工科数分习题课七　导数

石岩

shiyan200245@163.com

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本节课的内容和要求

1.掌握导数的概念;

2.熟练掌握导数的运算法则, 会计算复合函数、反函数、隐函数、参数方程的导数及高阶导数.

基本概念和主要结论

导数

$$
\begin{aligned}
f'(x_0) &= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\\
&= \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}.
\end{aligned}
$$

单侧导数可类似定义.

基本求导法则

1) $(u\pm v)' = u' \pm v'$.

2) $(uv)' = u'v + uv'$.

3) $\left(\dfrac{u}{v}\right)' = \dfrac{u'v-uv'}{v^2}$.

4) 反函数 $\dfrac{\mathrm{d}x}{\mathrm{d}y} = \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$.

5) 复合函数 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u}\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}$.

6) 参变量方程 $\begin{cases} x=\varphi(t) \\ y=\psi(t)\end{cases}$ $\alpha\le t\le\beta$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\psi'(t)}{\varphi'(t)}$.

7) 隐函数 $F(x,y)=0$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y},\ \text{where } F_x = \frac{\partial F}{\partial x}, F_y = \frac{\partial F}{\partial y}.$$

高阶导数

$$[u\pm v]^{(n)} = u^{(n)} \pm v^{(n)},$$

$$(uv)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}u^{(n-k)}v^{(k)},\ \text{where } \binom{n}{k} = \frac{n!}{k!(n-k)!},$$

(`Leibniz's formula`).

1.求高阶导数

$$(1)\qquad y=\frac{ax+b}{cx+d},\quad a,b,c,d\text{均为实数};$$

$$(2)\qquad y=\mathrm{e}^{ax}\sin bx,\ a,b\text{均为实数}.$$

$$(3)\qquad y=x^2\sin 2x,\ \text{求}\ y^{(50)}.$$

2. 已知

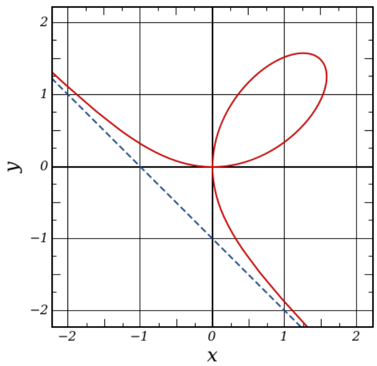
$$f(x)=\begin{cases}\exp\left(-\dfrac{1}{x^2}\right), & x\neq 0,\\ 0, & x=0.\end{cases}$$

求 (1) $f'(x)$; (2) $f^{(n)}(0), n\in\mathbb{N}$.

3.求由下列参量方程确定的函数$y=f(x)$的二阶导数

$$\begin{cases}x=\mathrm{e}^t\cos t,\\ y=\mathrm{e}^t\sin t.\end{cases}$$

4.求由笛卡尔叶形线(*Folium of Descartes*, 1638)



$$x^3 + y^3 - 3axy = 0$$

所确定的隐函数$y = f(x)$的一阶与二阶导数.

5. $y = y(x)$是由方程

$$\sqrt{x^2 + y^2} = \mathrm{e}^{\arctan \dfrac{y}{x}}$$

确定的隐函数, 求$\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Answers

$$1.(1)\ y' = \frac{1}{(cx+d)^2}(ad-cb),$$
$$y'' = \frac{-2c}{(cx+d)^3}(ad-cb),$$
$$y^{(n)} = \frac{(-1)^{n+1}n!\,c^{n-1}}{(cx+d)^{n+1}}(ad-cb).$$

$$(2)\ y' = \mathrm{e}^{ax}(a\sin bx + b\cos bx)$$
$$= \sqrt{a^2+b^2}\ \mathrm{e}^{ax}\sin(bx+\varphi), \quad \left(\varphi = \arctan\frac{b}{a}\right).$$
$$y'' = (\sqrt{a^2+b^2})^2\mathrm{e}^{ax}\sin(bx+2\cdot\varphi),$$
$$y^{(n)} = (\sqrt{a^2+b^2})^n\mathrm{e}^{ax}\sin(bx+n\cdot\varphi).$$

$$(3)\ y^{(50)} = 2^{50}(-x^2\sin 2x + 50x\cos 2x + \frac{1225}{2}\sin 2x).$$

$$2.(1) f'(x) = \begin{cases} \dfrac{2}{x^3}\exp\left(-\dfrac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \qquad (2) f^{(n)}(0) = 0.$$

$$3.\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\cos t + \sin t}{\cos t - \sin t}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{2}{\mathrm{e}^t\,(\cos t - \sin t)^3}.$$

$$4.\ y' = \frac{ay - x^2}{y^2 - ax},\ (y^2 - ax \neq 0), \quad y'' = -\frac{2a^3xy}{(y^2-ax)^3}.$$

$$5.\ y' = \frac{x+y}{x-y}.$$