Chapter 2

Mathematical Modeling of Control Systems

2-1 Introduction

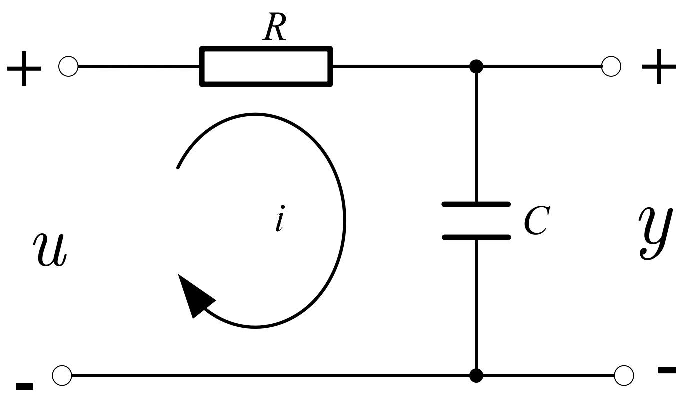
To understand and control complex systems, one must obtain quantitative mathematical models of these systems.

A mathematical model is defined as a set of equations that represents the dynamics of the system accurately, or at least fairly well.

The dynamics of many systems, whether they are mechanical, electrical, thermal, and so on, may be described in terms of differential equations—using physical laws governing a particular system such as **Newton's** laws and **Krichhoff's** laws.

1. Mathematical models

Example. RC Network



Mathematical model: by using the relationship

$$i = C\frac{dy}{dt}$$

we have

$$RC\frac{dy}{dt} + y = u$$

which is a first-order differential equation.

Mathematical models may include many different forms. For example:

1. Systems described by linear differential equations;
2. Systems described by nonlinear differential equations;
3. Systems described by partial differential equations;
4. ....

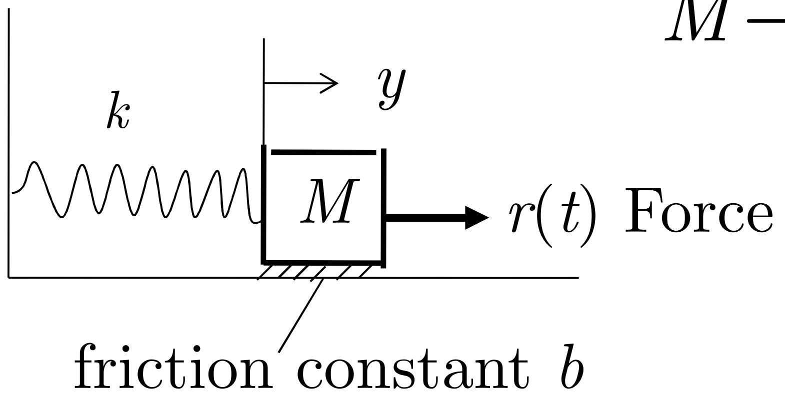
Which model is better depends on the specific system and the specific circumstances.

2. Linear systems

Example. Spring-mass-damper mechanical system. Such a system could represent an automobile shock absorber. We assume the friction force is proportional to the velocity of the mass and

$$v(t) = \frac{dy(t)}{dt}$$

$$M\frac{d^2y(t)}{dt^2} = r(t) - b\frac{dy(t)}{dt} - ky(t)$$



Rearranging the above differential equation yields

$$M\frac{d^2y(t)}{dt^2}+b\frac{dy(t)}{dt}+ky(t)=r(t)$$

It is clear that both sides are the linear functions of either y, y' ,y'' or r. Such a system is called a **linear system**.

In classical control theory, a linear system can be generally described by the following differential equation

$$\begin{aligned}&a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}\dot{y}+a_ny\\&=b_0x^{(m)}+b_1x^{(m-1)}+\cdots+b_{m-1}\dot{x}+b_mx,\quad m\leq n\end{aligned}$$

Then we say the system is an nth order linear system.

3. Linear time invariant systems and linear time varying systems

Consider the following linear differential equation

$$
\begin{aligned}
& a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y \\
& = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_m x, \quad m \le n
\end{aligned}
$$

1) The system is said to be **linear time invariant** if all its coefficients are constants;

2) The system is said to be **linear time varying** if at least one of its coefficients is the function of time.

Example. Consider the following two systems:

$$y^{(3)} + 2y'' + 3y' + 4y = r(t)$$

and

$$y^{(3)} + \sin(t)y'' + t^2 y' + 4ty = r(t)$$

By the above definition with respect to LTI and LTV systems, it is obvious that the latter is an LTV system.

2-2 Transfer function and impulse-response function

1.1 Laplace Transformation

Given a real function $f(t)$ with $f(t)=0$ for $t<0$. Then the **Laplace** transform of $f(t)$ is defined as

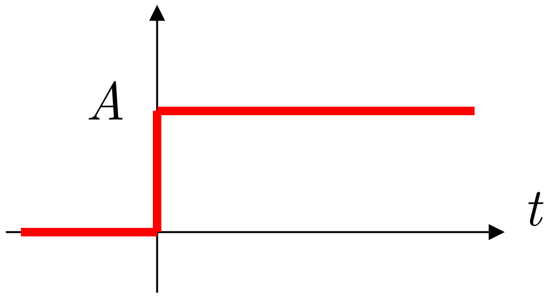
$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

where s is a complex variable.

Example. Let $f(t)=e^{-\alpha t}$, $t\geq 0$. Find $\mathcal{L}[f(t)]$.

Example. Step function

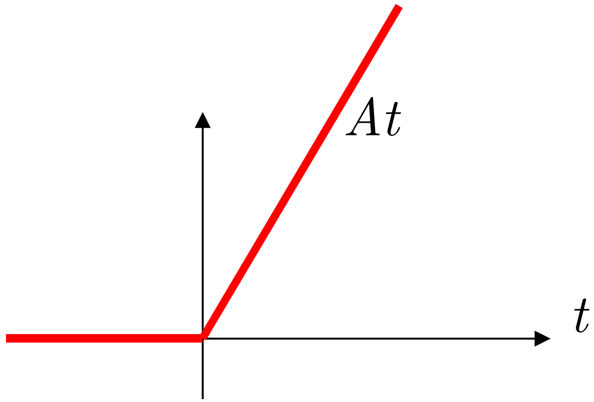
$$f(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Find $\mathcal{L}[f(t)]$.

Example. Ramp function

$$f(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Find $\mathcal{L}[f(t)]$.

Example. Sinusoidal function

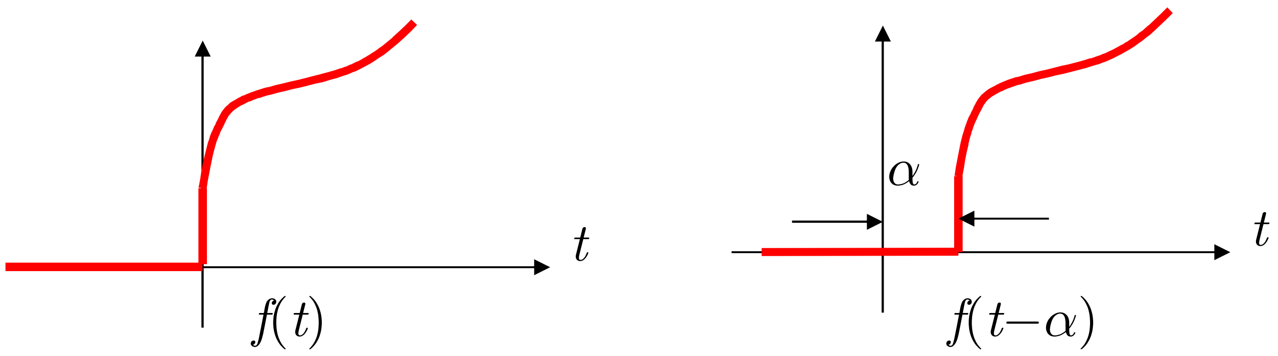
$$f(t)=\begin{cases} A\sin\omega t,\ t\geq 0 \\ 0,\ t<0 \end{cases}$$

Find $\mathcal{L}[f(t)]$.

Similarly, for $\mathcal{L}[A\cos\omega t]$

Find $\mathcal{L}[f(t)]$.

Example. Translated function $f(t-\alpha)$:

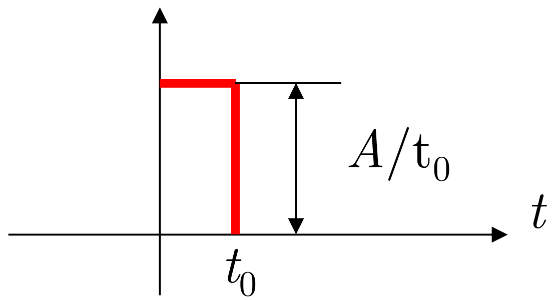


Let $\mathcal{L}[f(t)]=F(s)$. Then

$$\mathcal{L}[f(t-\alpha)]=e^{-\alpha s}F(s),\ \text{for}\ \alpha\geq 0.$$

Example. Pulse function

$$f(t)=\begin{cases}\dfrac{A}{t_0}, 0<t<t_0\\ 0, t<0, t>t_0\end{cases}$$

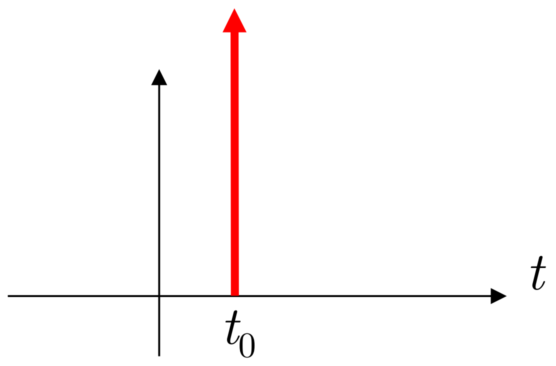


f(t) can be expressed as

$$f(t)=\frac{A}{t_0}1(t)-\frac{A}{t_0}1(t-t_0)$$

$$\mathcal{L}[f(t)]=\frac{A}{t_0 s}-\frac{A}{t_0 s}e^{-st_0}$$

Example. Impulse function



$$\delta(t-t_0)=\begin{cases}\infty, & t=t_0\\ 0, & t\neq t_0\end{cases}$$

$$\int_{t_0-}^{t_0+}\delta(t-t_0)dt=1$$

In particular,

$$\int_{0-}^{0+}\delta(t)dt=1$$

We have

$$\mathcal{L}[\delta(t-t_0)]=e^{-st_0}$$

and

$$\mathcal{L}[\delta(t)]=1$$

1.2. Inverse Laplace transformation

Given the **Laplace** transform $F(s)$, the operation of obtaining $f(t)$ is called the **inverse Laplace transformation**, and is denoted by

$$f(t)=\mathcal{L}^{-1}[F(s)]$$

The inverse **Laplace** transform integral is given as

$$f(t)=\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$$

where c is a real constant that is greater than the real parts of all the singularities of $F(s)$.

A simpler method for finding inverse Laplace transforms is based on the fact that the **unique** correspondence of a time function and its inverse Laplace transform holds for any continuous time function.

Example. Given $F(\mathrm{s})=1/(s+1)$. Find $f(t)$.

Example. Given $F(\mathrm{s})=1/s$. Find $f(t)$.

1.3. Laplace Transform Theorems

Theorem 1. Let k be a constant, and $F(s)$ be the **Laplace** transform of $f(t)$. Then

$$\mathcal{L}[kf(t)] = kF(s)$$

Theorem 2. Let $F_1(s)$ and $F_2(s)$ be the Laplace transform of $f_1(t)$ and $f_2(t)$, respectively. Then

$$\mathcal{L}[\alpha_1 f_1(t) + \alpha_2 f_2(t)] = \alpha_1 F_1(s) + \alpha_2 F_2(s), \forall \alpha_1, \alpha_2 \in R$$

Theorem 3. (**Differentiation Theorem**)

Let $F(s)$ be the **Laplace** transform of $f(t)$, and $f(0)$ is the limit of $f(t)$ as t approaches 0. The **Laplace** transform of the time derivative of $f(t)$ is

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = sF(s) - \lim_{t\to 0} f(t) = sF(s) - f(0)$$

In general, for higher-order derivatives of $f(t)$,

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \lim_{t\to 0}\left[s^{n-1} f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots + \frac{d^{n-1} f(t)}{dt^{n-1}}\right]$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

Theorem 4. (**Integration Theorem**)
The **Laplace** transform of the first integral of $f(t)$ is the **Laplace** transform of $f(t)$ divided by s; that is,

$$\mathcal{L}\left[\int_o^t f(\tau)d\tau\right]=\frac{F(s)}{s}$$

For n-lay integration,

$$\mathcal{L}\left(\underbrace{\int_0^t\cdots\int_0^t f(u)(du)^n}_{n}\right)=\frac{F(s)}{s^n}$$

Theorem 5. (**Complex Shifting Theorem**)

The **Laplace** transform of $f(t)$ multiplied by $e^{\mp\alpha t}$, where α is a constant, is equal to the *Laplace* transform $F(s)$ with s replaced by $s\pm\alpha$; that is,

$$\mathcal{L}[e^{\mp\alpha t}f(t)] = F(s\pm\alpha)$$

Theorem 6. (**Real Convolution Theorem**)

Let $F_1(s)$ and $F_2(s)$ be the **Laplace** transform of $f_1(t)$ and $f_2(t)$, respectively, and $f_1(t)=0$, $f_2(t)=0$, for $t<0$, then

$$F_1(s)F_2(s) = \mathcal{L}[f_1(t)*f_2(t)] = \mathcal{L}\left[\int_0^t f_1(\tau)f_2(t-\tau)d\tau\right]$$

1.4. Inverse Laplace transformation: Partial-fraction expansion

Consider the function

$$F(s) = \frac{B(s)}{A(s)}$$

where $A(s)$ and $B(s)$ are real polynomials of s. It is assumed that $\deg(A(s)) \geq \deg(B(s))$. Write $A(s)$ as

$$A(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$

whose roots are called **poles**; write

$$B(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m$$

whose roots are called **zeros**.

- **Partial-fraction expansion when $F(s)$ involves distinct poles only.** In this case, $F(s)$ can be written as

$$F(s) = \frac{B(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots \frac{a_n}{s+p_n}$$

where

$$a_i = \left[(s+p_i)F(s)\right]\Big|_{s=-p_i}$$

$$f(t) = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t}$$

Example.

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

Example.

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

$$= \frac{10+2(s+1)}{(s+1)^2+2^2}$$

$$= \frac{10}{(s+1)^2+2^2} + \frac{2(s+1)}{(s+1)^2+2^2}$$

$$= 5\frac{2}{(s+1)^2+2^2} + 2\frac{(s+1)}{(s+1)^2+2^2}$$

Therefore,

$$f(t) = 5e^{-t}\sin 2t + 2e^{-t}\cos 2t, \quad t \geq 0$$

- **Partial-fraction expansion when $F(s)$ involves Multiple poles.** If r of the n roots of $A(s)$ are identical, $G(s)$ is written

$$G(s) = \frac{B(s)}{(s+p_1)(s+p_2)\cdots(s+p_{n-r})(s+p_i)^r}, \quad (i \neq 1, 2, \cdots, n-r)$$

then $G(s)$ can be expanded as

$$G(s) = \underbrace{\frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \quad \cdots \quad + \frac{a_{n-r}}{s+p_{n-r}}}_{|\leftarrow\, n-r \text{ terms of simple roots } \rightarrow|}$$

$$\underbrace{+ \frac{A_1}{s+p_i} + \frac{A_2}{(s+p_i)^2} + \cdots + \frac{A_r}{(s+p_i)^r}}_{|\leftarrow\, r \text{ terms of repeated roots } \rightarrow|}$$

where

$$a_j = \left[(s + p_j) G(s) \right] \Big|_{s=-p_j} \qquad (j = 1, 2, \cdots, n - r)$$

The determination of the coefficients that correspond to the multiple-poles is described as follows.

$$\begin{cases} A_r = \left[(s + p_i)^r G(s) \right] \Big|_{s=-p_i} \\ A_{r-k} = \dfrac{1}{k!} \dfrac{d^k}{ds^k} \left[(s + p_i)^r G(s) \right] \Big|_{s=-p_i} \qquad (k = 1, 2, \cdots r - 1) \end{cases}$$

Example.

$$G(s) = \frac{1}{s(s+1)^3(s+2)}$$

$G(s)$ can be written as

$$G(s) = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3}$$

The coefficients corresponding to the simple roots and those of the third-order root are

$$\begin{cases} a_1 = \left[sG(s) \right] \Big|_{s=0} = \dfrac{1}{2} \\ a_2 = \left[(s+2)G(s) \right] \Big|_{s=-2} = \dfrac{1}{2} \end{cases} \qquad \begin{cases} A_3 = \left[(s+1)^3 G(s) \right] \Big|_{s=-1} = -1 \\ A_2 = \dfrac{d}{ds}\left[(s+1)^3 G(s) \right] \Big|_{s=-1} = 0 \\ A_1 = \dfrac{1}{2}\dfrac{d^2}{ds^2}\left[(s+1)^3 G(s) \right] \Big|_{s=-1} = -1 \end{cases}$$

1.5. Solving LTI differential equations

- Transform the differential equation to the s-domain by **Laplace** transform.
- Manipulate the transformed algebraic equation and solve for the output variable.
- Perform partial-fraction expansion to the transformed algebraic equation.
- Obtain the inverse **Laplace** transform.

Example.

$$\ddot{y} + 3\dot{y} + 2y = 5u(t)$$

where

$$u(t) = 1(t), y(0) = -1, \dot{y}(0) = 2$$

Taking the Laplace transform on both sides:

$$s^2Y(s) - sy(0) - \dot{y}(0) + 3sY(s) - 3y(0) + 2Y(s) = 5/s$$

Substituting the values of the initial conditions into the last equation. Then solving for $Y(s)$ and expanding by partial-fraction, we get

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{5}{2s} - \frac{5}{s+1} + \frac{3}{2(s+2)}$$

Taking the inverse Laplace transform, we obtain

$$y(t) = 2.5 - 5e^{-t} + 1.5e^{-2t} \qquad t \geq 0$$

Example. Solve the following differential equation:

$$y^{(3)} + 3y'' + 3y' + y = 1, \quad y(0) = y'(0) = y''(0) = 0$$

Solution:

$$F(s) = \frac{1}{s(s+1)^3} = \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3} + \frac{c_4}{s}$$

$$b_3 = [(s+1)^3 \frac{1}{s(s+1)^3}]_{s=-1} = -1$$

$$b_2 = \left\{ \frac{d}{ds} [\frac{1}{s(s+1)^3} (s+1)^3] \right\}_{s=-1} = [\frac{d}{ds} (\frac{1}{s})]_{s=-1}$$

$$= (-s^{-2})\Big|_{s=-1} = -1$$

$$b_1 = \frac{1}{2!}(2s^{-3})\Big|_{s=-1} = -1$$

$$c = \frac{1}{s(s+1)^3}s\Big|_{s=0} = 1$$

$$F(s) = \frac{1}{s} + \frac{-1}{(s+1)^3} + \frac{-1}{(s+1)^2} + \frac{-1}{s+1}$$

$$y = 1 - \frac{1}{2}t^2e^{-t} - te^{-t} - e^{-t}, \quad t \geq 0$$