(12). $\begin{cases} \dot{x}_1 = x_1^2 + x_2 \\ \dot{x}_2 = x_2^2 + x_3 \\ \dot{x}_3 = x_3^2 + u. \end{cases}$

$\dot{x}_1 = x_1^2 + x_2$

$= -x_1 + \underbrace{(x_1 + x_1^2 + x_2)}_{\Rightarrow z_2}$

$= -x_1 - z_2.$

$\dot{z}_2 = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1 = x_2^2 + x_3 + \dot{x}_1 + 2x_1\dot{x}_1$

$= -z_2 + x_1 + \underbrace{(x_2^2 + x_3 + \dot{x}_1 + 2x_1\dot{x}_1 + z_2 - x_1)}_{\Rightarrow z_3}.$

$\dot{z}_3 = u + 2x_2\dot{x}_2 + \ddot{x}_1 + 2\dot{x}_1^2 + 2x_1\ddot{x}_1 + \dot{z}_2 - \dot{x}_1$

$= -z_3 - z_2 + (u + 2x_2\dot{x}_2 + \ddot{x}_1 + 2\dot{x}_1^2 + 2x_1\ddot{x}_1 + \dot{z}_2 - x_1 + z_2 + z_3)$

$\therefore$ 设计 $u = -(2x_2\dot{x}_2 + \ddot{x}_1 + 2\dot{x}_1^2 + 2x_1\ddot{x}_1 + \dot{z}_2 - x_1 + z_2 + z_3)$

$\therefore \begin{cases} \dot{x}_1 = -x_1 - z_2 \\ \dot{z}_2 = -z_2 + x_1 + z_3 \\ \dot{z}_3 = -z_3 - z_2 \end{cases}$

令 $V(x) = \frac{1}{2}(x_1^2 + z_2^2 + z_3^2)$. $\dot{V}(x) = x_1\dot{x}_1 + z_2\dot{z}_2 + z_3\dot{z}_3$

$= -x_1^2 - x_1z_2 - z_2^2 + x_1z_2 + z_2z_3 - z_3^2 - z_2z_3$

$= -x_1^2 - z_2^2 - z_3^2 < 0$

$\therefore$ 系统 G.A.S.

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(12). $\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ x_3 = x_1x_2^2. \end{cases}$

$\dot{x}_3 = \dot{x}_1x_2^2 = (-x_3 + \underbrace{x_1 + x_3}_{\Rightarrow z_1})(x_2^2 + \underbrace{x_2^2 - x_3^2}_{z \Rightarrow z_2})$

$= (-x_3 + z_1)(x_3^2 + z_2)$. $z_1 = x_1 + x_3$. $z_2 = x_2^2 - x_3^2$.

$= -x_3^3 + z_1x_3^2 - z_2x_3 + z_1z_2 = -x_3^3 + z_1(x_3^2 + z_2) - z_2x_3.$

$\dot{z}_1 = u_1 + \dot{x}_3$，为消除 $z_1(x_3^2 + z_2)$，引入：

$\dot{z}_1 = u_1 + \dot{x}_3 + [-z_1 - k_1(x_3^2 + z_2) + z_1 + k_1(x_3^2 + z_2)]$

$\dot{z}_1 = -z_1 - k_1(x_3^2 + z_2) + [u_1 + \dot{x}_3 + z_1 + k_1(x_3^2 + z_2)]$.

$\dot{z}_2 = 2\dot{x}_2 x_2 - 2\dot{x}_3 x_3 = 2x_2 \cdot u - 2\dot{x}_3 x_3$. 为消去 $-z_2 x_3$，引入：

$$\dot{z}_2 = 2x_2 \cdot u_2 - 2x_3\dot{x}_3 + (-z_2 + x_3^2 + z_2 - x_3^2).$$
$$= -z_2 + x_3^2 + [2x_2 \cdot u_2 - 2x_3\dot{x}_3 + z_2 - x_3^2].$$

∴ 设计 $u_1 = -[\dot{x}_2 + z_1 + x_3(x_3^2 + z_2)]$

$$u_2 = -\frac{2x_3\dot{x}_3 + z_2 - x_3^2}{2x_2}$$

∴ $\begin{cases} \dot{x}_3 = -x_3^2 + z_1(x_3^2 + z_2) - z_2 x_3 \\ \dot{z}_1 = -z_1 - x_3(x_3^2 + z_2) \\ \dot{z}_2 = -z_2 + x_3^2. \end{cases}$ 令 $V(x) = \frac{1}{2}(x_3^2 + z_1^2 + z_2^2)$

$$\dot{V} = -x_3^4 + z_1 x_3(x_3^2 + z_2) - z_2 x_3^2 - z_1^2 - z_1 x_3(x_3^2 + z_2) - z_2^2 + z_2 x_3^2$$
$$= -x_3^4 - z_1^2 - z_2^2 < 0.$$ 系统 G.A.S.

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(37). $\begin{cases} \dot{x}_1 = x_1^2 x_2 + a x_1^3 \cos(x_1) \\ \dot{x}_2 = x_2 + u. \end{cases}$ $|a| \le 1$.

$$\dot{x}_1 = x_1^2 x_2 + a x_1^3 \cos(x_1) = x_1^2 x_2 + x_1^3 - (1 - a\cos x_1) x_1^3.$$
$$= x_1^2(x_2 + x_1) - (1 - a\cos x_1) x_1^3$$
$$= x_1^2(-x_1 + x_2 + 2x_1) - (1 - a\cos x_1) x_1^3$$
$$= -x_1^3 - (1 - a\cos x_1) x_1^3 + x_1^2 z_2, \quad z_2 = x_2 + 2x_1.$$

$$\dot{z}_2 = \dot{x}_2 + 2\dot{x}_1 = u + x_1 + 2\dot{x}_1$$
$$= -z_2 - x_1^3 + [u + x_1 + 2(x_1^2 x_2 + a x_1^3 \cos(x_1)) + z_2 + x_1^3].$$

设计 $u = -(x_1 + 2x_1^2 x_2 + z_2 + x_1^3 + v)$. v 为新的输入.

$\begin{cases} \dot{x}_1 = -x_1^3 - (1 - a\cos x_1) x_1^3 + x_1^2 z_2 \\ \dot{z}_2 = -z_2 - x_1^3 + 2a\cos(x_1) \cdot x_1^3 + v \end{cases}$

令 $V=\frac{1}{2}(x_1^2+z_2^2)$. $\dot{V}=-x_1^4-(1-a\cos x_1)x_1^4+x_1^3z_2-z_2^2-x_1^3z_2$

$+z_2(2a\cos(x_1)\cdot x_1^3+v)$.

$\dot{V}=-x_1^4\underline{+}(1-a\cos x_1)x_1^4-z_2^2+z_2(\cdots)$

选 $v=-2|x_1^3|\operatorname{sgn}(z_2)$.

$\because |a|<1$, $|\cos x_1|<1$

$\therefore \dot{V}=-x_1^4-z_2^2<0$. 系统 G.A.S.

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(4). $\begin{cases}\dot{x}_1=\sin x_2\\ \dot{x}_2=\cos x_2+u.\end{cases}$ $\dot{x}_1=\sin x_2=\sin(-\alpha(x_1)+x_2+\alpha(x_1))$.

令 $z_2=x_2+\alpha(x_1)$

$\alpha(x_1)$ 为一函数, $x_1\alpha(x_1)>0$; $\alpha(0)=0$; $|\alpha(x_1)|<\pi$.

设 $f(x_1,z_2)=\begin{cases}\frac{\sin(z_2-\alpha(x_1))-\sin(-\alpha(x_1))}{z_2}\cdot z_2\neq 0.\\ \cos(\alpha(x_1))\cdot z_2=0\end{cases}$ $\dot{x}_1=\sin(-\alpha(x_1))+\sin(-\alpha(x_1)+z_2)-\sin(-\alpha(x_1))$

$\dot{x}_1=-\sin(\alpha(x_1))+\sin(-\alpha(x_1)+z_2)-\sin(-\alpha(x_1))$

$=-\sin(\alpha(x_1))+f(x_1,z_2)z_2$.

$\begin{cases}\dot{x}_1=-\sin(\alpha(x_1))+f(x_1,z_2)z_2.\\ \dot{z}_2=-z_2-f(x_1,z_2)x_1\end{cases}$ $\dot{z}_2=\dot{x}_2+\dot{\alpha}(x_1)=\cos x_2+u+\dot{\alpha}(x_1)$

$\Leftarrow$ 令 $u=-z_2-\dot{\alpha}(x_1)-\cos x_2-f(x_1,z_2)x_1$ 为设

$\therefore$ 令 $V=\frac{1}{2}(x_1^2+z_2^2)$.

$\dot{V}=-x_1\sin(\alpha(x_1))+f(x_1,z_2)z_2x_1-z_2^2-f(x_1,z_2)x_1z_2$.

$=-x_1\sin(\alpha(x_1))-z_2^2<0$

$\therefore$ 系统 G.A.S.