# 工科数分习题课十四　定积分的应用

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Dec.28.2012

**本节课的内容和要求**

1.掌握用定积分的思想求平面图形的面积、旋转曲面的面积、旋转体的体积、曲线的弧长.

**基本概念和主要结论**

1)求平面图形的面积

$\circ$ 直角坐标　$y=f(x)$ (或 $x=g(y)$).

$$S=\int_a^b[f_2(x)-f_1(x)]\mathrm{d}x.\quad \left(\text{或 } S=\int_c^d[g_2(y)-g_1(y)]\mathrm{d}y.\right)$$

$\circ$ 参数方程　$x=x(t),y=y(t),t\in[\alpha,\beta]$.

$$S=\int_\alpha^\beta|y(t)x'(t)|\mathrm{d}t.\quad \left(\text{或 } S=\int_\alpha^\beta|x(t)y'(t)|\mathrm{d}t.\right)$$

若曲线封闭,则$S=\left|\int_\alpha^\beta y(t)x'(t)\mathrm{d}t\right|$ $\left(\text{或 } S=\left|\int_\alpha^\beta x(t)y'(t)\mathrm{d}t\right|.\right)$

$\circ$ 极坐标　$r=r(\theta),\theta\in[\alpha,\beta]$.

$$S=\frac{1}{2}\int_\alpha^\beta r^2(\theta)\mathrm{d}\theta.$$

2)由平行截面面积求体积

$$V=\int_a^b A(x)\mathrm{d}x.$$

特别地, 旋转体 $\Omega$ : $f(x)$绕$x$轴

$$V=\pi\int_a^b[f(x)]^2\mathrm{d}x.$$

3)求平面曲线的弧长

$\circ$ 直角坐标　$y=f(x)$.

$$s=\int_a^b\sqrt{1+[f'(x)]^2}\,\mathrm{d}x.$$

$\circ$ 参数方程　$x=x(t),y=y(t),t\in[\alpha,\beta]$.

$$s=\int_\alpha^\beta\sqrt{[x'(t)]^2+[y'(t)]^2}\,\mathrm{d}t.$$

○ 极坐标　$r=r(\theta),\theta\in[\alpha,\beta]$.

$$s = \int_\alpha^\beta \sqrt{[r(\theta)]^2+[r'(\theta)]^2}\mathrm{d}\theta.$$

4)求旋转曲面的面积

○ 直角坐标　$y=f(x)$绕$x$轴.

$$S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\mathrm{d}x.$$

○ 参数方程　$x=x(t),y=y(t),t\in[\alpha,\beta]$绕$x$轴.

$$S=2\pi\int_\alpha^\beta y(t)\sqrt{[x'(t)]^2+[y'(t)]^2}\mathrm{d}t.$$

○ 极坐标方程　$r=r(\theta),\theta\in[\alpha,\beta]$绕极轴.

$$S=2\pi\int_\alpha^\beta r(\theta)\sin\theta\sqrt{r^2(\theta)+[r'(\theta)]^2}\mathrm{d}\theta.$$

习题

1.证明曲边梯形$0 \leqslant y \leqslant f(x),\ a \leqslant x \leqslant b$绕$y$轴所得立体的体积公式为

$$V = 2\pi \int_a^b x f(x) \mathrm{d}x.$$

2.星形线(Astroid)

$$x = a\cos^3 t, y = a\sin^3 t, a > 0, t \in [0, 2\pi).$$

求: (1)弧长,

(2)曲线所围成图形的面积,

(3)曲线绕$x$轴旋转所得图形的体积,

(4)曲线绕$x$轴旋转所得图形的曲面面积.

$(1) 6a; (2) \dfrac{3}{8}\pi a^2; (3) \dfrac{32}{105}\pi a^3; (4) \dfrac{12}{5}\pi a^2.$