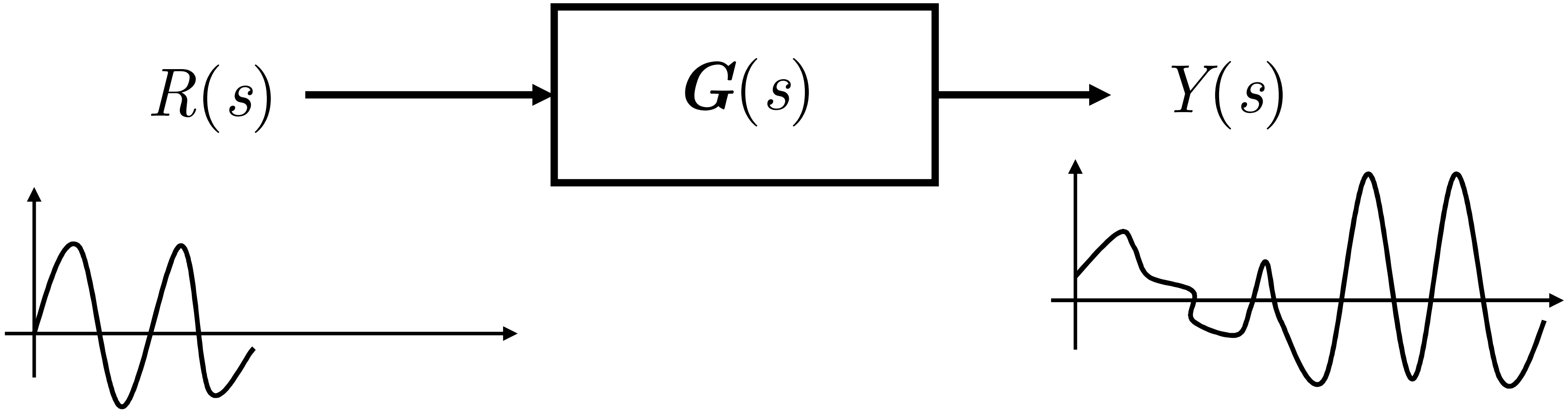
Chapter 7

Control System Analysis and Design by the Frequency Response Method

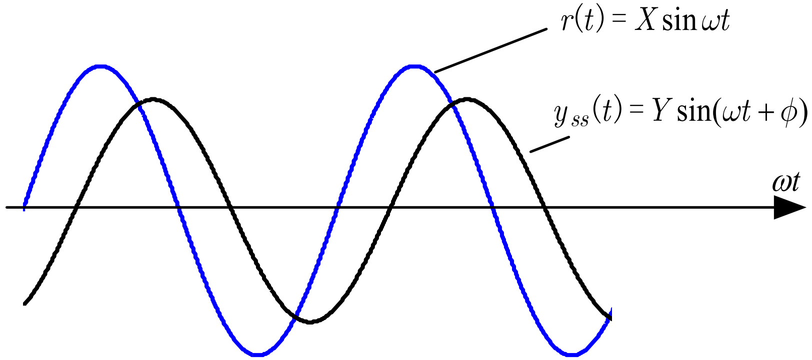
7-1 Introduction

1. The concept of frequency response



$$r(t) = X\sin\omega t$$

$$y_{ss}(t) = Y(\omega)\sin(\omega t + \phi(\omega))$$



Frequency response:

The frequency response of a system is defined as the **steady-state response** of the system to a sinusoidal input signal.

For a stable LTI system, the resulting output signal, as well as signals throughout the system, is also sinusoidal in the steady-state; **it differs from the input waveform only in amplitude and phase angle.**

2. Obtaining steady-state outputs to sinusoidal inputs

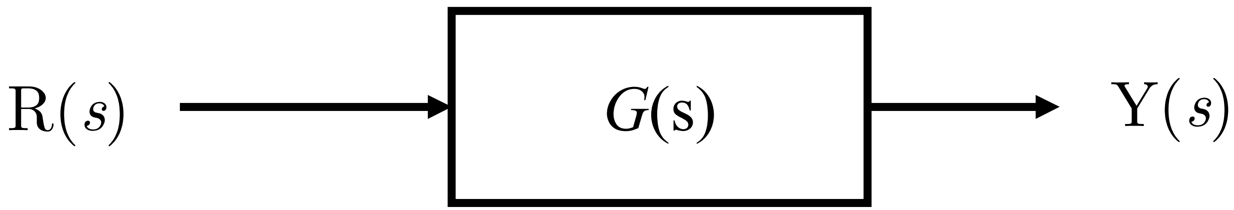
For a given stable LTI system with a sinusoidal input, **the steady state output can be completely characterized by its transfer function with s being replaced by $j\omega$**. That is, if the system transfer function is $G(s)$, then from the steady-state response

$$y_{ss}(t) = Y(\omega)\sin(\omega t + \phi),$$

we have

$$Y(\omega) = X\left|G(j\omega)\right|, \quad \phi(\omega) = \angle G(j\omega).$$

To prove this conclusion, consider the following controlled plant:



$$G(s)=\frac{p(s)}{q(s)}=\frac{p(s)}{\prod_{i=1}^{n}(s+s_i)} \qquad r(t)=X\sin\omega t \qquad R(s)=\frac{X\omega}{s^2+\omega^2}$$

$$Y(s)=\frac{p(s)}{\prod_{i=1}^{n}(s+s_i)}\cdot\frac{X\omega}{s^2+\omega^2}=\frac{k_1}{s+s_1}+\frac{k_2}{s+s_2}+\cdots+\frac{k_n}{s+s_n}$$
$$+\frac{a}{s+j\omega}+\frac{\bar{a}}{s-j\omega}$$

$$\mathrm{R}(s) \longrightarrow \boxed{\boldsymbol{G}(s)} \longrightarrow \mathrm{Y}(s)$$

$$Y(s) = \frac{k_1}{s+s_1} + \frac{k_2}{s+s_2} + \cdots + \frac{k_n}{s+s_n} + \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}$$

$$y(t) = \underbrace{k_1 e^{-s_1 t} + k_2 e^{-s_2 t} + \cdots + k_n e^{-s_n t}}_{\textbf{Transient response}} + \underbrace{L^{-1}\left\{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}\right\}}_{\textbf{Steady-state response}}$$

If the system is stable, then the transient response

$$y_t(t) = \sum_{i=1}^{n} k_1 e^{-s_i t} \rightarrow 0$$

The steady state response

$$y_{ss}(t) = L^{-1}\left\{\frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}\right\}$$

$$\boxed{a = ? \quad \bar{a} = ?}$$

Note that

$$Y(s) = G(s)\cdot\frac{X\omega}{s^2+\omega^2} = \sum_{i=1}^{n}\frac{k_i}{s+s_i} + \frac{a}{s+j\omega} + \frac{\bar{a}}{s-j\omega}$$

Hence,

$$a = G(s)\cdot\frac{X\omega}{s^2+\omega^2}(s+j\omega)\bigg|_{s=-j\omega} = -G(-j\omega)\cdot\frac{X}{2j}$$

$$\bar{a} = G(s)\cdot\frac{X\omega}{s^2+\omega^2}(s-j\omega)\bigg|_{s=j\omega} = G(j\omega)\cdot\frac{X}{2j}$$

Write

$$G(j\omega) = \left|G(j\omega)\right| e^{j\phi}$$

where

$$\phi = \angle G(j\omega) = \arctan\left[\frac{\mathrm{Im}\, G(j\omega)}{\mathrm{Re}\, G(j\omega)}\right]$$

Similarly,

$$G(-j\omega) = \left|G(-j\omega)\right| e^{-j\phi} = \left|G(j\omega)\right| e^{-j\phi}$$

Therefore,

$$a = -G(-j\omega) \cdot \frac{X}{2j} = -\frac{X\left|G(j\omega)\right| e^{-j\phi}}{2j}$$

$$\overline{a} = G(j\omega) \cdot \frac{X}{2j} = \frac{X\left|G(j\omega)\right| e^{j\phi}}{2j}$$

Now, the steady-state response can be written as

$$y_{ss}(t) = ae^{-j\omega t} + \overline{a}e^{j\omega t}$$

$$= X\left|G(j\omega)\right|\frac{e^{j(\omega t+\phi)} - e^{-j(\omega t+\phi)}}{2j} = Y(\omega)\sin(\omega t + \phi(\omega))$$

with

$$Y(\omega) = X\left|G(j\omega)\right|$$

and

$$\phi(\omega) = \angle G(j\omega) = \arctan\left[\frac{\operatorname{Im} G(j\omega)}{\operatorname{Re} G(j\omega)}\right]$$

based on which, we obtain the following important results:

$$\left|G(j\omega)\right| = \frac{Y}{X} =$$ Amplitude ratio of the output sinusoid to the input sinusoid

$$\angle G(j\omega) =$$ Phase shift of the output sinusoid with respect to the input sinusoid

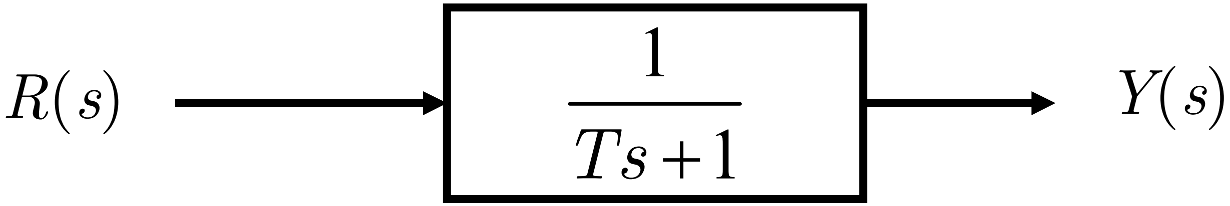
Noting that

$$Y(s) = G(s)R(s).$$

It follows that the **frequency response characteristics** of a stable LTI system to a sinusoidal input can be obtained directly from its transfer function $G(s)$ with s being replaced by $j\omega$.

Remark: Though we have deduced the frequency response under the assumption that $G(s)$ is stable, it can be extended to the case that $G(s)$ is unstable.

Example. The plant to be controlled is



Let $r(t)$=Xsinωt. Find its frequency response $y_{\mathrm{ss}}(t)$.

Solution: By definition of frequency response, $y_{\mathrm{ss}}(t)$ can be completely characterized by $G(j\omega)$, that is,

$$y_{ss}(t) = Y\sin(\omega t + \phi)$$

with

$$Y = X\left|G(j\omega)\right| = X\frac{1}{\sqrt{T^2\omega^2+1}}$$

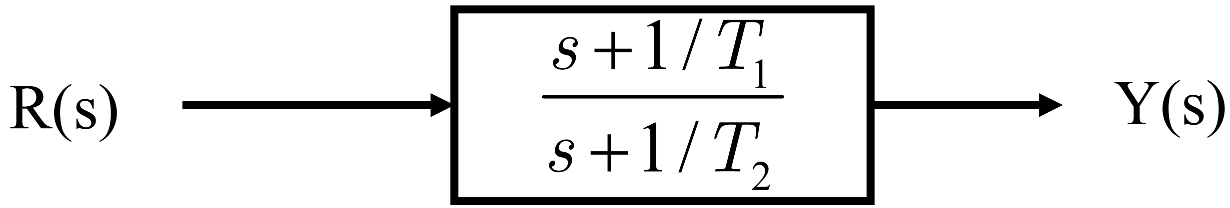
$$\phi = \angle G(j\omega) = -\tan^{-1}(T\omega).$$

Hence,

$$y_{ss}(t) = Y\sin(\omega t + \phi)$$
$$= X\frac{1}{\sqrt{T^2\omega^2+1}}\sin(\omega t - \tan^{-1}T\omega),$$

where $\phi<0$.

Example. The controlled plant is



Let $r(t)=X\sin\omega t$ and suppose $T_1>T_2$. Find its frequency response $y_{ss}(t)$.

Solution: By definition,

$$\begin{aligned} y_{ss}(t) &= Y\sin(\omega t + \phi) \\ &= X\left|G(j\omega)\right|\sin(\omega t + \angle G(j\omega)) \end{aligned}$$

Since in this example,

$$\begin{aligned} G(j\omega) &= \left.\frac{s + 1/T_1}{s + 1/T_2}\right|_{s=j\omega} = \frac{T_2(T_1 j\omega + 1)}{T_1(T_2 j\omega + 1)} \\ &= \underbrace{\frac{T_2\sqrt{(T_1^2\omega^2 + 1)}}{T_1\sqrt{(T_2^2\omega^2 + 1)}}}_{\left|G(j\omega)\right|} \angle \underbrace{(\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega))}_{\phi = \angle G(j\omega)}, \end{aligned}$$

$$y_{ss}(t) = Y\sin(\omega t + \phi)$$

$$= X\frac{T_2\sqrt{(T_1^2\omega^2+1)}}{T_1\sqrt{(T_2^2\omega^2+1)}}\sin(\omega t + [\tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega)])$$

Since by assumption, $T_1 > T_2$, it follows that $\phi > 0$.

Definition: A positive phase angle is called **phase lead** and a negative phase angle is called **phase lag**.

By this definition, the fist example is a phase lag system, while the second one is a phase lead system.

3. Why is frequency response important?

Since for an LTI system, its frequency response can be fully characterized by $G(j\omega)$, or by

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

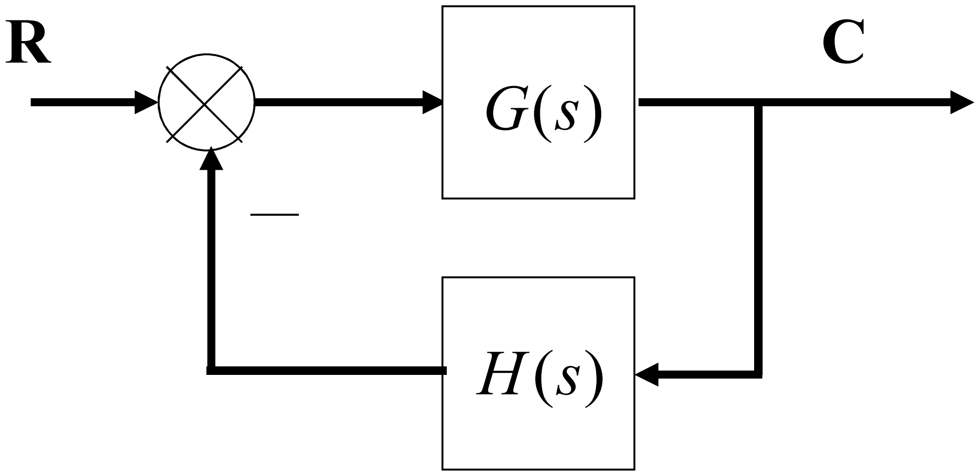
our later analysis will show that by using the so called **Bode** diagram and **Nyquist** plot, one is able to

- Determine how the system responds at different frequencies;
- Find stability properties of the system in closed-loop control;
- Design compensating controllers.

7-2 Bode Diagrams or Logarithmic Plots

1. Concept of Bode diagram

A Bode diagram consists of two graphs: One is a plot of the logarithm of the magnitude of a sinusoidal transfer function; the other is a plot of phase angle; both are plotted against the frequency on a logarithmic scale.



Bode Logarithmic magnitude :

$$L(\omega) = 20\log_{10}\left|G(j\omega)H(j\omega)\right| = 20\lg\left|G(j\omega)H(j\omega)\right|$$

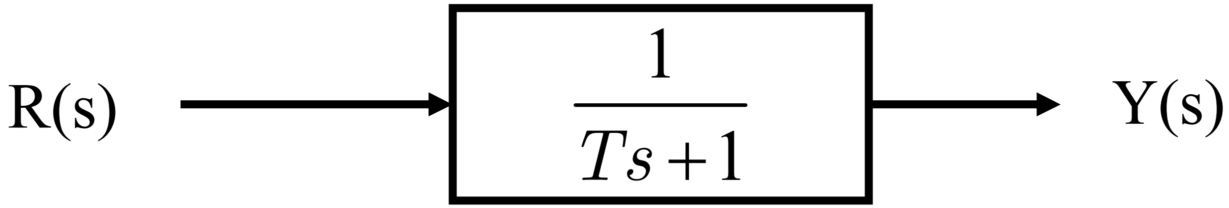
where the base of the logarithm is 10, the unit is (dB) and the horizontal axis is logarithmic scale $\lg\omega$.

Bode Phase angle:

$$\angle G(j\omega)H(j\omega)$$

with the horizontal axis being also $\lg\omega$.

Example. Given a first-order system



Plot its Bode diagram.

Solution: Write

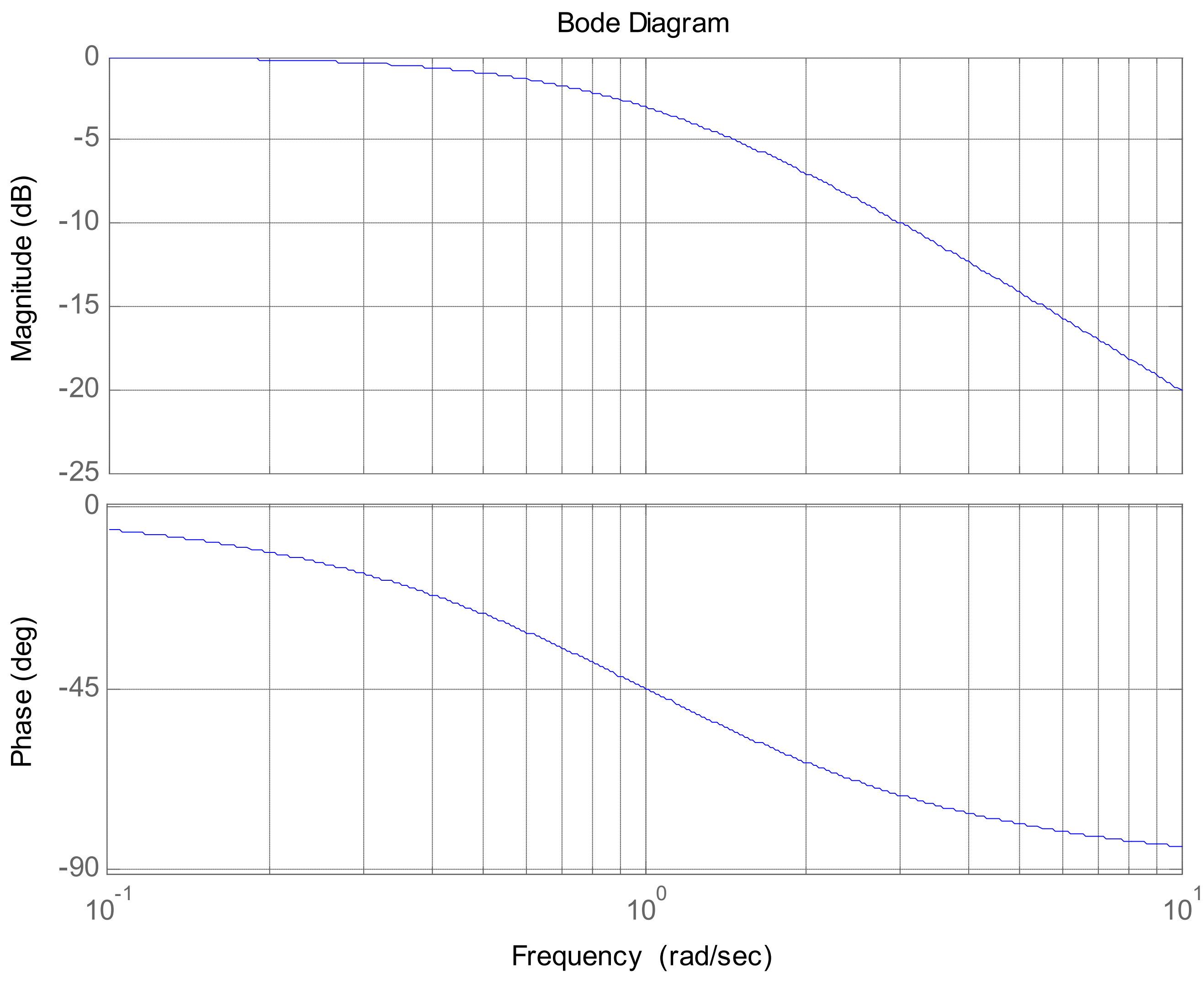
$$L = 20 \cdot \lg |G(j\omega)| = 20 \cdot \lg \left| \frac{1}{1 + j\omega T} \right|$$

$$= 20 \cdot \lg \frac{1}{\sqrt{[1 + (T\omega)^2]}} = -20 \lg \sqrt{[1 + (T\omega)^2]}$$

and

$$\angle G(j\omega) = -\tan^{-1}(T\omega)$$

Hence, both Bode magnitude and phase angle diagrams are shown below:



Example. An open-loop transfer function is given below:

$$G(s) = \frac{K}{s(T_1 s + 1)(T_2 s + 1)}$$

Plot its Bode diagram.

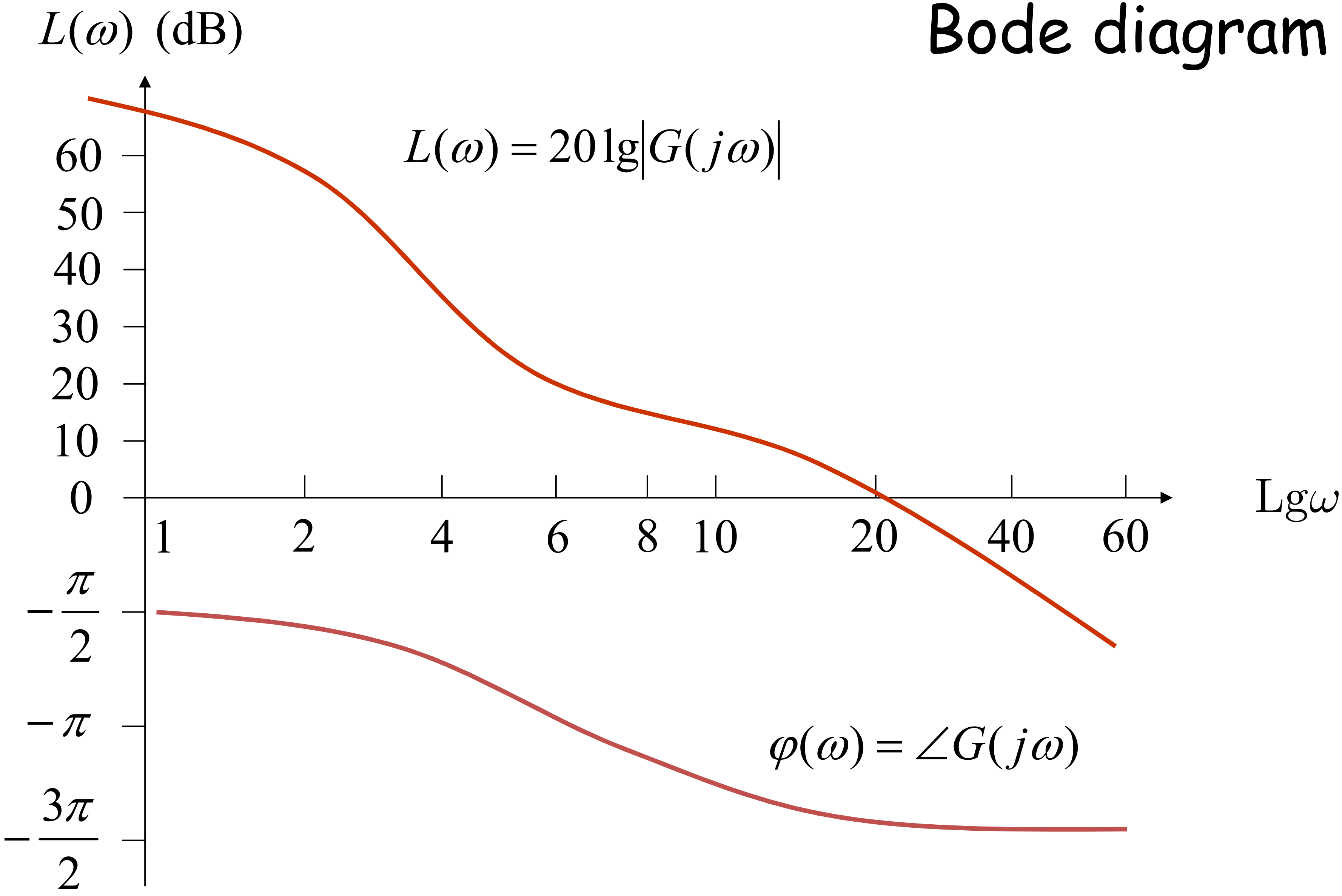
Solution: Write

$$L = 20 \cdot \lg|G(j\omega)| = 20 \cdot \lg K - 20 \cdot \lg \omega$$
$$-20 \cdot \lg\sqrt{[1 + (T_1\omega)^2]} - 20 \cdot \lg\sqrt{[1 + (T_2\omega)^2]}$$

and

$$\angle G(j\omega) = -90^0 - \tan^{-1}(T_1\omega) - \tan^{-1}(T_2\omega)$$

Bode diagram



From the aforementioned examples, it is clear that one of the main advantages of Bode diagram is that multiplication of magnitude can be converted into addition, which makes our analysis much simpler.

2. Bode diagrams for basic factors of G(jω)H(jω)

The basic factors that very frequently occur in an arbitrary transfer function $\boldsymbol{G}(j\omega)\boldsymbol{H}(j\omega)$ are

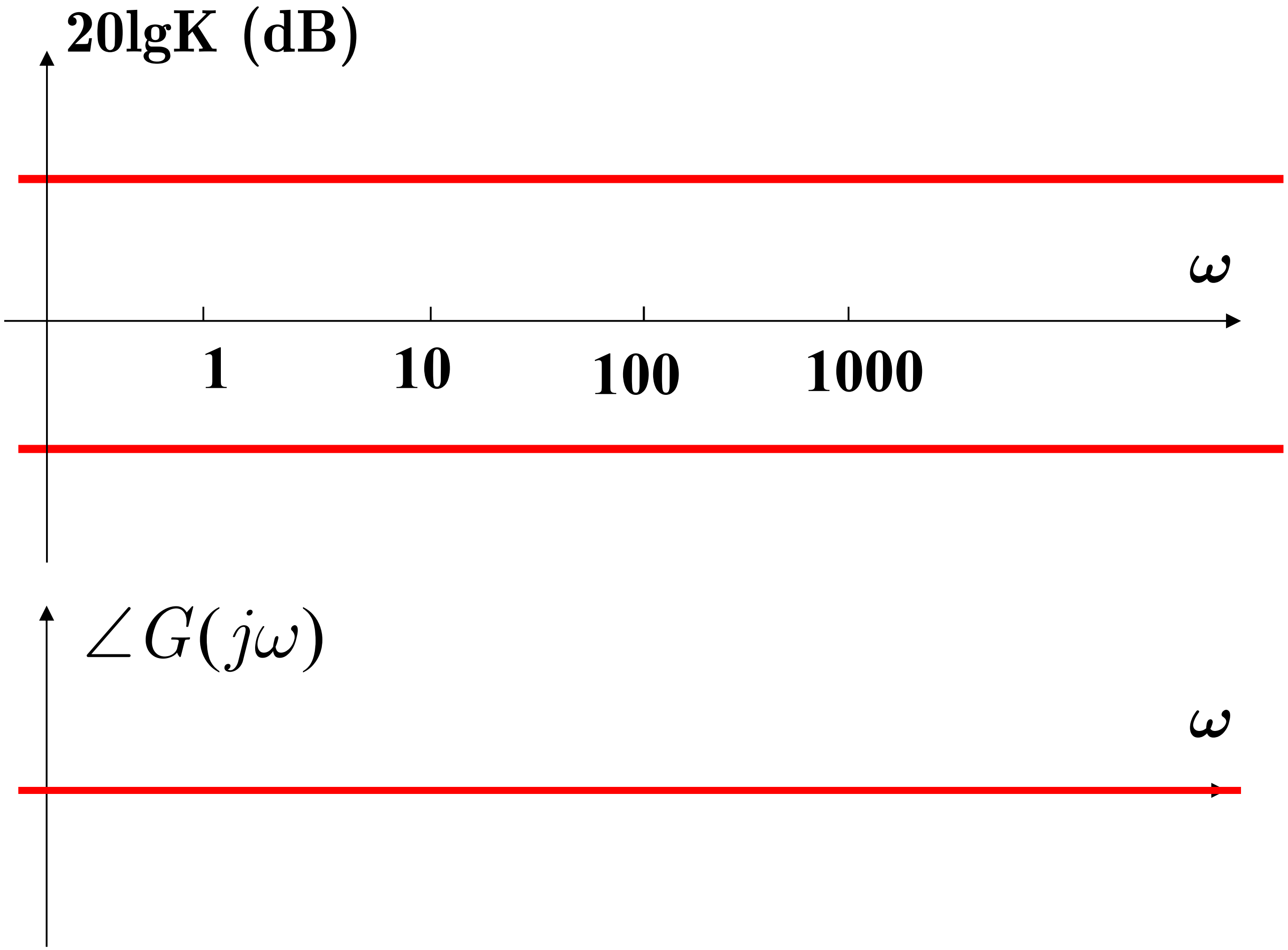
1). Gain K

2). Integral and derivative factors $(j\omega)^{\pm 1}$

3). First-order factors $(1+j\omega T)^{\pm 1}$

4). Quadratic factors $[(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1]^{\pm 1}$

1). Gain K



2). Integral and derivative factors $(j\omega)^{\pm 1}$

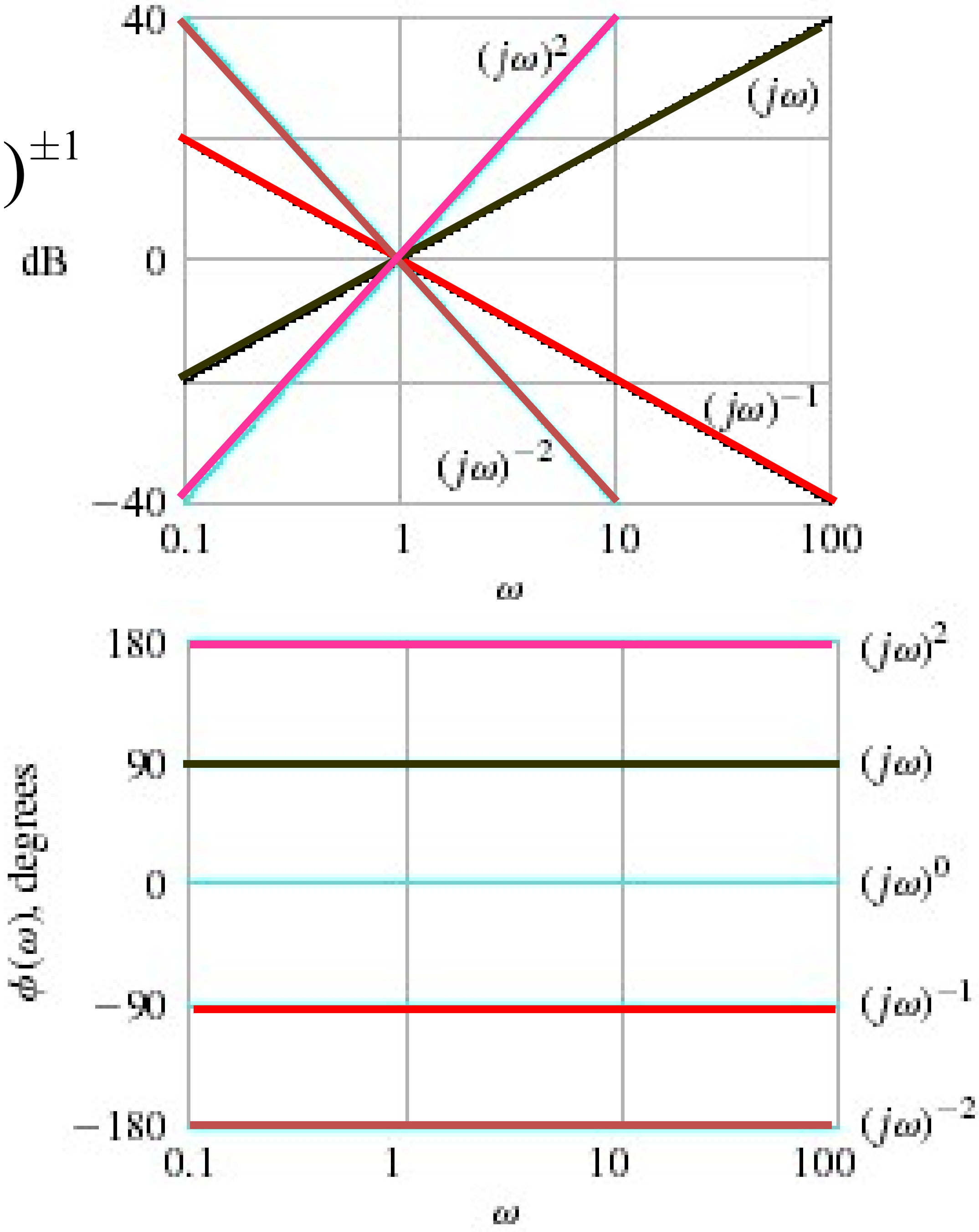
$$G(j\omega) = \frac{1}{j\omega}$$

$$G(j\omega) = j\omega$$

$$G(j\omega) = \frac{1}{(j\omega)^2}$$

$$\Rightarrow \frac{1}{(j\omega)} \times \frac{1}{(j\omega)}$$

$$G(j\omega) = (j\omega)^2$$



Example. Open-loop transfer functions are

$$G(s) = \frac{10}{s}$$

$$G(s) = \frac{10}{s^2}$$

Plot their Bode diagrams.

3). First - order factor: $G(s)=\dfrac{1}{Ts+1}$

• Bode magnitude:

$$L=20\cdot\lg|G|=-20\lg\sqrt{[1+(T\omega)^2]}$$

• Phase angle:

$$\angle G=\varphi(\omega)=-\tan^{-1}T\omega=\begin{cases}0, & \omega=0\\ -45^0, & \omega=1/T\\ -90^0, & \omega=+\infty\end{cases}$$

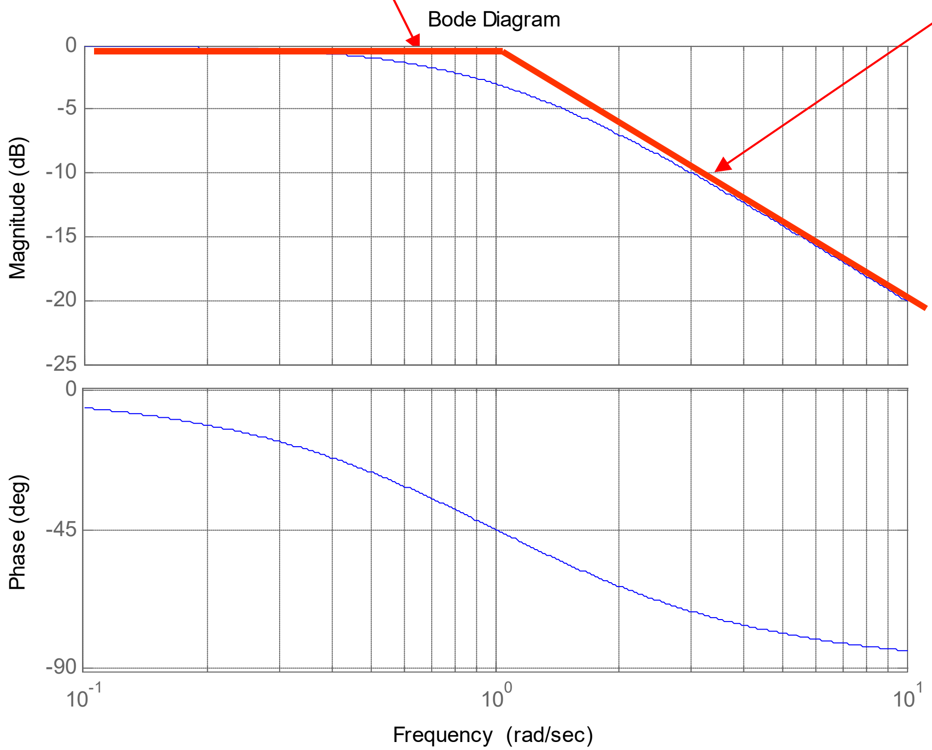
• Asymptotic curves for Bode magnitude:

$$\omega<<\frac{1}{T},\ \ L\approx 0(dB),$$

$$\omega>>\frac{1}{T},\ \ L\approx -20\cdot\lg T\omega=-20\lg T-20\lg\omega$$

$$L \approx 0(dB)$$

$$L \approx -20 \cdot \lg T\omega$$



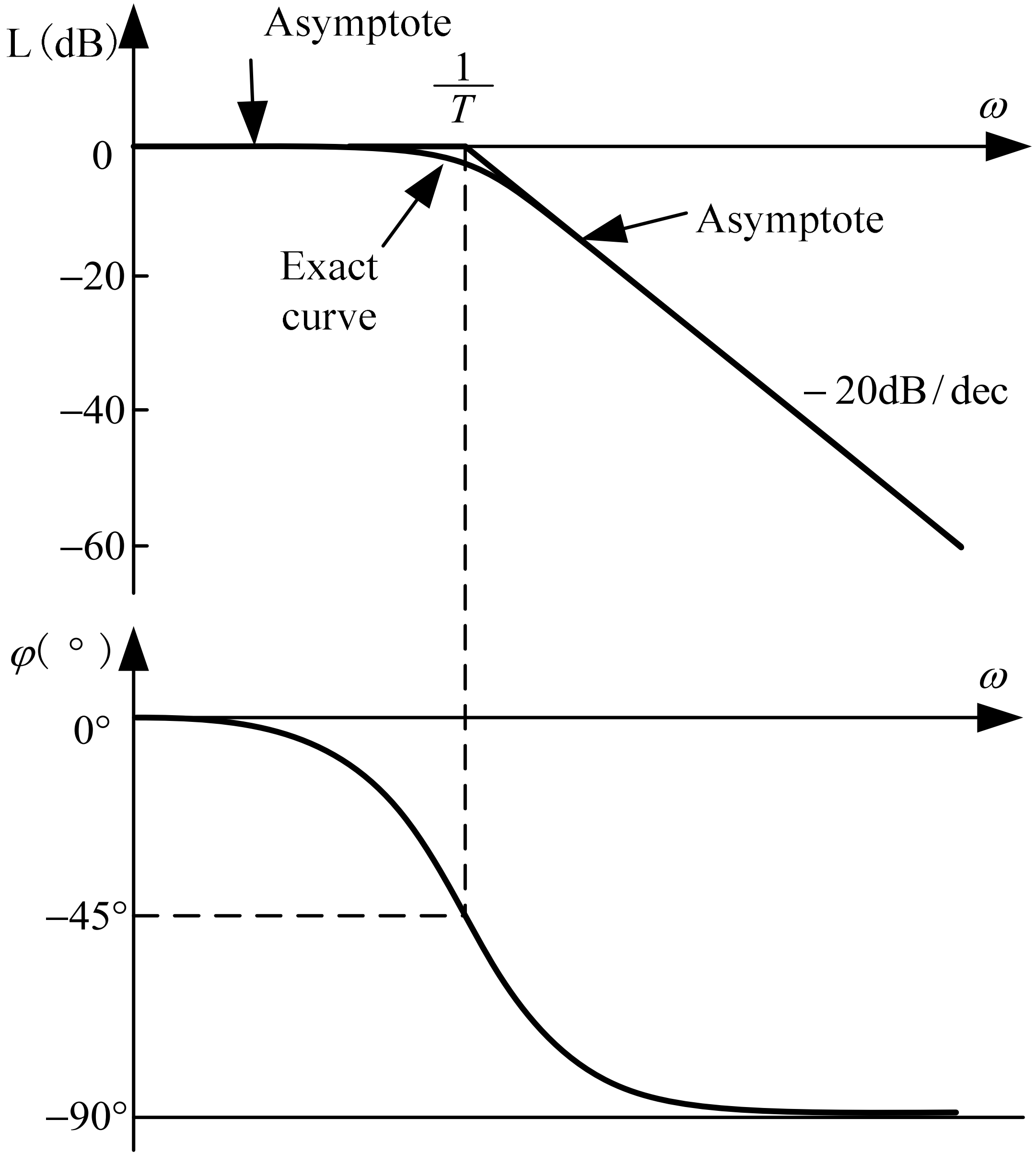
•The frequency $1/T$ is called **corner (break) frequency**, at which

$$\angle G = -\tan^{-1} T\omega \Big|_{\omega=1/T}$$

$$= -45^0$$

•The maximum error in the magnitude curve caused by the use of asymptotes occurs at $1/T$ is approximately equal to -3 dB since

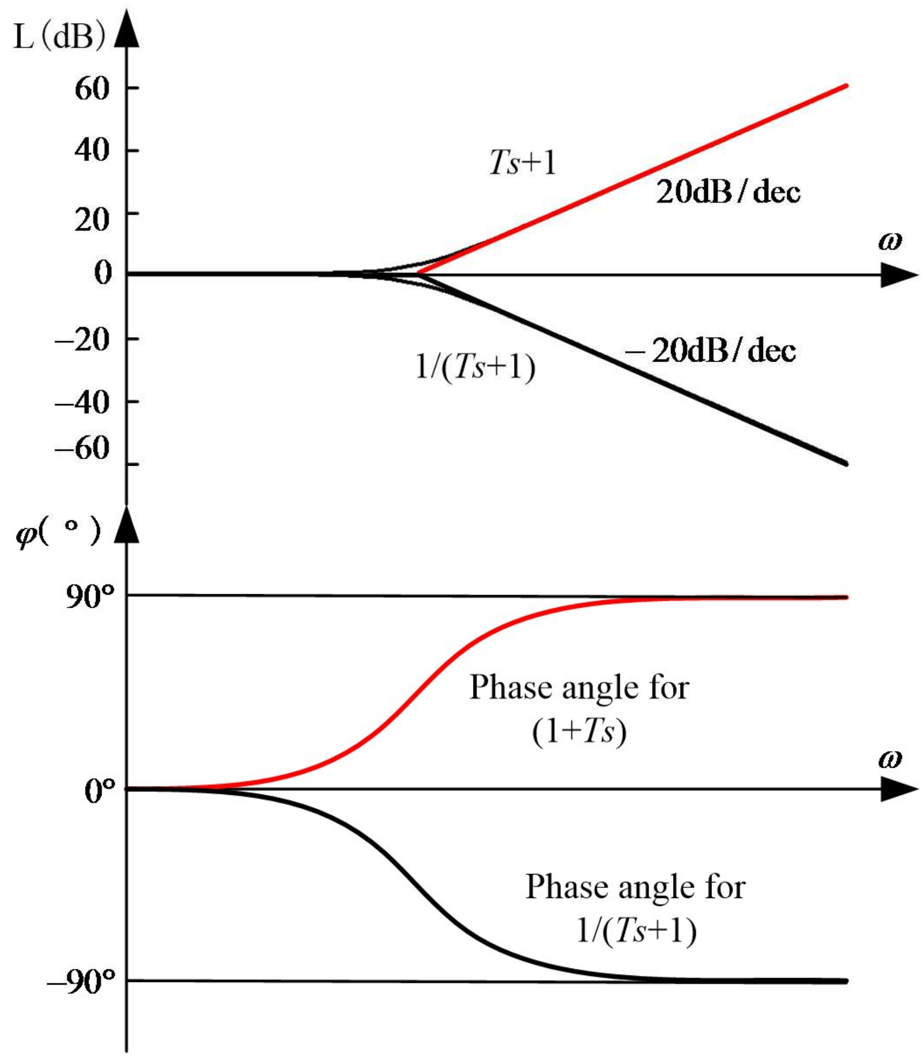
$$-20\lg\sqrt{[1+(T\omega)^2]}\Big|_{\omega=1/T} = -10\lg 2 = -3.03\text{dB}$$



L (dB)
Asymptote
1/T
ω
0
Exact curve
Asymptote
−20
−40
−60
−20dB/dec
φ(°)
ω
0°
−45°
−90°

• First-order derivative factor $G(s) = Ts + 1$:

whose Bode magnitude and phase angle curves, due to the property that $Ts+1$ is the reciprocal factor of $1/(Ts+1)$, *need only be changed in sign.*



Corner frequency: $1/T$

Example. Open-loop transfer functions are

$$G(s) = \frac{10}{Ts+1}$$

$$G(s) = \frac{10}{s(Ts+1)}$$

$$G(s) = \frac{10(T_2 s+1)}{s(T_1 s+1)}$$

Plot, for $T_1 > T_2$ and $T_1 < T_2$ cases, Bode diagrams.

4). Quadratic factor with $0<\zeta<1$:

$$G(j\omega)=\frac{1}{1+(2\zeta/\omega_n)j\omega+(j\omega/\omega_n)^2}$$

•The magnitude of $G(j\omega)$ is

$$|G(j\omega)|=\frac{1}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}$$

•The phase angle of $G(j\omega)$ is

$$\angle G=\varphi(\omega)=-\tan^{-1}\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

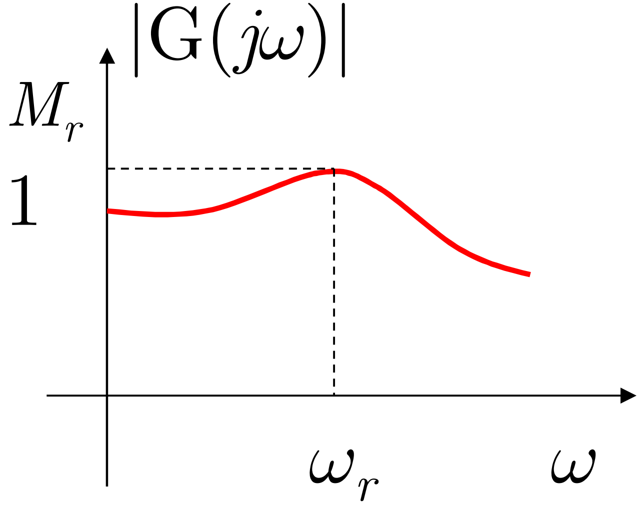
From $$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}}$$

we have

$$\lim_{\omega\to 0} G(j\omega) = 1$$

$$\lim_{\omega\to\infty} |G(j\omega)| = 0$$

$$|G(j\omega_n)| = \frac{1}{2\zeta}$$



• *Resonant frequency*:

$$\frac{d}{d\omega}|G| = 0 \quad \Rightarrow \quad \omega_r = \omega_n\sqrt{1-2\zeta^2},\ 0<\zeta\le 0.707$$

• *Resonant peak value*:

$$M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

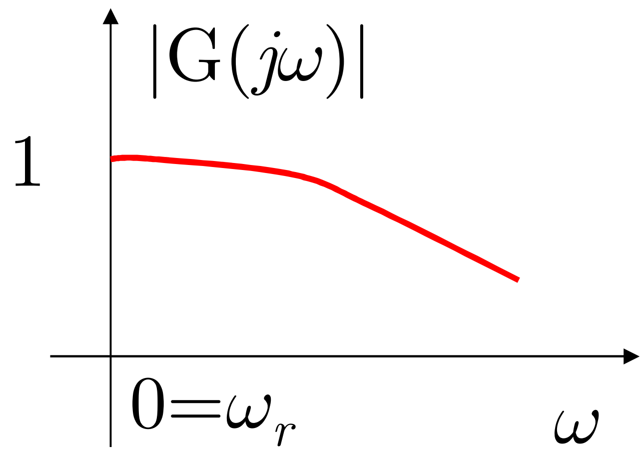
For $0 < \zeta \leq 0.707$,

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} < \omega_n$$

and

$$\lim_{\zeta \to 1/\sqrt{2}} \omega_r = 0$$

$$\lim_{\zeta \to 0} \omega_r \to \omega_n$$



from which it can be seen that for $\zeta > 0.707$, **there is no resonant peak** and the magnitude $|G(j\omega)|$ decreases monotonically as the frequency ω increases ($20\lg|G(j\omega)| < 0$ dB, for all values of $\omega > 0$. Recall that, for $0.707 < \zeta < 1$, the step response is oscillatory, but the oscillations are well damped and are hardly perceptible).

- Asymptotic curves for **Bode magnitude:**

$$20\lg|G(j\omega)| = -20\lg\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

When $\omega << \omega_n$, the Bode magnitude becomes

$$20\lg|G(j\omega)| \approx -20\lg 1 = 0\,\text{dB}$$

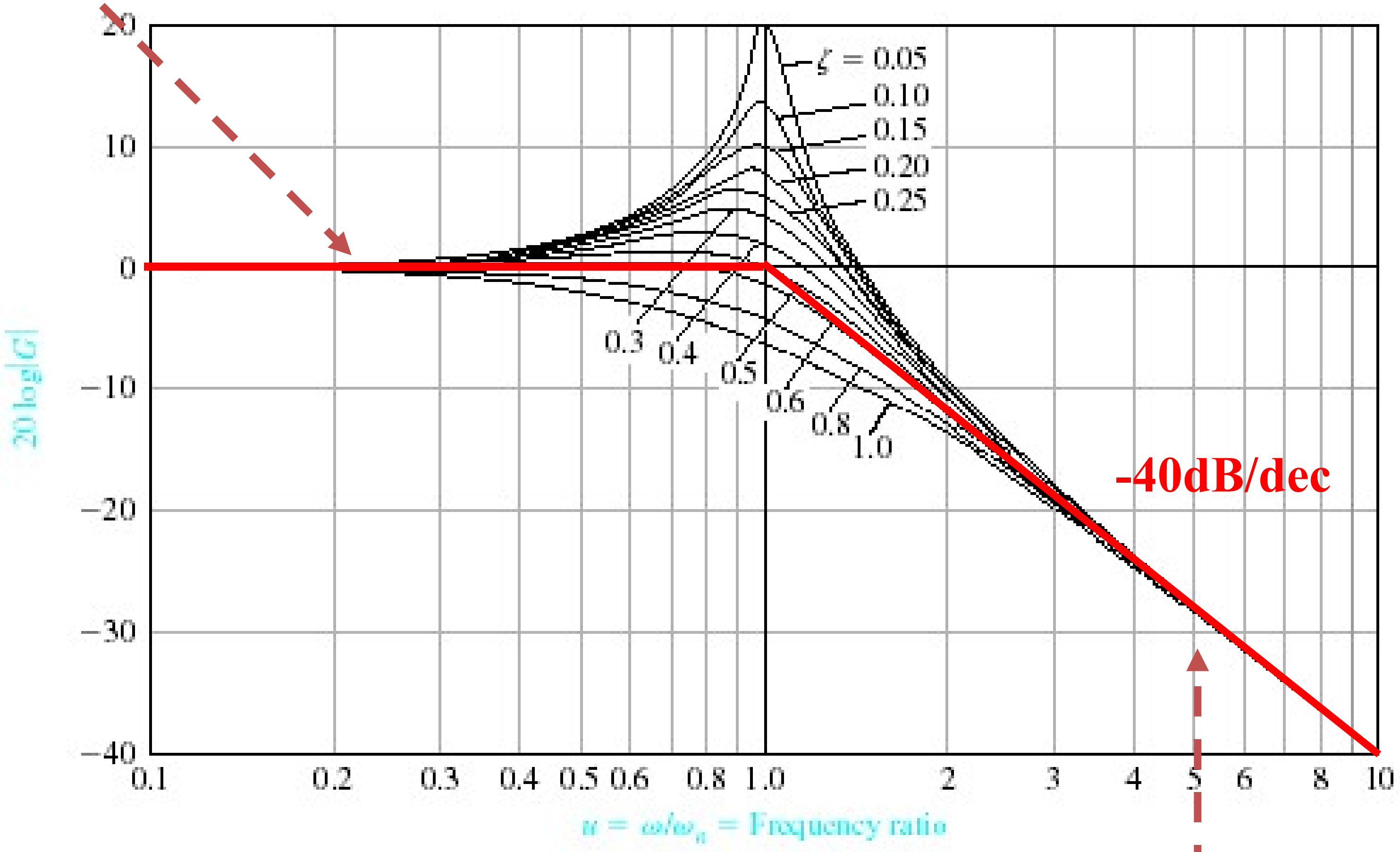
When $\omega >> \omega_n$, the Bode magnitude becomes

$$20\lg|G(j\omega)| \approx -20\lg\sqrt{\left(\frac{\omega^2}{\omega_n^2}\right)^2} = -40\lg\frac{\omega}{\omega_n}$$

Therefore, the corner frequency $=\omega_n$.

$$20\lg|G(\omega)| = -10\lg((1-u^2)^2 + 4\zeta^2 u^2), \qquad u = \omega/\omega_n$$

$$u << 1 \quad \rightarrow \quad 20\lg|G| \approx -10\lg 1 = 0\text{dB}$$



$$u >> 1 \rightarrow 20\lg|G| \approx -40\lg u, \qquad u = \omega/\omega_n$$

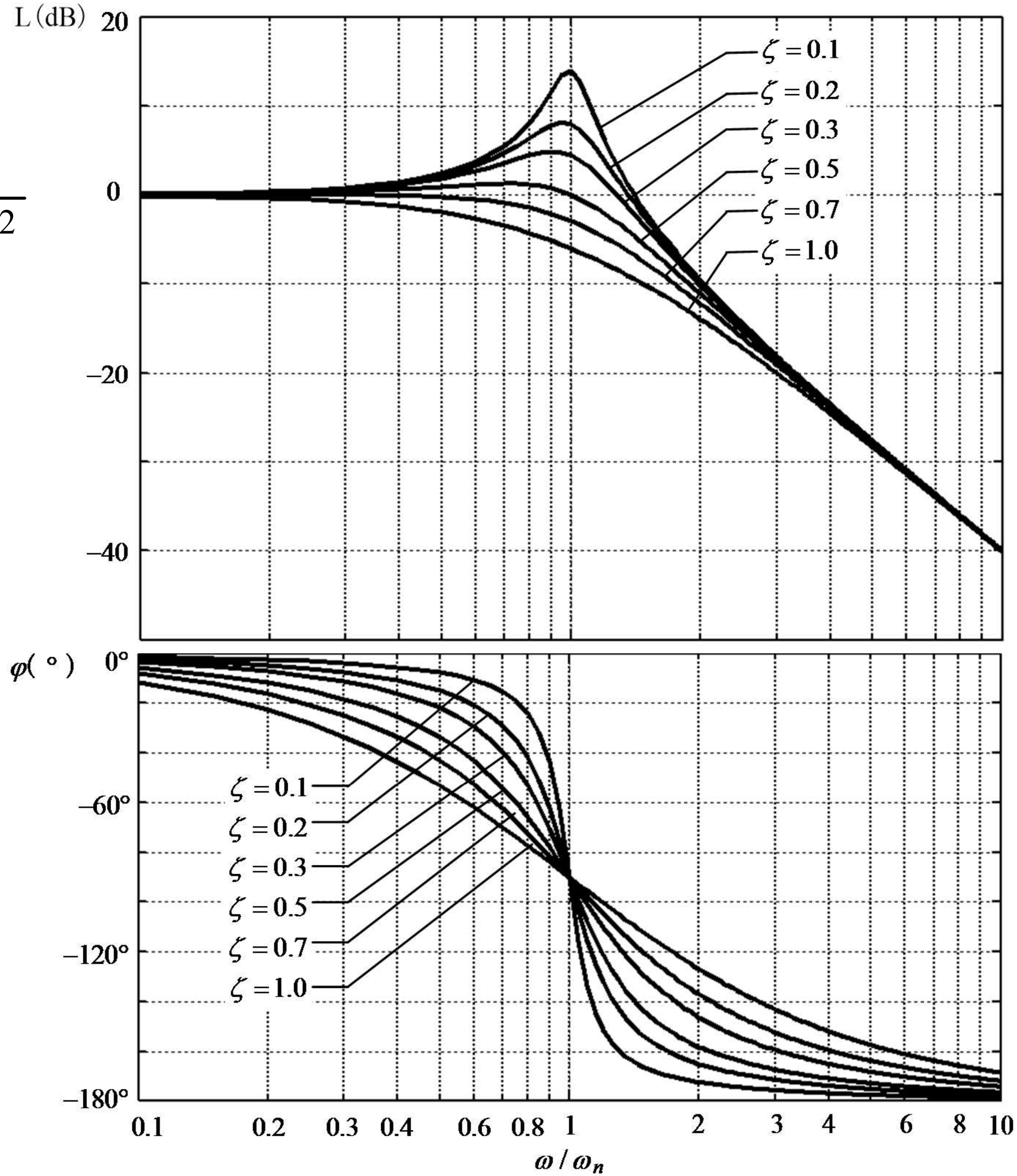
$$\angle G$$

$$= -\tan^{-1}\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

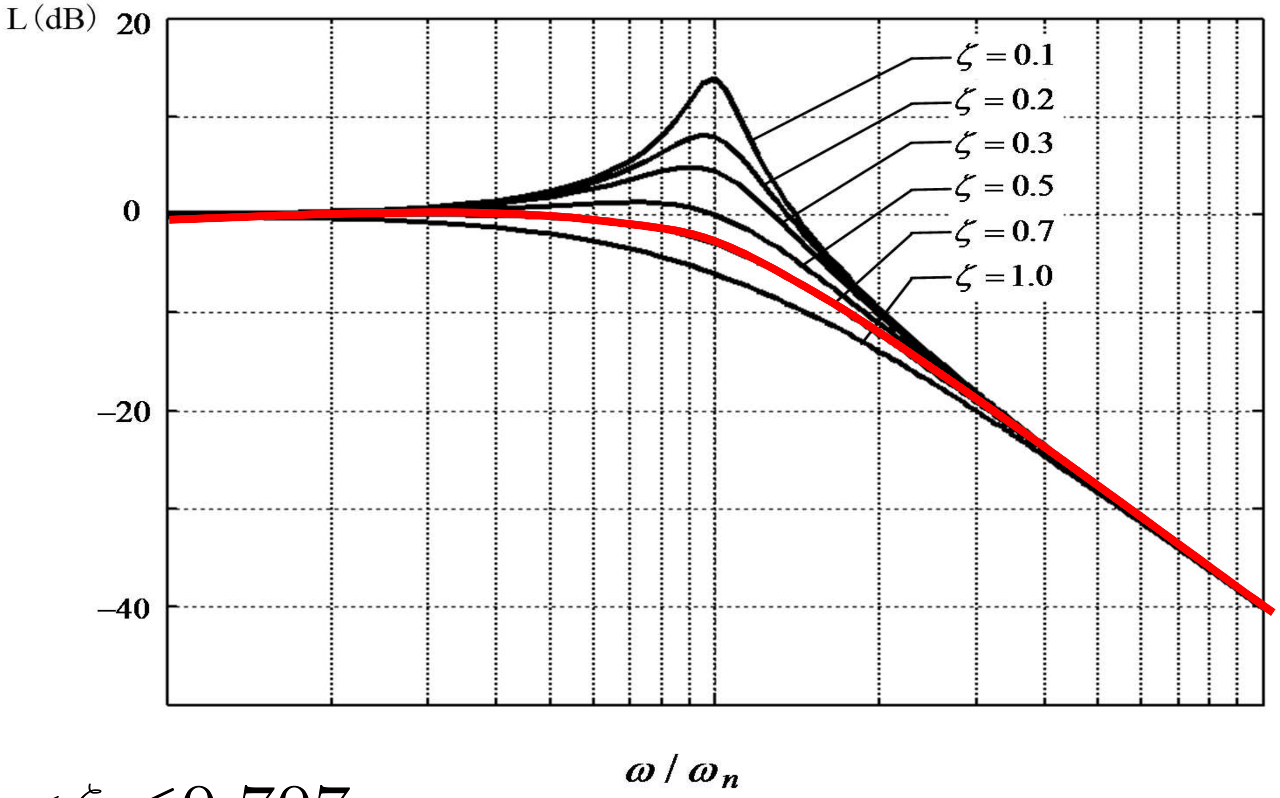
$$= \begin{cases} 0^0, & \omega = 0 \\ -90^0, & \omega = \omega_n \\ -180^0, & \omega = \infty \end{cases}$$

Note that at ω_n, the phase angle is -90^0 regardless of ζ since

$$\angle G = -\tan^{-1}\frac{2\zeta}{0}$$



Comment on the resonant peak value M_r



For $0 < \zeta \leq 0.707$,

$$M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

For $\zeta > 0.707$,

$$M_r = 1$$

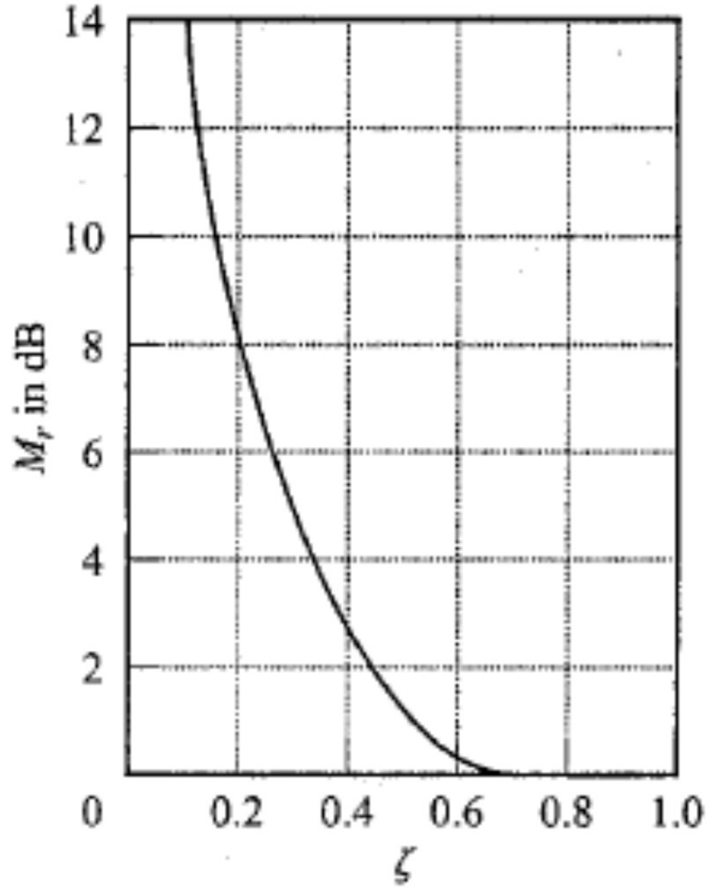
Since

$$M_r = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

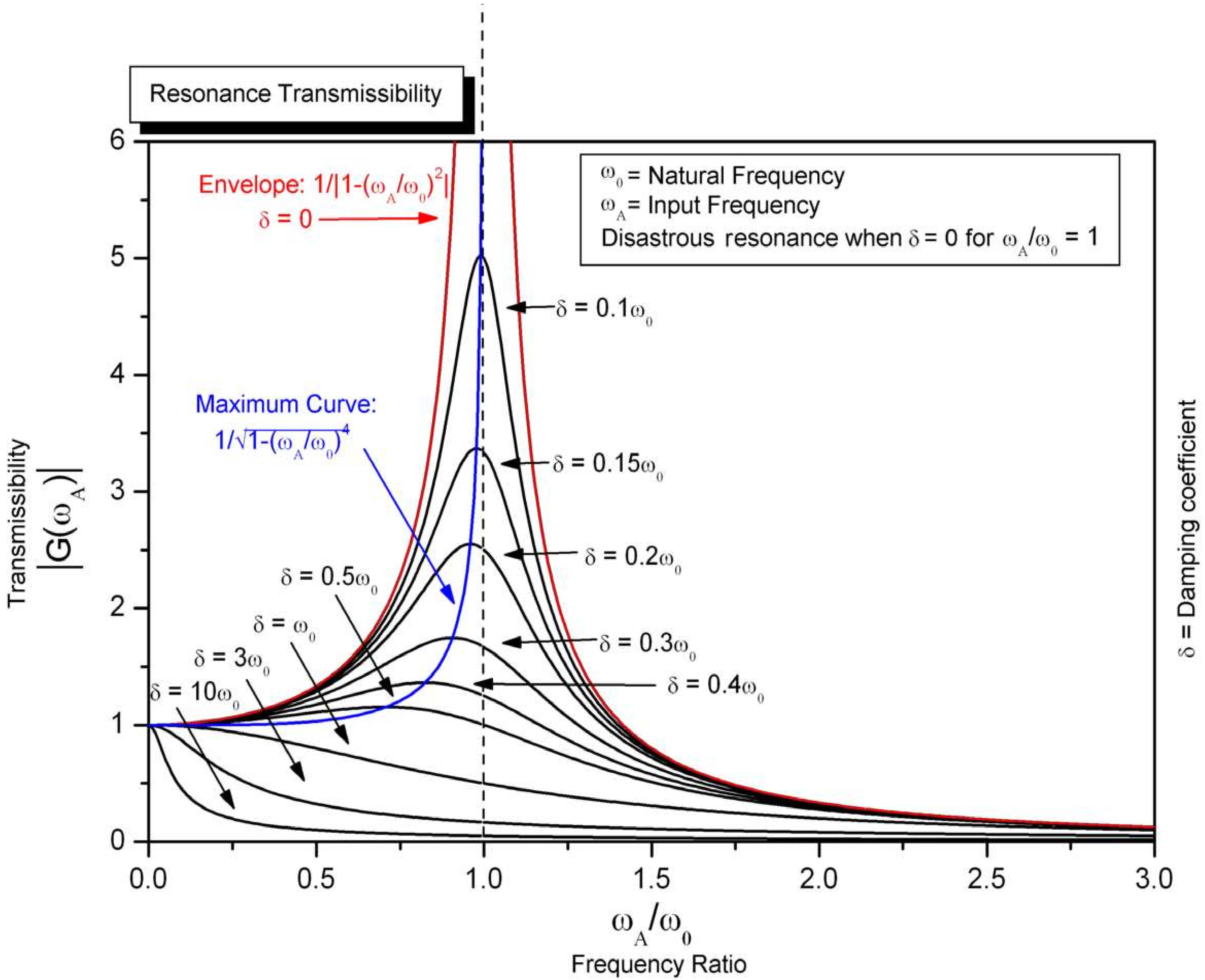
it follows that

$$\lim_{\zeta \to 0} \omega_r \to \omega_n$$

$$\lim_{\zeta \to 0} M_r \to \infty$$

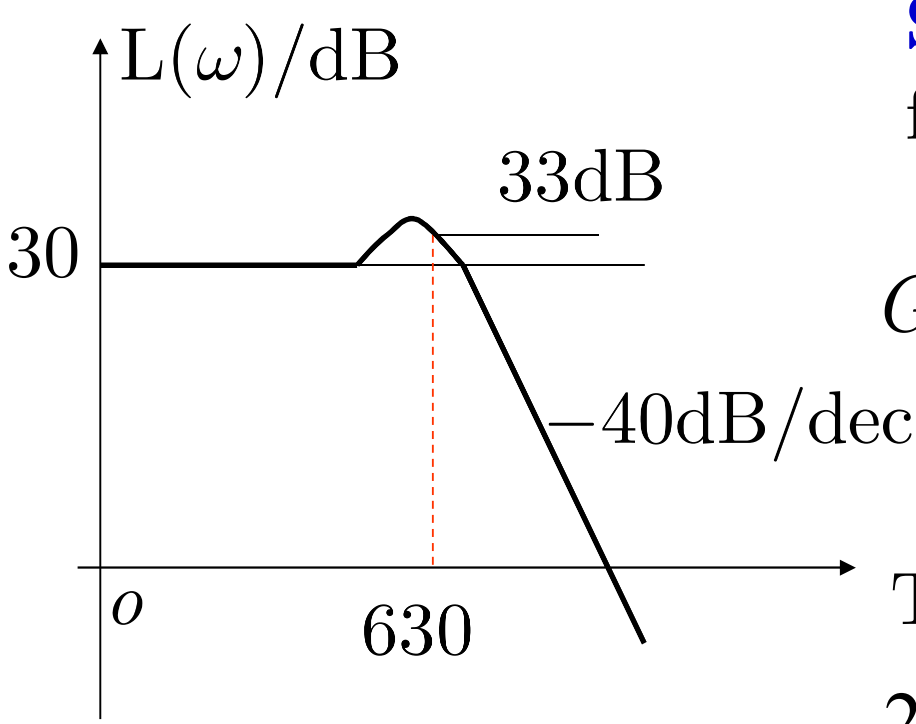


This means that if the undamped system is excited at its natural frequency, the magnitude of $G(j\omega)$ becomes infinite.



Resonance Transmissibility
Envelope: 1/|1-(ω_A/ω_0)^2|
δ = 0
ω_0 = Natural Frequency
ω_A = Input Frequency
Disastrous resonance when δ = 0 for ω_A/ω_0 = 1
δ = 0.1ω_0
δ = 0.15ω_0
δ = 0.2ω_0
δ = 0.3ω_0
δ = 0.4ω_0
δ = 0.5ω_0
δ = ω_0
δ = 3ω_0
δ = 10ω_0
Maximum Curve:
1/√1-(ω_A/ω_0)^4
Transmissibility
|G(ω_A)|
δ = Damping coefficient
ω_A/ω_0
Frequency Ratio
0
1
2
3
4
5
6
0.0
0.5
1.0
1.5
2.0
2.5
3.0

Example. The Bode asymptotic magnitude curve of a second-order system is shown below. Determine its transfer function and resonant peak value.



Solution: The transfer function is of the form:

$$G(j\omega)=\frac{K}{1+2\zeta\frac{\omega}{\omega_n}j+\left(\frac{j\omega}{\omega_n}\right)^2}$$

Therefore,

$$20\lg K=30\ \text{dB}\Rightarrow K=31.6$$

$$20\lg\frac{K}{2\zeta}=33\ \text{dB}\Rightarrow\zeta=\frac{1}{2\sqrt{2}}=0.35$$

3. General procedure for plotting Bode diagrams

1). Write $G(\mathrm{s})H(\mathrm{s})$ in the following normalized form (for instance):

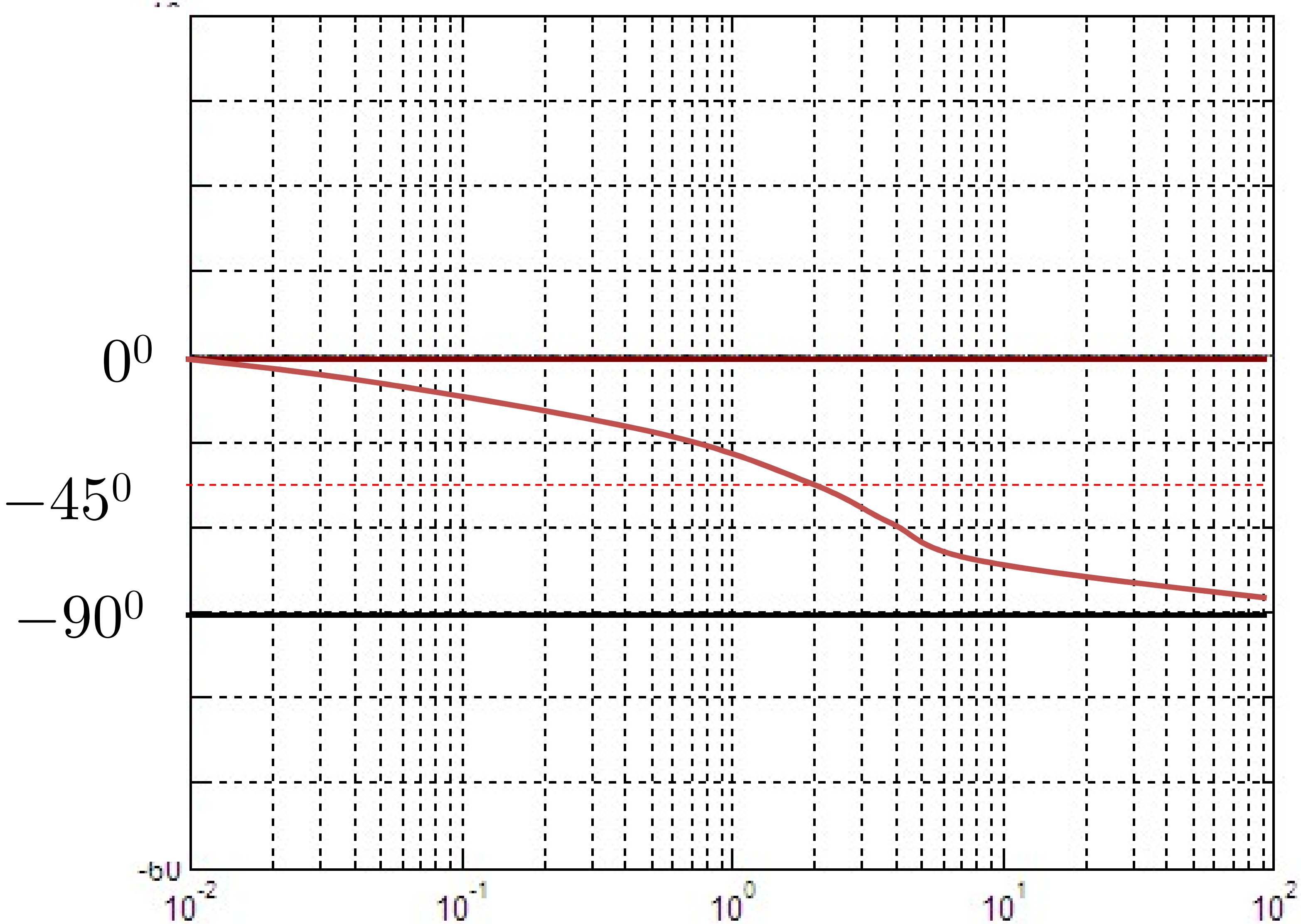
$$G(s)H(s)=\frac{K(\tau_1 s+1)(\tau_2 s+1)}{s(T_1 s+1)(T_2 s+1)[(s/\omega_n)^2+2\zeta(s/\omega_n)+1]}$$

from which the basic factors that compose the transfer function can be obtained.

2). Identify the corner frequencies associated with these basic factors (for instance): $1/\tau_1$, $1/\tau_2$, $1/T_1$, $1/T_2$, ω_n, and so on.

3). Draw the separate asymptotic magnitude curves for each of the factors:

Correspondingly, draw the separate phase angle curves for each of the factors:



4). Draw the composite curves by algebraically adding the individual curves.

Example. The controlled plant is

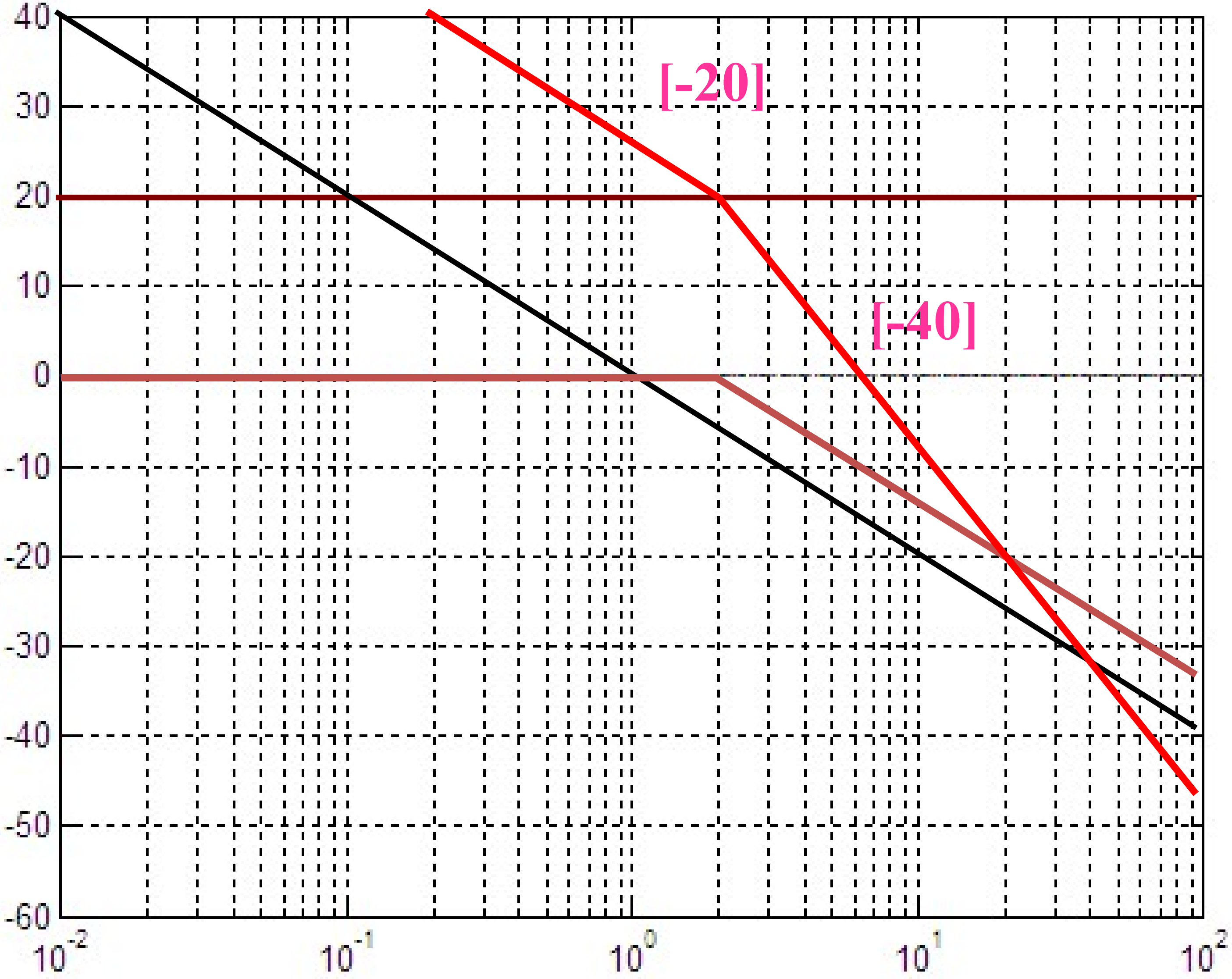
$$G(s) = \frac{10}{s(0.5s+1)}$$

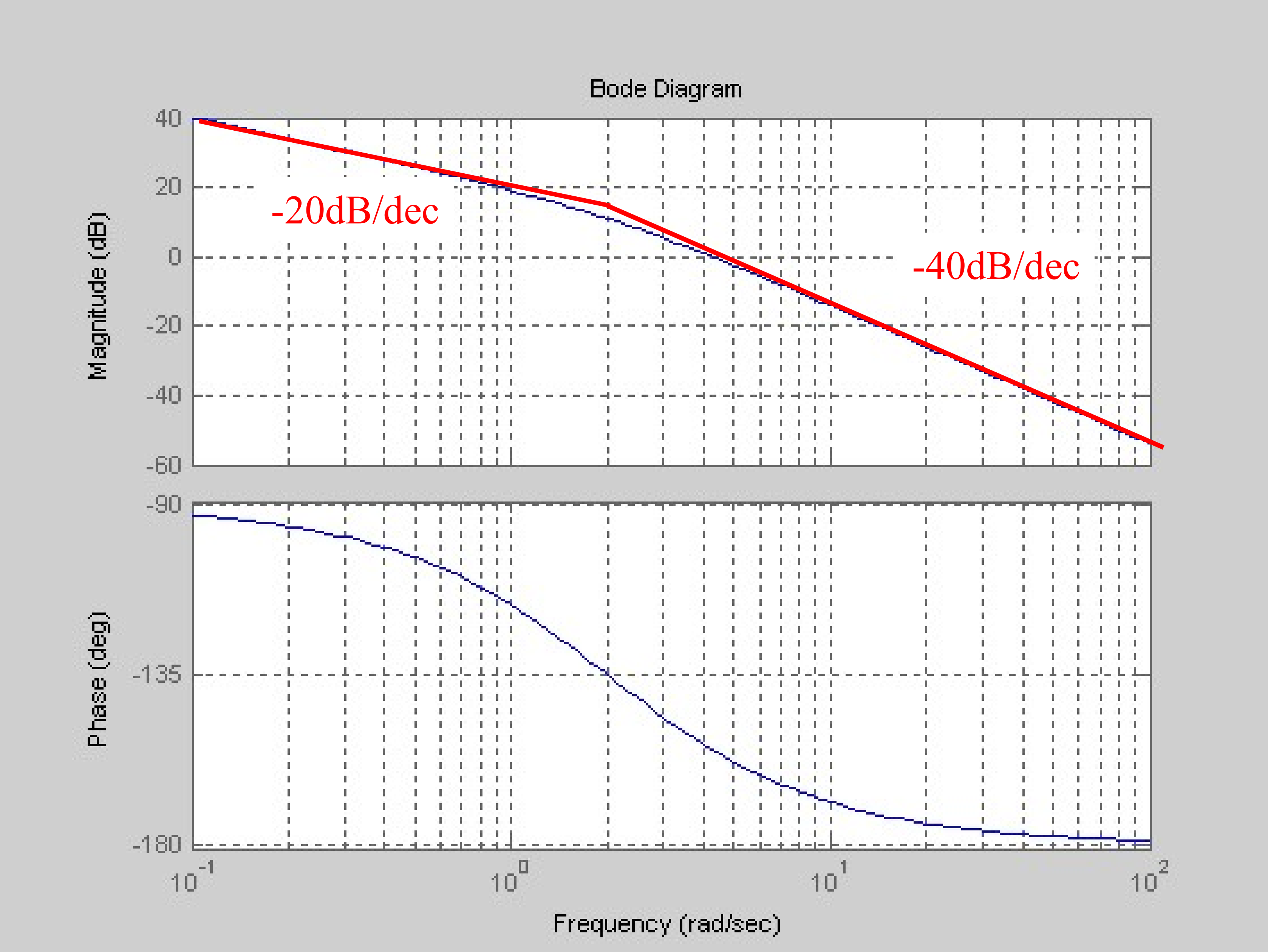
Draw its asymptotic Bode diagram.

Solution: Note that the system consists of

- Constant gain K =10 (20lg10=20 dB)
- An integral factor
- A first-order factor with corner frequency 2rad/s

$$10 \qquad \frac{1}{s} \qquad \frac{1}{0.5s+1}$$





Bode Diagram
Magnitude (dB)
40
20
0
-20
-40
-60
-20dB/dec
-40dB/dec
Phase (deg)
-90
-135
-180
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Example. The controlled plant is

$$G(j\omega)=\frac{5(0.1s+1)}{s(0.5s+1)[(s/50)^2+0.6(s/50)+1]}\Bigg|_{s=j\omega}$$

Draw its asymptotic Bode diagram.

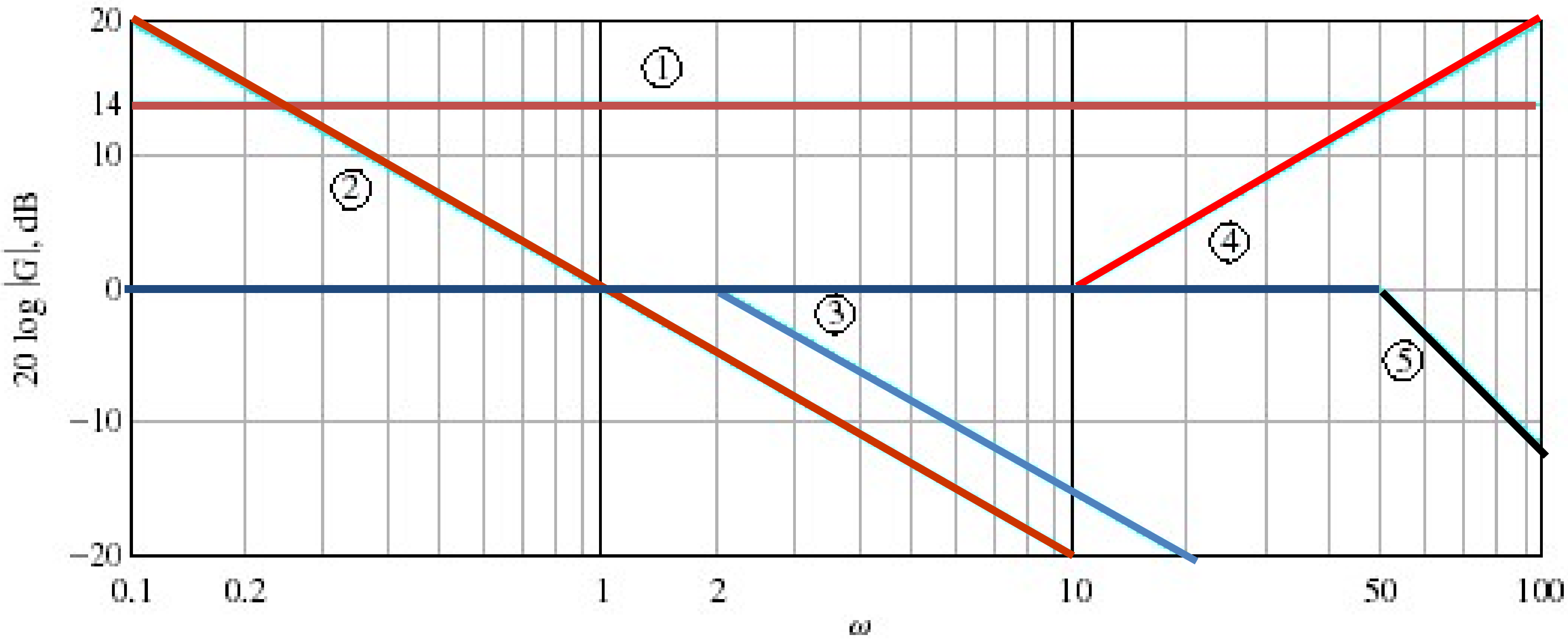
A constant gain $K=5$ $(20\lg K=14\text{ dB})$

An integral factor $1/s$

A first-order factor with corner frequency 2rad/s

A first-order derivative factor with corner frequency 10rad/s

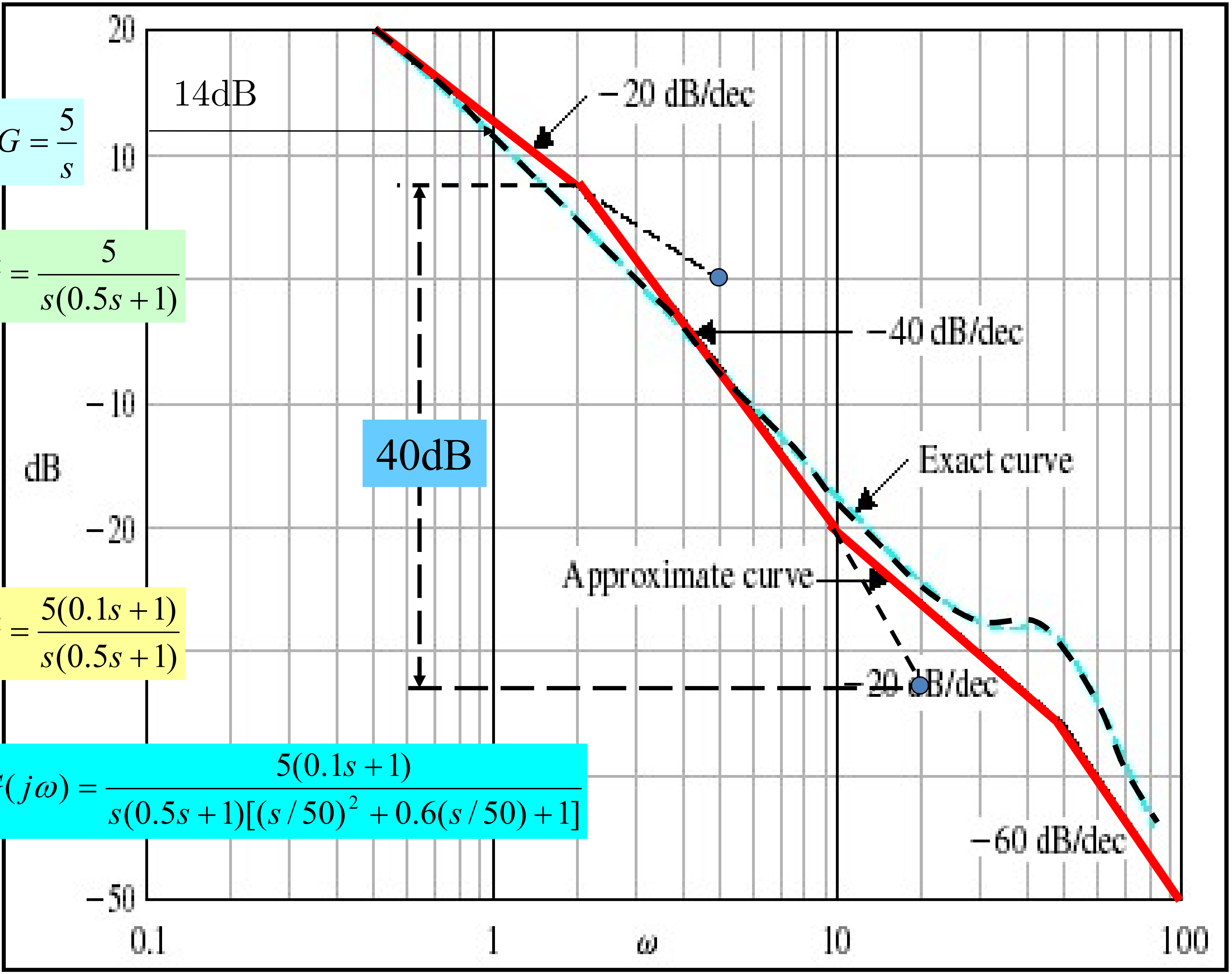
A pair of complex poles with corner frequency 50rad/s



$$G(j\omega)=\left.\frac{5(0.1s+1)}{s(0.5s+1)[(s/50)^2+0.6(s/50)+1]}\right|_{s=j\omega}$$

$$G_1(j\omega)=5 \qquad G_2(j\omega)=\frac{1}{j\omega} \qquad G_3(j\omega)=\frac{1}{0.5j\omega+1}$$

$$G_4(j\omega)=0.1j\omega+1 \qquad G_5(j\omega)=\frac{1}{(j\omega/50)^2+0.6(j\omega/50)+1}$$

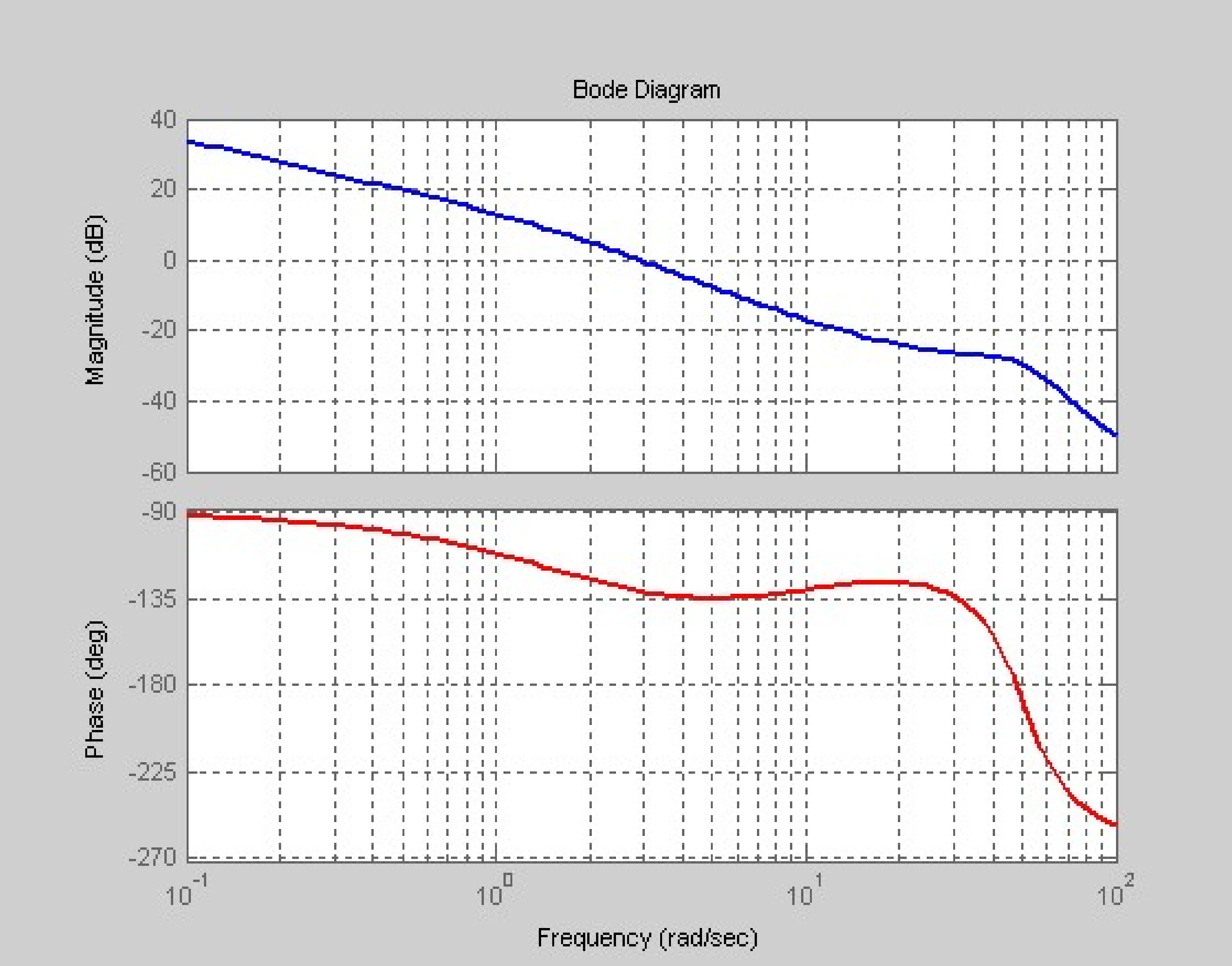


$$G = \frac{5}{s}$$

$$G = \frac{5}{s(0.5s+1)}$$

$$G = \frac{5(0.1s+1)}{s(0.5s+1)}$$

$$G(j\omega) = \frac{5(0.1s+1)}{s(0.5s+1)[(s/50)^2+0.6(s/50)+1]}$$



Bode Diagram
Magnitude (dB)
40
20
0
-20
-40
-60
Phase (deg)
-90
-135
-180
-225
-270
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Example. The controlled plant is

$$G(s) = \frac{10(s+3)}{s(s+2)(s^2+s+2)}$$

Draw its asymptotic Bode diagram.

Note that we must rewrite the above transfer function in a normalized form.

4. Minimum phase and nonminimum phase transfer functions

Definition: A transfer function is called a minimum phase transfer function if all its zeros and poles lie in the left-half s-plane, whereas those having poles and zeros in the right-half plane are called nonminimum phase transfer functions.

Example. We investigate the range of the phase angles for the following two plants over the entire frequency range from zero to infinity. By definition, $G_1(\mathrm{s})$ is a minimum phase transfer function, while $G_2(\mathrm{s})$ is a nonminimum phase one.

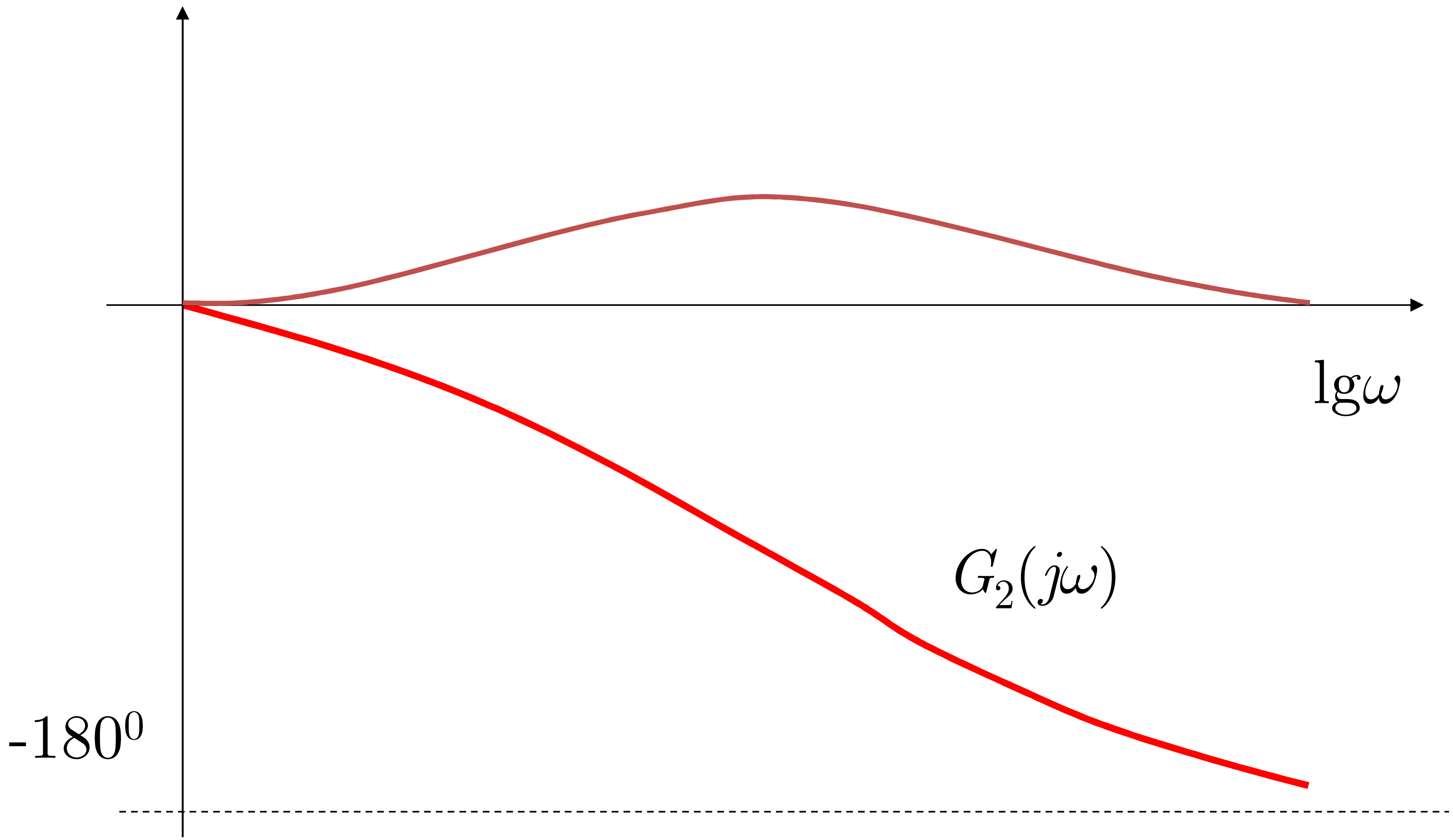
$$G_1(s) = \frac{1+Ts}{1+T_1 s} \Rightarrow G_1(j\omega) = \frac{1+Tj\omega}{1+T_1 j\omega}$$

$$G_2(s) = \frac{1-Ts}{1+T_1 s} \Rightarrow G_2(j\omega) = \frac{1-Tj\omega}{1+T_1 j\omega}$$

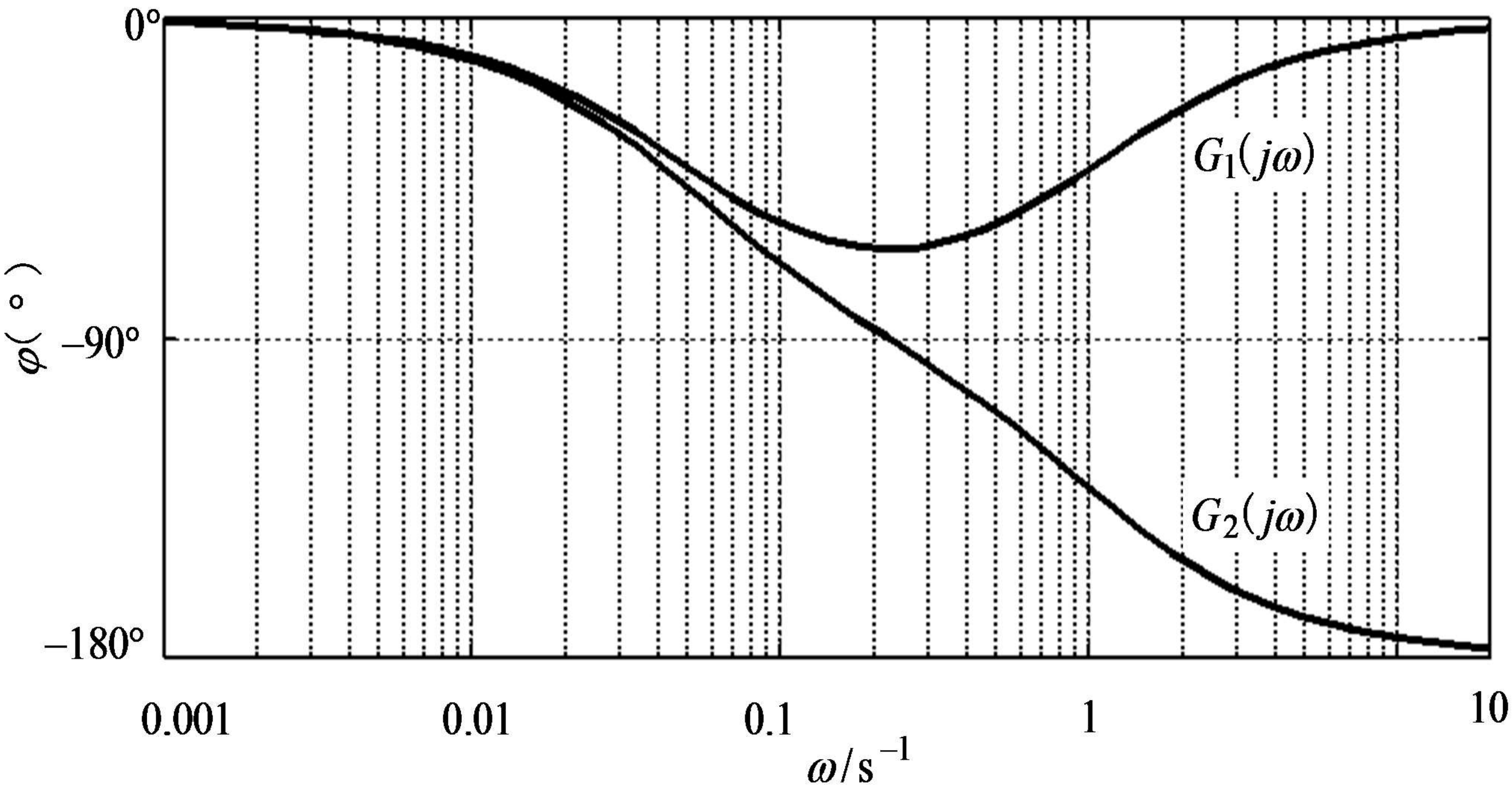
We investigate two cases:

$$0 < T_1 < T \text{ and } 0 < T < T_1.$$

In case of $T > T_1$, $G_1(s)$ and $G_2(s)$ have same Bode magnitude, but their phase angles are different:



In case of $T < T_1$, $G_1(\mathrm{s})$ and $G_2(\mathrm{s})$ have same Bode magnitude, but their phase angles are different:

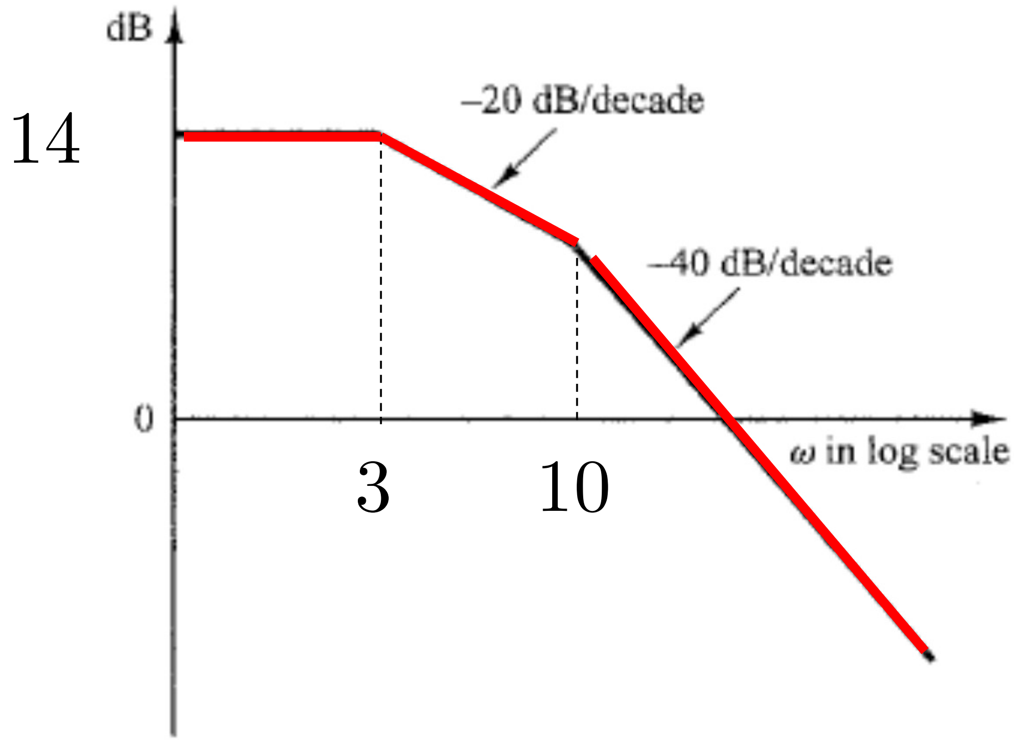


In general, we have the following conclusions:

- For a minimum phase system, the transfer function can be **uniquely determined from the magnitude curve alone**, while for nonminimum system, this is not the case.
- For systems with the same magnitude characteristics, the range in phase angle of the minimum-phase transfer function is minimum among all such systems, while the range in phase angle of any nonminimum-phase transfer function is greater than this minimum.

Hendrik Wade Bode (1905–1982) was born in Madison, Wisconsin. He received his B.A. degree in 1924 and M.A. Degree in 1926, both from the Ohio State University. While employed at Bell Laboratories, he attended Columbia University Graduate School, and received his Ph.D. degree in 1935. While working on the design of a variable equalizer, he discovered the relationship between gain and phase of a stable and minimum phase transfer function. Bode retired from Bell Telephone Laboratories in October 1967 at the age of 61, after 41 years of distinguished service in his career. He was immediately elected Gordon McKay Professor of Systems Engineering at Harvard University. Bode is a fellow of a number of scientific and engineering societies, including the IEEE, the American Physical Society, and the American Academy of Arts and Sciences. He has also achieved further accolade by being elected to the National Academy of sciences and the National Academy of Engineering. He was awarded the 1969 IEEE Edison Medal “For fundamental contributions to the arts of communication, computation, and control; for leadership in bringing mathematical science to bear on engineering problems; and for guidance and creative counsel in systems engineering.”

Example. The Bode magnitude plot of a minimum phase system is shown below:



Determine its transfer function and static position error constant K_p.

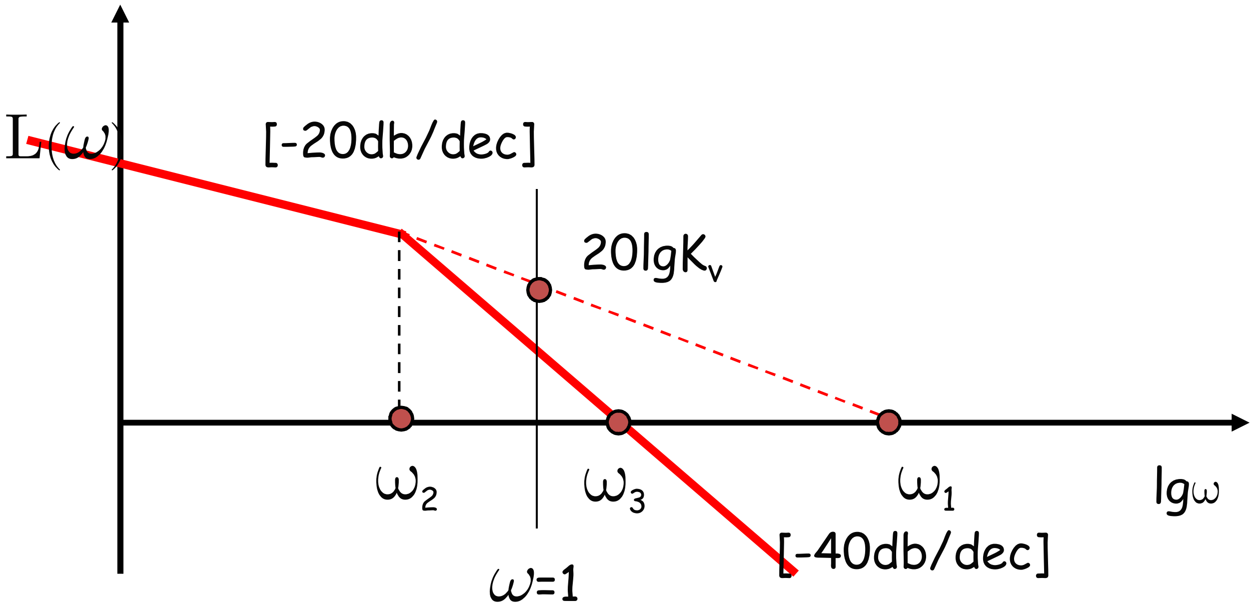
Solution: The minimum phase assumption implies that the magnitude plot alone suffices to determine its transfer function. The system consists of three factors: gain K, and two first-order factors with corner frequencies 3 rad/s and 10rad/s, respectively. Therefore, the transfer function is

$$G(s) = \frac{5}{(\frac{1}{3}s+1)(0.1s+1)}$$

The static position error constant is

$$K_p = \lim_{s\to 0} G(s) = 5$$

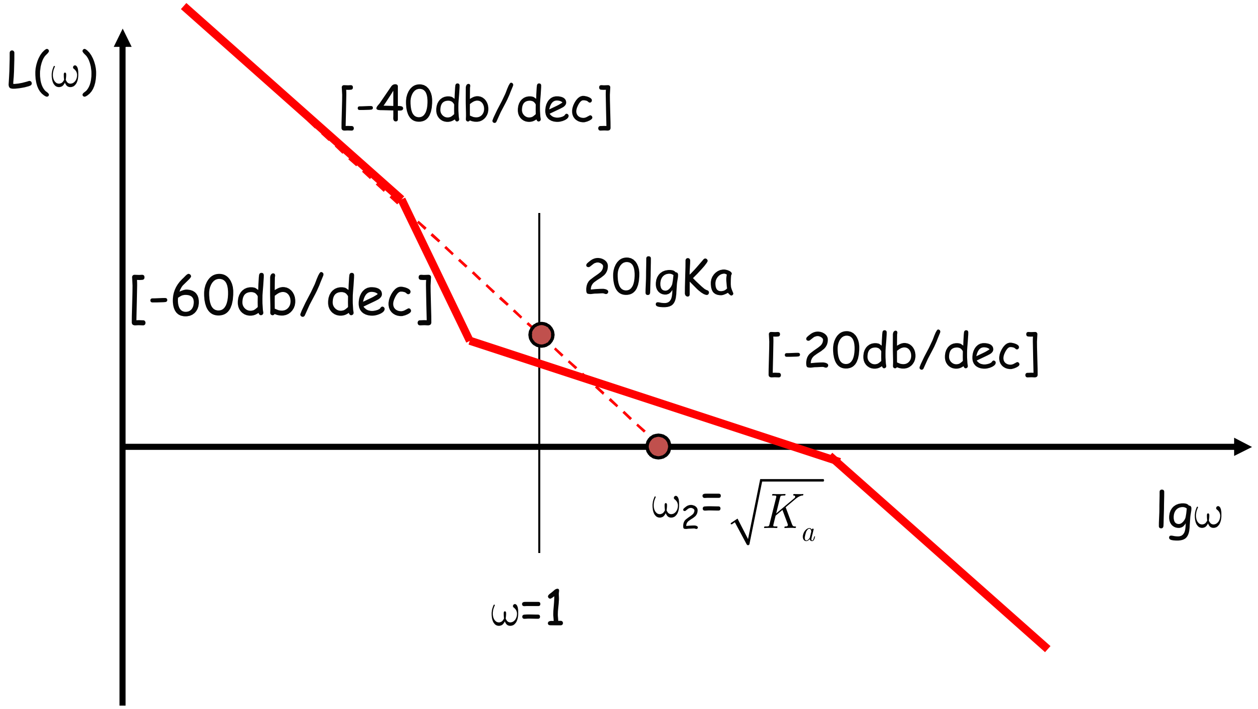
Example. The Bode magnitude plot for Type 1 minimum phase system is shown below:



Determine its transfer function and prove that

$$\omega_1 = K_v$$

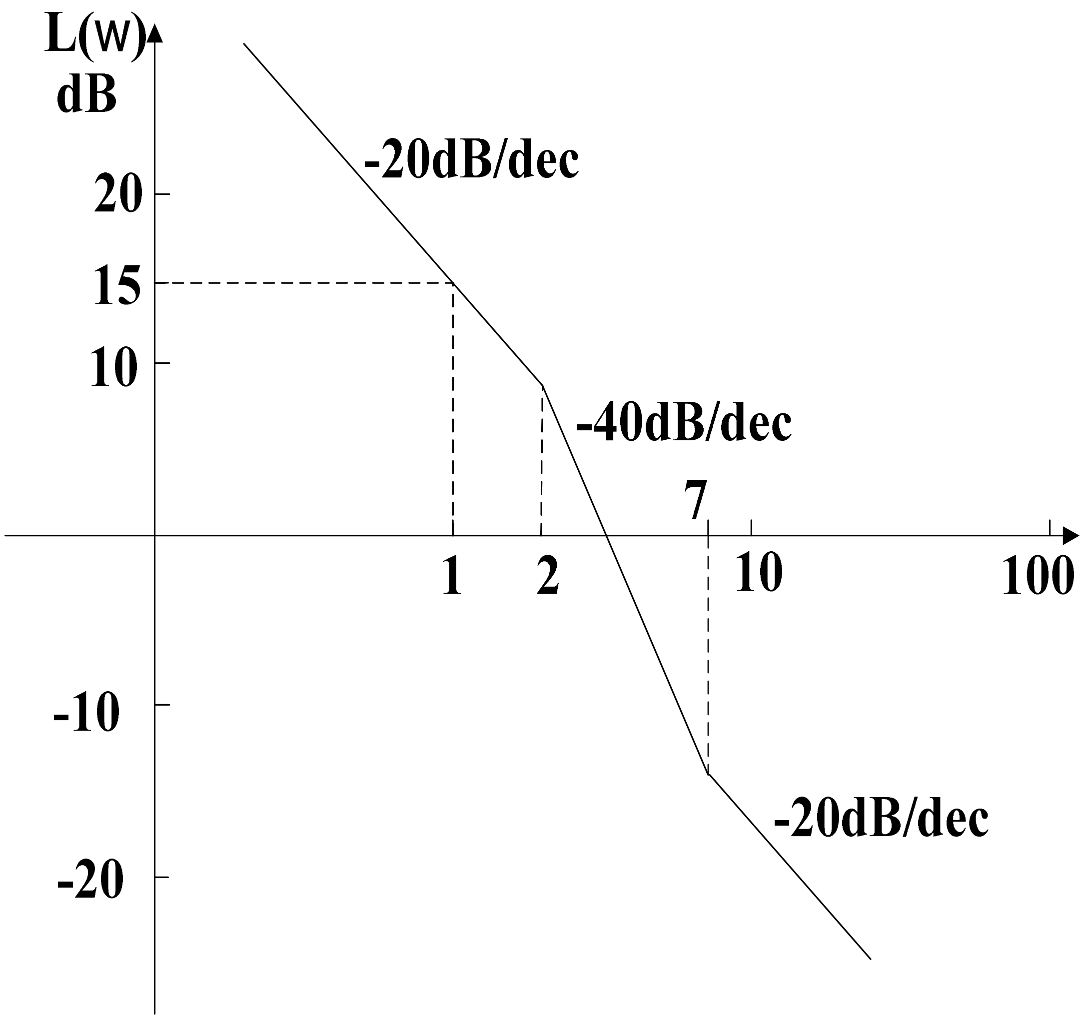
Example. The Bode magnitude plot for type 2 minimum phase system is shown below:



Prove that

$$\omega_2 = \sqrt{K_a}$$

Example. The asymptotical Bode magnitude plot of a minimum phase system is shown below. Determine its transfer function.



Solution: The minimum phase assumption implies that the transfer function is of the form:

$$G(s) = \frac{K(\tau s + 1)}{s(Ts + 1)}$$

The slope of the leftmost curve is −20dB/dec. Therefore, the system has an integral factor.

When ω=1, the y-axis is 15dB, at which the gain is

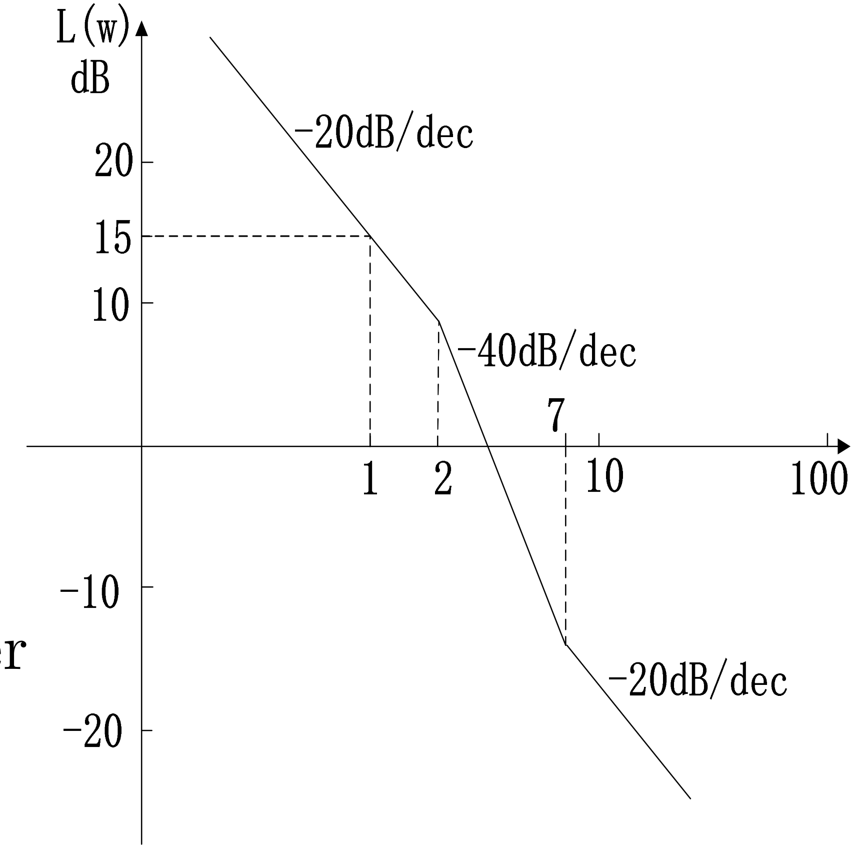
$$20\lg K=15,\ K=5.6$$

When 2≤ω≤7, the slope changes from −20dB/dec to −40dB/dec, which shows that ω=2 is the corner frequency of a first-order system.

ω=7 is the corner frequency of a first-order derivative factor.

Therefore, the transfer function is

$$G(s)=\frac{5.6(s/7+1)}{s(s/2+1)}$$



7-3 Polar Plots

1. The concept of Nyquist curve

The **Nyquist curve** of $G(j\omega)$ is a plot of the magnitude of $G(j\omega)$ versus the phase angle of $G(j\omega)$ on the polar coordinates as ω is varied from zero to infinity. Write

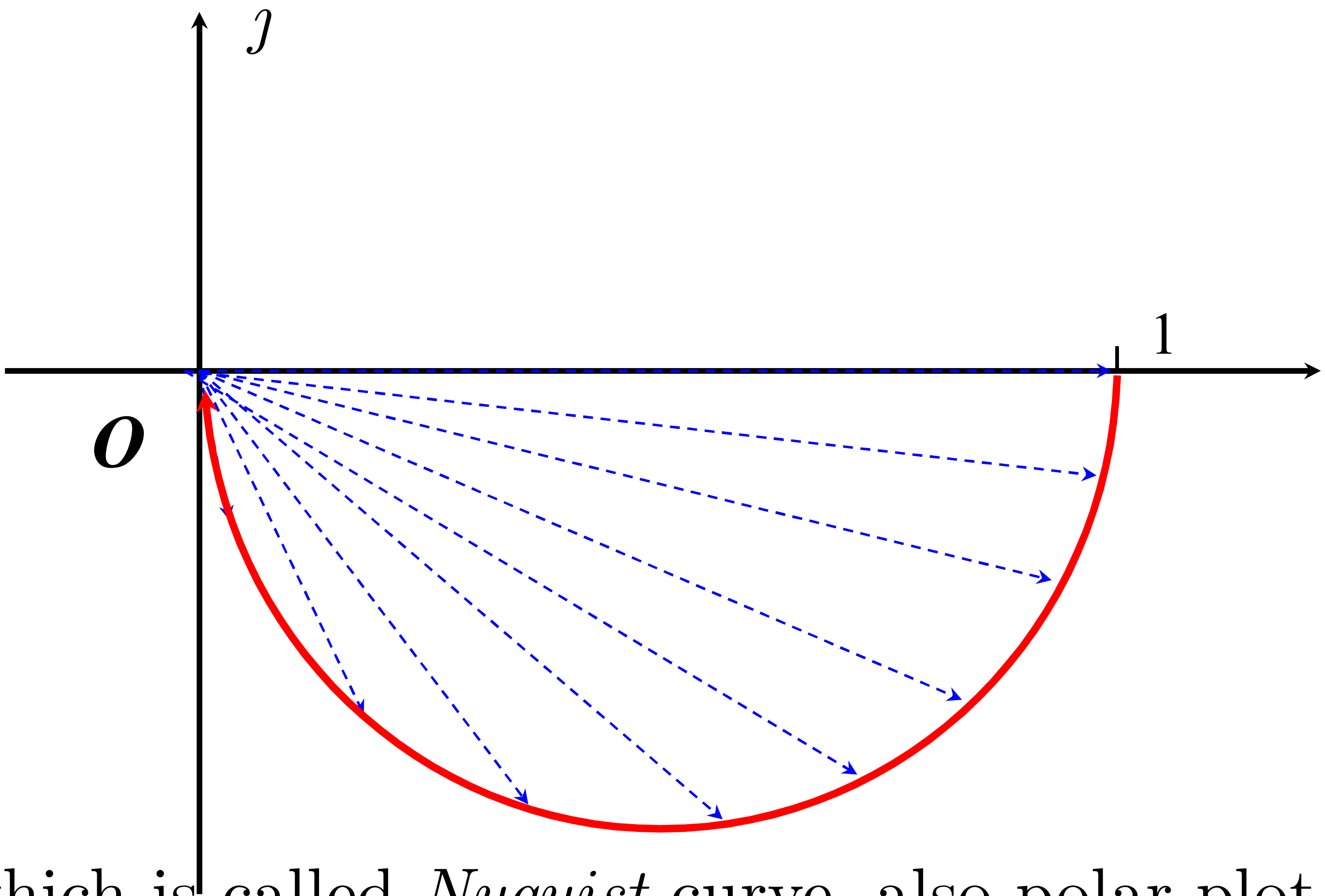
$$G(j\omega) = G(s)\Big|_{s=j\omega} = \mathrm{Re}[G(\omega)] + j\,\mathrm{Im}[G(\omega)]$$

or

$$G(j\omega) = \left|G(j\omega)\right| e^{j\phi(j\omega)} = \left|G(j\omega)\right| \angle G(j\omega)$$

Let $$r(\omega) = |G(j\omega)| \qquad \phi(\omega) = \angle G(j\omega)$$

Then, as ω varies from 0 to $+\infty$, the terminal of the vector function traces out a curve



which is called *Nyquist* curve, also polar plot.

2. Nyquist plots for basic factors of G(jω)H(jω)

1). Gain K

2). Integral and derivative factors $(j\omega)^{\pm 1}$

3). First-order factors $(1 + j\omega T)^{\pm 1}$

4). Quadratic factors $[(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1]^{\pm 1}$

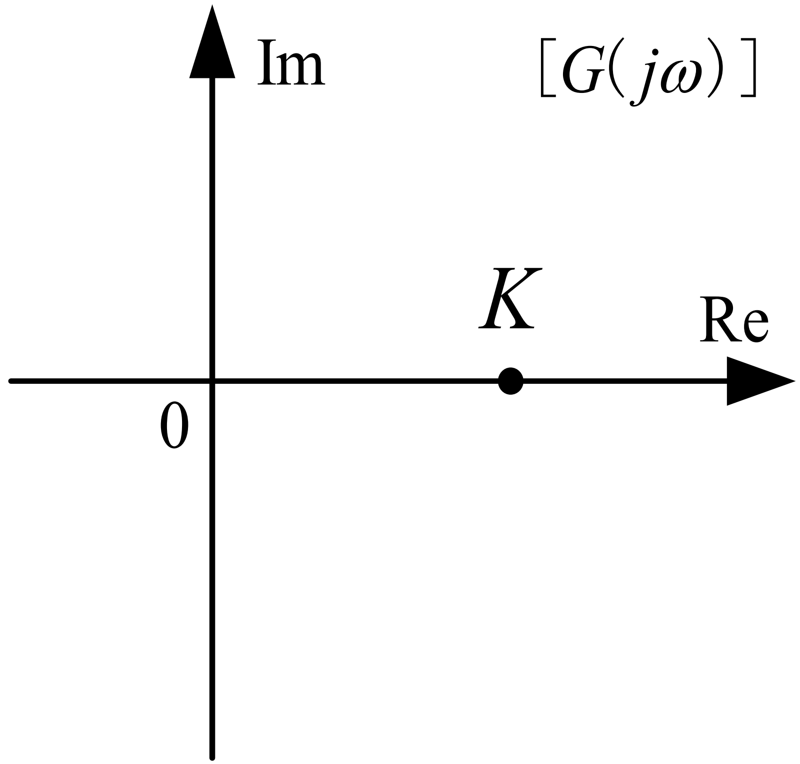
1). Gain K

$$G(s) = K$$

$$G(j\omega) = K$$

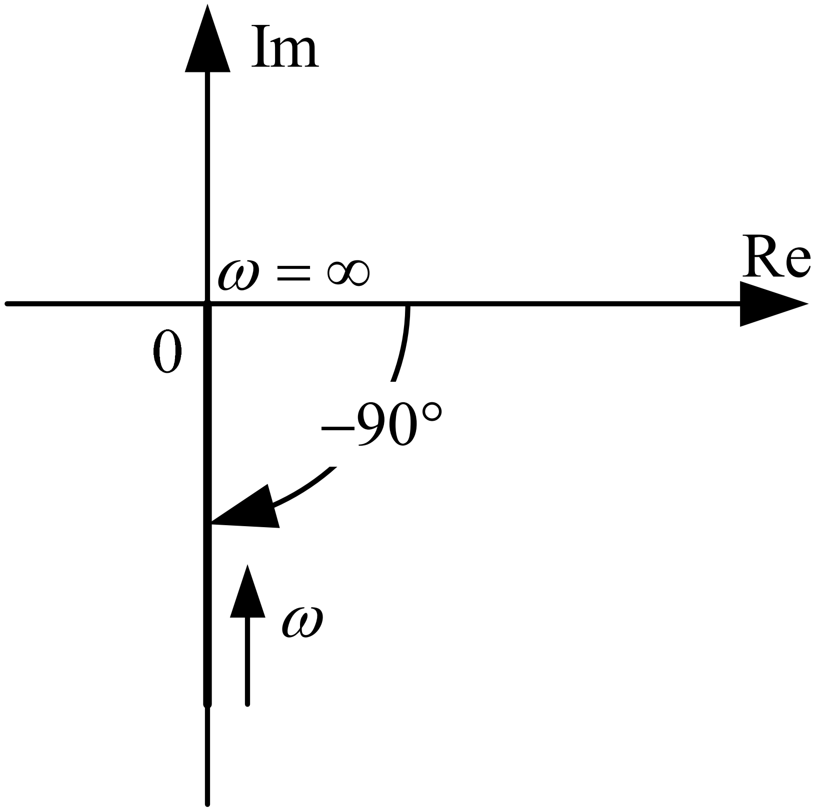
$$|G(j\omega)| = K$$

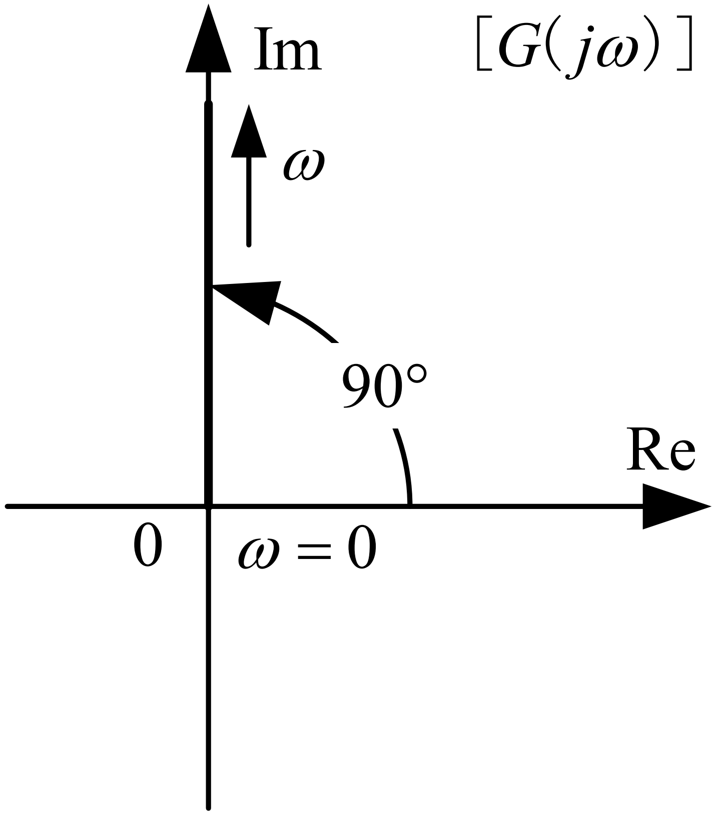
$$\phi = \angle |G(j\omega)| = 0^\circ$$



2). Integral and derivative factors $(j\omega)^{\pm 1}$

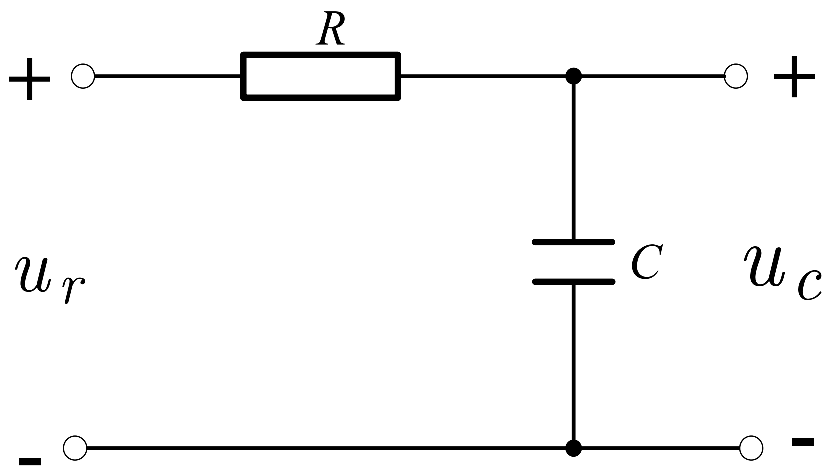
$$G(j\omega) = \frac{1}{j\omega} = \frac{1}{\omega} e^{-j\frac{\pi}{2}} \qquad\qquad G(j\omega) = j\omega = \omega e^{j\frac{\pi}{2}}$$





3). First-order factors $(1+j\omega T)^{\pm 1}$

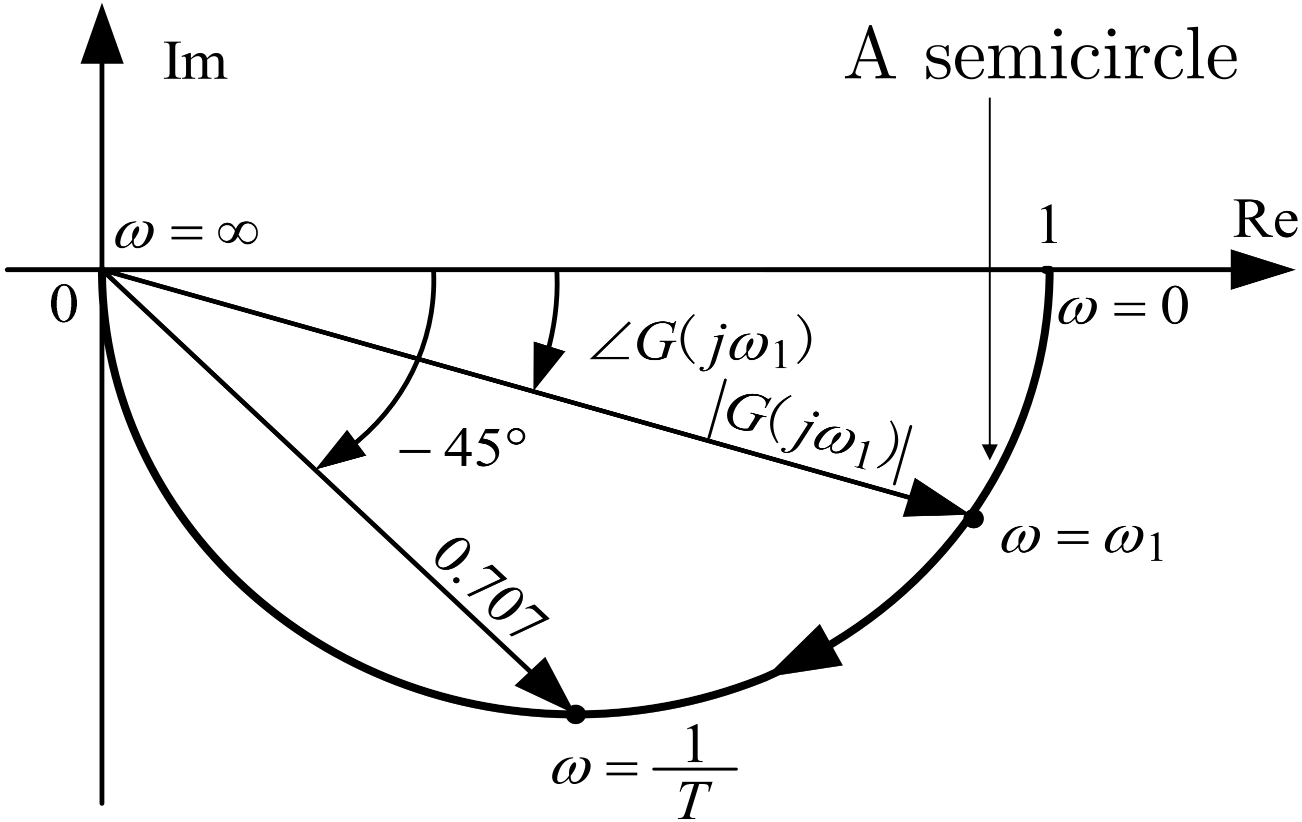
•Consider the following network:



$$G(s)=\frac{U_c(s)}{U_r(s)}=\frac{1}{RCs+1}=\frac{1}{Ts+1}$$

Plot its Nyquist curve.

$$G(j\omega)=\frac{1}{\sqrt{T^2\omega^2+1}}\angle-\tan^{-1}T\omega,\qquad \omega_1=\frac{1}{RC}=\frac{1}{T}$$



Notice: To draw the Nyquist curve, one should choose some typical points, for instance, ω=0, ω=1/T and ω=∞.

Example. Plot the *Nyquist* curve of the following transfer function:

$$G(s)=\frac{1}{s(Ts+1)}$$

Since

$$G(j\omega)=\frac{1}{j\omega(Tj\omega+1)}=-\frac{T}{(T^2\omega^2+1)}-j\frac{1}{\omega(T^2\omega^2+1)}$$

$$=\frac{1}{\omega\sqrt{T^2\omega^2+1}}\angle(-90^0-\tan^{-1}T\omega)$$

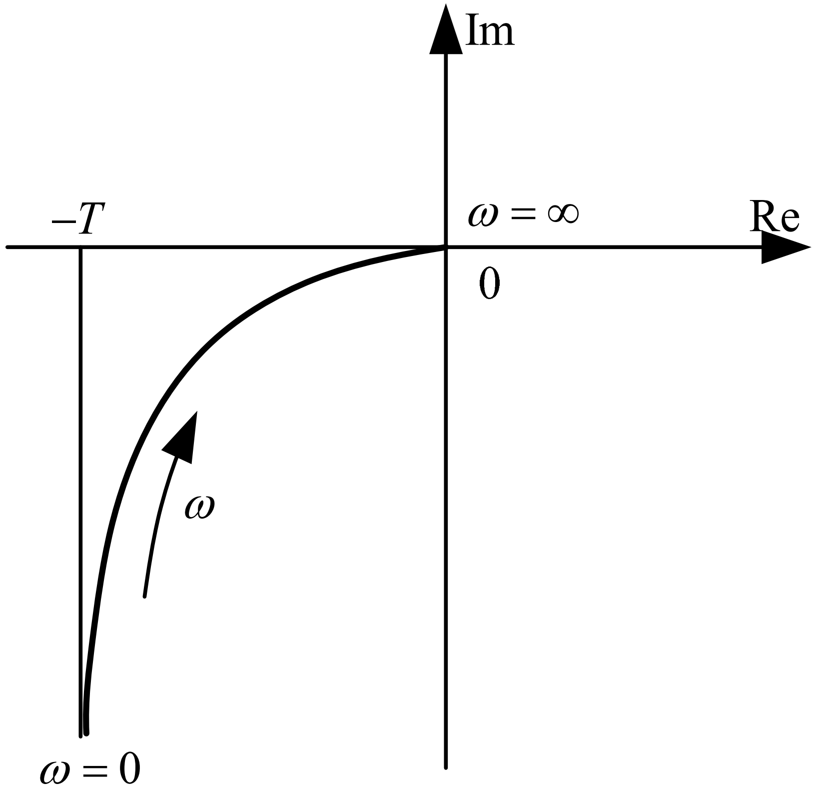
The low-frequency portion of the curve is

$$\lim_{\omega\to 0}G(j\omega)=-T-j\infty=-\infty\angle-90^0$$

The high-frequency portion is

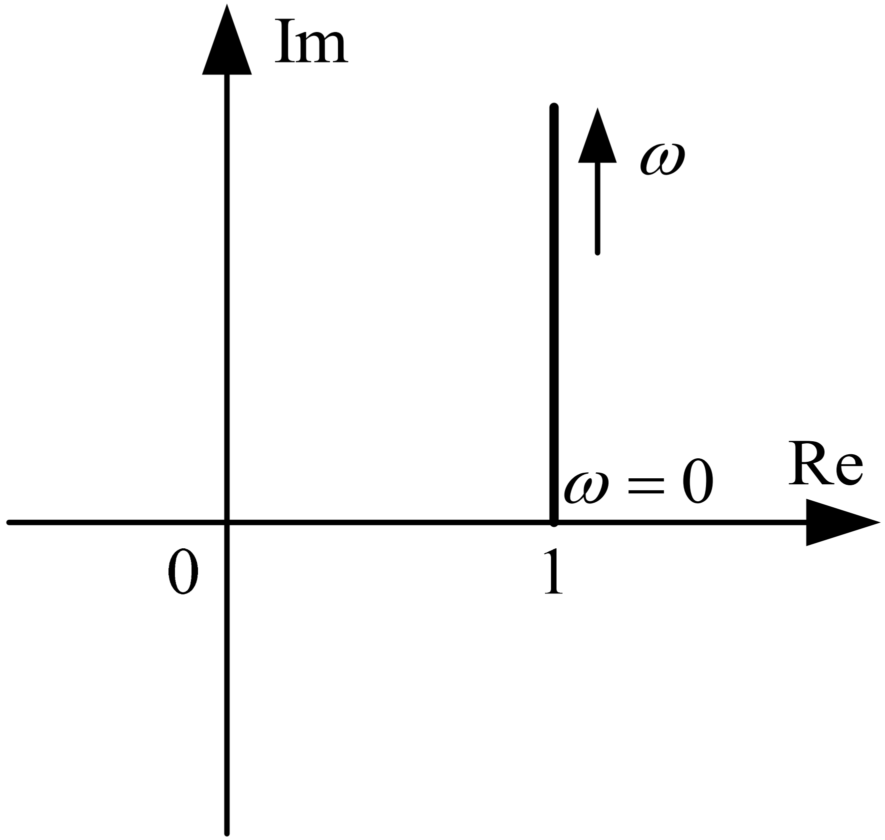
$$\lim_{\omega \to \infty} G(j\omega) = 0\angle -180^0$$

and is tangent to the negative real axis.

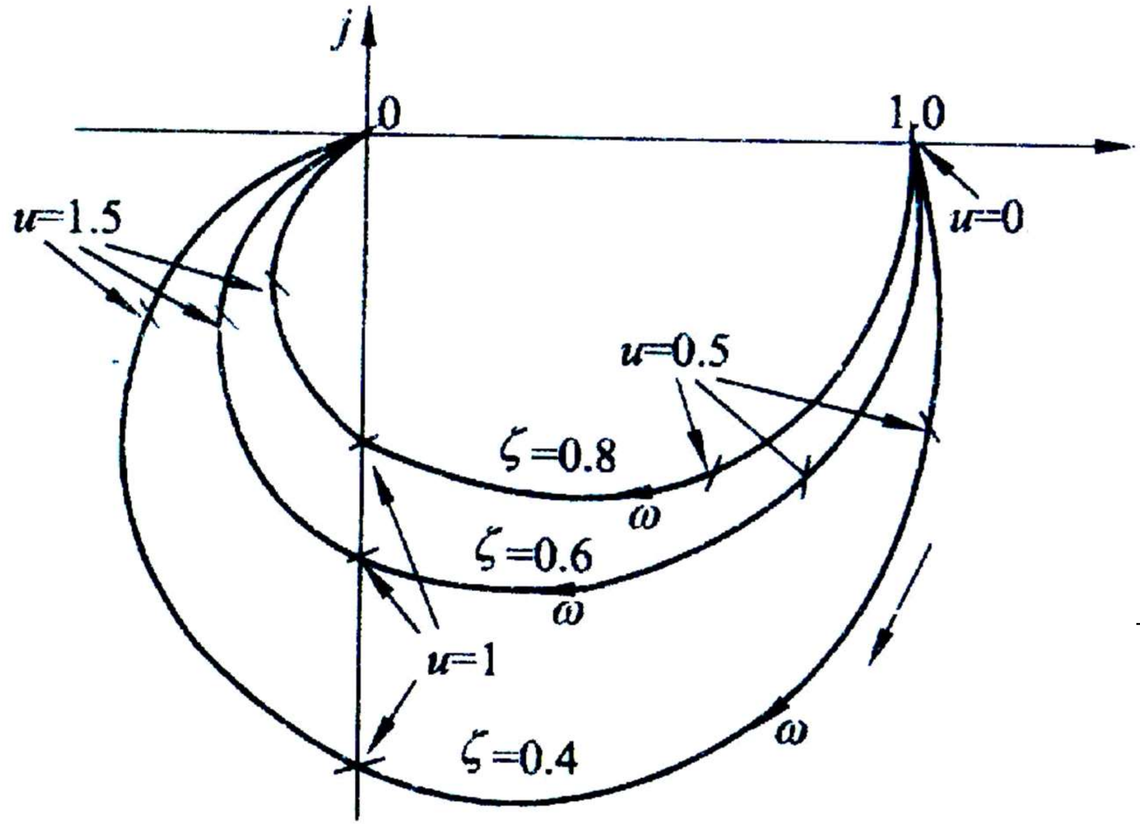


•First-order derivative factor:

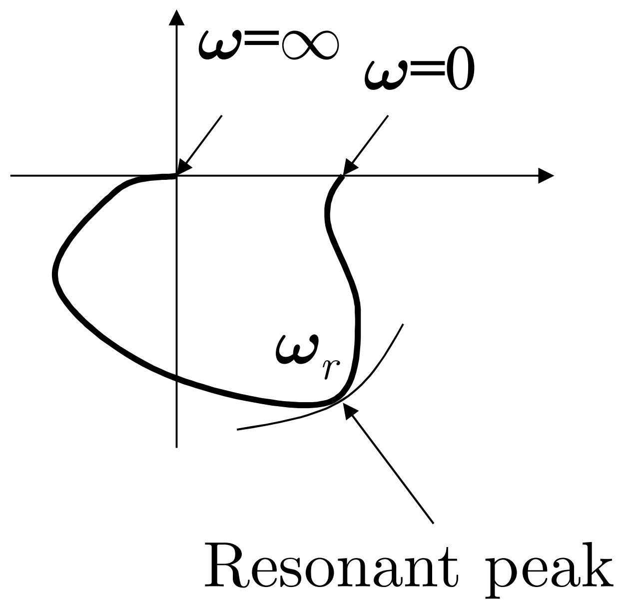
$$G(s) = Ts + 1 \Rightarrow G(j\omega) = Tj\omega + 1$$



4). Quadratic factor: $G(j\omega) = \dfrac{1}{(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1}$

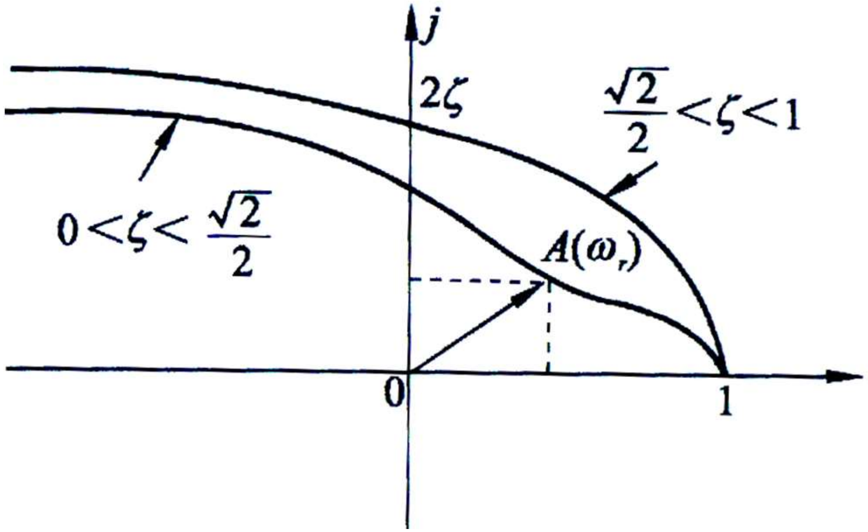


$$u = \frac{\omega}{\omega_n}, 0 < \zeta < 1$$



Quadratic factor: $[(j\omega/\omega_n)^2 + j2\zeta\omega/\omega_n + 1]$

$$= \left(1 - \frac{\omega^2}{\omega_n^2}\right) + j2\zeta\frac{\omega}{\omega_n}$$



The low-frequency portion of the curve is

$$\lim_{\omega \to 0} G(j\omega) = 1\angle 0^\circ$$

and the high-frequency portion is

$$\lim_{\omega \to \infty} G(j\omega) = \infty\angle 180^0$$

3. General shapes of polar plots

We consider the polar plots of general form of transfer function ($n>m$):

$$G(j\omega)=\frac{K\prod_{j=1}^{m}(1+j\omega\tau_j)}{(j\omega)^{\lambda}\prod_{i=1}^{n-\lambda}(1+j\omega T_i)} \qquad \lambda=0\ ,1\ ,2$$

- $\lambda=0$, Type 0 systems:

The low-frequency portion of the plot is

$$\lim_{\omega\to 0} G(j\omega)=\lim_{\omega\to 0}\frac{K(1+j\omega\tau_1)(1+j\omega\tau_2)\cdots}{(1+j\omega T_1)(1+j\omega T_2)\cdots}=Ke^{j0}$$

The high-frequency portion of the plot is

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{K(1 + j\omega\tau_1)(1 + j\omega\tau_2)\cdots}{(1 + j\omega T_1)(1 + j\omega T_2)\cdots}$$
$$= 0 e^{-j(n-m)\frac{\pi}{2}}$$

Example. Consider the following transfer function

$$G(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)}$$

Sketch its Nyquist plot, where $T_1 > 0$, $T_2 > 0$.

Solution:

$$G(j\omega) = \frac{K}{(T_1 j\omega + 1)(T_2 j\omega + 1)}$$

$$= \frac{K}{\sqrt{(T_1\omega)^2 + 1} \cdot \sqrt{(T_2\omega)^2 + 1}} \angle -(\tan^{-1} T_1\omega + \tan^{-1} T_2\omega)$$

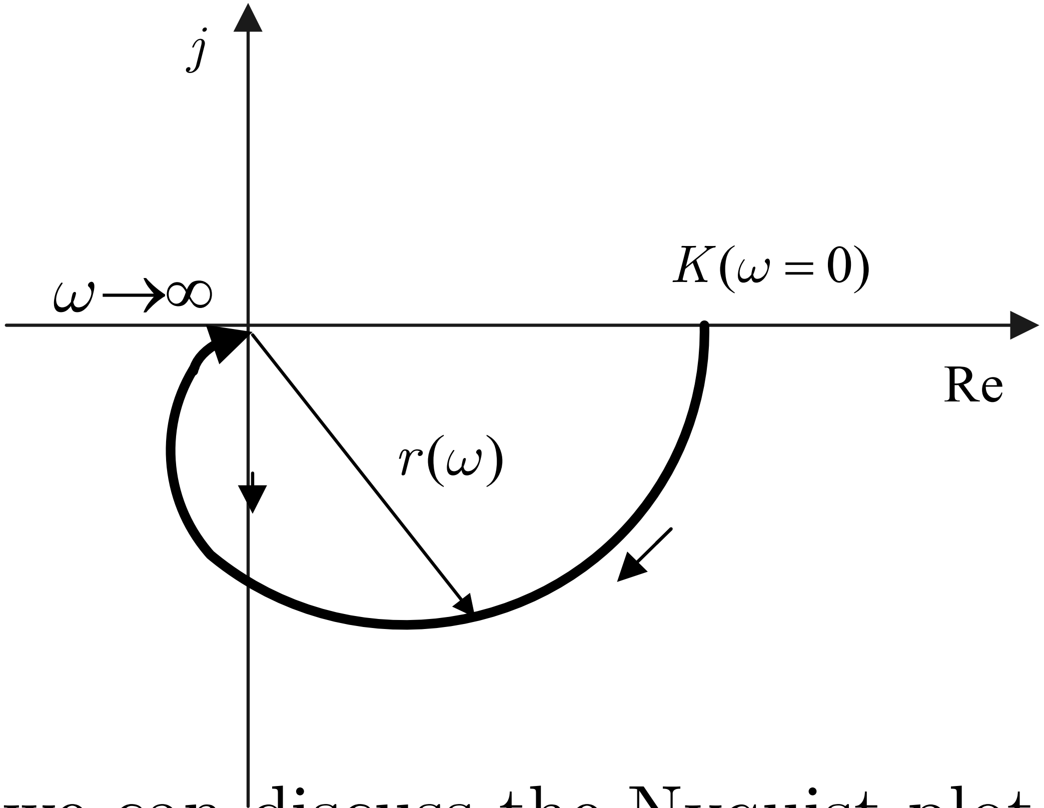
The low-frequency portion of the plot is

$$\lim_{\omega \to 0} G(j\omega) = Ke^{j0^\circ} = K\angle 0^\circ$$

The high-frequency portion of the plot is

$$\lim_{\omega \to \infty} G(j\omega) = 0e^{-j180^\circ} = 0\angle -180^0$$

and is tangent to the negative real axis.



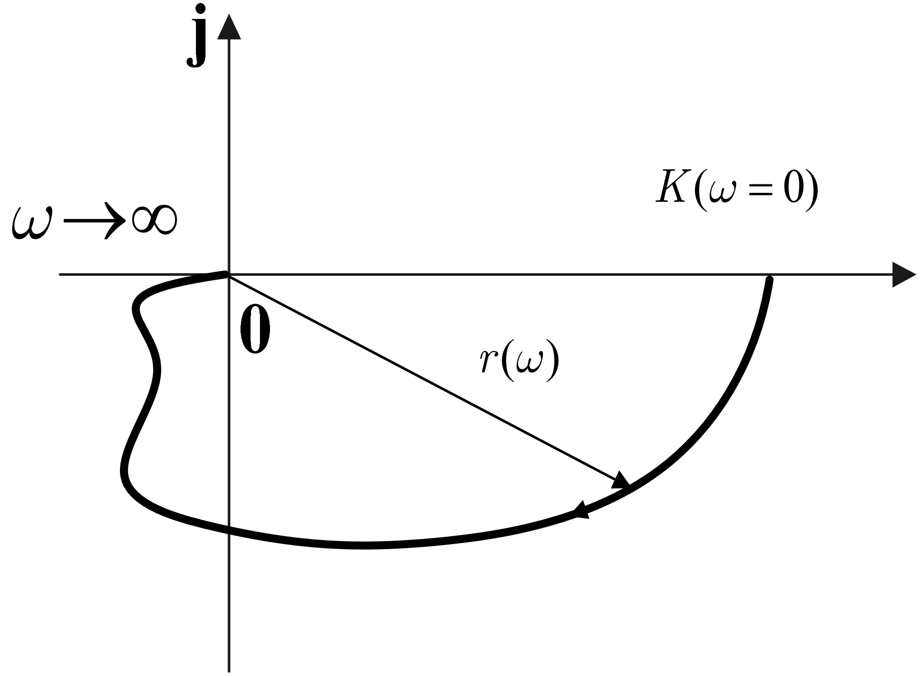
Similarly, we can discuss the Nyquist plot for

$$G(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}$$

Example. Consider the following transfer function:

$$G(s) = \frac{K(\tau s+1)}{(T_1 s+1)(T_2 s+1)(T_3 s+1)}$$

Sketch its Nyquist plot.



$G(j0) = K\angle 0^0;$

$G(j\infty) = 0\angle[-(3-1)90^0]$

$= 0\angle -180^0$

If $T_1,\ T_2 > \tau$ and $\tau > T_3$,

$$\angle\phi = -\tan^{-1} T_1\omega - \tan^{-1} T_2\omega + (\tan^{-1}\tau\omega - \tan^{-1} T_3\omega)$$

• λ =1, Type 1 systems:

The low frequency portion of the plot is

$$\lim_{\omega \to 0} G(j\omega)$$

$$= \lim_{\omega \to 0} \frac{K(1 + j\omega\tau_1)(1 + j\omega\tau_2)\cdots}{j\omega(1 + j\omega T_1)(1 + j\omega T_2)\cdots} = \lim_{\omega \to 0} \frac{K}{\omega} e^{-j\frac{\pi}{2}}$$

The high frequency portion of the plot is

$$\lim_{\omega \to \infty} G(j\omega) = \lim_{\omega \to \infty} \frac{K\prod_{j=1}^{m} \tau_j}{\prod_{i=1}^{n-1} T_i} (j\omega)^{m-n} = 0 \times e^{-j(n-m)\frac{\pi}{2}}$$

Example. Consider the following transfer function:

$$G(s) = \frac{K(\tau s+1)}{s(T_1 s+1)(T_2 s+1)}$$

Sketch its Nyquist plots for two cases: $\tau < T_1,\ T_2$ and $\tau > T_1,\ T_2$.

- $\lambda = 2$, Type 2 systems:

$$\lim_{\omega\to 0} G(j\omega) = \frac{K}{\omega^2} e^{-j\pi} = \frac{K}{\omega^2}\angle -180^0$$

$$\lim_{\omega\to\infty} G(j\omega) = 0 \times e^{-j(n-m)\frac{\pi}{2}} = 0\angle[-(n-m)90^0]$$

Example. Let the controlled transfer function be given as

$$G(s) = \frac{K}{s^2(Ts+1)} (T > 0)$$

Sketch its Nyquist curve.

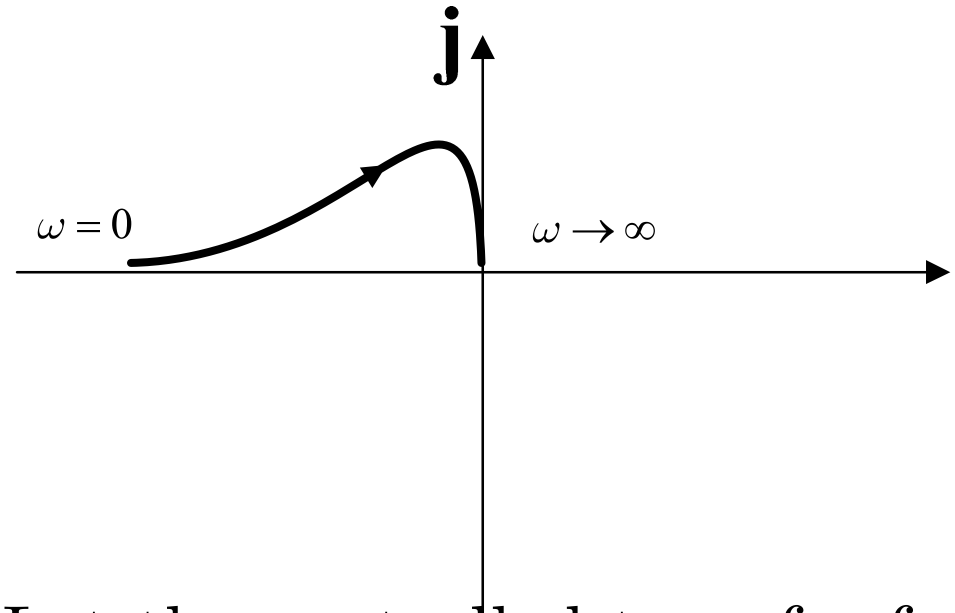
Solution: The system consists of four factors: Gain K and two integral factors and a first-order factor. The low-frequency portion of the plot is

$$G(0) = \infty, \phi(0) = \angle -180^0$$

and the high-frequency portion of the plot is

$$G(j\infty) = 0, \phi(\infty) = 2 \times (-90^0) - 90^0 = -270^0$$

The sketch of the Nyquist plot is



Example. Let the controlled transfer function be given as

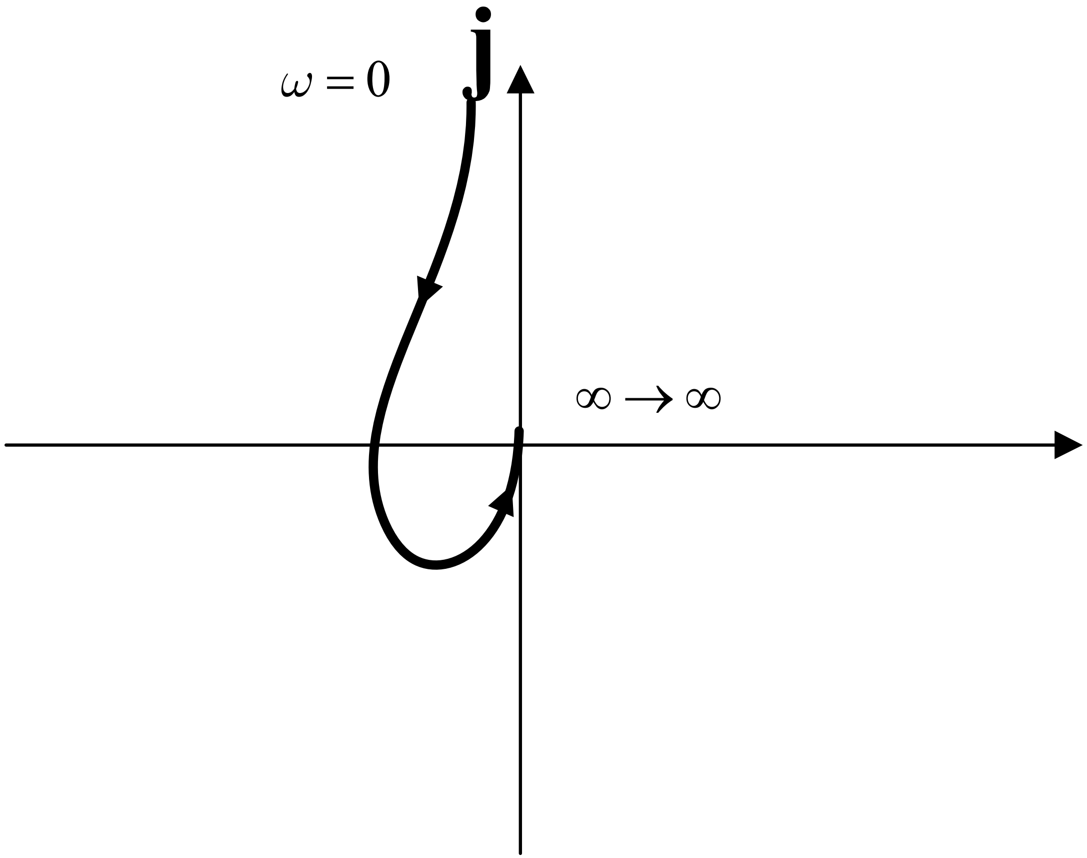
$$G(s) = \frac{K(\tau s + 1)}{s^2(Ts + 1)} (T > 0, \tau > 0)$$

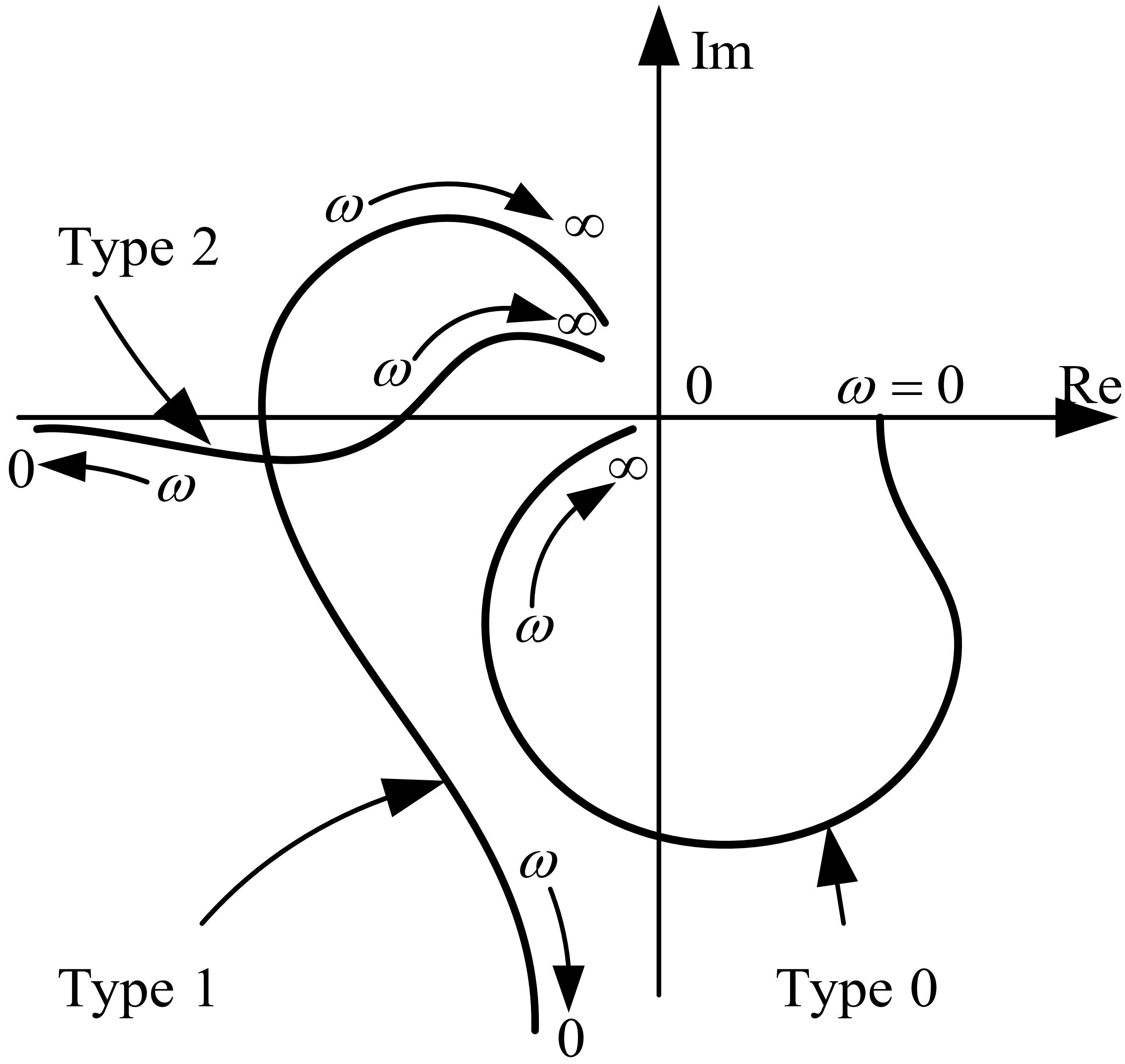
Sketch its Nyquist plots for two cases: $\tau < T$ and $\tau > T$.

Example. Let the controlled transfer function be given as

$$G(s) = \frac{K(\tau_1 s + 1)(\tau_2 s + 1)}{s^3}, (\tau_i > 0)$$

Sketch its Nyquist curve.



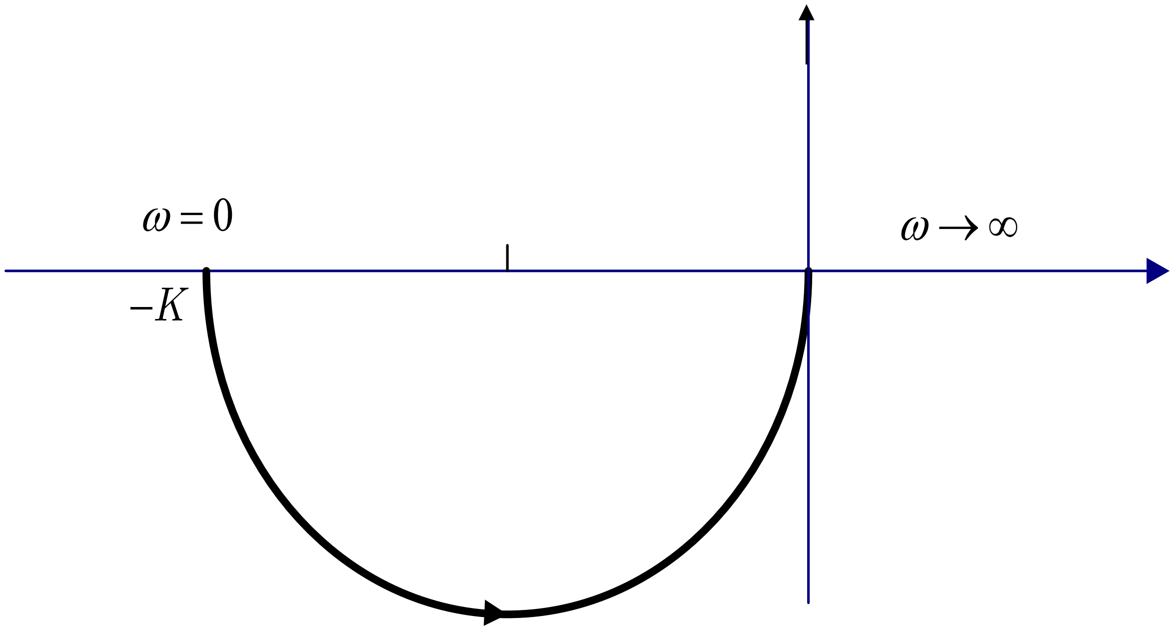


Im
Re
0
$\omega = 0$
Type 2
ω
∞
ω
∞
0
ω
∞
ω
ω
0
Type 1
Type 0

Example. Let the controlled transfer function be given as

$$G(s) = \frac{K}{(Ts-1)}, (T>0, K>0)$$

Sketch its Nyquist curve.



Example. Let the controlled transfer function be given as

$$G(s) = \frac{K}{s(Ts-1)}, (T > 0, K > 0)$$

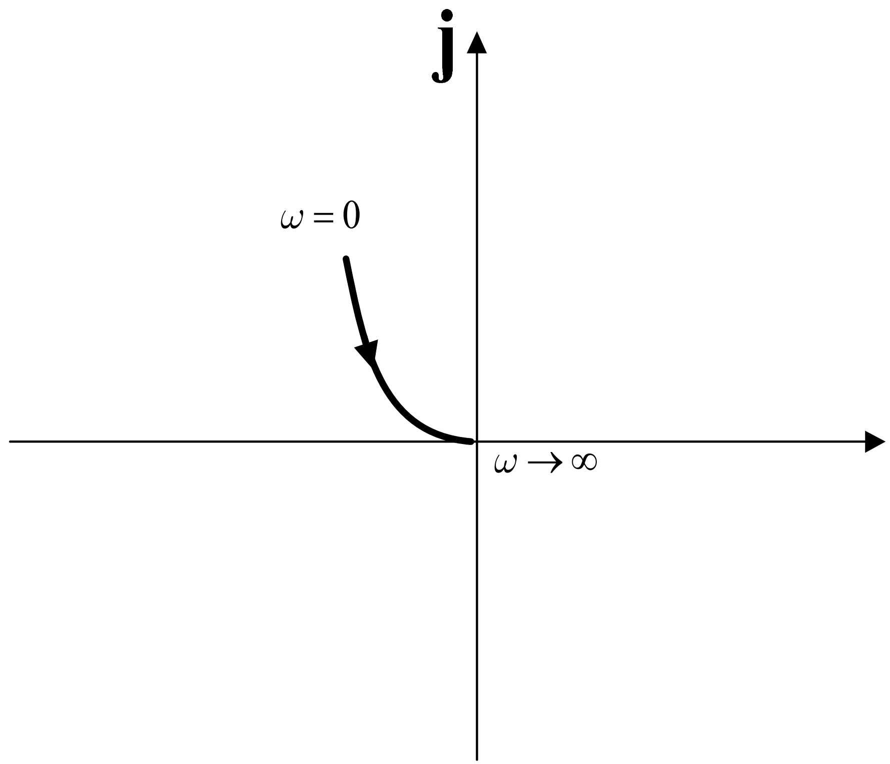
Sketch its Nyquist curve.

Solution: The system is nonminimum phase. The low-frequency portion of the plot is

$$G(0) = \infty, \phi(0) = -180^0 - 90^0 = -270^0$$

and the high-frequency portion of the plot is

$$G(j\infty) = 0, \phi(\infty) = -270^0 + 90^0 = -180^0$$


j
$\omega = 0$
$\omega \to \infty$