AUTOMATIC CONTROL

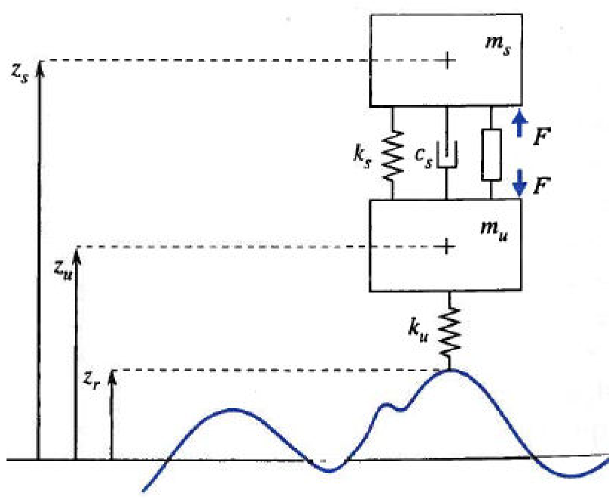
Computer, Electronic and Telecommunications Engineering

Laboratory practice n. 2

Objectives: modal analysis, internal and BIBO stability of LTI systems, steady state behaviour, step response of prototype models.

Problem 1 (analysis of LTI dynamical systems)

Consider the quarter car model[1] of a vehicle active suspension:



$z_s \rightarrow$ sprung mass position, [m]
$z_u \rightarrow$ unsprung mass position, [m]
$z_r \rightarrow$ road profile, [m]
$F \rightarrow$ suspension active force, [N]
$m_s \rightarrow$ sprung mass (1/4 vehicle), [kg]
$m_u \rightarrow$ unsprung mass (tyre + suspension), [kg]
$k_s \rightarrow$ suspension stiffness, [N/m]
$c_s \rightarrow$ suspension damping coefficient, [Ns/m]
$k_u \rightarrow$ tyre stiffness, [N/m]

The system inputs are:

$z_r(t) \rightarrow$ road profile (disturbance)
$F(t) \rightarrow$ suspension active force (to be provided by the control)

The suspension performance is evaluated using the following outputs:

$\ddot{z}_s(t) \rightarrow$ sprung mass acceleration (comfort)
$z_u(t) - z_r(t) \rightarrow$ tyre deflection (handling)

state, input and output vectors are defined as:

[1] This example and the picture are taken from, P. Bolzern et al. “Fondamenti di Controlli Automatici”, 3rd Ed., McGraw-Hill, Italia, 2008.

$$x(t)=\begin{bmatrix} z_s(t) \\ z_u(t) \\ \dot{z}_s(t) \\ \dot{z}_u(t) \end{bmatrix}=\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}, u(t)=\begin{bmatrix} z_r(t) \\ F(t) \end{bmatrix}=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}, y(t)=\begin{bmatrix} \ddot{z}_s(t) \\ z_u(t)-z_r(t) \end{bmatrix}=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

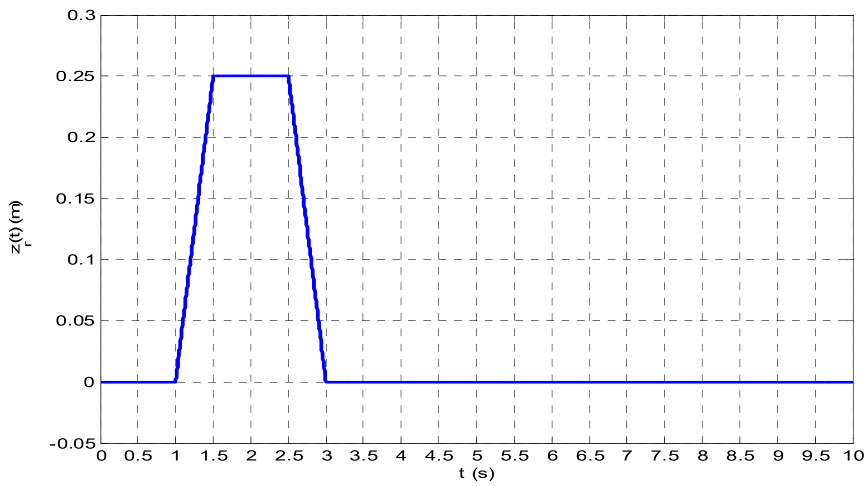
The state space representation describing the dynamic variation with respect to the static load condition in the presence of the gravity action is given by:

$$\dot{x}(t)=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & k_s/m_s & -c_s/m_s & c_s/m_s \\ k_s/m_u & -(k_s+k_u)/m_u & c_s/m_u & -c_s/m_u \end{bmatrix} x(t)+\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1/m_s \\ k_u/m_u & -1/m_u \end{bmatrix} u(t)$$

$$y(t)=\begin{bmatrix} -k_s/m_s & k_s/m_s & -c_s/m_s & c_s/m_s \\ 0 & 1 & 0 & 0 \end{bmatrix} x(t)+\begin{bmatrix} 0 & 1/m_s \\ -1 & 0 \end{bmatrix} u(t)$$

The numeric values of the physical parameters are: $m_s = 400$ kg, $m_u = 50$ kg, $k_s = 2\cdot10^4$ N/m, $k_u = 2.5\cdot10^5$ N/m, $c_s = 2\cdot10^3$ Ns/m.

1. Perform the system modal analysis and study the internal stability properties.
2. Compute the analytical expression of the sprung mass acceleration $\ddot{z}_s(t)$ in the presence of zero initial conditions and F(t) = 0 and $z_r(t) = 0.08\varepsilon(t)$ (m). (Hint: note that here only the response between the 1st input and the 1st output is considered…).
3. Consider the time output response defined at the previous point. Evaluate, if possible, the steady state value.
4. Build a suitable Simulink model for the simulation of the given system and simulate the time behaviour of both the outputs when x(0)=0, F(t) = 0 and the following time course for $z_r(t)$:



(Hints. In order to speed up the Simulink model construction, define the given system as an `ss` object and use the Simulink block "LTI System" in the library "Control system toolbox". Note also that the given system is 2-inputs 2-outputs (→ use "mux" and "demux" blocks in "Signal routing" library respectively to provide inputs and obtain outputs). The signal above can be defined by means of the block "Signal Builder" in the library "Sources", use blocks "To Workspace" to show simulation results. All the simulation procedures should be handled through a suitable MatLab script. In order to avoid numerical problems in the residue computation for MatLab version 2016 2017 … use the statement `\` instead of `inv` in the computation of the analytical expression of the output response, i.e. `Y=C*(s*eye(4)-A)\(B*U)` instead of `Y=C*inv(s*eye(4)-A)*B*U`)

(Answer: 1. the natural modes are: $m_1(t) = e^{-20.3t}\cos(69.3t+\varphi)$ and $m_2(t) = e^{-2.2t}\cos(6.55t+\varphi)$ both exponentially convergent, the system is internally stable.
2. $\ddot{z}_s(t) = 28.64e^{-20.3t}\cos(69.36t-1.65) + 4.21e^{-2.2t}\cos(6.55t+0.977)$, 3. $\ddot{z}_{s_{ss}}(t) = 0$)

Problem 2 (internal stability of LTI systems)

Study the internal stability properties of an LTI system characterized by the following dynamic matrix A:

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(Answer: the system is unstable)

Problem 3 (internal stability of LTI systems)

Suppose that $p \in \mathbb{R}$, then discuss the internal stability properties of the LTI system characterized by the following dynamic matrix A

$$A = \begin{bmatrix} p^2 - 1 & 0 & 0 \\ 0 & p - 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(Answer: $-1 < p < 1$ → asymptotically stable, $p = \pm 1$ → stable, $p > 1$, $p < -1$ → unstable)

Problem 4 (Internal and BIBO stability of LTI systems)

An LTI system is described by the following state space representation:

$$\dot{x}(t) = \begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} x(t)$$

Study its internal and BIBO stability properties.

(Answer: internal stability → unstable, BIBO stability → stable)

Problem 5 (BIBO stability of LTI systems)

Analyse the BIBO stability properties of an LTI system having the following dynamic transfer function in the presence of variations of the real parameter p:

$$H(s) = \frac{4}{s^2 + (p+1)s + 4p - 2}$$

(Answer: the system is BIBO stable for $p > 0.5$)

Problem 6 (step response of 2nd order systems)

Consider the following 2nd order system transfer functions:

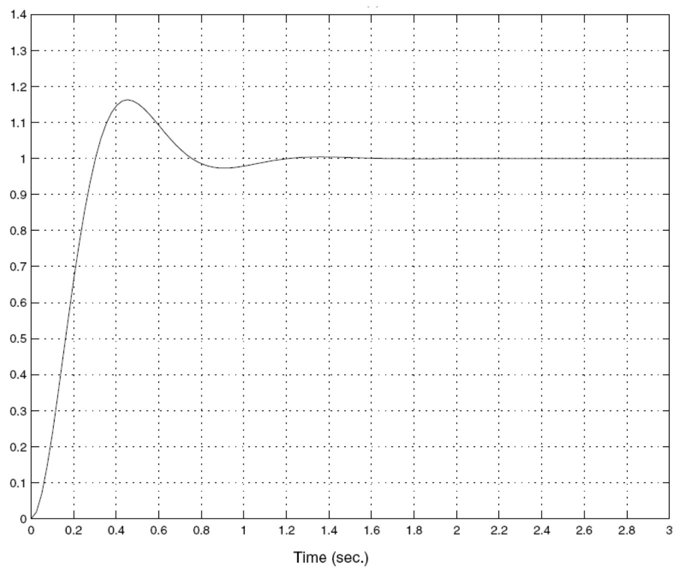
1. $H(s) = \dfrac{10}{s^2 + 1.6s + 4}$
2. $H(s) = \dfrac{20}{s^2 + 6s + 25}$

For each of them:
- evaluate the natural frequency ω_n and the damping coefficient ζ
- define H(s) in the MatLab environment and use the statement `step(H)` (see the online help for more details) in order to plot the unit step output response and, on the basis of the obtained graphical behaviour, evaluate:
  - Steady state value y_∞
  - Maximum overshoot $\hat{s}$ and peak time $\hat{t}$
  - Rise time t_r
  - 5% settling time $t_{s,5\%}$

(Answer: 1. $\omega_n = 2, \zeta = 0.4, y_\infty = 2.5, \hat{s} = 25.38\%, \hat{t} \simeq 1.715\,s, t_r \simeq 1.08\,s, t_{s,5\%} \simeq 3.8\,s$
2. $\omega_n = 5, \zeta = 0.6, y_\infty = 0.8, \hat{s} = 9.48\%, \hat{t} \simeq 0.785\,s, t_r \simeq 0.55\,s, t_{s,5\%} \simeq 1.046\,s$)

Problem 7 (step response of 2nd order systems)

The (zero state) output response in the presence of a step of amplitude 5 of an LTI system is reported in the picture below:



Determine parameters K, ω_n and ζ of a second order transfer function $H(s)$ of the forml:

$$H(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

whose step response matches the given time course.

(Answer: $K = 0.2, \omega_n = 8, \zeta = 0.5 \rightarrow H(s) = \dfrac{12.8}{s^2 + 8s + 64}$)

Problem 8 (steady state properties of LTI systems)
Given the LTI system is described by the following transfer function:

$$H(s) = \frac{1}{s^3 + 2s^2 + 5.25s + 4.25}$$

1. Compute, if possible, the steady state output response $y_{ss}(t)$ in the presence of the following input

$$u(t) = \left(3\sin(0.1t) + 2\right)\varepsilon(t)$$

2. Compute, if possible, the maximum amplitude of a sinusoidal input of the form

$$u(t) = A_u \sin(3t)\varepsilon(t)$$

   so that, at steady state, the maximum output amplitude satisfies $|\, y_{ss}(t) \,| < 1$.

(Answer: 1. $y_{ss}(t) = (0.7038 \sin(0.1t - 0.1232) + 0.4706)\,\varepsilon(t)$ 2. $|A_u| \le 17.7658$)

Problem 9 (Conceptual question on steady state properties of LTI dynamical systems)

Given a strictly proper, asymptotically stable and minimal LTI SISO dynamical system described by the transfer function:

$$H(s) = \frac{N_H(s)}{D_H(s)} = \frac{N_H(s)}{(s-p_1)^{\mu_1}(s-p_2)^{\mu_2}\cdots(s-p_h)^{\mu_h}}$$

For such a system, consider the zero state output response $y(t)$ in the presence of an input $u(t)$ with Laplace transform of the form:

$$U(s) = \frac{N_U(s)}{D_U(s)} = \frac{N_U(s)}{(s-q_1)^{m_1}(s-q_2)^{m_2}\ldots(s-q_{n_u})^{m_u}}$$

Suppose that the polynomials $D_H(s)$ and $D_U(s)$ have not common roots.
Show that the output response $y(t)$ is the sum of two contributions $y_{tr}(t)$ and $y_{ss}(t)$, such that:

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

with the following properties:

$y_{tr}(t) \to 0$, as $t \to \infty$
$y(t) \to y_{ss}(t)$, as $t \to \infty$

Discuss also the case when polynomials $D_H(s)$ and $D_U(s)$ have common roots.