工科数分习题课十一　不定积分

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Dec.7.2012

本节课的内容和要求

1.理解不定积分的概念;

2.熟练掌握不定积分的基本公式;

3.熟练运用换元积分法和分部积分法求不定积分.

基本概念和主要结论

1.原函数与不定积分

$$F'(x) = f(x),\ x \in I. \Rightarrow \int f(x)\mathrm{d}x = F(x) + C,\ x \in I.$$

◇ 基本性质

$$\begin{aligned}
&(1)\ \left[\int f(x)\mathrm{d}x\right]' = f(x);\\
&(2)\ \mathrm{d}\left[\int f(x)\mathrm{d}x\right] = f(x)\mathrm{d}x;\\
&(3)\ \int f'(x)\mathrm{d}x = \int \mathrm{d}f(x) = f(x) + C;\\
&(4)\ \int \sum_{i=1}^{n} \alpha_i f_i(x)\,\mathrm{d}x = \sum_{i=1}^{n}\left(\alpha_i \int f_i(x)\,\mathrm{d}x\right).
\end{aligned}$$

■ 定理　区间I上的连续函数存在原函数.

■ 命题　含有第一类间断点的函数没有原函数.(为什么?)

基本积分公式(略.详见教材.)

2.换元积分法与分部积分法

第一换元公式

$$\int f(x)\mathrm{d}x = \int g(\varphi(x))\varphi'(x)\mathrm{d}x = \int g(\varphi(x))\mathrm{d}\varphi(x) = G(\varphi(x)) + C.$$

第二换元公式

令 $u = \varphi(x)$, 则 $\displaystyle\int g(u)\mathrm{d}u = \int g(\varphi(x))\varphi'(x)\mathrm{d}x = \int f(x)\mathrm{d}x$

$$= F(x) + C = F(\varphi^{-1}(u)) + C.$$

分部积分公式

简记　$\displaystyle\int u\mathrm{d}v = uv - \int v\mathrm{d}u.$

练习

1.求不定积分

$$(1)\int \sec t\,\mathrm{d}t; \quad (2)\int \sec^2 t\,\mathrm{d}t; \quad (3)\int \sec^3 t\,\mathrm{d}t.$$

2.求不定积分$(a>0)$.

$$(1)\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} \qquad (2)\int \frac{\mathrm{d}x}{\sqrt{x^2\pm a^2}}$$

$$(3)\int \sqrt{a^2-x^2}\,\mathrm{d}x \qquad (4)\int \sqrt{x^2\pm a^2}\,\mathrm{d}x$$

$$(5)\int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} \qquad (6)\int \frac{\mathrm{d}x}{x^2\sqrt{x^2\pm a^2}}$$

3.求不定积分

$$\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$$

4.应用分部积分法求不定积分

$$\begin{aligned}
&(1) \qquad \int x^2 \cos x \,\mathrm{d}x; \\
&(2) \qquad \int x^2 \mathrm{e}^{-2x} \,\mathrm{d}x; \\
&(3) \qquad \int x \arctan x \,\mathrm{d}x; \\
&(4) \qquad \int \sqrt{x} \ln^2 x \,\mathrm{d}x.
\end{aligned}$$

思考　对于乘积形式的被积函数,应用分部积分法,u, v如何选取?

特别地,设$P_n(x)$是n阶多项式,思考以下形式的不定积分u, v该如何选取.

$$\begin{aligned}
&(1) \qquad \int P_n(x)\mathrm{e}^x \,\mathrm{d}x; \quad (2) \int P_n(x) \sin x \,\mathrm{d}x; \quad (3) \int P_n(x) \cos x \,\mathrm{d}x; \\
&(4) \qquad \int P_n(x) \ln x \,\mathrm{d}x; \ (5) \int P_n(x) \arcsin x \,\mathrm{d}x; \ (6) \int P_n(x) \arctan x \,\mathrm{d}x;
\end{aligned}$$

答案

1.

$$
\begin{aligned}
&(1) \quad \int \sec t \,\mathrm{d}t = \ln|\sec t + \tan t| + C;\\
&(2) \quad \int \sec^2 t \,\mathrm{d}t = \tan t + C;\\
&(3) \quad \int \sec^3 t \,\mathrm{d}t = \frac{1}{2}\left(\sec t \tan t + \ln|\sec t + \tan t|\right) + C.
\end{aligned}
$$

2. $a > 0$.

$$
\begin{aligned}
&(1) \quad \int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C;\\
&(2) \quad \int \frac{\mathrm{d}x}{\sqrt{x^2 \pm a^2}} = \ln|x + \sqrt{x^2 \pm a^2}| + C;\\
&(3) \quad \int \sqrt{a^2-x^2}\,\mathrm{d}x = \frac{1}{2}(x\sqrt{a^2-x^2} + a^2 \arcsin\frac{x}{a}) + C;\\
&(4) \quad \int \sqrt{x^2 \pm a^2}\,\mathrm{d}x = \frac{1}{2}(x\sqrt{x^2 \pm a^2} \pm a^2 \ln|x + \sqrt{x^2 \pm a^2}|) + C;\\
&(5) \quad \int \frac{\mathrm{d}x}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2 x} + C;\\
&(6) \quad \int \frac{\mathrm{d}x}{x^2\sqrt{x^2 \pm a^2}} = \mp\frac{\sqrt{x^2 \pm a^2}}{a^2 x} + C.
\end{aligned}
$$

3.

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{x\sqrt{x^2-1}} \qquad &= -\arcsin\frac{1}{x} + C\\
&= \arccos\frac{1}{x} + C\\
&= \arctan\sqrt{x^2-1} + C.
\end{aligned}
$$

4.

$$
\begin{aligned}
&(1) \qquad \int x^2 \cos x \, \mathrm{d}x = (x^2 - 2)\sin x + 2x\cos x + C; \\
&(2) \qquad \int x^2 \mathrm{e}^{-x} \, \mathrm{d}x = -\frac{1}{2}\mathrm{e}^{-2x}(x^2 + x + \frac{1}{2}) + C; \\
&(3) \qquad \int x \arctan x \, \mathrm{d}x = \frac{1}{2}(x^2 + 1)\arctan x - \frac{1}{2}x + C; \\
&(4) \qquad \int \sqrt{x} \ln^2 x \, \mathrm{d}x = \frac{2}{3}x^{\frac{3}{2}}\left(\ln^2 x - \frac{4}{3}\ln x + \frac{8}{9}\right) + C.
\end{aligned}
$$