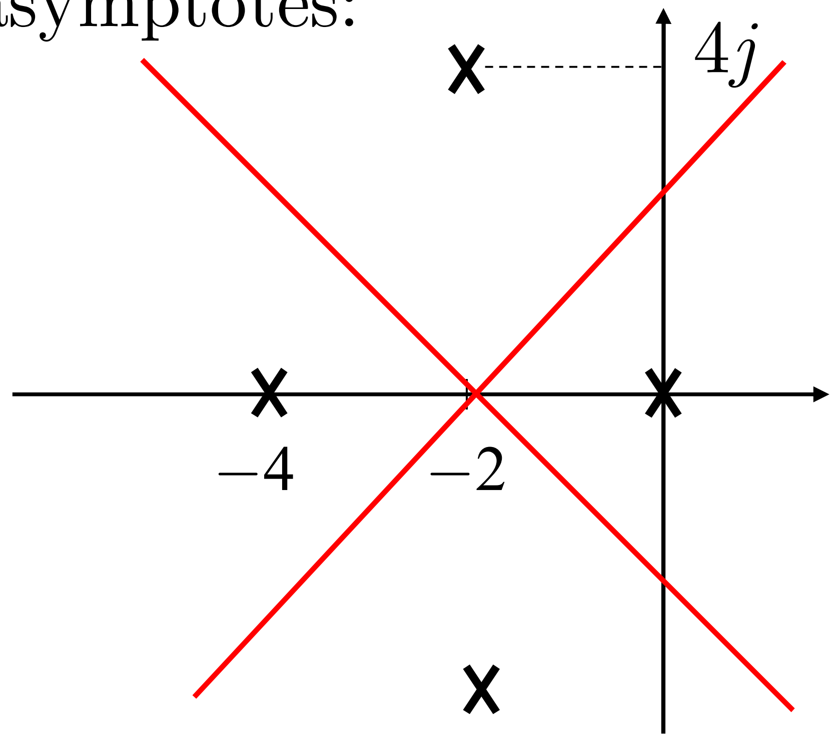
four open-loop poles are: $0, -4, -2 \pm 4j$

Sketch its root locus plot.

- By rule 2, root locus exists on $[-4, 0]$.
- By rule 3, we obtain the asymptotes:

$$\sigma_a = \frac{-4-2+4j-2-4j-0}{4} = -2$$

$$\varphi_a = 180^0 \times \frac{2k+1}{4} = \begin{cases} 45^0 & (k=0) \\ 135^0 & (k=1) \\ -45^0 & (k=-1) \\ -135^0 & (k=-2) \end{cases}$$

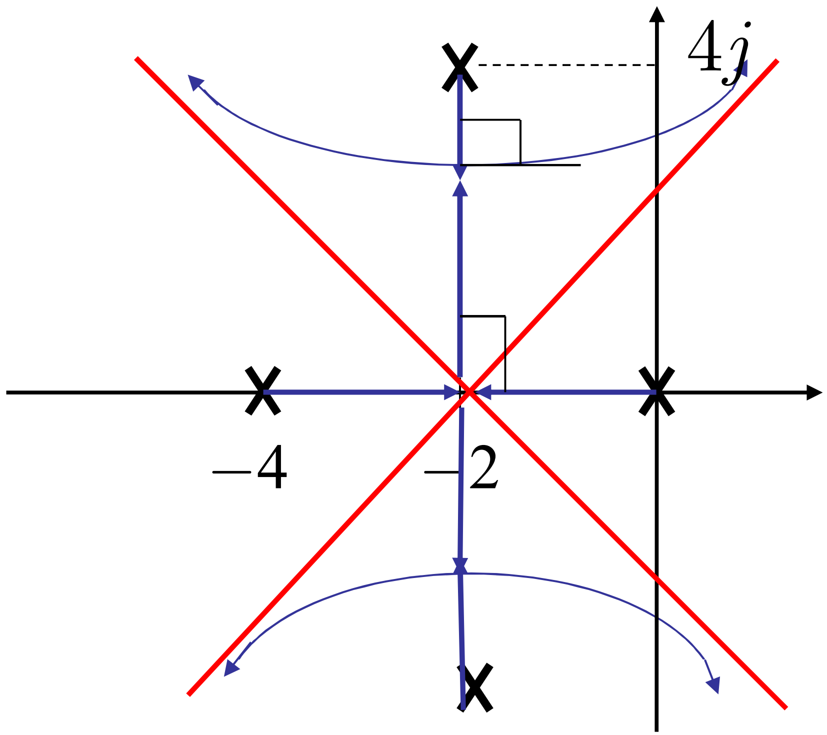


- By rule 4, the breakaway points satisfy

$$\frac{1}{d}+\frac{1}{d+4}+\frac{1}{d+2+4j}+\frac{1}{d+2-4j}=0$$

$$d^3+6d^2+18d+20=0$$

Solving, $\begin{cases} d_1=-2 \\ d_2=-2+2.45j \\ d_3=-2-2.45j \end{cases}$

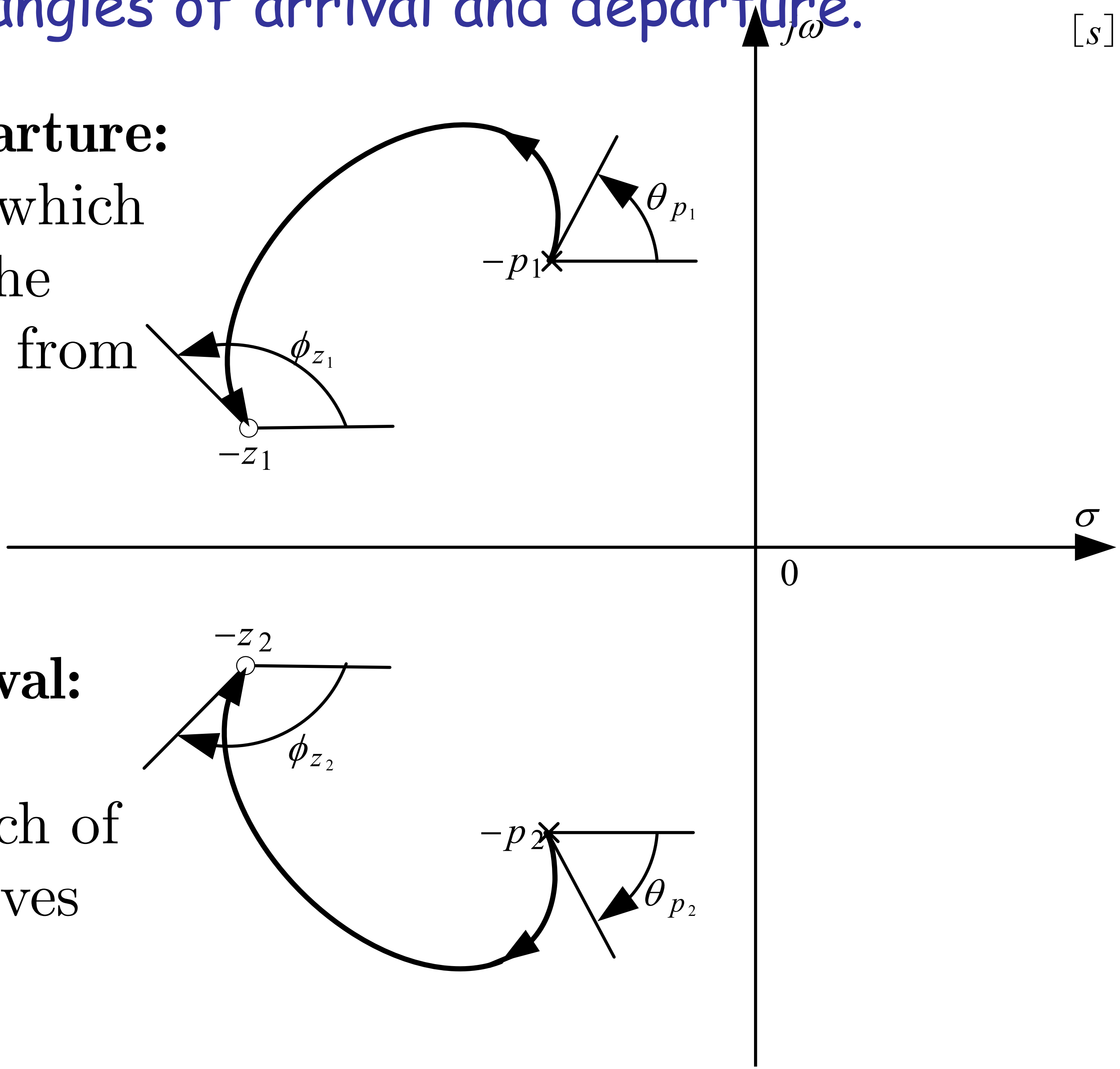


The angle of breakaway for each pair of intersecting point is $180^0/2=90^0$.

Rule 5. The angles of arrival and departure.

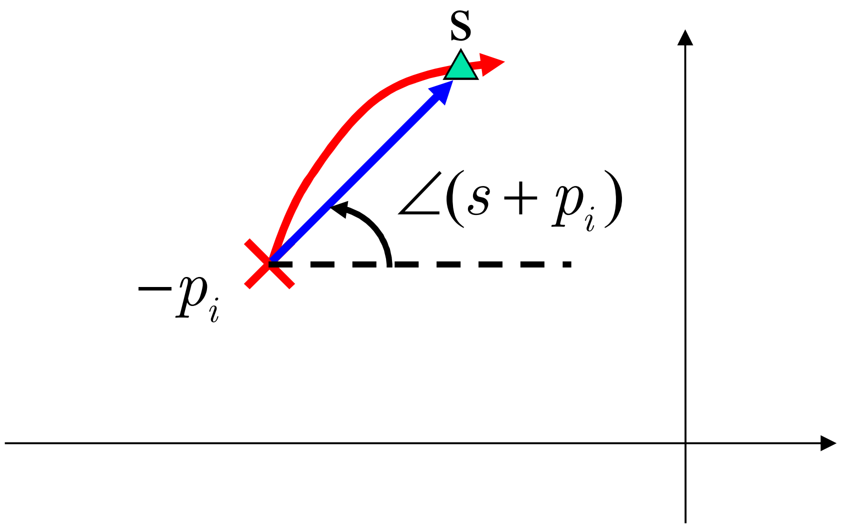
Angle of departure: the angle by which a branch of the locus departs from one pole.

Angle of arrival: the angle by which a branch of the locus arrives at one zero.



1) Angle of departure

Choose a test point s and move it in the very vicinity of $-p_i$. Then, if s is on the root locus, the angle condition must be satisfied:



$$\sum_{j=1}^{m} \angle(s+z_j) - \sum_{k=1}^{n} \angle(s+p_k) = 180^0 \times (2k+1)$$

Rewrite the above equation as

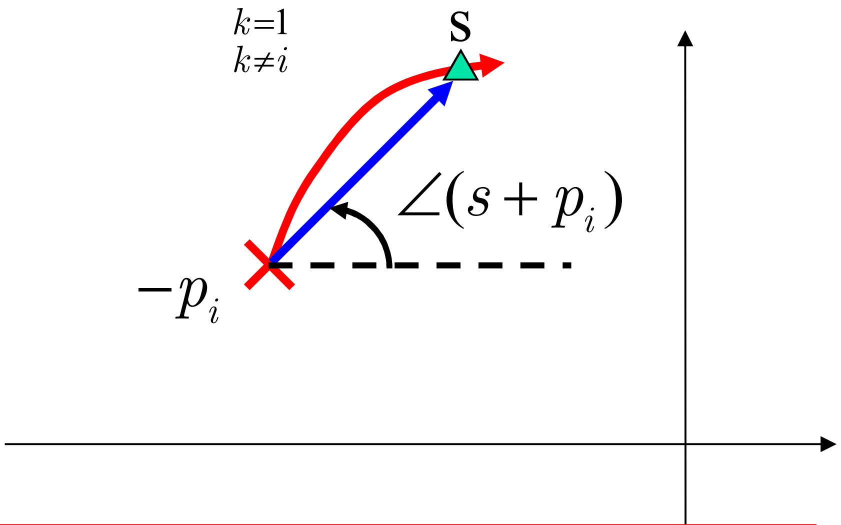
$$\sum_{j=1}^{m} \angle(s+z_j) - \sum_{\substack{k=1\\k\neq i}}^{n} \angle(s+p_k) - \angle(s+p_i) = 180^0 \times (2k+1)$$

Therefore,

$$\angle(s+p_i) = 180^0 \times (2k+1) + \sum_{j=1}^{m} \angle(s+z_j) - \sum_{\substack{k=1\\k\neq i}}^{n} \angle(s+p_k)$$

Let

$$\theta_{p_i} := \lim_{s\to -p_i} \angle(s+p_i)$$



Then

$$\boxed{\theta_{p_i} = \lim_{s\to -p_i} \angle(s+p_i) = 180^0 \times (2k+1) + \sum_{j=1}^{m} \angle(-p_i+z_j) - \sum_{\substack{k=1\\k\neq i}}^{n} \angle(-p_i+p_k)}$$

Example. The open-loop transfer function

$$G(s)H(s)=\frac{K^*}{s(s^2+2s+1)}=\frac{K^*}{s(s+1+j)(s+i-j)}$$

Therefore,

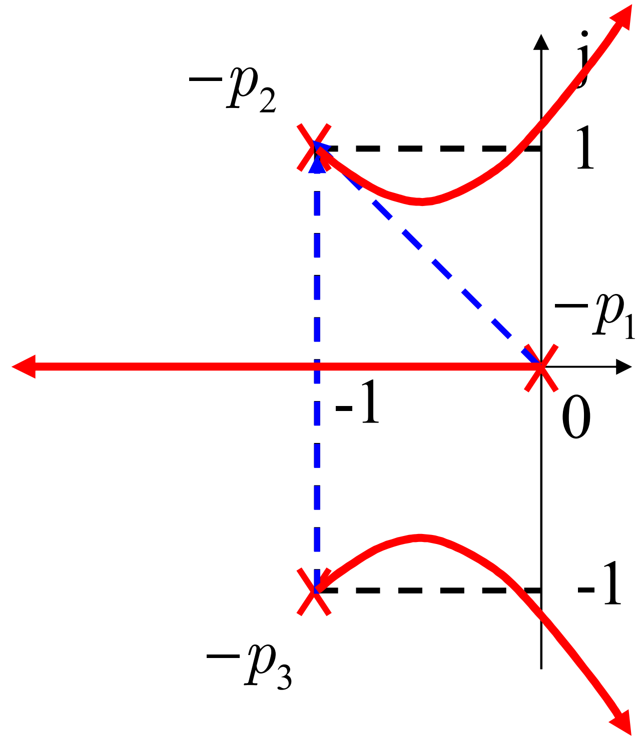
$$-p_1=0,\quad -p_2=-1+j,\quad -p_3=-1-j$$

By using the departure angle formula,

$$\theta_{p_2}=180^0-135^0-90^0=-45^0$$

By the symmetry property of the root locus,

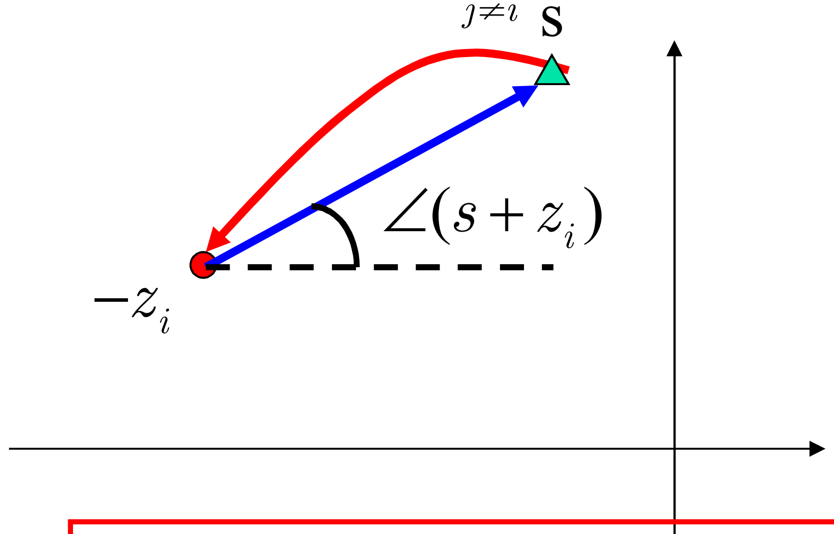
$$\theta_{p_3}=45^0$$



2) Angle of arrival: Similar to the deduction of the departure angle, for the test point s we have

$$\sum_{j=1}^{m} \angle(s+z_j) - \sum_{j=1}^{n} \angle(s+p_j)$$

$$= \angle(s+z_i) + \sum_{\substack{j=1 \\ j\neq i}}^{m} \angle(s+z_j) - \sum_{j=1}^{n} \angle(s+p_j) = 180^0 \times (2k+1)$$



Let $s \longrightarrow -z_i$ and

$$\phi_{z_i} := \lim_{s \to -z_i} \angle(s+z_i)$$

Then it follows that

$$\boxed{\phi_{z_i} = 180^0 \times (2k+1) - \sum_{\substack{j=1 \\ j\neq i}}^{m} \angle(-z_i + z_j) + \sum_{k=1}^{n} \angle(-z_i + p_k)}$$

Example. The open-loop transfer function

$$G(s)H(s) = \frac{K^*(s^2 + 4.5s + 5.625)}{s(s+1)(s+2)}$$

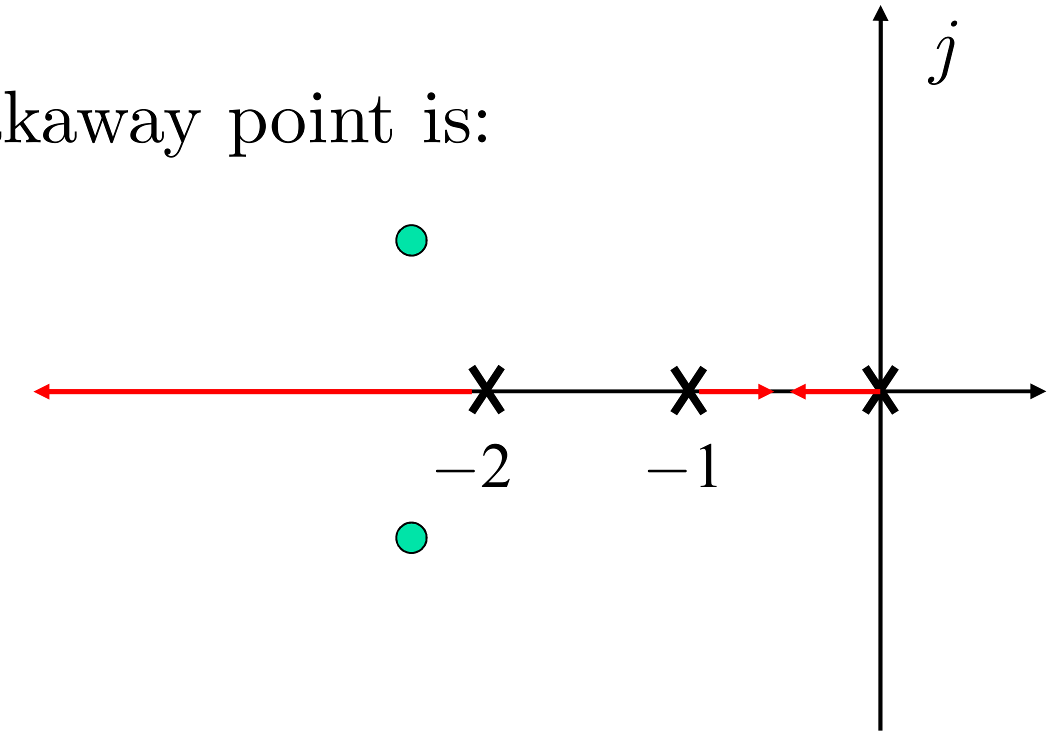
$$z_1 = -2.25 + j0.75, \; z_2 = -2.25 - j0.75$$

- By rule 2, root loci exist on $[-1, 0]$ and $(-\infty, -2]$. By rule 1, the root locus from -2 to $-\infty$ can be determined.
- By rule 4, the breakaway point is:

$$\frac{1}{d} + \frac{1}{d+1} + \frac{1}{d+2}$$

$$= \frac{1}{d + 2.25 + 0.75j}$$

$$+ \frac{1}{d + 2.25 - 0.75j}$$

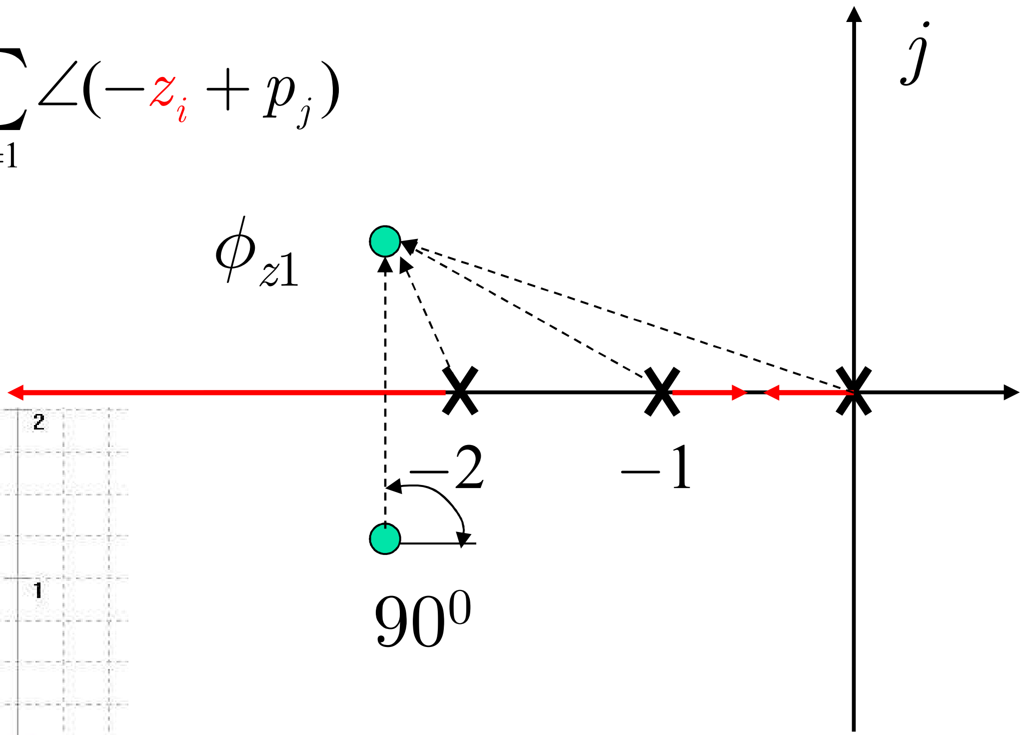


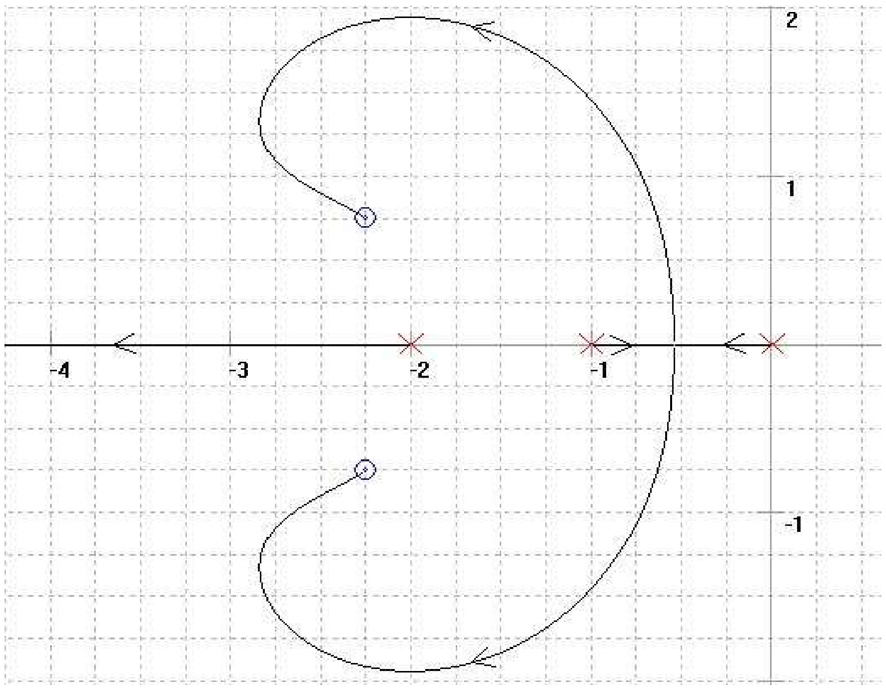
The two points that breakaway at 90^0.

By rule 5, the angle of arrival is

$$\phi_{z_1} = 180^0 - \sum_{\substack{j=1 \\ j \neq i}}^{m} \angle(-z_i + z_j) + \sum_{j=1}^{n} \angle(-z_i + p_j)$$

$$= 180^0 - 90^0 + \theta_1 + \theta_2 + \theta_3$$





Example. Verify that in the case of $-p_i$ being l repeated poles, the angle of departure is (the same conclusion can also be extended to the angle of arrival)

$$\boxed{\theta_{p_i} = \frac{1}{l}\left[180^0 \times (2k+1) + \sum_{j=1}^{m} \angle(-p_i + z_j) - \sum_{\substack{k=1 \\ k \neq i}}^{n} \angle(-p_i + p_k)\right]}$$

Proof. In the case that $-p_i$ is l repeated poles, we have

$$\left[\sum_{j=1}^{m} \angle(s+z_j) - \sum_{\substack{k=1 \\ k \neq i}}^{n} \angle(s+p_k) - l\angle(s+p_i)\right]_{s=-p_i} = 180^0 \times (2k+1)$$

Hence,

$$l\theta_{p_i} = 180^0 \times (2k+1) + \sum_{j=1}^{m} \angle(-p_i + z_j) - \sum_{\substack{k=1 \\ k \neq i}}^{n} \angle(-p_i + p_k)$$

Rule 6. Intersection of the root loci with the imaginary axis.

$$G(s)=\frac{K^*}{s(s+1)(s+2)}$$

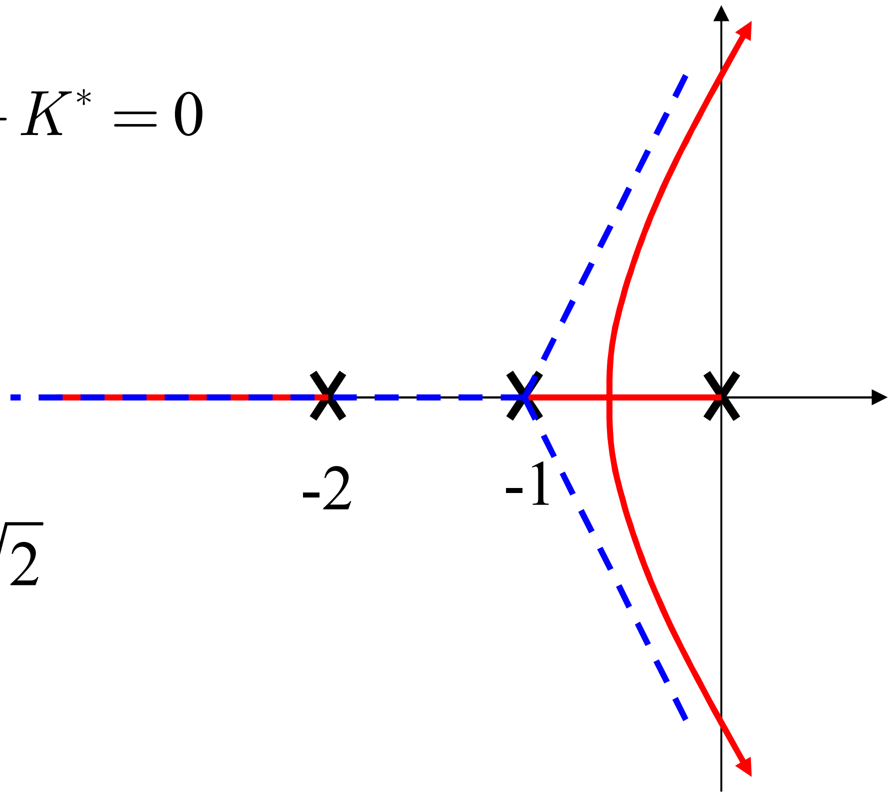
The closed-loop characteristic equation is

$$s^3+3s^2+2s+K^*=0$$

Let $s = j\omega$. Then,

$$-j\omega^3-3\omega^2+2j\omega+K^*=0$$

$$\begin{cases}-\omega^3+2\omega=0\\ -3\omega^2+K^*=0\end{cases}\Rightarrow\begin{cases}\omega=0, \quad \pm\sqrt{2}\\ K^*=0, \quad 6\end{cases}$$



Rule 7. The sum of closed loop poles

Let

$$G(s)H(s) = K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

Then, if $n \geq m+2$, the sum of the open-loop poles =the sum of the closed-loop poles; that is, **the sum of poles remains unchanged** as K^* varies form 0 to $+\infty$.

Example. Let

$$G(s)H(s) = \frac{K^*}{(s+p_3)(s+p_1)(s+p_2)}$$

which satisfies Rule 7. The closed-loop characteristic equation is

$$(s+p_1)(s+p_2)(s+p_3)+K^* = (s+s_1)(s+s_2)(s+s_3)$$

where $-s_i$ denote the closed-loop poles. Moreover,

$$\begin{aligned}
&(s+p_1)(s+p_2)(s+p_3)+K^* \\
&= s^3 + (p_1+p_2+p_3)s^2 + (p_1p_2+p_1p_2+p_2p_3)s + p_1p_2p_3 + K^* \\
&= (s+s_1)(s+s_2)(s+s_3) \\
&= s^3 + (s_1+s_2+s_3)s^2 + (s_1s_2+s_1s_2+s_2s_3)s + s_1s_2s_3
\end{aligned}$$

It is clear that

$$(p_1+p_2+p_3) = (s_1+s_2+s_3)$$

no matter what value of K^* is taken.

In general,

From
$$1 + K^* \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

the closed-loop characteristic equation is

$$\prod_{j=1}^{n}(s+p_j) + K^* \prod_{i=1}^{m}(s+z_i) = \prod_{i=1}^{n}(s+s_i)$$

$$= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1}s + a_n = 0$$

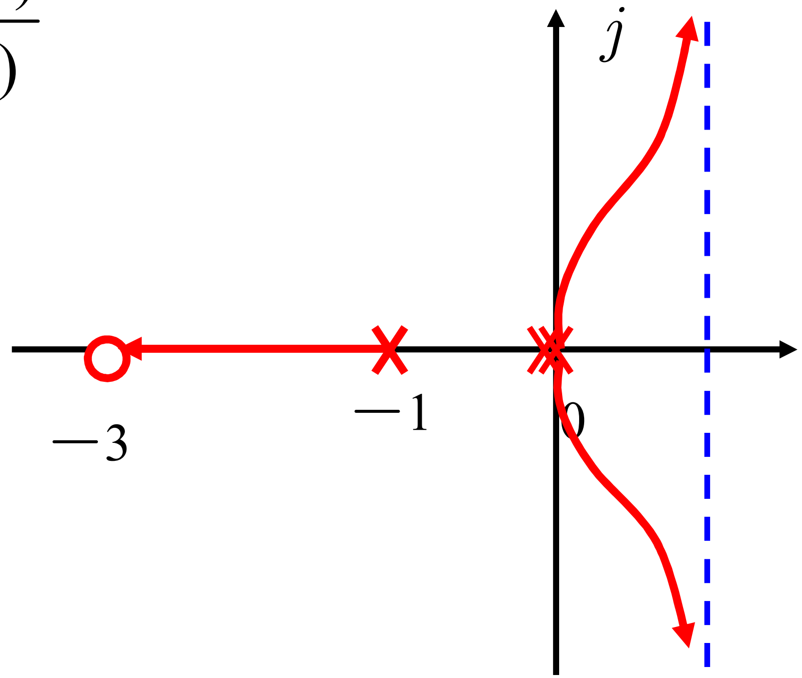
where $-s_i$ denote the closed-loop poles. If $n \geq m+2$,

$$a_1 = (p_1 + p_2 + \cdots + p_n) = (s_1 + s_2 + \cdots + s_n) = \text{const}$$

That is, the sum of the open-loop poles =the sum of the closed-loop poles and therefore **the sum of poles remains unchanged** as K^* varies form 0 to $+\infty$.

Example. An open-loop transfer function of a unity-feedback system is given below:

$$G(s)=\frac{K^*(s+3)}{s^2(s+1)}$$



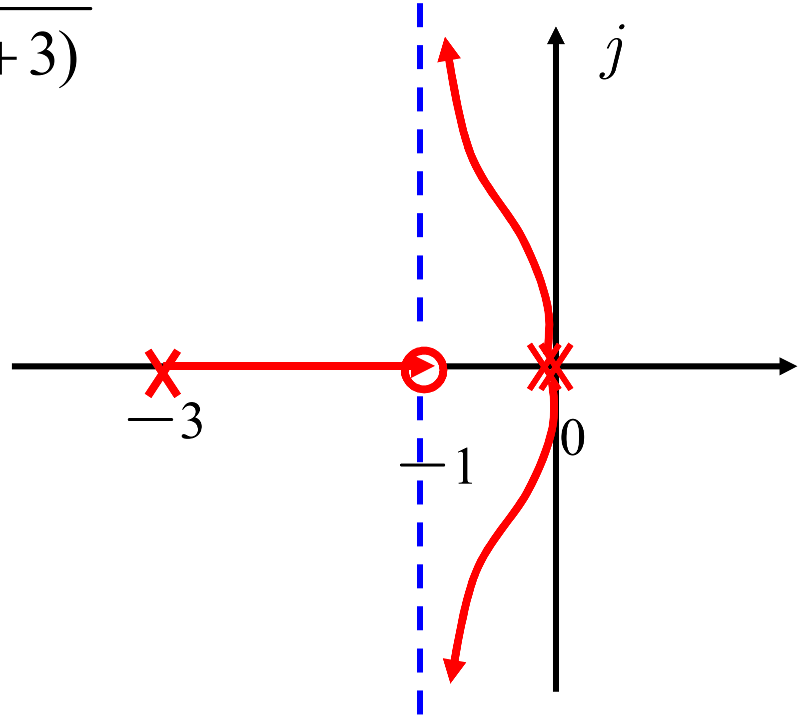
Sketch the root locus plot.

- By rule 2, root locus exists on $[-3,\ -1]$.
- By rule 3, for the asymptotes,

$$\sigma_a=\frac{-1+3}{2}=1 \qquad \varphi_a=180^0\times\frac{2k+1}{2}=\begin{cases}90^0 & (k=0)\\ -90^0 & (k=-1)\end{cases}$$

Example. An open-loop transfer function of a unity-feedback system is given below:

$$G(s) = \frac{K^*(s+1)}{s^2(s+3)}$$



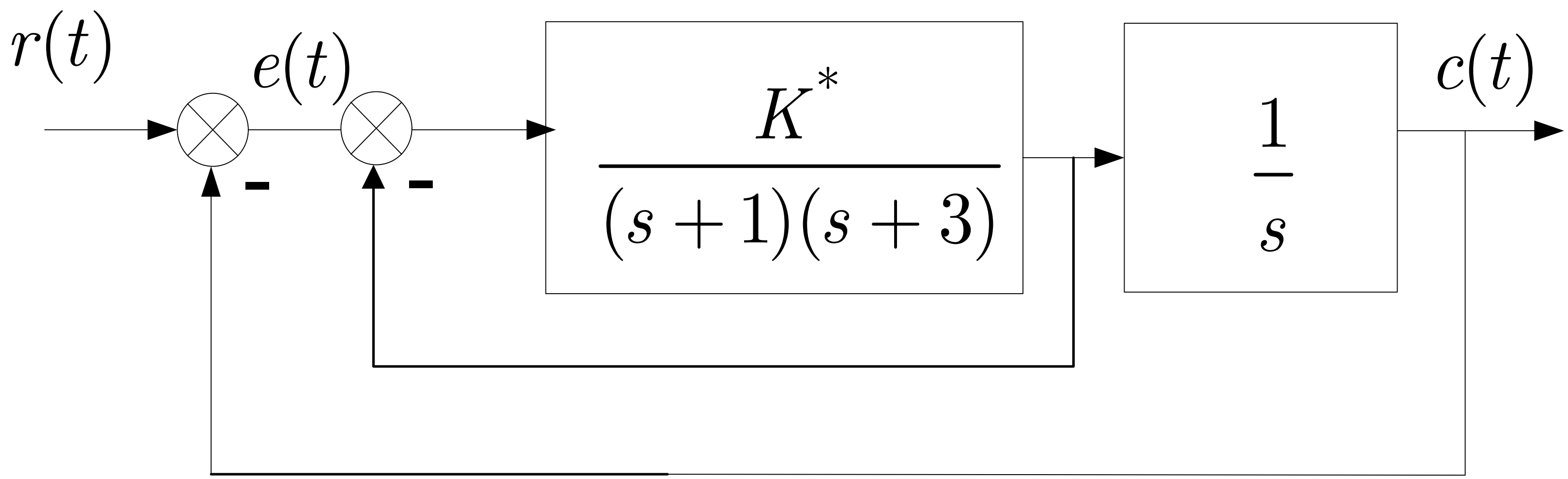
Sketch the root locus plot.

- By rule 2, root locus exists on $[-3, -1]$.
- By rule 3, for the asymptotes,

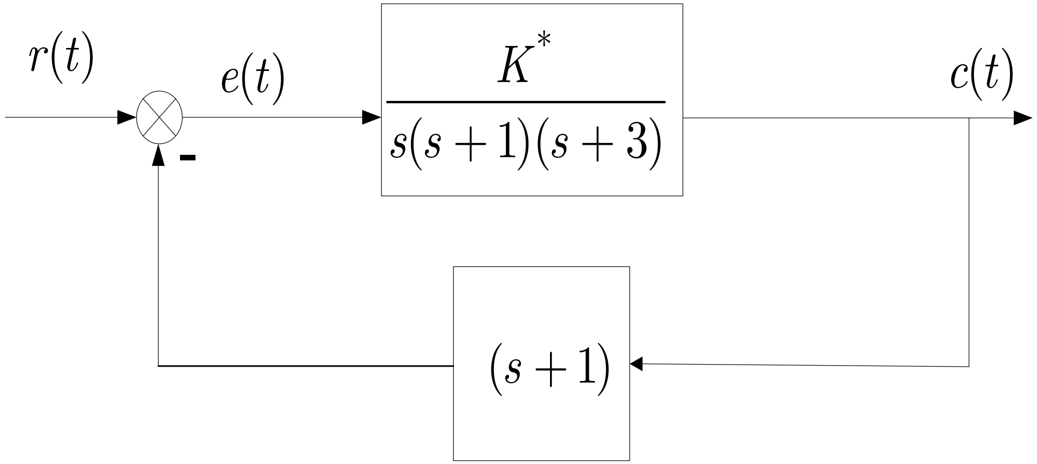
$$\sigma_a = \frac{-3+1}{2} = -1 \qquad \varphi_a = 180^0 \times \frac{2k+1}{2} = \begin{cases} 90^0 & (k=0) \\ -90^0 & (k=-1) \end{cases}$$

Cancellation of Poles of G(s) with Zeros of H(s)

If the denominator of $G(s)$ and the numerator of $H(s)$ involve common factors then the corresponding open-loop poles and zeros will cancel each other, reducing the degree of the characteristic equation by one or more. Consider the following system:



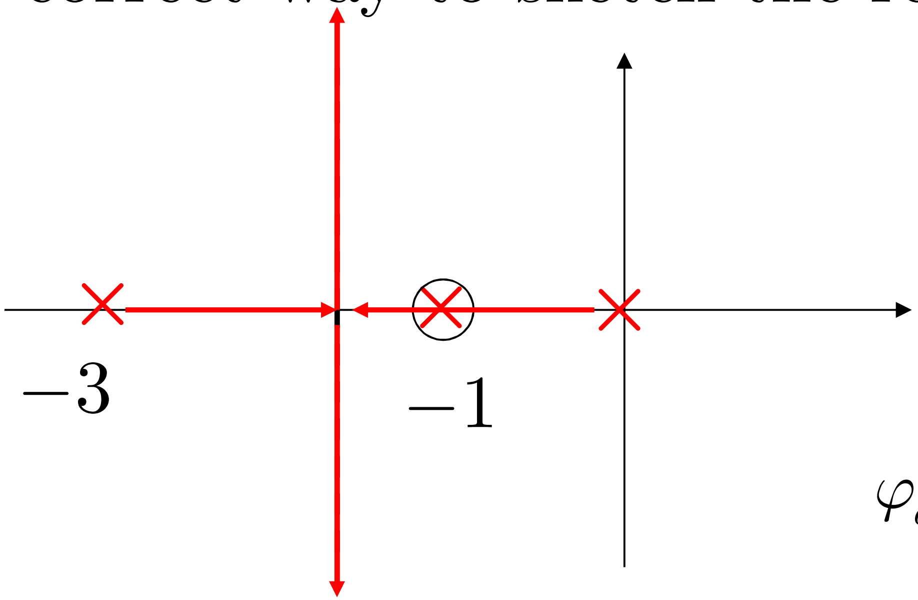
which can be transformed into the following form:



Therefore,

$$G(s)H(s) = \frac{K^*(s+1)}{s(s+1)(s+3)}$$

Noting the closed-loop characteristic equation, the correct way to sketch the root loci is



$$\sigma_a = \frac{-1-3+1}{2} = -1.5$$

$$\varphi_a = 180^0 \times \frac{2k+1}{2} = \begin{cases} 90^0 & (k=0) \\ -90^0 & (k=-1) \end{cases}$$

Appendix 1

Proof of Locus start and end points:

(1)The root loci start from the open loop poles. In fact, from

$$1+K^{*}\frac{\prod_{i=1}^{m}(s+z_{i})}{\prod_{j=1}^{n}(s+p_{j})}=0$$

the close-loop characteristic polynomial is

$$\prod_{j=1}^{n}(s+p_{j})+K^{*}\prod_{i=1}^{m}(s+z_{i})=0$$

Therefore, when $K^{*}=0$,

$$\prod_{j=1}^{n}(s+p_{j})=0 \Rightarrow s=-p_{j},\ \ j=1,2,\cdots n$$

(2) The root loci end at the open loop zeros.

Write

$$1+K^*\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=0 \Rightarrow \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=-\frac{1}{K^*}$$

When $K^*\to\infty$, s must tend to $-z_i$. Moreover, if $n>m$, we can write

$$\lim_{s\to\infty}\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=\lim_{s\to\infty}\frac{1}{s^{n-m}}=0$$

That is, there are $n-m$ root locus branches tend to infinity.

Appendix 2

Proof for the intersection point of asymptotes on real axis and the corresponding angles:

Write

$$1+GH=\frac{K^{*}\prod^{m}(s+z_{j})}{\prod^{n}(s+p_{i})}=1+K^{*}\frac{s^{m}+b_{1}s^{m-1}+\cdots+b_{m-1}s+b_{m}}{s^{n}+a_{1}s^{n-1}+\cdots+a_{n-1}s+a_{n}}=0$$

Hence,

$$-K^{*}=\frac{s^{n}+a_{1}s^{n-1}+\cdots+a_{n-1}s+a_{n}}{s^{m}+b_{1}s^{m-1}+\cdots+b_{m-1}s+b_{m}}$$

$$=\frac{s^{n}-\left(\sum_{i=1}^{n}-p_{i}\right)s^{n-1}+\cdots+a_{n-1}s+a_{n}}{s^{m}-\left(\sum_{,=1}^{m}-z_{j}\right)s^{m-1}+\cdots+b_{m-1}s+b_{m}}$$

Applying long division method to the above fraction, we have

$$-K^{*}=s^{n-m}-\left((\sum_{i=1}^{n}-p_{i})-(\sum_{j=1}^{m}-z_{j})\right)s^{n-m-1}+\cdots \quad (1)$$

On the other hand, let σ_a be the intersection point of the asymptotes on the real axis. Since

$$s+p_i=s-\sigma_a-(-p_i-\sigma_a)$$
$$s+z_j=s-\sigma_a-(-z_j-\sigma_a)$$

it follows that when s is very large,

$$s+p_i\approx s-\sigma_a$$
$$s+z_j\approx s-\sigma_a$$

Substituting the above relationships into the characteristic equation yields

$$-K^* = \frac{\prod^{n}(s+p_i)}{\prod^{m}(s+z_j)} \approx \frac{(s-\sigma_a)^n}{(s-\sigma_a)^m} = (s-\sigma_a)^{n-m}$$

$$= s^{n-m} - \sigma_a(n-m)s^{n-m-1} + \cdots \qquad (2)$$

where binomial theorem has been used. Compared the coefficients of (1) and (2), it follows that

$$\sigma_a(n-m) = \left((\sum_{i=1}^{n} -p_i) - (\sum_{j=1}^{m} -z_j) \right)$$

Hence,

$$\sigma_a = \frac{(\sum_{i=1}^{n} -p_i) - (\sum_{j=1}^{m} -z_j)}{(n-m)}$$

For the angles with respect to the real axis, we note that from (2),

$$-K^{*} = (s-\sigma_{a})^{n-m}$$

That is,

$$180^{0} \times (2k+1) = \angle(s-\sigma_{a})^{n-m} = (n-m)\angle(s-\sigma_{a})$$

Let

$$\varphi_{a} \coloneqq \angle(s-\sigma_{a})$$

We have

$$\varphi_{a} = 180^{0} \times \frac{(2k+1)}{(n-m)}$$

This completes the proof.

Appendix 3

Proof of break away point:

From

$$1+\frac{K^{*}\prod^{m}(s+z_{j})}{\prod^{n}(s+p_{i})}=0$$

the close-loop characteristic equation is

$$\prod^{n}(s+p_{i})+K^{*}\prod^{m}(s+z_{j})=0 \quad (1)$$

Since the break away point corresponds to multiple poles of (1), we have

$$\left[\prod^{n}(s+p_{i})+K^{*}\prod^{m}(s+z_{j})\right]'=\left[\prod^{n}(s+p_{i})\right]'+\left[K^{*}\prod^{m}(s+z_{j})\right]'=0 \quad (2)$$

Therefore, from (1) and (2), it follows that

$$\frac{\left[\prod^{n}(s+p_i)\right]'}{\prod^{n}(s+p_i)} = \frac{\left[\prod^{m}(s+z_j)\right]'}{\prod^{m}(s+z_j)}$$

That is,

$$\left\{\ln\left[\prod^{n}(s+p_i)\right]\right\}' = \left\{\ln\left[\prod^{m}(s+z_i)\right]\right\}'$$

or

$$\sum^{n}\left[\ln(s+p_i)\right]' = \sum^{m}\left[\ln(s+z_j)\right]' \quad (3)$$

from which we obtain

$$\sum^{n}\frac{1}{s+p_i} = \sum^{m}\frac{1}{s+z_j}$$

The result thus follows by letting $s=d$.