Chapter 5

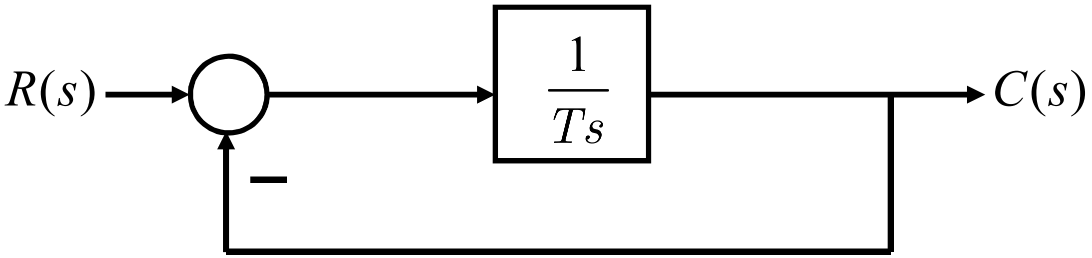
Time-Domain Analysis of Control Systems (2)

5-2 First-Order Systems

1. Mathematical model

$$T\frac{dc}{dt}+c(t)=r(t)$$

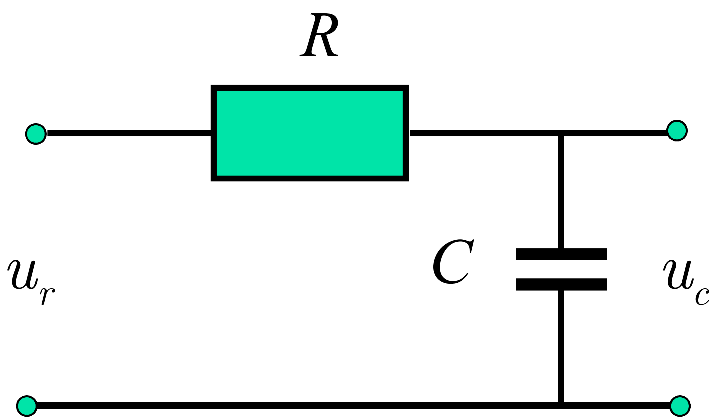
where T is time constant. Many physical systems can be described by first-order system, such as an RC circuit, thermal system, or the like. A simplified block diagram:



$$\frac{C(s)}{R(s)}=\frac{1}{Ts+1}$$

Example. Electrical first-order system

$$RC\frac{du_c}{dt}+u_c=u_i$$



Taking the *Laplace* transform of the equation with zero initial condition yields

$$\frac{U_c(s)}{U_r(s)}=\frac{1}{RCs+1}=\frac{1}{Ts+1}$$

where the time constant $T=RC$.

Example. A simplified mathematical model of a missile in linear motion can be described by a first-order system:

$$m\frac{dv}{dt} = f(t) - kv$$

where $f(t)$ is the force, v is the velocity, kv denotes the force of air resistance which is proportional to the velocity, and m, the mass.

Taking the *Laplace* transform of the equation with zero initial condition yields

$$V(s) = \frac{1}{ms+k}F(s) = \frac{1}{k}\frac{1}{(m/k)s+1}F(s)$$

2. Unit-Step Response of First-order Systems

$$R(s) \longrightarrow \boxed{\dfrac{1}{Ts+1}} \longrightarrow C(s)$$

The input signal: $R(s)=\dfrac{1}{s}$

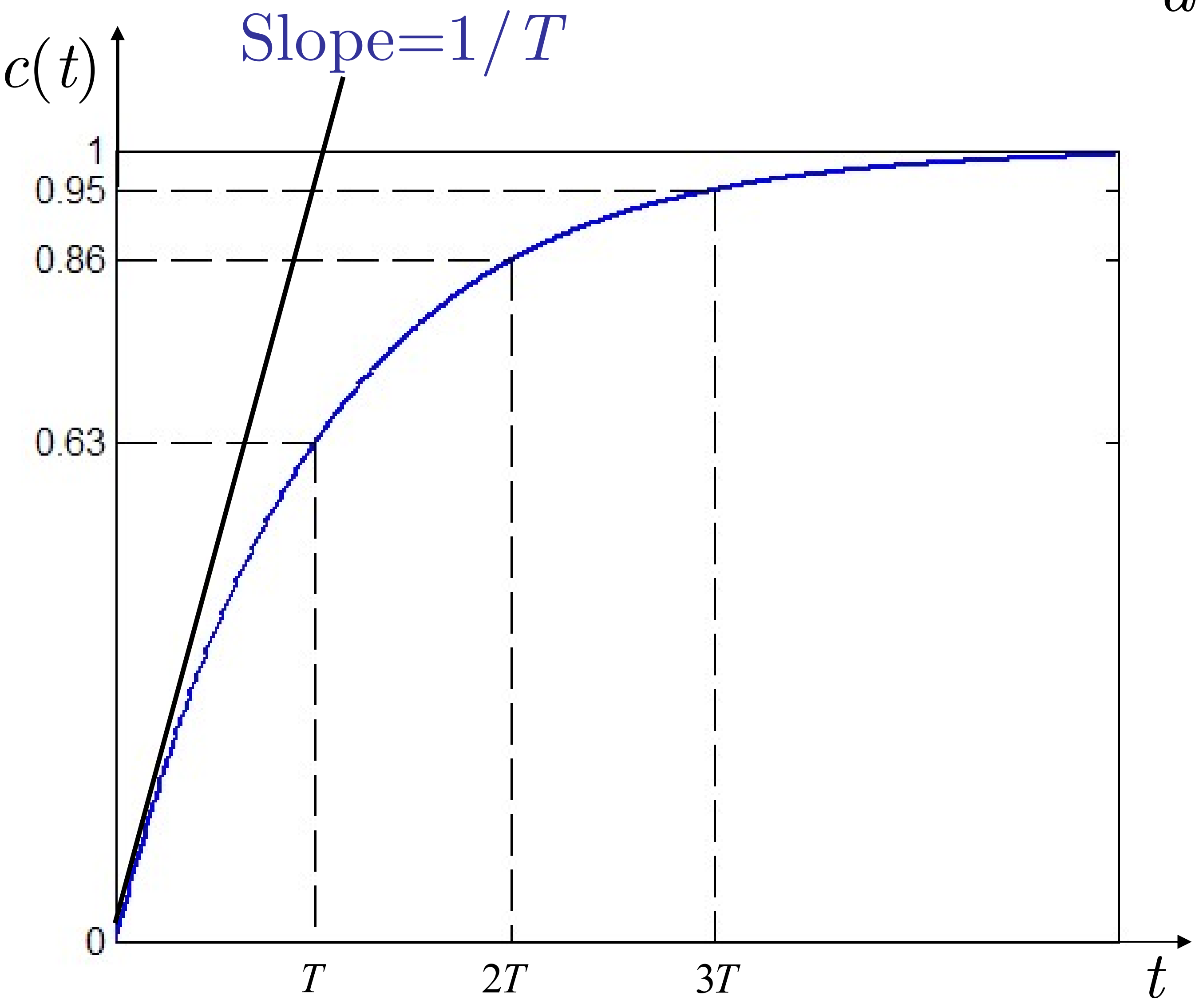
Hence, the output

$$C(s)=\frac{1}{s(Ts+1)}=\frac{1}{s}-\frac{T}{Ts+1}=\frac{1}{s}-\frac{1}{s+1/T}$$

from which, we obtain the time response

$$c(t)=1-e^{-t/T},t\geq 0$$

Step response $c(t) = 1 - e^{-t/T}$, $\left.\frac{dc}{dt}\right|_{t=0} = \frac{1}{T}$



$c(T) = 0.632$

$c(2T) = 0.865$

$c(3T) = 0.950$

$c(4T) = 0.982$

$t_s = 3T$

$\sigma\% = 0$

3. Unit-Ramp response of First-order Systems

$$R(s) \longrightarrow \boxed{\frac{1}{Ts+1}} \longrightarrow C(s)$$

If the input is the unit ramp $r(t) = t \cdot 1(t)$

then $R(s) = \dfrac{1}{s^2}$

Expanding $C(s)$ into partial fractions gives

$$C(s) = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s + 1/T}$$

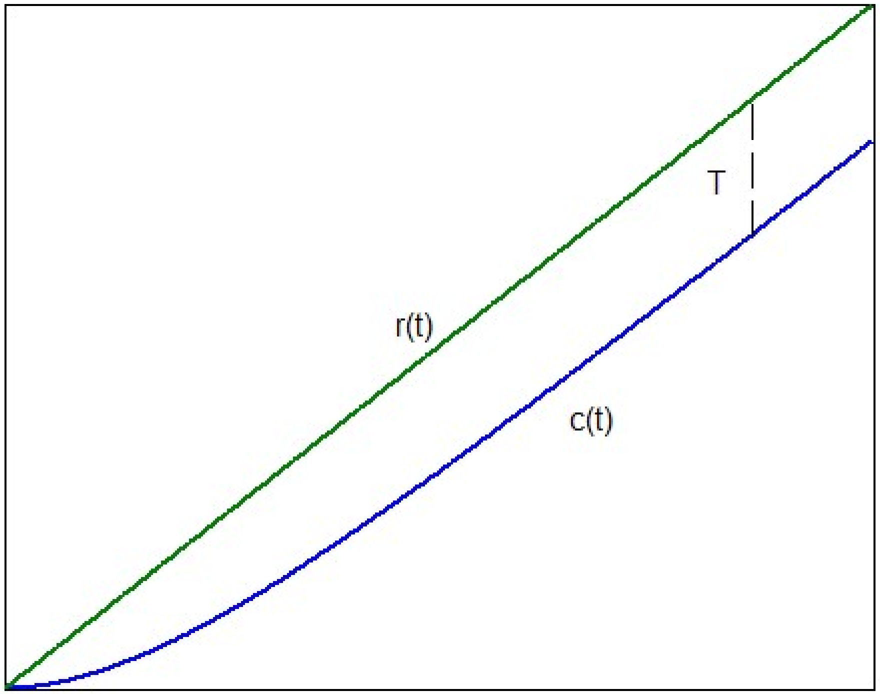
Thus, the time-domain response is

$$c(t) = t - T + Te^{-\frac{t}{T}}, t \geq 0$$

The error signal $e(t)$ is then

$$e(t) = r(t) - c(t) = T(1 - e^{-\frac{t}{T}}), t \geq 0$$
$$\Rightarrow e(\infty) = T$$

The error in following the unit-ramp input is equal to T for sufficiently large t. The smaller the time constant T, the smaller the steady-state error is.



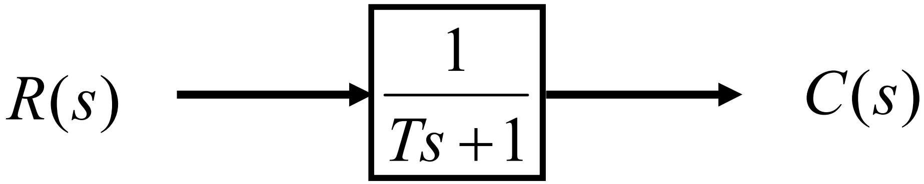
Notice that for first-order systems, the steady-state errors are different when input signals are unit-step function and unit-ramp function, respectively. For unit-step function,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}[r(t) - c(t)] = \lim_{t\to\infty} e^{-t/T} = 0$$

while for unit-ramp function,

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty}[r(t) - c(t)] = \lim_{t\to\infty} T(1 - e^{-\frac{t}{T}}) = T$$

4. Unit-Impulse response of First-order Systems



If the input is a unit-impulse, then $R(s)=1$

Therefore,

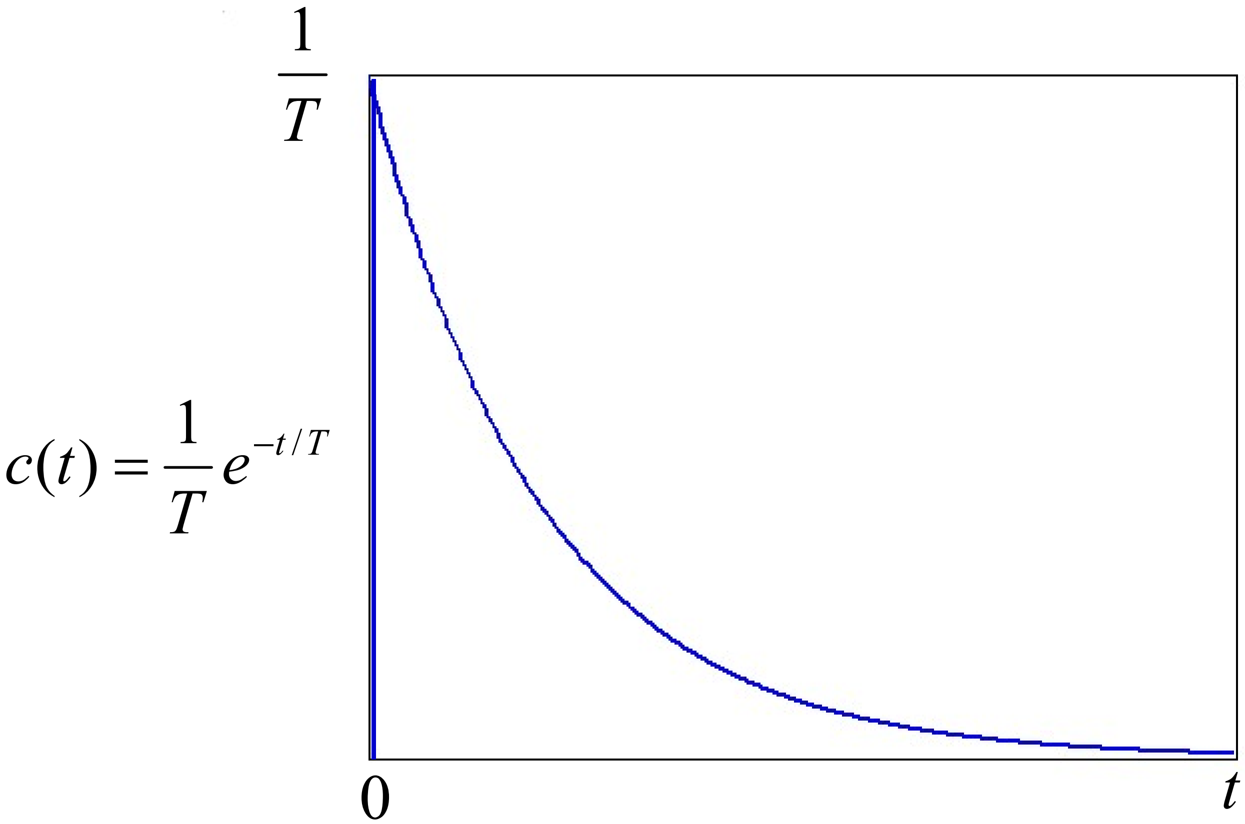
$$C(s)=\frac{1}{Ts+1}$$

Taking the inverse Laplace transform gives

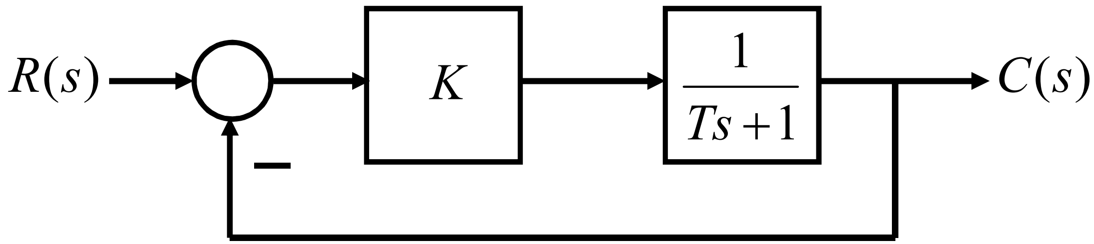
$$c(t)=\frac{1}{T}e^{-t/T}, t\geq 0$$

Unit-Impulse response

Since we have assumed zero initial conditions, the output must change instantaneously from 0 at t=0⁻ to $1/T$ at t=0⁺.



Example. Determine the unit-step response of the following closed-loop system:



Solution: The output $C(s)$ is:

$$C(s)=\frac{K}{s(Ts+1+K)}=\frac{K/T}{s\left[s+(1+K)/T\right]}=\frac{a}{s}+\frac{b}{s+(1+K)/T}$$

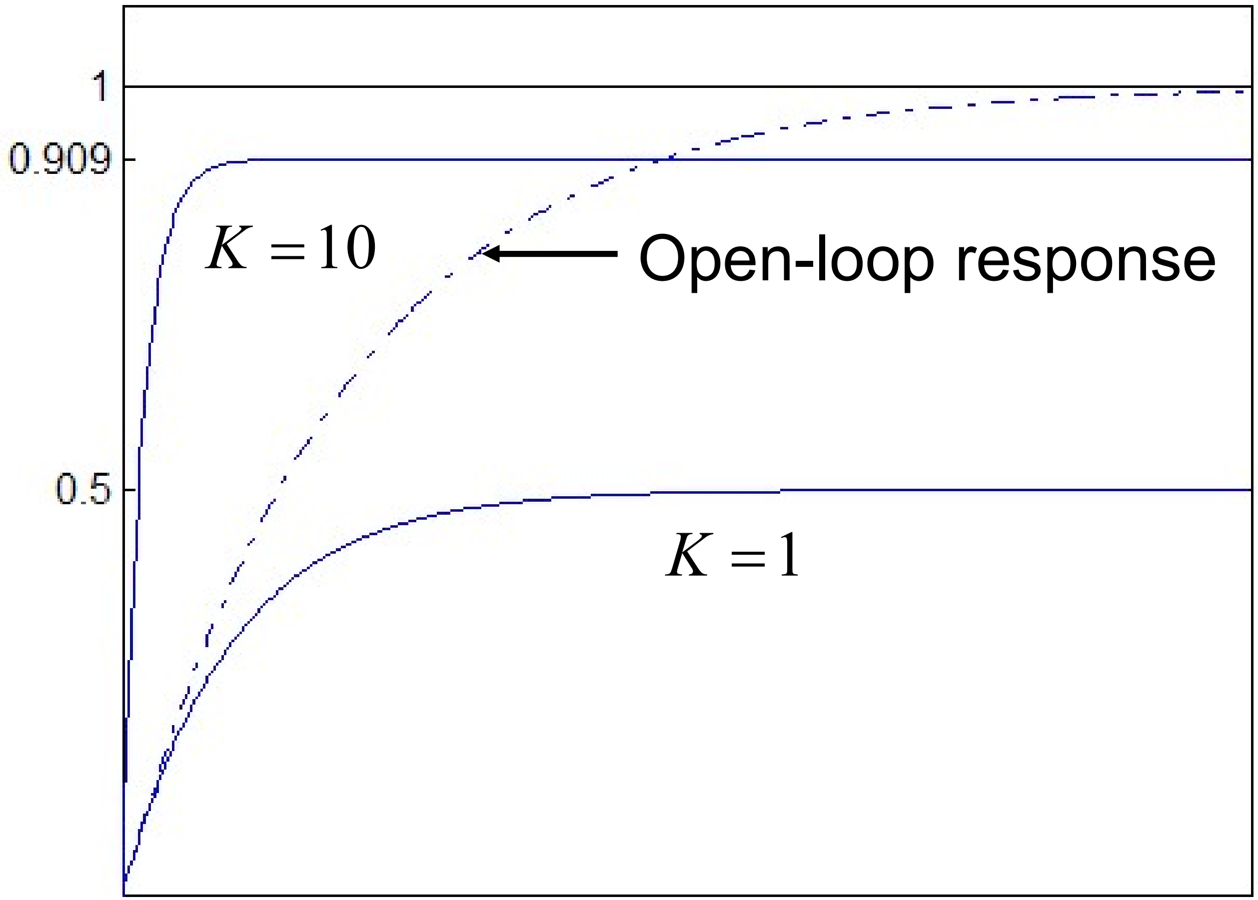
Evaluating the partial-fraction coefficients yields

$$a=\frac{K}{1+K}, \quad b=-\frac{K}{1+K}$$

Hence, the time response

$$c(t)=\frac{K}{1+K}\left(1-e^{-[(1+K)/T]t}\right), t\geq 0$$

$$c(t) = \frac{K}{1+K}\left(1 - e^{-[(1+K)/T]t}\right), \quad t \geq 0$$



5. An important property of LTI systems

The responses of first-order systems to the three inputs are given below:

$$c_t = t - T + Te^{-\frac{t}{T}}, t \geq 0$$

$$c_{1(t)} = 1 - e^{-t/T}, t \geq 0$$

$$c_\delta = \frac{1}{T} e^{-t/T}, t \geq 0$$

which have the following properties:

$$\frac{dc_t}{dt} = c_{1(t)}$$

$$\frac{dc_{1(t)}}{dt} = c_\delta$$

In complex domain:

$$C_t(s) = \frac{1}{Ts+1}\frac{1}{s^2} \Rightarrow sC_t(s) = s\frac{1}{Ts+1}\frac{1}{s^2} = \frac{1}{Ts+1}\frac{1}{s} = C_{1(t)}(s)$$

$$C_{1(t)}(s) = \frac{1}{Ts+1}\frac{1}{s} \Rightarrow sC_{1(t)}(s) = s\frac{1}{Ts+1}\frac{1}{s} = \frac{1}{Ts+1} = C_\delta(s)$$

Conclusion: For unit **ramp, step and impulse inputs**, the response to the derivative of an *input* signal can be obtained by differentiating the response of the system to the original signal.

Such a conclusion can be readily extended to any LTI systems. In fact, let

$$\frac{1}{s^2} \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

Then

$$C(s) = G(s)\frac{1}{s^2}$$

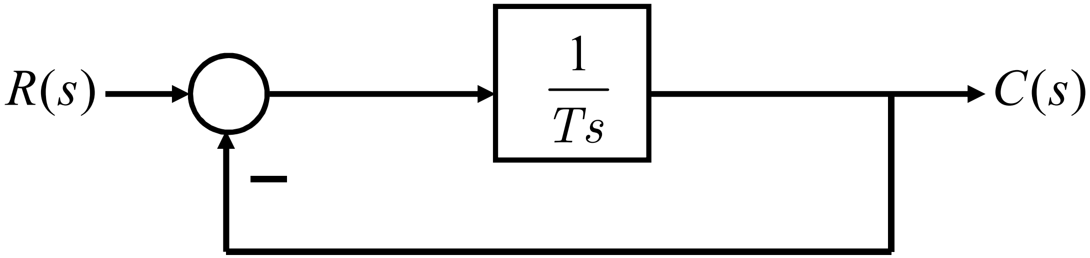
Obviously,

$$sC(s) = sG(s)\frac{1}{s^2} = G(s)\left(s\frac{1}{s^2}\right) = G(s)\frac{1}{s}$$

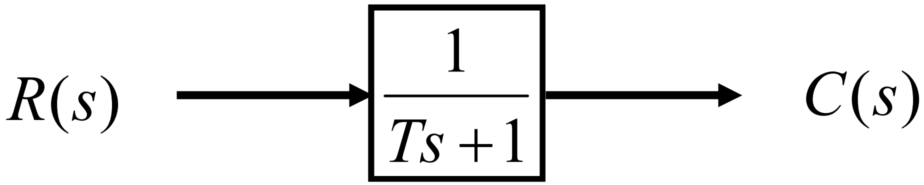
The left hand side is the derivative of the *output* and the right hand side is the response to the derivative of the *input.*

Summary of first-order systems

1. Standard first-order system:



$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

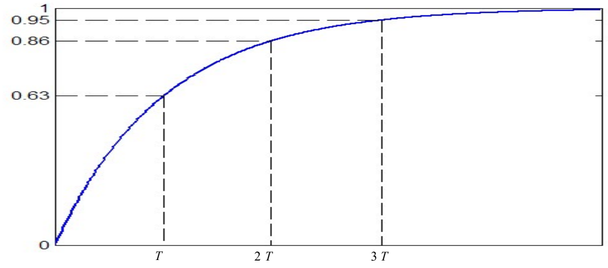


2. The main feature of first-order systems: Only one parameter: T, **which determines everything**:

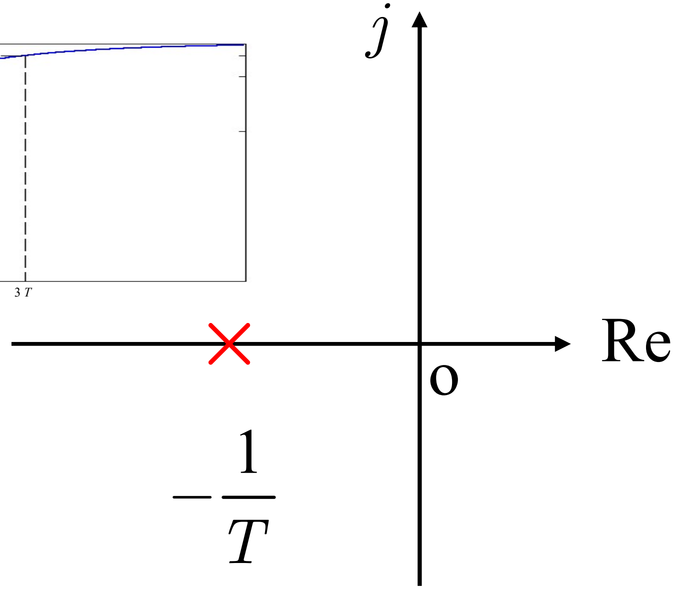
$$c_{1(t)} = 1 - e^{-t/T}, t \geq 0$$

$$c_t = t - T + Te^{-\frac{t}{T}}, t \geq 0$$

$$c_\delta = \frac{1}{T} e^{-t/T}, t \geq 0$$



S-plane:



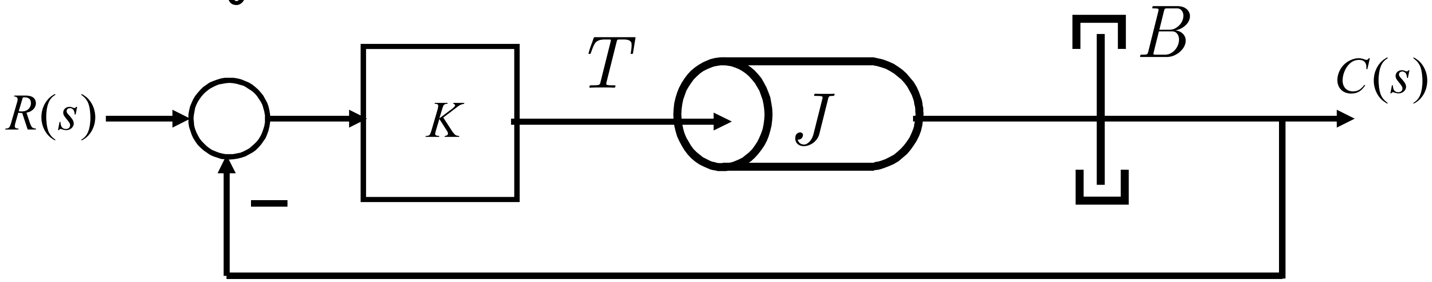
- T is the unique measure of the speed of **unit step response**: $t_s=3T$ (5%) or $t_s=4T$ (2%).

- Only one pole, $s=-1/T$, whose distance to the imaginary axis determines the speed of response.

5-3. Second-order systems

Many physical control systems can be described by or approximated to a second-order differential equation.

1. Examples of second-order systems: Servo control system:



The system consists of a proportional control and load elements (inertia and viscous friction elements). Control objective: output position C tracks the desired position R.

The transfer function with respect to the torque T, the moment of inertia J and viscous friction referred to the motor shaft B is

$$\frac{C(s)}{T(s)} = \frac{1}{s(Js+B)}$$

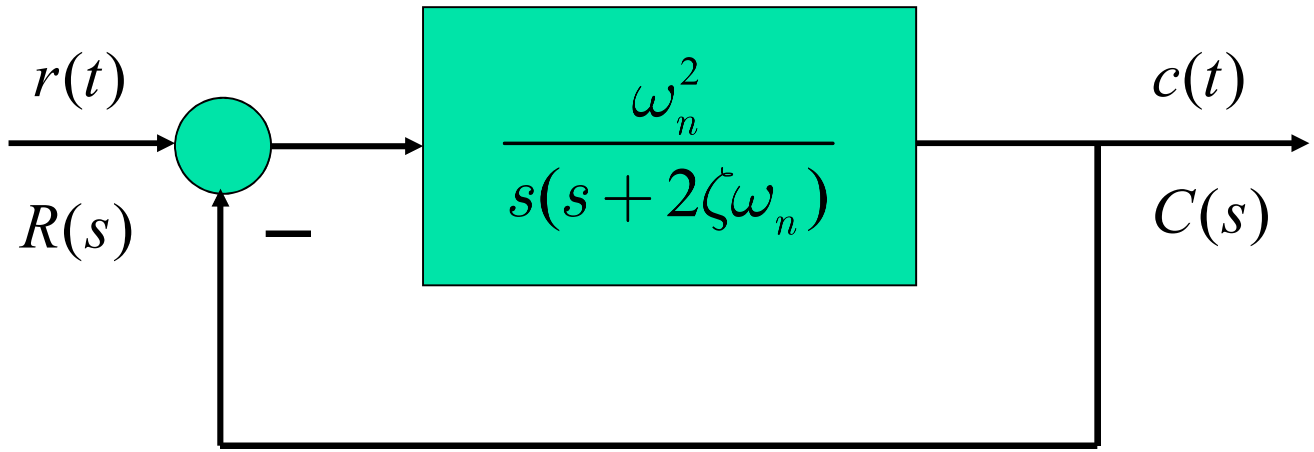
Therefore, the closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2+(B/J)s+K/J}$$

The **standard form** of second order systems is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

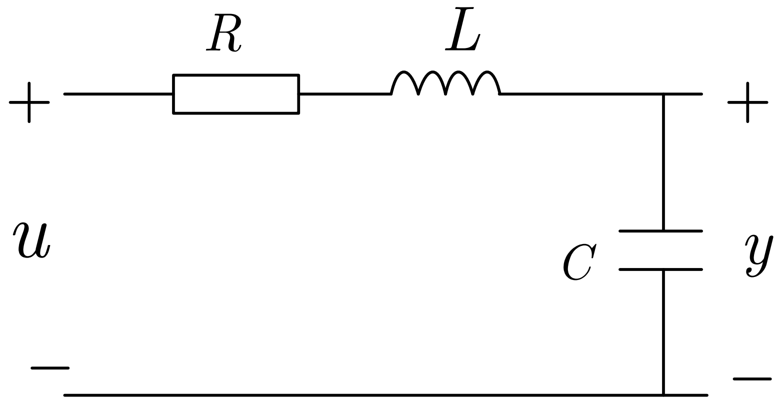
where ω_n is undamped natural frequency and ζ, the damping ratio.



The characteristic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

RLC circuit: A second-order system example.

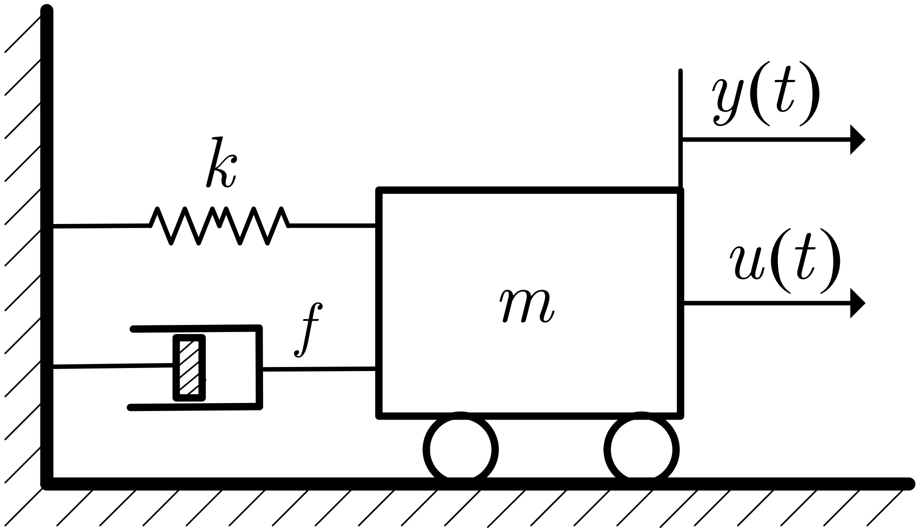


$$\frac{Y(s)}{U(s)} = \frac{1}{LCs^2 + RCs + 1} = \frac{1/LC}{s^2 + (R/L)s + 1/LC}$$

$$\omega_n = \sqrt{\frac{1}{LC}}$$

$$2\zeta\omega_n = \frac{R}{L} = 2\left(\frac{1}{2}R\sqrt{\frac{C}{L}}\right)\sqrt{\frac{1}{LC}}$$

Mass-Spring-Damper System: A second-order system example.



k : Spring constant

f : Damping coefficient

$$m\ddot{y} + f\dot{y} + ky = u$$

$$\frac{Y(s)}{U(s)} = \frac{1}{ms^2 + fs + k} = \frac{1}{k}\left(\frac{k/m}{s^2 + (f/m)s + k/m} \right)$$

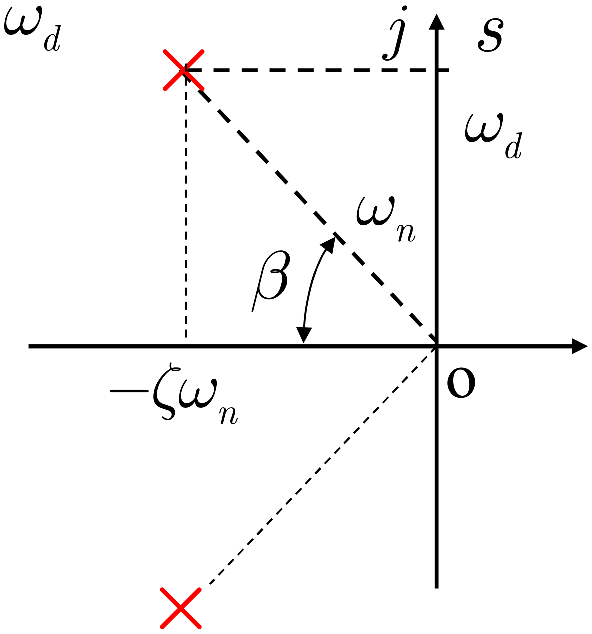
2. Step response of second-order systems

(1) Underdamped case ($0<\zeta<1$): From

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

The two poles can be expressed as

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$$

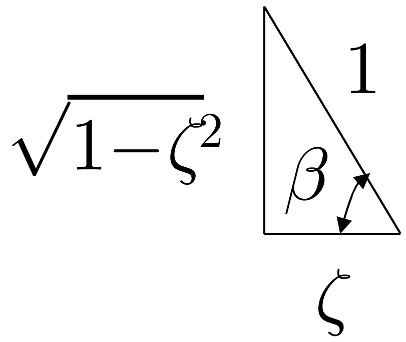


where

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\boldsymbol{\omega_d}$ is called **damped frequency**.

The step response



$$C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}=\frac{1}{s}-\frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$=\frac{1}{s}-\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}-\frac{\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}$$

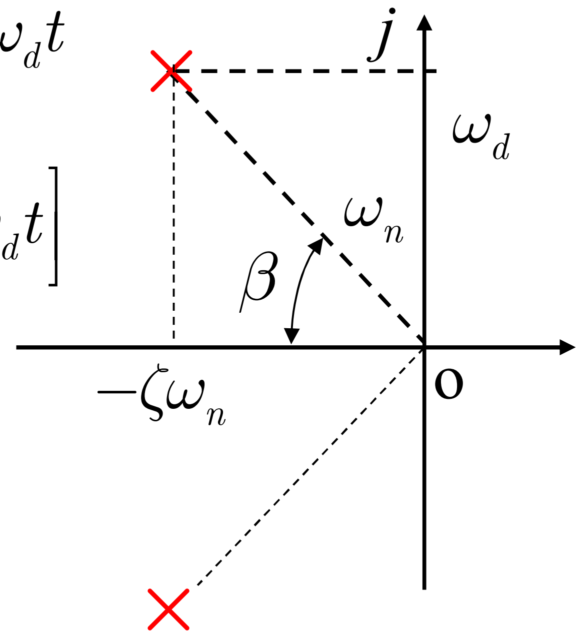
Hence,

$$c(t)=1-e^{-\zeta\omega_n t}\cos\omega_d t-e^{-\zeta\omega_n t}\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t$$

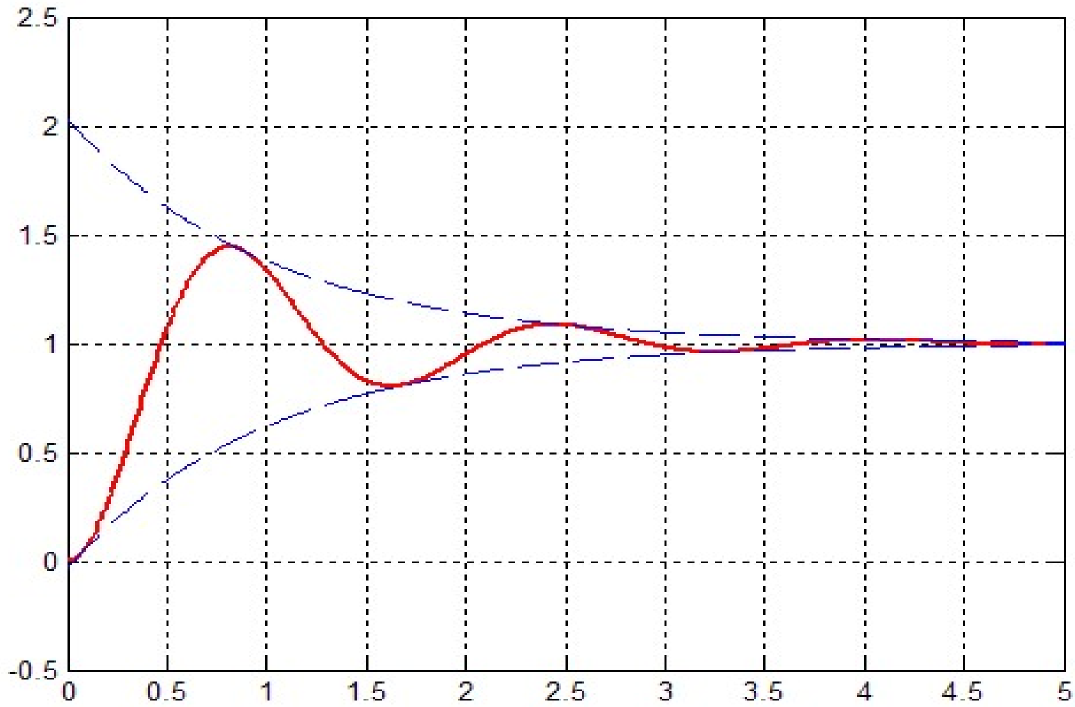
$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sqrt{1-\zeta^2}\cos\omega_d t+\zeta\sin\omega_d t\right]$$

$$=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\beta\right),t\geq 0$$

where $\cos\beta=\zeta$ or $\mathrm{tg}\beta=\sqrt{1-\zeta^2}\,/\zeta$.



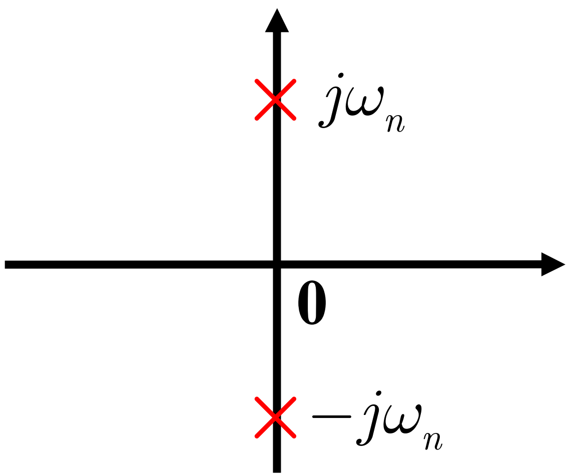
$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\beta\right),t\geq 0$$

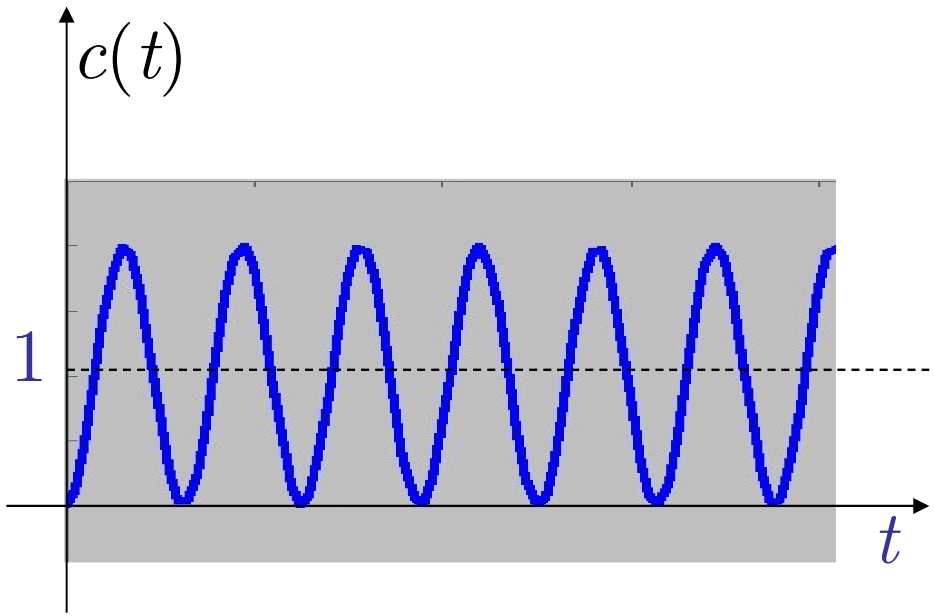


The response oscillatorily decays to one ($0<\zeta<1$)

In particular, if the damping ratio ζ=0, the response becomes **undamped** and oscillations continue indefinitely:

$$c(t) = 1 - \cos\omega_n t, \quad t \geq 0$$





This is why we call ω_n as undamped natural frequency, which is in fact cannot be observed experimentally; what we are able to observe is damped frequency

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

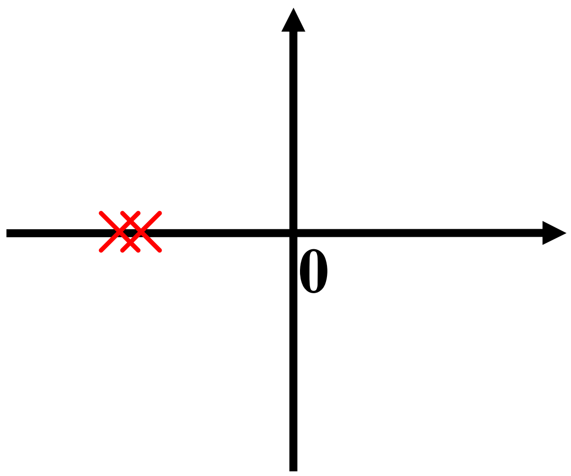
which is always lower than ω_n when $0<\zeta<1$. If ζ is increased beyond unity, that is, $\zeta > 1$, the response becomes **overdamped** and will not oscillate.

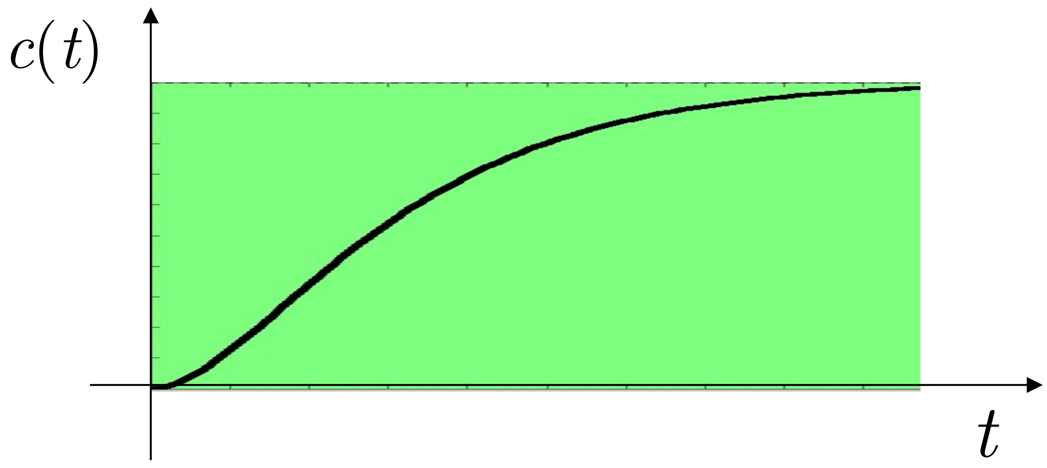
(2) Critical damped case (ζ=1): The unit-step response

$$C(s)=\frac{\omega_n^2}{s(s+\omega_n)^2}=\frac{1}{s}-\frac{1}{s+\omega_n}-\frac{\omega_n}{(s+\omega_n)^2}$$

Taking the inverse Laplace transform of both sides of the above equation yields

$$c(t)=1-e^{-\omega_n t}(1+\omega_n t),\quad t\geq 0$$





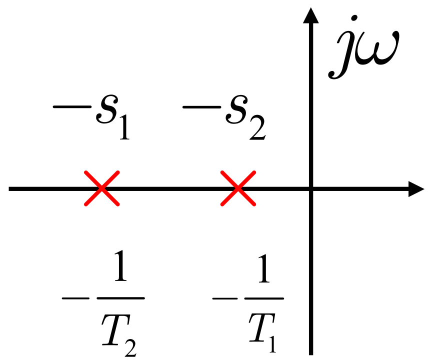
(3) Overdamped Case (ζ >1):

- **Unit-step response:** The characteristic equation has two negative real poles:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = (s+s_1)(s+s_2) = \left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)$$

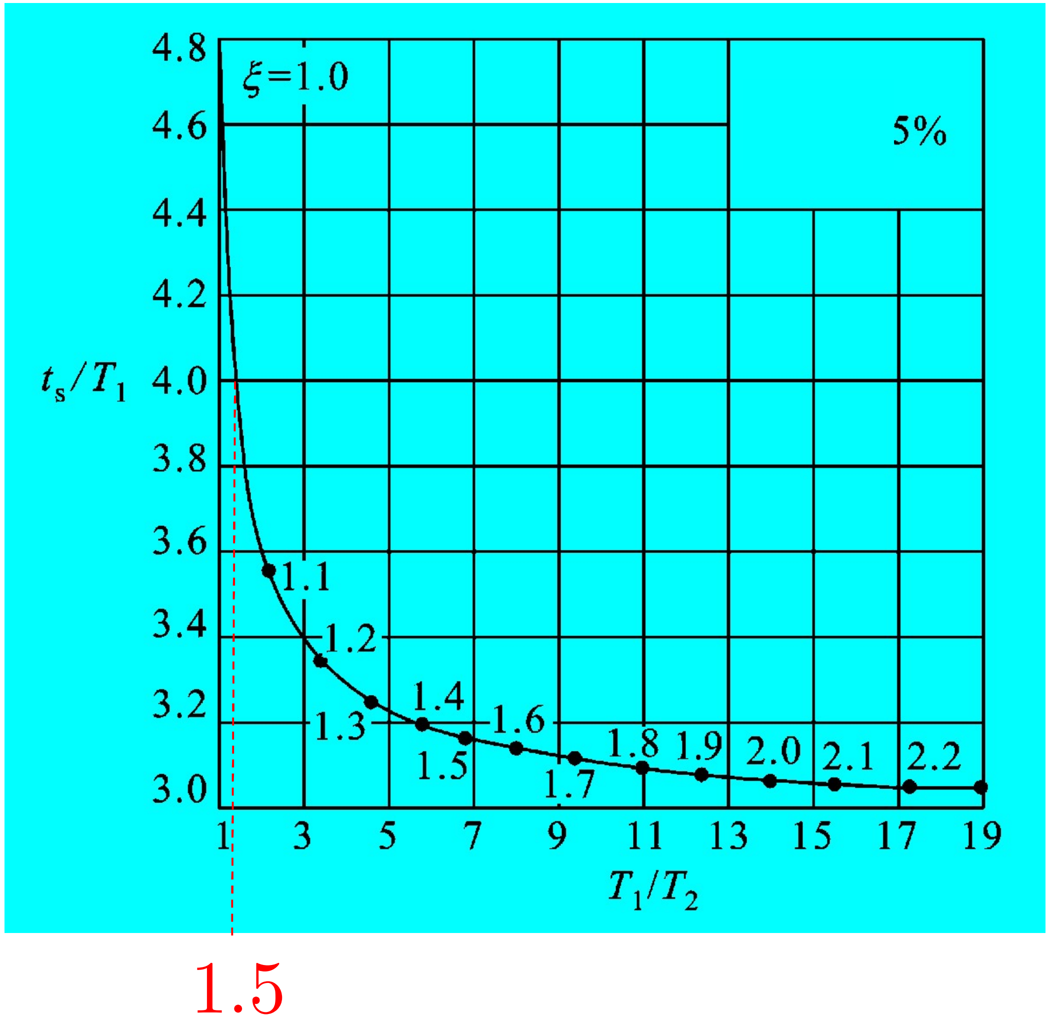
where

$$\begin{cases} T_1 = \dfrac{1}{\underbrace{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}_{s_2}} := \dfrac{1}{s_2} \\[2ex] T_2 = \dfrac{1}{\underbrace{\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}}_{s_1}} := \dfrac{1}{s_1} \end{cases}$$



Therefore,

$$T_1 > T_2 \quad (s_2 < s_1),\ \ \omega_n^2 = \frac{1}{T_1 T_2}$$



ξ=1.0
5%
t_s/T_1
4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
3.0
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2.0
2.1
2.2
1
3
5
7
9
11
13
15
17
19
T_1/T_2
1.5

Hence,

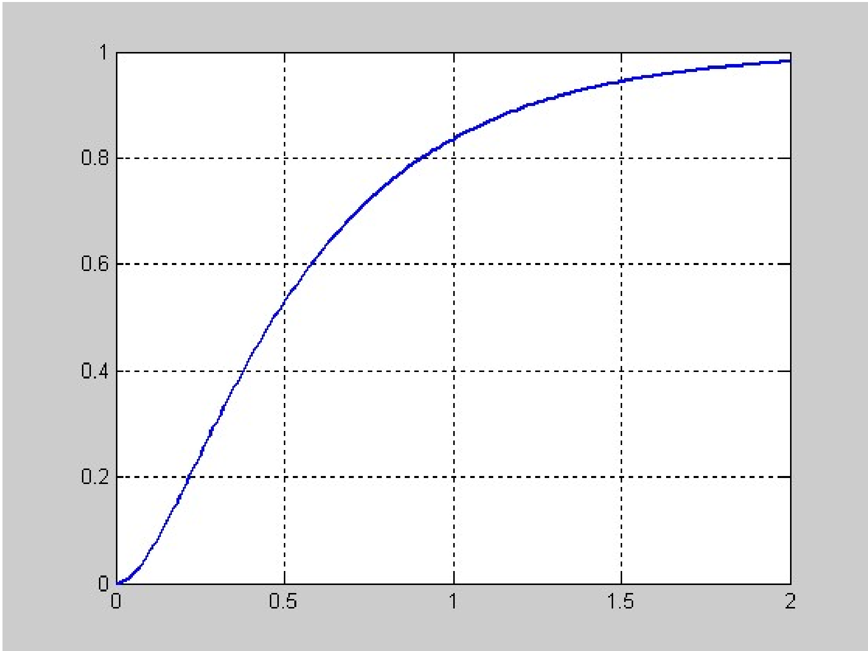
$$C(s)=\frac{\frac{1}{T_1T_2}}{\left(s+\frac{1}{T_1}\right)\left(s+\frac{1}{T_2}\right)}\frac{1}{s}=\frac{1}{s(T_1s+1)(T_2s+1)}$$

$$=\frac{1}{s}+\frac{1}{\frac{T_2}{T_1}-1}\times\frac{1}{\left(s+\frac{1}{T_1}\right)}+\frac{1}{\frac{T_1}{T_2}-1}\times\frac{1}{\left(s+\frac{1}{T_2}\right)}$$

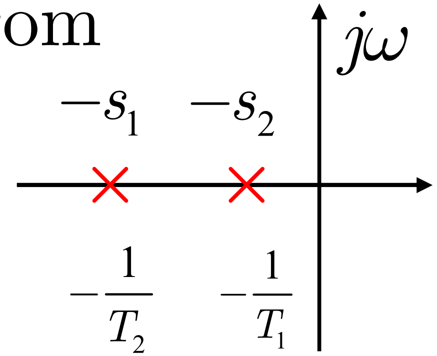
The unit-step response is

$$c(t)=1+\frac{1}{\frac{T_2}{T_1}-1}e^{-\frac{1}{T_1}t}+\frac{1}{\frac{T_1}{T_2}-1}e^{-\frac{1}{T_2}t}$$

$$=1+\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{e^{-s_1t}}{s_1}-\frac{e^{-s_2t}}{s_2}\right),\ t\geq 0$$

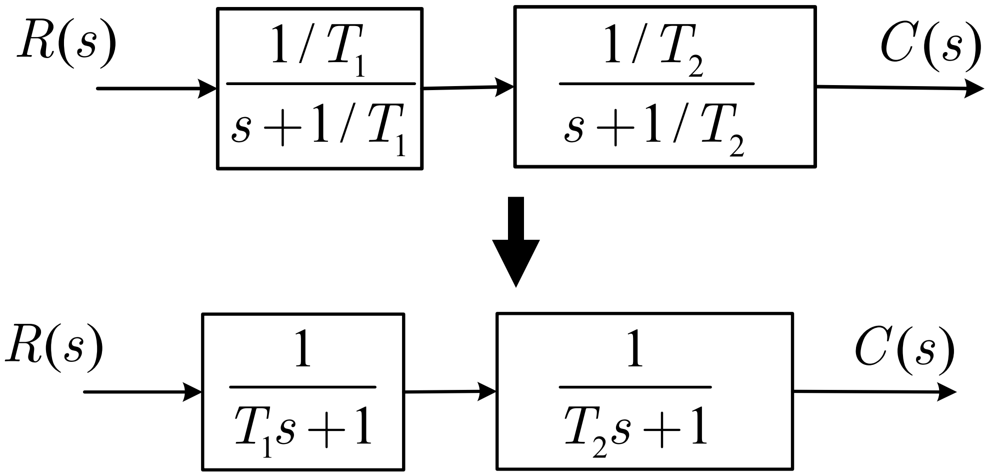


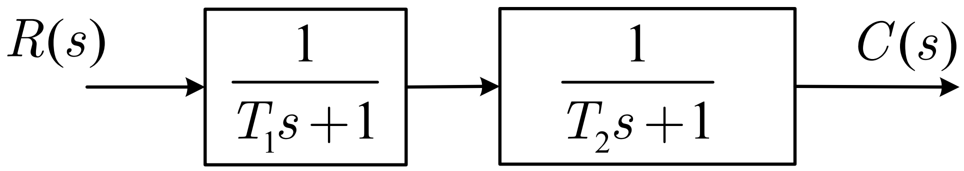
- **Reduction to a first-order system.** From



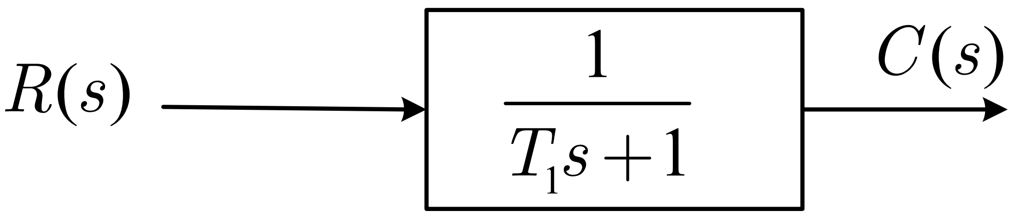
$$C(s) = \frac{\frac{1}{T_1 T_2}}{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)} R(s)$$

it can be seen that for overdamped case, the second-order system can be expressed as two first-order systems connected in cascade:

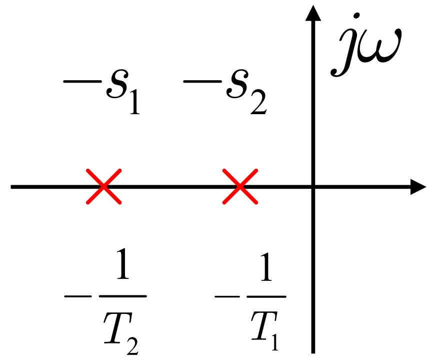




In particular, if $-1/T_1$ is located very much closer to the $j\omega$ axis than $-1/T_2$, (which means $1/T_1 << 1/T_2$), then for an approximate solution we may neglect $-1/T_2$ and the second-order system can be reduced to a first-order system:



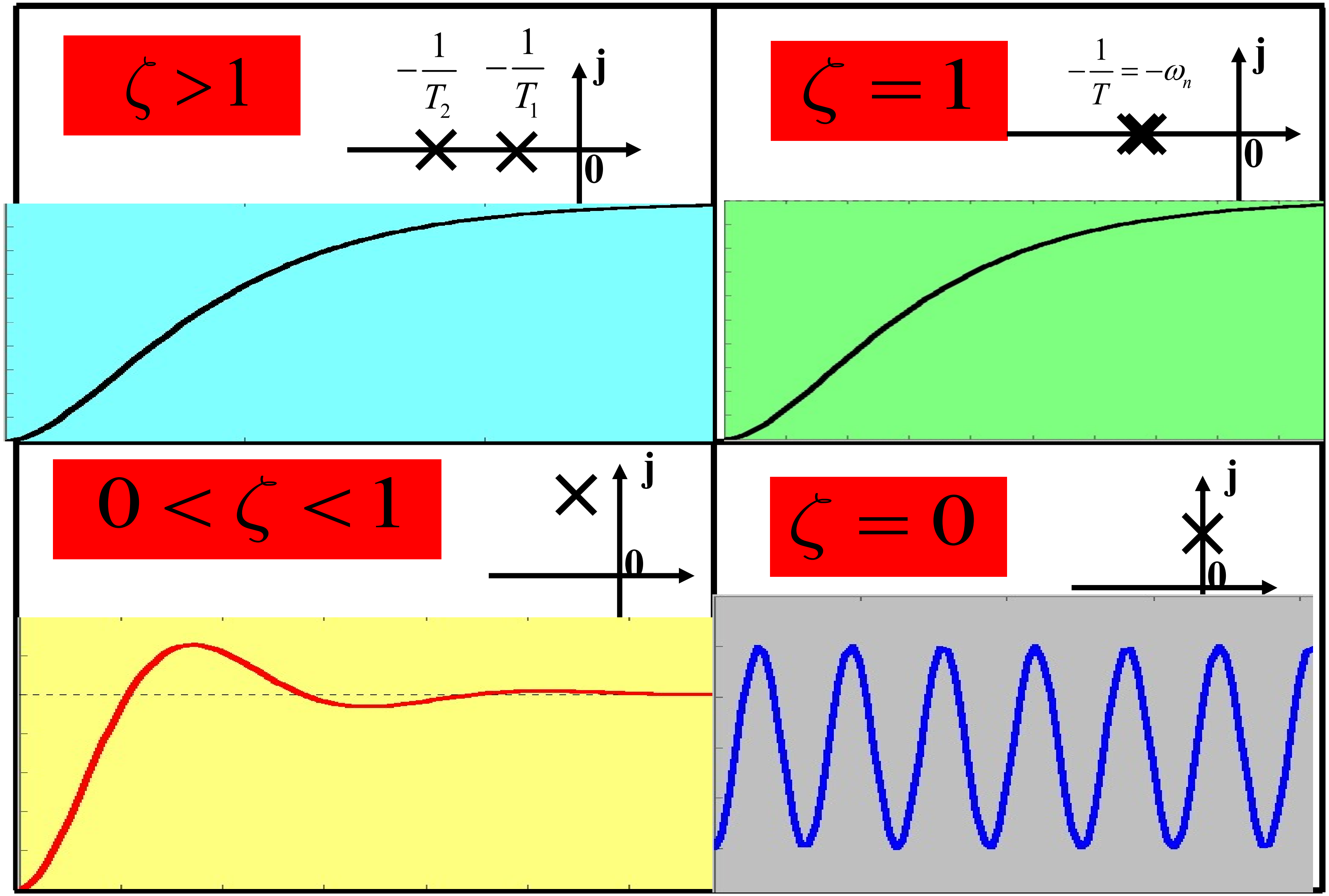
or
$$\frac{C(s)}{R(s)} = \frac{s_2}{s+s_2} = \frac{\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}{s+\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}}$$



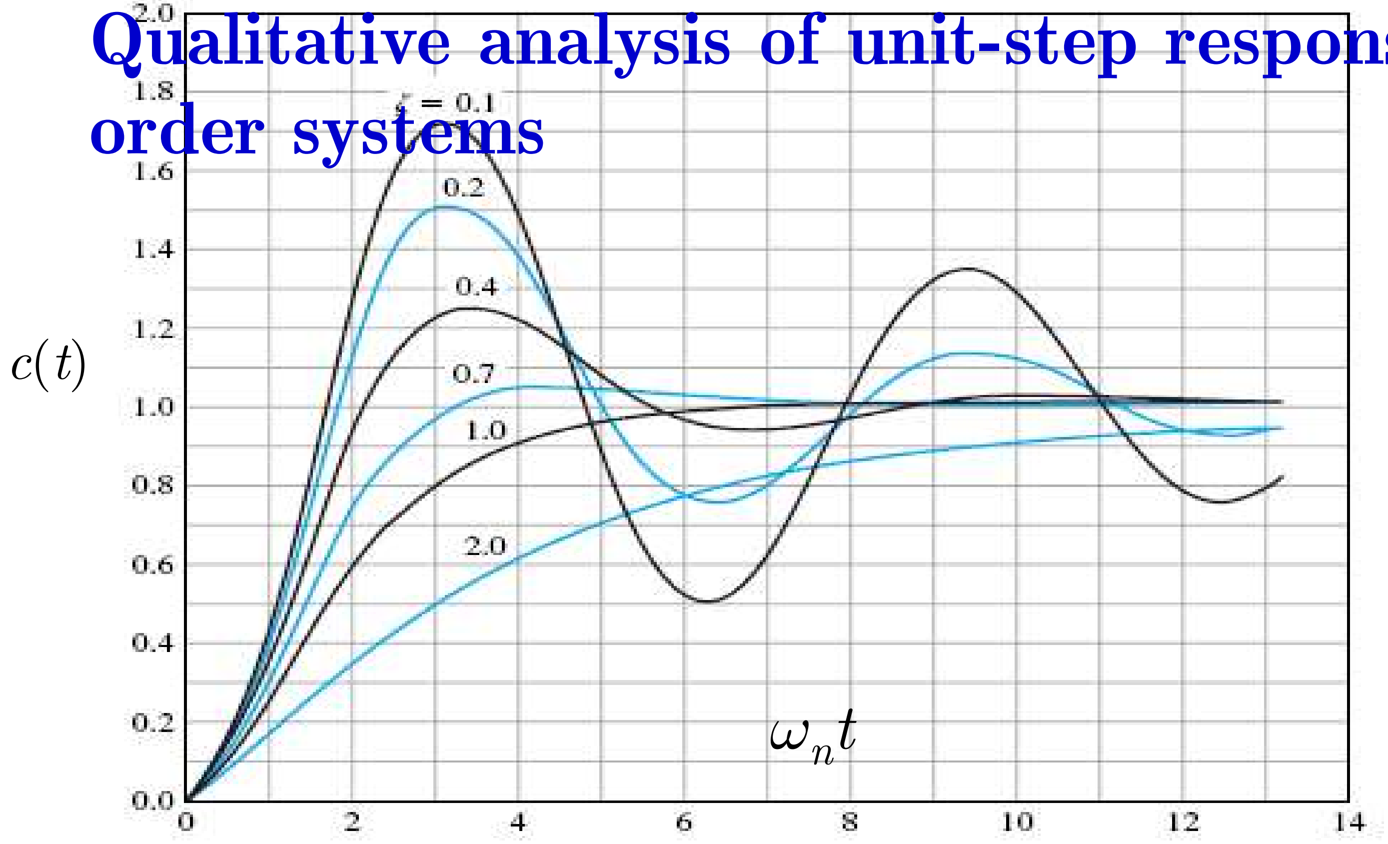
$$c(t) = 1 - e^{-(\zeta-\sqrt{\zeta^2-1})\omega_n t}, \quad t \ge 0$$

Second-order system

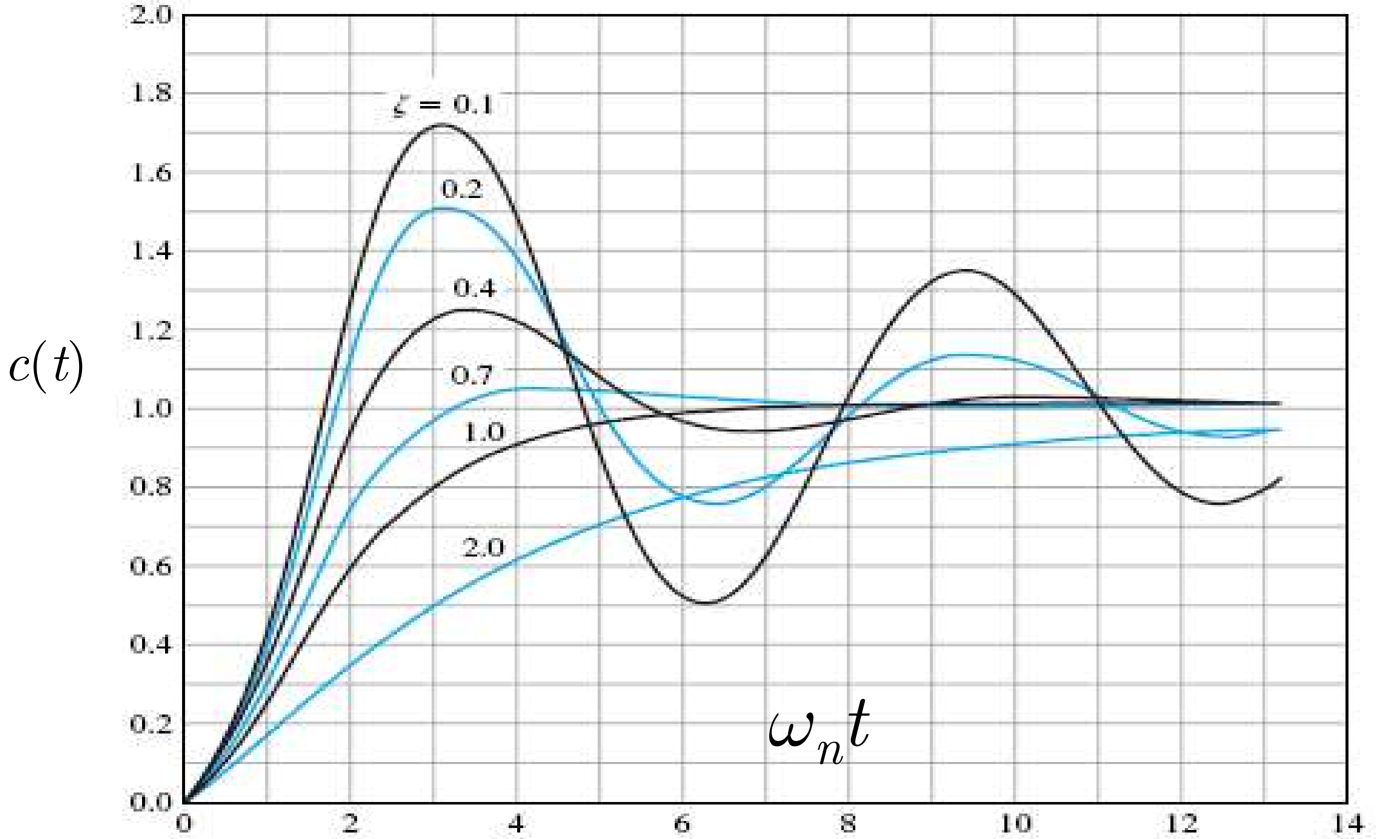
$$\Phi(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Qualitative analysis of unit-step response of second-order systems

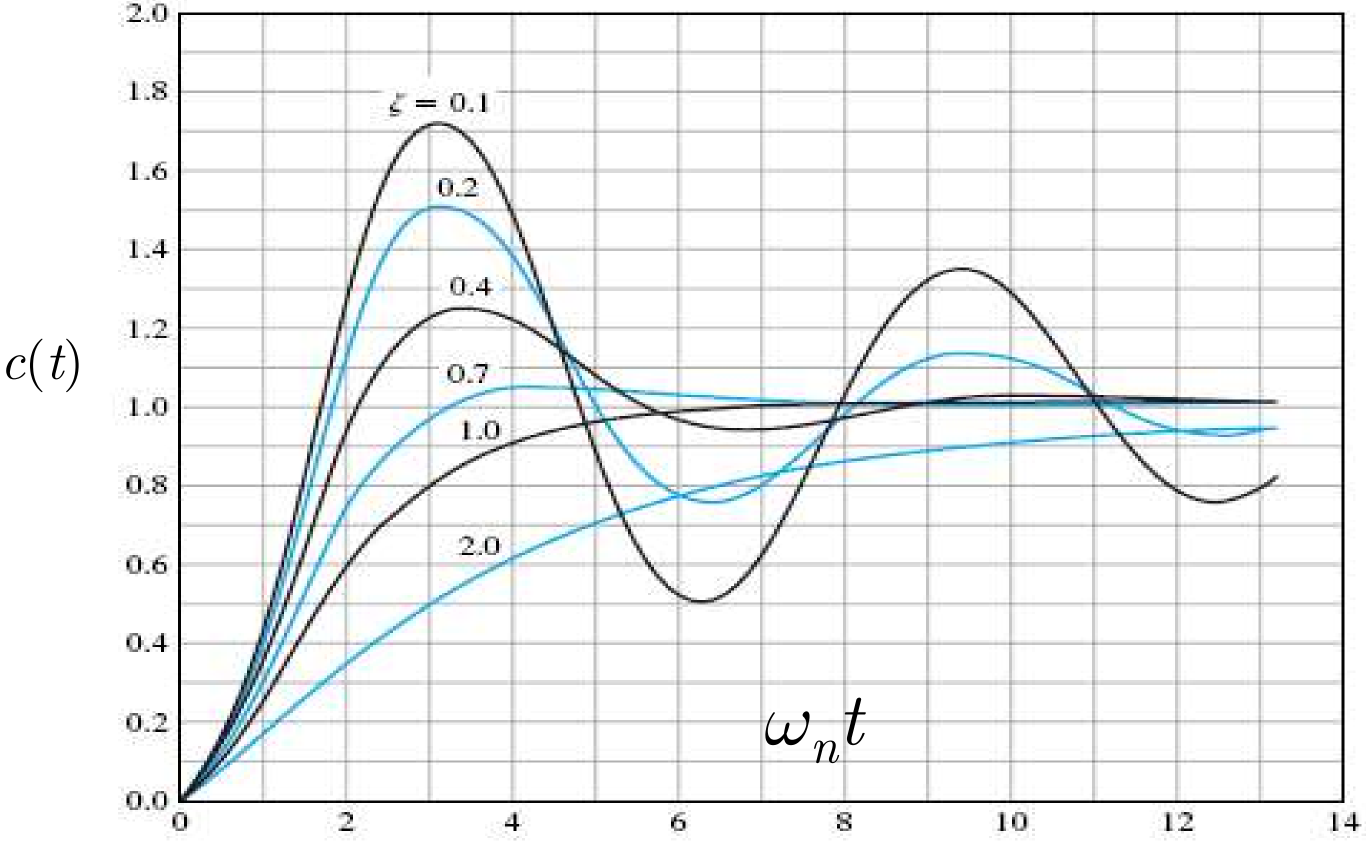


(1) When $0.4 \leq \zeta \leq 0.8$, the system gets close to the final value more rapidly than a critically damped or overdamped systems. Small values of ζ ($\zeta < 0.4$) yield excessive overshoot in the transient response, and a system with a large value of ζ ($\zeta > 0.8$) responds sluggishly.



(a)

In particular, when $\zeta \approx 0.707$, the system exhibits fastest response with a nice overshoot ($=4\%$).



(a)

(2) Among the systems without oscillation ($\zeta \geq 1$), the critical damped system exhibits the fastest response.

(3) An overdamped system is always sluggish.

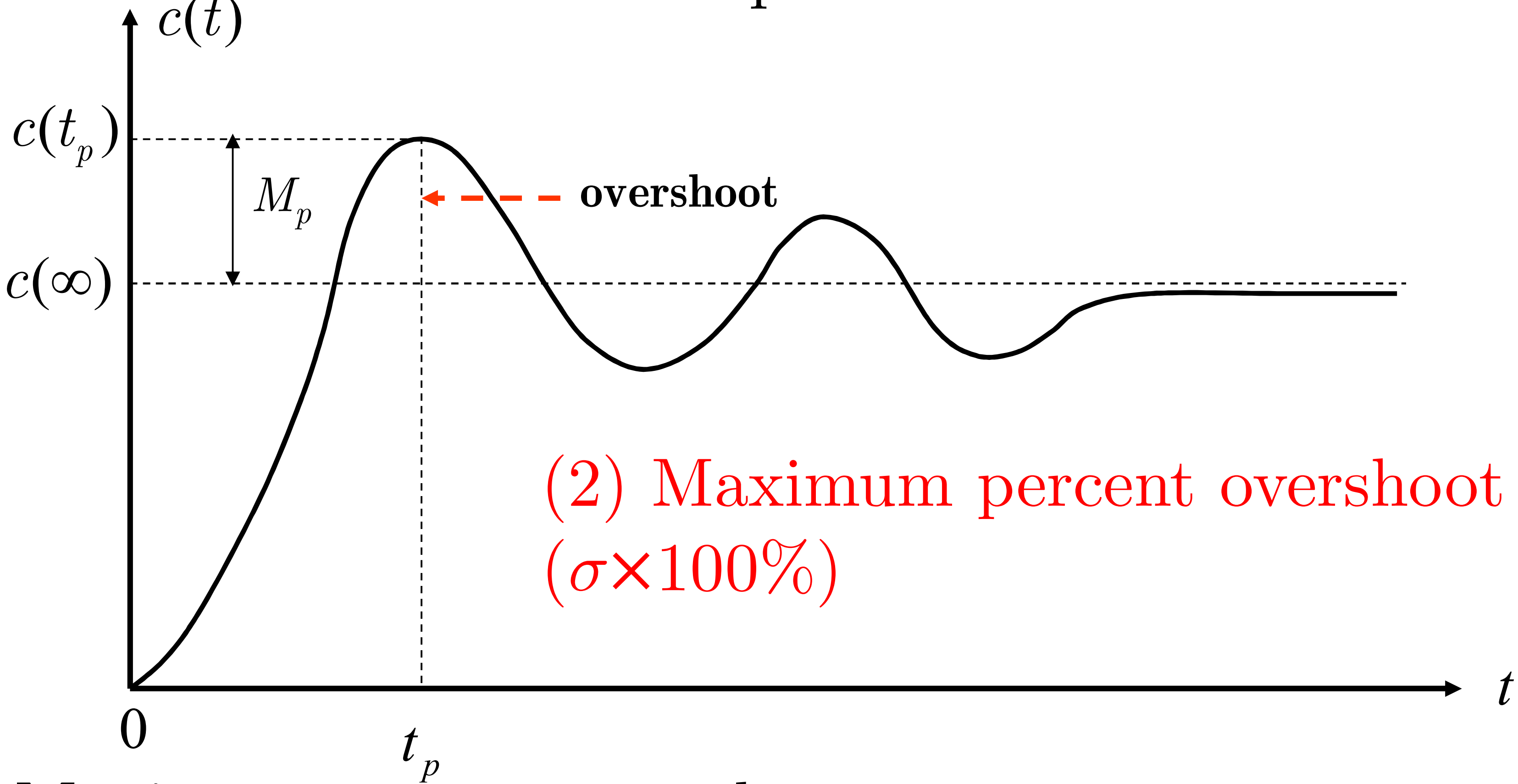
3. Definitions of Transient Response Specifications

- In many practical cases, the desired performance characteristics are specified in terms of **time-domain quantities**.

- Usually, the performance characteristics of a control system are specified in terms of the transient response to a **unit-step input** since it is easy to generate and is sufficiently drastic.

(1) Peak time t_p and maximum percent overshoot M_p

t_p is the time required for the response to reach the **first** peak of the overshoot.



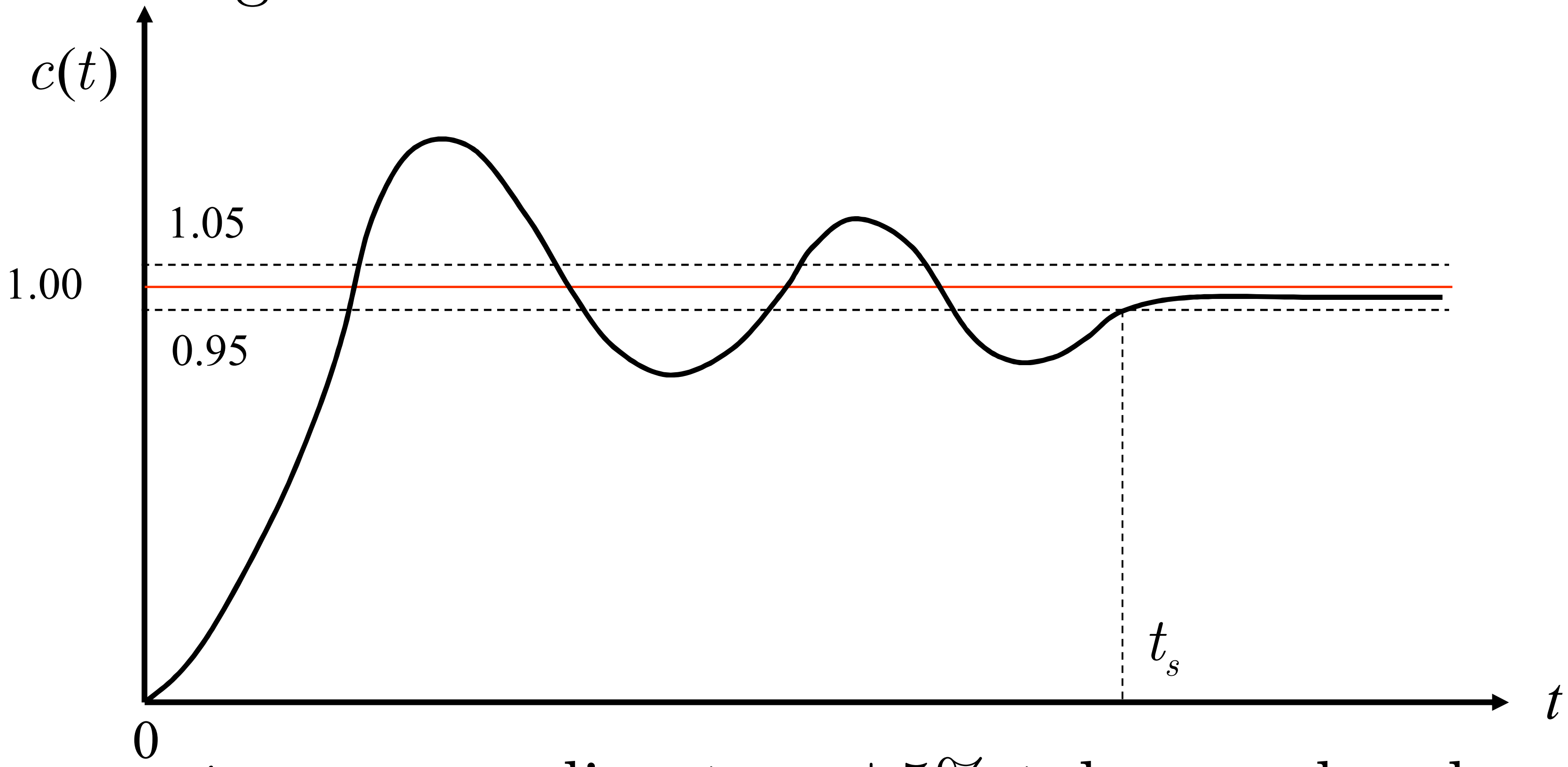
(2) Maximum percent overshoot M_p ($\sigma \times 100\%$)

Maximum percent overshoot:

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

(3) Settling time t_s

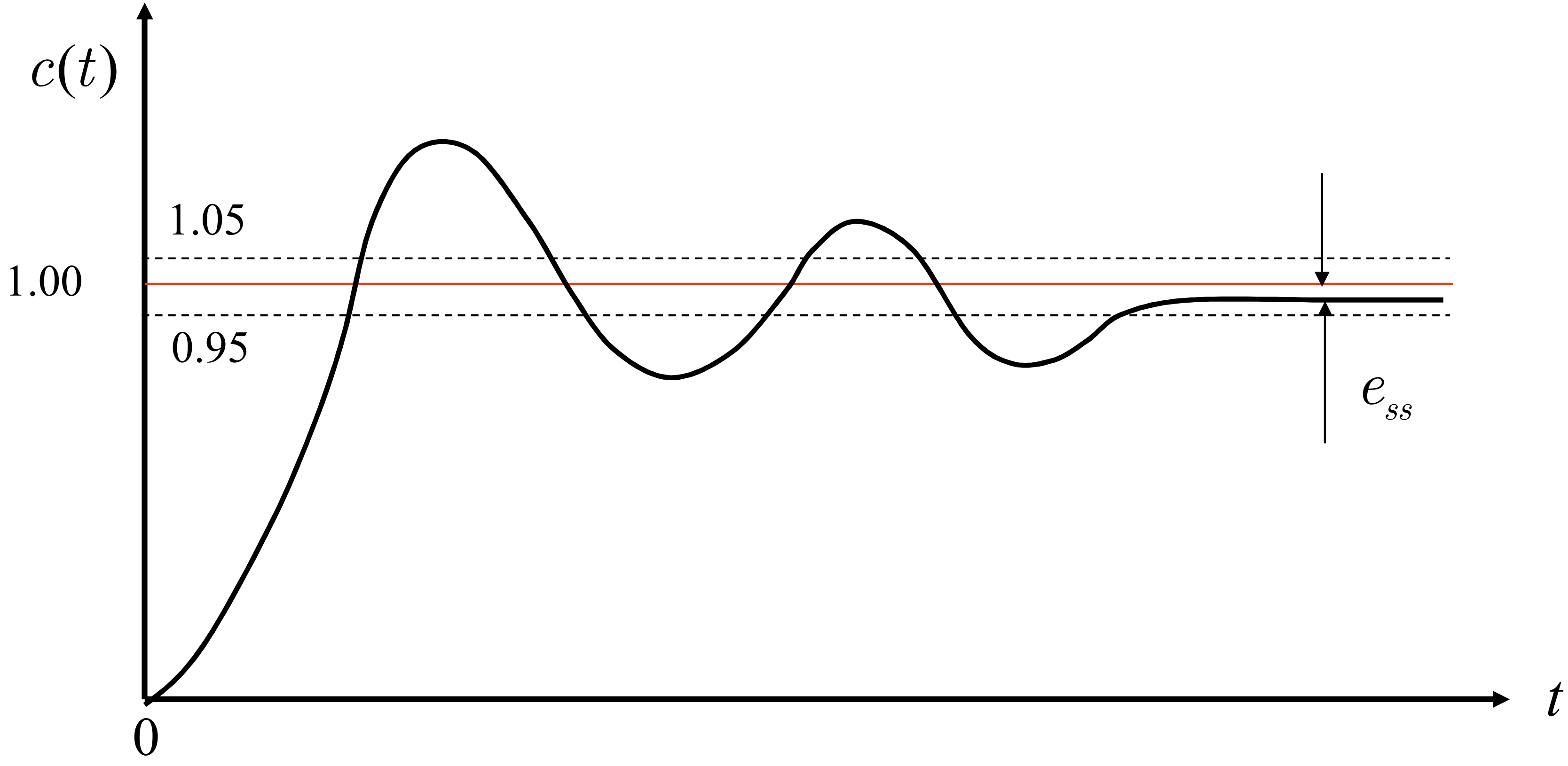
The settling time is defined as the time required for the step response to reach and stay within a specified percentage of its final value. A frequently used figure is 5% or 2% tolerance band.



t_s corresponding to a $\pm 5\%$ tolerance band

(4) Steady-state error e_{ss}

$$e_{ss} := \lim_{t\to\infty}(r(t)-c(t)) \;=\lim_{t\to\infty}(1-c(t))$$



Steady-state error

4. Transient Response Specifications for second-order systems

(1) Peak time t_p ($0<\zeta<1$):

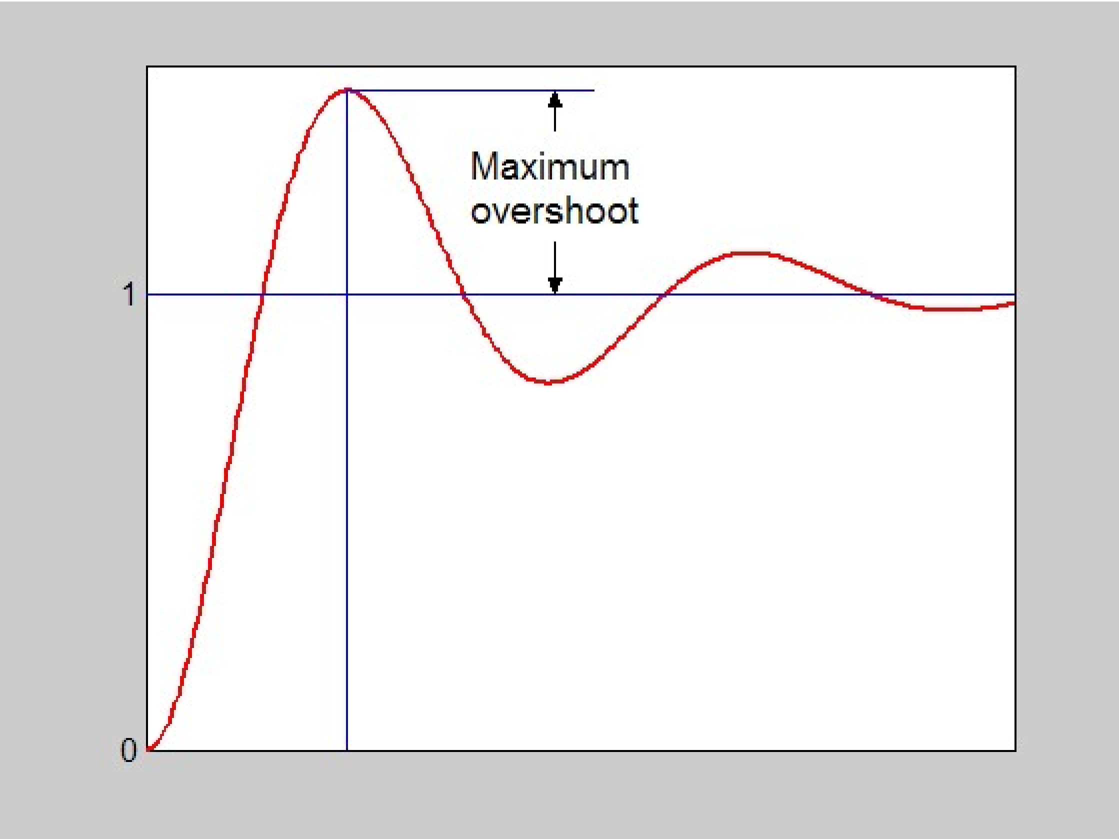
$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\beta\right)$$

$$\frac{dc(t)}{dt}=\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_d t=0\Rightarrow\omega_d t=0,\pi,2\pi,\ldots$$

$$t_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

(2) Maximum Percent Overshoot (0<ζ<1):

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \qquad c(t_p) = 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

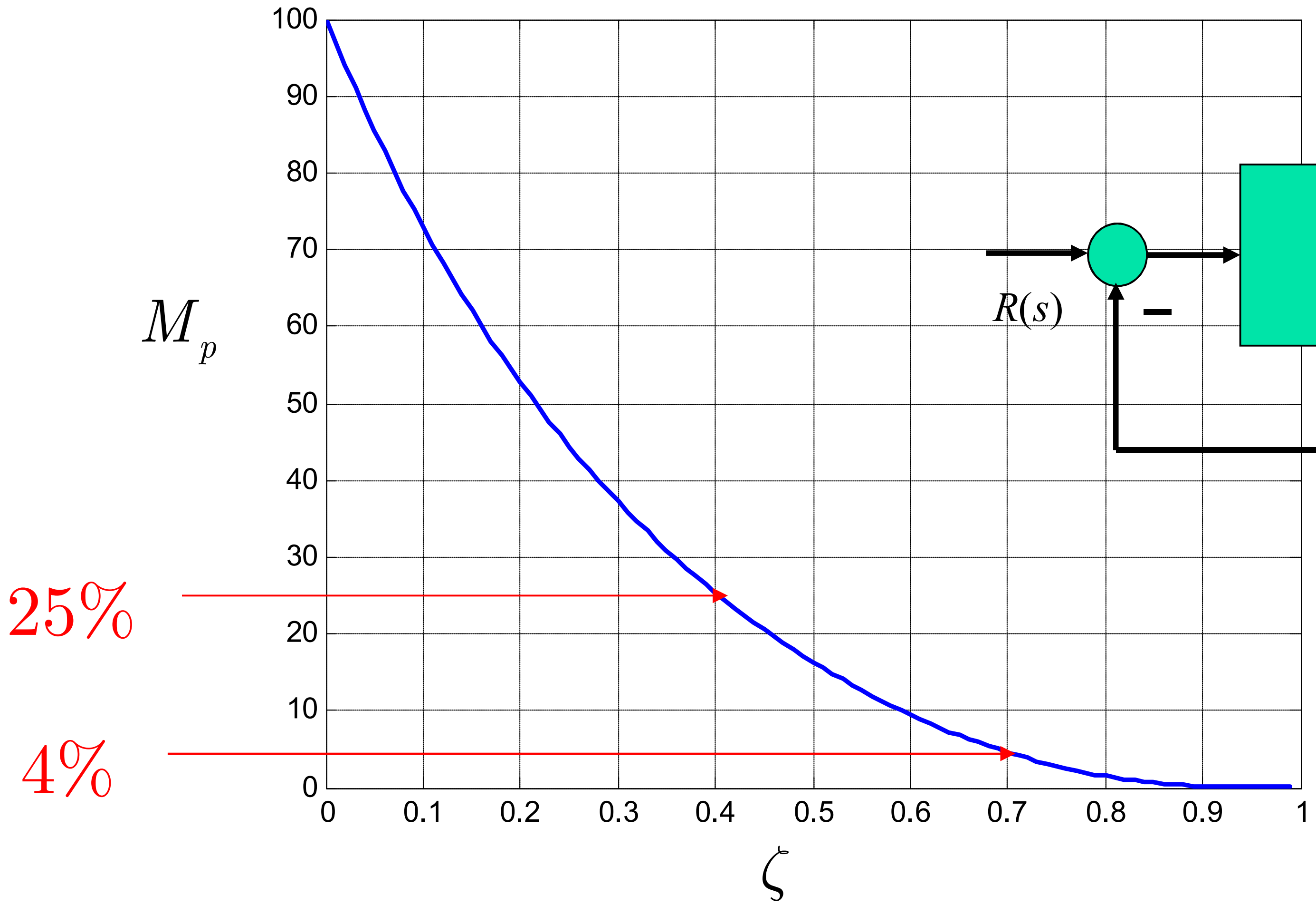


Hence,

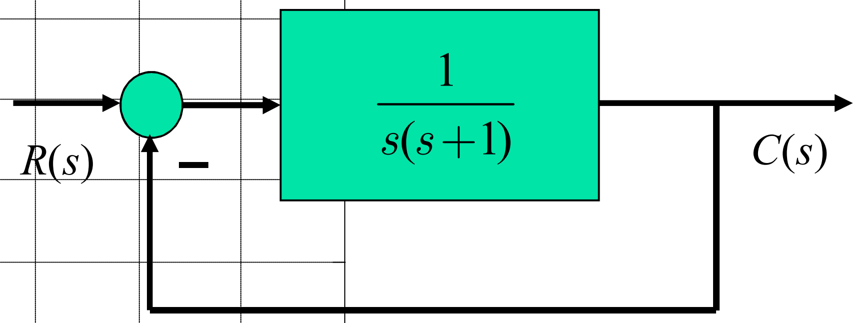
$$M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

The relationship between ζ and M_p is given below. Note that if ζ is between 0.4 and 0.7, then M_p is between 25% and 4%.

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

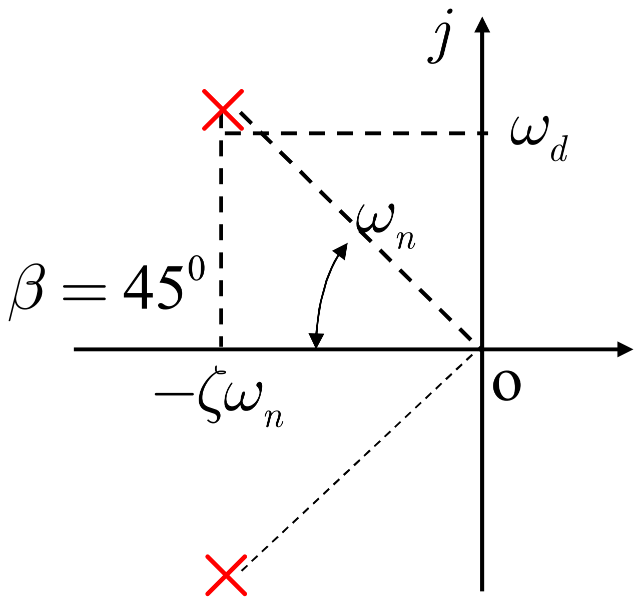


Example:



Find t_p and M_p

In particular, when $\zeta = 1/\sqrt{2} = 0.707$, which corresponds to $\beta = 45^0$, $M_p = 4\%$! Such a ζ is called **optimal damping ratio**.

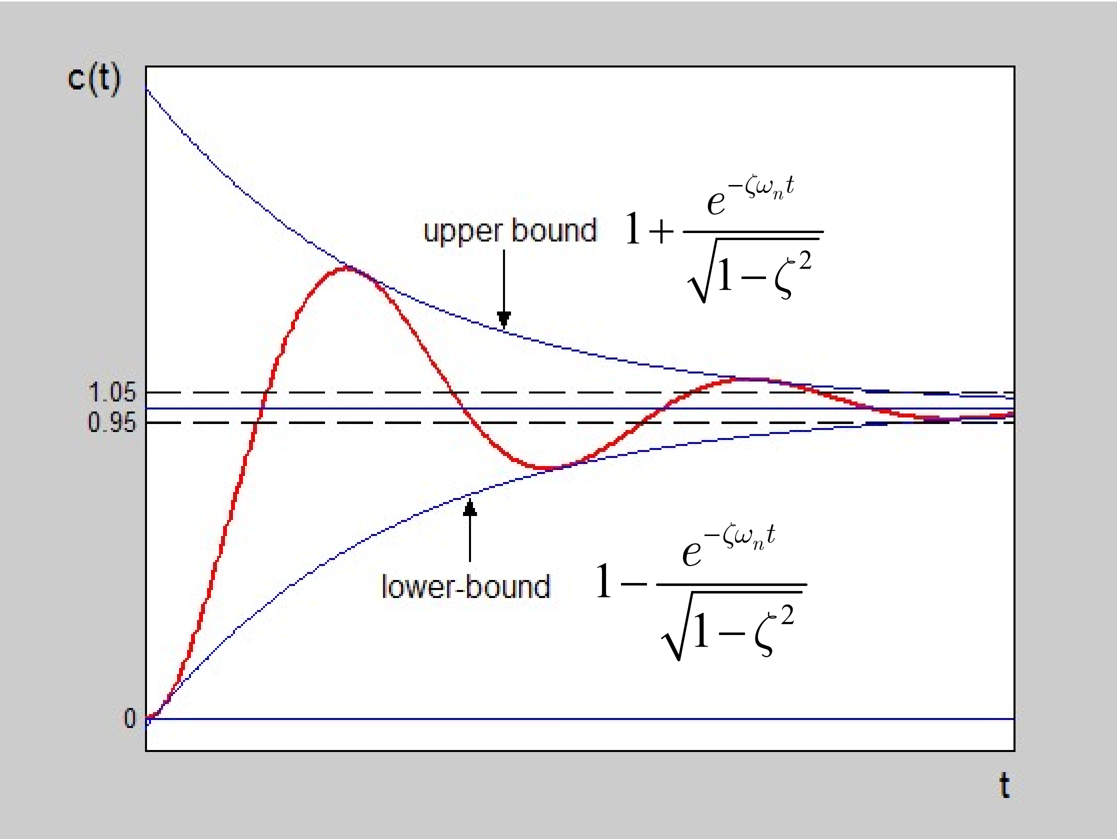


(3) Settling time ($0<\zeta<1$):

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \beta)$$

$$1 \pm \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} = \text{envelope curves}$$

whose time constant is $T = 1/\zeta\omega_n$.



Therefore,

$t_s \approx 4T = 4/\zeta\omega_n$, (2% criterion)

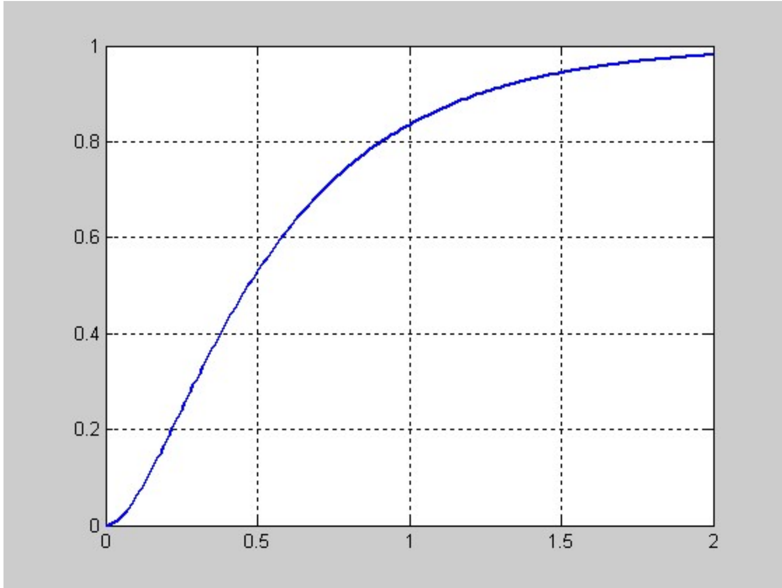
$t_s \approx 3T = 3/\zeta\omega_n$, (5% criterion)

(4) Settling time t_s for ($\zeta \geq 1$):

From

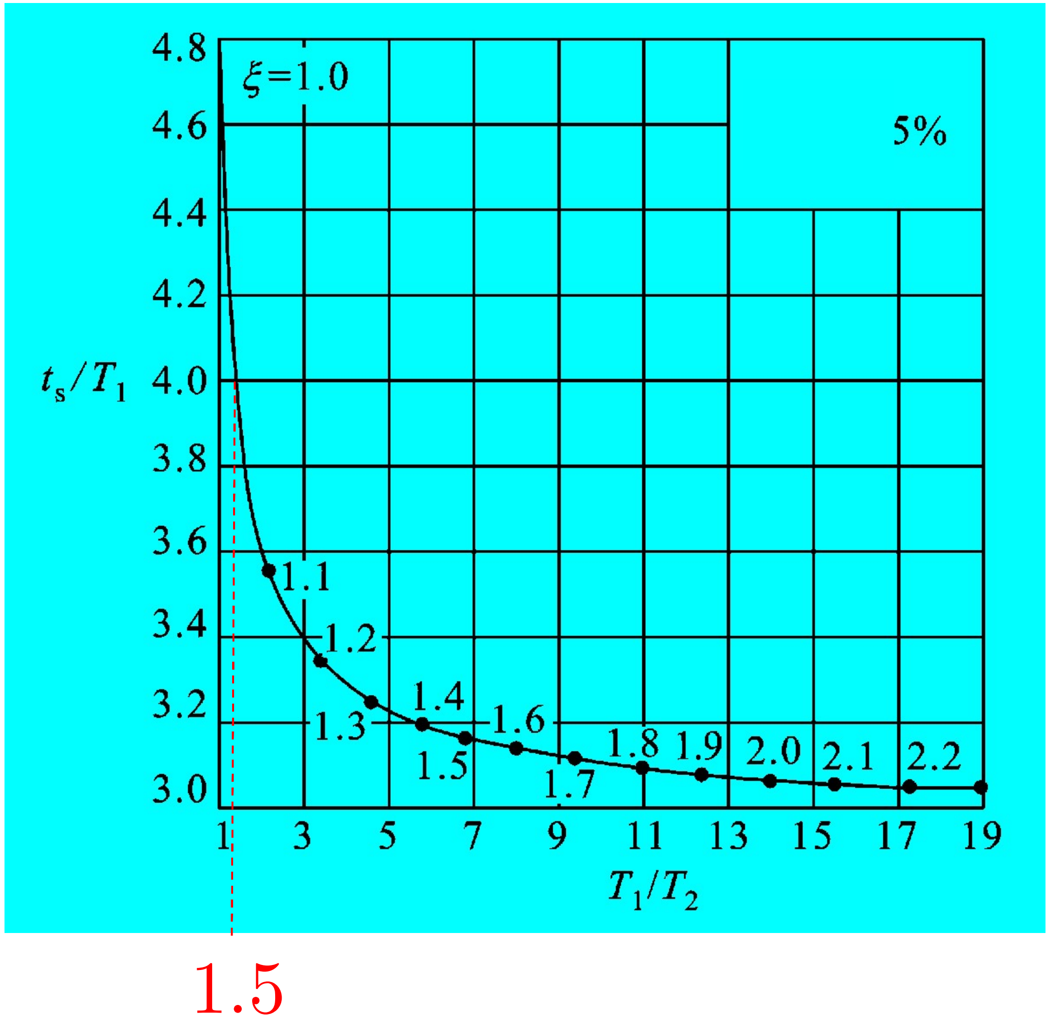
$$C(s) = \frac{\frac{1}{T_1 T_2}}{\left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right)} \frac{1}{s}$$

$$= \frac{1}{s(T_1 s + 1)(T_2 s + 1)}$$



the system can be considered as two first-order subsystems connected in cascade. Since no oscillation arises, only settling time t_s is concerned.

When $\zeta \geq 1$, the settling time t_s can be obtained by looking up the following table (for 5% tolerance band):



For example:

1) $T_1 = T_2 \Leftrightarrow \zeta = 1$

$$t_s = 4.75T_1$$

2) $T_1 / T_2 = 1.5 \Rightarrow \zeta = 1.02$

$$t_s = 4T_1$$

3) $T_1 / T_2 = 4 \Rightarrow \zeta = 1.25$

$$t_s \approx 3.3T_1$$

4) $T_1 / T_2 > 4 \Rightarrow \zeta > 1.25$

$$t_s \approx 3T_1$$

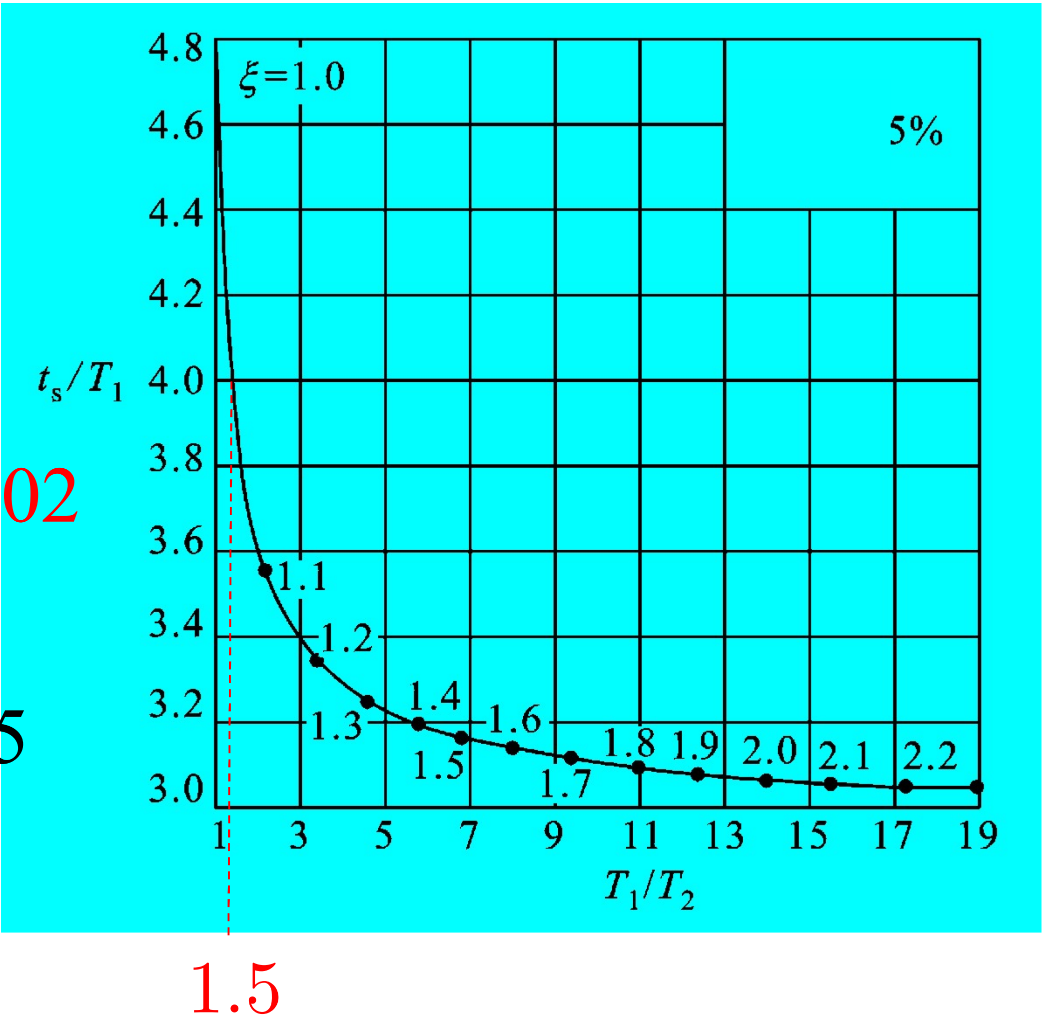
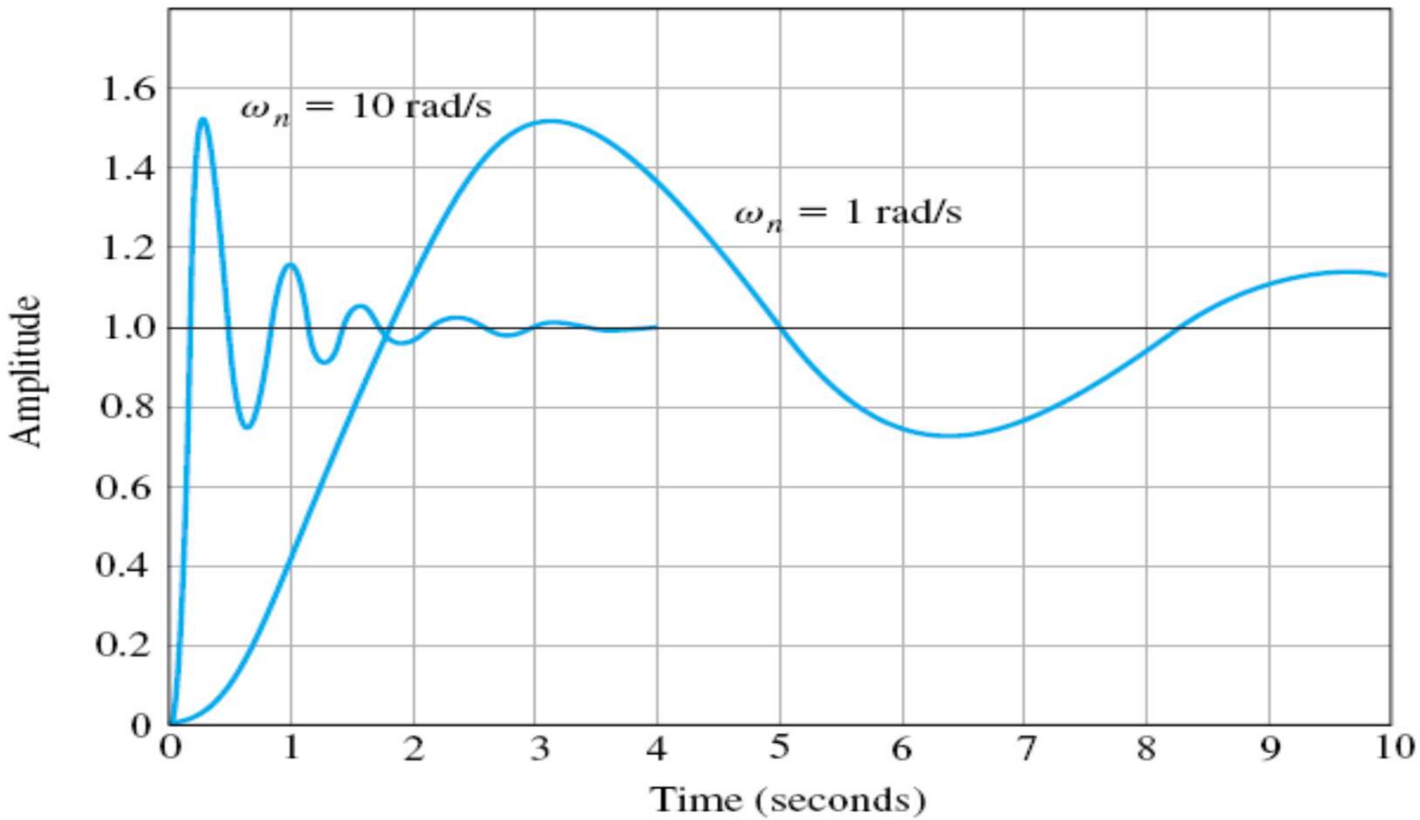


Table: t_s/ T_1 versus T_1/ T_2

Comments on settling time

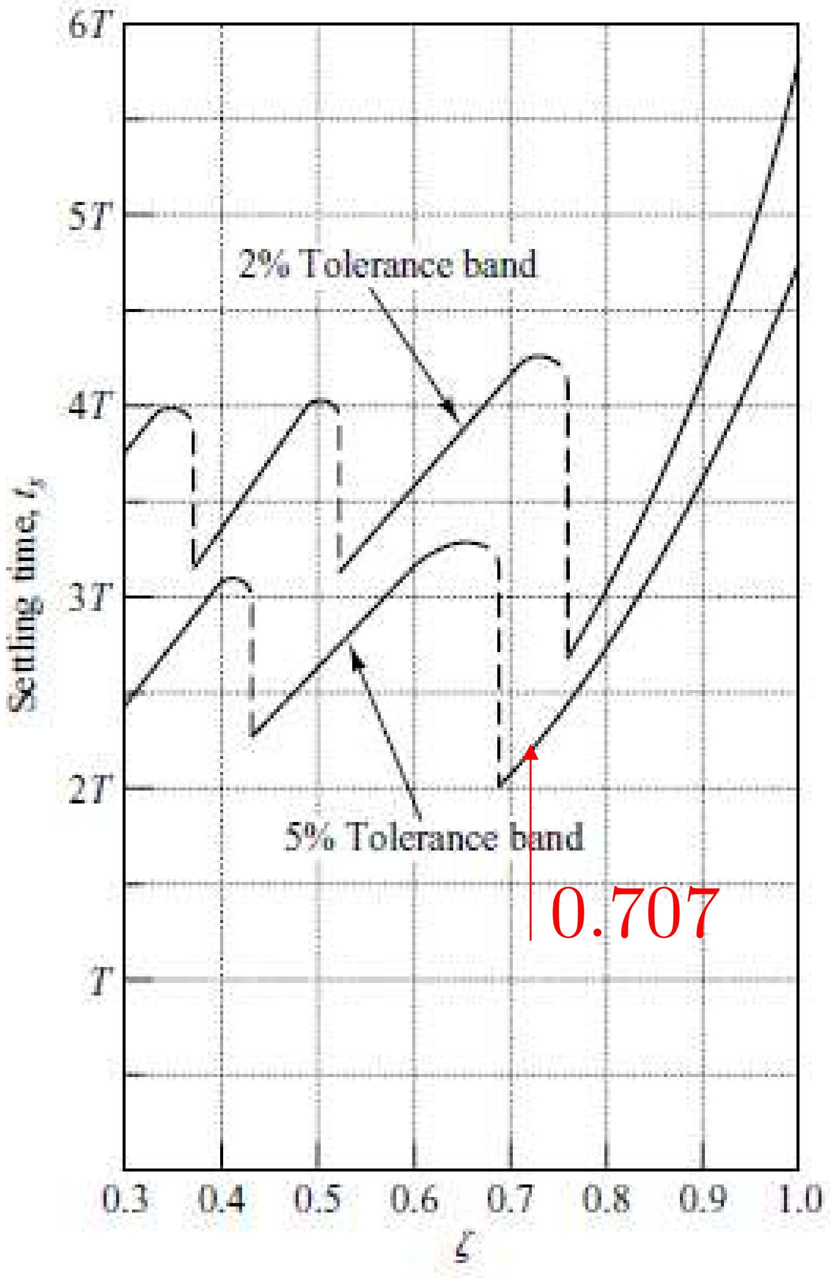
- t_s is inversely proportional to the product of ζ and ω_n. Since ζ is usually given by designer from the requirement of M_p, t_s is mainly determined by ω_n.

Example. The step responses for ζ=0.2 with ω_n=1 and ω_n=10, respectively, are shown below.

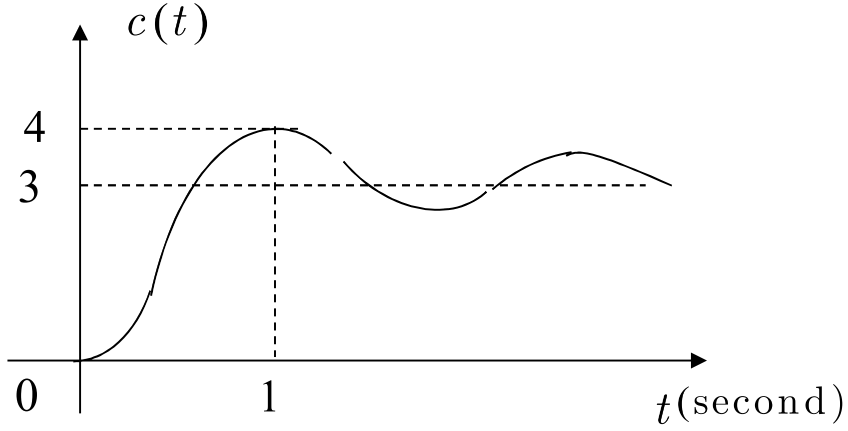


For a given ζ, the response is faster for larger ω_n. Note that the overshoot is independent of ω_n.

On the other hand, with fixed ω_n, t_s reaches a minimum value around $\zeta = 0.76$ (for the 2% criterion) or $\zeta = 0.68$ (for the 5% criterion) and then increases almost linearly for large values of ζ, where $T=1/\zeta\omega_n$. Note that $\zeta=0.707$ implies that $\beta=45^0$. Also, discontinuities arise.



Example. The unit-step response of a second-order system is shown below, where $\lim_{t\to\infty} c(t)=3$. Determine its transfer function.



Solution: The transfer function must have the following form:

$$\Phi(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

with $K=3$. From the response, it is clear that $t_p=1$s and $M_p=(1/3)100\%=33.3\%$. Hence, by utilizing the formulas

$$M_p = e^{-\pi\zeta/\sqrt{1-\zeta^2}} \times 100\%$$

and

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

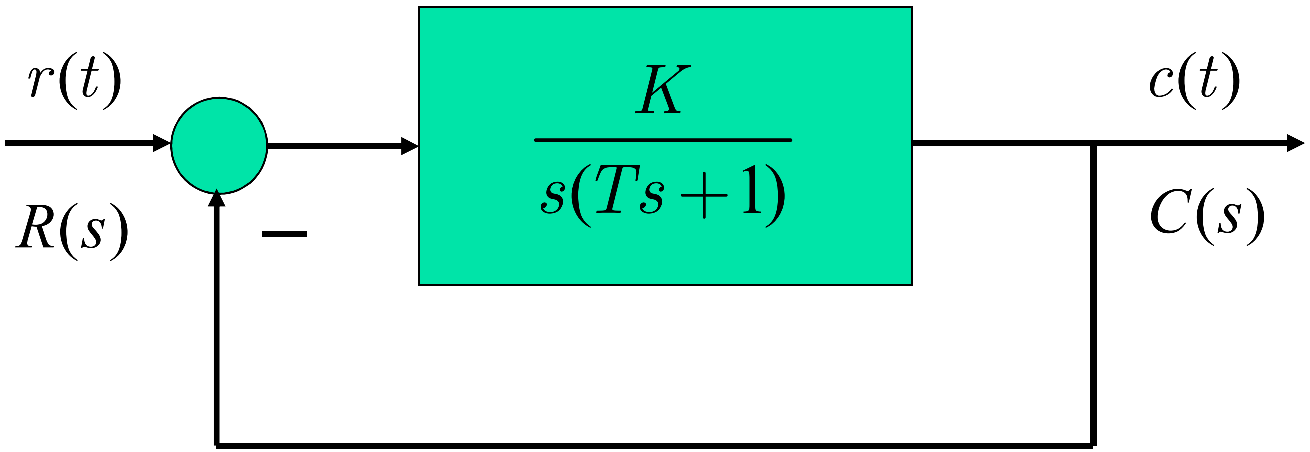
one can obtain that

$$\zeta = 0.33$$

and

$$\omega_n = 3.33\,\mathbf{rad/s}$$

Example. Consider the following system:



where T=0.1s, and K is the open-loop gain. To determine K such that no overshoot and t_s=1s.

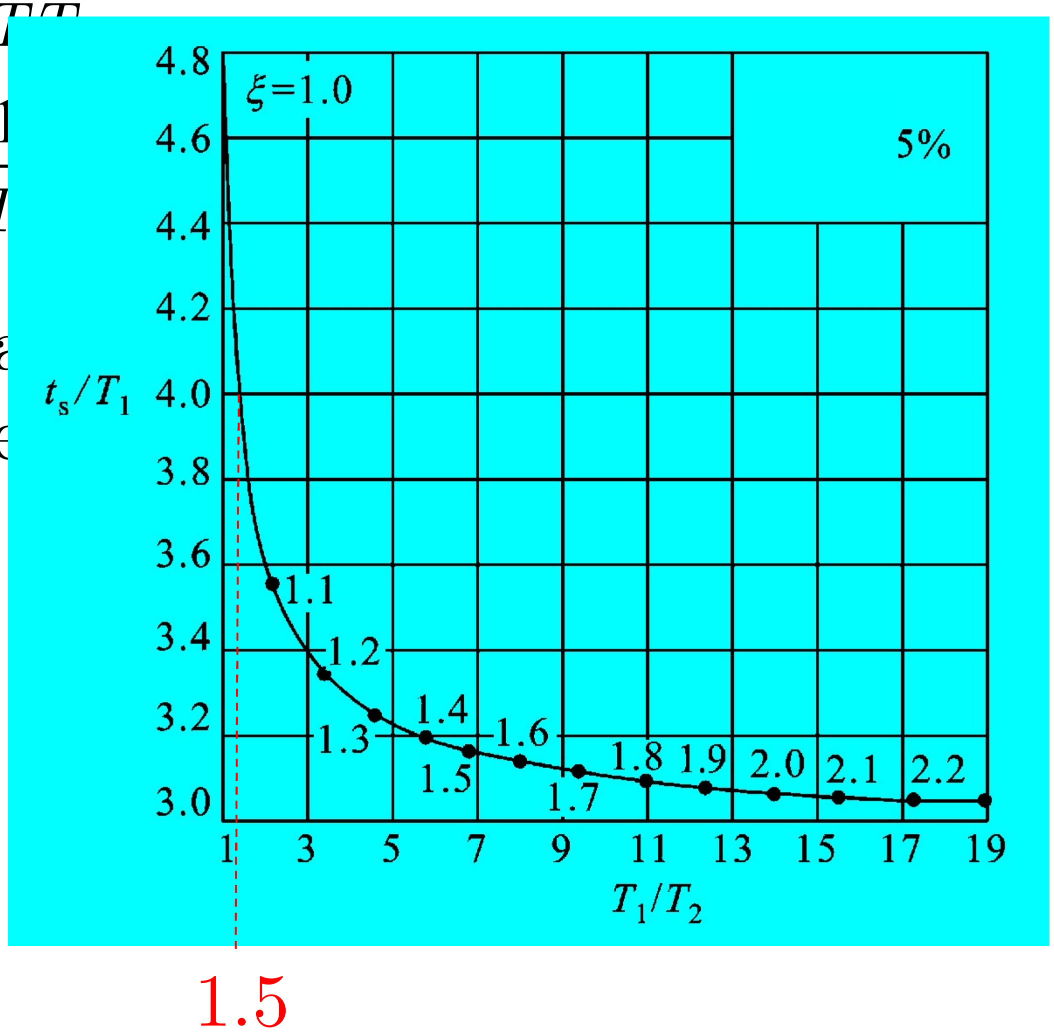
Solution: By the requirements, ζ should satisfy $\zeta \geq 1$. Therefore, the closed-loop characteristic equation can be expanded as

$$d(s) = s^2 + \frac{1}{T}s + \frac{K}{T} = \left(s + \frac{1}{T_1}\right)\left(s + \frac{1}{T_2}\right) = s^2 + \left(\frac{1}{T_1} + \frac{1}{T_2}\right)s + \frac{1}{T_1 T_2} = 0$$

Equating the coefficients for the same power of s yields:

$$\begin{cases} \frac{K}{T} = \frac{1}{TT\ldots} \\ \frac{1}{T} = \frac{1}{T\ldots} \end{cases}$$

To make the response a…
required that ζ be close…
we obtain that when



we have $$\frac{T_1}{T_2} = 1.5$$

$$\frac{t_s}{T_1} = 4$$

In that case, ζ=1.02, which is very much close to 1 and therefore, possesses a fast transient response. Since t_s=1s, we have

Therefore, $$T_1 = \frac{1}{4} t_s = 0.25\ \mathrm{s}$$

$$\frac{T_1}{T_2} = 1.5 \Rightarrow T_2 = \frac{T_1}{1.5} = \frac{0.25}{1.5} = 0.167\ \mathrm{s}$$

To determine K, notice that

$$\begin{cases} \dfrac{K}{T} = \dfrac{1}{T_1 T_2} \\ \dfrac{1}{T} = \dfrac{1}{T_1} + \dfrac{1}{T_2} \end{cases}$$

Therefore,

$$K = \left.\frac{T}{T_1 T_2}\right|_{T=0.1} = \frac{0.1}{0.25 \times 0.167_2} = 2.4 s^{-1}$$

Finally, we must check that

$$\frac{1}{T_1} + \frac{1}{T_2} = \frac{1}{T} = \frac{1}{0.1} = 10$$

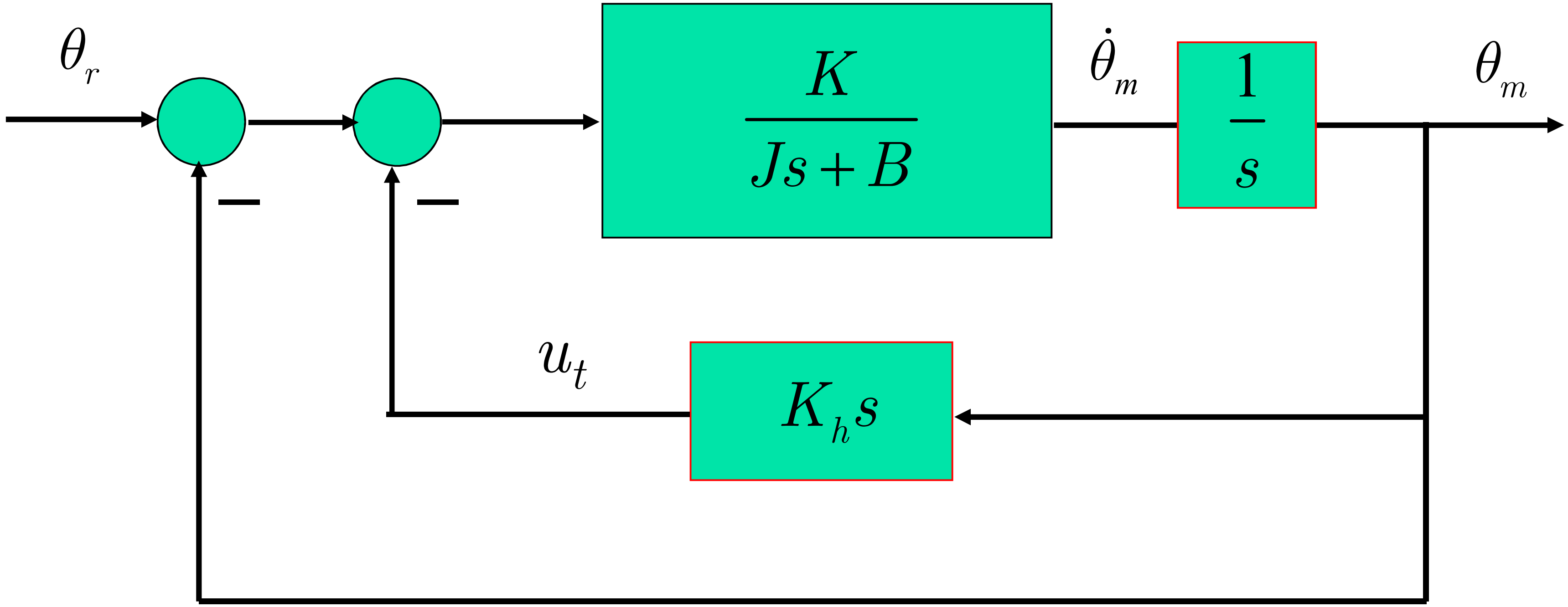
should be satisfied. Otherwise, K should be redesigned. Fortunately, in this example, from

$$\frac{1}{T_1} = 4, \quad \frac{1}{T_2} = 6$$

such a condition is satisfied.

5. Servo system with velocity feedback

The derivative of the output signal can be used to improve system performance :



Note that without the derivative feedback (K_h=0), system may exhibit excessive overshoot. Indeed, the closed-loop transfer function in that case is

$$\frac{\Theta_m(s)}{\Theta_r(s)} = \frac{K/J}{s^2 + (B/J)s + K/J} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where

$$\omega_n = \sqrt{K/J}$$

$$\zeta = \frac{B}{2\sqrt{JK}}$$

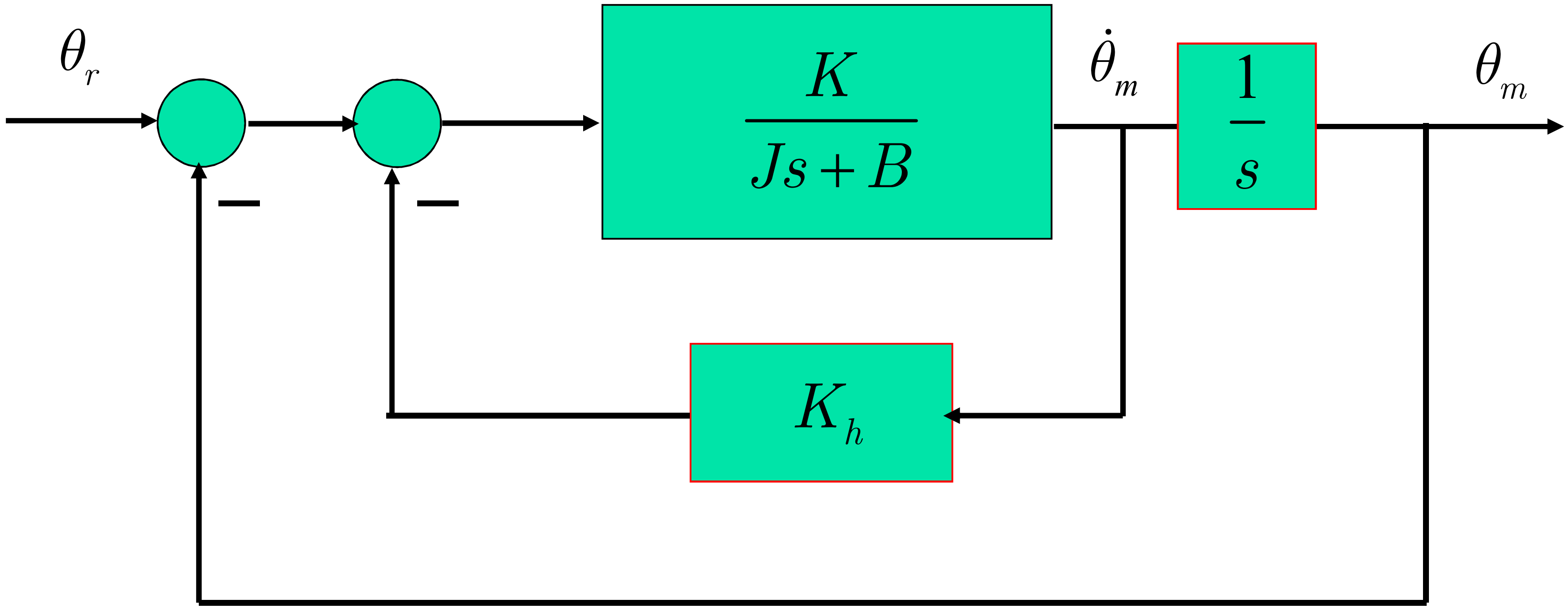
However, taking the derivative feedback into account, the closed-loop transfer function becomes

$$\frac{\Theta_m(s)}{\Theta_r(s)} = \frac{\omega_n^2}{s^2 + 2(\zeta + K_h\omega_n/2)\omega_n s + \omega_n^2}$$

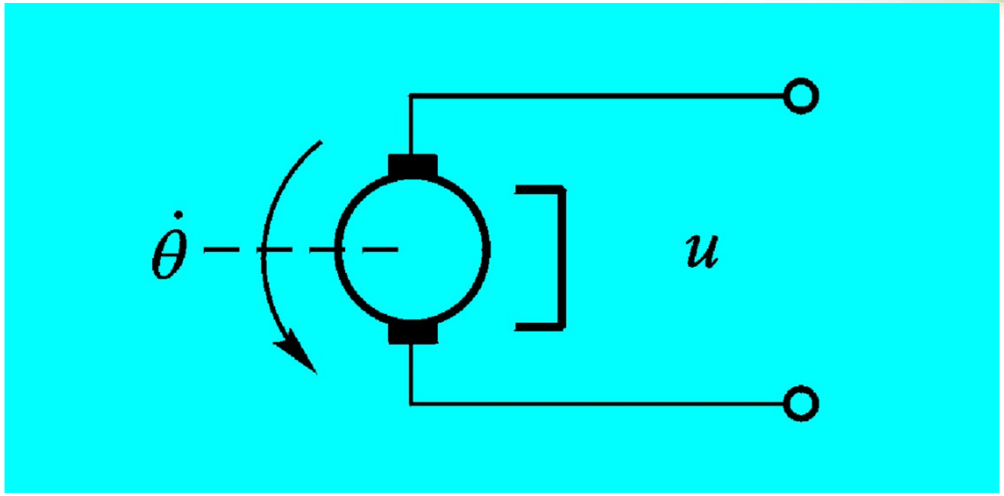
The improved damping ratio is:

$$\zeta_t = \zeta + \frac{1}{2}K_h\omega_n$$

For a servo control system, in obtaining the derivative of the output position signal, it is desirable to use a tachometer generator instead of physically differentiating the output signal (noise effect, p.175).



Servo-Tek Tachometer Generators provide a convenient means of converting rotational speed into an isolated analog voltage signal suitable for control applications.



Mathematical model:

$$u(t) = K_h \dot{\theta}(t)$$

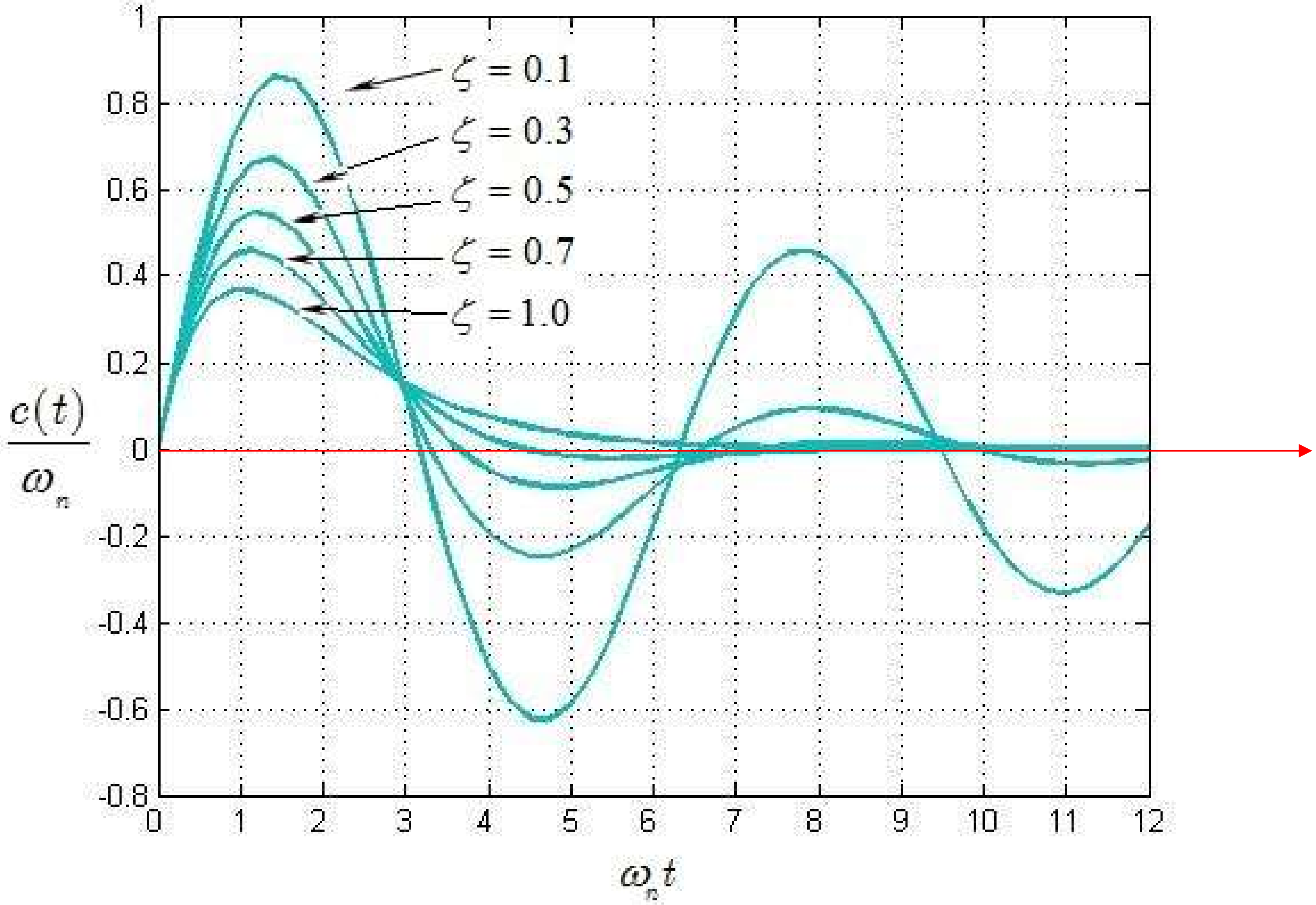
6. Impulse response of second-order systems

By the property of LTI systems, differentiating the corresponding unit-step response of the second-order system (or directly taking the inverse Laplace transform) yields

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin\left(\omega_d t\right), t \ge 0, \quad 0 \le \zeta < 1$$

$$c(t) = \omega_n^2 t e^{-\omega_n t}, \quad t \ge 0, \quad \zeta = 1$$

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t}, t \ge 0, \quad \zeta > 1$$

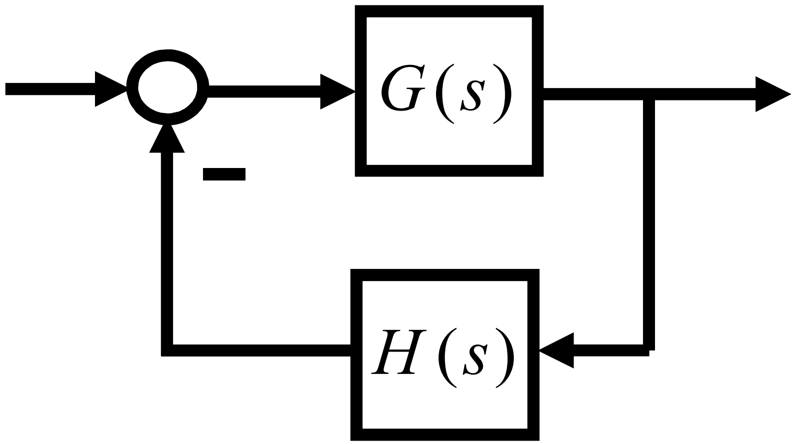


For $\zeta < 1$, $c(t)$ oscillates and takes both positive and negative values.

5-4 Higher-Order Systems

In this section, we shall present a transient response analysis of higher-order systems in general forms. We shall show that the response of higher-order systems is the sum of the responses of first-order and second-order systems.

1. Transient response of higher-order systems



Consider the closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{b_0 \prod_{i=1}^{m}(s+z_i)}{\prod_{k=1}^{n}(s+p_k)}, m \leq n$$

Assumption 1: $m \leq n$.

Almost any physical control system satisfies this condition. A transfer function satisfies Assumption 1 is called a **proper transfer function**. If $m < n$, the system is called **strictly proper**. We only deal with proper or strictly proper systems.

Assumption 2: All the closed-loop poles lie in the left-half s-plane.

Assumption 3: All the poles are *distinct*.

Unit-step response: Let the input signal be a unit-step function. Then the output is

$$C(s) = \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{k=1}^{n} (s + p_k)} \cdot \frac{1}{s} = \frac{a}{s} + \sum_{k=1}^{n} \frac{a_k}{s + p_k}$$

where a and a_k are the residues of the poles at $s=0$ and $s=-p_k$, respectively:

$$a = \lim_{s \to 0} \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{k=1}^{n} (s + p_k)} = \frac{b_0 \prod_{i=1}^{m} (z_i)}{\prod_{k=1}^{n} (p_k)} = \Phi(0)$$

$$a_k = \lim_{s \to -p_k} \frac{b_0 \prod_{i=1}^{m} (s + z_i)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (s + p_i)} \frac{1}{s} = \frac{b_0 \prod_{i=1}^{m} (-p_k + z_i)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (-p_k + p_i)} \frac{1}{-p_k}$$

We consider the general case that $C(s)$ consists of real poles and pairs of complex-conjugate poles. Then,

$$C(s)=\frac{a}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k)+c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

where $n=q+2r$. For example,

$$C(s)=\frac{-s^3+3s+4}{(s+1)(s^2+2s+2)}\frac{1}{s}=\frac{2}{s}-\frac{2}{s+1}-\frac{s+1}{s^2+2s+2}$$

Therefore, for the general case, the unit-step response can be written as

$$c(t)=a+\sum_{i=1}^{q}a_i e^{-p_i t}+\sum_{k=1}^{N}b_k e^{-\zeta_k\omega_k t}\cos\omega_k\sqrt{1-\zeta_k^2}\,t$$

$$+\sum_{k=1}^{N}c_k e^{-\zeta_k\omega_k t}\sin\omega_k\sqrt{1-\zeta_k^2}\,t$$

$$=a+\sum_{i=1}^{q}a_i e^{-p_i t}+\sum_{k=1}^{N}d_k e^{-\zeta_k\omega_k t}\sin(\omega_k\sqrt{1-\zeta_k^2}\,t+\theta_k)$$

Some useful concepts: Residues and dipoles:

• Under the **Assumption 1**, a pair of closely located closed-loop pole and zero is called a **dipole** and can be neglected (p.181).

• On the other hand, if a closed-loop pole is located very far from the imaginary axis, the transient associated with the pole lasts a short time and therefore may be neglected.

Example. A system's transfer function is

$$\Phi(s) = \frac{(s+1.01)}{(s+1)(s+2)}$$

By partial fraction expansion,

$$\Phi(s)=\frac{(s+1.01)}{(s+1)(s+2)}=\frac{0.01}{(s+1)}+\frac{1-0.01}{(s+2)}$$

Taking the inverse Laplace transform yields

$$c(t)=0.01e^{-t}+(1-0.01)e^{-2t}, \quad t\geq 0$$

Obviously, the contribution of the term $0.01e^{-t}$ to the response is small and therefore, can be neglected.

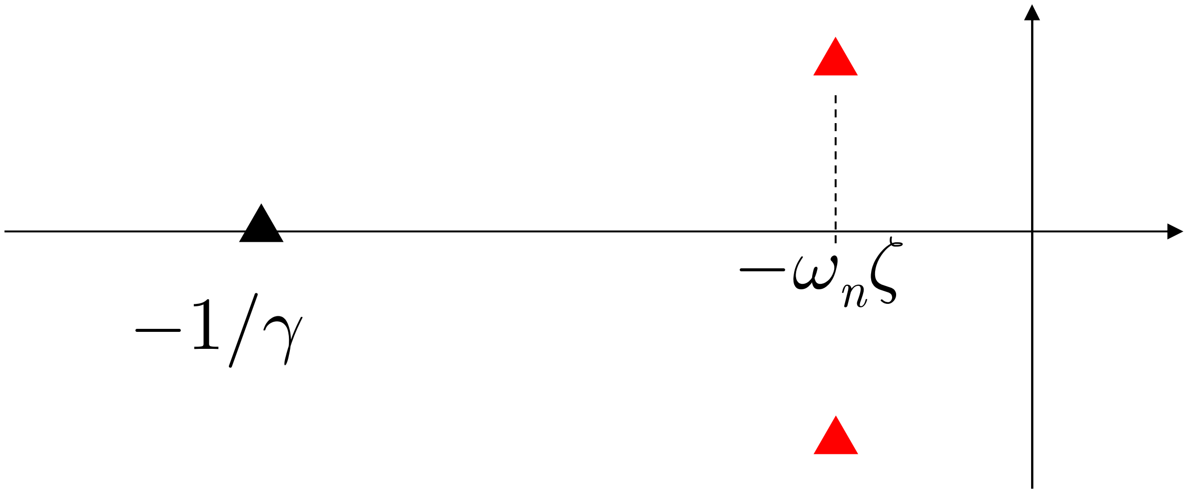
Example. A system's transfer function is

$$\Phi(s)=\frac{1/5}{(s+1)(\frac{1}{5}s+1)}\approx\frac{1}{(s+1)}$$

2. Dominant poles

Example. Consider the following third-order closed-loop system:

$$\Phi(s)=\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(\gamma s+1)}$$



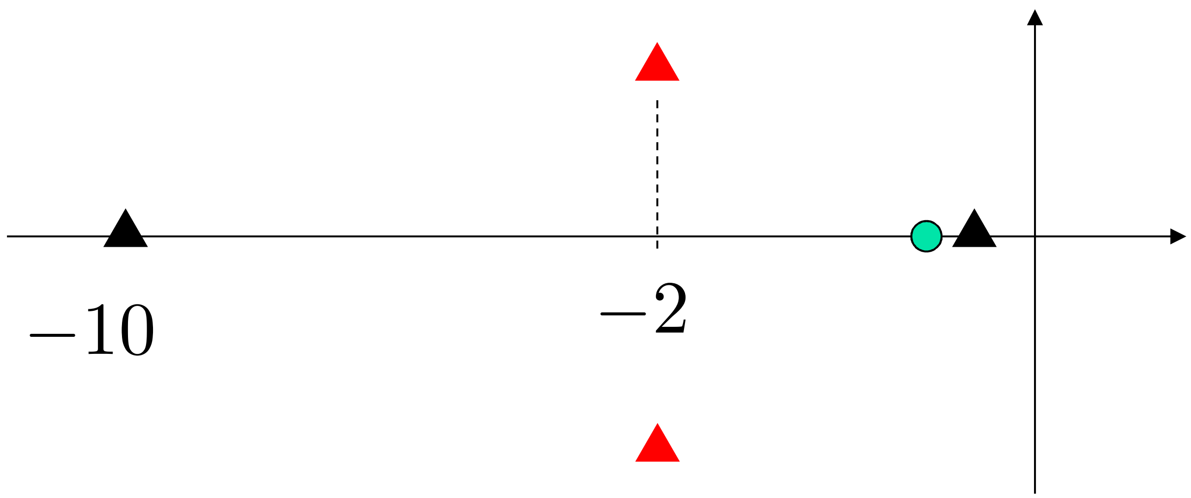
If the *real* parts satisfy $1/\gamma \geq 5\omega_n\zeta$ $(1/\gamma \geq 4\omega_n\zeta)$, then the third-order system can be approximated by a second-order system

$$\Phi(s) \approx \frac{\omega_n^2}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

whose two poles are called **dominant poles**.

In general, if the ratios of real parts exceed 5 (or 4) and there are no zeros nearby, then the closed-loop poles nearest the $j\omega$ axis will dominate in the transient response and are called **dominant poles**.

Example. The locations of closed-loop poles and zero of a higher order system are shown below. Determine its dominant poles.



The pair of zero-pole nearest the $j\omega$ axis is a dipole and therefore can be neglected. By the rule of determining the dominant poles introduced above, the system can be approximated by a second-order system with a pair of complex-conjugate dominant poles.

3. Stability analysis in complex plane

(1). Concept of stability

Definition: A signal $x(t)$ is said to be bounded if there is a positive real number M such that

$$|x(t)| \leq M$$

for all $t \in [0, \infty)$.

Definition: A system is said to be bounded-input-bounded-output (BIBO) stable if for each bounded input the corresponding output is bounded.

(2). Stability criterion

Theorem: An LTI system with **closed-loop** transfer function $G(s)$ is said to be stable *if and only if* $G(s)$ is proper and all its poles lie in the left-half s-plane ($\Leftrightarrow$*have negative real parts*).

Example. Consider the following three systems:

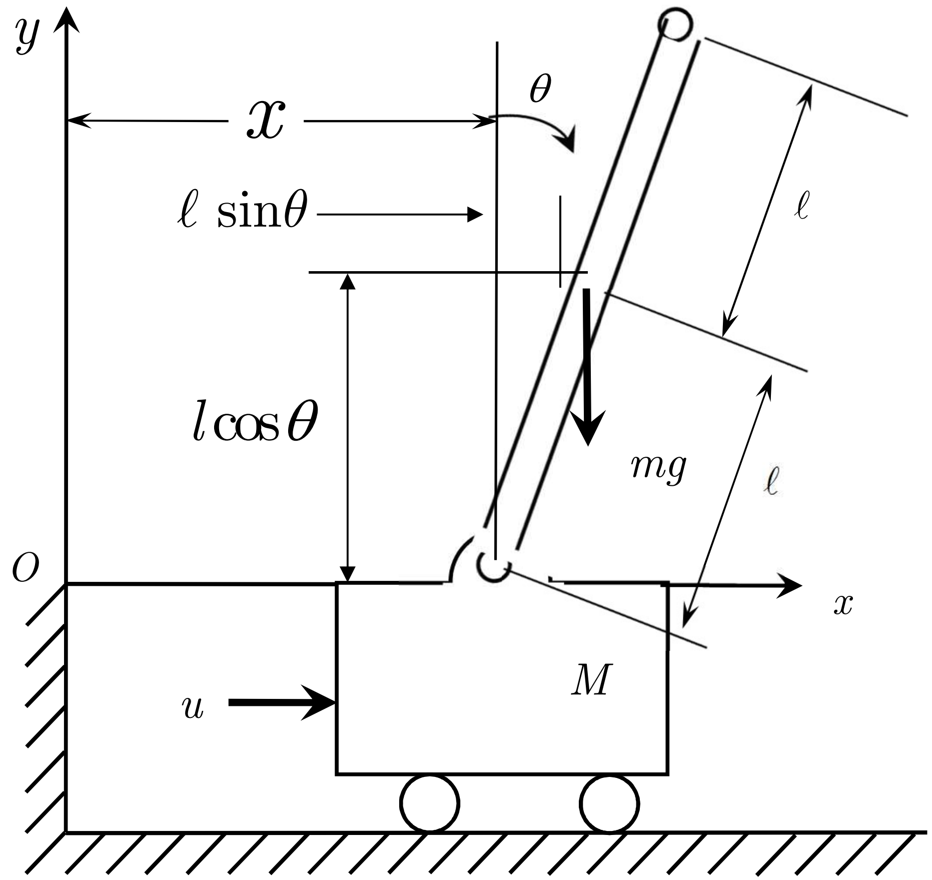
$$\Phi(s) = \frac{1}{(s+1)};$$

$$\Phi(s) = \frac{1}{(s-1)};$$

$$\Phi(s) = \frac{1}{(s^2+1)}.$$

Investigate their unit-step responses.

Example. An inverted pendulum mounted on a motor-drive cart is shown below. The objective is to keep the rod in a vertical position.



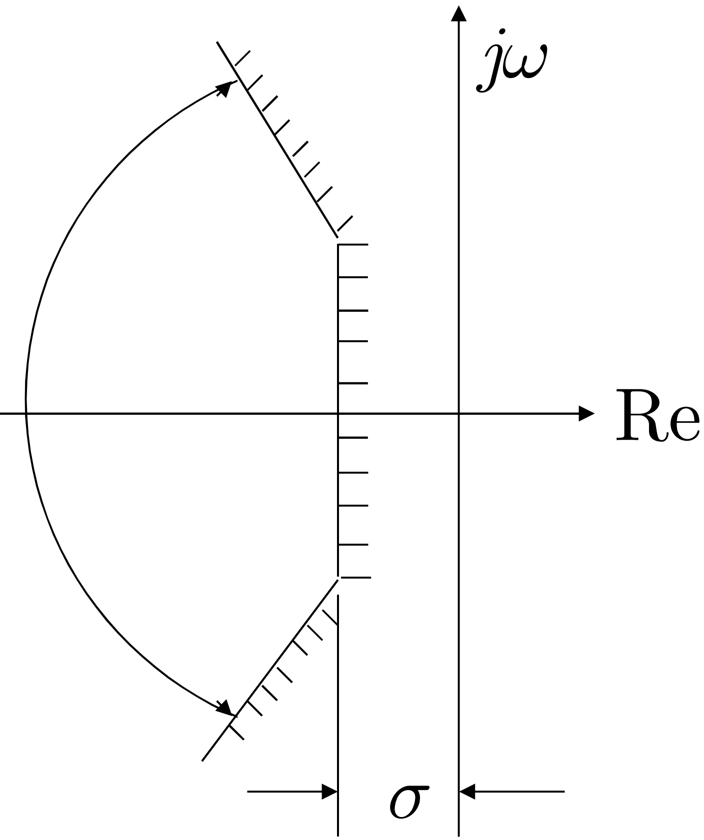
The system is unstable.

$$\frac{\Theta(s)}{U(s)}$$

$$= \frac{lm}{\left[(M+m)mgl-(MI+mMl^2+mI)s^2\right]}$$

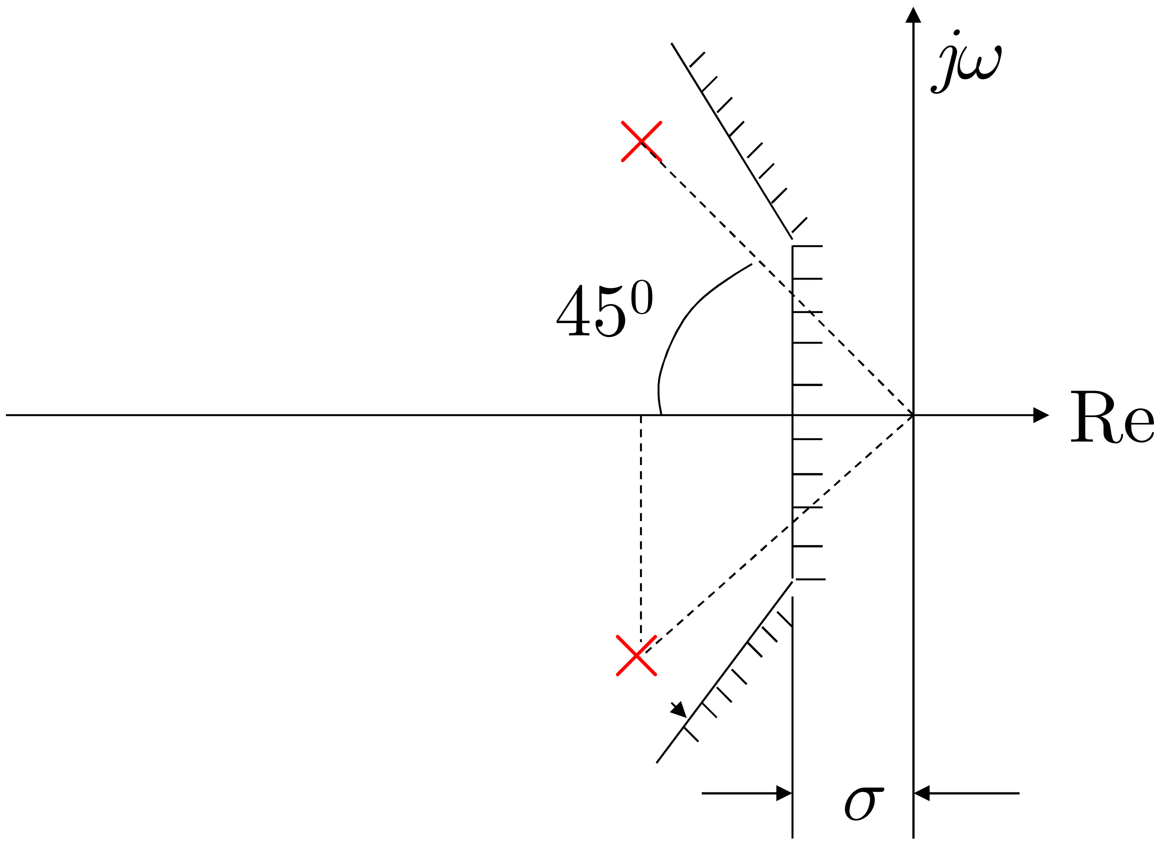
(3). Relative Stability

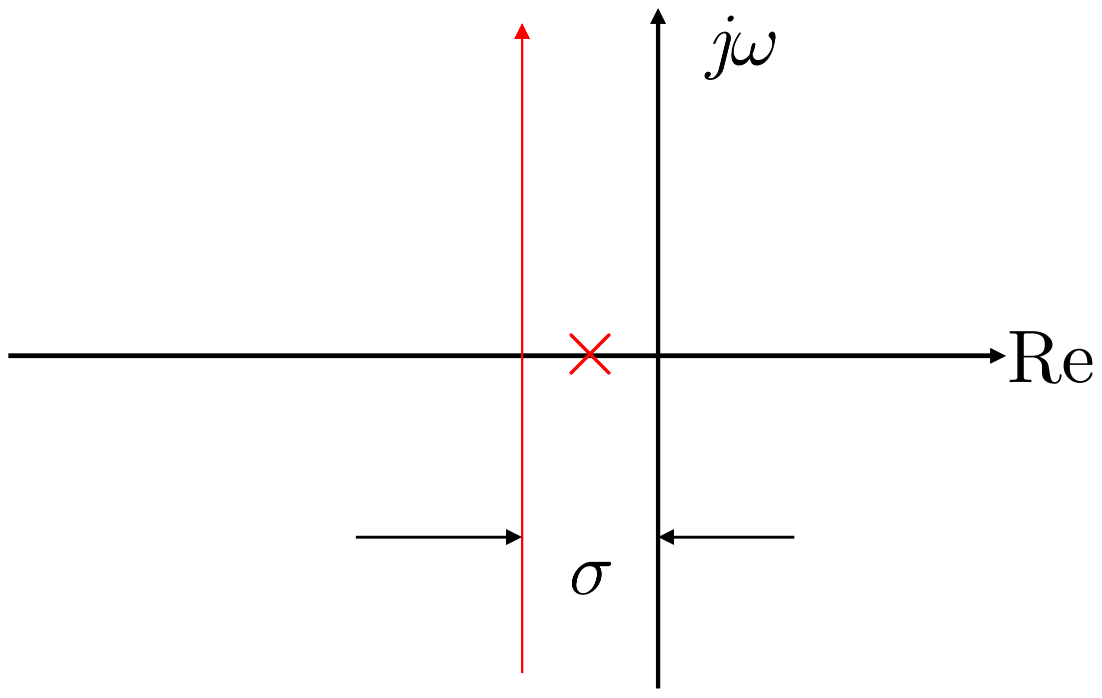
If dominant poles lie closed to $j\omega$ axis, the transient may exhibit excessive oscillation or very slow. Therefore, to guarantee fast, yet well



damped, transient response characteristics, it is necessary that the closed-loop poles lie in this region.

For example, if the two dominant poles are located in the region with β=45^0 as shown in the following figure. Then, ζ=0.707. The system possesses a satisfactory transient response.





Another relative stability region