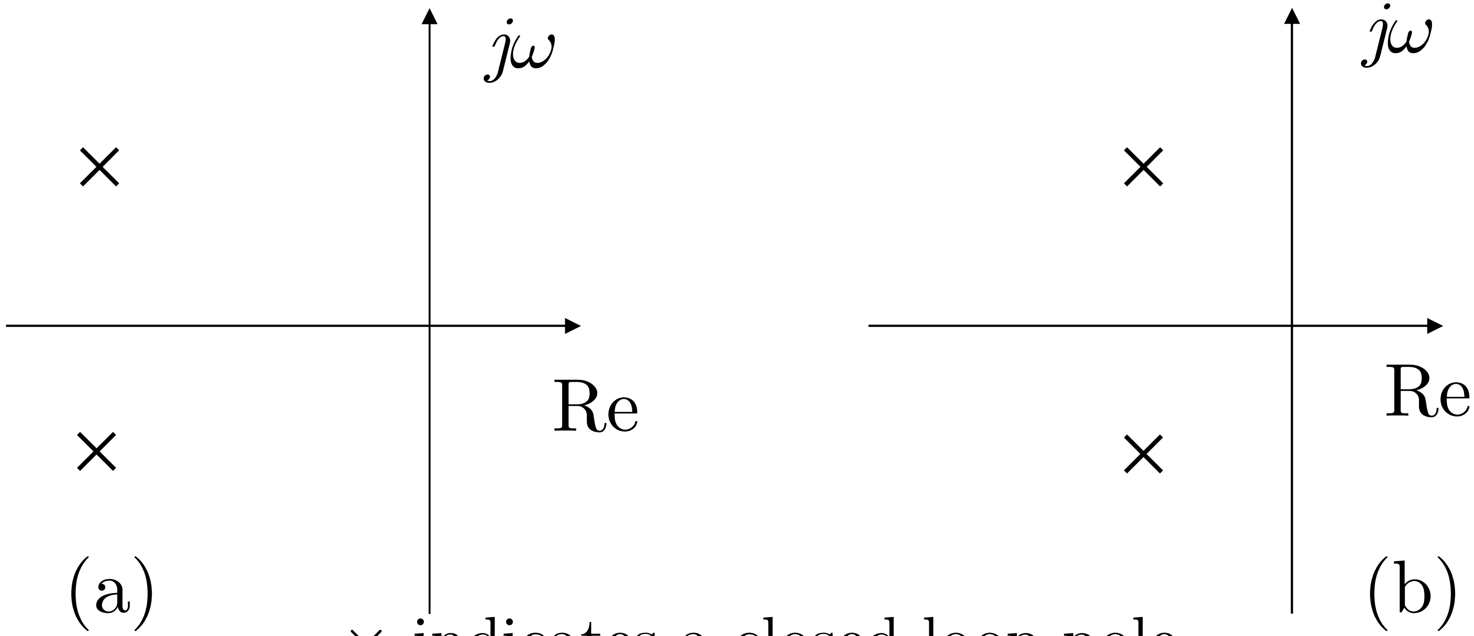
7-5 Relative Stability

1. Concept of Relative Stability

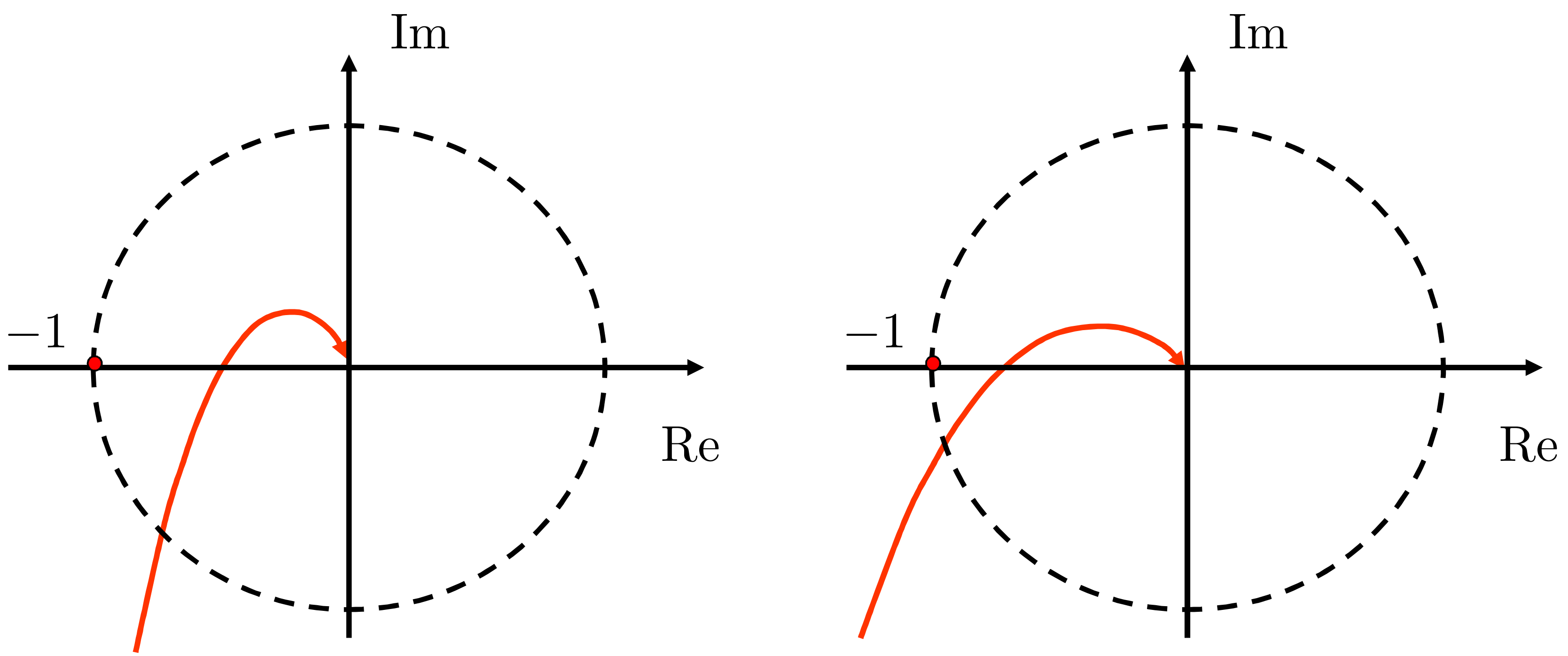
In designing a control system, it is required that the system be stable. Furthermore, it is necessary that the system have **adequate relative stability**.

Consider the two systems shown below:



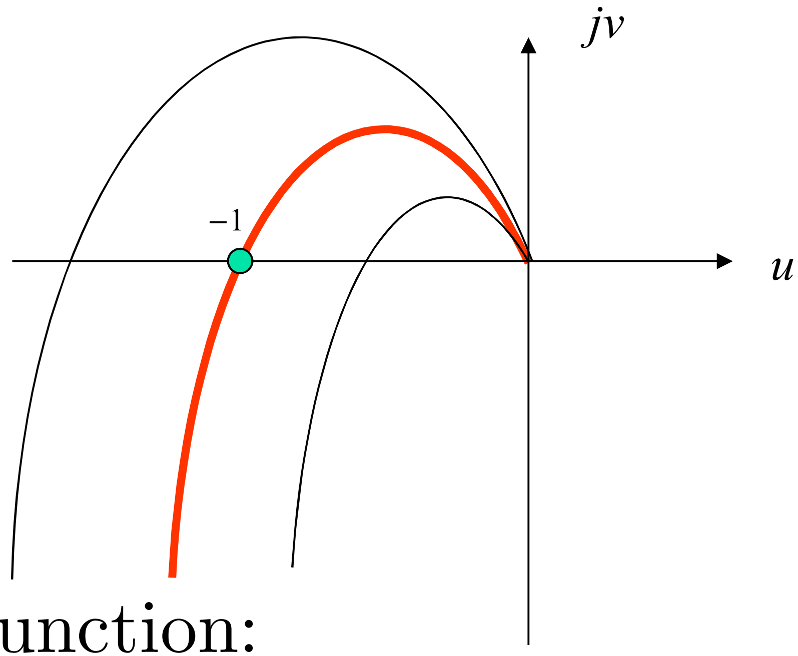
$\times$ indicates a closed-loop pole.

System (a) is obviously more stable than system (b) because the closed-loop poles of system (a) are located farther left than those of system (b). The Nyquist curves of the two systems show that the closer the closed-loop poles are located to the $j\omega$ axis, the closer the **Nyquist** curve is to the -1 point.



2. Phase and Gain Margins

We can use the closeness of approach of the $G(j\omega)$ locus to the -1 point to measure the margin of stability.



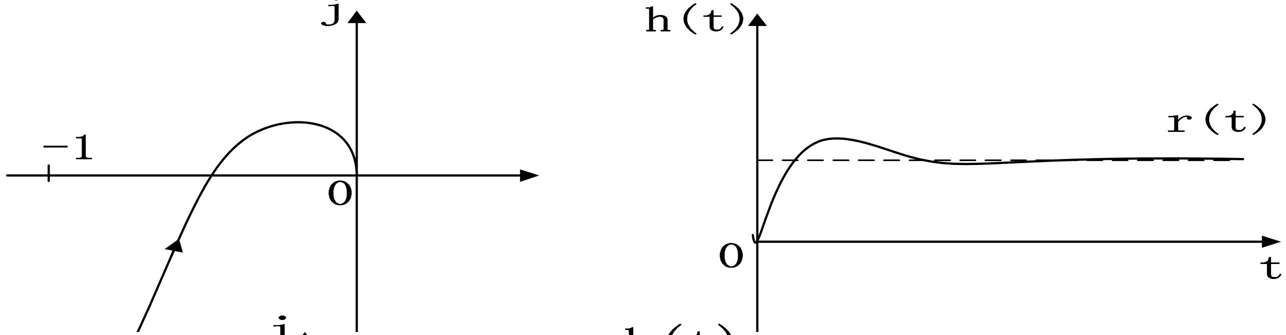
Example. Open loop transfer function:

$$G(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}, \quad T_i > 0, K > 0$$

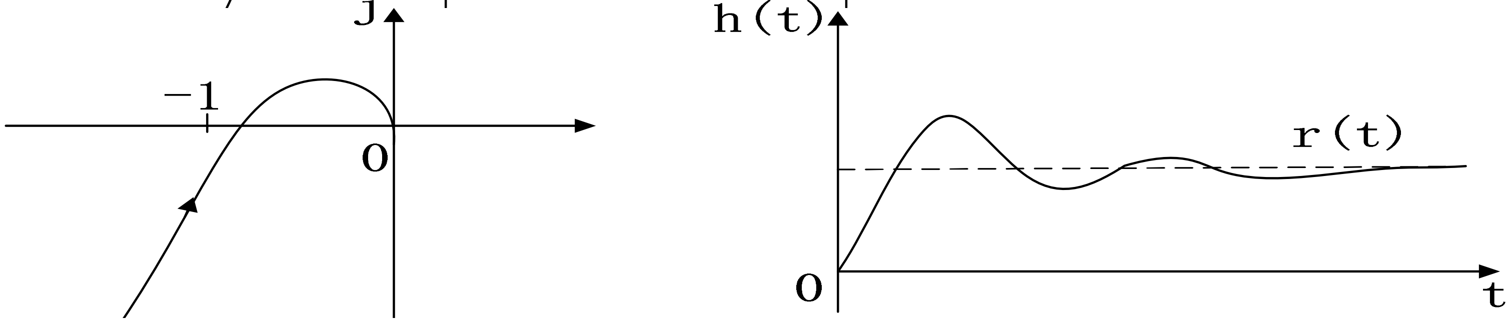
For large value of K, the system is unstable. As the gain is decreased to a certain value, the $G(j\omega)$ locus passes through the -1 point, which implies that with this gain value, the system is on the verge of instability and will exhibit sustained oscillations.

The step responses of the third order open-loop transfer function with four different values of K:

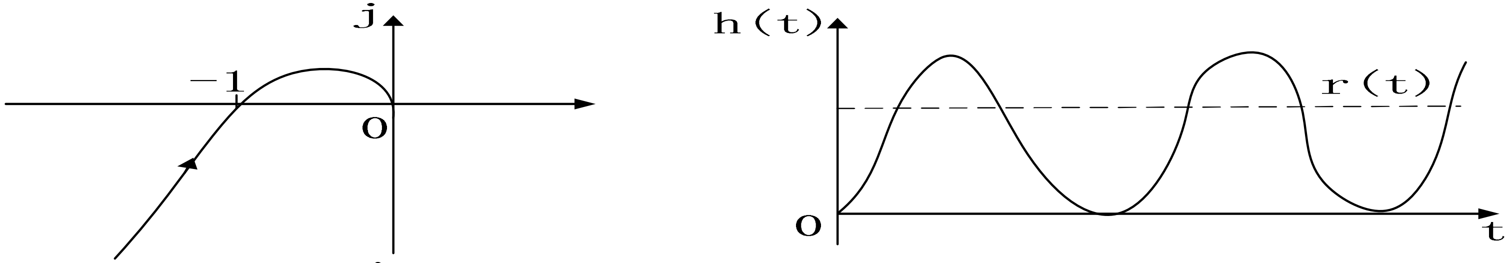
(a)



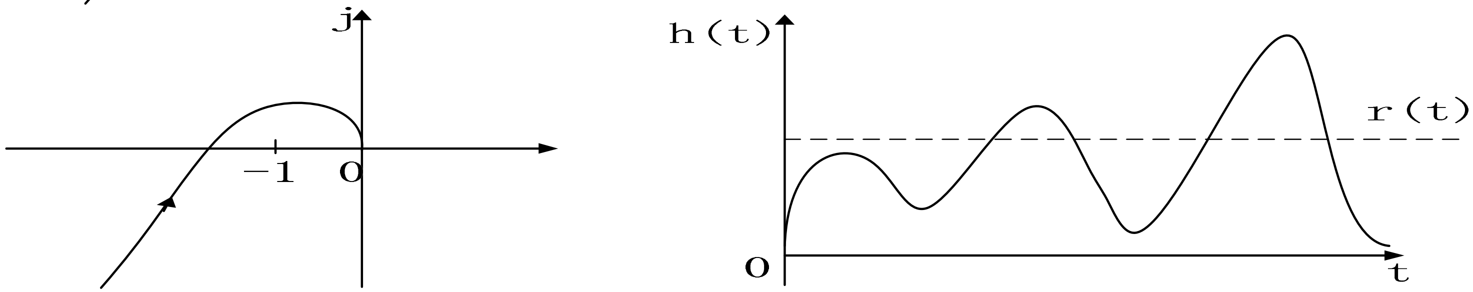
(b)



(c)



(d)

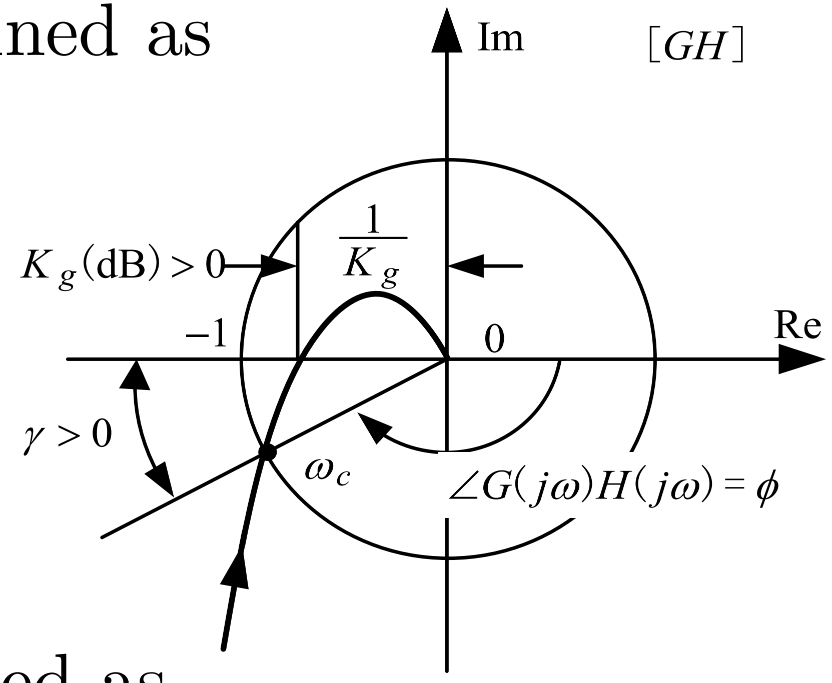


Definitions of Phase and gain margins

1)The phase margin is defined as

$$\gamma = 180^0 + \angle G(j\omega_c)H(j\omega_c)$$

where ω_c is the **gain crossover frequency** at which $|GH(j\omega_c)|=1$.

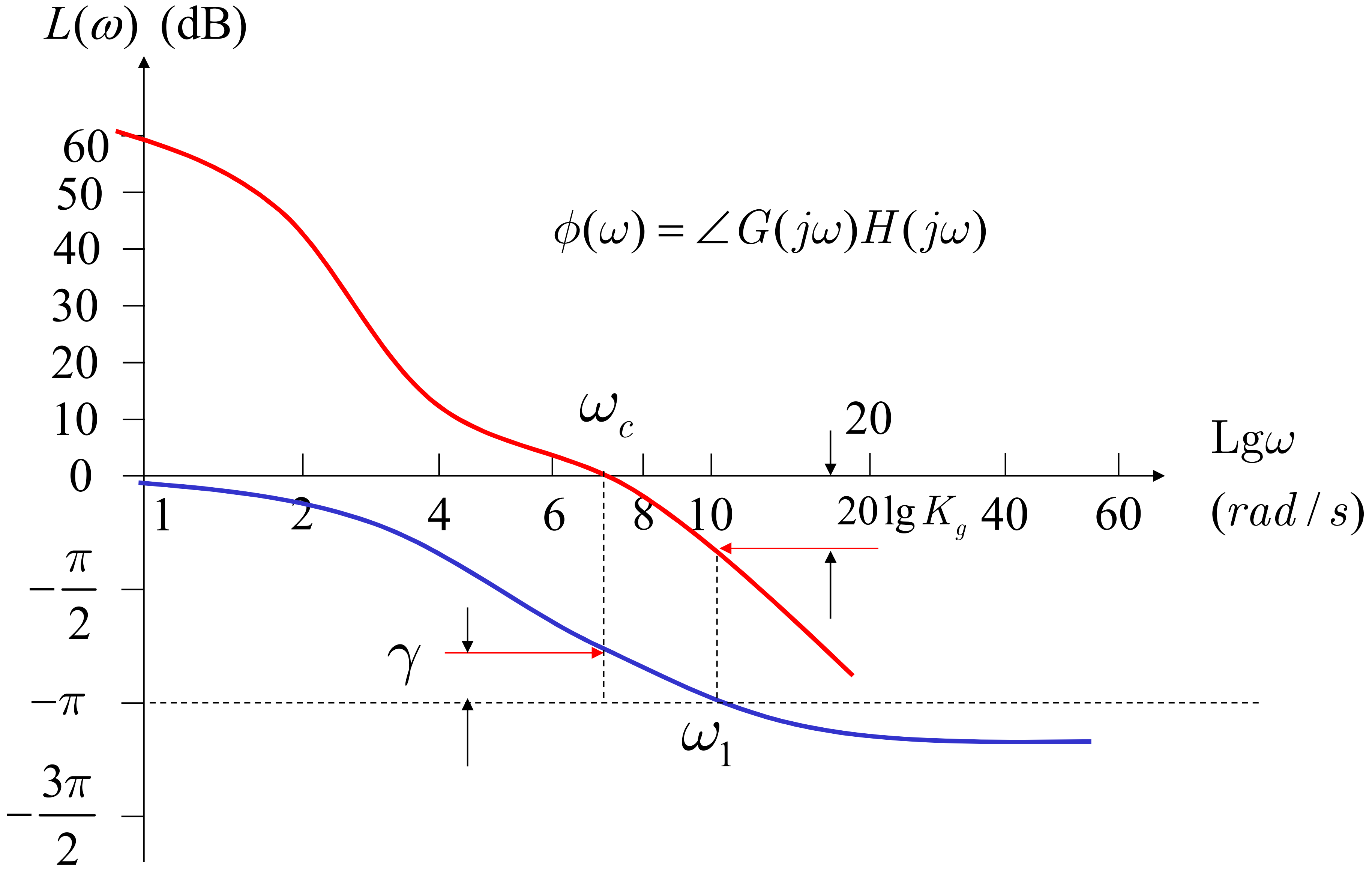


2)The gain margin is defined as

$$K_g = \frac{1}{|G(j\omega_1)H(j\omega_1)|}$$

where ω_1 is the **phase crossover frequency** at which $\angle GH(j\omega_1)=-180^0$.

In terms of decibels $20\lg K_g = -20\lg|G(j\omega_1)H(j\omega_1)|$ (dB)



Example. Given the open-loop transfer function as

$$G(s) = \frac{2}{s(s+1)(\frac{1}{5}s+1)}$$

Determine its phase and gain margins.

Solution: From

$$|G(j\omega_c)| = 1$$

$$\omega_c = 1.2247$$

$$\varphi(\omega_c) = -90^\circ - \tan^{-1}\omega_c - \tan^{-1}\frac{\omega_c}{5} = -154.53^0$$

$$\boxed{\gamma = 180^\circ + \varphi(\omega_c) = 180^0 \text{-} 154.53^0 = 25.47^0}$$

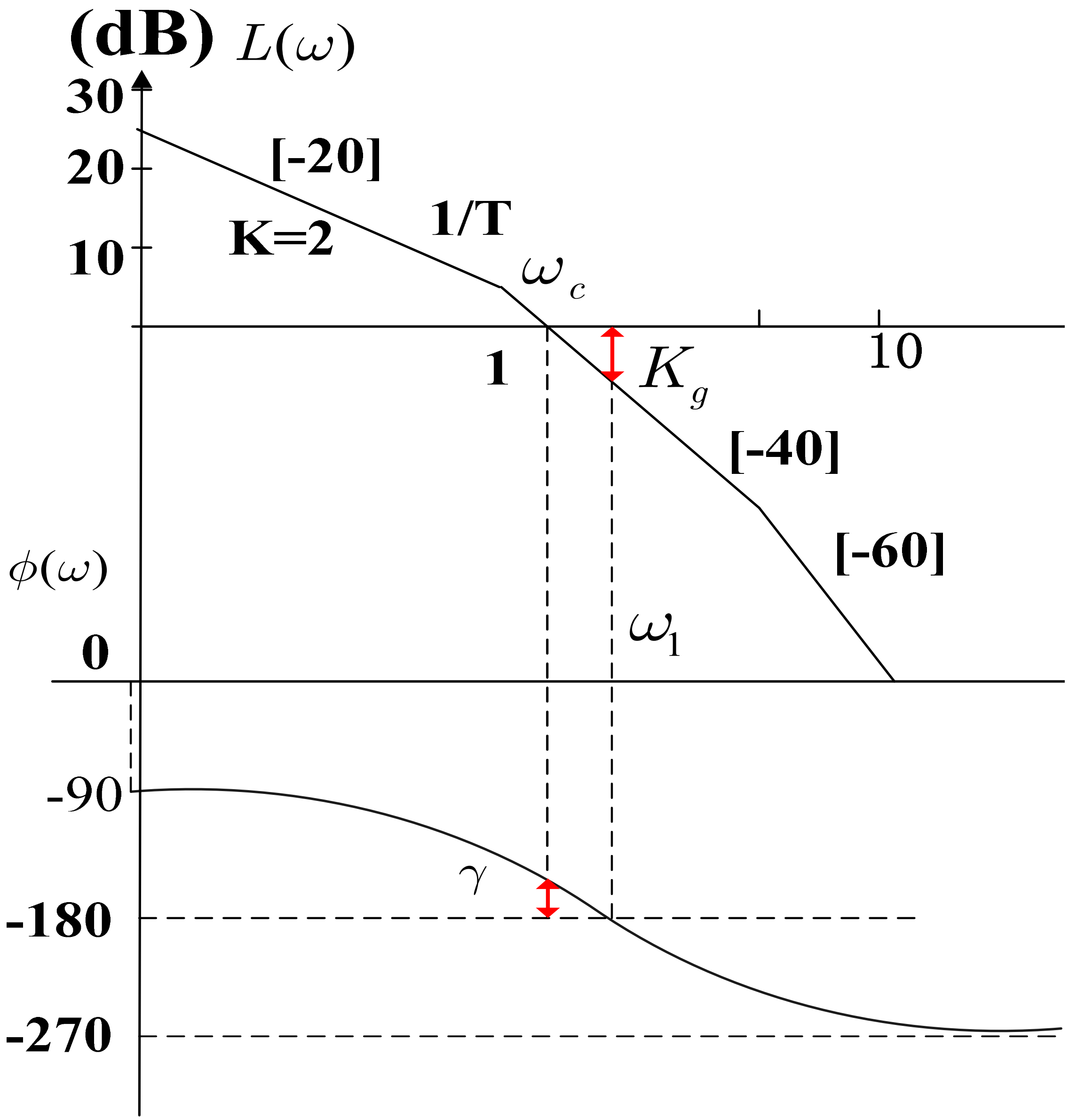
Gain margin:

$$K_g = \frac{1}{|G(j\omega_1)|}$$

$$\omega_1 = \sqrt{5}$$

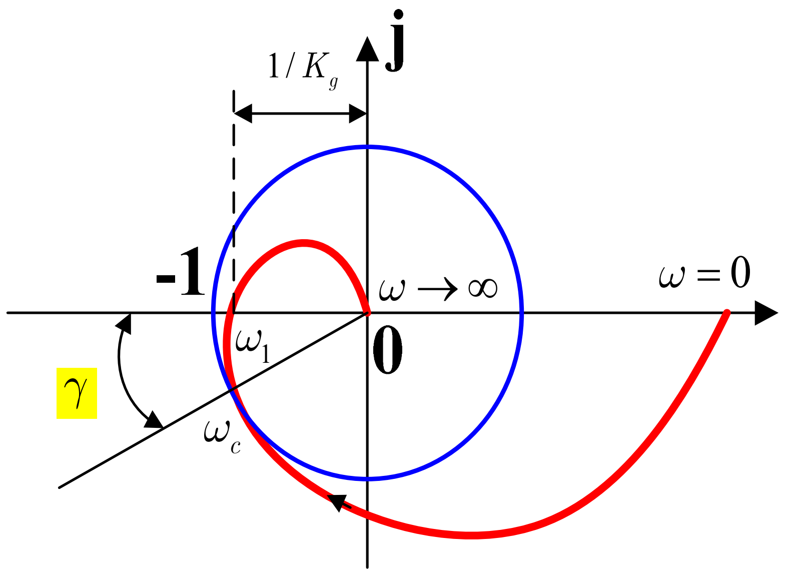
$$\boxed{K_g = 9.54}$$

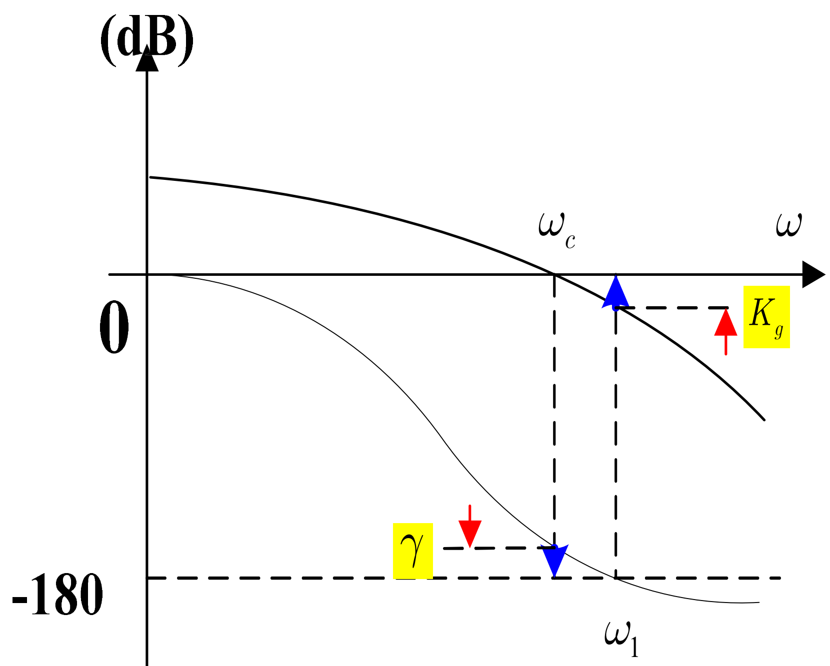
The gain and phase margins can also be measured from Bode diagram:



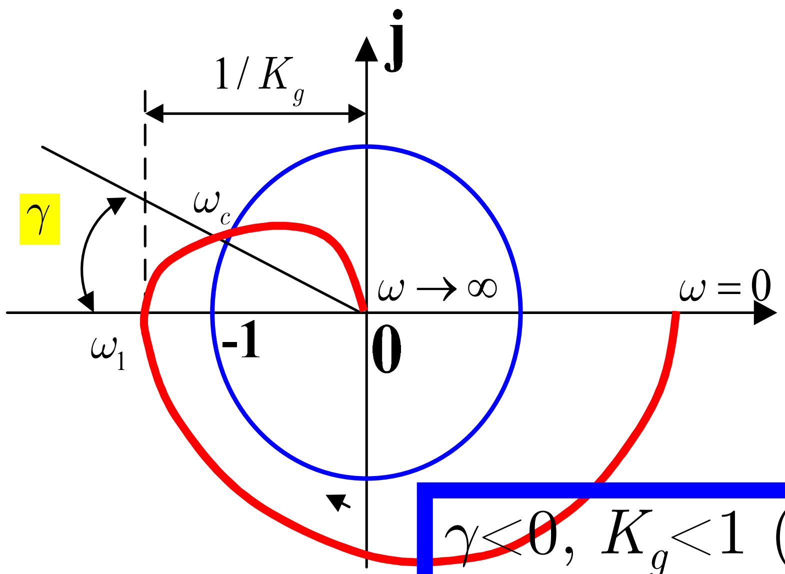
3. A few Comments on Phase and Gain margins

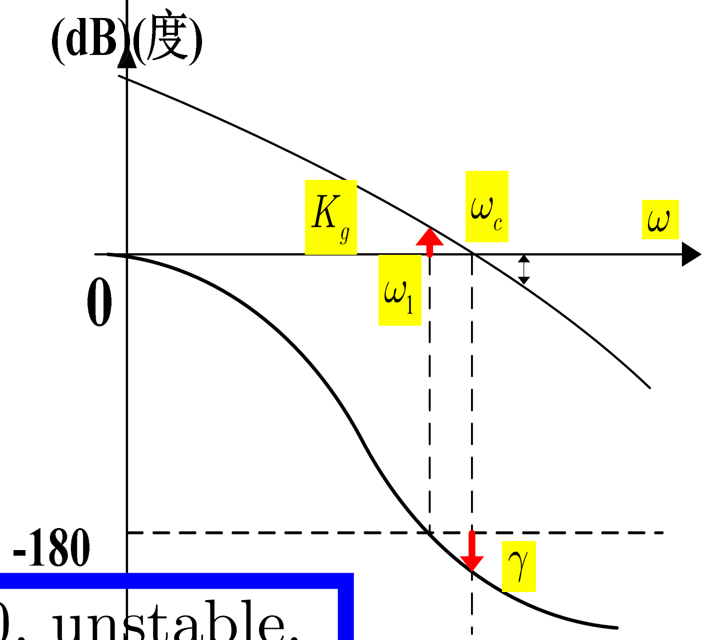
1) Phase and gain margins, as a measure of the closeness to the -1 point, can be used as design criteria.
2) Both margins should be used in the determination of relative stability (that is, gain or phase margin alone is not enough).
3) For minimum phase systems, both γ and K_g must be positive for the system to be stable. **Negative margins indicate instability**.





$\gamma>0,\ K_g>1\ (\mathrm{dB})>0$, stable.

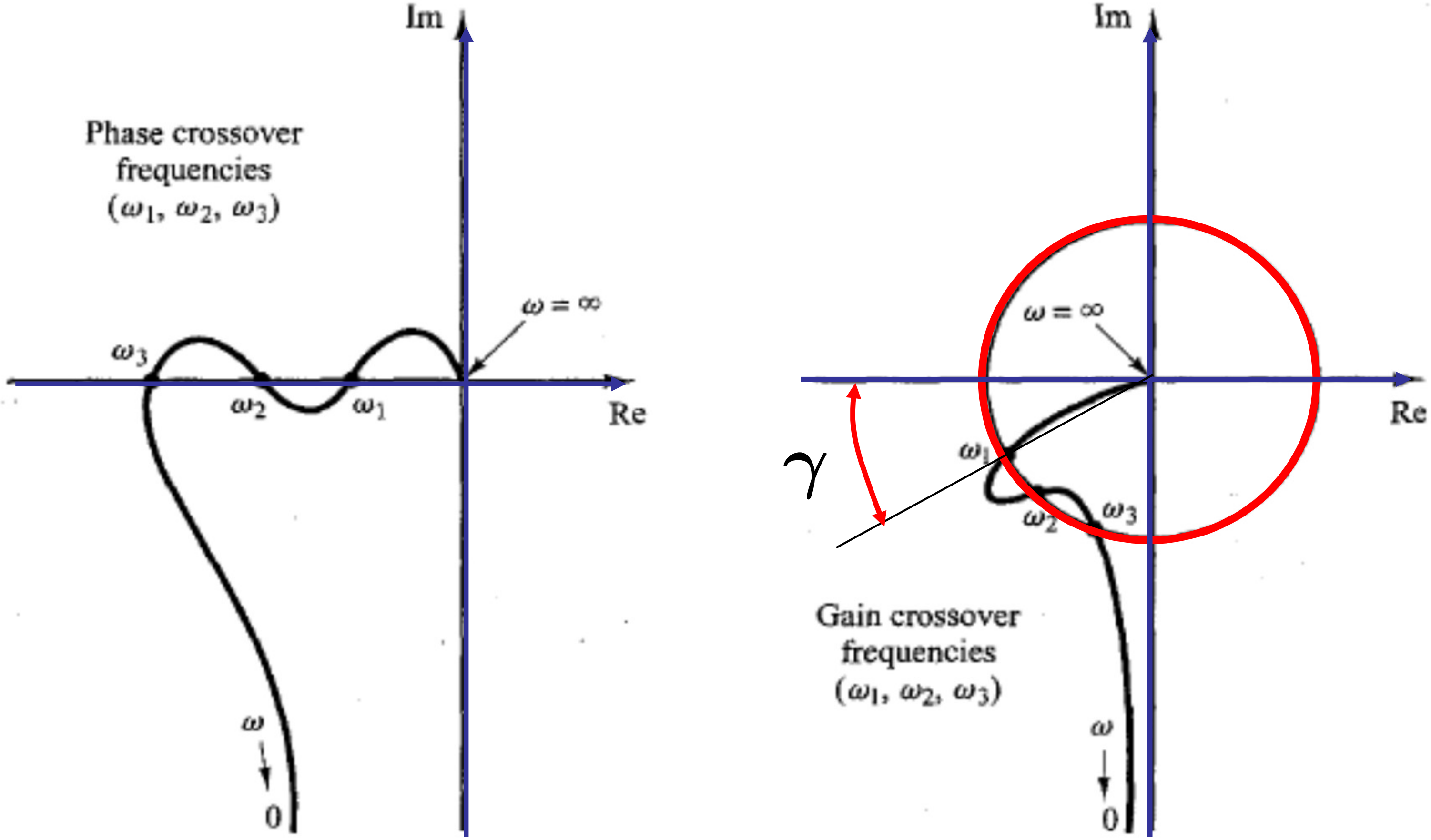




$\gamma<0,\ K_g<1\ (\mathrm{dB})<0$, unstable.

4) Proper γ and K_g ensure us against variations in the system components and bound the behavior of the closed-loop system near resonant frequency. In general, $30^0 \leq \gamma \leq 60^0$ and $K_g \geq 6$ dB. With these values, a minimum-phase system has guaranteed stability, even if the open-loop gain and time constants of the components vary to a certain extent.

5) Conditionally stable systems will have two or more *phase crossover frequencies*; some higher-order systems with complicated numerator dynamics may also have two or more *gain crossover frequencies*.



For stable systems having two or more gain crossover frequencies, the phase margin is measured at the **highest** gain crossover frequency.

Example. Given the open-loop transfer function as

$$G(s)=\frac{K}{(s+1)^3} \Rightarrow G(j\omega)=\frac{K}{(j\omega+1)^3}$$

Determine its phase and gain margins when K is chosen as 4 and 10, respectively.

Solution: When K=4,

$$\left|G(j\omega)\right|=\frac{4}{\sqrt{(1+\omega^2)^3}}, \quad \angle G(j\omega)=-3\tan^{-1}\omega$$

Letting $|G(j\omega)|$=1, it can be solved that

$$\omega_c=\sqrt{16^{\frac{1}{3}}-1}=1.233$$

Therefore,

$$\gamma=180^0+\angle G(j\omega_c)=180^0-152.9^\circ=27.1^\circ$$

On the other hand, from

$$\angle G(j\omega_1) = -3\tan^{-1}\omega_1 = -180^0$$

we have

$$\omega_1 = \sqrt{3}$$

$$|G(j\omega_1)| = \frac{4}{\sqrt{(1+3)^3}} = \frac{1}{2}$$

Therefore,

$$K_g = 2$$

When K=10,

$$|G(j\omega)| = \frac{10}{\sqrt{(1+\omega^2)^3}}, \quad \angle G(j\omega) = -3\tan^{-1}\omega$$

Letting $|G(j\omega)|=1$, it can be solved that

$$\omega_c = \sqrt{100^{\frac{1}{3}} - 1} = 1.91,$$

Therefore,

$$\gamma = 180^0 + \angle G(j\omega_c) = 180^0 - 187^0 = -7^0 < 0^0$$

Similarly, from

$$\angle G(j\omega_1) = -3\tan^{-1}\omega_1 = -180^0 \Rightarrow \omega_1 = \sqrt{3}$$

we have

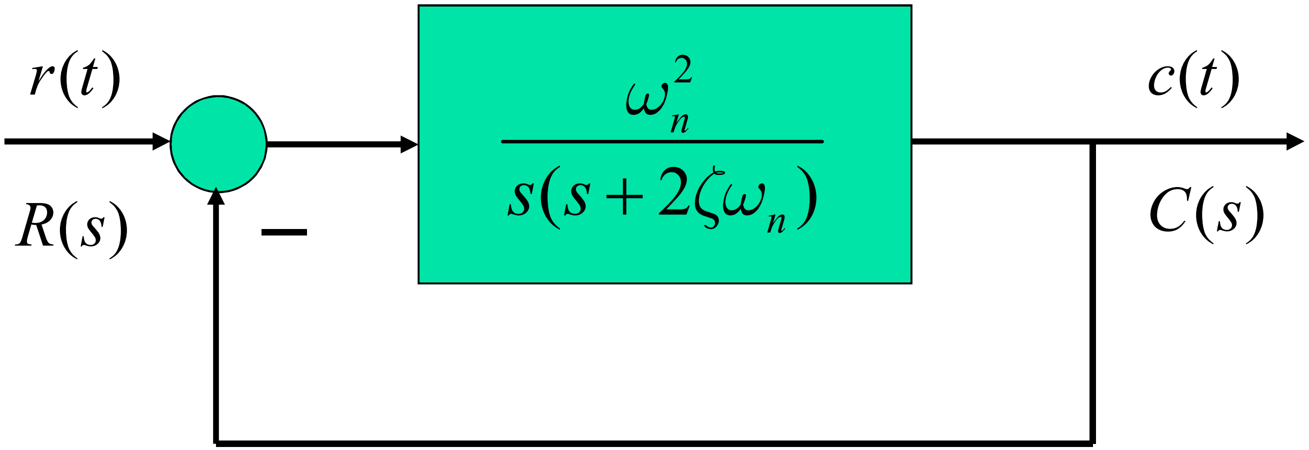
$$\left|G(j\omega_1)\right| = \frac{10}{\sqrt{(1+3)^3}} = \frac{5}{4} > 1$$

Therefore,

$$K_g = \frac{4}{5} < 1$$

The system is unstable!

4. Correlation between Step Transient Response and Frequency Response in the Standard Second-Order System



1) For unit-step input, the output is

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \beta\right)$$

with

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

The maximum percent overshoot is

$$M_p = e^{-\pi\varsigma/\sqrt{1-\varsigma^2}} \times 100\%$$

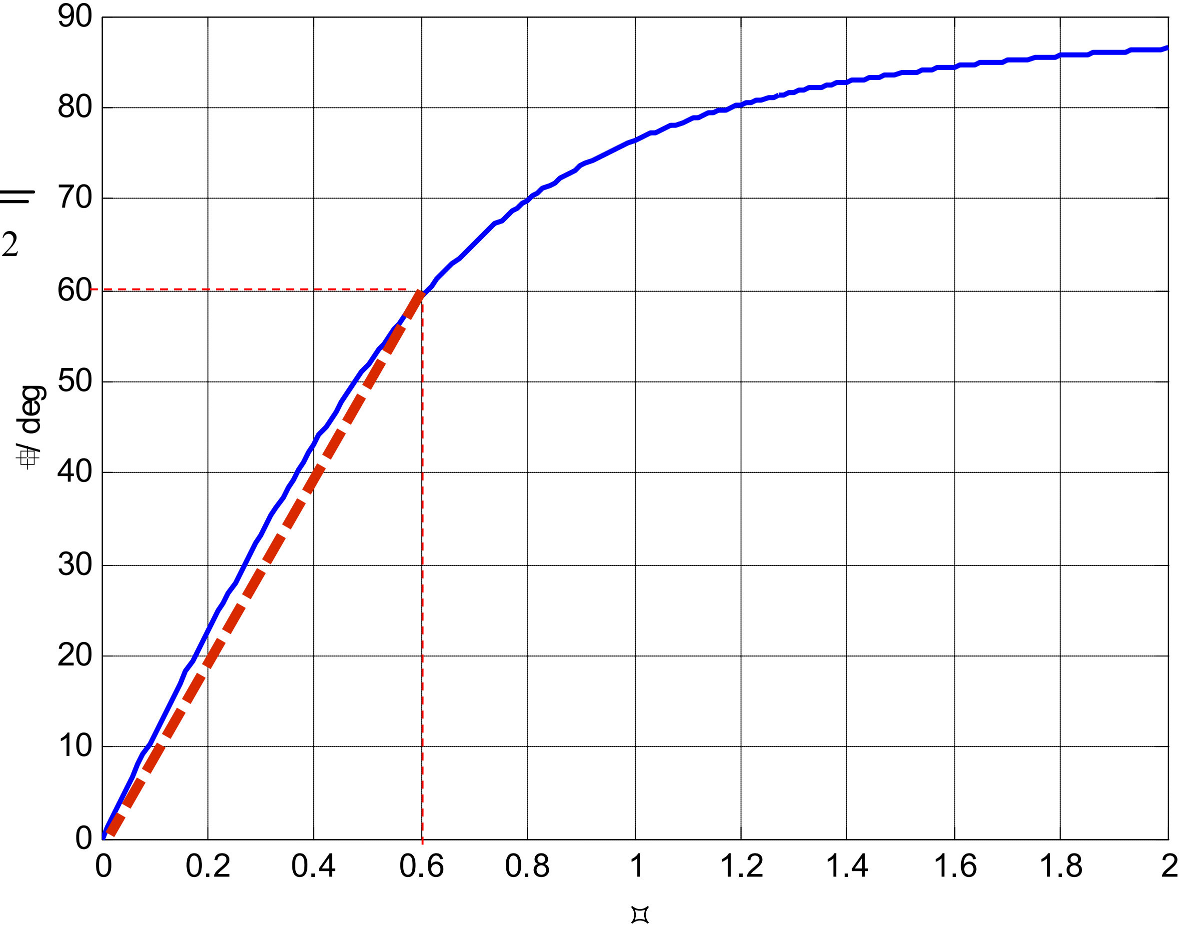
Note that the overshoot becomes excessive for values of $\zeta<0.4$.

2) For the open-loop transfer function, the phase margin is

$$\gamma = 180^0 + \angle G(j\omega_c) = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{(4\zeta^4+1)}-2\zeta^2}}$$

See the appendix for the deduction of the above equation.

$$\gamma = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{(4\zeta^4+1)}-2\zeta^2}}$$



Curve γ versus ζ

Thus a phase margin of 60^0 corresponds to a damping ratio of 0.6.

The correlation between the step transient response and frequency response of the standard second-order system is summarized below.

1) γ and ζ are related approximately by a straight line for $0 \leq \zeta \leq 0.6$, as follows:

$$\zeta = \gamma / 100.$$

For higher order systems having a dominant pair of closed-loop poles, *this relationship may be used as a rule of thumb in estimating the damping ratio from the frequency response.*

2) From

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$\omega_r = \omega_n \sqrt{1-2\zeta^2}$$

it is clear that ω_r and ω_d are almost the same for small values of ζ. Therefore, ω_r is indicative of the speed of the transient response of the system [t_s=3.5/(ζ ω_n)].

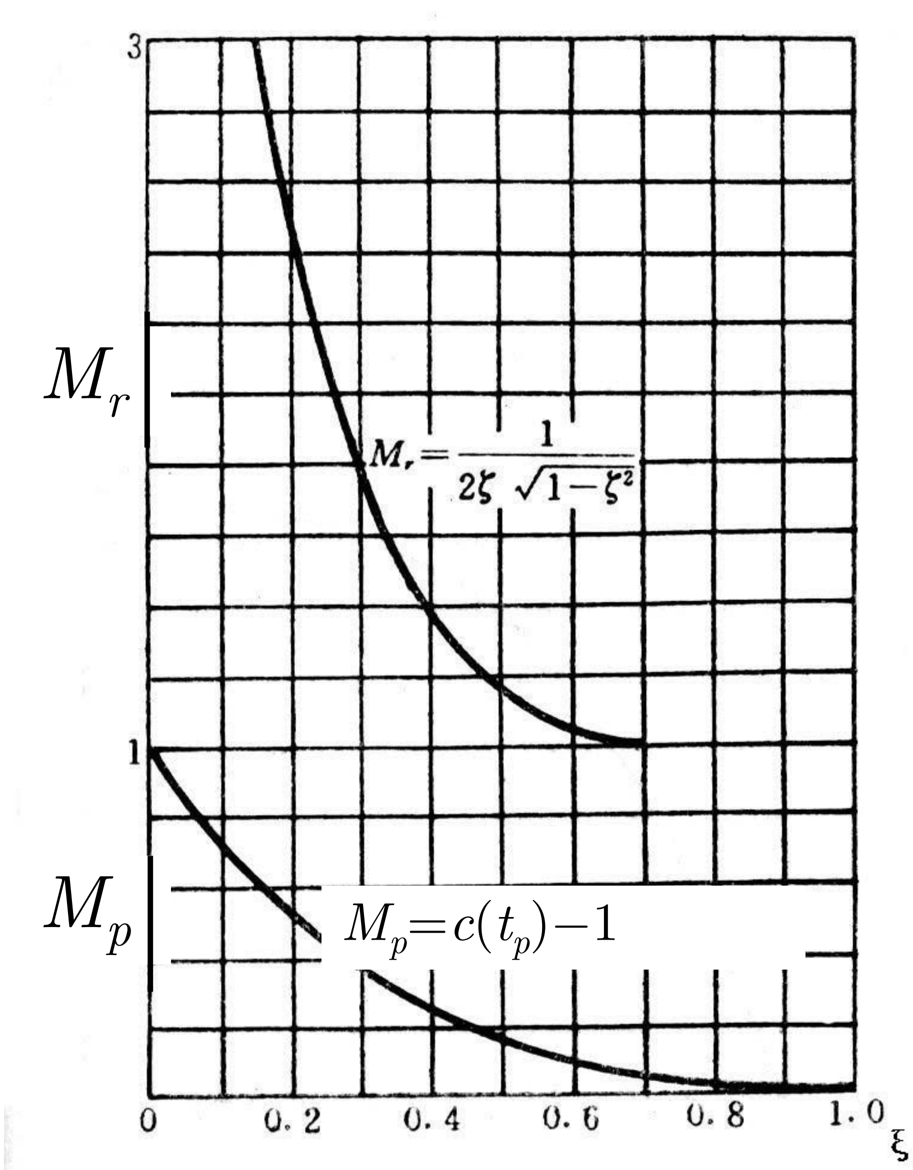
3) From

$$M_p = e^{-\pi\varsigma/\sqrt{1-\varsigma^2}} \times 100\%$$

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, 0 < \zeta \le 0.707$$

The smaller the value of ζ is, the larger the values of M_r and M_p are.

This figure clearly shows the correlation between M_p and M_r. Note that if ζ >0.707, there is no resonant peak; however, oscillations always exist in time domain for $0<\zeta<1$.



5. Correlation between Step Transient Response and Frequency Response in general systems (time domain and frequency domain)

For higher-order systems having a dominant pair of complex conjugate closed-loop poles, the following relationships generally exist between the step transient response and frequency response:

1) The value of M_r is indicative of the relative stability. Satisfactory transient performance should be $1.0 < M_r < 1.4$ (0 dB$< M_r <$3dB, $0.4 < \zeta < 0.7$);
For values of $M_r > 1.5$, the step transient response may exhibit excessive overshoot.

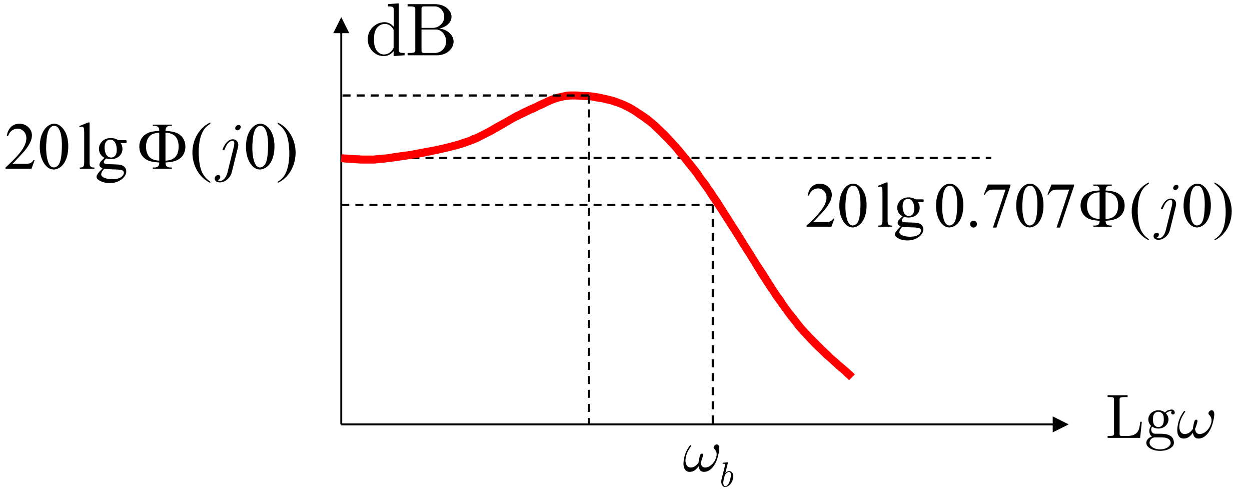
2) The magnitude of the resonant frequency ω_r is indicative of the speed of the transient response. The larger the value of ω_r, the faster the time response is.

3) The resonant peak frequency ω_r and the damped natural frequency ω_d for the step transient response are very close to each other for lightly damped systems.

6. Cutoff Frequency and Bandwidth

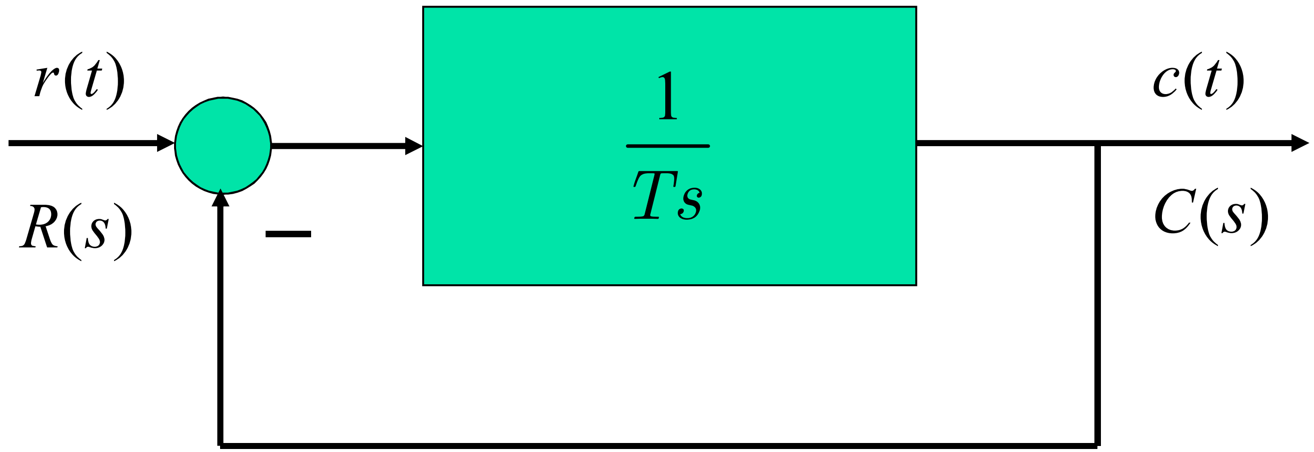
(Performance index for **closed-loop systems**)

1) Definition



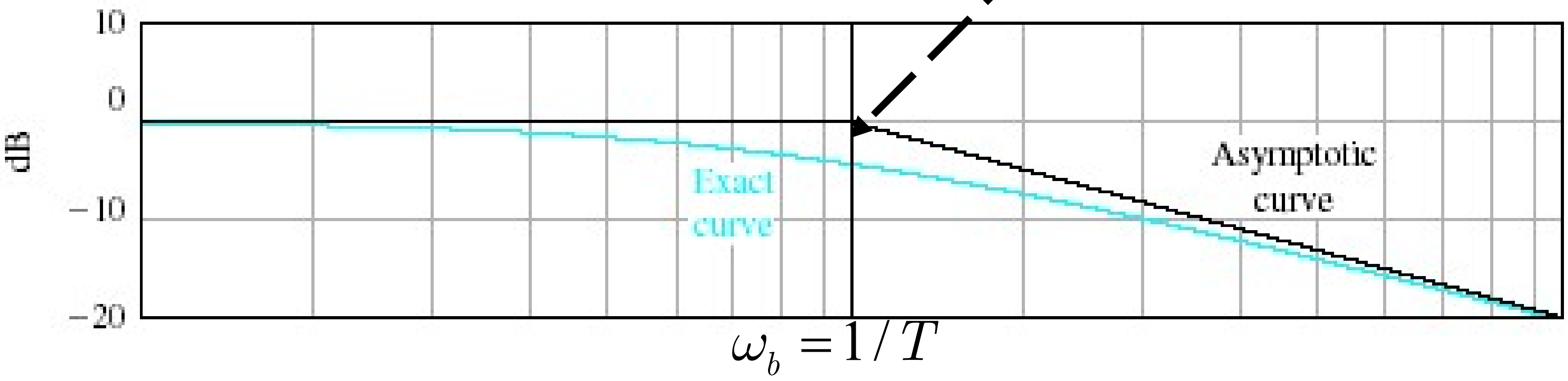
The bandwidth is the frequency range $0 \leq \omega \leq \omega_b$, where ω_b (called cutoff frequency) is the frequency at which the magnitude of the **closed-loop** frequency response is 3 dB below its zero-frequency value $\Phi(j0)$.

Example. Given a first-order system

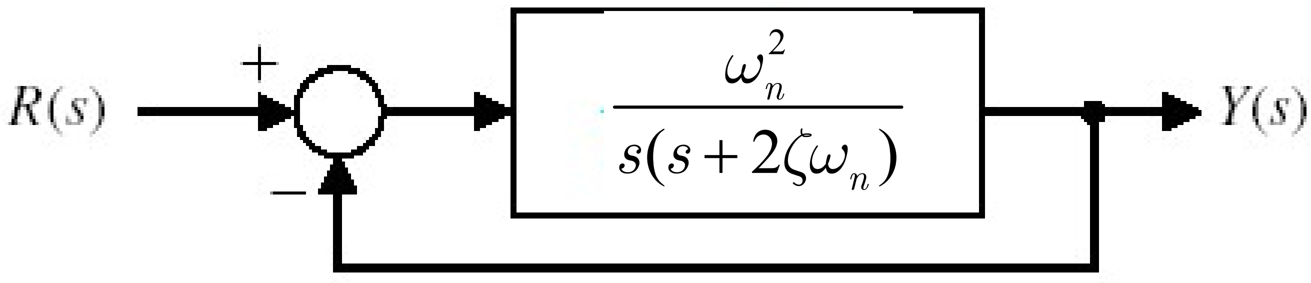


Determine its cutoff frequency ω_b.

$\omega_b = 1/T$! $t_s = 3T$, $T\downarrow \Rightarrow \omega_b\uparrow$



Example. Given a second-order system



Determine its cutoff frequency ω_b.

Since

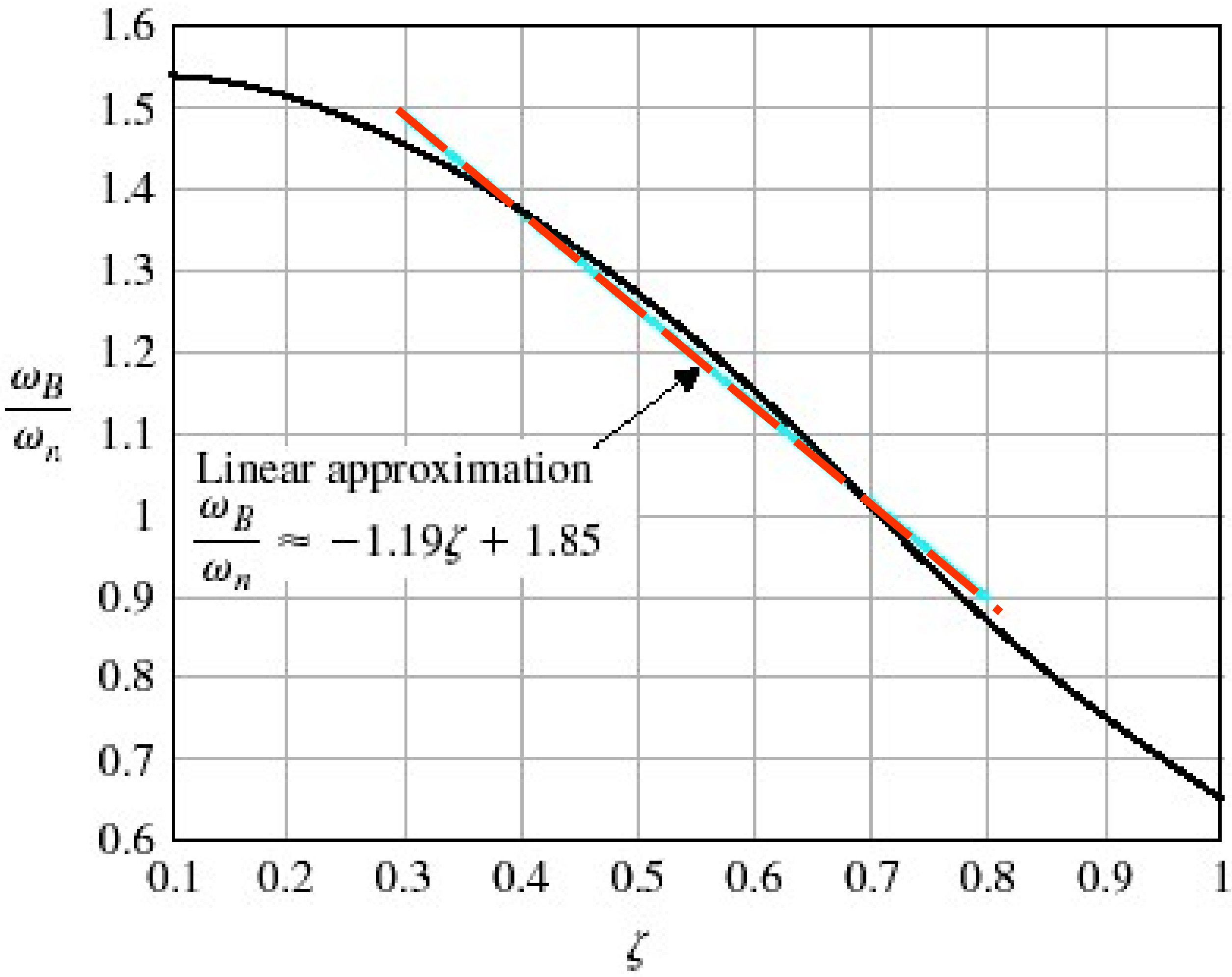
$$|\Phi(j\omega)| = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + 4\zeta^2\omega^2/\omega_n^2}}$$

$$|\Phi(j0)| = 1$$

we obtain that

$$\sqrt{(1-\omega_b^2/\omega_n^2)^2 + 4\zeta^2\omega_b^2/\omega_n^2} = \sqrt{2}$$

$$\omega_b = \omega_n[(1-2\zeta^2) + \sqrt{(1-2\zeta^2)^2+1}]^{\frac{1}{2}}$$



The bandwidth ω_b can be approximately related to the natural frequency ω_n of the system: t_s=$3/\zeta\omega_n$, ω_b=$K\omega_n$; hence, $\omega_n \uparrow \Rightarrow \omega_b \uparrow$.

2) General case

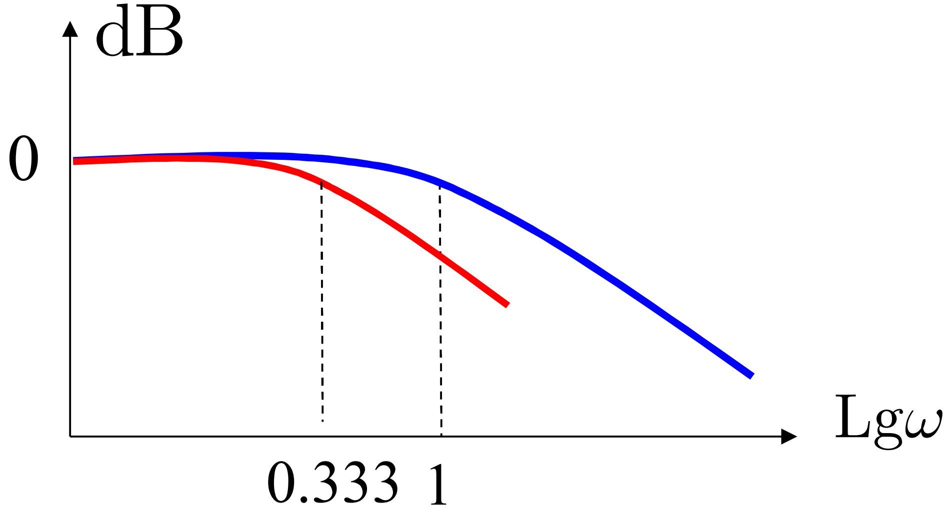
- A large bandwidth corresponds to a fast response;

- The bandwidth ω_b can be approximately related to the natural frequency ω_n of a system;

- For the system to follow arbitrary inputs accurately, it must have a large bandwidth. From the viewpoint of noise, however, the bandwidth should not be too 1arge. Thus there are conflicting requirements on the bandwidth, and a compromise is usually necessary for good design.

Example. Consider the following systems:

$$\frac{C(s)}{R(s)} = \frac{1}{s+1} \qquad \frac{C(s)}{R(s)} = \frac{1}{3s+1}$$



In time domain, the step response for system 1 is faster than that of system 2.

In general, a larger bandwidth implies a faster time response.

Summary of Frequency Response Analysis

1. Definition of frequency response

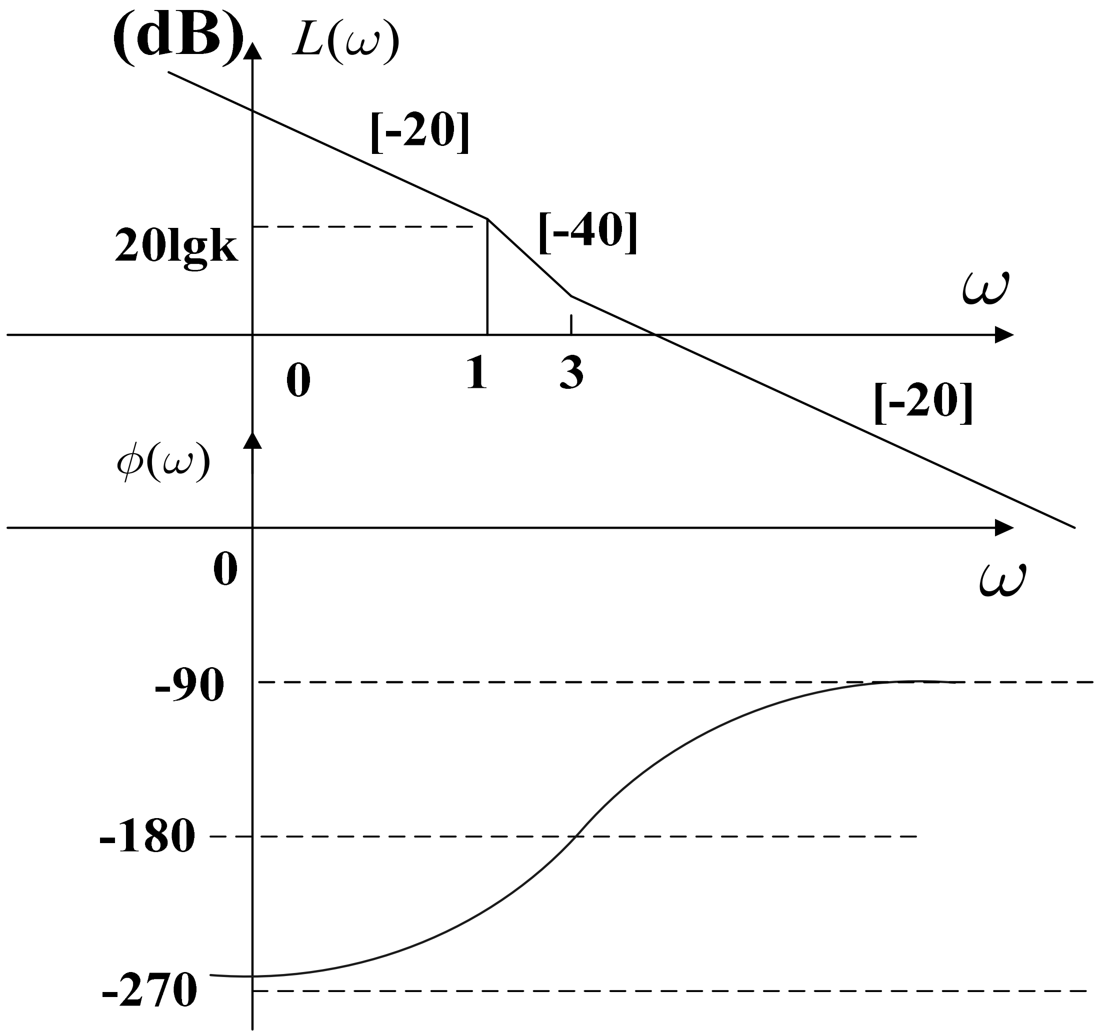
The frequency response of a system is defined as the steady-state response of the system to a sinusoidal input signal, which, as we have proved, can be fully characterized by its sinusoidal transfer function:

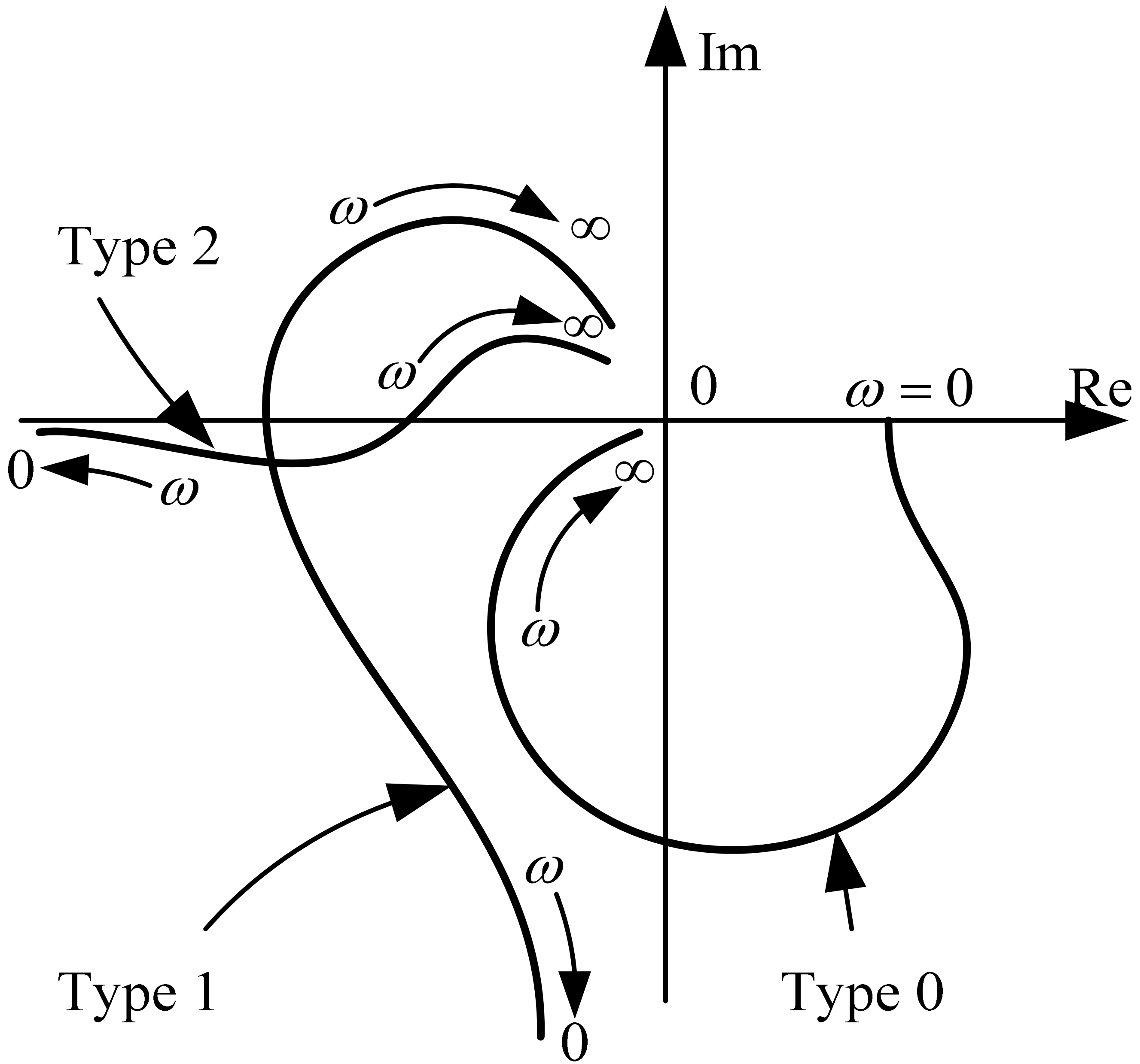
$$G(j\omega) = \left|G(j\omega)\right| \angle G(j\omega)$$

$$\left|G(j\omega)\right| = \frac{Y}{X} =$$ Amplitude ratio of the output sinusoid to the input sinusoid

$$\angle G(j\omega) =$$ Phase shift of the output sinusoid with respect to the input sinusoid

2. Bode and Nyquist plots





Im
Re
0
ω = 0
ω
∞
Type 2
Type 1
Type 0

3. Nyquist stability criterion

A system is stable if and only if from

$$\boldsymbol{N=(P-Z)/2}$$

we can obtain that $Z=0$, where P is the number of the open-loop poles in the right-hand half s-plane and N is the number of encirclements of the point $(-1, j0)$ of $G(j\omega)H(j\omega)$ *in the counterclockwise direction* as ω varies from 0 to $+\infty$.

Its Bode diagram counterpart is:

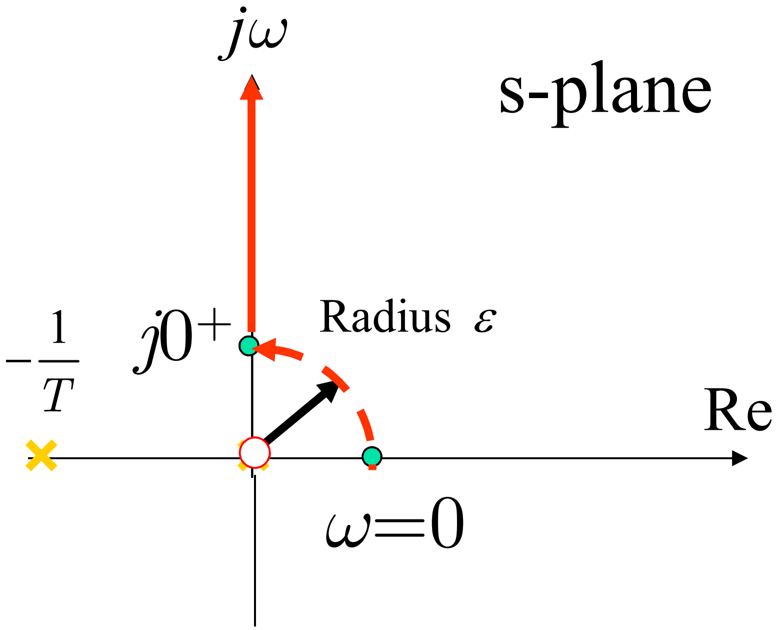
The system is stable if and only if from

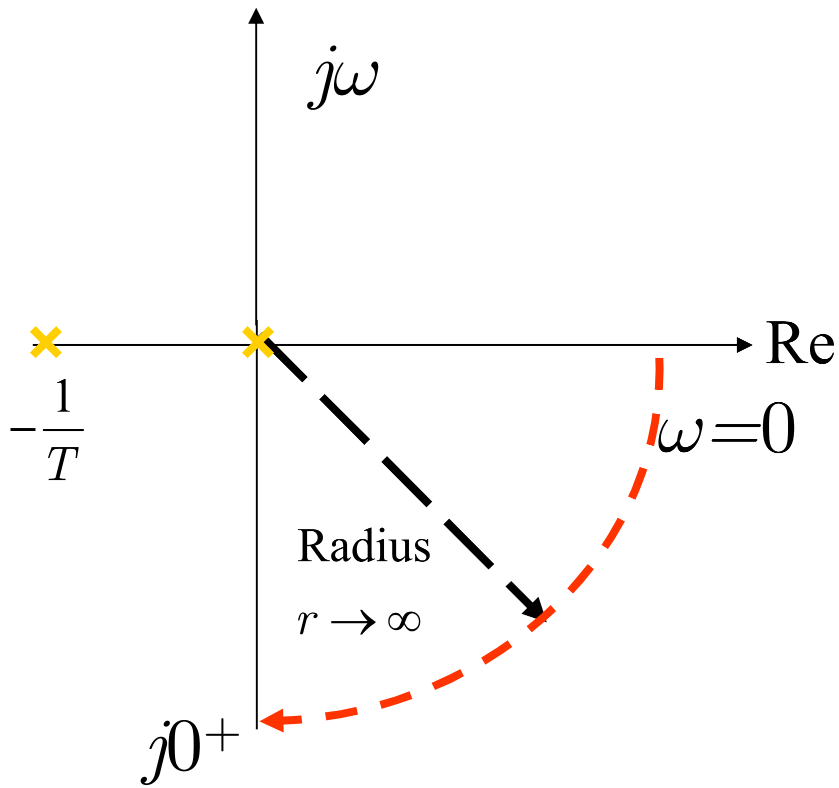
$$\boldsymbol{N_+ - N_- = (P - Z)/2}$$

we can obtain that $Z=0$ ($20\lg|\mathrm{G}(j\omega)|>0$).

4. Extension to the cases when $G(s)H(s)$ involves poles and zeros at the origin

s-plane





Example:

$$G(s) = \frac{Ks}{(T_1 s + 1)(T_2 s + 1)}$$

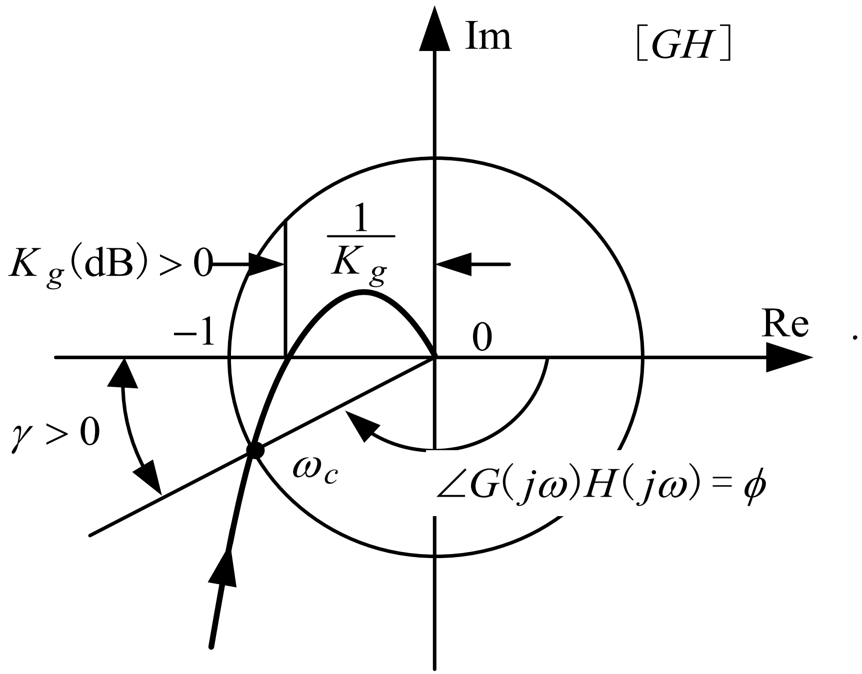
5. Gain and phase margins

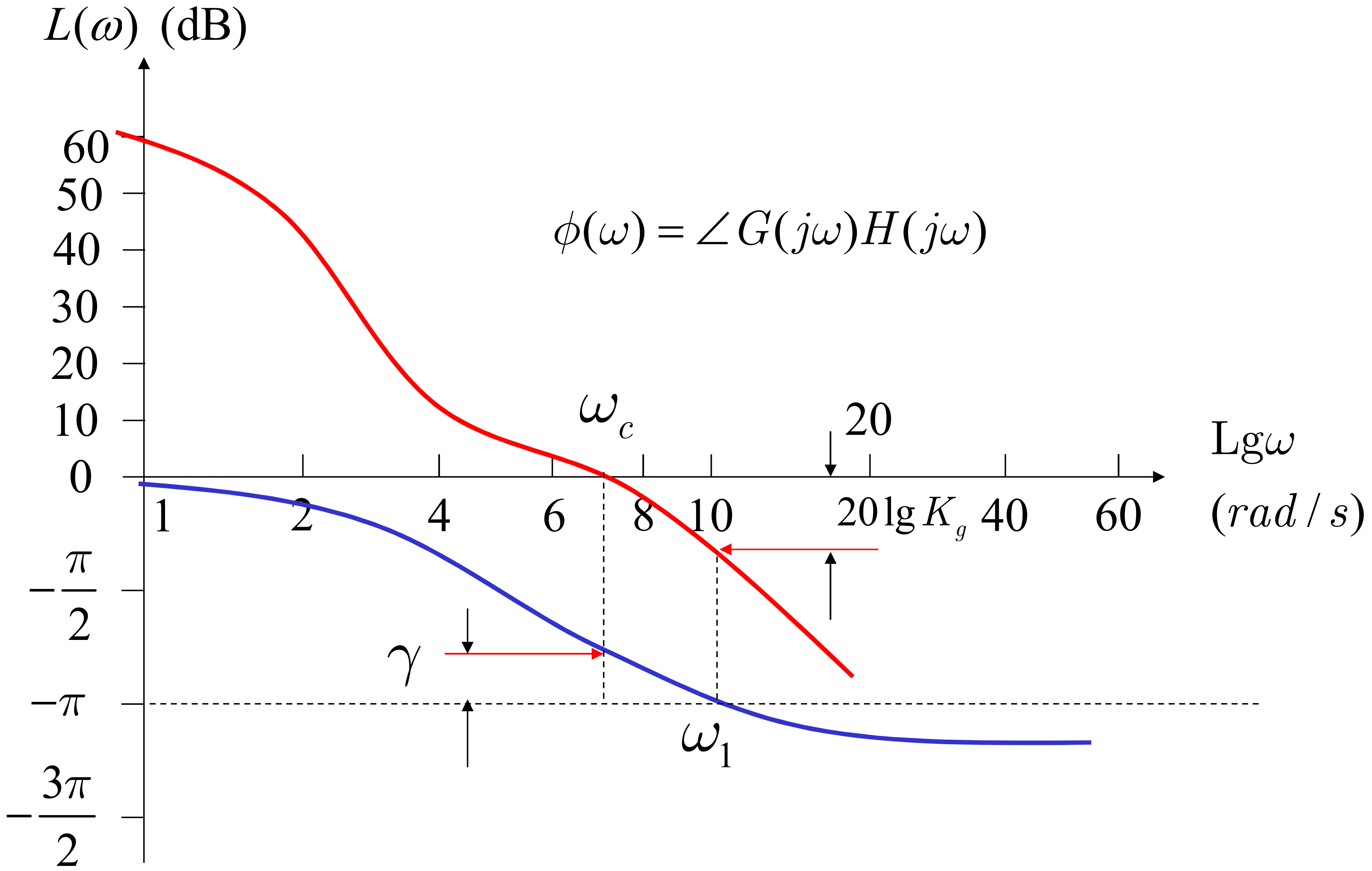
$$\gamma = 180^0 + \angle G(j\omega_c)H(j\omega_c)$$

with ω_c : $|GH(j\omega_\mathrm{c})| = 1.$

$$K_g = \frac{1}{|G(j\omega_1)H(j\omega_1)|}$$

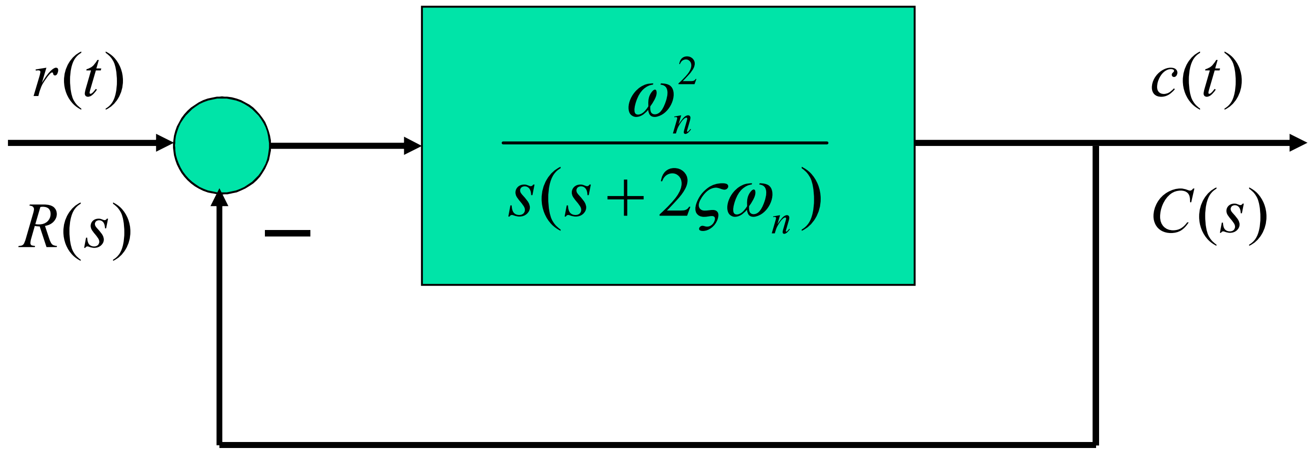
with ω_1: $\angle GH(j\omega_1) = -180^0$.




$L(\omega)$ (dB)
60
50
40
30
20
10
0
$\phi(\omega) = \angle G(j\omega)H(j\omega)$
ω_c
20
$\mathrm{Lg}\omega$
1
2
4
6
8
10
$20\lg K_g$
40
60
(rad/s)
$-\frac{\pi}{2}$
γ
$-\pi$
ω_1
$-\frac{3\pi}{2}$

Appendix

The deduction of phase margin for the standard second-order system:



Letting

$$|G(j\omega_c)|=1 \Leftrightarrow \frac{\omega_n^2}{\omega_c\sqrt{\omega_c^2+4\zeta^2\omega_n^2}}=1$$

Hence,

$$\omega_c = \omega_n \sqrt{\sqrt{(4\zeta^4 + 1)} - 2\zeta^2}$$

At this frequency,

$$\angle G(j\omega_c) = -\angle j\omega - \angle(j\omega_c + 2\zeta\omega_n)$$

$$= -90^0 - \tan^{-1} \frac{\sqrt{\sqrt{(4\zeta^4 + 1)} - 2\zeta^2}}{2\zeta}$$

Therefore,

$$\gamma = 180^0 + \angle G(j\omega_c) = 90^0 - \tan^{-1} \frac{\sqrt{\sqrt{(4\zeta^4 + 1)} - 2\zeta^2}}{2\zeta}$$

$$= \tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{(4\zeta^4 + 1)} - 2\zeta^2}}$$