①. $\begin{cases} \dot{x}_1 = \sin x_2 - u \\ \dot{x}_2 = u. \end{cases}$ $\dot{y} = \dot{x}_1 = \sin x_2 - u.$

$y = x_1$ $\rho = 1.$

$\gamma(x) = -1.$ $\alpha(x) = \dfrac{-\sin x_2}{-1} = \sin x_2.$

$\xi = h(x) = x_1.$ ∴ x_1为可控观部分. $x_1 \equiv 0.$ $u \equiv \alpha(x)\Big|_{x = T^{-1}\begin{bmatrix}\eta \\ 0\end{bmatrix}}$

∴ $u \equiv \sin x_2.$ $\dot{x}_2 = \sin x_2$. 显然不是A.S.

∴ 非最小相位系统.

$T(x) = \begin{bmatrix} \eta \\ \hline \xi \end{bmatrix} = \begin{bmatrix} \phi(x) \\ x_1 \end{bmatrix}.$ $g_1(x) = -1.$ $g_2(x) = 1$

$$\frac{\partial\phi(x)}{\partial x_1} \cdot -1 + \frac{\partial\phi(x)}{\partial x_2} \cdot 1 = 0 \Rightarrow \frac{\partial\phi(x)}{\partial x_2} - \frac{\partial\phi(x)}{\partial x_1} = 0.$$

令 $\phi(x) = x_1 + x_2$, $\dot{\eta} = \dot{\phi}(x) = \dot{x}_1 + \dot{x}_2 = \sin x_2 - u + u = \sin x_2.$

$T(x) = \begin{bmatrix} x_1 + x_2 \\ x_1 \end{bmatrix}$ $\begin{cases} x_1 = \xi \\ x_2 = \eta - \xi \end{cases} \Leftrightarrow \begin{cases} \eta = x_1 + x_2. \\ \xi = x_1. \end{cases}$

∴ 正则型为 $\dot{\eta} = \sin(\eta - \xi).$

$\dot{\xi} = \sin(\eta - \xi) - u.$

$y = \xi.$

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②. $\begin{cases} \dot{x}_1 = \sin x_2 \\ \dot{x}_2 = u \\ \dot{x}_3 = -\sin x_3 + (1 + \cos x_2) u. \end{cases}$ $\dot{y} = \dot{x}_1 = \sin x_2.$

$\ddot{y} = \cos x_2 \cdot \dot{x}_2 = \cos x_2 \cdot u.$

$y = x_1$ ∴ $\rho = 2$, $x \in D_0 = \{x \in R^3 \mid x_2 \neq k\pi\}.$

$\gamma(x) = \cos x_2.$ $\alpha(x) = -\dfrac{0}{\cos x_2} = 0.$ $\xi = \begin{bmatrix} x_1 \\ \sin x_2 \end{bmatrix}.$

$\xi \equiv 0 \Rightarrow x_1 \equiv 0, x_2 \equiv 0$ $u \equiv 0$, $\dot{x}_3 = -\sin x_3$. 显然不是A.S.

∴ 非最小相位系统

$$T(x)=\begin{bmatrix}\eta\\ \xi\end{bmatrix}=\begin{bmatrix}\phi(x)\\ x_1\\ \sin x_2\end{bmatrix}. \quad g_1(x)=0 \quad g_2(x)=1 \quad g_3(x)=1+\cos x_2.$$

$$\therefore \frac{\partial\phi(x)}{\partial x_2}+\frac{\partial\phi(x)}{\partial x_3}(1+\cos x_2)=0.$$

令 $\phi(x)=x_3-x_2-\sin x_2$. $\dot\eta=\dot\phi(x)=\dot x_3-\dot x_2-\cos x_2\cdot\dot x_2$

$$\xi=\begin{bmatrix}x_1\\ \sin x_2\end{bmatrix} \quad \begin{cases}\xi_1=x_1\\ \xi_2=\sin x_2.\\ \eta=x_3-x_2-\sin x_2\end{cases}$$

$\dot\eta=-\sin x_3+(1+\cos x_2)u-u-\cos x_2\cdot u.$

$\dot\eta=-\sin x_3+\underline{u}+\underline{\cos x_2\cdot u}-\underline{u}-\underline{\cos x_2\cdot u}$

$\therefore \dot\eta=-\sin x_3$

$$\Downarrow$$

$$\begin{cases}x_1=\xi_1.\\ x_2=\arcsin\xi_2.\\ x_3=\eta+\xi_2+\arcsin\xi_2\end{cases}$$

$\therefore$ 正则型为
$$\begin{cases}\dot\eta=-\sin(\eta+\xi_2+\arcsin\xi_2).\\ \dot\xi_1=\xi_2\\ \dot\xi_2=\cos(\sin^{-1}\xi_2)\cdot u.\\ y=\xi_1.\end{cases}$$

~~[illegible]~~ ①. (1). $\dot x_1=-\sin x_2.$

$\dot x_2=u$

$\dot x_3=-\sin x_3+(1+\cos x_3)u.$

$$f(x)=\begin{bmatrix}-\sin x_2\\ 0\\ -\sin x_3\end{bmatrix},\ g(x)=\begin{bmatrix}0\\ 1\\ 1+\cos x_3\end{bmatrix}$$

$$ad_f g=\frac{\partial g}{\partial x}f-\frac{\partial f}{\partial x}g=\begin{bmatrix}0&0&0\\0&0&0\\0&0&-\sin x_3\end{bmatrix}\begin{bmatrix}-\sin x_2\\0\\-\sin x_3\end{bmatrix}-\begin{bmatrix}0&-\cos x_2&0\\0&0&0\\0&0&-\cos x_3\end{bmatrix}\begin{bmatrix}0\\1\\1+\cos x_3\end{bmatrix}$$

$$ad_f g.=\begin{bmatrix}0\\0\\\sin^2 x_3\end{bmatrix}-\begin{bmatrix}-\cos x_2\\0\\-\cos^2 x_3-\cos x_3\end{bmatrix}=\begin{bmatrix}\cos x_2\\0\\1+\cos x_3\end{bmatrix}.$$

$$ad_f^2 g=\frac{\partial\, ad_f g}{\partial x}f-\frac{\partial f}{\partial x}ad_f g=\begin{bmatrix}0&-\sin x_2&0\\0&0&0\\0&0&-\sin x_3\end{bmatrix}\begin{bmatrix}-\sin x_2\\0\\-\sin x_3\end{bmatrix}-\begin{bmatrix}0&-\cos x_2&0\\0&0&0\\0&0&-\cos x_3\end{bmatrix}\begin{bmatrix}\cos x_2\\0\\1+\cos x_3\end{bmatrix}$$

$$ad_f^2 g=\begin{bmatrix}0\\0\\\sin^2 x_3\end{bmatrix}-\begin{bmatrix}0\\0\\-\cos^2 x_3-\cos x_3\end{bmatrix}=\begin{bmatrix}0\\0\\1+\cos x_3\end{bmatrix}.$$

$$\therefore \mathcal{G}(x) = \begin{bmatrix} 0 & \cos x_2 & 0 \\ 1 & 0 & 0 \\ 1+\cos x_3 & 1+\cos x_3 & 1+\cos x_3 \end{bmatrix}. \quad \forall \cos x_2 \neq 0,\ 1+\cos x_3 \neq 0,\ r(\mathcal{G}(x)) = 3.$$

$\therefore$ ~~$\forall x_2 \neq 0,$~~ $\forall x_3 \neq k\pi.\ x_2 \neq \frac{k}{2}\pi$，$\mathcal{G}(x)$ 满秩.

$$\mathbb{D} = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1+\cos x_3 \end{bmatrix}, \begin{bmatrix} \cos x_2 \\ 0 \\ 1+\cos x_3 \end{bmatrix} \right\}.$$

$$\left[\begin{bmatrix} 0 \\ 1 \\ 1+\cos x_3 \end{bmatrix}, \begin{bmatrix} \cos x_2 \\ 0 \\ 1+\cos x_3 \end{bmatrix} \right] = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin x_3 \end{bmatrix} \begin{bmatrix} \cos x_2 \\ 0 \\ 1+\cos x_3 \end{bmatrix} + \begin{bmatrix} 0 & -\sin x_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sin x_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1+\cos x_3 \end{bmatrix}.$$

$$= \begin{bmatrix} -\sin x_2 \\ 0 \\ -\sin x_3 - \sin x_3 \cos x_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ -\sin x_3 - \cos x_3 \sin x_3 \end{bmatrix} = \begin{bmatrix} -\sin x_2 \\ 0 \\ 0 \end{bmatrix}.$$

$\therefore \mathbb{D}$ 不是对合的，系统不能全状态反馈线性化.

(2). $f(x) = \begin{bmatrix} 0 \\ x_3 \\ \sin x_2 \end{bmatrix} \quad g(x) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$

$$ad_f g = -\frac{\partial f}{\partial x} g = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \cos x_2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}.$$

$$ad_f^2 g = -\frac{\partial f}{\partial x} ad_f g = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \cos x_2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cos x_2 \end{bmatrix}.$$

$$\therefore \mathcal{G}(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & \cos x_2 \end{bmatrix}. \quad \forall \cos x_2 \neq 0 \Rightarrow \forall x_2 \neq \frac{k}{2}\pi,\ r(\mathcal{G}(x)) = 3.$$

$\mathbb{D} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$. $f_1 . f_2$ 为常数，$[f_1 . f_2] = 0 \in \mathbb{D}$

$\therefore \mathbb{D}$ 为对合的. 该系统可全状态～～.

$\exists y = h(x)$. $\frac{\partial h}{\partial x} g = 0$, $\frac{\partial L_f h}{\partial x} g = 0$. $\frac{\partial L_f^2 h}{\partial x} g \neq 0$.

①. $\frac{\partial h}{\partial x} g = 0 = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \frac{\partial h}{\partial x_1} + \frac{\partial h}{\partial x_3} = 0.$

②. $\frac{\partial L_f h}{\partial x} g = 0$. $L_f h = \frac{\partial h}{\partial x_1} \cdot 0 + \frac{\partial h}{\partial x_2} \cdot x_3 + \frac{\partial h}{\partial x_3} \cdot \sin x_2 = \frac{\partial h}{\partial x_2} x_3 + \frac{\partial h}{\partial x_3} \sin x_2$

$\therefore L_f h$ 与 x_1 无关，$\frac{\partial L_f h}{\partial x_1} = 0 \Rightarrow \frac{\partial L_f h}{\partial x_3} = \frac{\partial h}{\partial x_2} = 0$，$h(x)$ 与 x_2 无关.

$\therefore$ 取 $h(x) = x_1 - x_3$.

$$\therefore T(x) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \begin{matrix} = h(x) = x_1 - x_3. \\ = L_f h = -\sin x_2. \\ = L_f^2 h = -\cos x_2 \cdot \dot{x}_2 = -x_3 \cos x_2. \end{matrix}$$

(3). $f(x) = \begin{bmatrix} x_2 \\ 0 \\ \sin x_1 \end{bmatrix}$ $\quad g(x) = \begin{bmatrix} 0 \\ 1 \\ 1+\cos x_2 \end{bmatrix}$

$$ad_f g = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sin x_2 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \\ \sin x_1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \cos x_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1+\cos x_2 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

$$ad_f^2 g = -\frac{\partial f}{\partial x} ad_f g = -\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ \cos x_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cos x_1 \end{bmatrix}.$$

$$G(x) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1+\cos x_2 & 0 & \cos x_1 \end{bmatrix}, \ \forall \cos x_1 \neq 0 \Rightarrow \forall x_1 \neq \frac{k}{2}\pi, \ r(G(x)) = 3.$$

$$D = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 1+\cos x_2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

$$[f_1, f_2] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\sin x_2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = 0 \in D$$，D是对合的.

∴ 系统可全状态 ～. $\exists y = h(x)$, $\frac{\partial h}{\partial x} g = 0$ $\quad \frac{\partial L_f h}{\partial x} g = 0$, $\quad \frac{\partial L_f^2 h}{\partial x} g \neq 0$.

①. $\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} + \frac{\partial h}{\partial x_3} (1+\cos x_2) = 0.$

②. $\frac{\partial L_f h}{\partial x} g = 0.$ $\qquad L_f h = \frac{\partial h}{\partial x_1} \cdot x_2 + \frac{\partial h}{\partial x_3} \cdot \sin x_1.$

$\frac{\partial L_f h}{\partial x} \cdot g = \frac{\partial L_f h}{\partial x_2} + \frac{\partial L_f h}{\partial x_3} (1+\cos x_2) = 0. \Rightarrow \frac{\partial h}{\partial x_1} + 0 = 0 \Rightarrow$ $h(x)$与x_1无关.

取 $h(x) = x_3 - x_2 - \sin x_2.$

$$\therefore T(x) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h \\ L_f^2 h \end{bmatrix} = \begin{bmatrix} x_3 - x_2 - \sin x_2 \\ \sin x_1 \\ x_2 \cos x_1 \end{bmatrix}.$$