5.3　分部积分公式

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$$\int xe^{x}dx = ?$$

设函数$u=u(x)$和$v=v(x)$可导,

$$(uv)' = u'v + uv',$$

$$uv' = (uv)' - u'v,$$

$$\int uv'dx = uv - \int u'vdx,$$ 分部积分公式

选u和v的总原则: 1. v易求;

2. $\int v\,du$比$\int u\,dv$易求.

定理 5.3.1

设函数 $u(x)$ 和 $v(x)$ 可导，且存在原函数, 则 $u(x)v'(x)$ 存在原函数, 并有

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx$$

例1 求积分 $\int x\cos x dx$.

解（一） 令 $u=\cos x$, $xdx = d\left(\frac{x^2}{2}\right) = dv$

$$\int x\cos x dx = \frac{x^2}{2}\cos x + \int \frac{x^2}{2}\sin x dx$$

显然，u, v' 选择不当，积分更难.

解（二） 令 $u = x$, $\cos x dx = d(\sin x) = dv$

$$\int x\cos x dx = x\sin x - \int \sin x dx$$

$$= x\sin x + \cos x + C.$$

例2 求积分 $\int x^2 e^x dx.$

解 $u = x^2,\qquad e^x dx = de^x = dv,$

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x dx$$

(再次使用分部积分法) $u = x,\quad e^x dx = dv$

$$= x^2 e^x - 2(xe^x - e^x) + C.$$

总结:

若被积函数是幂函数和正(余)弦函数或幂函数和指数函数的乘积，就考虑设幂函数为u，使其降幂一次(假定幂指数是正整数)

例3 求积分 $\int x\arctan x dx$.

解 令 $u=\arctan x$， $xdx=d\frac{x^2}{2}=dv$

$$\int x\arctan x dx=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}d(\arctan x)$$

$$=\frac{x^2}{2}\arctan x-\int\frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$=\frac{x^2}{2}\arctan x-\int\frac{1}{2}\cdot(1-\frac{1}{1+x^2})dx$$

$$=\frac{x^2}{2}\arctan x-\frac{1}{2}(x-\arctan x)+C.$$

例4 求积分 $\int x^3 \ln x dx$.

解 $u = \ln x$, $x^3 dx = d\dfrac{x^4}{4} = dv$,

$$\int x^3 \ln x dx = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C.$$

总结 若被积函数是幂函数和对数函数或幂函数和反三角函数的乘积，就考虑设对数函数或反三角函数为 u.

例5 求积分 $\int \sin(\ln x)dx$. 造循环

解 $\boxed{\int \sin(\ln x)dx} = x\sin(\ln x) - \int x d[\sin(\ln x)]$

$$= x\sin(\ln x) - \int x\cos(\ln x)\cdot\frac{1}{x}dx$$

$$= x\sin(\ln x) - x\cos(\ln x) + \int x d[\cos(\ln x)]$$

$$= x[\sin(\ln x) - \cos(\ln x)] - \boxed{\int \sin(\ln x)dx}$$

$$\therefore \int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C.$$

例6 求积分 $\int e^x \sin x dx$.

解 $\boxed{\int e^x \sin x dx} = \int \sin x de^x$

$= e^x \sin x - \int e^x d(\sin x)$

$= e^x \sin x - \int e^x \cos x dx = e^x \sin x - \int \cos x de^x$

$= e^x \sin x - (e^x \cos x - \int e^x d \cos x)$

$= e^x (\sin x - \cos x) - \boxed{\int e^x \sin x dx}$

$\therefore \int e^x \sin x dx = \dfrac{e^x}{2}(\sin x - \cos x) + C.$

例7 计算$\int \cos^n x dx$, $\int \sin^n x dx$, 其中$n \in N^*$.

解: $\because \cos^n x = \cos x \cos^{n-1} x = (\sin x)' \cos^{n-1} x$,

$$\therefore \int \cos^n x dx = \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x \sin^2 x dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)dx$$

$$= \sin x \cos^{n-1} x + (n-1)\int \cos^{n-2} x dx - (n-1)\int \cos^n x dx$$

$$\therefore \int \cos^n x dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

递推法

而$\int \cos x dx = \sin x + C$，$\int dx = x + C$是确定的.

类似可求得:

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx.$$

例8 求积分 $\int \frac{x\arctan x}{\sqrt{1+x^2}}dx.$

解 $\because \quad \left(\sqrt{1+x^2}\right)' = \frac{x}{\sqrt{1+x^2}},$

$$\therefore \quad \int \frac{x\arctan x}{\sqrt{1+x^2}}dx = \int \arctan x d\sqrt{1+x^2}$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}d(\arctan x)$$

$$= \sqrt{1+x^2}\arctan x - \int \sqrt{1+x^2}\cdot\frac{1}{1+x^2}dx$$

$$=\sqrt{1+x^2}\arctan x-\boxed{\int\frac{1}{\sqrt{1+x^2}}dx}$$

令 $x=\tan t$

$$\int\frac{1}{\sqrt{1+x^2}}dx=\int\frac{1}{\sqrt{1+\tan^2 t}}\sec^2 t\,dt=\int\sec t\,dt$$

$$=\ln|\sec t+\tan t|+C=\ln(x+\sqrt{1+x^2})+C$$

$$\therefore\quad\int\frac{x\arctan x}{\sqrt{1+x^2}}dx$$

$$=\sqrt{1+x^2}\arctan x-\ln(x+\sqrt{1+x^2})+C.$$

例 9　已知 $f(x)$ 的一个原函数是 e^{-x^2}，求 $\int xf'(x)dx$.

解　$\int xf'(x)dx = \int xdf(x) = xf(x) - \int f(x)dx,$

$\because \left(\int f(x)dx\right)' = f(x), \quad \therefore \int f(x)dx = e^{-x^2} + C,$

两边同时对 x 求导, 得　$f(x) = -2xe^{-x^2},$

$\therefore \int xf'(x)dx = xf(x) - \int f(x)dx$

$= -2x^2e^{-x^2} - e^{-x^2} + C.$

例10: $\int \ln\left(x+\sqrt{1+x^2}\right)dx$

$$= x\ln\left(x+\sqrt{1+x^2}\right)-\int x\left(\ln\left(x+\sqrt{1+x^2}\right)\right)' dx$$

$$= x\ln\left(x+\sqrt{1+x^2}\right)-\int \frac{x}{\sqrt{1+x^2}}dx$$

$$= x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+c$$

例 11: $\int \frac{x e^{\arctan x}}{(1+x^2)^{\frac{3}{2}}} dx$

造循环

$$\int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx = \int \left(e^{\arctan x}\right)' \frac{x}{\sqrt{1+x^2}}dx = e^{\arctan x}\frac{x}{\sqrt{1+x^2}} - \int e^{\arctan x}\left(\frac{1}{1+x^2}\right)^{\frac{3}{2}}dx$$

$$= e^{\arctan x}\frac{x}{\sqrt{1+x^2}} - \int \left(e^{\arctan x}\right)' \frac{1}{\sqrt{1+x^2}}dx$$

$$= e^{\arctan x}\frac{x}{\sqrt{1+x^2}} - \left(e^{\arctan x}\right)\frac{1}{\sqrt{1+x^2}} - \int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx$$

$$\Rightarrow \int \frac{xe^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}dx = \frac{1}{2}\left(e^{\arctan x}\frac{x}{\sqrt{1+x^2}} - \left(e^{\arctan x}\right)\frac{1}{\sqrt{1+x^2}}\right) + c$$

例12：建立递推关系式

$$I_n = \int \frac{dx}{\sin^n x}, \quad D_n = \int \sin^n x dx$$

解：

$$I_n = \int \frac{dx}{\sin^n x} = -\int \frac{d\cot x}{\sin^{n-2} x}$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (n-2)\int \frac{\cos^2 x}{\sin^n x} dx$$

$$= -\frac{\cot x}{\sin^{n-2} x} + (n-2) I_n - (n-2) I_{n-2}$$

$$\Rightarrow I_n = -\frac{\cot x}{(n-1)\sin^{n-2} x} + \frac{(n-2)}{(n-1)} I_{n-2}$$

$$D_n = \int \sin^n x dx = -\int \sin^{n-1} x d\cos x$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x \cos^2 x dx$$

$$= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x\left(1-\sin^2 x\right) dx$$

$$= -\cos x \sin^{n-1} x + (n-1) D_{n-2} - (n-1) D_n$$

$$\Rightarrow D_n = \frac{-\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} D_{n-2}$$

选择 u 的有效方法:LIATE选择法

L----对数函数；　　　　**I**----反三角函数；

A----代数函数；　　　　**T**----三角函数；

E----指数函数；

哪个在前哪个选作 u.

作业：

习题5.3　(5) (6) (8) (10)