MODERN CONTROL ENGINEERING

Tutorial Class I

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Assignment 1

1. Find the Laplace transform of the given time functions

(Note that $f(t) = 0$ for $t < 0$)

Typical solutions:

- By definition $L[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

- Laplace Transform Table

f(*t*)	*F*(*s*)	*f*(*t*)	*F*(*s*)
Unit impulse $\delta(t)$	1	$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
Unit step $1(t)$	$\dfrac{1}{s}$	$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
t	$\dfrac{1}{s^2}$	e^{-at}	$\dfrac{1}{s+a}$
$\dfrac{1}{2}t^2$	$\dfrac{1}{s^3}$	…	…

- By properties of Laplace Transforms (6 Theorems)

(1) $f(t) = (t+1)(t+5)$

Solution: $f(t) = (t+1)(t+5) = t^2 + 6t + 5$

$$F(s) = \frac{2}{s^3} + \frac{6}{s^2} + \frac{5}{s}$$

(2) $f(t) = e^{-0.5t} \cos 10t$

Solution: $f_1(t) = \cos 10t \Rightarrow F_1(s) = \dfrac{s}{s^2 + 100}$

By Complex Shifting Theorem

$$f(t) = e^{-0.5t} f_1(t)$$

$$\Rightarrow F(s) = F_1(s+a) = \frac{s+0.5}{(s+0.5)^2 + 100}$$

(3) $f(t)=\sin\left(5t+\dfrac{\pi}{3}\right)$

Solution: $f_1(t)=\sin 5t \Rightarrow F_1(s)=\dfrac{5}{s^2+25}$

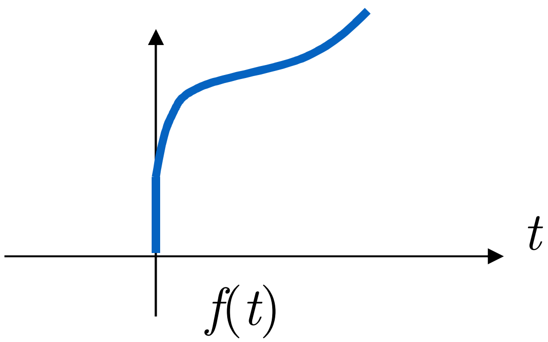
By Time Delay Theorem

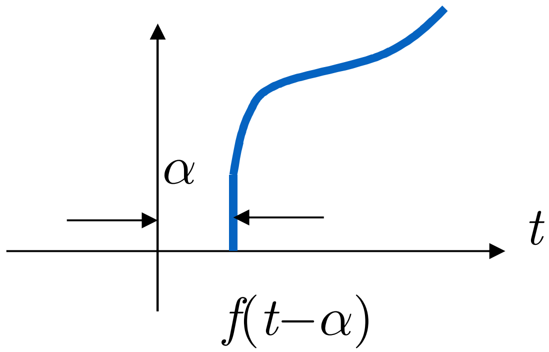
$$f(t)=f_1\left(t+\frac{\pi}{15}\right)\Rightarrow F(s)=e^{\frac{\pi}{15}s}F_1(s)=\frac{5e^{\frac{\pi}{15}s}}{s^2+25}$$

■ **Because Time Delay Theorem is not applicable here.**

If $L\left[f(t)\right]=F(s)$, then there exists a real number $\alpha>0$ such that

$$L\left[f(t-\alpha)\cdot 1(t-\alpha)\right]=e^{-\alpha s}F(s)$$





(3) $f(t) = \sin\left(5t + \frac{\pi}{3}\right)$

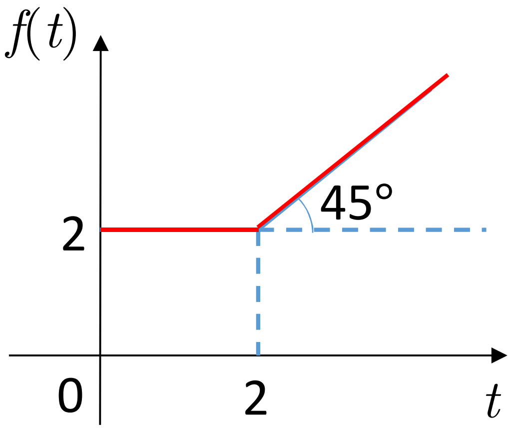
Solution:

$$f(t) = \sin 5t \cos\frac{\pi}{3} + \cos 5t \sin\frac{\pi}{3} = \frac{1}{2}\sin 5t + \frac{\sqrt{3}}{2}\cos 5t$$

$$\Rightarrow F(s) = \frac{5}{2(s^2+25)} + \frac{\sqrt{3}s}{2(s^2+25)} = \frac{\sqrt{3}s+5}{2(s^2+25)}$$

Thus pay attention to the conditions of the Laplace Transform Theorems before you apply them.

2. (1) Find the Laplace transform of the following time function shown in the figure below



Solution:

$$f(t) = 2[1(t) - 1(t-2)] + t \cdot 1(t-2)$$
$$= 2 \cdot 1(t) + (t-2) \cdot 1(t-2)$$

$$f_1(t) = 2 \cdot 1(t) \Rightarrow F_1(s) = \frac{2}{s}$$

$$f_2(t) = (t-2) \cdot 1(t-2)$$
$$\Rightarrow F_2(s) = e^{-2s}\frac{1}{s^2} \text{ (Time Delay Theorem)}$$

It follows that

$$F(s) = \frac{2}{s} + e^{-2s}\frac{1}{s^2}$$

3. Find the inverse Laplace transform of the following rational functions

(3) $F(s) = \dfrac{s+7}{(s+1)(s^2+3s+2)}$

Solution: The partial-fraction expansion of $F(s)$ is

$$F(s) = \frac{s+7}{(s+1)(s^2+3s+2)} = \frac{s+7}{(s+1)^2(s+2)}$$

$$= \frac{A}{s+1} + \frac{B}{\left(s+1\right)^2} + \frac{C}{s+2}$$

where A, B, C are derived as (**Residue computation formula**)

$$A = \frac{d}{ds}\left[\left(s+1\right)^2 F(s)\right]\Big|_{s=-1} = -5 \qquad B = \left[\left(s+1\right)^2 F(s)\right]\Big|_{s=-1} = 6$$

$$C = \left[\left(s+2\right) F(s)\right]\Big|_{s=-2} = 5$$

$$\Rightarrow F(s)=\frac{-5}{s+1}+\frac{6}{\left(s+1\right)^2}+\frac{5}{s+2}$$

$$f(t)=-5e^{-t}+6te^{-t}+5e^{-2t}, t\geq 0$$

(**Complex shifting theorem**)

- **Residue computation formula**

Partial-fraction expansion of $G(s)$ involves multiple poles

$$G(s)=\frac{A_1}{s+p_i}+\frac{A_2}{(s+p_i)^2}+\cdots+\frac{A_r}{(s+p_i)^r}+\frac{B_1}{s+p_j}+\frac{C_1}{s+p_k}+\cdots$$

$|\leftarrow$ r terms of repeated roots $\rightarrow|$

$$\begin{cases} A_r=\left[(s+p_i)^r G(s)\right]\Big|_{s=-p_i} \\ A_{r-k}=\dfrac{1}{k!}\dfrac{d^k}{ds^k}\left[(s+p_i)^r G(s)\right]\Big|_{s=-p_i} \qquad (k=1,2,\cdots r-1) \end{cases}$$

4. Solve the following differential equations by using Laplace transform:

(1) $\ddot{x}(t)+\dot{x}(t)+x(t)=\delta(t),\ \dot{x}(0)=x(0)=0$

Solution: By the Differentiation Theorem, taking the Laplace transform on both sides of the ODE yields

$$s^2X(s)+sX(s)+X(s)=1$$

$$\Rightarrow X(s)=\frac{1}{s^2+s+1}$$

$$X(s)=\frac{1}{s^2+s+1}=\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}=\frac{2}{\sqrt{3}}\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

- Taking the inverse Laplace transform, we obtain

$$x(t) = \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) e^{-t/2}, \ t > 0$$

- **Differentiation Theorem**

$$\mathcal{L}\left[\frac{d^n f(t)}{dt^n}\right] = s^n F(s) - \lim_{t \to 0}\left[s^{n-1} f(t) + s^{n-2}\frac{df(t)}{dt} + \cdots + \frac{d^{n-1} f(t)}{dt^{n-1}}\right]$$

$$= s^n F(s) - s^{n-1} f(0) - s^{n-2} f^{(1)}(0) - \cdots - f^{(n-1)}(0)$$

Pay attention to the initial conditions when the Differentiation Theorem is applied.

Assignment 2

- **Diagram simplification**

Note that only the movement between two summing points (two branch points) is valid.

Example 1. A system is described by the following equations:

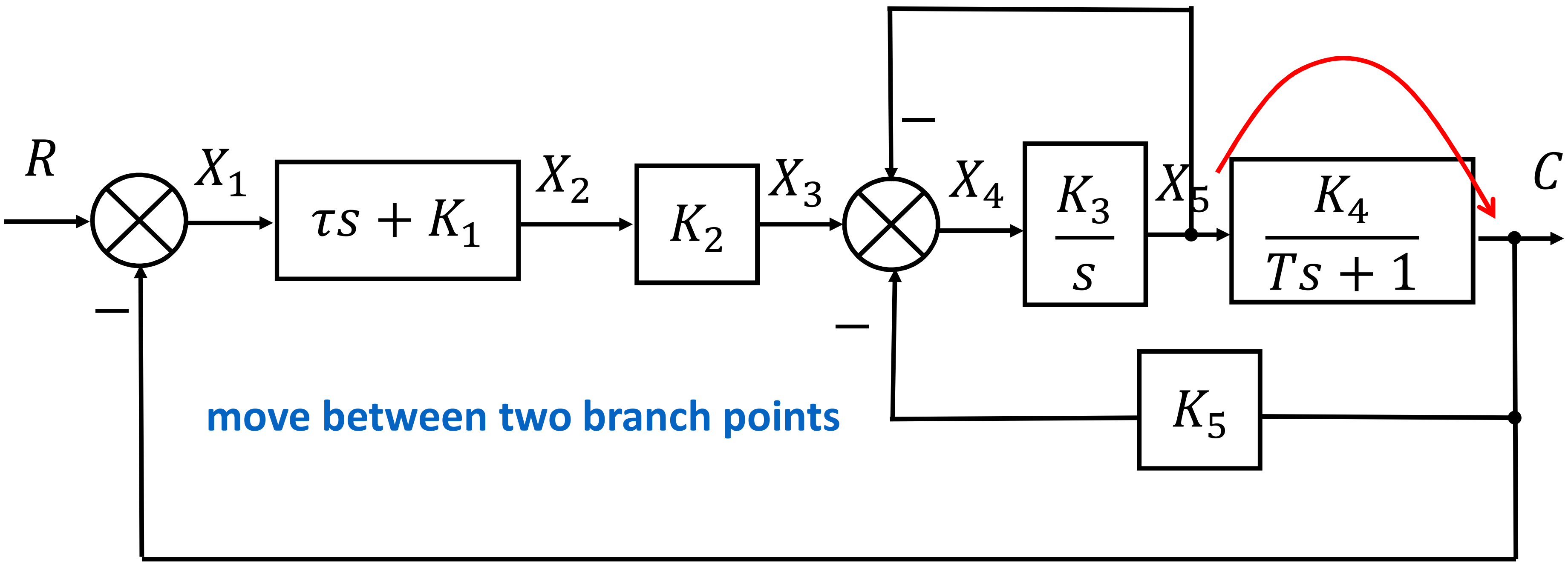
$$\begin{cases} x_1 = r - c \\ x_2 = \tau \dot{x}_1 + K_1 x_1 \\ x_3 = K_2 x_2 \\ x_4 = x_3 - x_5 - K_5 c \\ \dot{x}_5 = K_3 x_4 \\ K_4 x_5 = T\dot{c} + c \end{cases}$$

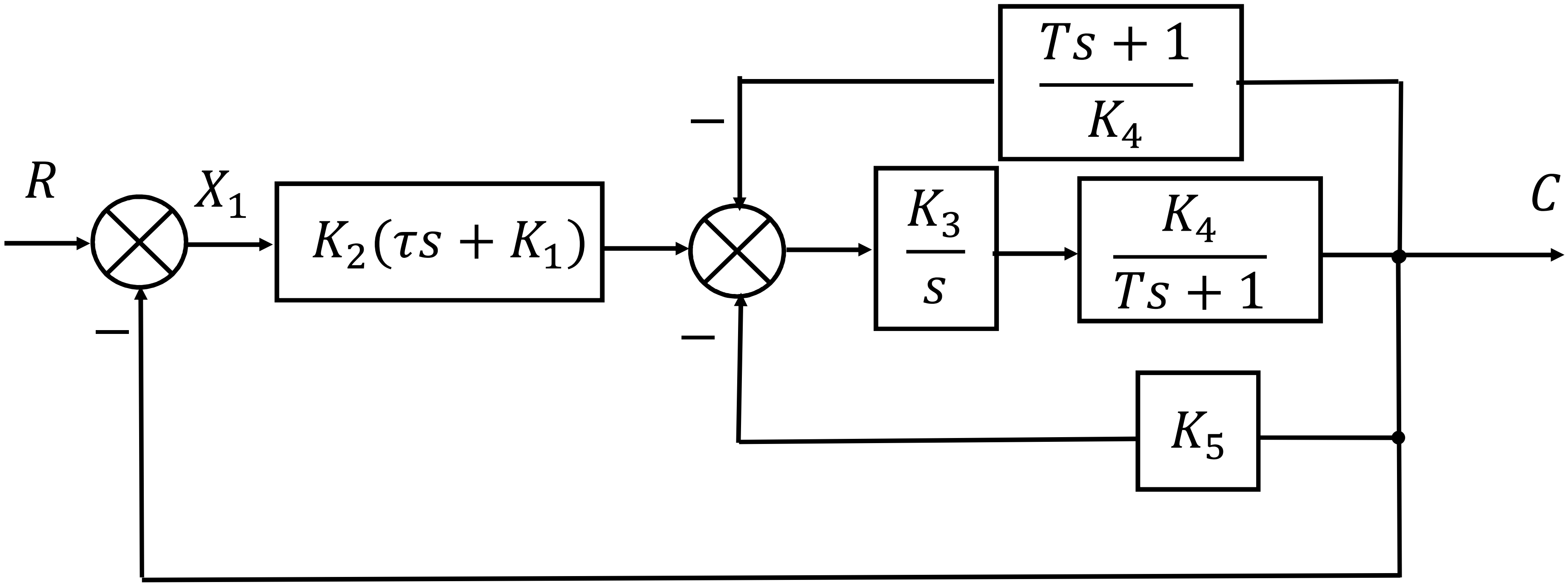
where τ, K_1, K_2, K_3, K_4, K_5, and T are positive constants, $r(t)$ is the input signal and $c(t)$ is the output signal. Draw its block diagram and obtain the transfer function $C(s)/R(s)$.

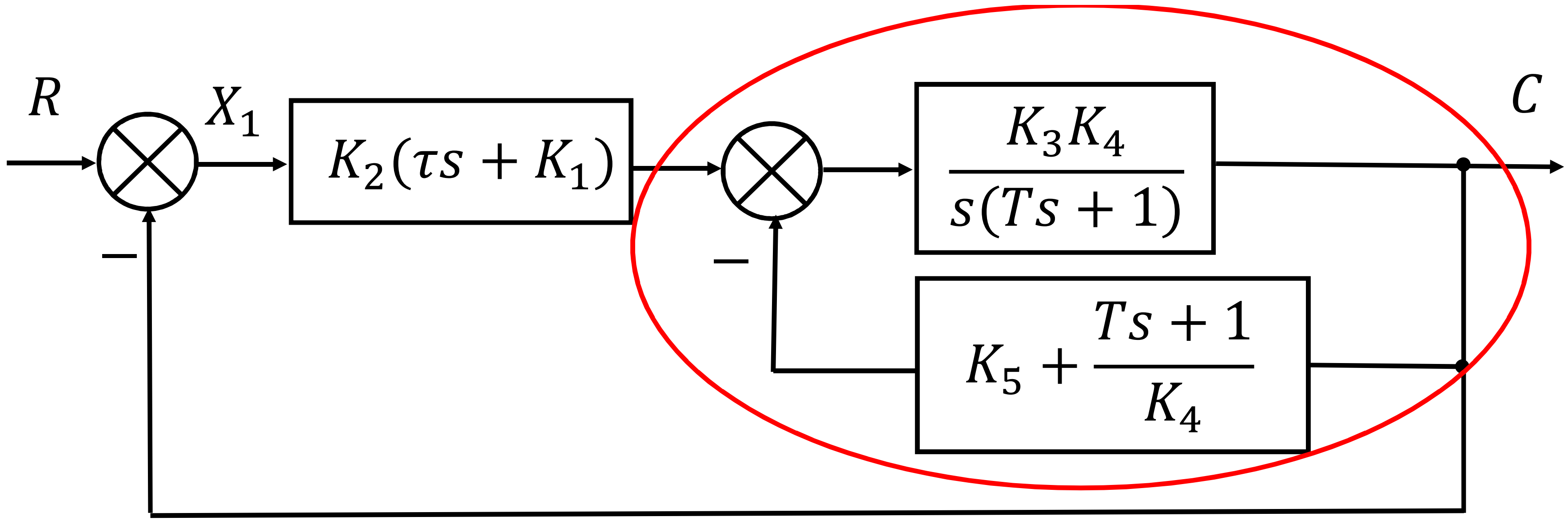
Solution: Taking the Laplace transform for each equation

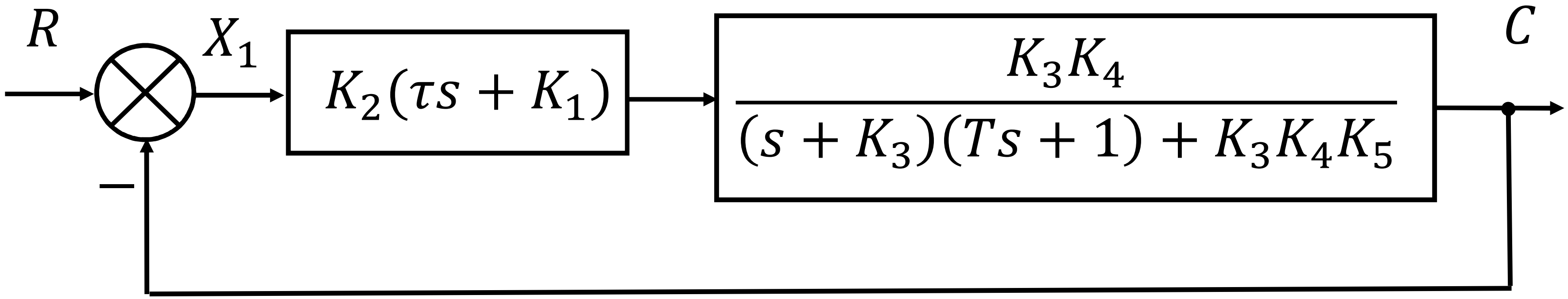
$$X_1 = R - C,\ X_2 = \left(\tau s + K_1\right) X_1,\ X_3 = K_2 X_2$$

$$X_4 = X_3 - X_5 - K_5 C,\ X_5 = \frac{K_3}{s} X_4,\ C = \frac{K_4}{Ts+1} X_5$$


R
X_1
$\tau s + K_1$
X_2
K_2
X_3
X_4
$\frac{K_3}{s}$
X_5
$\frac{K_4}{Ts+1}$
C
K_5
move between two branch points

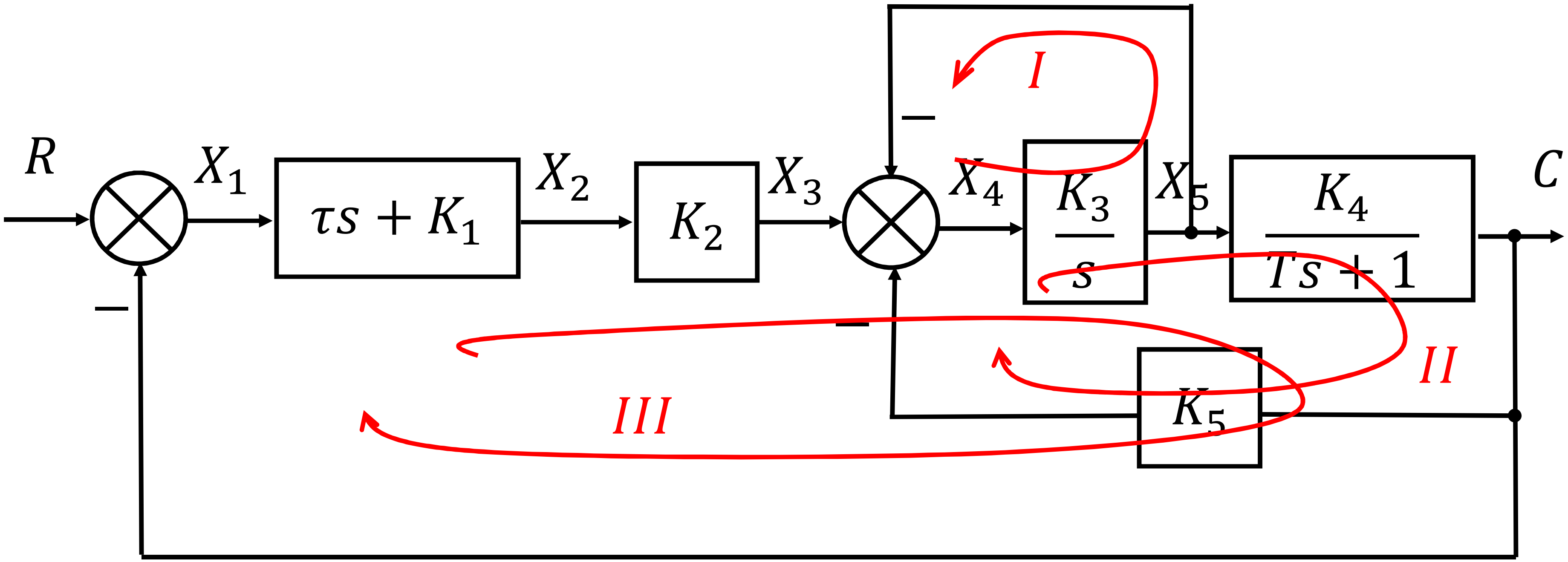

R
X_1
$K_2(\tau s + K_1)$
$\frac{Ts+1}{K_4}$
$\frac{K_3}{s}$
$\frac{K_4}{Ts+1}$
C
K_5





$$
\begin{aligned}
\frac{C(s)}{R(s)} &= \frac{K_2K_3K_4(\tau s+K_1)}{(s+K_3)(Ts+1)+K_2K_3K_4(\tau s+K_1)+K_3K_4K_5} \\
&= \frac{K_2K_3K_4(\tau s+K_1)}{Ts^2+(K_3T+1+K_2K_3K_4\tau)s+K_3+K_1K_2K_3K_4+K_3K_4K_5}
\end{aligned}
$$

Alternative solution. Mason's Formula



- One forward path: $$P_1 = (\tau s + K_1)K_2 \frac{K_3 K_4}{s(Ts+1)}$$

- Three individual loops: $$L_1 = -\frac{K_3}{s} \qquad L_2 = -\frac{K_3 K_4 K_5}{s(Ts+1)}$$

$$L_3 = -\frac{K_2 K_3 K_4 (\tau s + K_1)}{s(Ts+1)}$$

- No non-touching loops, therefore,

$$\Delta = 1-(L_1+L_2+L_3)$$
$$= 1+\frac{K_3}{s}+\frac{K_3K_4K_5}{s(Ts+1)}+\frac{K_2K_3K_4(\tau s+K_1)}{s(Ts+1)}$$

- The cofactors are:

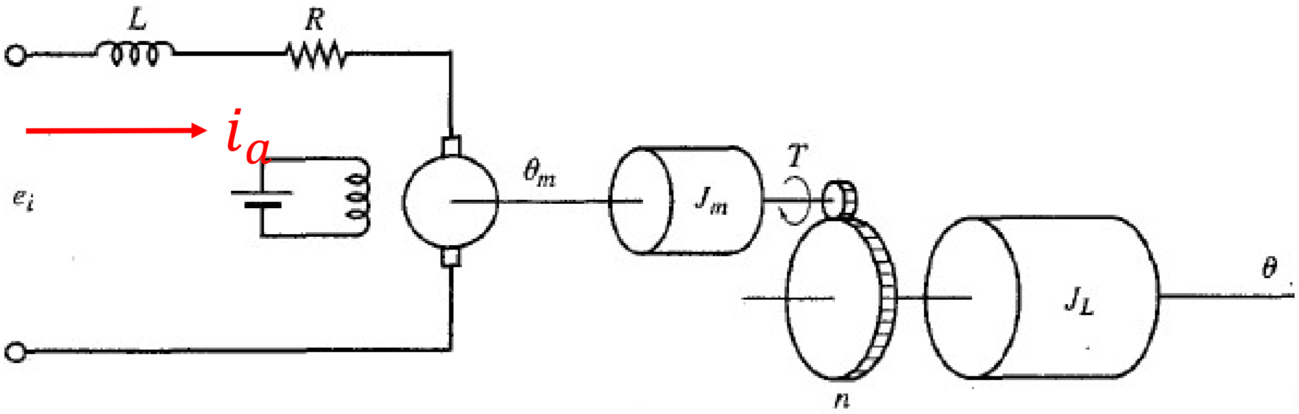
$$\Delta_1 = 1$$

Consequently,

$$\frac{C}{R}=\frac{P_1\Delta_1}{\Delta}=\frac{(\tau s+K_1)K_2\dfrac{K_3K_4}{s(Ts+1)}}{1+\dfrac{K_3}{s}+\dfrac{K_3K_4K_5}{s(Ts+1)}+\dfrac{K_2K_3K_4(\tau s+K_1)}{s(Ts+1)}}$$

$$=\frac{K_2K_3K_4(\tau s+K_1)}{\left(s+K_3\right)(Ts+1)+K_3K_4K_5(Ts+1)+K_2K_3K_4(\tau s+K_1)}$$

B-3-13. Consider the system shown. An armature-controlled dc servomotor drives a load consisting of the moment of inertia J_L. The torque developed by the motor is T. The moment of inertia of the motor rotor is J_m. The angular displacements of the motor rotor and the load element are θ_m and θ, respectively. The gear ratio is $n = \theta/\theta_m$. Obtain the transfer function $\Theta(s)/E_i(s)$.



Solution:

- The electromagnetic torque is

$$T_m = K_m i_a \tag{1}$$

where K_m is the motor torque constant, i_a is the armature current.

- The equation for the armature circuit is

$$L\frac{di_a}{dt} + Ri_a + e_b = e_i \tag{2}$$

where e_b is the back emf voltage.

- e_b in terms of the shaft's rotational velocity $\frac{d\theta_m}{dt}$ is

$$e_b = K_b \frac{d\theta_m}{dt} \tag{3}$$

where K_b is the back emf constant.

- The torque equations are

$$J_m \frac{d^2\theta_m}{dt} + T = T_m \tag{4}$$

$$J_L \frac{d^2\theta}{dt} = T_L \tag{5}$$

Moreover, there is

$$T = T_L \frac{\theta}{\theta_m} = nT_L \tag{6}$$

- Combining the Laplace transform of (1)-(5), we have

$$\begin{cases} (Ls+R)I_a + K_b s\Theta_m = E_I \\ J_m s^2 \Theta_m + nJ_L s^2 \Theta = K_m I_a \end{cases} \tag{7}$$

- Substituting $\Theta_m = \Theta / n$ into (7), it follows that

$$\frac{\Theta}{E_i} = \frac{K_m n}{s\left[\left(Ls+R\right)\left(J_m + n^2 J_L\right)s + K_m K_b\right]} \tag{8}$$

Assignment 6

B-5-2. Consider the unit-step response of a unity-feedback control system whose open-loop transfer function is

$$G(s) = \frac{1}{s(s+1)}$$

Obtain the peak time, maximum overshoot, and settling time.

Solution.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{1}{s^2+s+1}$$

$$\zeta = 0.5,\ \omega_n = 1$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{3}} \approx 3.63\,\text{sec}.$$

$$M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} = e^{-\frac{\pi}{\sqrt{3}}} \times 100\% = 16.3\%$$

$$t_s = \frac{3.5}{\zeta\omega_n} = \frac{3.5}{0.5} = 7\,\text{sec. (5\% tolerance band)}$$

$$t_s = \frac{4.5}{\zeta\omega_n} = \frac{4.5}{0.5} = 9\,\text{sec. (2\% tolerance band)}$$

$$C(s) = \frac{1}{s^2+s+1}\cdot\frac{1}{s} = \frac{1}{s} - \frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} - \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\cdot\frac{1}{\sqrt{3}}$$

Oops, I forgot the formulas!

$$c(t) = 1 - e^{-\frac{t}{2}}\cdot\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}e^{-\frac{t}{2}}\cdot\sin\frac{\sqrt{3}}{2}t$$

$$= 1 - e^{-\frac{t}{2}}\cdot\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t + \frac{1}{2}\sin\frac{\sqrt{3}}{2}t\right)$$

$$= 1 - e^{-\frac{t}{2}}\cdot\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t + \frac{\pi}{3}\right)$$

$$\frac{dc(t)}{dt} = -\frac{2}{\sqrt{3}}\left[-\frac{1}{2}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)+e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\frac{\sqrt{3}}{2}\right]$$

$$= -\frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\left[-\frac{1}{2}\sin\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)+\frac{\sqrt{3}}{2}\cos\left(\frac{\sqrt{3}}{2}t+\frac{\pi}{3}\right)\right]$$

$$= \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t = 0$$

$$\boxed{t_p = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}} \approx 3.63\,\text{sec.}}$$

$$c(t_p) = 1 - e^{-\frac{\pi}{\sqrt{3}}}\cdot\frac{2}{\sqrt{3}}\sin\left(\pi+\frac{\pi}{3}\right) = 1 + e^{-\frac{\pi}{\sqrt{3}}}$$

$$\boxed{M_p = e^{-\frac{\pi}{\sqrt{3}}}\times 100\% = 16.3\%}$$

Assignment 7

1. Determine the stability of the following closed-loop polynomials by utilizing Routh's stability criterion. If the system is unstable, determine the poles that lie in the right-half s plane.

4) $s^3 + 10s^2 + 16s + 160 = 0$

Solution. Routh array:

$$
\begin{array}{llll}
s^3 & 1 & 16 & 0 \\
s^2 & 10 & 160 & \\
\boxed{s^1 \quad 0 \quad 0} & & & \\
s^0 & & &
\end{array}
$$

The system is unstable!

- The auxiliary polynomial is then formed from the coefficients of the second row:

$$P(s) = 10s^2 + 160$$

Hence,

$$dP(s)/ds = 20s$$

- Let the term in s row be replaced by 20s. The array becomes

$$\begin{array}{llll} s^3 & 1 & 16 & 0 \\ s^2 & 10 & 160 & \\ s^1 & 20 & 0 & \\ s^0 & 160 & & \end{array}$$

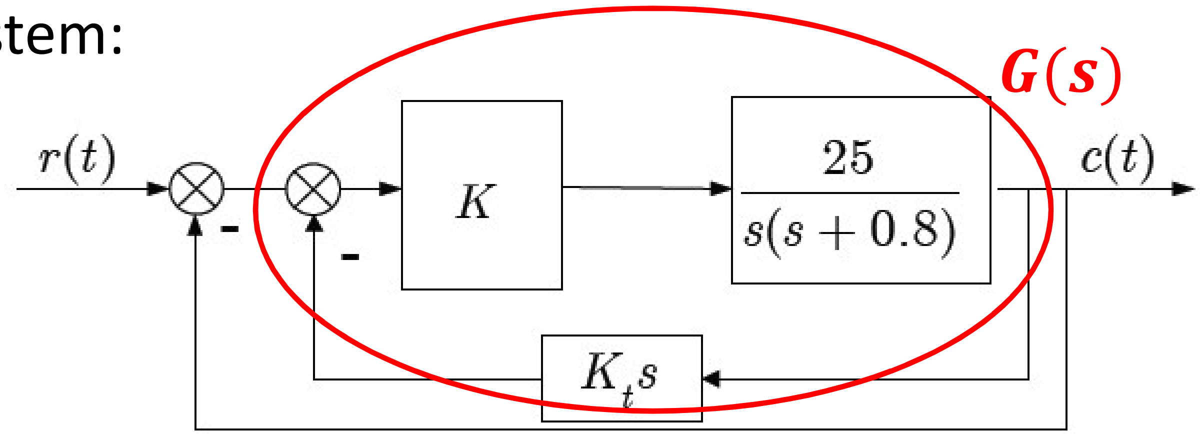
- The indicates that the system has **no pole in the right-half s-plane, but two poles on the $j\omega$ axis**.

- In fact,

$$P(s) = s^2 + 16 = 0$$
$$\Rightarrow s_{1,2} = \pm j4$$

Assignment 7

2. The following block diagram represents a simplified aircraft control system:



Determine K and K_h, so that $\omega_n = 6 rad/s$ and $\zeta = 1$. Based on the evaluated K and K_h, determine the system settling time t_s.

Solution:

- The open loop transfer function $G(s)$ is

$$G(s) = \frac{\dfrac{25K}{s(s+0.8)}}{1+\dfrac{25K}{s(s+0.8)} \cdot K_t s} = \frac{25K}{s^2+(0.8+25KK_t)s}$$

- The closed-loop transfer function $\Phi(s)$ is

$$\Phi(s)=\frac{G(s)}{1+G(s)}=\frac{25K}{s^2+\left(0.8+25KK_t\right)s+25K}$$

- Hence, we have

$$\omega_n{}^2=25K=36 \Rightarrow K=1.44$$

$$0.8+25KK_t=2\zeta\omega_n=12$$
$$\Rightarrow K_t=0.31$$

- Regarding the settling time, since $\Phi(s)$ can be rewritten as

$$\Phi(s)=\frac{36}{s^2+12s+36}=\frac{1}{\left(\frac{1}{6}s+1\right)^2}$$

$$T_1=T_2=\frac{1}{6}, \boxed{t_s=4.75T_1\approx 0.79\,\text{sec.}}$$

by looking up the Table (for $\zeta \geq 1$)

For example:

1) $T_1 = T_2 \Leftrightarrow \zeta = 1$

$$t_s = 4.75T_1$$

2) $T_1 / T_2 = 1.5 \Rightarrow \zeta = 1.02$

$$t_s = 4T_1$$

3) $T_1 / T_2 = 4 \Rightarrow \zeta = 1.25$

$$t_s \approx 3.3T_1$$

4) $T_1 / T_2 > 4 \Rightarrow \zeta > 1.25$

$$t_s \approx 3T_1$$

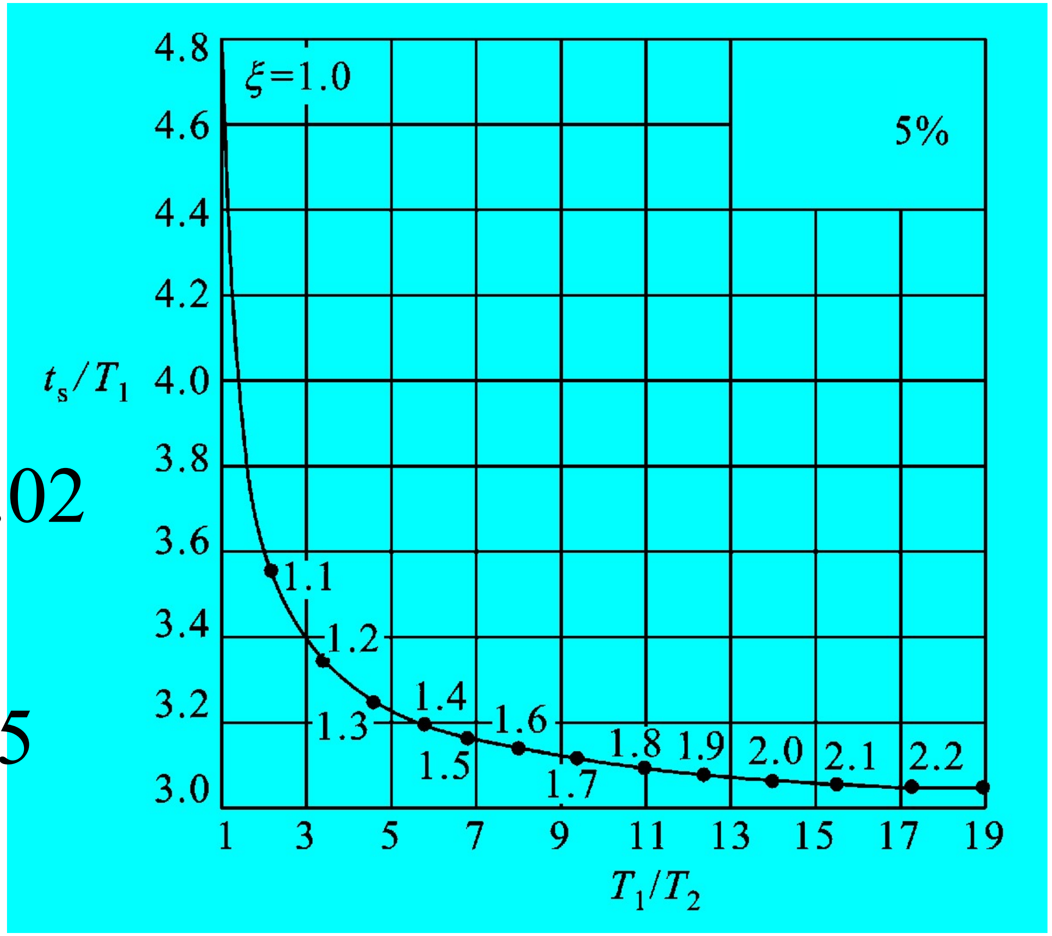
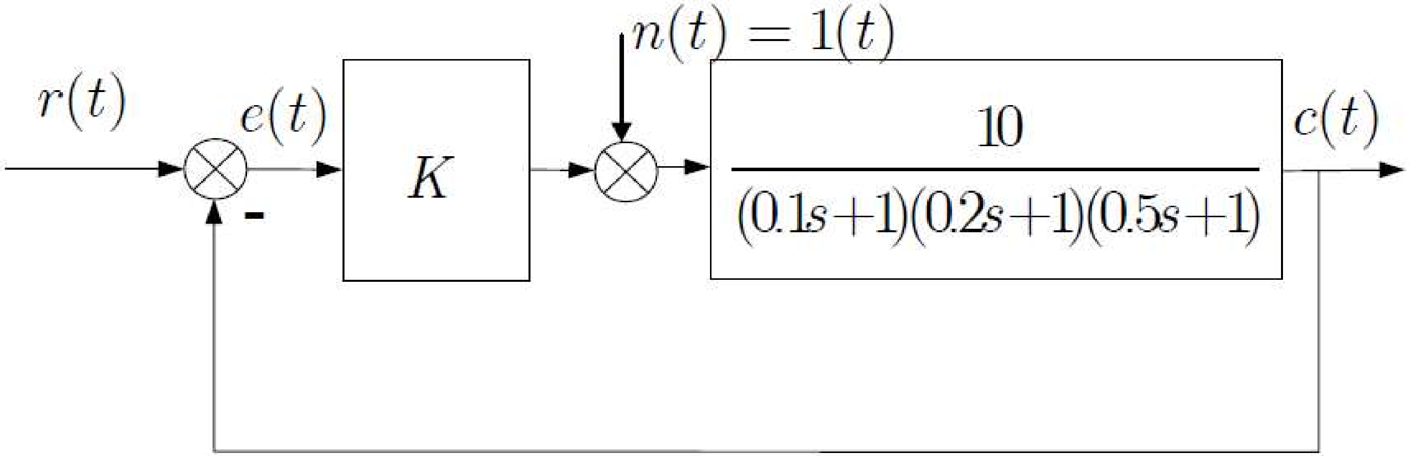


Table: t_s/ T_1 versus T_1/ T_2

Assignment 9

1. A system's block diagram is shown below



Is it possible to choose an appropriate value of K such that the steady state error e_{ssn}=−0.099 when the system is subjected to a disturbance $n(t)$?

Solution:

$$\Phi_{EN}(s)=\frac{-\dfrac{10}{(0.1s+1)(0.2s+1)(0.5s+1)}}{1+\dfrac{10K}{(0.1s+1)(0.2s+1)(0.5s+1)}}$$

$$=\frac{-10}{(0.1s+1)(0.2s+1)(0.5s+1)+10K}$$

$$=\frac{-10}{0.01s^3+0.17s^2+0.8s+1+10K}$$

- Find the range of K **to ensure the stability of the closed-loop system**.

$$\begin{cases}1+10K>0\\0.17\times0.8-0.01\times(10K+1)>0\end{cases}\Rightarrow\boxed{-0.1<K<1.26}$$

- **Assume that all the poles of $s\frac{1}{s}\Phi_{EN}(s)$ lie in the left-half *s*-plane**, find the value of K to ensure that the e_{ssn}=−0.099

$$e_{ssn} = \lim_{s\to 0} s\frac{1}{s}\frac{-10}{0.01s^3+0.17s^2+0.8s+1+10K} = \frac{-10}{1+10K} = 0.099$$

Hence,

$$\boxed{K = 10.001}$$

which beyond the range of K determined to ensure the system stability.

In other words, **it is impossible to choose an appropriate value of K** such that the steady state error e_{ssn}=−0.099 when the system is subjected to a disturbance $n(t)$.

Assignment 10

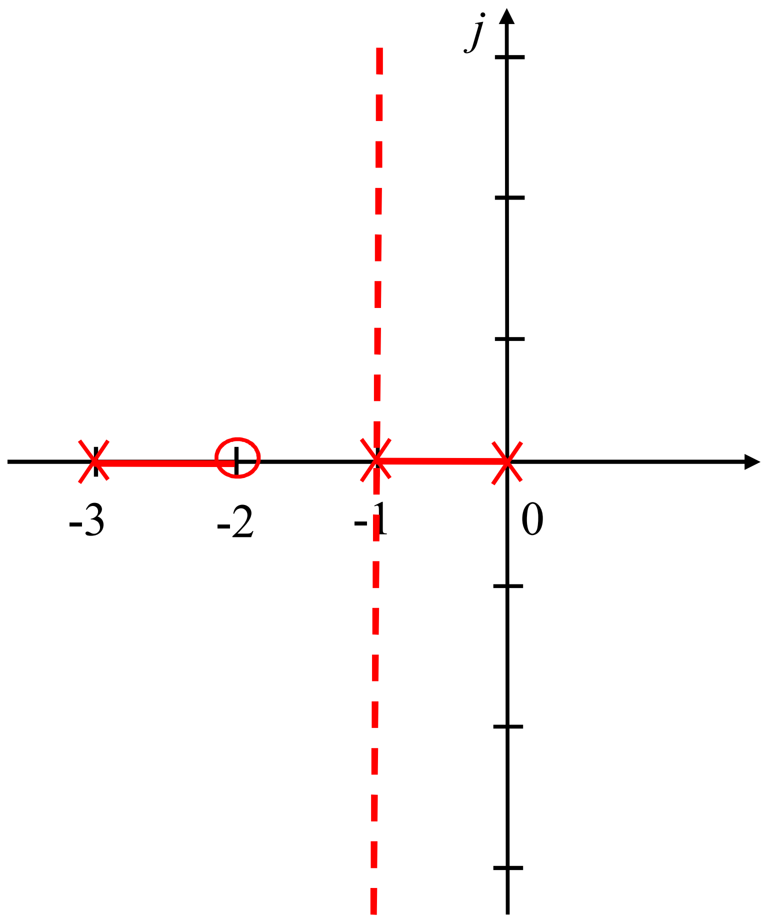
1. The open-loop transfer function of a unity feedback is

$$G(s)=\frac{K^*(s+2)}{s(s+1)(s+3)}$$

1) Sketch the root loci;
2) Determine a pair of dominant poles and corresponding open-loop gain K when $\zeta = 0.5$.

Solution. 1)

- Open-loop zero $z_1 = -2$,
  poles $p_1 = 0$, $p_2 = -1$, $p_3 = -3$
- By **Rule 2**, root locus exists on $[-3, -2]$ and [-1, 0]
- By **Rule 3**, for the linear asymptotes



$$\sigma_a=\frac{-1-3+2}{2}=-1$$

$$\varphi_a=180^0\times\frac{(2k+1)}{2},(k=0,-1)$$

- By **Rule 4**, the breakin/breakaway points satisfy

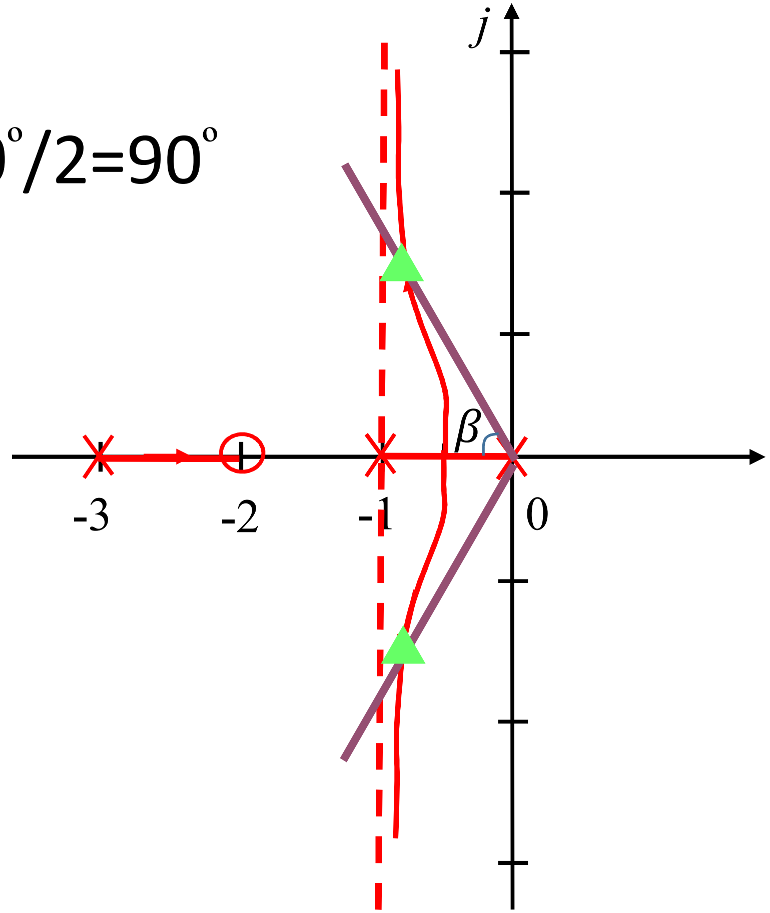
$$\frac{1}{d}+\frac{1}{d+1}+\frac{1}{d+3}=\frac{1}{d+2}$$

Solving $d^3+5d^2+8d+3=0$, yields

$$\begin{cases} d_1=-0.53 \\ d_{2,3}=-2.23\pm 0.79j \end{cases}$$

The angle of breakaway is 180°/2=90°

2) Draw the straight line corresponding to the damping ratio $\zeta = 0.5$, i.e.



$$\cos\beta = 0.5 \Rightarrow \beta = 60^{\circ}$$

- Find the intersection points, i.e. the dominant poles $s_d = \sigma_d \pm j\omega_d = 0.7 \pm 1.2j$. Clearly, the following equation should be satisfied.

$$\angle(\sigma_d + j\omega_d + 2) - \angle(\sigma_d + j\omega_d) - \angle(\sigma_d + j\omega_d + 1)$$
$$-\angle(\sigma_d + j\omega_d + 3) = (2k+1)\pi$$

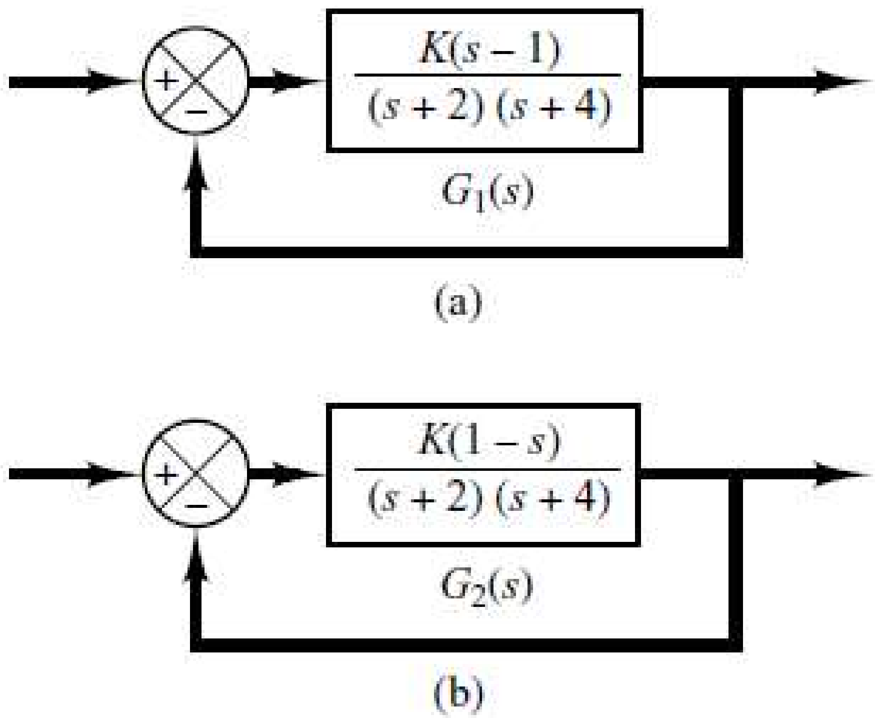
- The corresponding open-loop gain K can thus be determined.

$$\frac{K^* \left|s_d + 2\right|}{\left|s_d\right|\left|s_d + 1\right|\left|s_d + 3\right|} = 1$$

$$K^* \approx 3.81,\ K = K^* \times \frac{2}{3} \approx 2.54$$

Assignment 11

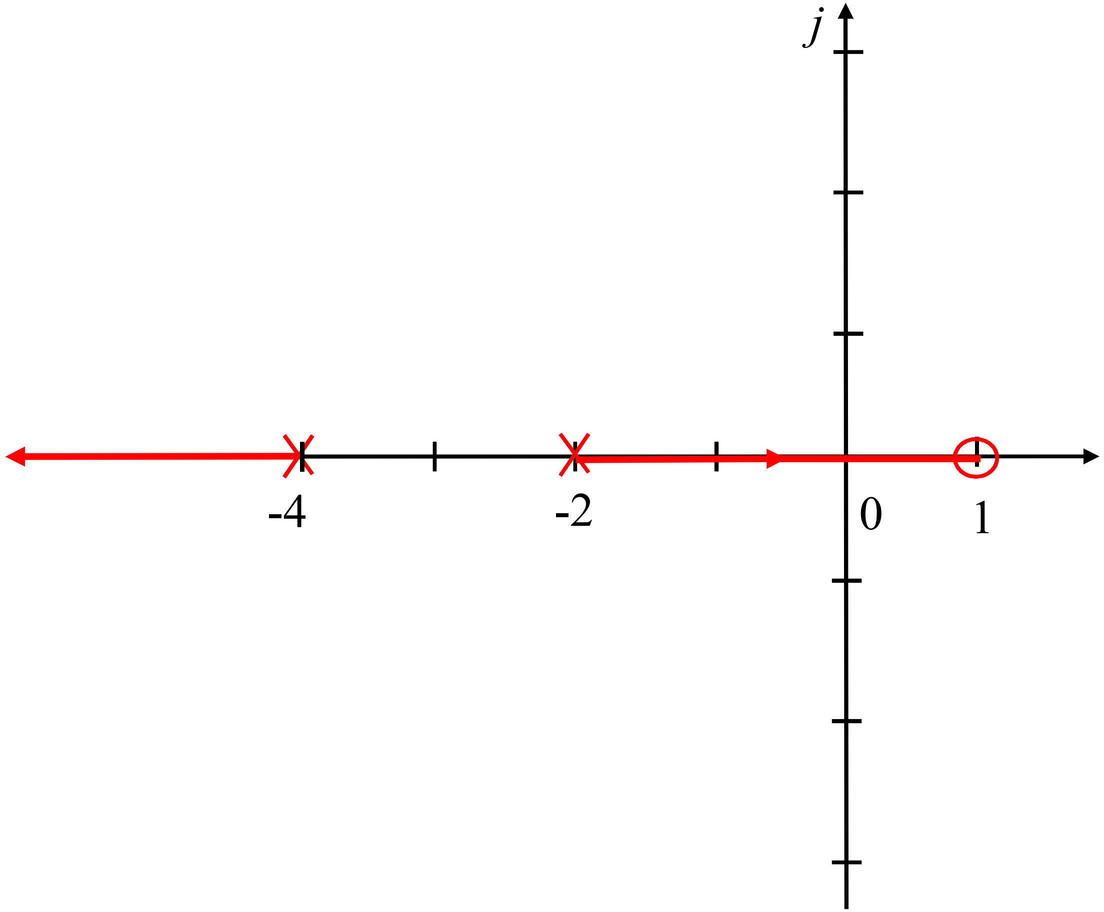
B-6-12. Plot root-locus diagrams for the nonminimum-phase systems shown in Figures (a) and (b), respectively.



Solution.

(a) $G(s)$ is in standard form, thus the rules for plotting root loci of negative feedback systems should be applied.

- Open-loop zero $z_1 = 1$, poles p_1 = -2 , p_2 = -4
- By **Rule 2**, root locus exists on $(-\infty, -4]$ and [-2, 1]



(b) Rewrite $G(s)$ as the standard form

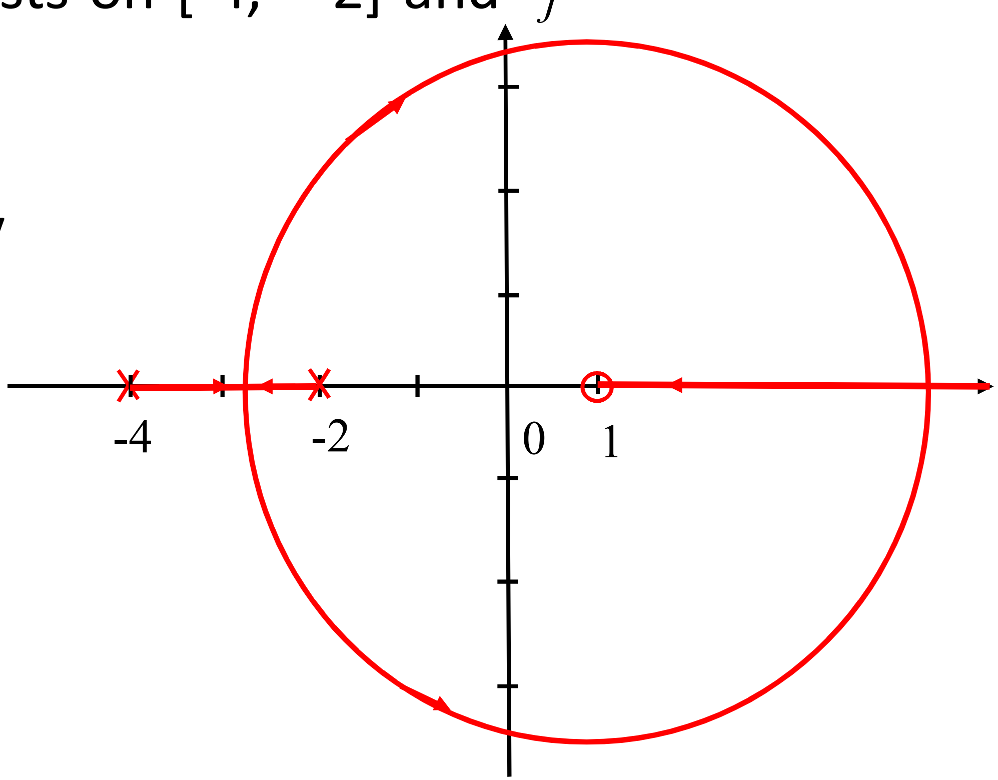
$$G(s) = \frac{-K(s-1)}{(s+2)(s+4)}$$

thus the rules for plotting root loci of positive feedback systems should be applied.

- Open-loop zero $z_1 = 1$, poles p_1 = -2 , p_2 = -4
- By **Rule 2**, root locus exists on [-4, $-$2] and [1, ∞)
- By **Rule 4**, the break-in/breakaway points satisfy

$$\frac{1}{d+2} + \frac{1}{d+4} = \frac{1}{d-1}$$

$$\begin{cases} d_1 = -2.87 \\ d_2 = 4.87 \end{cases}$$



It can be shown that the root loci involves the circular locus where the enter of the circle is (1,0) and the radius equal to $\sqrt{15}$.

Proof: Substituting $s = \sigma + j\omega$ into

$$D(s) = s^2 + (6-K)s + 8 + K = 0$$

It follows that

$$(\sigma + j\omega)^2 + (6-K)(\sigma + j\omega) + 8 + K = 0$$

$$\Rightarrow \begin{cases} 2\sigma\omega + (6-K)\omega = 0 \\ \sigma^2 - \omega^2 + (6-K)\sigma + 8 + K = 0 \end{cases}$$

$$\Rightarrow \begin{cases} 2\sigma = K - 6 \\ \sigma^2 - \omega^2 + (6-K)\sigma + 8 + K = 0 \end{cases}$$

$$\Rightarrow \quad (\sigma - 1)^2 + \omega^2 = 15$$

$\blacksquare$

Assignment 12

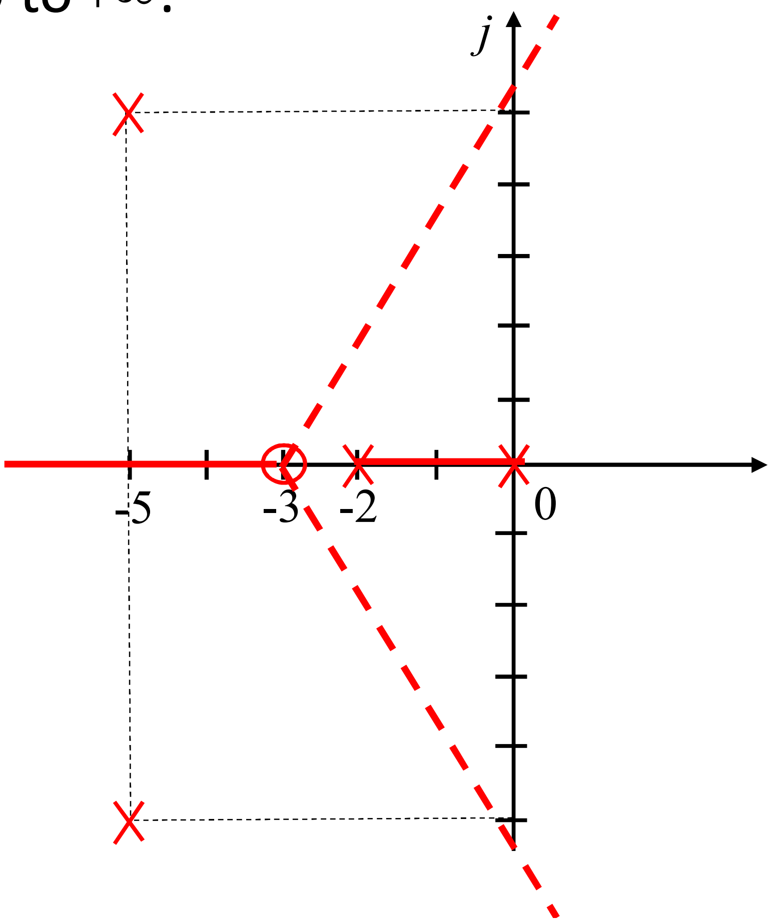
3. The open-loop transfer function of a unity-feedback system is

$$G(s) = \frac{K^*(s+3)}{s(s+2)(s^2+10s+50)}$$

Sketch root loci when K^* varies from 0 to $+\infty$.

Solution:

- Open-loop zero $z_1 = -3$,
  poles $p_1 = 0$, $p_2 = -2$, $p_{3,4} = -5 \pm 5j$
- By Rule 2, root locus exists on
  $(-\infty, -3]$ and $[-2, 0]$
- By Rule 3, for the linear asymptotes



$$\sigma_a = \frac{-5-5j-5+5j-2+3}{3} = -3$$

$$\varphi_a = 180^0 \times \frac{(2k+1)}{3}, (k = 0, \pm 1)$$

- By Rule 4, the breakaway points satisfy

$$\frac{1}{d}+\frac{1}{d+2}+\frac{1}{d+5-5j}+\frac{1}{d+5+5j}=\frac{1}{d+3}$$

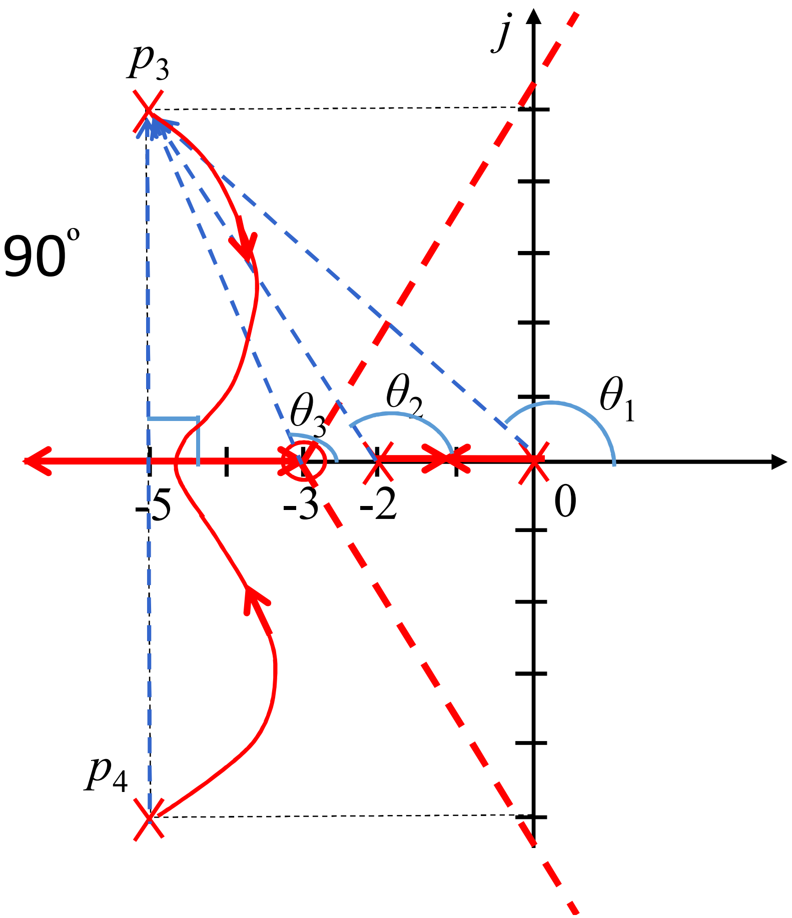
Solving $3d^4+36d^3+178d^2+420d+300=0$, yields

$$\begin{cases} d_1=-4.83 \\ d_2=-1.17 \\ d_{3,4}=-3\pm 2.94j \end{cases}$$

The angle of breakaway is $180^{\circ}/2=90^{\circ}$

- By Rule 5, the angle of departure for p_3 is

$$\begin{aligned}\theta_{p_3} &= 180°-90°-\theta_1-\theta_2+\theta_3 \\ &= 90°-135°-121°+111.8° \\ &= -54.2°\end{aligned}$$



- By Rule 6, the intersection of the root loci with the imaginary axis is computed as follows

The closed-loop characteristic equation is

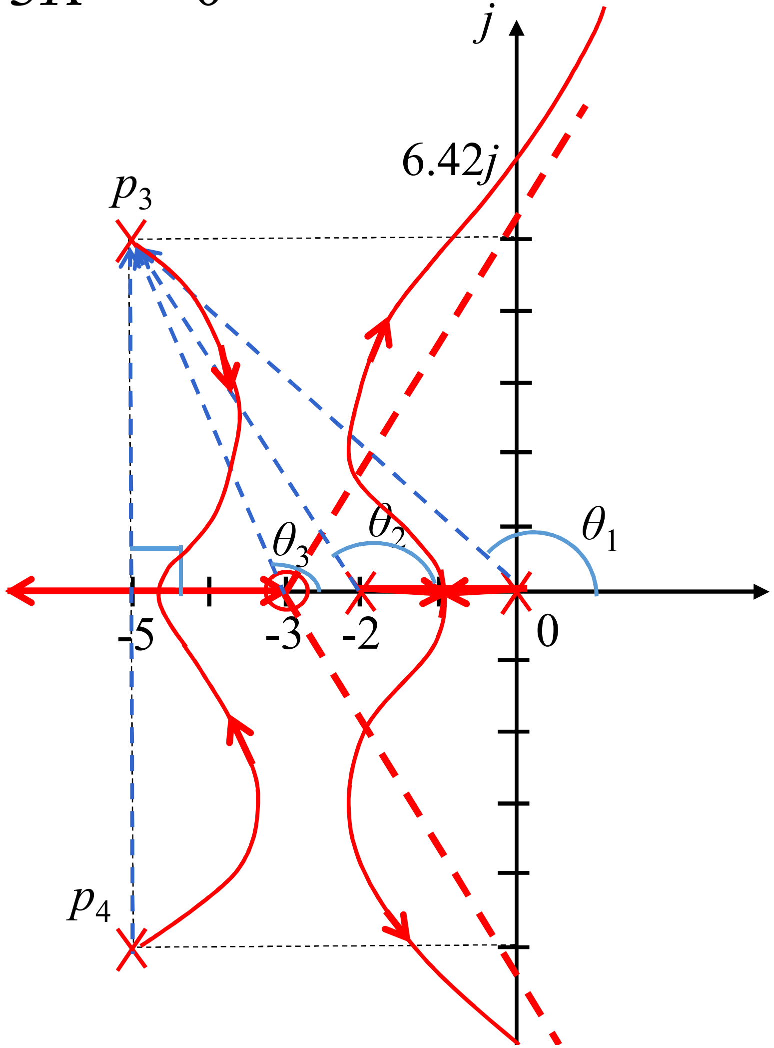
$$s^4 + 12s^3 + 70s^2 + (K^* + 100)s + 3K^* = 0$$

Let $s = j\omega$. Then,

$$\omega^4 - 12j\omega^3 - 70\omega^2 + (K^* + 100)j\omega + 3K^* = 0$$

$$\begin{cases} \omega^4 - 70\omega^2 + 3K^* = 0 \\ -12\omega^3 + (100 + K^*)\omega = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \omega = 0, \quad \pm 6.42 \\ K^* = 0, \quad 395.2 \end{cases}$$



4. The open-loop transfer function of a unity-feedback system is

$$G(s)=\frac{K^*(s+1)}{s(s+0.5)(s^2+10s+50)}$$

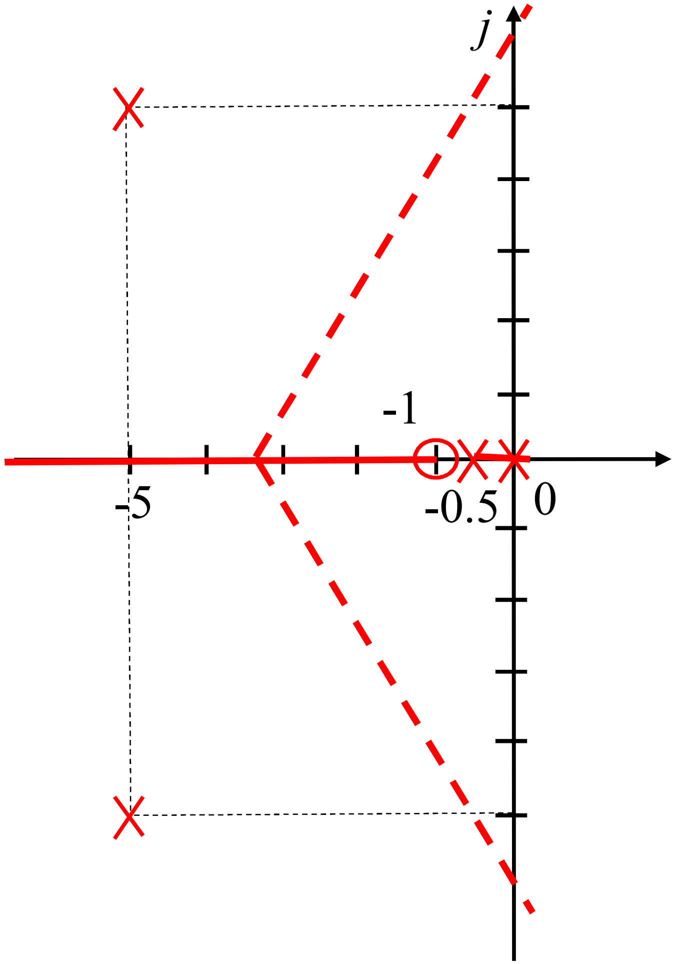
Sketch root loci when K^* varies from 0 to $+\infty$.

Solution:

- Open-loop zero $z_1 = -1$,
  poles $p_1 = 0$, $p_2 = -0.5$, $p_{3,4} = -5 \pm 5j$
- By Rule 2, root locus exists on $(-\infty, -1]$ and [-0.5, 0]
- By Rule 3, for the linear asymptotes

$$\sigma_a = \frac{-5-5j-5+5j-0.5+1}{3} = -3.17$$

$$\varphi_a = 180^0 \times \frac{(2k+1)}{3}, (k=0,\pm 1)$$



- By Rule 4, the breakaway points satisfy

$$\frac{1}{d}+\frac{1}{d+0.5}+\frac{1}{d+5-5j}+\frac{1}{d+5+5j}=\frac{1}{d+1}$$

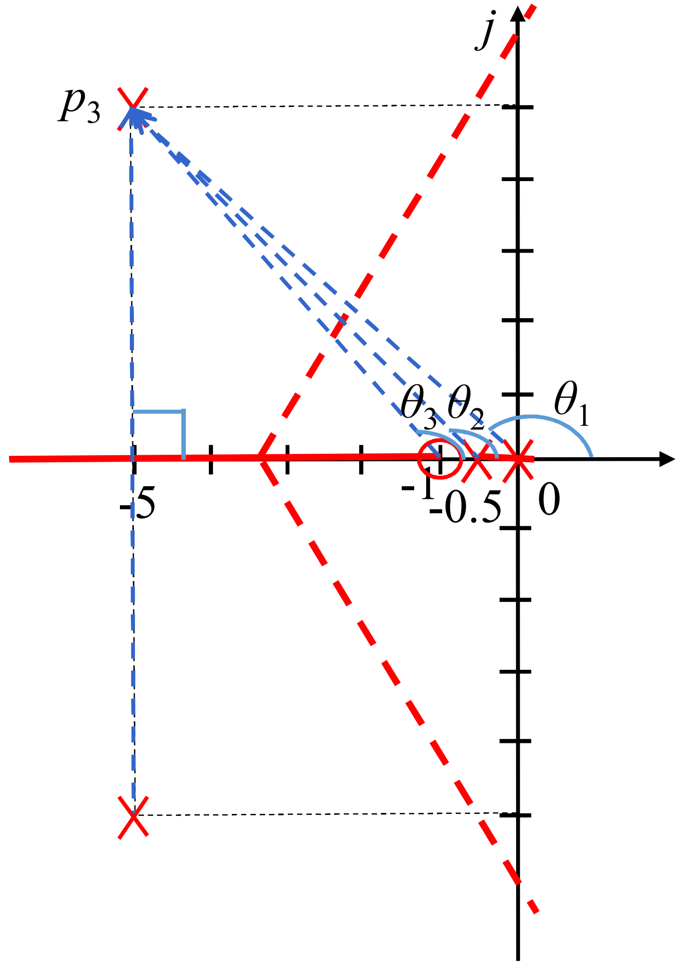
Solving $3d^4+25d^3+86d^2+110d+25=0$, yields

$$\begin{cases} d_1=-2.1 \\ d_2=-0.29 \\ d_{3,4}=-2.97\pm 2.24j \end{cases}$$

The angle of breakaway is $180^{\circ}/2=90^{\circ}$

- By Rule 5, the angle of departure for p_3 is

$$\begin{aligned}\theta_{p_3}&=180°-90°-\theta_1-\theta_2+\theta_3\\ &=90°-135°-132°+128.7°\\ &=-48.3°\end{aligned}$$

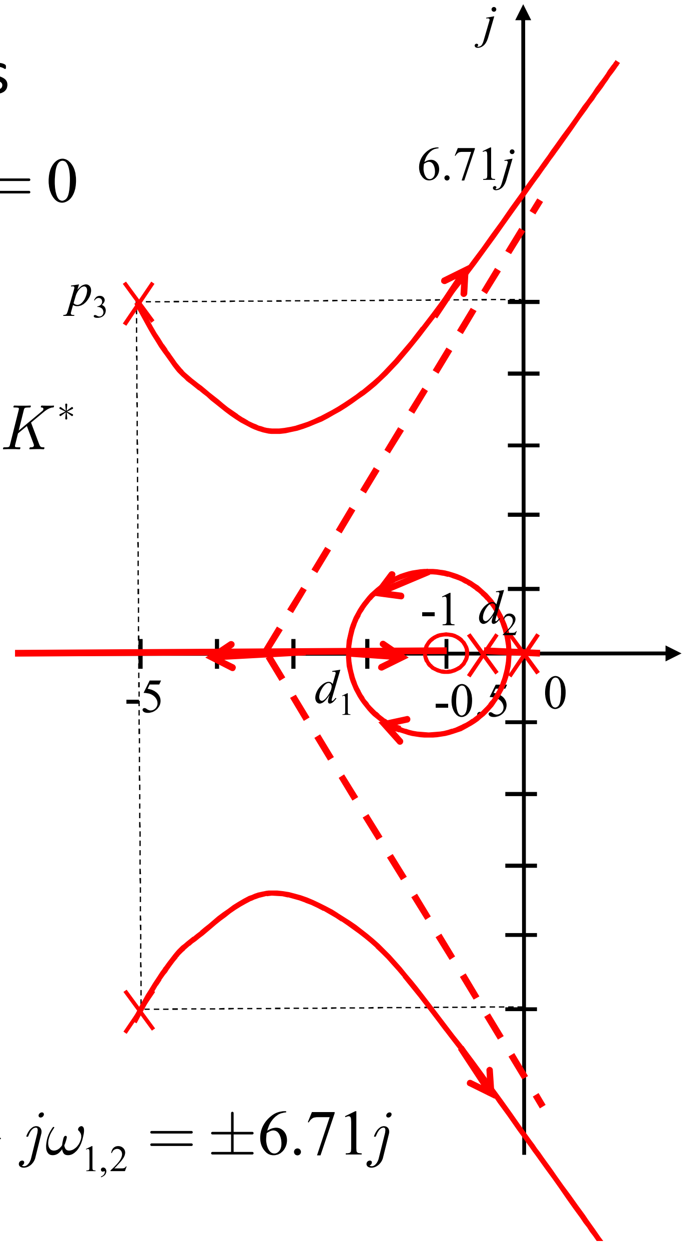


- By Rule 6, the intersection of the root loci with the imaginary axis is computed as follows

  The closed-loop characteristic equation is

$$s^4+10.5s^3+55s^2+(K^*+25)s+K^*=0$$

  Routh array

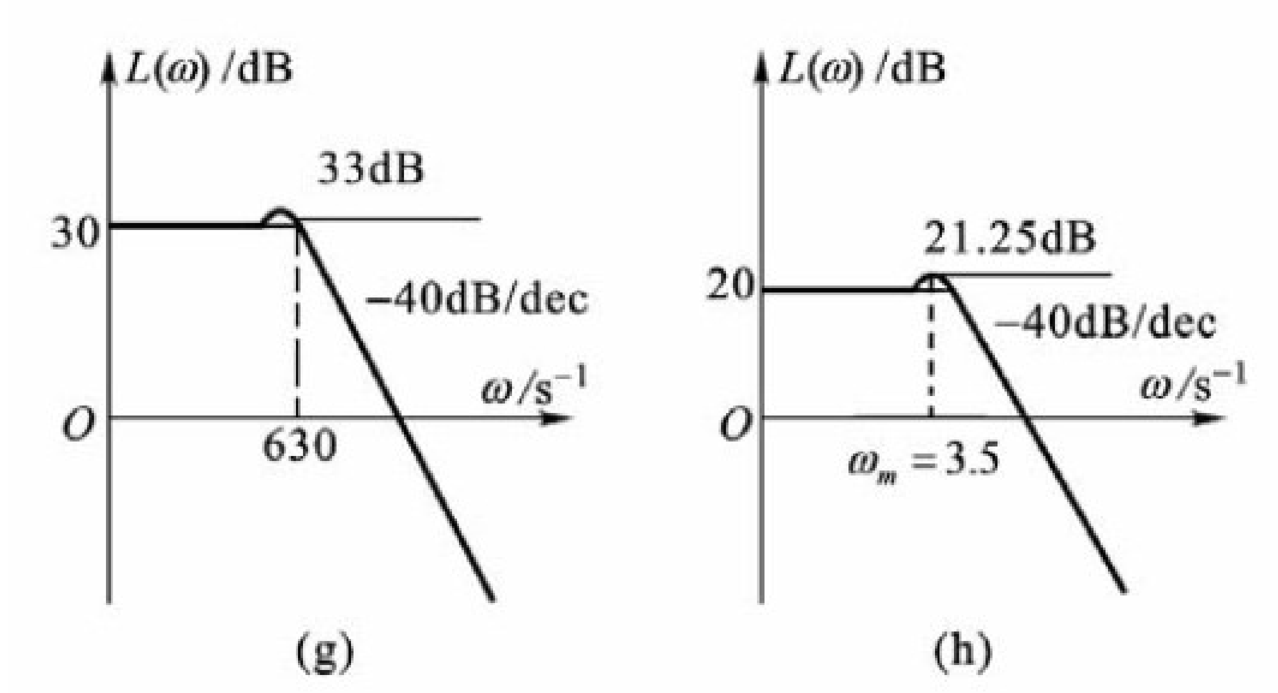


$$\begin{array}{c|ccc}
s^4 & 1 & 55 & K^* \\
s^3 & 10.5 & K^*+25 & \\
s^2 & 55-\dfrac{K^*+25}{10.5} & K^* & \\
s^1 & K^*+25-\dfrac{10.5K^*}{55-\dfrac{K^*+25}{10.5}} & 0 & \\
s^0 & & &
\end{array}$$

$$=0$$

$$\Rightarrow K^*=448 \ \Rightarrow 9.95s^2+448=0 \ \Rightarrow j\omega_{1,2}=\pm 6.71j$$

A13-2. The Bode asymptotic magnitude curves of some minimum phase transfer functions are shown below. Determine their transfer functions.



Solution:

(g) $\omega_n = 630\,rad/s$ (h) $\omega_m = \omega_n\sqrt{1-2\zeta^2} = 3.5\,rad/s$

(g) $G(s)$ has the following form

$$G(s) = \frac{K\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{K \times 630^2}{s^2 + 2\zeta \times 630s + 630^2}$$

- Determine the open-loop gain K

$$20\lg K = 30 \Rightarrow K = 31.6$$

- Determine the damping ratio ζ

$$20\lg \left. \frac{\left| G(j\omega) \right|}{K} \right|_{\omega=630} = 20\lg \frac{630^2}{2\zeta \times 630^2} = 20\lg \frac{1}{2\zeta} = 3dB$$

Hence,

$$\zeta = 0.35$$

$$G(s) = \frac{1.25 \times 10^7}{s^2 + 441s + 39690}$$

(h) $G(s)$ has the following form

$$G(s) = \frac{K\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

- Determine the open-loop gain K

$$20\lg K = 20 \Rightarrow K = 10$$

- Determine the damping ratio ζ

$$20\lg \left.\frac{|G(j\omega)|}{K}\right|_{\omega=\omega_m} = 20\lg \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.25dB$$

Hence,

$$\zeta = 0.5$$

- From $\omega_n\sqrt{1-2\zeta^2} = 3.5$, we obtain ω_n=4.95 rad/s

$$G(s) = \frac{245}{s^2 + 4.95s + 24.5}$$

Assignment 14

B-7-7. Sketch the polar plots of the open-loop transfer function

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)}{s^2(Ts+1)}$$

for the following two cases:

(a) $T_a > T > 0,\ T_b > T > 0$

(b) $T > T_a > 0,\ T > T_b > 0$

Solution:

$$G(j\omega)H(j\omega)$$

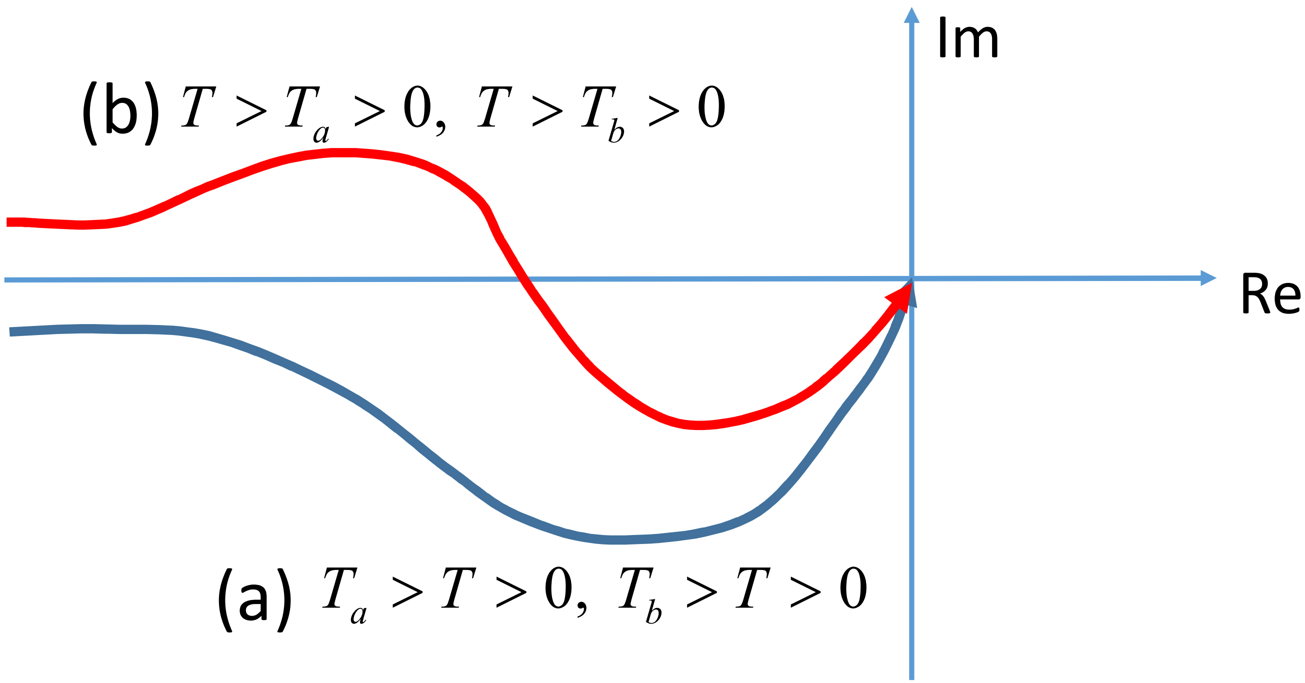
$$= \frac{K\sqrt{T_a{}^2\omega^2+1}\cdot\sqrt{T_b{}^2\omega^2+1}}{\omega^2\sqrt{T^2\omega^2+1}} \angle -180^\circ + \tan^{-1} T_a\omega + \tan^{-1} T_b\omega - \tan^{-1} T\omega$$

- The low-frequency portion of the plot is

$$G(j\omega)H(j\omega)\Big|_{\omega=0} = \infty\angle -180^\circ$$

- The high-frequency portion of the plot is

$$G(j\omega)H(j\omega)\Big|_{\omega=\infty}$$
$$= 0\angle -180^\circ + 90^\circ + 90^\circ - 90^\circ = 0\angle -90^\circ$$



Assignment 15

1. Determine the closed-loop stability of the following open-loop transfer functions by using Nyquist stability criterion or Bode stability criterion (Logarithm frequency stability criterion).

(5) $$G(s) = \frac{10}{s(0.2s+1)(s-1)}$$

Solution: when $\omega = 0$, $G(j\omega) = \infty\angle 0^\circ - 180^\circ = \infty\angle -180^\circ$

when $\omega = 0^+$, $G(j\omega) = \infty\angle -90^\circ - 180^\circ = \infty\angle -270^\circ$

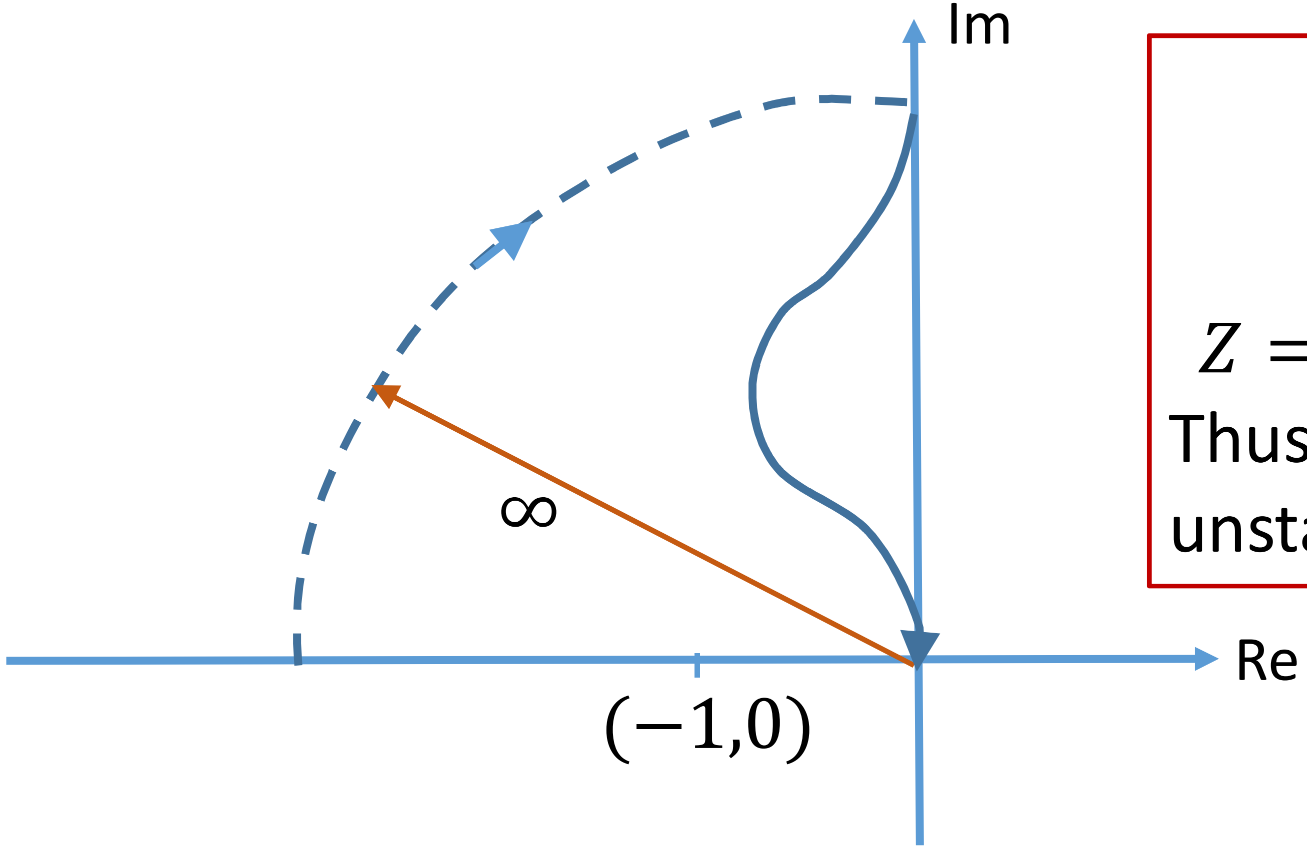
when $0^+ < \omega < \infty$,

$$G(j\omega) = \frac{10}{\omega\sqrt{0.04\omega^2+1}\sqrt{\omega^2+1}}\angle -270^\circ - \tan^{-1} 0.2\omega + \tan^{-1}\omega$$

$$\angle G(s) > \angle -270^{\circ}$$

when $\omega \rightarrow \infty$,

$$G(s) = 0\angle -270^{\circ}$$



$$N = -\frac{1}{2}$$

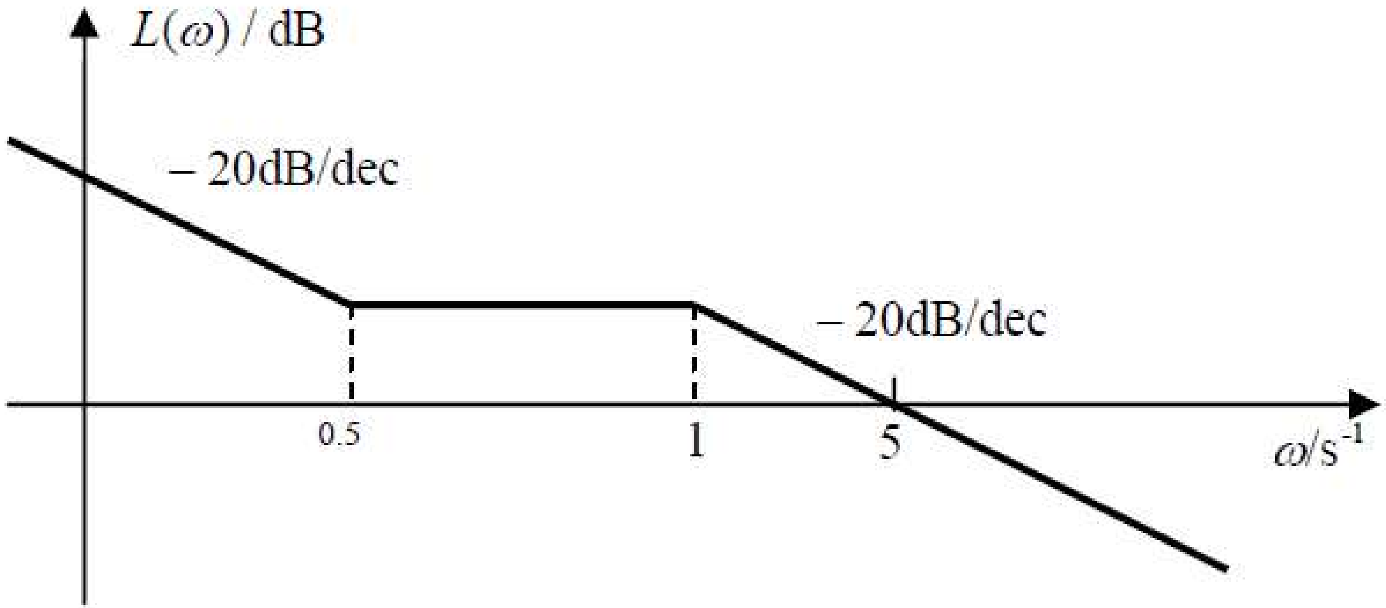
$$P = 1$$

$$Z = P - 2N = 2$$

Thus the system is unstable.

4. The Bode asymptotic magnitude curve of the open-loop transfer function of a unity-feedback system is shown below. If it is known that a zero of the open-loop transfer function lies in the right half s-plane.

1. Determine the open-loop transfer function;
2. Determine the stability of the closed-loop system by using Bode stability criterion.



Solution:

(1) $G(s)$ has the following form

$$G(s) = \frac{K(\tau s - 1)}{s(Ts + 1)}$$

- Determine K, τ, T

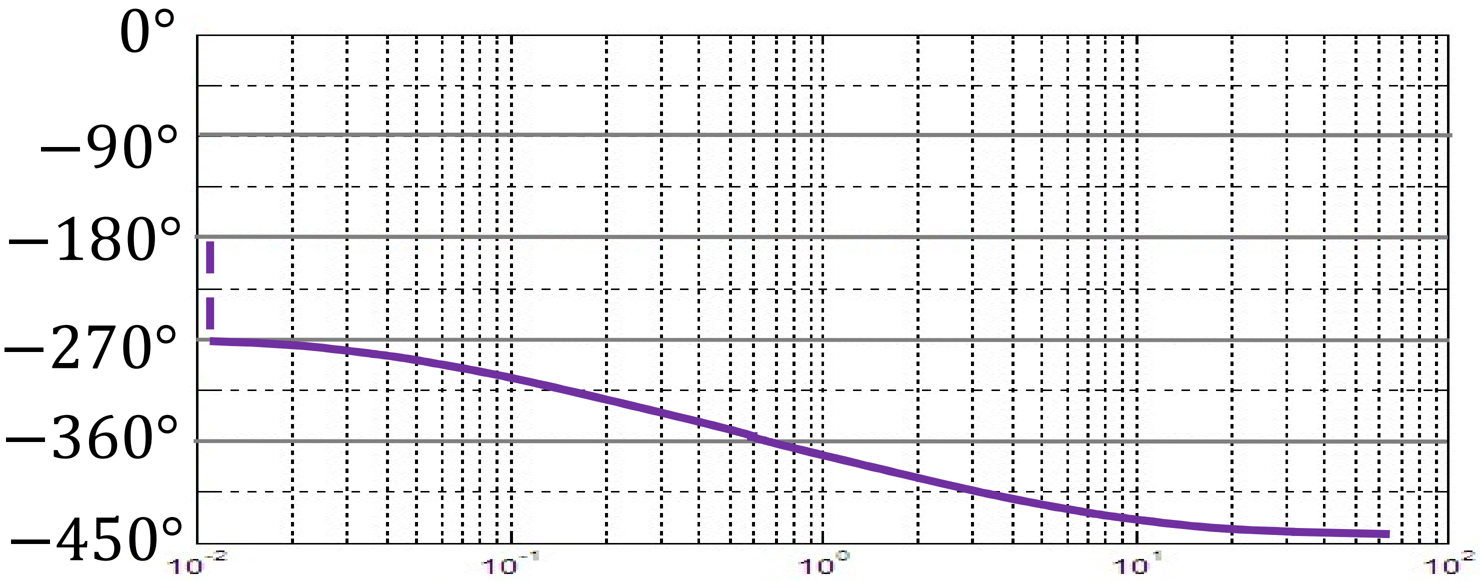
$$\tau = \frac{1}{0.5} = 2,\ \ T = 1$$

$$\frac{K \times 2 \times 5}{5 \times 5} = 1 \Rightarrow K = 2.5$$

- Hence, there is

$$G(s) = \frac{2.5(2s - 1)}{s(s + 1)}$$

(1) Plot the phase-angle curve

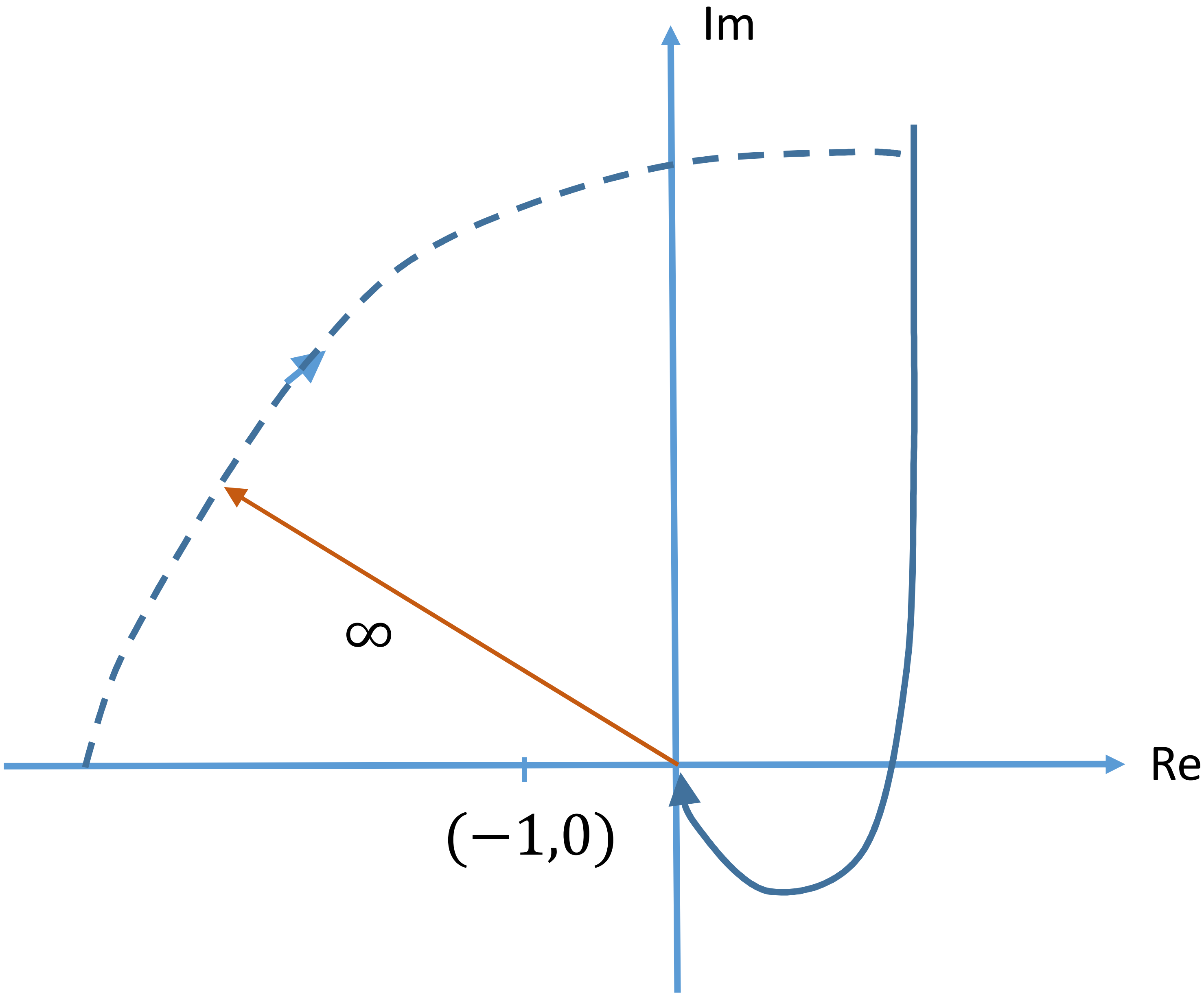


$$N = -\frac{1}{2}$$

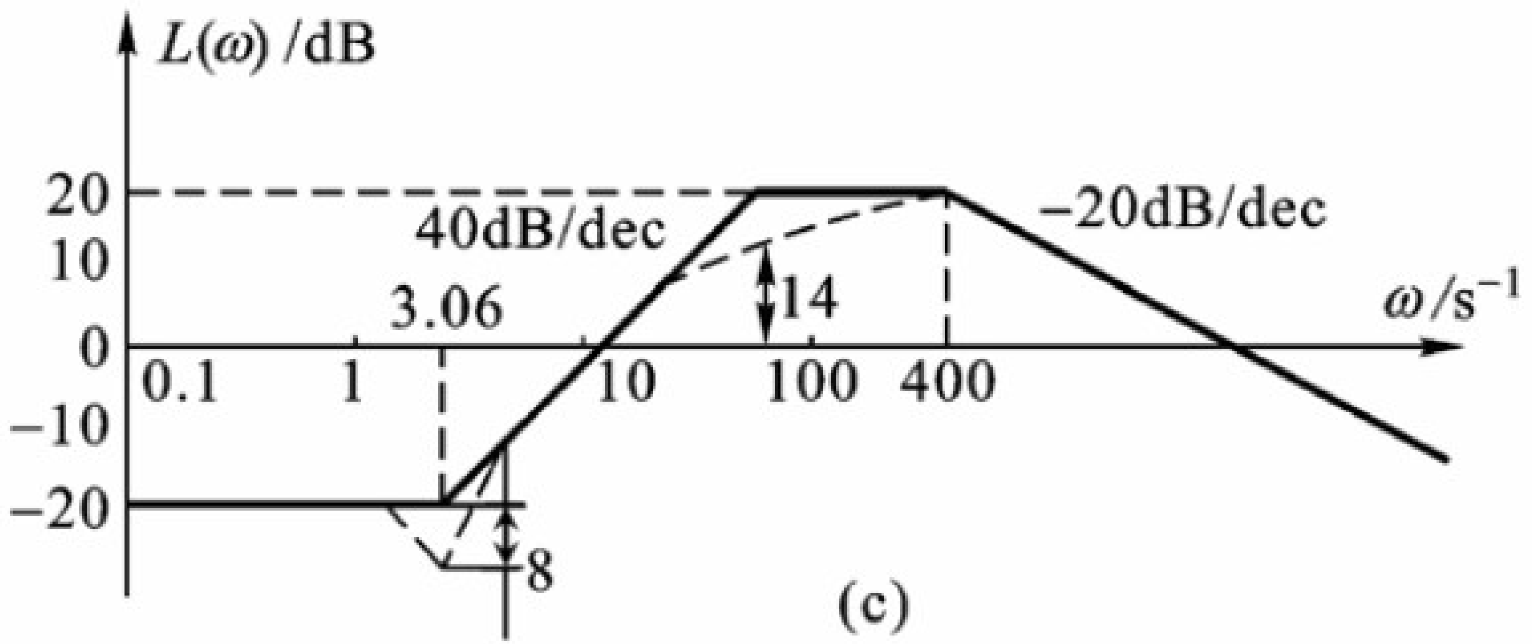
$$P = 0$$

$$Z = P - 2N = 1$$

Thus the system is unstable.


Im
Re
∞
(−1,0)

5. Bode diagram of minimum phase systems are shown below. Determine their transfer functions and draw the corresponding phase angle plots. Then, determine their stability.



(c)

Solution: $G(s) = KG_1(s)G_2(s)G_3(s)$

(1) Determine the open-loop gain K

$$20\lg K = -20 \Rightarrow K = 0.1$$

(2) Determine the quadratic derivative factor

$$G_1(s) = \left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n}s + 1$$

$$\omega_n = 3.06 \qquad 20\lg 2\zeta = -8 \Rightarrow \zeta = 0.2$$

(3) Determine the three first-order factors

$$G_2(s) = \frac{1}{\left(T_1 s + 1\right)^2}, G_3(s) = \frac{1}{T_2 s + 1}$$

$$\frac{1}{T_1} = 30.6 \Rightarrow T_1 = 0.033 \qquad \frac{1}{T_2} = 400 \Rightarrow T_1 = 0.0025$$

Assignment 16

1. The open-loop transfer function of a unity-feedback system is as follows.

$$G(s)=\frac{16}{s(s+4\sqrt{2})}$$

Evaluate $M_p, t_s, t_p, \omega_c, M_r, \omega_r, \omega_b, \gamma$, and K_g.

Solution:

$$\Phi(s)=\frac{G(s)}{1+G(s)}=\frac{16}{s^2+4\sqrt{2}s+16}$$

$$\Rightarrow \omega_n=4rad/s,\ \zeta=\frac{\sqrt{2}}{2}\approx 0.707$$

$$1.\ M_p=e^{-\pi\zeta/\sqrt{1-\zeta^2}}\times 100\%=4.32\%$$

$$2.\ t_s=\frac{3.5}{\zeta\omega_n}=\frac{3.5}{2\sqrt{2}}\approx 1.24s$$

$$3.\ t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \approx 1.11s$$

$$4.\ \left|G(j\omega_c)\right| = \frac{16}{\omega_c\sqrt{32+\omega_c^{\ 2}}} = 1$$

$$\Rightarrow \omega_c = 2.57 rad/s$$

$$5.\ M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1$$

$$6.\ \omega_r = \omega_n\sqrt{1-2\zeta^2} = 0$$

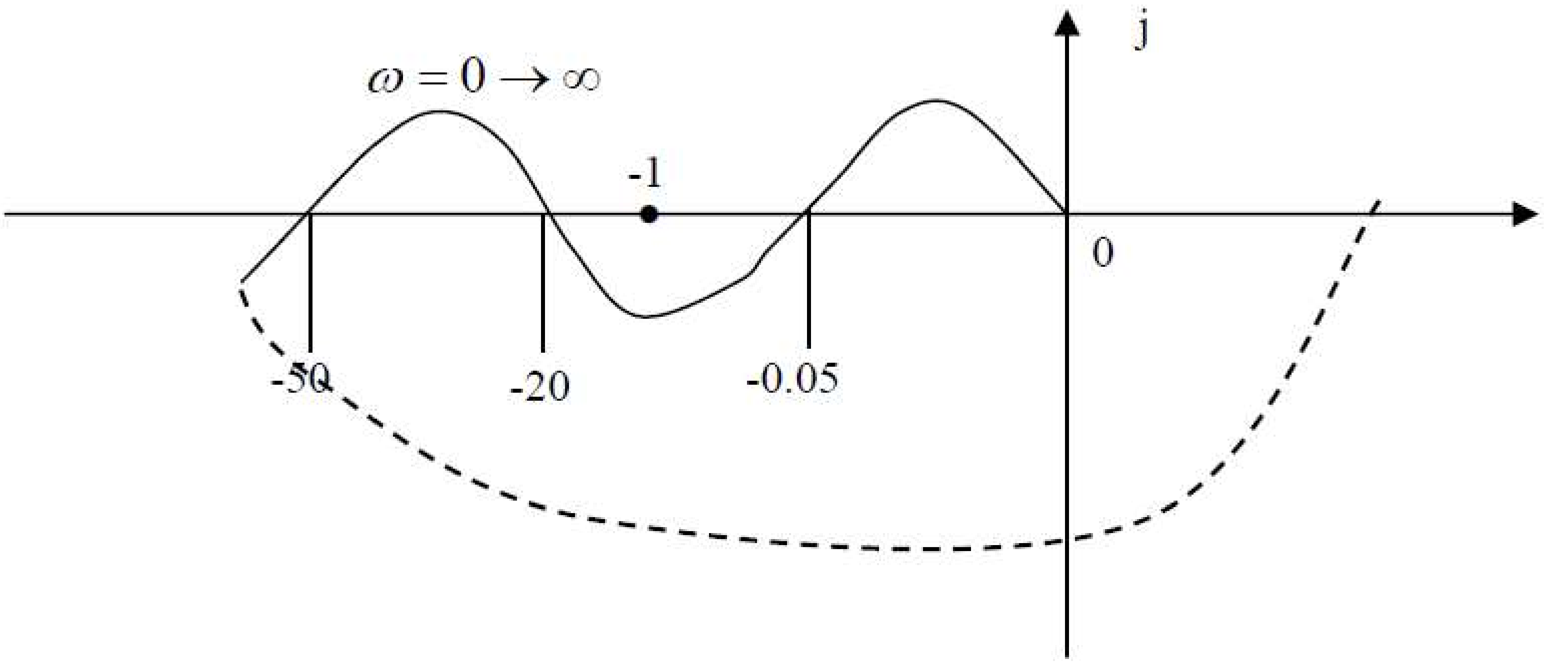
$$7.\ 20\lg\left|\Phi(j\omega_b)\right| = 20\lg\frac{16}{\sqrt{(16-\omega_b^{\ 2})^2+32\omega_b^{\ 2}}} = -3dB$$

$$\Rightarrow \omega_b = 4 rad/s$$

$$8.\ \gamma = \text{-}90^\circ - \tan^{-1}\left(\frac{2.57}{4\sqrt{2}}\right) + 180^\circ \approx 65.6^\circ$$

9. Since the phase-angle curve lies above the -180^{O} for all ω, $K_g = \infty dB$.

5. A Nyquist curve of an open-loop transfer function is shown below with open-loop gain $K = 500, p = 0$, where p is the number of the unstable poles of the open loop transfer function. Determine the range of K for which the system is stable.



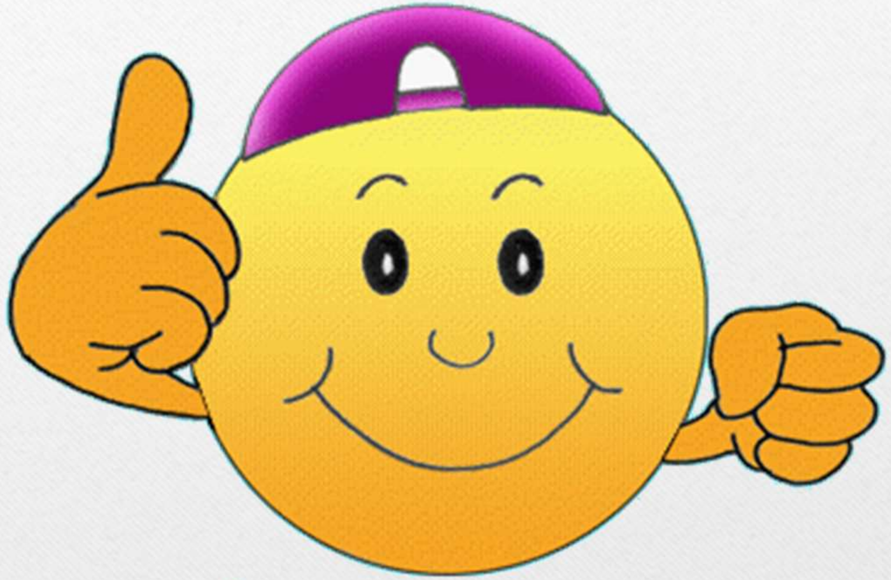
Solution: When K is changed, $|G(j\omega)|$ will be adjusted whereas $\angle G(j\omega)$ is left unchanged.

- $|G(j\omega)|: 0.05 \rightarrow 1, K: 500 \rightarrow 100000$
- $|G(j\omega)|: 50 \rightarrow 1, K: 500 \rightarrow 10$
- $|G(j\omega)|: 20 \rightarrow 1, K: 500 \rightarrow 25$

Hence, the range of K to ensure the stability of the closed-loop system is

$$K < 10 \quad \text{or} \quad 25 < K < 10000$$

GO FOR IT !



GOOD LUCK !