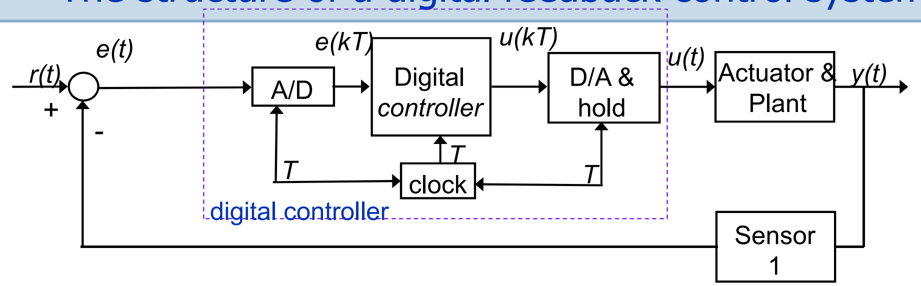
Automatic Control

Introduction to digital control

- **Analysis of digital control systems**

The structure of a digital feedback control system

The structure of a digital feedback control system



In order to study and design digital feedback control systems we need to describe the dynamic characteristics of

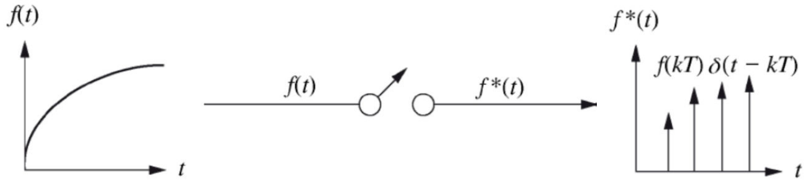
- A/D conversion
- controller
- D/A conversion

For simplicity, discrete time signals (i.e. discrete in time but continuous in amplitude) will be considered → quantization is performed with a number of levels so that the quantization error is negligible

A/D conversion

A/D conversion

Basically, A/D conversion can be described through an ideal sampler, made up by a switch that closes every T seconds for one instant of time:



f(t) is the signal to be sampled, f*(t) is the sampled signal,

T is the **sampling period** (s),

$\omega_s = 2\pi/T$ is the **sampling angular frequency** (rad/s)

A/D conversion

In order to understand the aliasing effect, we refer to the **spectrum** (i.e. the Fourier transform) $F(j\omega)$ of the signal to be sampled f(t)

$$F(j\omega) = \mathcal{F}\{f(t)\} \triangleq \int_0^{+\infty} f(t)e^{-j\omega t}dt$$

For a real valued signal f(t), we have

$$F(-j\omega) = -\bar{F}(j\omega)$$

Amplitude and phase spectrum are defined as:

$$|F(j\omega)| \rightarrow \textbf{amplitude spectrum}$$

$$\arg(F(j\omega)) \rightarrow \textbf{phase spectrum}$$

A/D conversion

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{F(j\omega)\} = f(t) = \frac{1}{2\pi}\int_0^{+\infty} F(j\omega)e^{j\omega t}d\omega =$$

$$= \frac{1}{\pi}\int_0^{+\infty} |F(j\omega)|\cos(\omega t + \arg(F(j\omega)))d\omega$$

Thus

- f(t) can be expressed as an infinite sum of harmonic components with angular frequency ω
- each harmonic component of f(t) is weighted by $|F(j\omega)|/(\pi\, d\omega)$
- Let's define

$$\omega_- = \min \omega : |F(j\omega)| \neq 0, \omega_0 = \max \omega : |F(j\omega)| \neq 0$$

the signal bandwidth is defined as

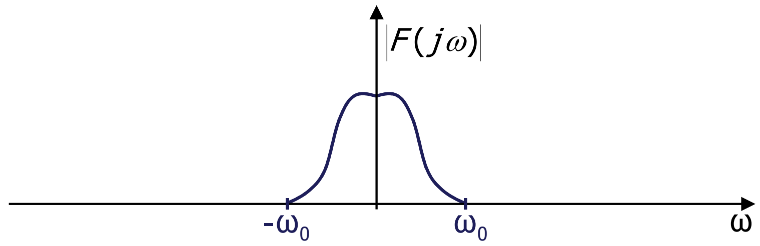
$$\omega_b = \omega_0 - \omega_-$$

if $\omega_0 < \infty \rightarrow$ **band limited signal**

A/D conversion

Then, a signal is said to be bandlimited if its amplitude spectrum goes to zero for all frequencies beyond some threshold called the cutoff frequency ω_0

In this situation, the amplitude spectrum can be represented as*



* The pictures of this part are taken from C. Greco, M. Indri, Controlli Automatici, Politecnico di Torino - CELM (2007)

A/D conversion

Now, let's study the characteristics of the spectrum of the sampled signal f*(t) on the basis of the spectrum of f(t).

The sampled signal f*(t) is given by:

$$f^*(t) = f(t)\sum_{k=0}^{\infty}\delta(t-kT) = f(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

Since $\sum_{k=-\infty}^{\infty}\delta(t-kT)$ is periodic, it can be expressed through the Fourier series:

$$\sum_{k=-\infty}^{\infty}\delta(t-kT) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_s}, \; C_k = \frac{1}{T}\int_{-T/2}^{T/2}\sum_{k=-\infty}^{\infty}\delta(t-kT)e^{jk\omega_s}dt = \frac{1}{T}$$

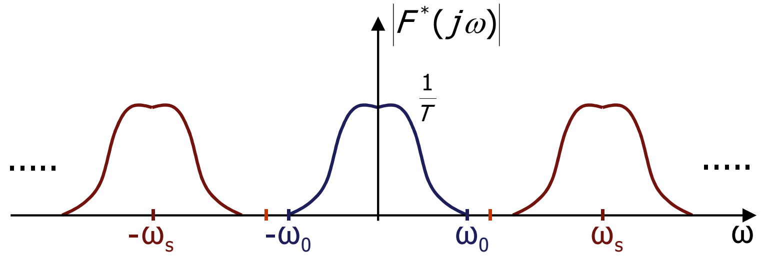
Then:

$$f^*(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} f(t)e^{jk\omega_s t} \rightarrow \mathcal{F}\left\{f^*(t)\right\} = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega-\omega_s))$$

A/D conversion

$$F^*(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F(j(\omega-k\omega_s))$$

Hence, the effect of ideal sampling is to replicate the original **central band spectrum** centered at 0 with **sideband spectra** centered at $\pm\omega_s$, at $\pm 2\omega_s$, …

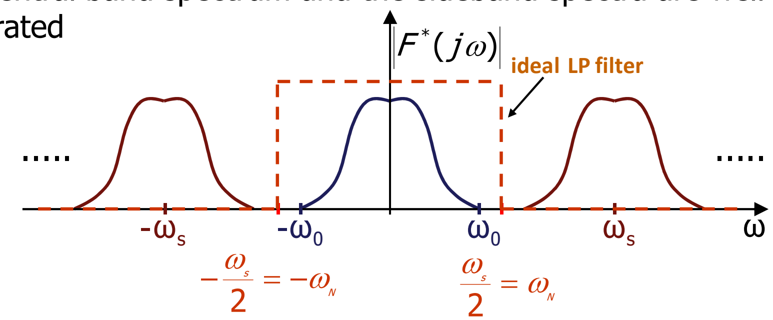


A/D conversion

If the following relation is satisfied (Shannon's sampling theorem):

$$\omega_s > 2\omega_0 \rightarrow \omega_0 < \frac{\omega_s}{2} = \omega_N \rightarrow \text{Nyquist frequency}$$

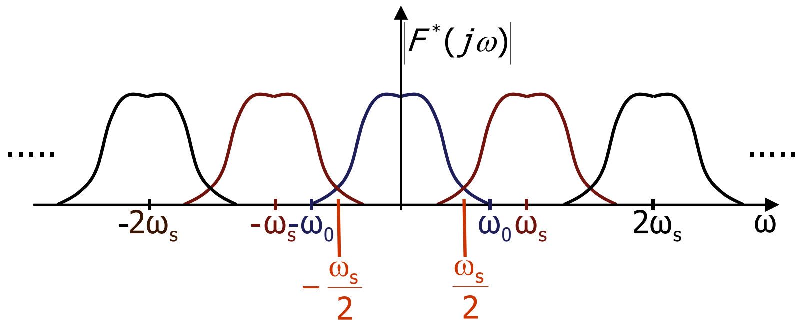
the central band spectrum and the sideband spectra are well separated



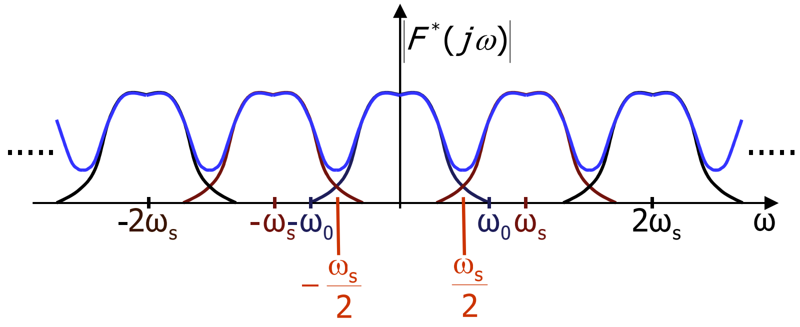
the sampled signal f(t) can be reconstructed through ideal low pass filtering (further details are provided later).

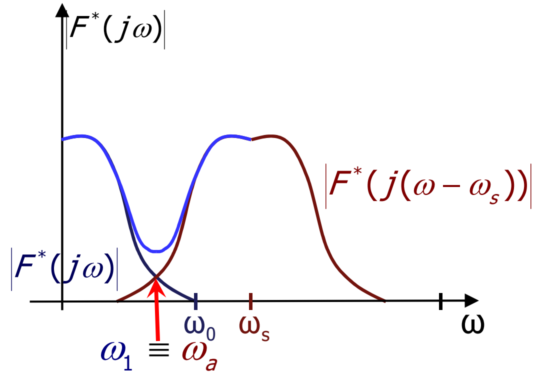
A/D conversion

If the hypotheses of the Shannon's sampling theorem are not satisfied, spectrum replications of f* overlap:



The resulting spectrum is the sum of the central band and the sideband spectra:

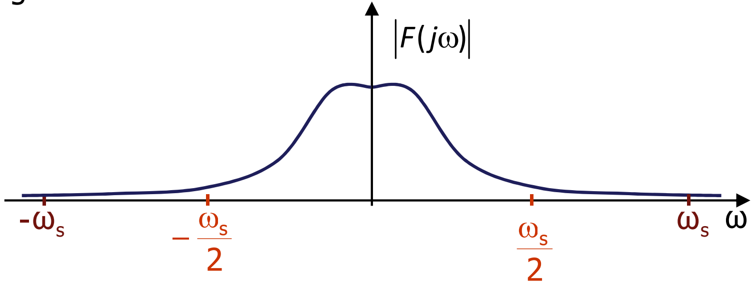




It happens that, at frequency ω_1, both the central band $|F^*(j\omega_1)|$, and the side band $|F^*(j(\omega_s-\omega_1))|$ contributions affect the computation of the spectrum of f*

In this case, the frequency $\omega_a = |\omega_s-\omega_1|$ of the side band spectrum is referred to as an **alias** of ω_1

Real signals are not band limited

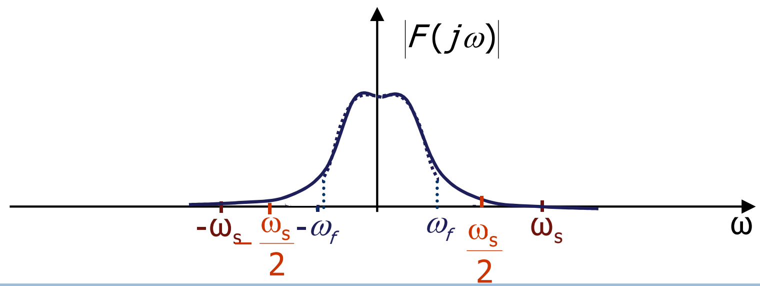


In such cases, spectra overlapping and aliasing effects can not be avoided.

Aliasing effect can be prevented by low pass filtering the signal to be sampled in order to remove its high frequency components

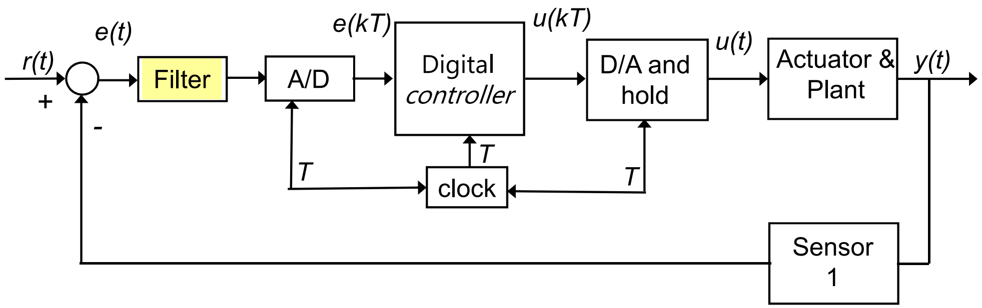
Low pass filters, such as Butterworth, Bessel, Chebychev, with cut-off frequency ω_f such that $\omega_f < \omega_s/2$ can be suitably employed to limit the bandwidth of the signal to be sampled

In this case the employed filter is referred to as the **anti-aliasing filter**



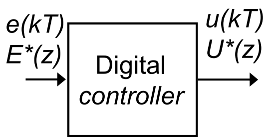
The structure of a digital feedback control system

Sampled data feedback control system with anti-aliasing filter



Digital controller

The digital controller



The digital controller is realized through a proper discrete time transfer function in the $\mathcal{Z}$-transform domain:

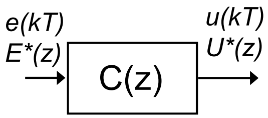
$$C(z) = \frac{b_n + b_{n-1}z^{-1} + \ldots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0 z^{-n}} =$$

$$= \frac{b_n z^n + b_{n-1}z^{n-1} + \ldots + b_1 z + b_0}{z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_0}$$

$$U^*(z) = C(z)E^*(z) =$$

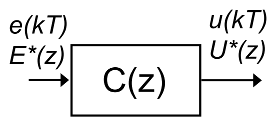
$$= \frac{b_n + b_{n-1}z^{-1} + \ldots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0 z^{-n}} E^*(z)$$

$\rightarrow$



The digital controller

Given the controller transfer function:



$$C(z) = \frac{b_n + b_{n-1}z^{-1} + \ldots + b_0 z^{-n}}{1 + a_{n-1}z^{-1} + a_{n-2}z^{-2} + \ldots + a_0 z^{-n}}$$

The control input u(k) at time kT is computed through the difference equation:

$$u(k) = -a_{n-1}u(k-1) - a_{n-2}u(k-2) - \ldots - a_0 u(k-n) +$$
$$+ b_n e(k) + b_{n-1}e(k-1) + \ldots + b_0 e(k-n)$$

The digital controller

$$u(k) = -a_{n-1}u(k-1) - a_{n-2}u(k-2) - \ldots - a_0 u(k-n) + \\ + b_n e(k) + b_{n-1}e(k-1) + \ldots + b_0 e(k-n)$$

The difference equation above can be easily implemented through a simple computer program (e.g. using C-code*):

- **n** ↔ n
- **va** ↔ $[a_{n-1} \ldots a_0]$
- **vb** ↔ $[b_{n-1} \ldots b_0]$
- **bn** ↔ b_n
- **vu** ↔ [u(k-1) … u(k-n)]
- **ve** ↔ [e(k-1) … e(k-n)]
- **ei** ↔ e(k)
- **ui** ↔ u(k)

```
ui=0;
for (j=1;j<n+1;j++) {
    ui=ui-va[j]*vu[j]+vb[j]*ve[j];
}
ui=ui+bn*ei;
for (j=1;j<n-2;j++) {
    vu[j+1]=vu[j];
    ve[j+1]=ve[j];
}
vu[1]=ui;
ve[1]=ei;
```

* see C. Greco, M. Indri, Controlli Automatici, Politecnico di Torino - CELM (2007)

D/A conversion

D/A conversion

In principle, an ideal low-pass filter is able to reconstruct exactly the original signal f(t) from the sampled signal f*(t) if the bandwidth of the filter is $\omega_s/2$ and the highest frequency component of f(t), ω_0 is less than $\omega_s/2$

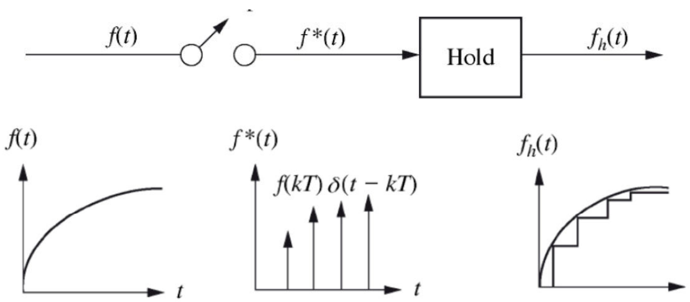
However, since ideal filters can not be realized by means of physically realizable systems, they can be approximated using data hold devices

Data holds are data reconstruction devices that approximate, in some sense, an ideal low-pass filter

D/A conversion

The simplest and most widely employed data reconstruction device is the **zero-order hold (ZOH) filter**

The ZOH clamps the output to a value equal to the input at the current sampling instant and keeps it until the next sampling:



D/A conversion

It is possible to derive the relationship between the sampled signal $f^*(t)$ and the ZOH output $f_h(t)$ in terms of Laplace transform

Consider as input of the ZOH is a unit impulse $\delta(t)$, the corresponding output is a rectangular impulse given by

$$f_h(t) = \varepsilon(t) - \varepsilon(t-T)$$

Thus:

$$F_h(s) = \mathcal{L}\{\varepsilon(t) - \varepsilon(t-T)\} = \frac{1-e^{-Ts}}{s}$$

The ZOH transfer function is then given by:

$$G_{ZOH}(s) = \frac{1-e^{-Ts}}{s}$$

D/A conversion

Let us analyze the frequency response of the ZOH filter:

$$G_{ZOH}(s) = \frac{1-e^{-Ts}}{s}$$

We have:

$$G_{ZOH}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} \underset{\substack{\text{collect } e^{-j\omega T/2} \\ \text{multiply num and den by } 2T}}{\overset{=}{\uparrow}} e^{-j\omega T/2}\frac{e^{j\omega T/2}-e^{-j\omega T/2}}{2j}\frac{2T}{\omega T} =$$

$$= T\frac{\sin(\omega T/2)}{\omega T/2}e^{-j\omega T/2}$$

D/A conversion

$$G_{ZOH}(j\omega) = \frac{1-e^{-j\omega T}}{j\omega} = T\frac{\sin(\omega T/2)}{\omega T/2}e^{-j\omega T/2}$$

$$|G_{ZOH}(j\omega)| = T\left|\frac{\sin(\omega T/2)}{\omega T/2}\right|, \angle G_{ZOH}(j\omega) = -\frac{\omega T}{2} + \angle(\sin(\omega T/2))$$

Note that

$$\frac{\omega T}{2} \approx 0 \Rightarrow \frac{\sin(\omega T/2)}{\omega T/2} \approx 1 \rightarrow G_{ZOH}(j\omega) \simeq Te^{-j\omega T/2}, \omega \in [0, \omega_N]$$

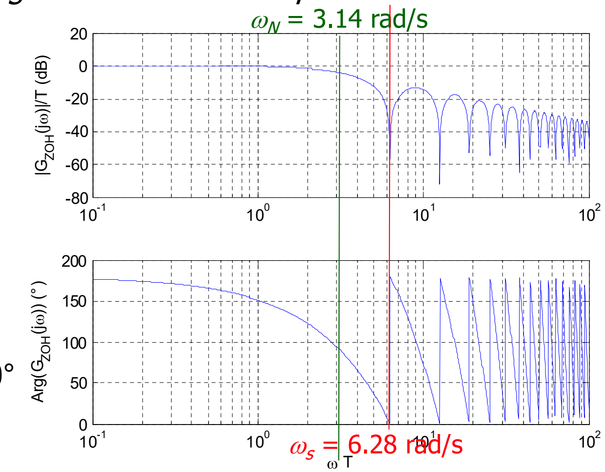
$$\frac{|G_{ZOH}(j\omega_N)|}{T} = \frac{|G_{ZOH}(j\omega_s/2)|}{T} \underset{\omega_s = \frac{2\pi}{T}}{\overset{=}{\uparrow}} \frac{1}{T}\frac{\sin\left(\frac{T}{2}\frac{2\pi}{2T}\right)}{\frac{2\pi}{2T}} = \frac{2}{\pi} \simeq -3 \text{ dB}$$

D/A conversion

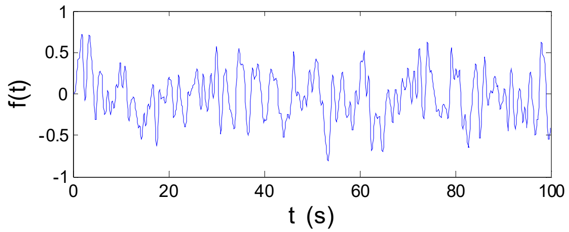
Example: $G_{ZOH}(s) = \frac{1-e^{-s}}{s} \rightarrow T = 1\,s, \omega_s = \frac{2\pi}{T} = 6.28\, rad/s$

The ZOH filter frequency behavior is low pass

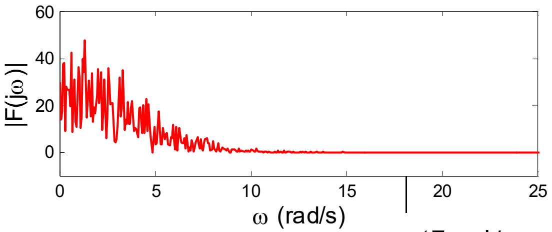
Note the phase lag of $-180°$ at ω_s



D/A conversion



← continuous-time signal

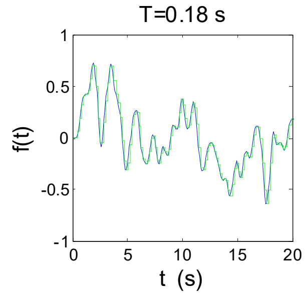


← spectrum

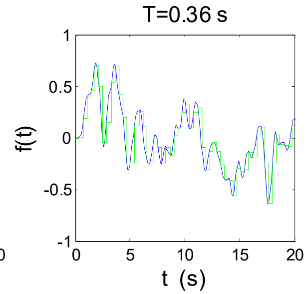
ω_0 = 17 rad/s, $\omega_s \geq 2\,\omega_0$ → $T \leq 2\pi/\omega_s = \pi/\omega_0 = 0.185$ s

D/A conversion

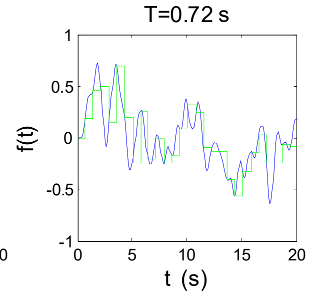
ω_0 =17 rad/s, $\omega_s \geq 2\,\omega_0$ → $T \leq \pi/\omega_0 = 0.185$ s



full information maintained



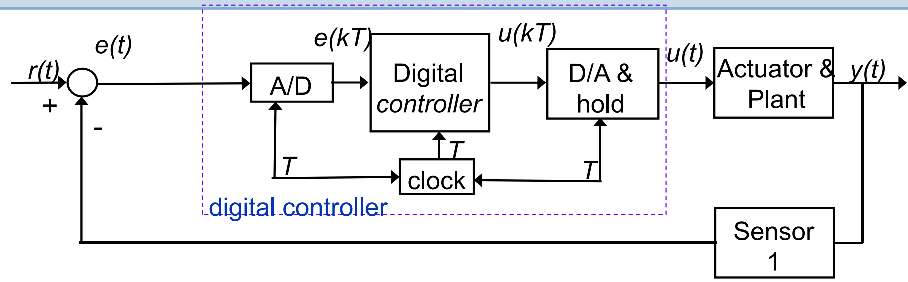
little loss of information



relevant loss of information

Analysis of a digital control system

Analysis of a digital control system

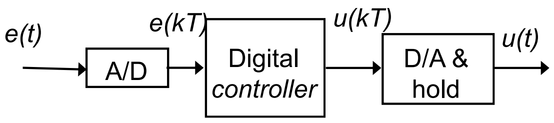


Sampled data control systems have been introduced using the following assumptions:

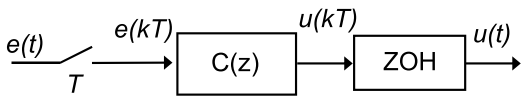
- A/D conversion → ideal impulsive sampler
- Digital controller → z-domain transfer function
- D/A conversion → ZOH filter

Analysis of a digital control system

Thus, the digital controller structure below

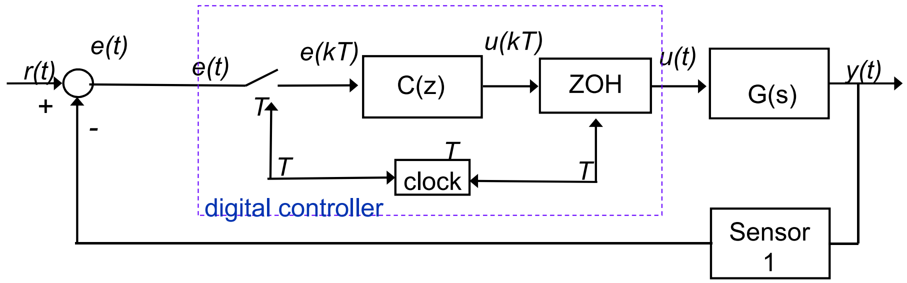


can be represented as:



Analysis of a digital control system

Then, the following sampled data control system structure can be considered:



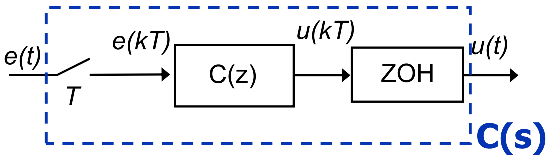
Note the presence of both continuous time and discrete time signals and dynamic systems (→hybrid system)

Analysis of a digital control system

In order to analyze the performance of the considered digital control system, there are different approaches:

1. Convert the system $G(s)$ to its discrete-time equivalent $G(z)$ ignoring its inter-sampling behavior → discrete-time analysis

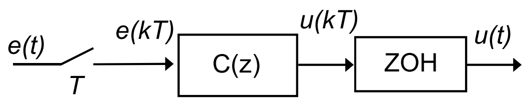
2. Model the digital controller in continuous time → continuous-time analysis



3. Use numerical simulation (e.g., Simulink) (only provides an answer for the specific case considered in the simulation)

Analysis of a digital control system

In order to obtain a continuous time representation of the digital controller

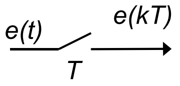


we recall the already introduced properties of each block:

- A/D conversion → ideal impulsive sampler
- Digital controller → z-domain transfer function
- D/A conversion → ZOH filter

Analysis of a digital control system

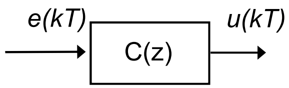
- A/D conversion



$$E^*(s) = E(z)\big|_{z=e^{Ts}} = E^*(e^{Ts}) \rightarrow E^*(e^{j\omega T})$$

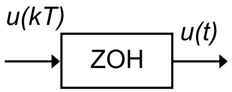
$$E^*(e^{j\omega T}) = \frac{1}{T}E_s(j\omega) = \frac{1}{T}\sum_{h=-\infty}^{\infty} E(j(\omega + h\omega_s))$$

- Digital controller



$$C(z) \rightarrow C(e^{j\omega T}) \rightarrow U^*(e^{j\omega T}) = C(e^{j\omega T})E^*(e^{j\omega T})$$

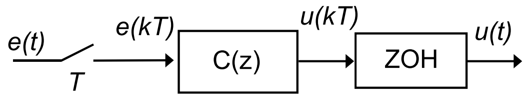
- D/A conversion



$$U(j\omega) = G_{ZOH}(j\omega)U^*(e^{j\omega T})$$

Analysis of a digital control system

We get



$$U(j\omega) = G_{ZOH}(j\omega)C(e^{j\omega T})\frac{1}{T}E_s(j\omega)$$

Supposing e(t) band limited and recalling that ZOH behaves like a low pass filter

$$U(j\omega) \underset{\substack{\uparrow \\ \omega \le \omega_N}}{=} G_{ZOH}(j\omega)C(e^{j\omega T})\frac{1}{T}E(j\omega) \rightarrow U(s) = \underbrace{G_{ZOH}(s)C(e^{sT})\frac{1}{T}}_{\tilde{C}(s)}E(s)$$

Thus, the digital controller is equivalent to an analog controller of the form:

$$\tilde{C}(s) = \underbrace{G_{ZOH}(s)}_{D/A}C(e^{sT})\underbrace{\frac{1}{T}}_{A/D} \rightarrow G_{A/D}(s) = \frac{1}{T}$$