Automatic Control

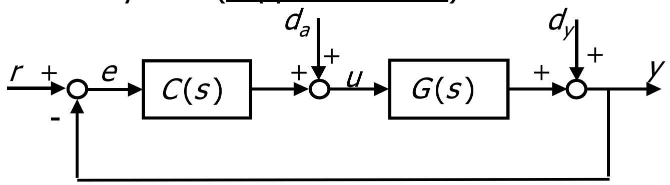
Steady state analysis of feedback control systems in the presence of polynomial references and disturbances

Steady state requirements and design

Steady state analysis

Introduction

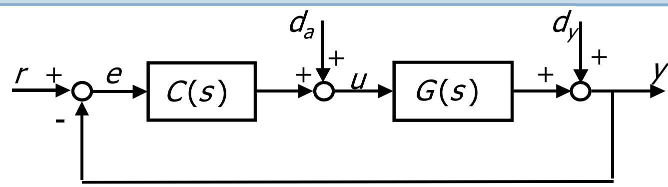
We want to analyse the steady state properties of the following feedback control system (<u>supposed stable</u>).



It is assumed that $L(s)$ has not any zero at the origin and it is expressed in the dc-gain form.

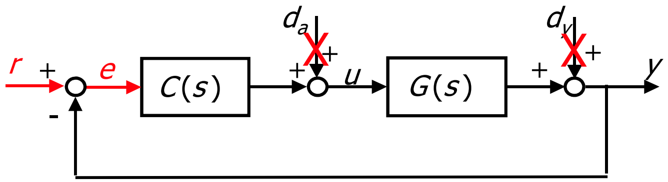
$$L(s)=\frac{K_g}{s^g}L'(s) \underset{\uparrow \atop L'(s)=\frac{N_L'(s)}{D_L'(s)}}{=} \frac{K_g}{s^g}\frac{N_L'(s)}{D_L'(s)}, \rightarrow \begin{cases} K_g=\lim\limits_{s\to 0} s^g L(s) \\ \lim\limits_{s\to 0} N_L'(s)=1, \lim\limits_{s\to 0} D_L'(s)=1 \\ \Rightarrow \lim\limits_{s\to 0} L'(s)=1 \end{cases}$$

Introduction



- We consider, in particular, the following issues:
  - steady state properties of the tracking error $e = r - y$ in the presence of the reference r
  - steady state properties of the output y in the presence of the disturbances d_a and d_y
- Moreover, we consider the special case when r, d_a and d_y are expressed as polynomials of t.

Steady state tracking error for polynomial *r(t)*



The **steady state tracking error** is defined as

$$|e_r^\infty| = \lim_{t\to\infty}|e(t)| = \lim_{t\to\infty}|r(t)-y(t)|$$

its properties are studied in the presence of a monomial reference with degree h and amplitude ρ.

$$r(t) = \rho\frac{t^h}{h!} \qquad\qquad r(s) = \frac{\rho}{s^{h+1}}$$

Steady state tracking error for polynomial *r(t)*

The problem is solved through the final value theorem (FVT).

$$|e_r^\infty| = \lim_{t\to\infty}|e(t)| \underset{\substack{\uparrow\\ \text{FVT}}}{=} \lim_{s\to 0} s\,|e(s)| = \lim_{s\to 0} s\,|S(s)r(s)| = \lim_{s\to 0} s\left|S(s)\frac{\rho}{s^{h+1}}\right| =$$

$$= \lim_{s\to 0}\left|\frac{1}{1+L(s)}\frac{\rho}{s^h}\right| = \lim_{s\to 0}\left|\frac{1}{1+K_g\dfrac{N_L^{'}(s)}{s^g D_L^{'}(s)}}\frac{\rho}{s^h}\right| =$$

$$= \lim_{s\to 0}\left|\frac{s^g D_L^{'}(s)}{s^g D_L^{'}(s)+K_g N_L^{'}(s)}\frac{\rho}{s^h}\right|$$

Steady state tracking error for polynomial *r(t)*

… we got:

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{s^g D_L^{'}(s)}{s^g D_L^{'}(s)+K_g N_L^{'}(s)}\frac{\rho}{s^h}\right|$$

Note that the hypothesis of FVT are met if $g \geq h$

If $g < h$ FVT can not be applied and the limit $\lim_{t\to\infty}|e(t)|$ must be computed directly (i.e. in time domain).

In this case, it results $\lim_{t\to\infty}|e(t)| = \infty$.

Steady state tracking error for polynomial *r(t)*

Then

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{s^g D_L^{'}(s)}{s^g D_L^{'}(s)+K_g N_L^{'}(s)}\frac{\rho}{s^h}\right|$$

g is the number of **zeros at the origin** of the sensitivity function S(s)

In the unitary feedback with cascade control scheme the number of **zeros at the origin** of the sensitivity function S(s) coincides with **the number of poles at the origin of the loop function *L(s)***

When $g \geq h$ the limit is:

- finite if $g = h$
- zero if $g > h$

Steady state tracking error for polynomial *r(t)*

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{1}{1+K_g\dfrac{N_L'(s)}{s^g D_L'(s)}}\frac{\rho}{s^h}\right|$$

$$L(s) = \frac{K_g}{s^g}\frac{N_L'(s)}{D_L'(s)}$$

$$\rightarrow \begin{cases} K_g = \lim_{s\to 0} s^g L(s) \\ \lim_{s\to 0} L'(s) = \lim_{s\to 0}\dfrac{N_L'(s)}{D_L'(s)} = 1 \end{cases}$$

Some examples:

- $g = h = 0$

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{\rho}{1+K_0\dfrac{N_L'(s)}{D_L'(s)}}\right| = \left|\frac{\rho}{1+K_0}\right|$$

Steady state tracking error for polynomial *r(t)*

- $g = h = 1$

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{1}{1+K_1\dfrac{N_L'(s)}{sD_L'(s)}}\frac{\rho}{s}\right| = \lim_{s\to 0}\left|\frac{sD_L'(s)}{sD_L'(s)+K_1N_L'(s)}\frac{\rho}{s}\right| = \left|\frac{\rho}{K_1}\right|$$

- $g = h = 2$

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{1}{1+K_2\dfrac{N_L'(s)}{s^2D_L'(s)}}\frac{\rho}{s^2}\right| = \lim_{s\to 0}\left|\frac{s^2D_L'(s)}{s^2D_L'(s)+K_1N_L'(s)}\frac{\rho}{s^2}\right| = \left|\frac{\rho}{K_2}\right|$$

Steady state tracking error for polynomial *r(t)*

- $g = 1 \;\; h = 0$

$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{1}{1+K_1\dfrac{N_L'(s)}{sD_L'(s)}}\rho\right| = \lim_{s\to 0}\left|\frac{sD_L'(s)}{sD_L'(s)+K_1N_L'(s)}\rho\right| = 0$$

- $g = 2 \;\; h = 1$

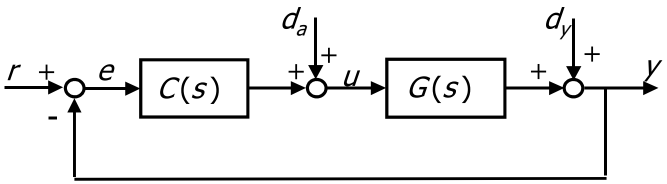
$$|e_r^\infty| = \lim_{s\to 0}\left|\frac{1}{1+K_2\dfrac{N_L'(s)}{s^2D_L'(s)}}\frac{\rho}{s}\right| = \lim_{s\to 0}\left|\frac{sD_L'(s)}{s^2D_L'(s)+K_1N_L'(s)}\rho\right| = 0$$

Steady state tracking error for polynomial *r(t)*

- *Definition (System type)*
A LTI feedback control system is a **type-*h* system** if its steady-state tracking error due to a polynomial reference input of degree *h* is bounded.

- *Result (System type and zero tracking error)*
The steady-state tracking error of an LTI feedback control system due to a polynomial reference input of degree *h* is zero if and only if the system type is greater than *h*.

Steady state tracking error for polynomial *r(t)*



▪ *Remark (System type and loop transfer function)*

In the unitary feedback with cascade compensation scheme the **system type** coincides with **the number of poles at the origin of the loop function $L(s)$.**

Steady state tracking error for polynomial *r(t)*

The steady state tracking error $|e^{\infty}_{r}|$ is finite iff:

- The system type g (i.e. the number of poles at the origin of L(s)) is such that $g \geq h$ (i.e. the degree of the monomial r(t))

In particular,

- if $g = h$, $|e^{\infty}_{r}|$ is bounded by a quantity that can be made arbitrarily small by increasing the value of the gain K_g
- if $g > h$, $|e^{\infty}_{r}|$ is zero

The steady state tracking error $|e^{\infty}_{r}|$ is unbounded iff $g < h$, (i.e. $|e^{\infty}_{r}| \to \infty$)

The following table resumes the obtained results

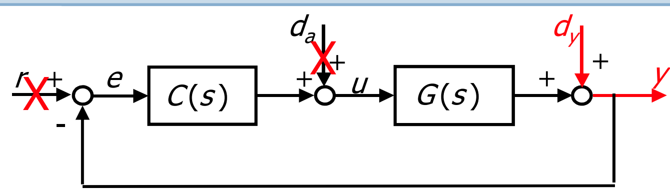
Steady state tracking error for polynomial *r(t)*

$\left|e_r^{\infty}\right|$

h / type g	0 (step) r(t)=ρε(t)	1 (linear ramp) r(t)=ρtε(t)	2 (parabolic ramp) r(t)=ρ(t²/2)ε(t)
0	$\left\lvert\frac{\rho}{1+K_0}\right\rvert$	∞	∞
1	0	$\left\lvert\frac{\rho}{K_1}\right\rvert$	∞
2	0	0	$\left\lvert\frac{\rho}{K_2}\right\rvert$

$K_g = \lim_{s \to 0} s^g L(s) \to$ generalized steady state gain of $L(s)$

Steady state output error for polynomial d_y



The **steady state output error due to d_y** is defined as

$$\left|y^{\infty}_{d_y}\right| = \lim_{t \to \infty} \left|y(t)\right|$$

its properties will be studied in the presence of a monomial disturbance d_y with degree h and amplitude δ_y:

$$d_y(t) = \delta_y \frac{t^h}{h!} \qquad d_y(s) = \frac{\delta_y}{s^{h+1}}$$

Steady state output error for polynomial d_y

The problem can be solved using FVT:

$$\left|y^{\infty}_{d_y}\right| = \lim_{t\to\infty}|y(t)| \underset{\uparrow \atop \text{FVT}}{=} \lim_{s\to 0} s|y(s)| = \lim_{s\to 0} s|S(s)d_y(s)| =$$

$$= \lim_{s\to 0} s\left|S(s)\frac{\delta_y}{s^{h+1}}\right| = \lim_{s\to 0}\left|S(s)\frac{\delta_y}{s^{h}}\right|$$

Note that the FVT procedure leads to the same result found for $|e^{\infty}_{r}|$

Therefore, we can conclude that steady state output error $|y^{\infty}_{dy}|$ due to a monomial disturbance d_y with degree h is:

- bounded if $g = h$
- zero if $g > h$
- unbounded if $g < h$

The following table resumes the obtained results…

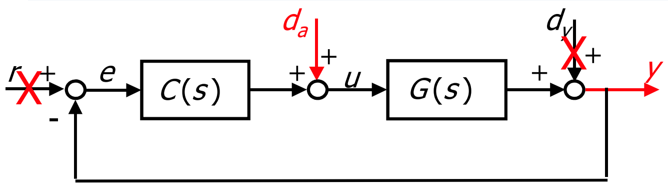
Steady state output error for polynomial d_y

$\left|y^{\infty}_{d_y}\right|$

type g \ h	0 (step) $d_y(t)=\delta_y\varepsilon(t)$	1 (linear ramp) $d_y(t)=\delta_y t\varepsilon(t)$	2 (parabolic ramp) $d_y(t)=\delta_y(t^2/2)\varepsilon(t)$
0	$\left\lvert\frac{\delta_y}{1+K_0}\right\rvert$	∞	∞
1	0	$\left\lvert\frac{\delta_y}{K_1}\right\rvert$	∞
2	0	0	$\left\lvert\frac{\delta_y}{K_2}\right\rvert$

$$K_g = \lim_{s\to 0} s^g L(s) \rightarrow \text{ generalized steady state gain of } L(s)$$

Steady state output error for polynomial d_a



The **steady state output error due to d_a** is defined as

$$\left|y^{\infty}_{d_a}\right| = \lim_{t\to\infty}|y(t)|$$

its properties will be studied in the presence of a monomial disturbance d_a with degree h and amplitude δ_a

$$d_a(t) = \delta_a\frac{t^h}{h!} \qquad d_a(s) = \frac{\delta_a}{s^{h+1}}$$

Steady state output error for polynomial d_a

The problem can be solved using FVT:

$$\left|y^{\infty}_{d_a}\right| = \lim_{t\to\infty}|y(t)| = \underset{\uparrow \atop \text{FVT}}{=} \lim_{s\to 0} s|y(s)| = \lim_{s\to 0} s|Q(s)d_a(s)| =$$

$$= \lim_{s\to 0} s\left|\frac{G(s)}{1+G(s)C(s)}\frac{\delta_a}{s^{h+1}}\right| = \lim_{s\to 0}\left|\frac{N_G(s)D_c(s)}{D_G(s)D_c(s)+N_G(s)N_c(s)}\frac{\delta_a}{s^{h}}\right|$$

Recall that, by assumption, polynomials $N_G(s)$ and $N_C(s)$ have not roots at $s = 0$ (unstable zero-pole cancellations are not allowed).

In order to apply FVT, C(s) and G(s) can be expressed as

$$C(s) = K_c\frac{N'_c(s)}{s^{g_c}D'_c(s)}, G(s) = K_G\frac{N'_G(s)}{s^{g_G}D'_G(s)}$$

$$\lim_{s\to 0} N'_c(s) = \lim_{s\to 0} D'_c(s) = \lim_{s\to 0} N'_G(s) = \lim_{s\to 0} D'_G(s) = 1$$

Steady state output error for polynomial d_a

… we get

$$= \lim_{s\to 0}\left|\frac{N_G(s)D_c(s)}{D_G(s)D_c(s)+N_G(s)N_c(s)}\frac{\delta_a}{s^h}\right| =$$

$$= \lim_{s\to 0}\left|\frac{s^{g_c}D'_c(s)K_G N'_G(s)}{s^{g_c+g_G}D'_c(s)D'_G(s)+K_G N'_G(s)K_c N'_c(s)}\frac{\delta_a}{s^h}\right|$$

If $g_c < h$, FVT can not be applied and the limit $\lim_{t\to\infty}|y(t)|$ must be computed directly (i.e. in time domain).

In this case it results $\lim_{t\to\infty}|y(t)| = \infty$ (i.e. $|y^\infty_{da}| = \infty$)

Steady state output error for polynomial d_a

The steady state output error $|y^\infty_{da}|$ is finite iff

- The number of poles at the origin g_c of the controller is such that $g_c \geq h$

In particular,

- if $g_c = h$ the steady state output error $|y^\infty_{da}|$ is bounded by a quantity that can be made arbitrarily small by increasing the controller gain K_c
- if $g_c > h$, $|y^\infty_{da}|$ is zero

The steady state output error $|y^\infty_{da}|$ is unbounded iff $g_c < h$, (i.e. $|y^\infty_{da}| \to \infty$)

The following table resumes the results that can be obtained in this case.

Steady state output error for polynomial d_a

$|y^\infty_{d_a}|$

g_c \ h	0 (step) $d_a(t)=\delta_a\varepsilon(t)$	1 (linear ramp) $d_a(t)=\delta_a t\varepsilon(t)$	2 (parabolic ramp) $d_a(t)=\delta_a t^2/2\varepsilon(t)$
0	$\left\lvert\frac{\delta_a}{K_0}\right\rvert$	∞	∞
1	0	$\left\lvert\frac{\delta_a}{K_1}\right\rvert$	∞
2	0	0	$\left\lvert\frac{\delta_a}{K_2}\right\rvert$

$$K_0 = \begin{cases} K_c & \text{if } G(s) \text{ has poles in } 0 \\ \dfrac{1+K_cK_G}{K_G} & \text{if } G(s) \text{ has not poles in } 0 \end{cases} \qquad K_{g_c} = \lim_{s\to 0} s^{g_c}C(s),\ g_c \geq 1$$

Steady state requirements and design

Steady state requirements

Steady state requirements (performance) can be expressed in the form

$$|e_r^\infty| \le 0.1,\ \text{when } r(t) = \varepsilon(t) \qquad |e_r^\infty| = 0,\ \text{when } r(t) = 2t\varepsilon(t)$$

$$\begin{cases} |e_r^\infty| \le 0.1,\ \text{when } r(t) = t\varepsilon(t) \\ |y_{d_y}^\infty| = 0,\ \text{when } d_y(t) = \delta_y\varepsilon(t), |\delta_y| \le 0.4 \end{cases}$$

$$\begin{cases} |e_r^\infty| \le 0.1,\ \text{when } r(t) = t\varepsilon(t) \\ |y_{d_y}^\infty| = 0,\ \text{when } d_y(t) = \delta_y\varepsilon(t), |\delta_y| \le 0.4 \\ |y_{d_a}^\infty| \le 0.01,\ \text{when } d_a(t) = \delta_a\varepsilon(t), |\delta_a| \le 2 \end{cases}$$

Steady state requirements

- According to the steady state analysis, the quantities

$$|e_r^\infty|, |y_{d_y}^\infty|, |y_{d_a}^\infty|$$

depend on

  - the number g of poles at the origin of the loop transfer function L(s)
  - the gain $K_g = \lim_{s\to 0} s^g L(s)$

- Therefore requirements on $|e_r^\infty|, |y_{d_y}^\infty|, |y_{d_a}^\infty|$.
- Can be translated into requirements on K_g and g.

Steady state design

- How requirements on K_g and g can be taken into account in the controller design ?
- To this end, let us express C(s) and G(s) as

$$C(s) = K_C \frac{N'_c(s)}{s^{g_c} D'_c(s)}, G(s) = K_G \frac{N'_G(s)}{s^{g_G} D'_G(s)}$$

$$\lim_{s\to 0} N'_c(s) = \lim_{s\to 0} D'_c(s) = \lim_{s\to 0} N'_G(s) = \lim_{s\to 0} D'_G(s) = 1$$

- We have:

$$L(s) = \frac{K_g}{s^g} \cdot \frac{N'_L(s)}{D'_L(s)} \rightarrow L(s) = C(s)G(s) = \frac{K_C K_G}{s^{g_c+g_G}} \cdot \frac{N'_c(s) N'_G(s)}{D'_c(s) D'_G(s)}$$

- Then:

$$K_g = K_C K_G, g = g_C + g_G$$

Steady state design

$$K_g = K_C K_G, g = g_C + g_G$$

- Now, since K_G and g_G are fixed, the required values of K_g and g can be obtained through suitable choices of K_C and g_C
- Thus, in order to proceed systematically with the design of C(s) we can express it as

$$C(s) = C_{SS}(s)\, C_T(s)$$

steady state controller **transient controller**

- where: $C_{SS}(s) = \dfrac{K_C}{s^{g_c}}, C_T(s) = \dfrac{N'_c(s)}{D'_c(s)}, \lim_{s\to 0} \dfrac{N'_c(s)}{D'_c(s)} = 1$

Steady state design

$$C_{SS}(s) = \frac{K_C}{s^{g_c}}$$

- Therefore $C_{SS}(s)$ must
  - add a suitable number of poles at the origin in order to get the required value of g (system type).
  - tune the value of K_c in order to get the required value of K_g (care has to be taken in the choice of sign of K_c to guarantee closed loop stability at the end of the design, in this regard, see the next Remark).
- At the end of the design of $C_{SS}(s)$ the closed loop system may result unstable → don't worry: the design is not finished.

The transient controller $C_T(s)$ will be designed in order to guarantee stability and to meet the desired performance on the transient behavior (to be defined) ensuring the condition

$$\lim_{s\to 0} C_T(s) = \lim_{s\to 0} \frac{N'_c(s)}{D'_c(s)} = 1$$

Steady state design: Remark 1

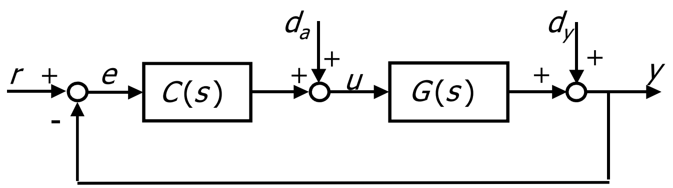
In order to choose the sign of K_c to ensure stability of the feedback system, the following procedure can be adopted:

1. draw the Nyquist diagram of the loop tf $L'(s) = C_{SS}(s)G(s)$ using the sign of K_c so that $K_g = K_G K_c > 0$
   a. if $L'(s)$ leads to a stable feedback system, then this is the correct sign choice, provided that $C_T(s)$ will be designed to avoid significant modifications of the frequency response of $L'(s)$ near the critical point
   b. elseif $L'(s)$ leads to an unstable feedback system, discuss whether a suitable choice of $C_T(s)$ may be able to stabilize the feedback system (e.g. through a phase lead action) around the critical point → in case of success, this is the correct sign choice
2. if both 1.a. and 1.b. fail, repeat the procedure changing the sign of K_c wrt to 1. to verify that this is the correct sign choice (see next Remark)

Steady state design: Remark 2

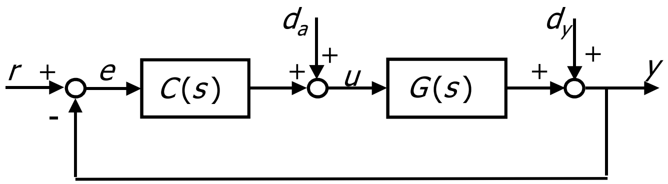
The procedure for the choice of the sign of K_c always gives a positive outcome in the view of the following Theorem (given without proof).

Theorem For every plant described by the minimal transfer function $G(s)$ of order n there exists at least a controller $C(s)$ of order n such that the feedback structure below is stable.



Steady state design: examples

Steady state design: example 1



$$G(s)=\frac{s+1}{(s+2)(s+4)} \qquad d_y(t)=\delta_y\varepsilon(t),\ |\delta_y|\le 0.1$$

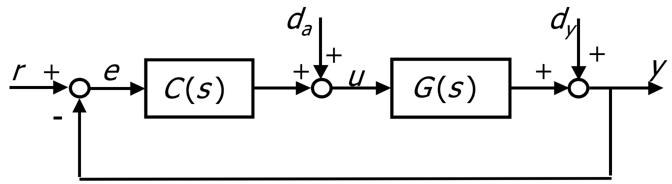
Design a steady state controller $C_{SS}(s)$ such that the following requirements are met:

$|e_r^\infty| \le 0.1$ for $r(t)=0.5t\varepsilon(t)$

$|y_{d_y}^\infty| \le 0.001$

(Result: $C_{SS}(s) = 40/s$)

Steady state design: example 2



$$G(s)=\frac{(s+3)(s+8)}{s(s+1)^2} \qquad d_y(t)=\delta\, t\varepsilon(t),\ |\delta|\le 0.1$$

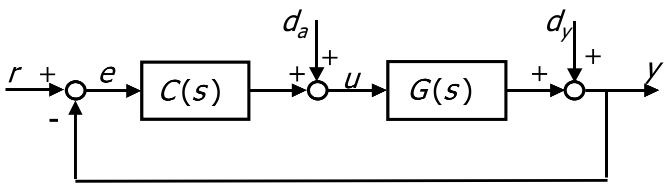
Design a steady state controller $C_{SS}(s)$ such that the following requirements are met:

$|e_r^\infty| \le 0.1$ for $r(t)=t^2\varepsilon(t)$

$|y_{d_y}^\infty| = 0$

(Result: $C_{SS}(s) = 0.9/s$)

Steady state design: example 3



$$G(s)=\frac{500}{s^2-900}$$

Design a steady state controller $C_{SS}(s)$ such that the following requirement is met:

$|e_r^\infty| = 0$ for $r(t)=\varepsilon(t)$

(Result: $C_{SS}(s) = 1/s$)