Automatic Control

Analysis of the step response of prototype 1st and 2nd order systems

Effect of additional poles, zeros and delays on the step response of prototype systems

Step response analysis of prototype 1st order systems

Transfer function

Consider a stable first order system described by the transfer function

$$H(s)=\frac{K^*}{s-p}\rightarrow\begin{cases}K^*\rightarrow \text{ gain}\\ p\rightarrow \text{ pole}\end{cases}$$

Let

$$\tau=\left|\frac{1}{p}\right|, K=-\frac{K^*}{p}$$

The transfer fuction can be rewritten in the following form

$$H(s)=\frac{K}{1+\tau s}\rightarrow\begin{cases}K\rightarrow \text{ dc gain}\\ \tau\rightarrow \text{ time constant}\end{cases}$$

Step response of a first order system: analytical form

In the presence of a step input $u(t)$ with amplitude $\bar{u}$:

$$u(t)=\bar{u}\varepsilon(t)\xrightarrow{\mathcal{L}}U(s)=\frac{\bar{u}}{s}$$

the output response of the first order system

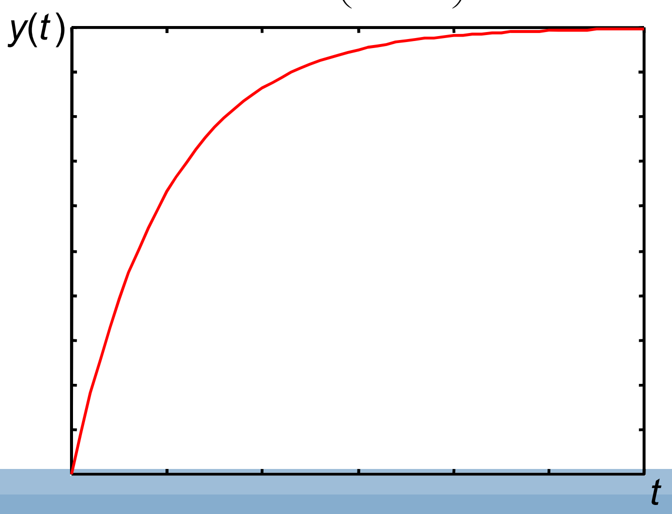
$$H(s)=\frac{K}{1+\tau s}$$

can be computed as:

$$Y(s)=H(s)U(s)=\frac{K}{1+\tau s}\frac{\bar{u}}{s}\xrightarrow{\mathcal{L}^{-1}}y(t)=\bar{u}K\left(1-e^{-\frac{t}{\tau}}\right), t\geq 0$$

Step response of a 1st order system: graphical course

$$y(t) = \bar{u} \cdot K\left(1 - e^{-\frac{t}{\tau}}\right), t \geq 0$$

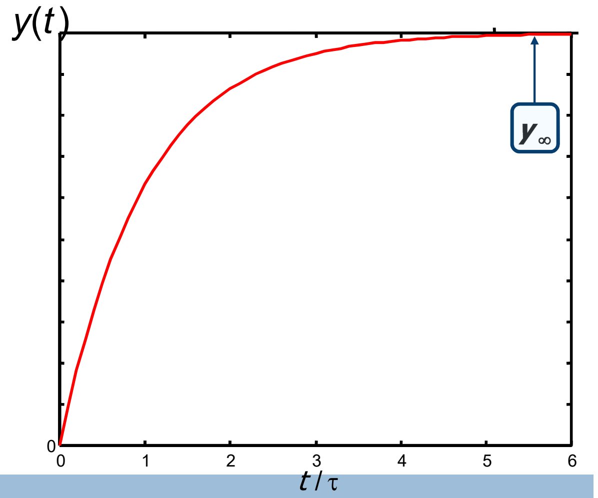


Steady state value

Steady state value

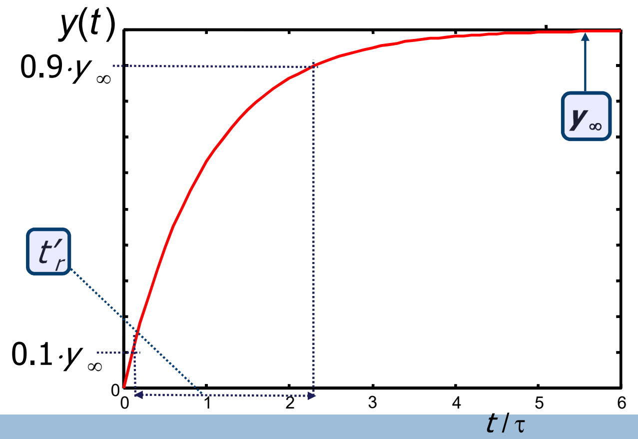
y_∞ is the asymptotic value of the output response $y(t)$ as $t \to \infty$

$$y_\infty = \lim_{t\to\infty} y(t) =$$
$$= \lim_{s\to 0} s \cdot Y(s) =$$
$$= \lim_{s\to 0} s \cdot \frac{K}{1+\tau s}\frac{\bar{u}}{s} =$$
$$= K \cdot \bar{u}$$



10% ÷ 90% rise time

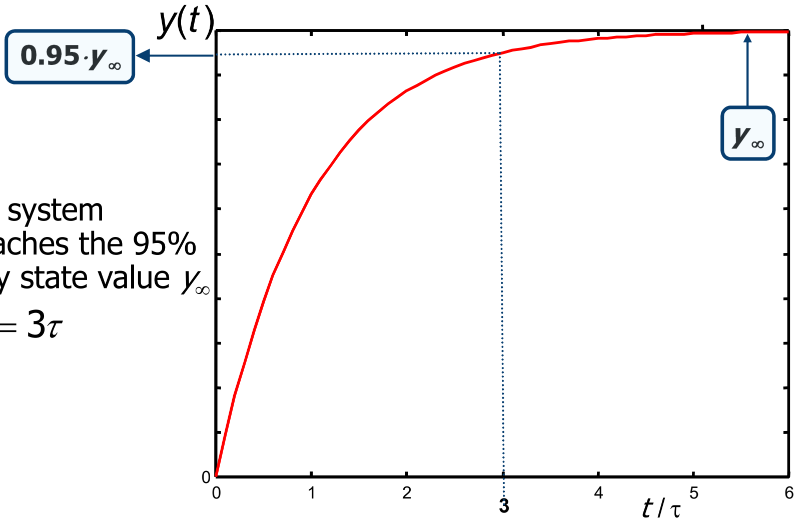
10% ÷ 90% rise time t'_r is the time required to the step response to go from the 10% to the 90% of the steady-state value $y = y_\infty$



Settling time

The **settling time** $\pm\alpha$ % $t_{s,\alpha\%}$ is the amount of time required to the step response to reach and stay within the $\pm\alpha$ % of the steady- state value y_∞. Typical values of α are: $\alpha = 1$, $\alpha = 2$, $\alpha = 5$

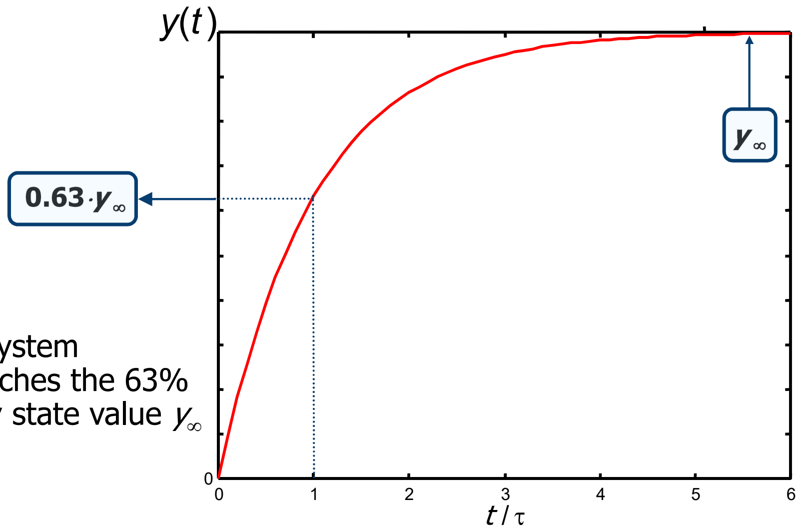
Settling time



Remark
after 3τ, the system response reaches the 95% of the steady state value y_∞

$$t_{s,5\%} = 3\tau$$

Time constant evaluation



Remark
after τ, the system response reaches the 63% of the steady state value y_∞

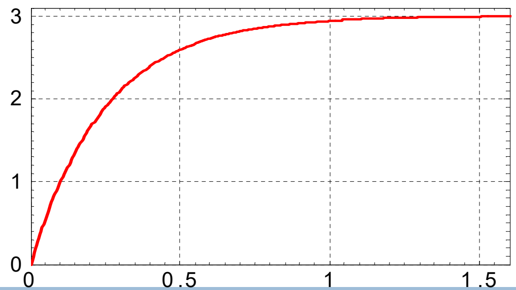
Derivation of a 1st order model through a graphical procedure

Problem formulation

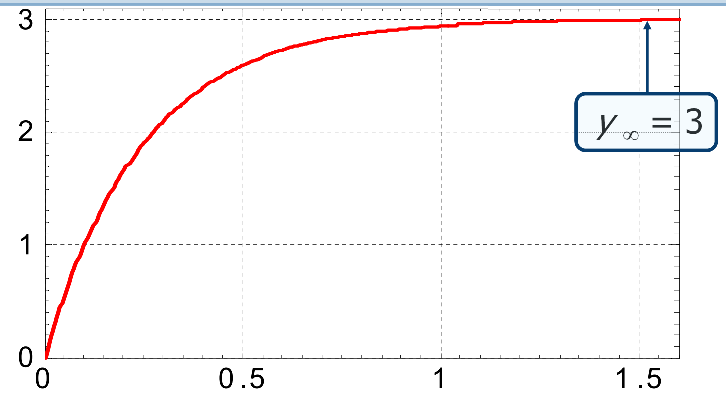
Consider the 1st order system

$$H(s) = \frac{K}{1+\tau s}$$

compute K and τ so that its output response in the presence of a step input of unitary amplitude ($\bar{u} = 1$) is the one reported in the picture below.



Computation of K



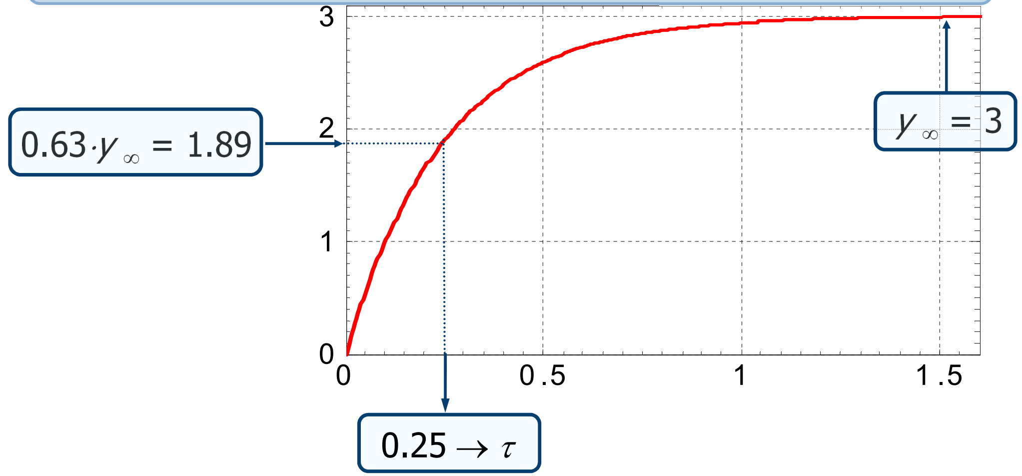
since

$$y_\infty = K \cdot \bar{u} = 3, \quad \bar{u} = 1$$

you get:

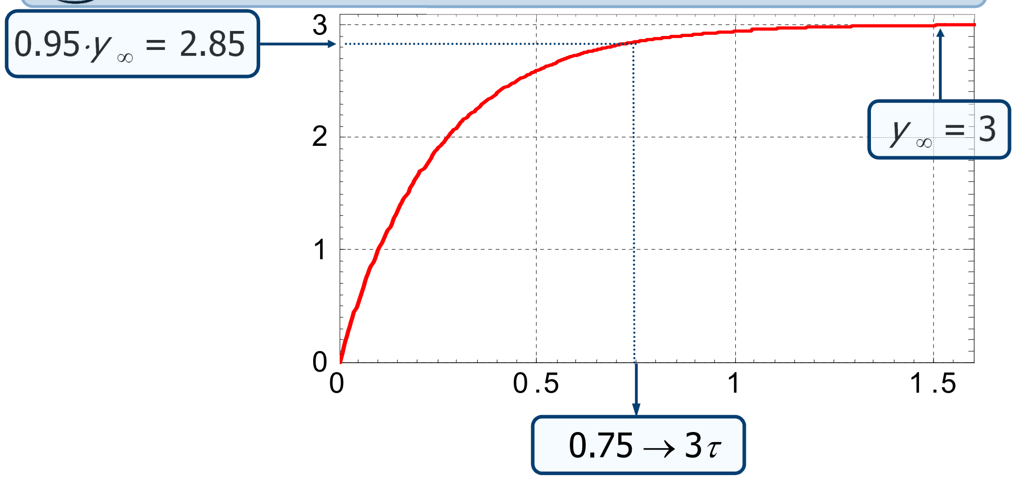
$$K = \frac{y_\infty}{\bar{u}} = 3$$

Computation of τ



$$\tau = 0.25 \text{ s}$$

Computation of τ: 2nd method



$$\tau = 0.75 / 3 = 0.25 \text{ s}$$

Step response analysis of prototype 2nd order systems

Transfer function

Consider a stable second order system described by the transfer function.

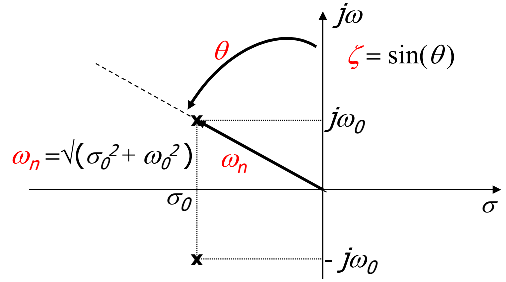
$$H(s) = K\frac{1}{1+2\frac{\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}} = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$\rightarrow \begin{cases} K \rightarrow \text{ gain} \\ \omega_n \rightarrow \text{ natural frequency} \\ 0<\zeta<1 \rightarrow \text{ damping ratio} \end{cases}$$

$$\tau = \frac{1}{\zeta\omega_n} \rightarrow \text{ time constant}$$

Natural frequency and damping coefficient

Natural frequency (ω_n) and damping coefficient (ζ) of a couple of complex conjugate poles $\sigma_0 \pm j\omega_0$ are defined as follows.



$$\sigma_0 = -\zeta\omega_n \qquad \omega_0 = \omega_n\sqrt{(1-\zeta^2)}$$

$$\omega_n = \sqrt{(\sigma_0^2+\omega_0^2)} \qquad \zeta = -\sigma_0/\sqrt{(\sigma_0^2+\omega_0^2)}$$

$$\omega_n > 0 \qquad |\zeta| < 1$$

Step response of a 2nd order system: analytical form

In the presence of a step input $u(t)$ with amplitude $\bar{u}$

$$u(t) = \bar{u}\varepsilon(t) \xrightarrow{\mathcal{L}} U(s) = \frac{\bar{u}}{s}$$

the output response of the second order system

$$H(s) = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

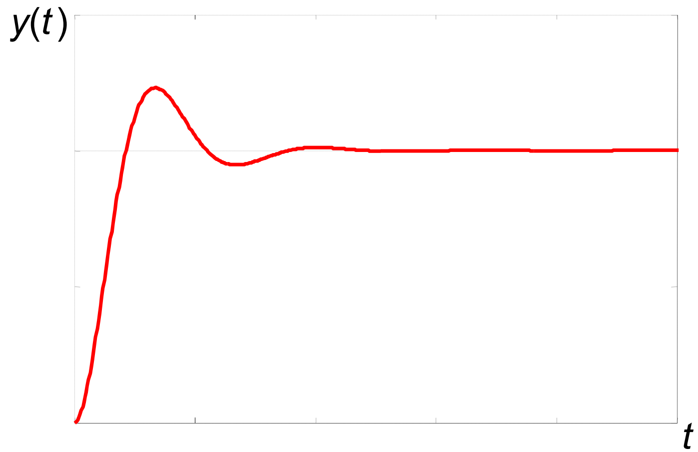
can be computed as:

$$Y(s) = H(s)U(s) = K\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\frac{\bar{u}}{s} \xrightarrow{\mathcal{L}^{-1}} y(t) =$$

$$= \bar{u}K\left(1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\arccos(\zeta)\right)\right), t \geq 0$$

Step response of a 2nd order system: graphical course

$$y(t) = \bar{u}K\left(1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\arccos(\zeta)\right)\right), t \geq 0$$



Steady state and peak values

Steady state value y_∞ is the asymptotic value of the output response $y(t)$ as $t \to \infty$

$$y_\infty = \lim_{t\to\infty} y(t) = \lim_{s\to 0} s\cdot Y(s) =$$

$$= \lim_{s\to 0} s\cdot K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{\bar{u}}{s} = K\cdot\bar{u}$$

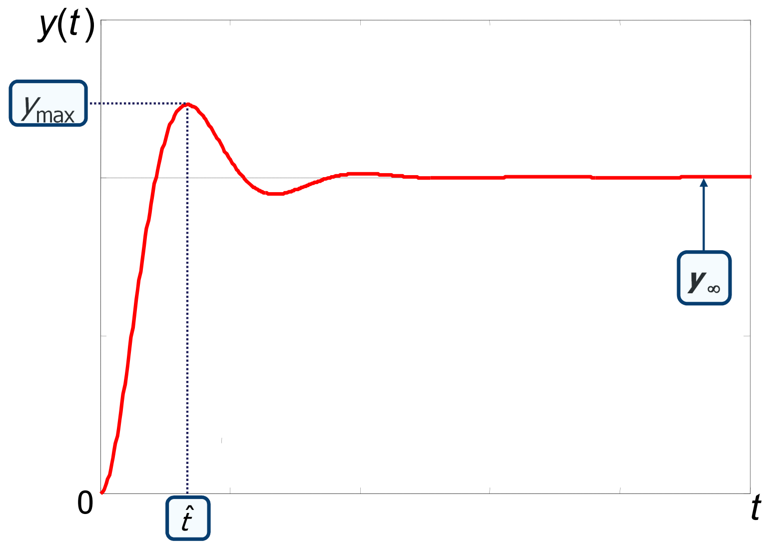
Peak value y_{max} is the maximum value of $y(t)$.

$$y_{max} = \max_t y(t)$$

Peak time $\hat{t}$ is the time instant for which

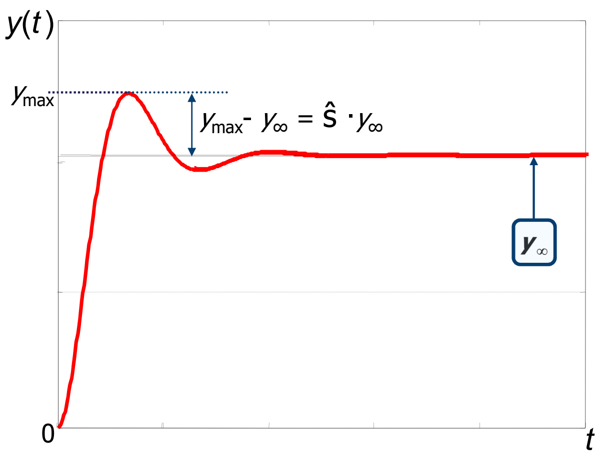
$$y_{max} = y(\hat{t})$$

Steady state and peak values



Maximum overshoot

- The **maximum overshoot $\hat{s}$** is defined as $\hat{s} = \dfrac{y_{max} - y_\infty}{y_\infty}$



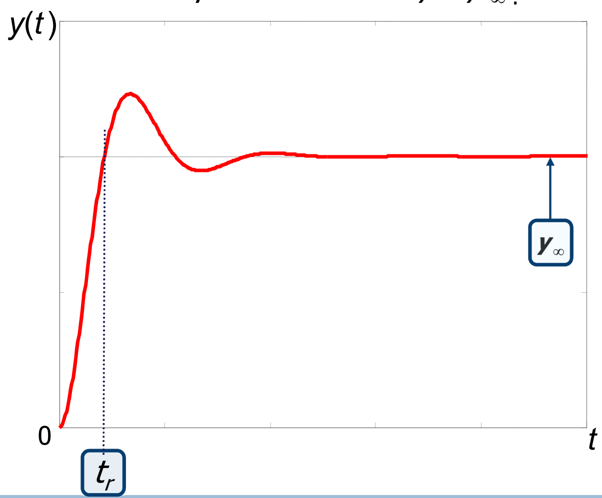
- The quantity $\hat{s}$ can be also expressed in percentual terms $\hat{s}_\%$

$$\hat{s}_\% = 100\cdot\hat{s}$$

- In practice, the same symbol $\hat{s}$ is used to indicate $\hat{s}_\%$

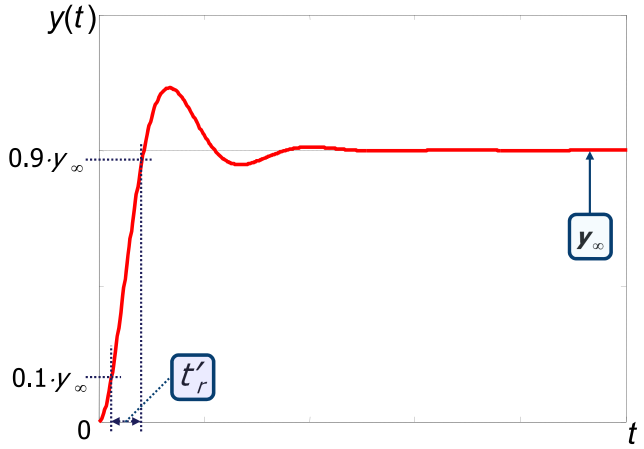
Rise time

- The **rise time t_r** is the time required to the step response to reach for the first time the steady state value → $y = y_\infty$.



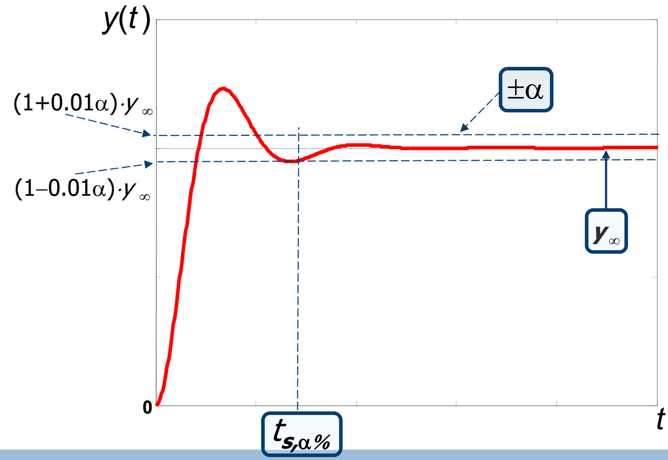
10-90% rise time

- The **10% ÷ 90% rise time t'_r** is the time required to the step response to go from the 10% to the 90% of the steady-state value $y = y_\infty$.



Settling time

- The **settling time** $\pm\alpha$ % $t_{s,\alpha\%}$ is the amount of time required to the step response to reach and stay within the $\pm\alpha$ % of the steady-state value y_∞. Typical values of α are: $\alpha = 1$, $\alpha = 2$, $\alpha = 5$.



Step response parameters vs. ω_n and ζ

In a second order system of the form

$$H(s) = K\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

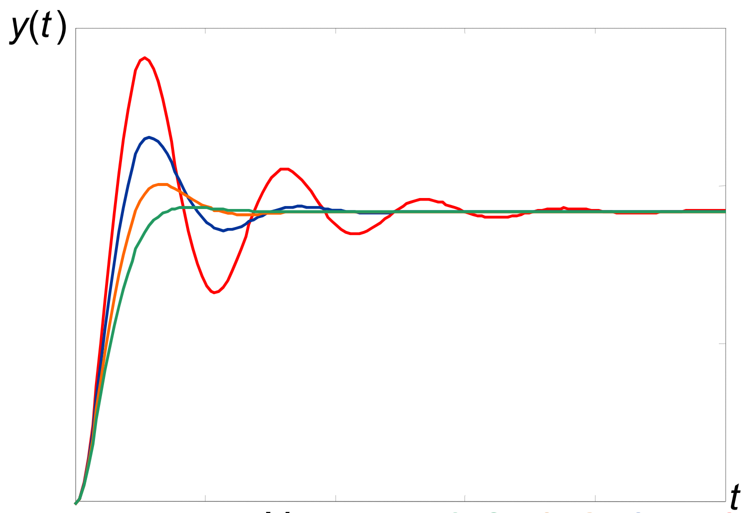
the parameters of the step response just defined can be expressed as functions of ω_n and ζ

$$\hat{S} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}, \quad \hat{t} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$t_r = \frac{1}{\omega_n\sqrt{1-\zeta^2}}\left(\pi - \arccos(\zeta)\right), \quad t_r' \approx \frac{2.16\zeta + 0.6}{\omega_n}$$

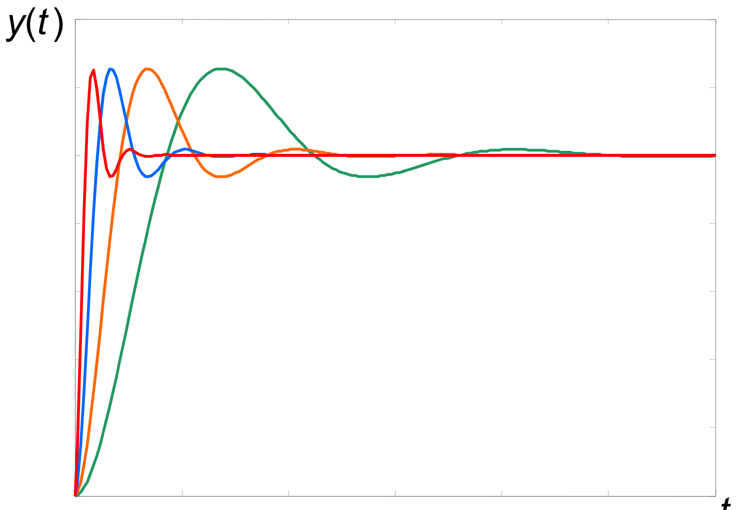
$$t_{s,\alpha\%} = \frac{1}{\omega_n\zeta}\ln(\alpha/100)^{-1}$$

Analysis vs. ζ



ω_n=1 rad/s ζ = 0.8, 0.6, 0.4, 0.2

Analysis vs. ω_n



$\zeta = 0.4 \quad \omega_n = 0.5, 1, 2, 4$ rad/s

Special case $\zeta = 1$ (1/3)

When $\zeta = 1$, the transfer function:

$$H(s) = K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

becomes

$$H(s) = \frac{K}{(1+\tau s)^2}, \tau = \frac{1}{\omega_n}$$

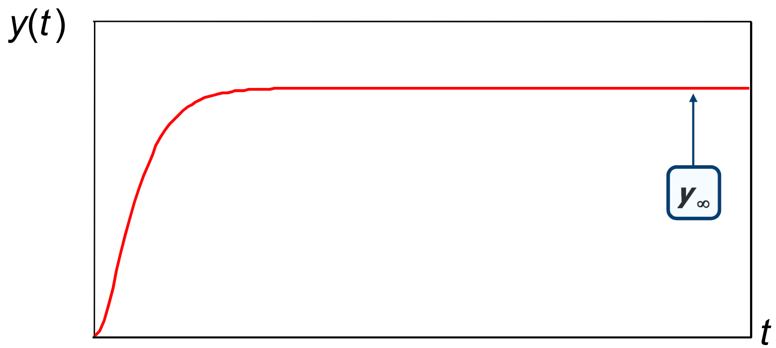
$\rightarrow$ two coincident $\mathbb{R}$ poles in $s = -1/\tau$

The output response in the presence of a step input of amplitude $\bar{u}$ is

$$y(t) = \bar{u} \cdot K \left(1 - e^{-\frac{t}{\tau}} - \frac{t}{\tau} e^{-\frac{t}{\tau}}\right), t \geq 0$$

Special case $\zeta = 1$ (2/3)

The graphical behavior is:



Note the absence of oscillations and overshoot in the transient phase before reaching the steady state value y_∞

Special case $\zeta = 1$ (3/3)

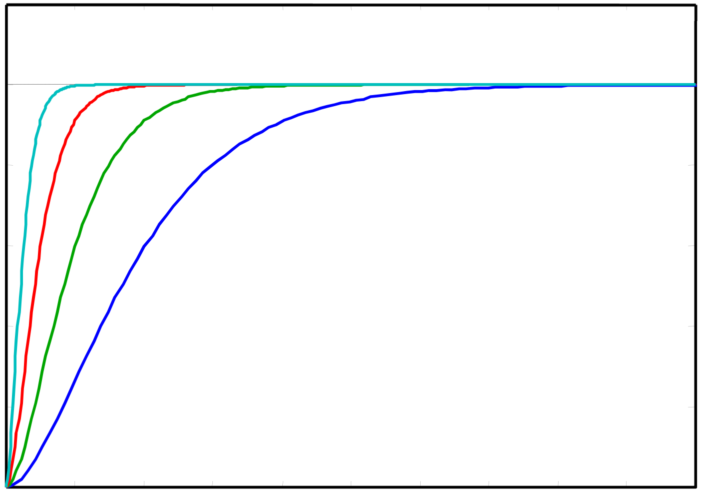
The response characteristics can be studied through the following parameters:

- Steady state value y_∞
- Rise time 10% - 90% t'_r
- Settling time $\pm\ \varepsilon\%$ $t_{s,\,\varepsilon\%}$

The table below provides approximate relationships between the response parameters y_∞, t'_r, $t_{s,\,\varepsilon\%}$ and the transfer function parameters K and τ

y_∞	t'_r	$t_{s,\,5\%}$	$t_{s,\,1\%}$
$\bar{u} \cdot K$	$\approx 3.36 \cdot \tau$	$\approx 4.74 \cdot \tau$	$\approx 6.64 \cdot \tau$

Case ζ = 1 graphical behavior vs. τ



$\zeta = 1 \quad \tau = 2, 1, 0.5, 0.25$ s

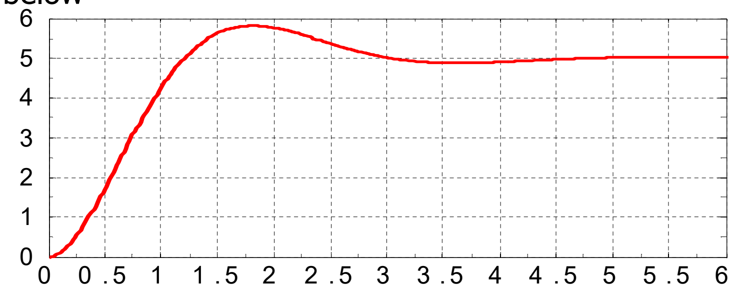
Derivation of a 2nd order model through a graphical procedure

Problem formulation

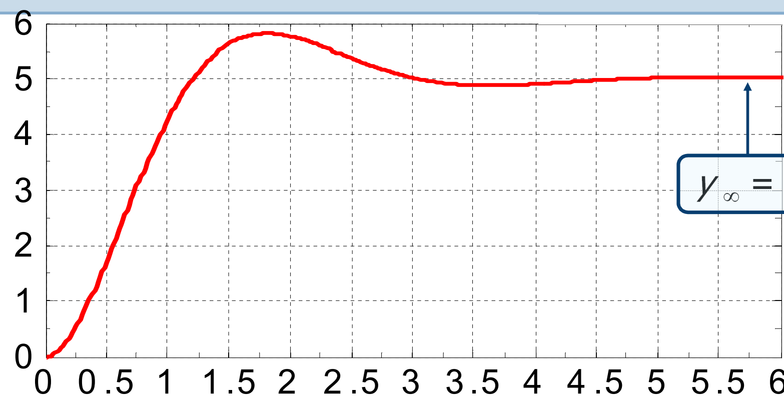
Consider the 2nd order system:

$$H(s) = K \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

compute K, ζ e ω_n so that its output response in the presence of a step input of unitary amplitude ($\bar{u} = 1$) is the one reported in the picture below



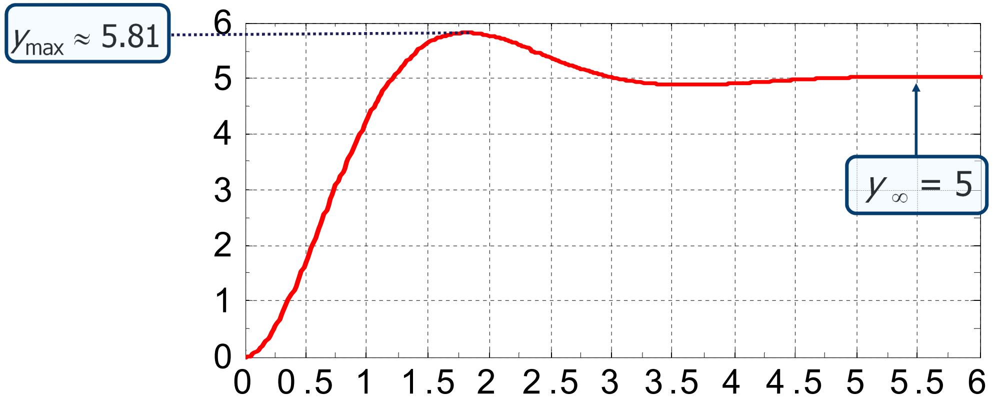
Computation of K



Since $y_\infty = K \cdot \bar{u} = 5, \quad \bar{u} = 1$

you get $$K = \frac{y_\infty}{\bar{u}} = 5$$

Computation of ζ (1/2)



Since $y_\infty = 5, \quad y_{max} = 5.81$

the maximum overshoot is given by: $\hat{s} = \dfrac{y_{max} - y_\infty}{y_\infty} = 0.162$

Computation of ζ (2/2)

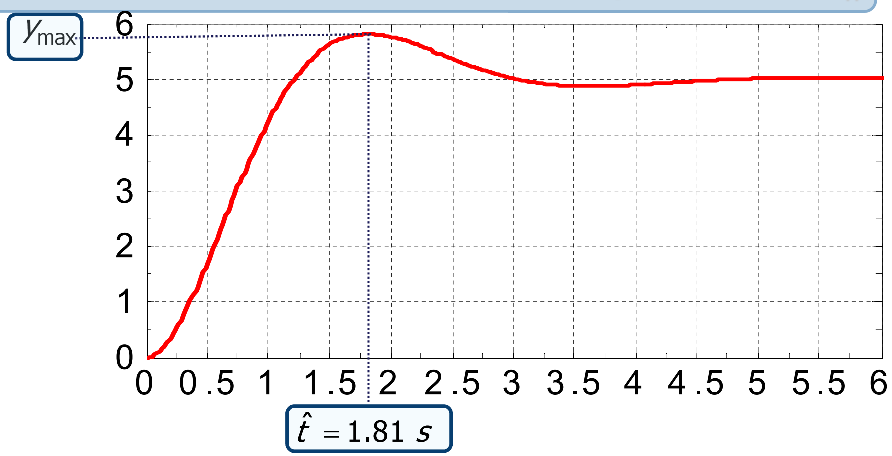
Recalling the maximum overshoot expression as a function of ζ:

$$\hat{s} = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}}$$

then:

$$\zeta = \frac{|\ln(\hat{s})|}{\sqrt{\pi^2 + \ln^2(\hat{s})}} \underset{\hat{s}=0.162}{\overset{\uparrow}{\approx}} 0.5$$

Computation of ω_n



Using the peak time expression as a function of of ω_n and ζ :

$$\hat{t} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \Rightarrow \omega_n = \frac{\pi}{\hat{t}\sqrt{1-\zeta^2}} \underset{\zeta=0.5,\hat{t}=1.81}{\overset{\uparrow}{=}} 2 \text{ rad/s}$$

Effect of additional poles, zeros and delays on the step response of prototype systems

Effect of an additional pole

2nd order system with an additional real negative pole

Consider the following prototype second order system

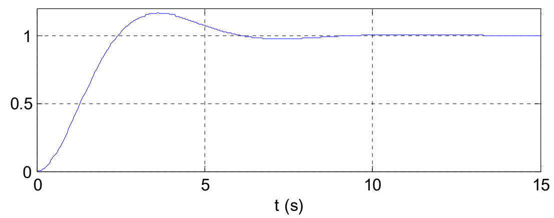
$$H(s)=\frac{K}{1+2\frac{\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}}\rightarrow\begin{cases}K=1\\ \omega_n=1\\ \zeta=0.5\end{cases},\tau=\frac{1}{\zeta\omega_n}=2\text{ s},\begin{cases}\sigma_0=-0.5\\ \omega_0=0.866=\frac{\sqrt{3}}{2}\end{cases}$$

The corresponding step response is given by

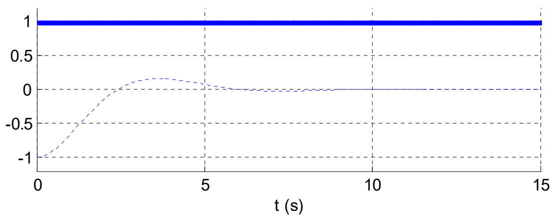
$$y(t)=\left(1+1.1547e^{-0.5t}\cos\left(0.866t+2.618\right)\right)\varepsilon(t)$$

2nd order system with an additional real negative pole

$$y(t)=\left(\boxed{1}+\boxed{1.1547e^{-0.5t}\cos\left(0.866t+2.618\right)}\right)\varepsilon(t)$$



Time course of the response



Time course of the response modes

2nd order system with an additional real negative pole

Example 1: a prototype second order system with an additional real negative pole with a bigger time constant

$$H(s)=\frac{K}{\left(1+2\frac{\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}\right)\left(1-\frac{s}{p}\right)}$$

$$\rightarrow\begin{cases}K=1\\ \omega_n=1\\ \zeta=0.5\end{cases},\tau=\frac{1}{\zeta\omega_n}=2\text{ s},\begin{cases}p=-0.2\\ \tau_p=\left|\frac{1}{p}\right|=5\text{ s}\end{cases}$$

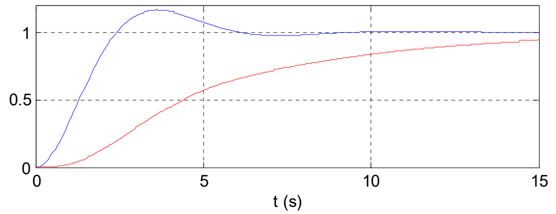
The corresponding step response is given by

$$y(t)=\left(1-1.19e^{-0.2t}+0.252e^{-0.5t}\cos\left(0.866t+0.7137\right)\right)\varepsilon(t)$$

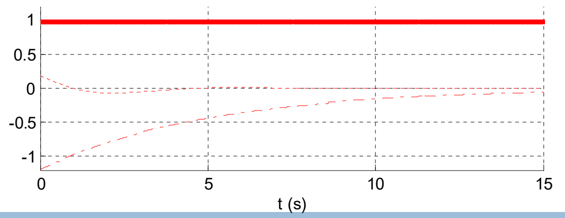
2^{nd} order system with an additional real negative pole

$$y(t) = \left(1 + 1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\right)\varepsilon(t)$$

$$y(t) = \left(\boxed{1} - \boxed{1.19e^{-0.2t}} + \boxed{0.252e^{-0.5t}\cos\left(0.866t + 0.7137\right)}\right)\varepsilon(t)$$



Time course of the response (note the presence of a significant **tail effect** during the transient extinction).



Time course of the response modes.

2^{nd} order system with an additional real negative pole

Example 2: a prototype second order system with an additional real negative pole with a similar time constant

$$H(s) = \frac{K}{\left(1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}\right)\left(1 - \frac{s}{p}\right)}$$

$$\rightarrow \begin{cases} K = 1 \\ \omega_n = 1 \\ \zeta = 0.5 \end{cases}, \tau = \frac{1}{\zeta\omega_n} = 2 \text{ s}, \begin{cases} p = -1 \\ \tau_p = \left|\frac{1}{p}\right| = 1 \text{ s} \end{cases}$$

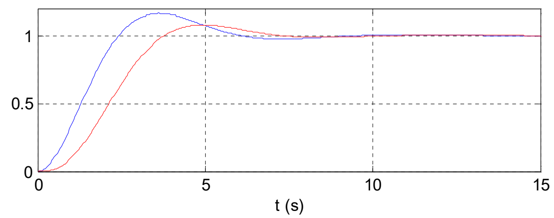
The corresponding step response is given by

$$y(t) = \left(1 - e^{-t} + 1.1547e^{-0.5t}\cos\left(0.866t + 1.5708\right)\right)\varepsilon(t)$$

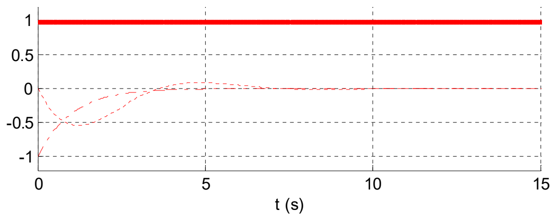
2^{nd} order system with an additional real negative pole

$$y(t) = \left(1 + 1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\right)\varepsilon(t)$$

$$y(t) = \left(\boxed{1} - \boxed{e^{-t}} + \boxed{1.1547e^{-0.5t}\cos\left(0.866t + 1.5708\right)}\right)\varepsilon(t)$$



Time course of the response (note the presence of a small **tail effect** during the transient extinction).



Time course of the response modes.

2^{nd} order system with an additional real negative pole

Example 3: a prototype second order system with an additional real negative pole with a smaller time constant

$$H(s) = \frac{K}{\left(1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}\right)\left(1 - \frac{s}{p}\right)}$$

$$\text{Example 3} \rightarrow \begin{cases} K = 1 \\ \omega_n = 1 \\ \zeta = 0.5 \end{cases}, \tau = \frac{1}{\zeta\omega_n} = 2 \text{ s}, \begin{cases} p = -10 \\ \tau_p = \left|\frac{1}{p}\right| = 0.1 \text{ s} \end{cases}$$

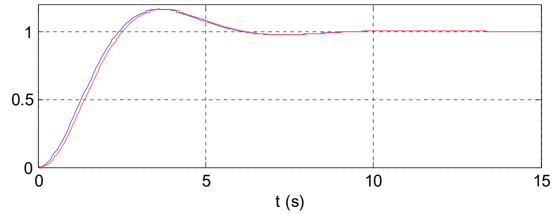
The corresponding step response is given by

$$y(t) = \left(1 - 0.011e^{-10t} + 1.2105e^{-0.5t}\cos\left(0.866t + 2.5271\right)\right)\varepsilon(t)$$

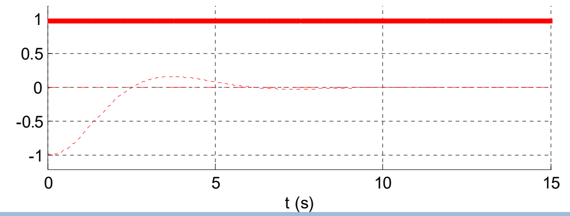
2nd order system with an additional real negative pole

$$\color{blue}{y(t)} = \left(1 + 1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\right)\varepsilon(t)$$

$$\color{red}{y(t)} = \left(\boxed{1} - 0.011e^{-10t} + 1.2105e^{-0.5t}\cos\left(0.866t + 2.5271\right)\right)\varepsilon(t)$$



Time course of the response (no tail effect).



Time course of the response modes.

Effect of an additional negative zero

2nd order system with an additional real negative zero

Example 1: a prototype second order system with an additional real negative zero placed at a lower frequency wrt ω_n

$$H(s) = \frac{K\left(1 - \frac{s}{z}\right)}{1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K = 1 \\ \omega_n = 1 \\ \zeta = 0.5 \end{cases}, \tau = \frac{1}{\zeta\omega_n} = 2 \text{ s}, z = -0.2$$

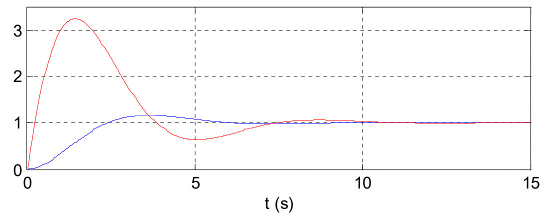
The corresponding step response is given by

$$y(t) = \left(1 + 5.2915e^{-0.5t}\cos\left(0.866t - 1.7609\right)\right)\varepsilon(t)$$

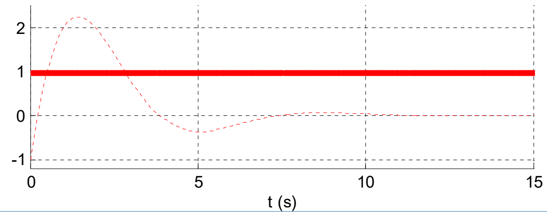
2nd order system with an additional real negative zero

$$\color{blue}{y(t)} = \left(1 + 1.1547e^{-0.5t}\cos\left(0.866t + 2.618\right)\right)\varepsilon(t)$$

$$\color{red}{y(t)} = \left(\boxed{1} + 5.2915e^{-0.5t}\cos\left(0.866t - 1.7609\right)\right)\varepsilon(t)$$



Time course of the response (note a significant overshoot increase).



Time course of the response modes.

2nd order system with an additional real negative zero

Example 2: a prototype second order system with an additional real negative zero placed at a similar frequency wrt ω_n

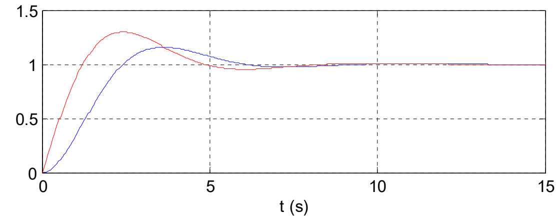
$$H(s)=\frac{K\left(1-\dfrac{s}{z}\right)}{1+2\dfrac{\zeta}{\omega_n}s+\dfrac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K=1 \\ \omega_n=1 \\ \zeta=0.5 \end{cases}, \tau=\frac{1}{\zeta\omega_n}=2\ \text{s}, z=-1$$

The corresponding step response is given by

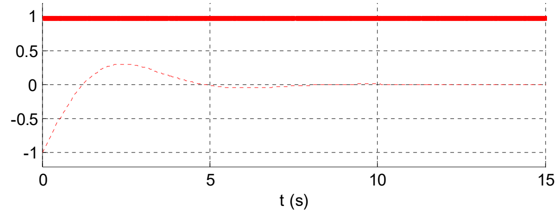
$$y(t)=\left(1+1.1547e^{-0.5t}\cos\left(0.866t-2.618\right)\right)\varepsilon(t)$$

2nd order system with an additional real negative zero

$$y(t)=\left(1+1.1547e^{-0.5t}\cos\left(0.866t+2.618\right)\right)\varepsilon(t)$$
$$y(t)=\left(\boxed{1}+\boxed{1.1547e^{-0.5t}\cos\left(0.866t-2.618\right)}\right)\varepsilon(t)$$



Time course of the response (note a slight overshoot increase).



Time course of the response modes.

2nd order system with an additional real negative zero

Example 3: a prototype second order system with an additional real negative zero placed at a higher frequency wrt ω_n

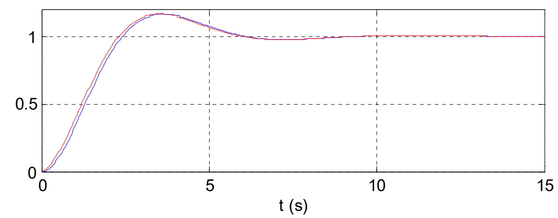
$$H(s)=\frac{K\left(1-\dfrac{s}{z}\right)}{1+2\dfrac{\zeta}{\omega_n}s+\dfrac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K=1 \\ \omega_n=1 \\ \zeta=0.5 \end{cases}, \tau=\frac{1}{\zeta\omega_n}=2\ \text{s}, z=-10$$

The corresponding step response is given by

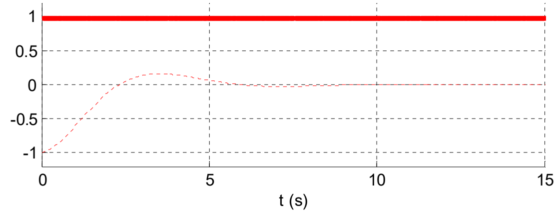
$$y(t)=\left(1+1.015e^{-0.5t}\cos\left(0.866t+2.7069\right)\right)\varepsilon(t)$$

2nd order system with an additional real negative zero

$$y(t)=\left(1+1.1547e^{-0.5t}\cos\left(0.866t+2.618\right)\right)\varepsilon(t)$$
$$y(t)=\left(\boxed{1}+\boxed{1.015e^{-0.5t}\cos\left(0.866t+2.7069\right)}\right)\varepsilon(t)$$



Time course of the response (note a negligible overshoot increase).



Time course of the response modes.

Effect of an additional positive zero

2nd order system with an additional real positive zero

Example 1: a prototype second order system with an additional real positive zero(*) placed at a lower frequency wrt ω_n

$$H(s)=\frac{K\left(1-\frac{s}{z}\right)}{1+2\frac{\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K=1 \\ \omega_n=1 \\ \zeta=0.5 \end{cases},\ \tau=\frac{1}{\zeta\omega_n}=2\text{ s}, z=0.2$$

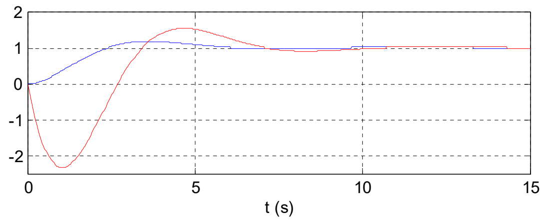
The corresponding step response is given by

$$y(t)=\left(1+6.4291e^{-0.5t}\cos\left(0.866t+1.7270\right)\right)\varepsilon(t)$$

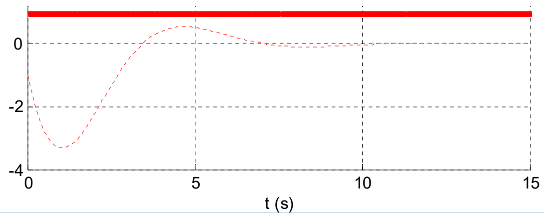
(*) zeros with positive real part are referred to as **nonminimum phase zeros**

2nd order system with an additional real positive zero

$$y(t)=\left(1+1.1547e^{-0.5t}\cos\left(0.866t+2.618\right)\right)\varepsilon(t)$$
$$y(t)=\left(\boxed{1}+\boxed{6.4291e^{-0.5t}\cos\left(0.866t+1.7270\right)}\right)\varepsilon(t)$$



Time course of the response (note the significant **inverse response** behavior).



Time course of the response modes.

2nd order system with an additional real positive zero

Example 2: a prototype second order system with an additional real positive zero placed at a similar frequency wrt ω_n

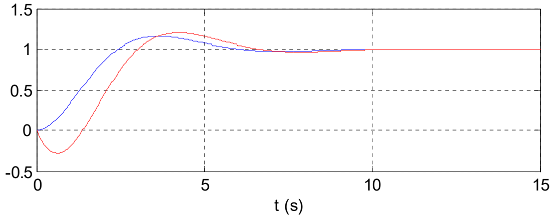
$$H(s)=\frac{K\left(1-\frac{s}{z}\right)}{1+2\frac{\zeta}{\omega_n}s+\frac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K=1 \\ \omega_n=1 \\ \zeta=0.5 \end{cases},\ \tau=\frac{1}{\zeta\omega_n}=2\text{ s}, z=1$$

The corresponding step response is given by

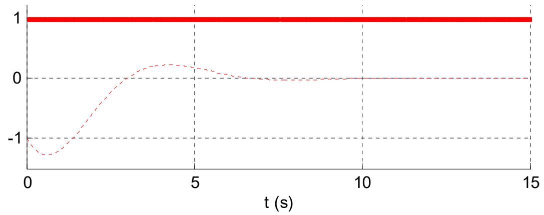
$$y(t)=\left(1+2e^{-0.5t}\cos\left(0.866t+2.0944\right)\right)\varepsilon(t)$$

2nd order system with an additional real positive zero

$$y(t) = \left(1 + 1.1547e^{-0.5t}\cos(0.866t + 2.618)\right)\varepsilon(t)$$
$$y(t) = \left(\boxed{1} + \boxed{2e^{-0.5t}\cos(0.866t + 2.0944)}\right)\varepsilon(t)$$



Time course of the response (note the inverse response behavior).



Time course of the response modes.

2nd order system with an additional real positive zero

Example 3: a prototype second order system with an additional real positive zero placed at a higher frequency wrt ω_n

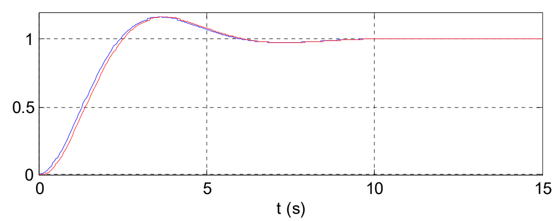
$$H(s) = \frac{K\left(1 - \frac{s}{z}\right)}{1 + 2\frac{\zeta}{\omega_n}s + \frac{s^2}{\omega_n^2}} \rightarrow \begin{cases} K = 1 \\ \omega_n = 1 \\ \zeta = 0.5 \end{cases}, \tau = \frac{1}{\zeta\omega_n} = 2 \text{ s}, z = 10$$

The corresponding step response is given by

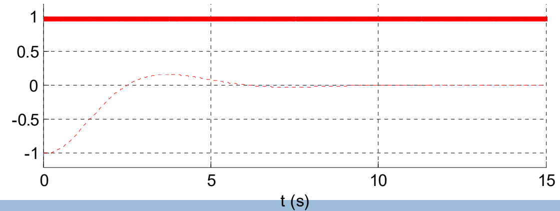
$$y(t) = \left(1 + 1.2166e^{-0.5t}\cos(0.866t + 2.5357)\right)\varepsilon(t)$$

2nd order system with an additional real positive zero

$$y(t) = \left(1 + 1.1547e^{-0.5t}\cos(0.866t + 2.618)\right)\varepsilon(t)$$
$$y(t) = \left(\boxed{1} + \boxed{1.2166e^{-0.5t}\cos(0.866t + 2.5357)}\right)\varepsilon(t)$$



Time course of the response (note the negligible inverse response behavior).



Time course of the response modes.

Effects of additional zeros on prototype step response

The step response of a stable LTI system with tf of the form

$$\tilde{H}(s) = H(s)\left(1 - \frac{s}{z}\right) = H(s) - \frac{s}{z}H(s)$$

is always given by the algebraic sum of the step response of $H(s)$ and of its time derivative multiplied by $-1/z$.

If $|z|$ is "big", the step response of $H(s)$ dominates over the one of $-s/z\,H(s)$...

at the opposite, if $|z|$ is "small", the step response of $-s/z\,H(s)$ dominates over the one of $H(s)$.

Effects of additional zeros on prototype step response

$$\tilde{H}(s) = H(s)\left(1 - \frac{s}{z}\right) = H(s) - \frac{s}{z}H(s)$$

when $|z|$ is "small" , the step response of $-s/z\,H(s)$ dominates over the one of $H(s)$.

In particular, when the response of $H(s)$ is strongly monotonic increasing during the transient:

- if $z < 0$ (i.e. real negative zero), the time derivative term introduces an increase of the maximum overshoot (the smaller is $|z|$, the bigger is this effect).
- if $z > 0$ (i.e. real positive zero), the behaviors of $-s/z\,H(s)$ and $H(s)$ have opposite signs and the time derivative term introduces an **inverse response** behavior during the initial part of the transient (the smaller is $|z|$, the bigger is this effect).

Effect of time delay

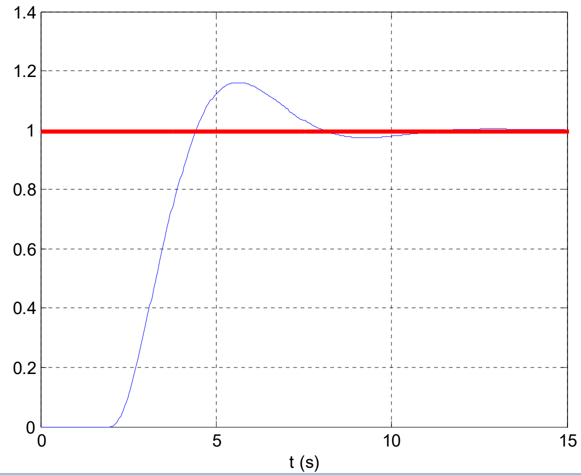
Time delay in dynamic systems

Time delay occurs in dynamic systems when there is a delay between the commanded input and the start of the output response.

For example, consider a heating system that operates by heating water for pipeline distribution to radiators at distant locations. Since the hot water must flow through the line, the radiators will not begin to get hot until after a certain time delay. Thus, the time between the command for more heat and the beginning of the rise in temperature at a distant location along the pipeline is the time delay.

Time delay in dynamic systems

In the picture below there is a delay of 2 s between the step commanded output (red) and the output response (blue).



Modeling time delay

An LTI system whose dynamic behavior is described by a tf *H(s)* in the presence of a time delay of θ s can be represented as

$$H_{delay}(s) = H(s)e^{-\theta s}$$

The corresponding state space representation is

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t-\theta) \\ y(t) = C\,x(t) + D\,u(t-\theta) \end{cases}$$

Transfer function with delay

- Definition of transfer function with delay in MatLab

$$H(s) = \frac{e^{-2s}}{s^2+s+1}$$

- Define the Laplace variabile **s** using **tf** statement

```
>> s=tf('s');
```

- Define

```
>> H=1/(s^2+s+1);
>> H.inputdelay=2;
 Transfer function:
                    1
exp(-2*s) * -------------
              s^2 + s + 1
```


Padé approximation of time delay

$$H_{delay}(s) = H(s)e^{-\theta s}$$

Note that the transfer function of an LTI system with delay is not real rational. Example

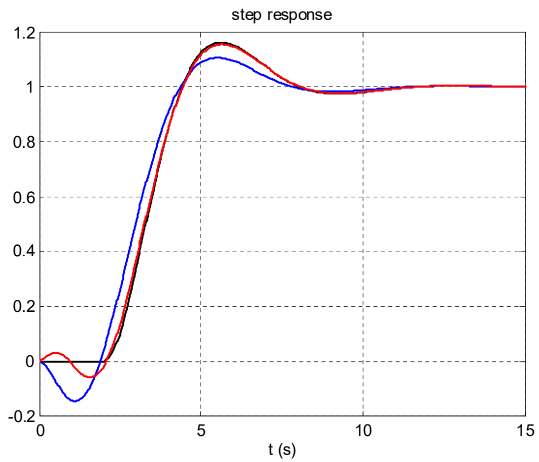
$$H_{delay}(s) = \frac{e^{-2s}}{s^2+s+1}$$

In order to obtain a real rational representation of a transfer function with delay, the following functions referred to as **Padé approximation** of 1st and 2nd order respectively are employed to approximate the time delay term $e^{-\theta s}$.

$$e^{-\theta s} \approx \frac{1-\frac{\theta}{2}s}{1+\frac{\theta}{2}s}, \quad e^{-\theta s} \approx \frac{1-\frac{\theta}{2}s+\frac{\theta^2 s^2}{12}}{1+\frac{\theta}{2}s+\frac{\theta^2 s^2}{12}}$$

Padé approximation of time delay

Example: $H(s) = \frac{e^{-2s}}{s^2+s+1}$



$$H_1(s) = \frac{1}{s^2+s+1}\cdot\frac{1-s}{1+s}$$

$$H_2(s) = \frac{1}{s^2+s+1}\cdot\frac{1-s+\frac{s^2}{3}}{1+s+\frac{s^2}{3}}$$

Padé approximation of time delay

```
PADE  Pade approximation of time delays.

SYSX = PADE(SYS,N) returns a delay-free approximation SYSX of
the continuous-time delay system SYS by replacing all delays
by their Nth-order Pade approximation.  The default is N=1.

>> s=tf('s');

>> H=1/(s^2+s+1);

>> H.inputdelay=2;

>> SYSX = zpk(PADE(H,1))

 Zero/pole/gain:

      - (s-1)

--------------------

(s+1) (s^2  + s + 1)
```