Chapter 3

Mathematical Modeling of Mechanical Systems and Electrical Systems

3-1 Introduction

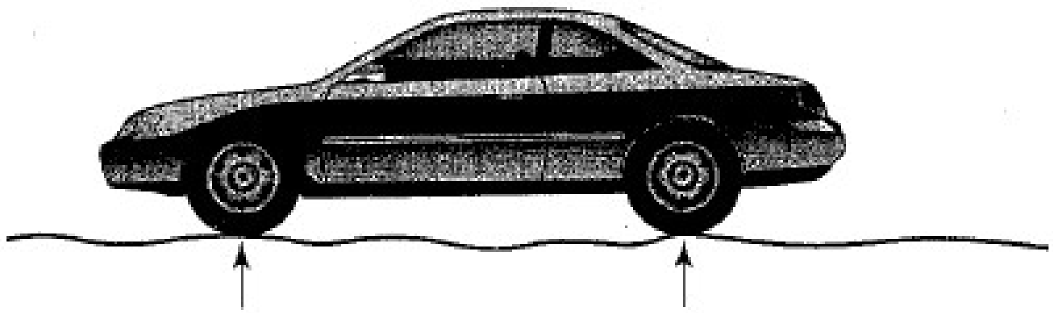
This chapter presents mathematical modeling of mechanical systems and electrical systems.

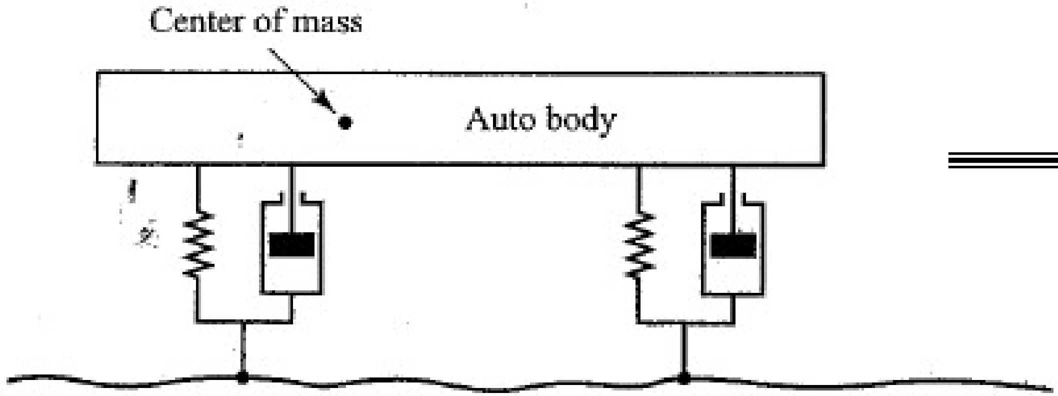
The mathematical law governing **mechanical** systems is **Newton's second law**, while the basic laws governing the **electrical** circuits are **Kirchhoff's laws**.

3-2 Mathematical modeling of mechanical systems

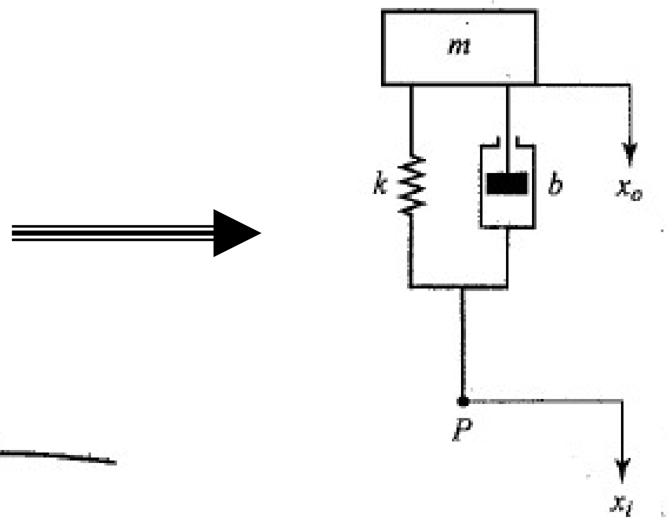
Typical mechanical systems may involve two kinds of motion: linear motion and rotational motion. Spring, mass, damper and inverted pendulum are widely used devices to describe a large class of mechanical systems.

Example (Automobile suspension system). As the car moves along the road, the vertical displacements at the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass and a rotational motion about the center of mass, and can be simplified as a system with springs, mass and dampers shown below.



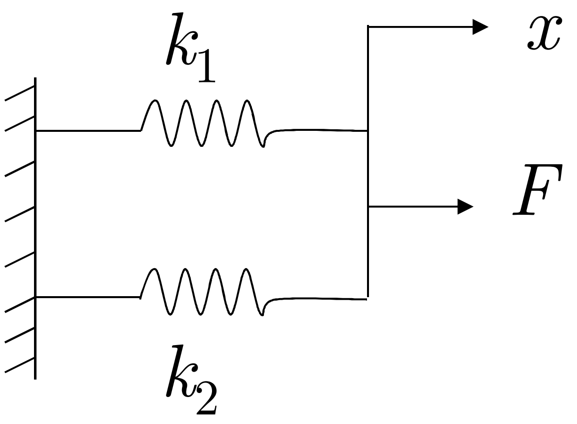


(a)

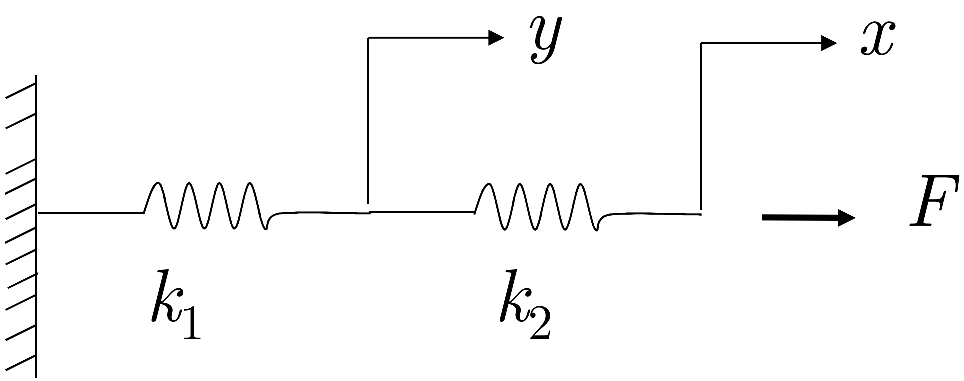


3-2 Mathematical modeling of mechanical systems

Example. Equivalent spring constants.



$$k_1x+k_2x=F \Rightarrow k_{eq}=\frac{F}{x}=k_1+k_2$$



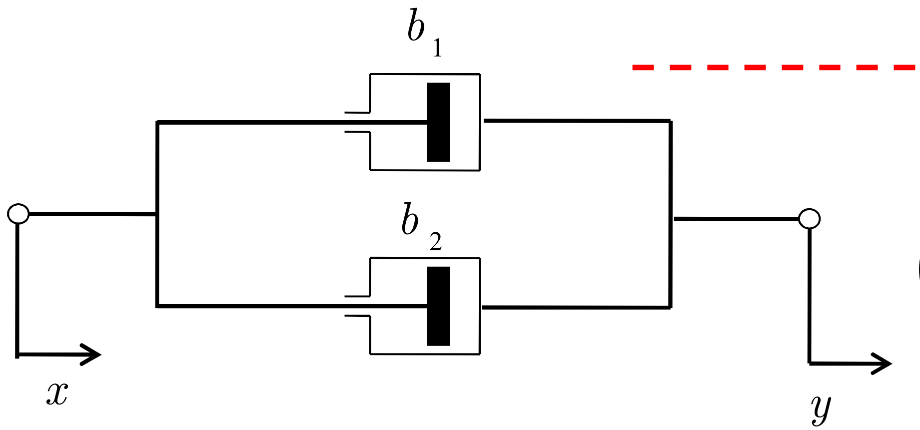
$$k_1y=F,\ k_2(x-y)=F$$

$$\Rightarrow k_2(x-\frac{F}{k_1})=F$$

$$k_{eq}=\frac{F}{x}=\frac{1}{\dfrac{1}{k_1}+\dfrac{1}{k_2}}$$

Systems consisting of two springs in parallel and series, respectively.

Example. Systems consisting of two dampers connected in parallel and series, respectively. Find their equivalent viscous-friction coefficients b_{eq} with respect to dy/dt and dx/dt.

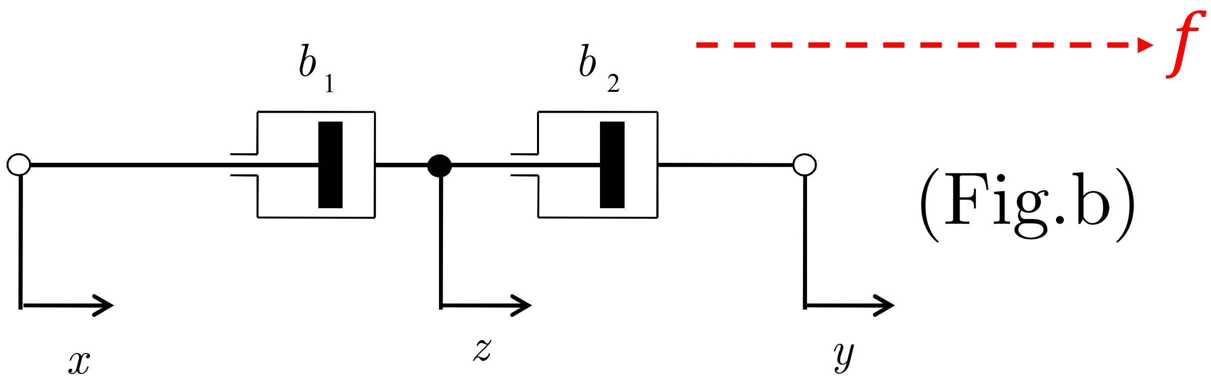


(Fig.a)

Solution: (a) The force f that causes the displacements is

$$\begin{aligned} f &= b_1(\dot{y}-\dot{x}) + b_2(\dot{y}-\dot{x}) \\ &= (b_1+b_2)(\dot{y}-\dot{x}) \\ &= b_{eq}(\dot{y}-\dot{x}) \end{aligned}$$

(b):



(Fig.b)

The same force f is transmitted through the shaft and therefore,

$$f = b_1(\dot{z} - \dot{x}) = b_2(\dot{y} - \dot{z}) \qquad (1)$$

from which we have

$$(b_1 + b_2)\dot{z} = b_2\dot{y} + b_1\dot{x} \Rightarrow \dot{z} = \frac{1}{(b_1 + b_2)}(b_2\dot{y} + b_1\dot{x}) \qquad (2)$$

Since we want to determine the relationship between dy/dt and dx/dt, that is,

$$f = b_{eq}(\dot{y} - \dot{x})$$

where b_{eq} is the equivalent viscous friction coefficient, by substituting (2) into (1) yields.

$$f = b_2\left(\dot{y} - \frac{1}{(b_1 + b_2)}(b_2\dot{y} + b_1\dot{x})\right)$$

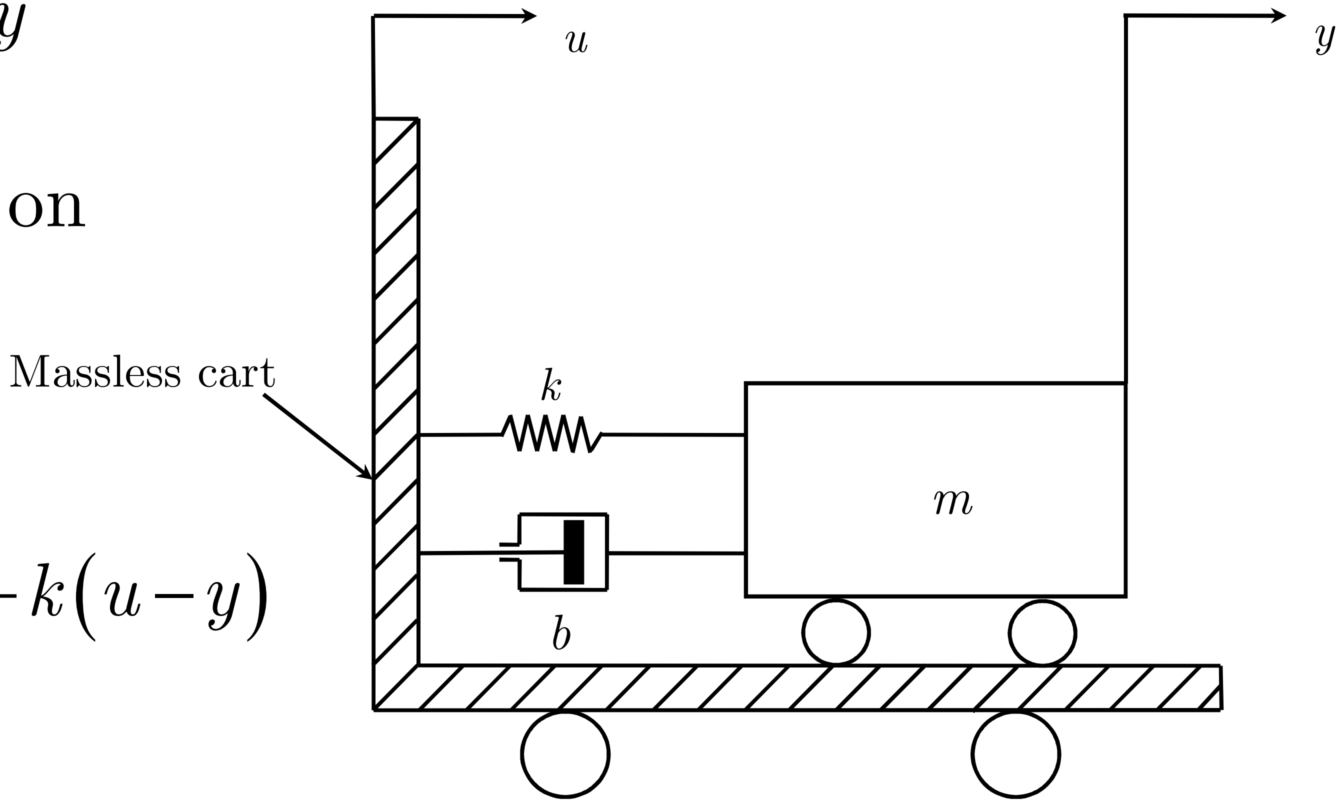
$$= \frac{b_1 b_2}{(b_1 + b_2)}(\dot{y} - \dot{x})$$

Hence, $$b_{eq} = \frac{b_1 b_2}{(b_1 + b_2)} = \frac{1}{\frac{1}{b_1} + \frac{1}{b_2}}$$

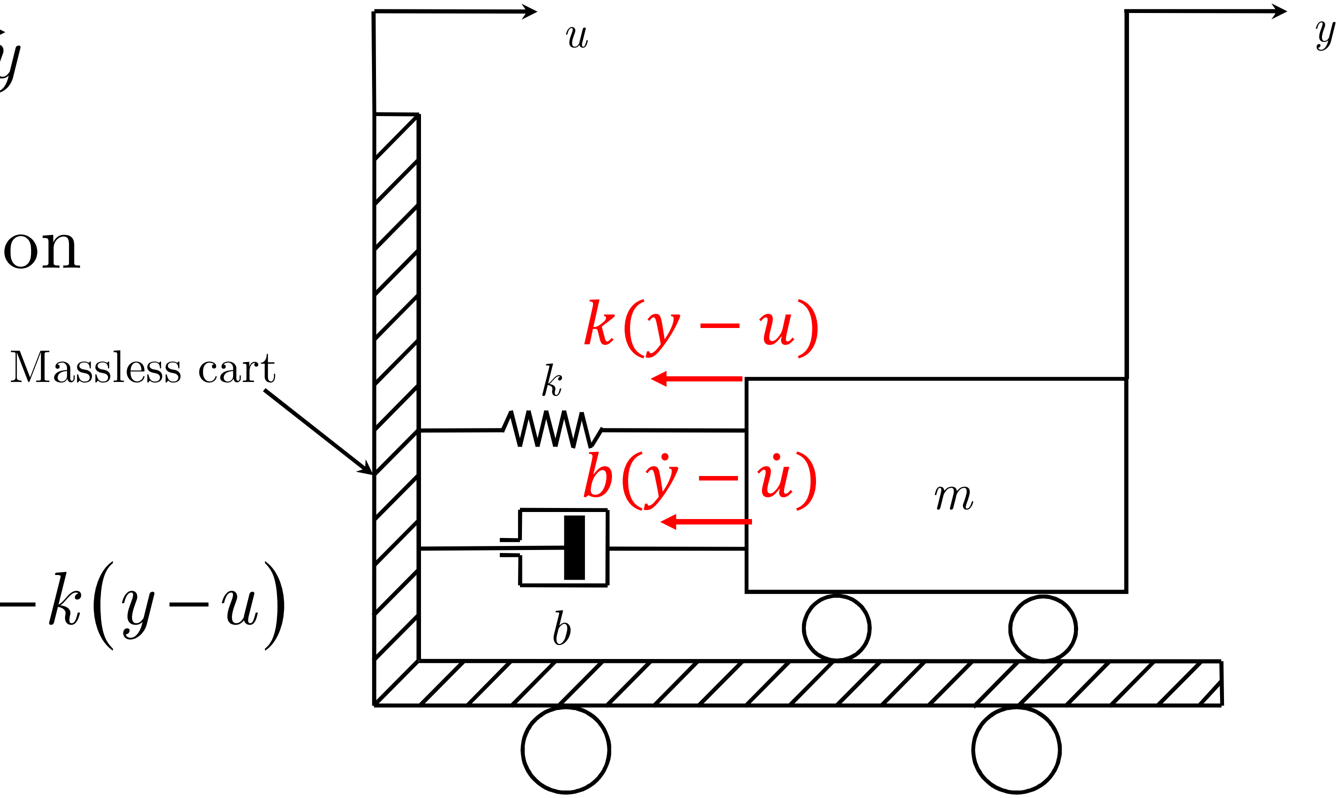
Example. Consider the spring-mass-dashpot system mounted on a *massless cart*. Let u, the displacement of the cart, be the input, and y, the displacement of the mass, be the output. Obtain the mathematical model of the SMD system.

Force acting on the mass:

$$b\left(\frac{du}{dt}-\frac{dy}{dt}\right)+k\left(u-y\right)$$



Example. Consider the spring-mass-dashpot system mounted on a *massless cart*. Let u, the displacement of the cart, be the input, and y, the displacement of the mass, be the output. Obtain the mathematical model of the SMD system.



Force acting on the mass:

$$-b\left(\frac{dy}{dt}-\frac{du}{dt}\right)-k\left(y-u\right)$$

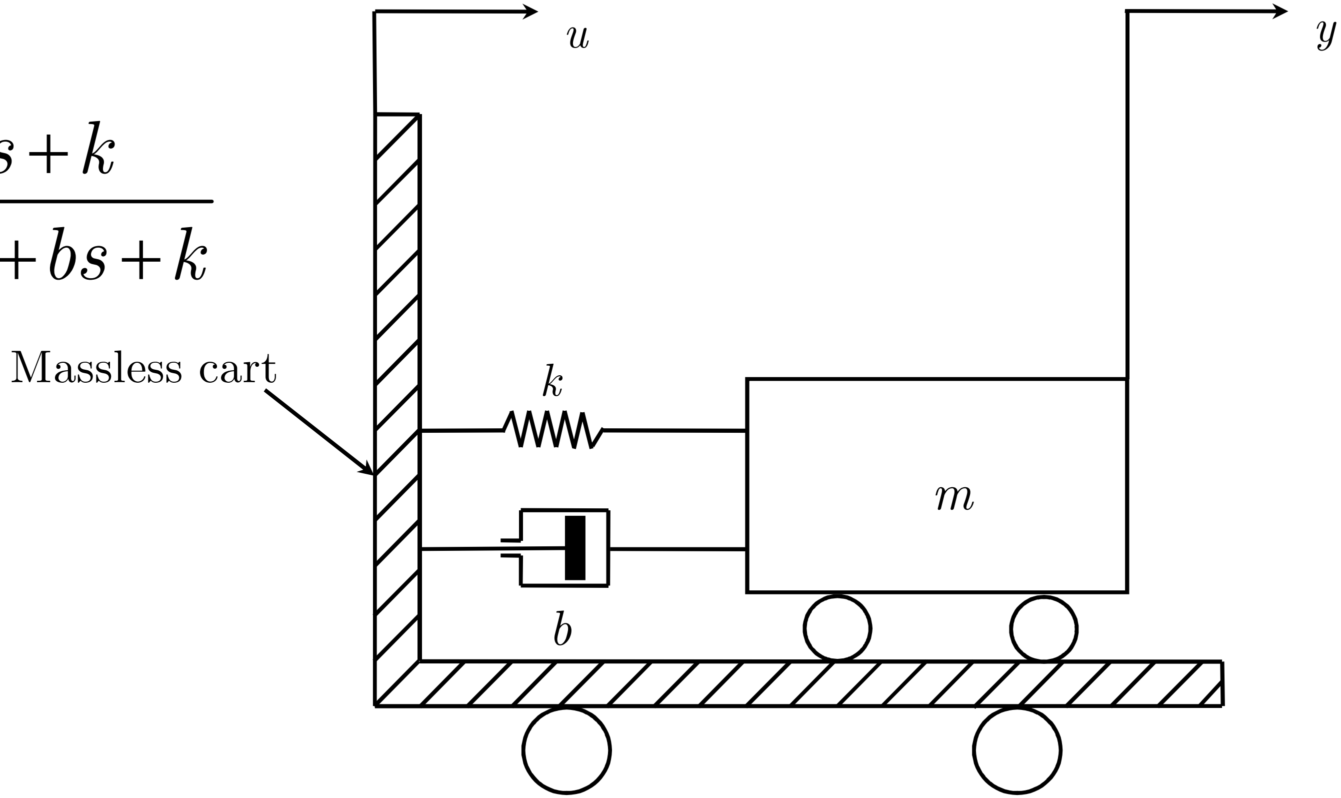
By Newton's law:

$$m\frac{d^2y}{dt}=-b\left(\frac{dy}{dt}-\frac{du}{dt}\right)-k\left(y-u\right)$$

$$\Rightarrow m\frac{d^2y}{dt}+b\frac{dy}{dt}+ky=b\frac{du}{dt}+ku$$

Therefore, the system transfer function can be obtained as

$$\frac{Y(s)}{U(s)}=\frac{bs+k}{ms^2+bs+k}$$

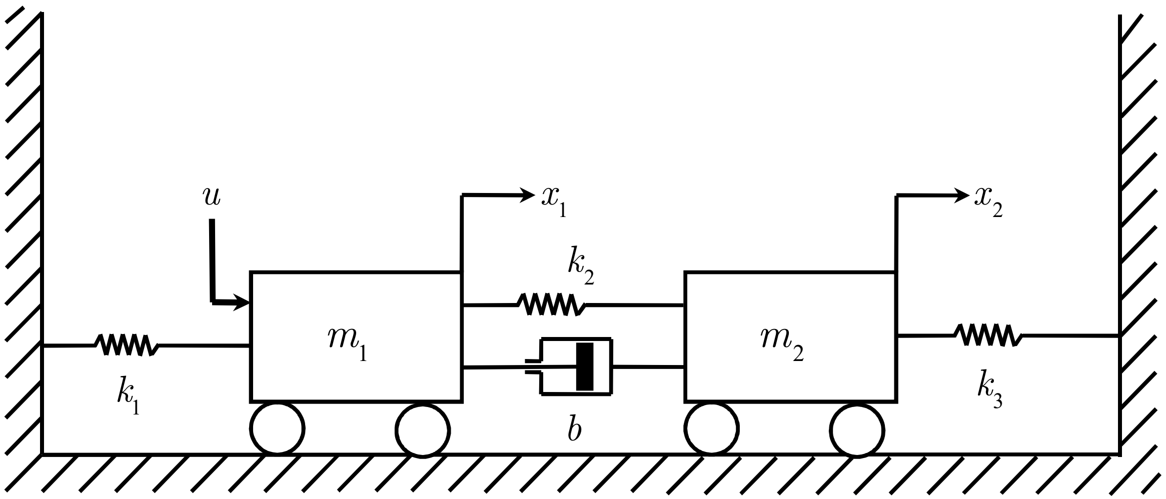


Example. Mechanical system is shown below. Obtain the transfer functions $X_1(s)/U(s)$ and $X_2(s)/U(s)$, where u denotes the force, and x_1 and x_2 denote the displacements of the two carts, respectively.

Solution: By Newton's second law, we have

$$m_1\ddot{x}_1 = u - k_1x_1 - k_2(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2)$$

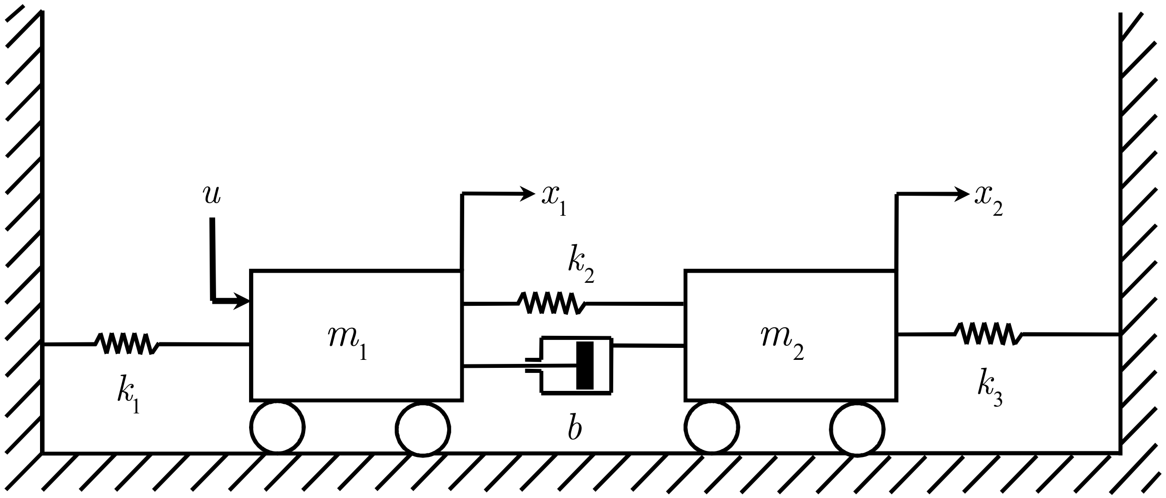
$$m_2\ddot{x}_2 = -k_3x_2 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$



Simplifying, we obtain

$$m_1\ddot{x}_1 + b\dot{x}_1 + (k_1 + k_2)x_1 = u + b\dot{x}_2 + k_2 x_2$$

$$m_2\ddot{x}_2 + b\dot{x}_2 + (k_2 + k_3)x_2 = b\dot{x}_1 + k_2 x_1$$



Taking the Laplace transforms of these two equations, and assuming zero initial conditions,

$$[m_1 s^2 + bs + (k_1 + k_2)]X_1(s) = U + (bs + k_2)X_2(s) \quad (1)$$

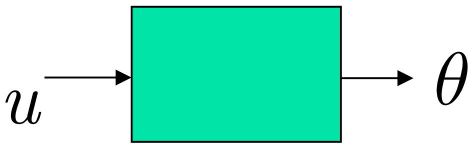
$$[m_2 s^2 + bs + (k_2 + k_3)]X_2(s) = (bs + k_2)X_1(s) \quad (2)$$

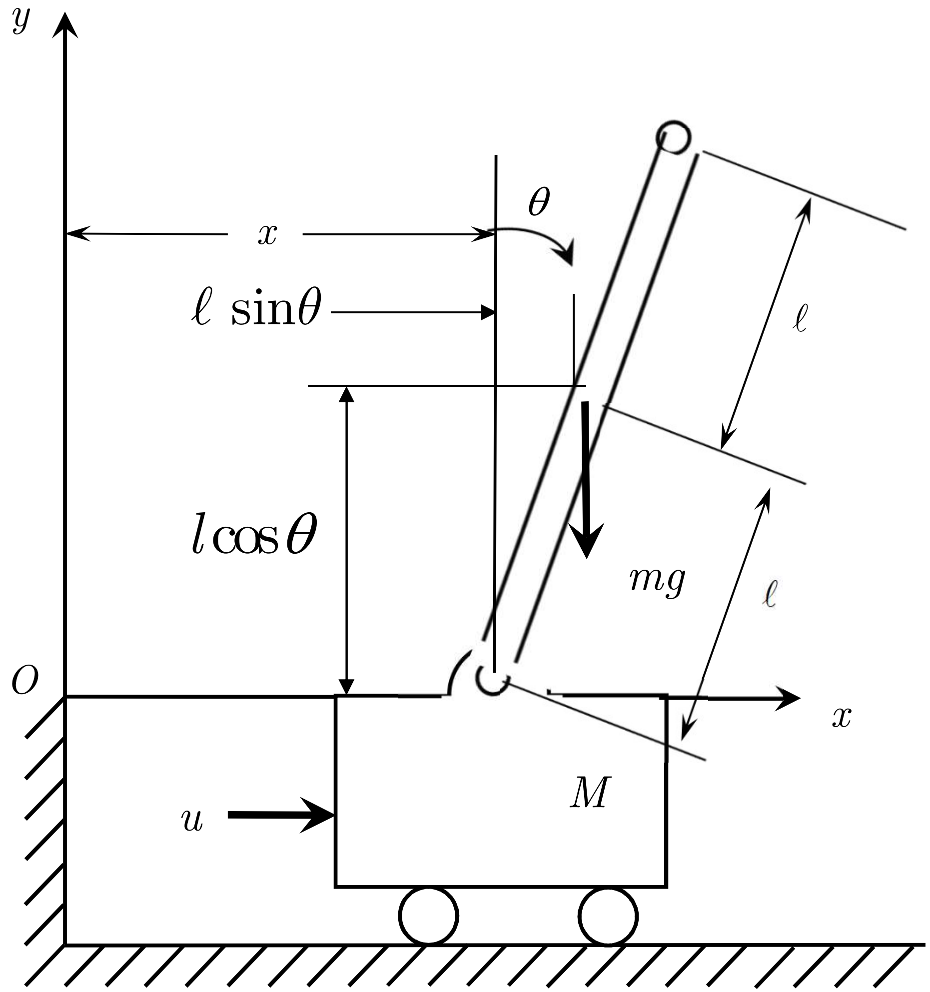
Solving (2) for $X_2(s)$ and substituting it into (1) and simplifying, we get

$$\frac{X_1(s)}{U(s)} =$$

$$\frac{[m_2 s^2 + bs + (k_2 + k_3)]}{[m_1 s^2 + bs + (k_1 + k_2)][m_2 s^2 + bs + (k_2 + k_3)] - (bs + k_2)^2}$$

Example. An inverted pendulum mounted on a motor-drive cart is shown below. This is a model of attitude control of a space booster on takeoff; that is, the objective of the attitude control problem is to keep the space booster in a vertical position.





Define:

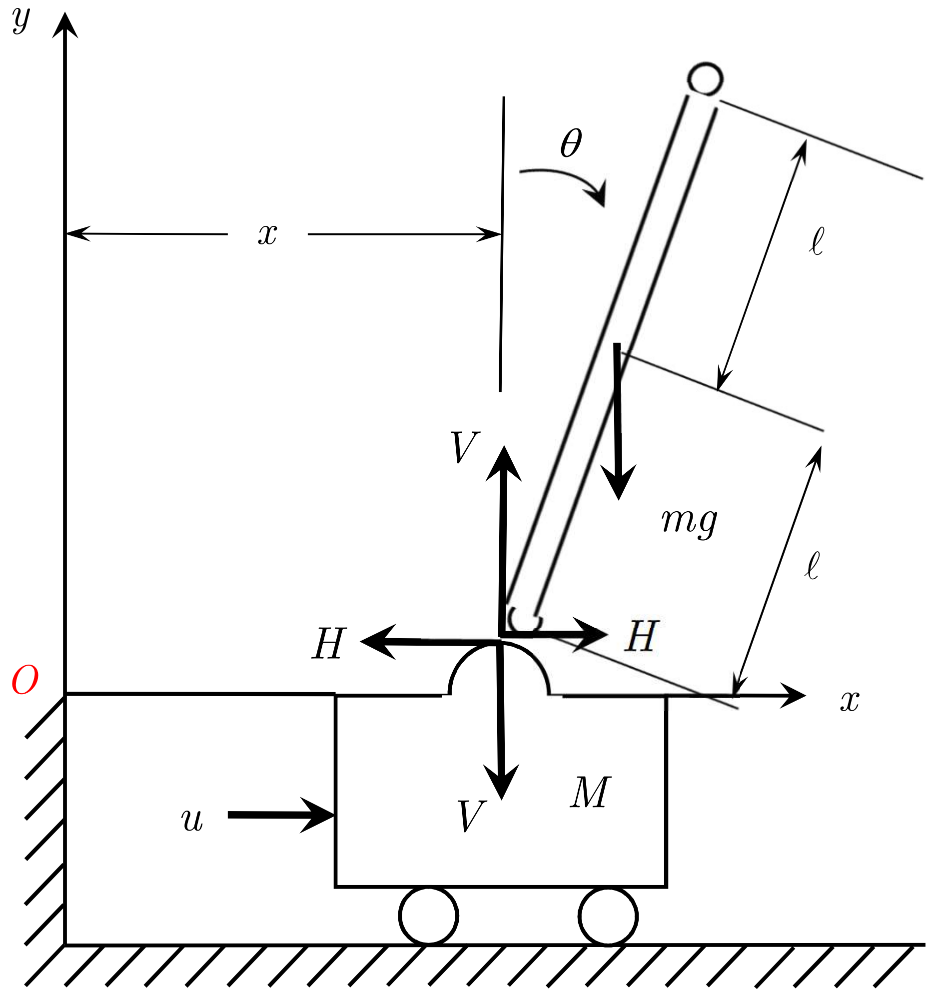
u: The control force to the cart;

θ: The angle of the rod from the vertical line;

$(x_G,\ y_G)$: The $(x,\ y)$ coordinates of the center of gravity of the rod. Hence,

$$x_G = x + l\sin\theta$$

$$y_G = l\cos\theta$$



1) The horizontal motion of cart is described by

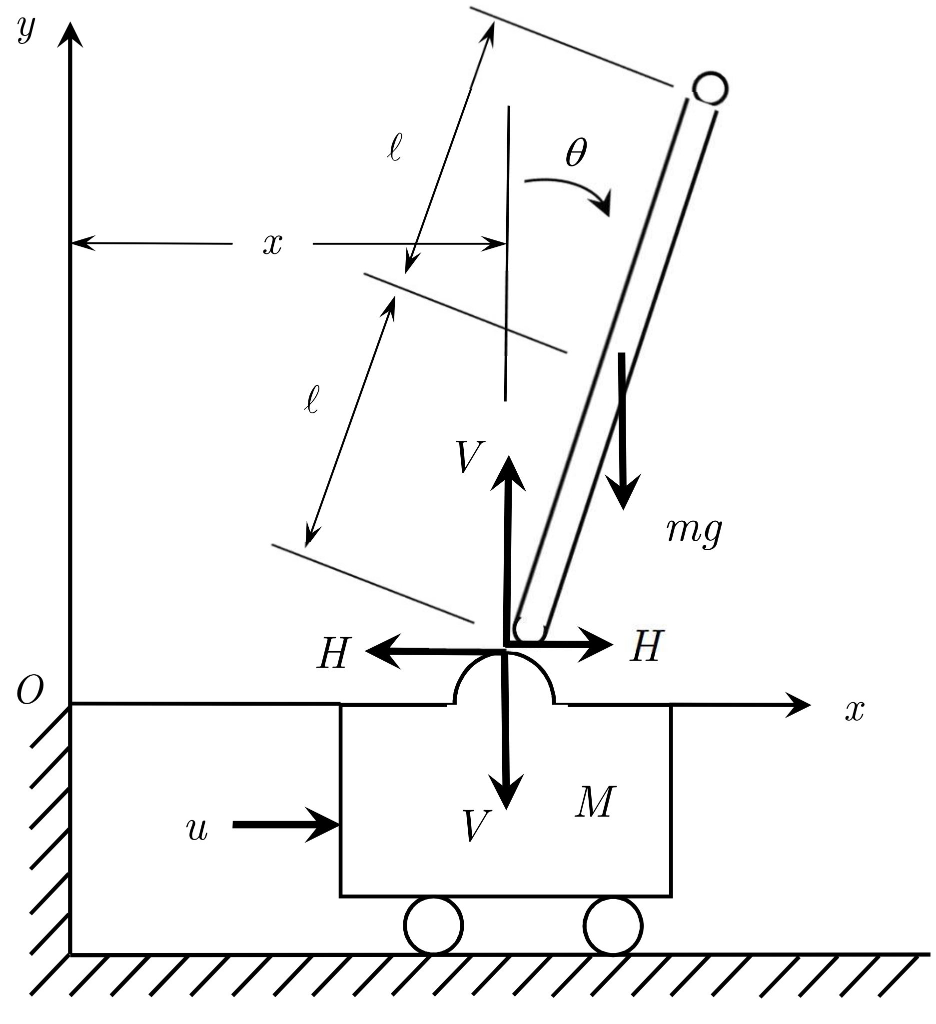
$$M\frac{d^2x}{dt^2} = u - H \quad (1)$$

2)The horizontal motion of center of gravity of the rod is

$$m\frac{d^2}{dt^2}(x + l\sin\theta) = H \quad (2)$$

3)The vertical motion of center of gravity of the pendulum rod is

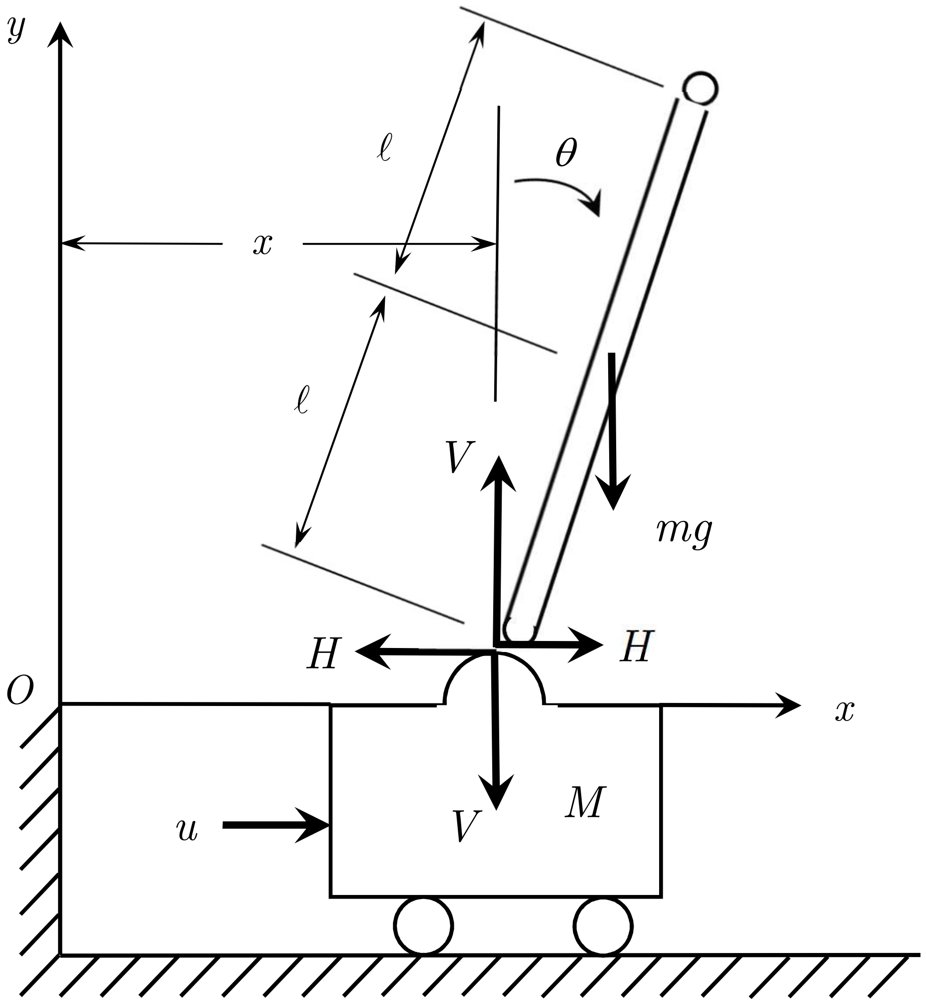
$$m\frac{d^2}{dt^2}(l\cos\theta) = V - mg \quad (3)$$



4) The rotational motion of the rod about its center of gravity can be described by

$$I\ddot{\theta} = V(l\sin\theta) - H(l\cos\theta) \quad (4)$$

where I is the moment of inertia of the rod about its center of gravity.



5) Linearization of the equations (2)-(4):

Since we must keep the inverted pendulum vertical, we can assume that θ and $d\theta/dt$ are small quantities so that $\sin\theta\approx\theta$, $\cos\theta\approx 1$ and $\theta(d\theta/dt)^2\approx 0$. Hence, the equations (2)-(4) can be linearized as

$$\begin{cases} m(\ddot{x}+l\ddot{\theta})=H & (2) \\ 0=V-mg & (3) \\ I\ddot{\theta}=V\theta l-Hl & (4) \end{cases} \qquad + \qquad M\ddot{x}=u-H \quad (1)$$

By replacing (1) into (2) yields

$$(M+m)\ddot{x}+ml\ddot{\theta}=u \quad (5)$$

From equations (2)-(4), we have

$$I\ddot{\theta}=mgl\theta-Hl=mgl\theta-l(m\ddot{x}+ml\ddot{\theta})$$

$$\Rightarrow (I+ml^2)\ddot{\theta}+lm\ddot{x}=mgl\theta \quad (6)$$

From (6), we have

$$\ddot{x} = \frac{mgl\theta - (I + ml^2)\ddot{\theta}}{lm} \quad (7)$$

Substituting (7) into (5) yields

$$(M+m)\frac{mgl\theta - (I + ml^2)\ddot{\theta}}{lm} + ml\ddot{\theta} = u \quad (8)$$

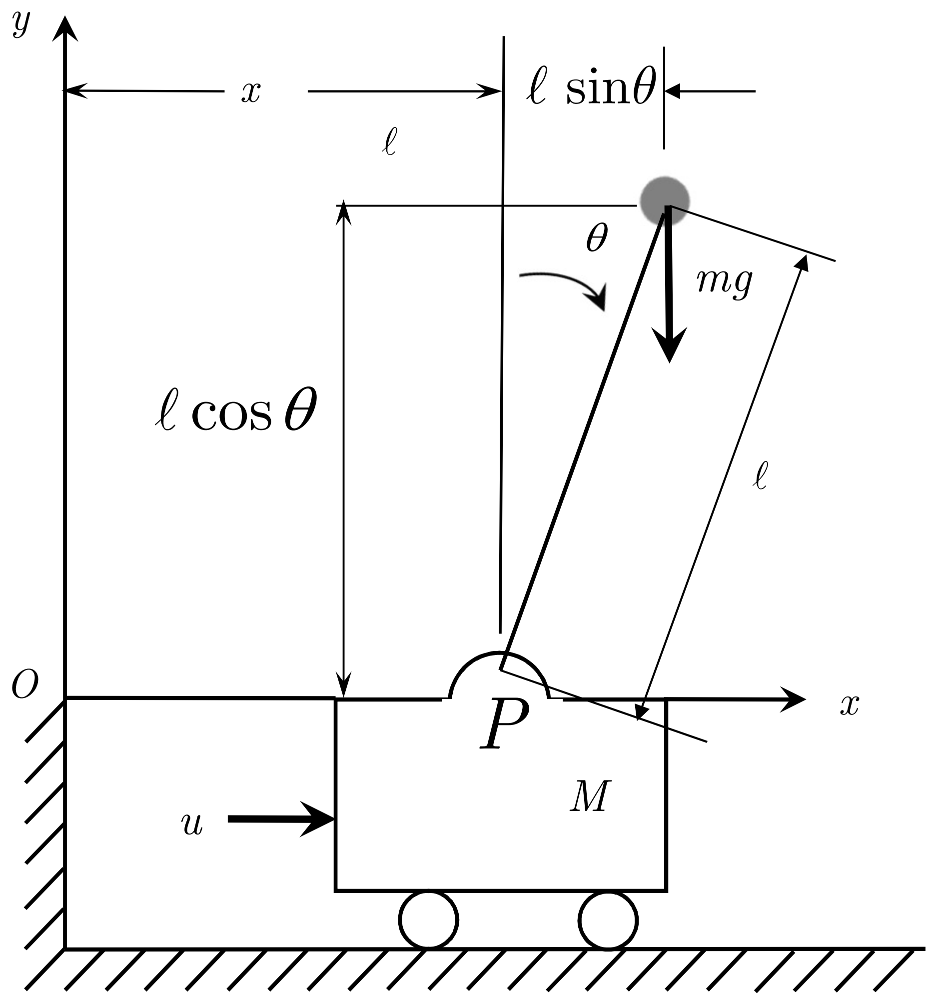
Taking the Laplace transform on both sides of (8) yields

$$(M+m)\frac{mgl\Theta - (I + ml^2)s^2\Theta}{lm} + mls^2\Theta = U(s) \quad (9)$$

Therefore,

$$\frac{\Theta(s)}{U(s)} = \frac{lm}{\left[(M+m)mgl - (MI + mMl^2 + mI)s^2\right]}$$

Example. An inverted pendulum is shown in the figure below. Similar to the previous example:

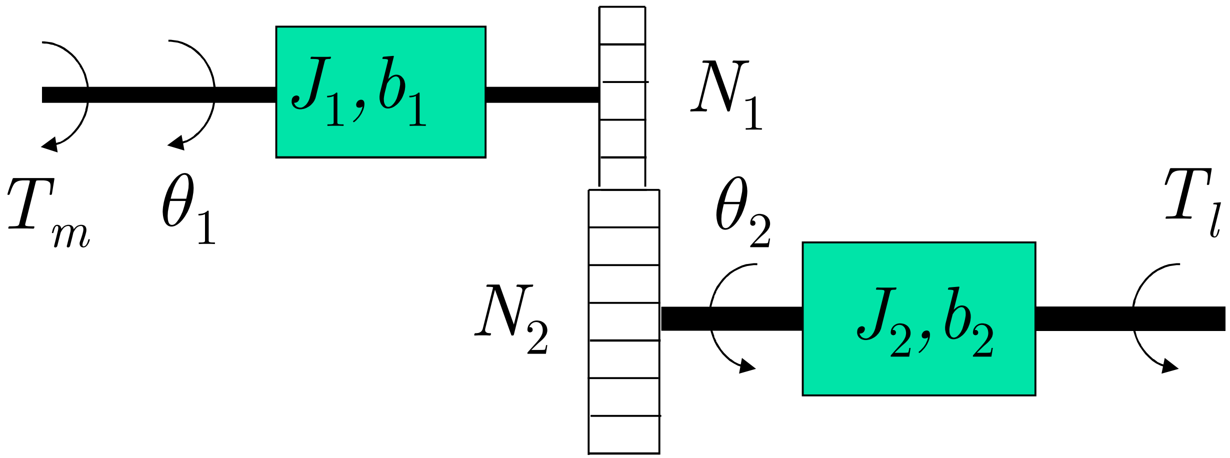


$$\frac{\Theta(s)}{U(s)} = \frac{lm}{\left[(M+m)mgl-(MI+mMl^2+mI)s^2\right]}$$

For this case, the moment of inertia of the pendulum about its center of gravity is small, and we assume $I = 0$. Then the mathematical model for this system becomes:

$$\frac{\Theta(s)}{U(s)} = \frac{1}{\left[(M+m)g - Mls^2\right]}$$

Example (see p.232 for detail). Gear Train: Rotational transformer. In servo control systems, the motor drives a load through gear train.

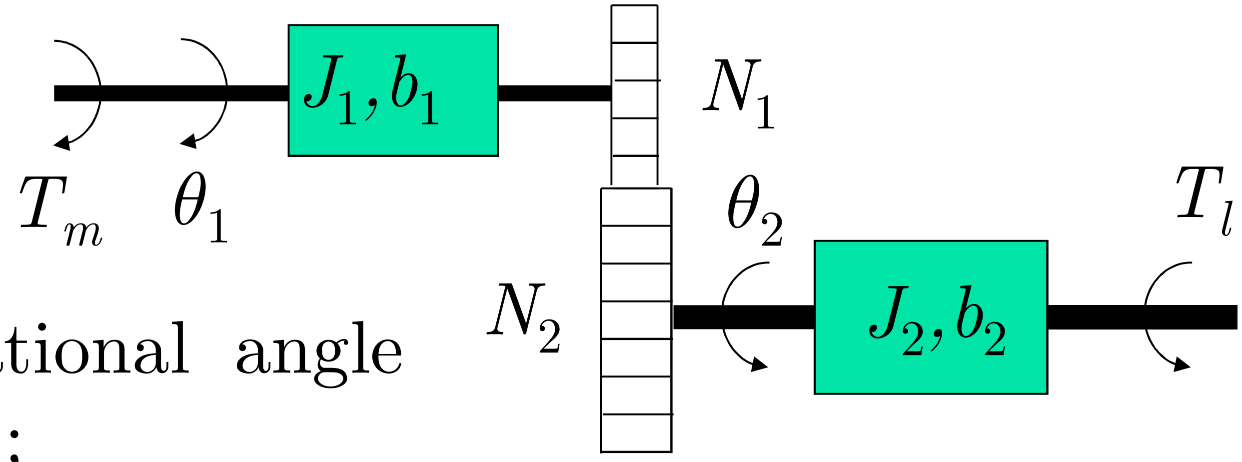


Define:

T_m: the torque developed by the motor;

T_l: the load torque;

$J_1(J_2), b_1(b_2)$: The moment of inertia and viscous-friction coefficient of each gear-train component;



$\theta_{1,}$ θ_2: the rotational angle of gears 1 and 2;

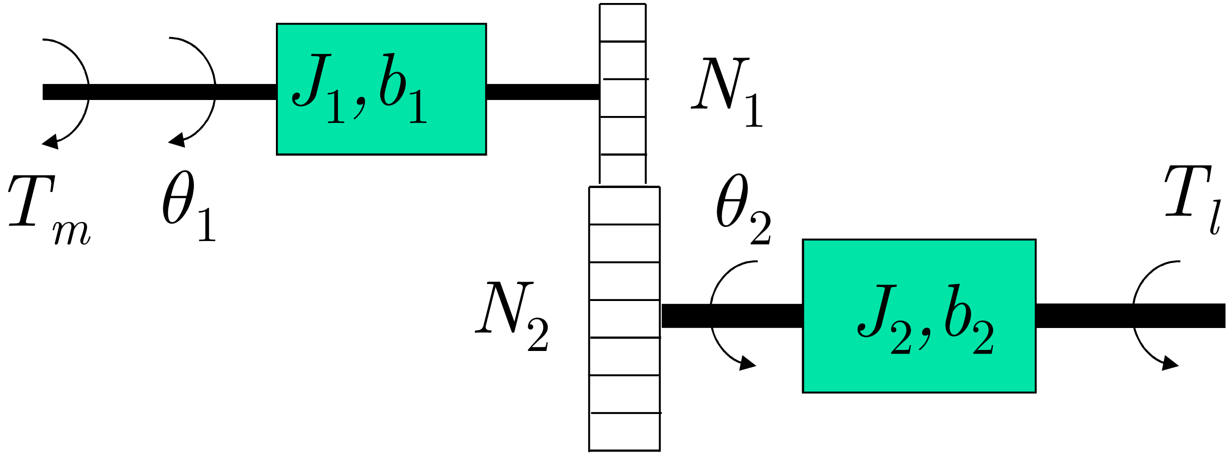
$N_{1,}$ N_2: the numbers of teeth on gears 1 and 2.

Gear ratio (transmission ratio): $$n = \frac{N_1}{N_2} = \frac{\theta_2}{\theta_1} = \frac{\omega_2}{\omega_1}$$

1) For Shaft 1

$$J_1\ddot{\theta}_1 = T_m - T_1 - b_1\dot{\theta}_1 \quad (1)$$

where T_1 is the load torque on gear 1 due to rest of the gear train.

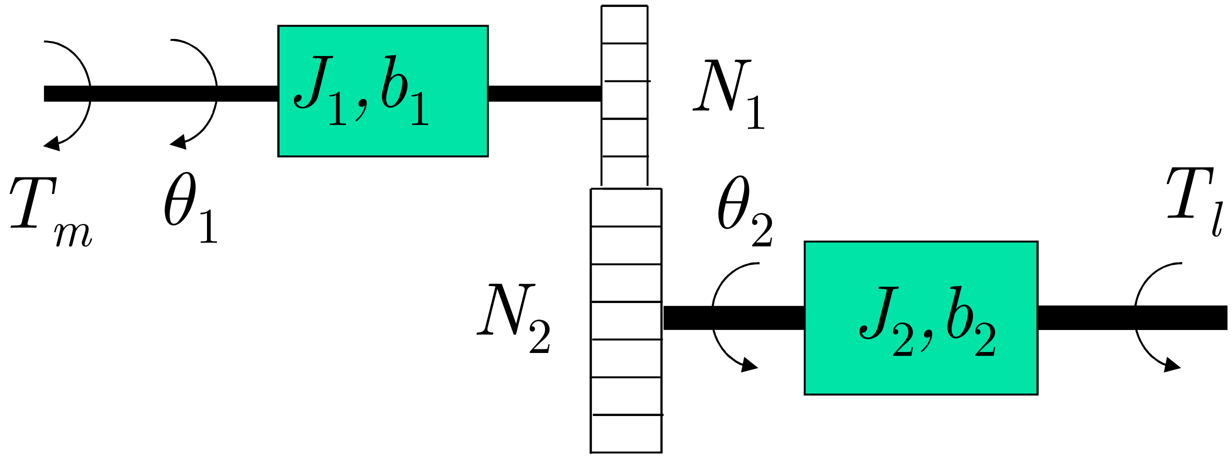


2) For Shaft 2

$$J_2\ddot{\theta}_2 = T_2 - T_l - b_2\dot{\theta}_2 \quad (2)$$

where T_2 is torque transmitted to gear 2. Since work done by gear 1 is equal to that of gear 2,

$$T_1\theta_1 = T_2\theta_2 \Rightarrow \frac{T_1}{T_2} = \frac{N_1}{N_2} \quad (3)$$



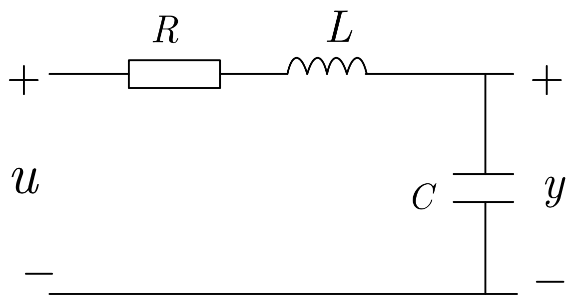
3) Eliminating T_1, T_2 , θ_2 and writing the resulting equation in terms of θ_1 and its time derivatives, we obtain

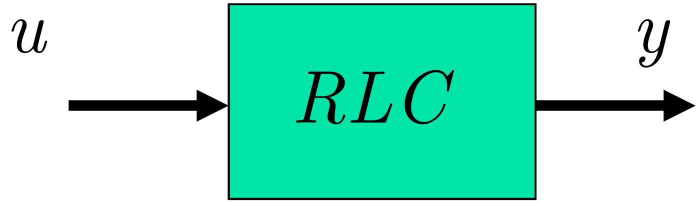
$$\left(J_1 + J_2 \frac{N_1^{\ 2}}{N_2^{\ 2}}\right)\ddot{\theta}_1 + \left(b_1 + b_2 \frac{N_1^{\ 2}}{N_2^{\ 2}}\right)\dot{\theta}_1 + \frac{N_1}{N_2} T_l = T_m \quad (4)$$

3-2 Mathematical modeling of electrical systems

1. RLC circuit

Example. RLC circuit





By using Kirchihoff's law, we have

$$u = iR + y + v_L$$

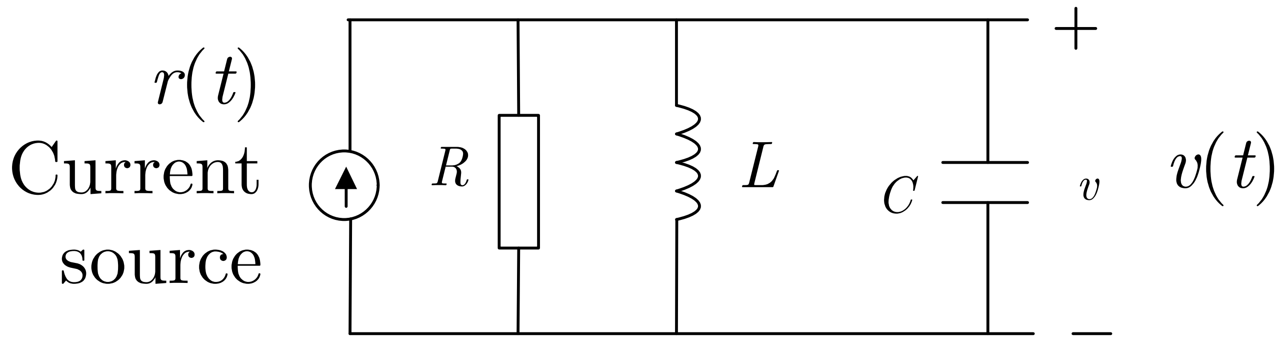
Since

$$i = C\frac{dy}{dt}, \quad v_L = L\frac{di}{dt} = LC\frac{d^2y}{dt^2}$$

we have

$$LC\frac{d^2y}{dt^2} + RC\frac{dy}{dt} + y = u$$

Example. RLC circuit. Find $V(s)/R(s)$.



$$\frac{v(t)}{R} + C\,\frac{dv(t)}{dt} + \frac{1}{L}\int_0^t v(t)dt = r(t)$$

Taking the Laplace transform of the above equation yields

$$\frac{V(s)}{R} + CsV(s) + \frac{1}{Ls}V(s) = R(s)$$

Hence,

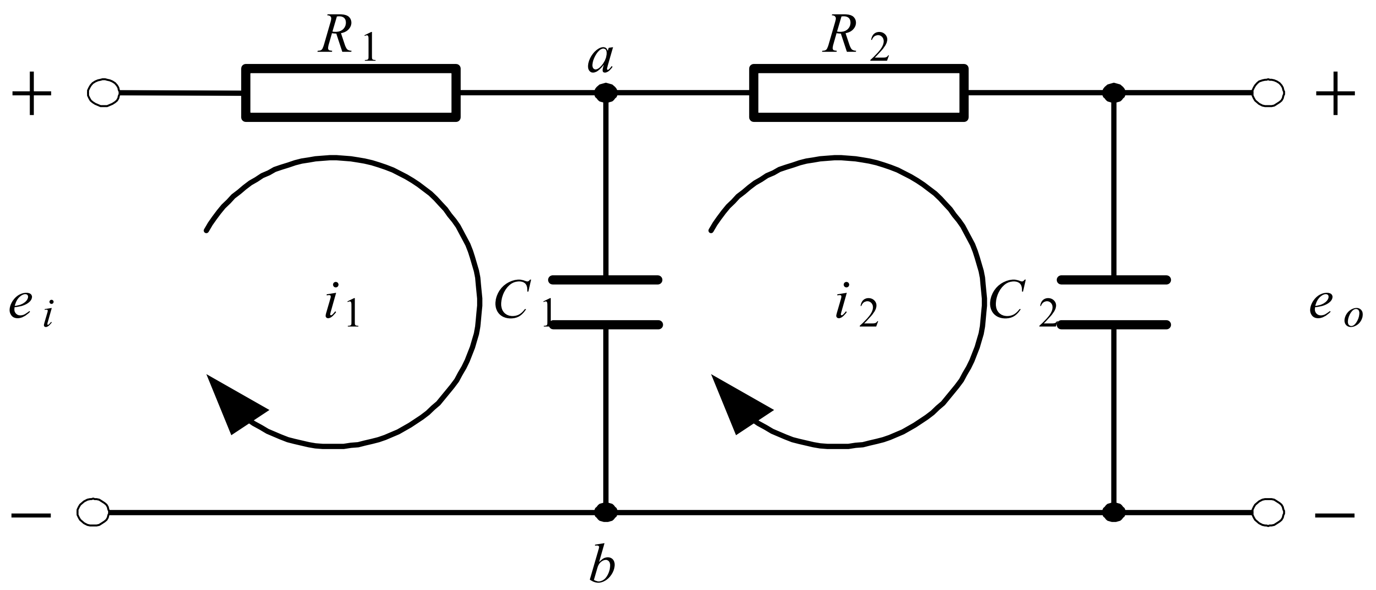
$$V(s)(\frac{1}{R} + Cs + \frac{1}{Ls}) = R(s)$$

$$\Rightarrow \frac{V(s)}{R(s)} = \frac{1}{(\frac{1}{R} + Cs + \frac{1}{Ls})}$$

$$= \frac{RLs}{RLCs^2 + Ls + R}$$

2. Transfer functions of cascaded elements

Example. Find $E_o(s)/E_i(s)$.



Note that in this circuit, the second portion (R_2C_2) produces a *loading effect* on the first stage (R_1C_1 portion); that is , we cannot obtain the transfer function as we did for transfer functions in cascade.

$$\begin{cases} e_i = i_1 R_1 + u_{c_1} \\ C_1 \dfrac{du_{c_1}}{dt} = i_1 - i_2 \Rightarrow u_{c_1} = \dfrac{1}{C_1}\int (i_1 - i_2)dt \end{cases}$$

and

$$\begin{cases} e_o + i_2 R_2 = \underbrace{\dfrac{1}{C_1}\int (i_1 - i_2)dt}_{u_{c1}} \\ e_o = \dfrac{1}{C_2}\int i_2 dt \end{cases}$$

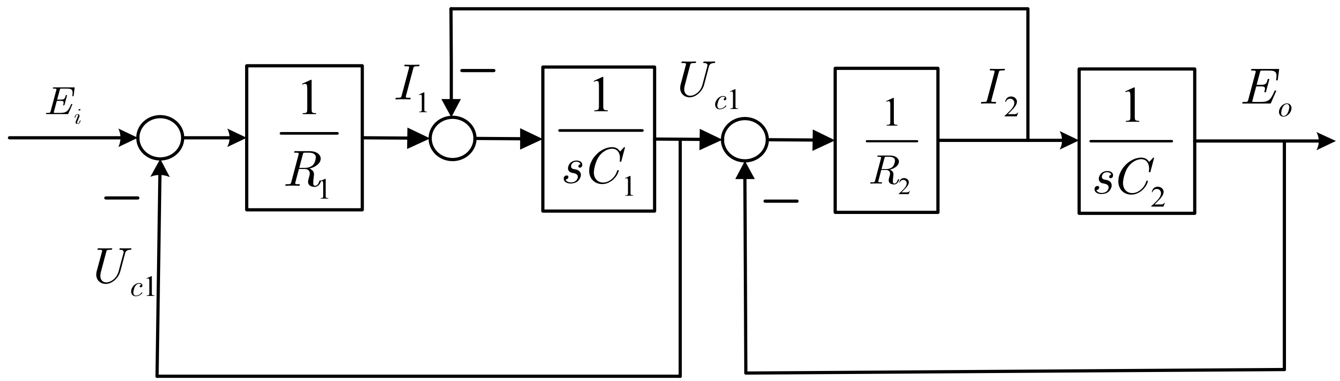
Taking the Laplace transforms of the above equations, we obtain

$$I_1 = \frac{1}{R_1}[E_i - U_{c1}]$$

$$U_{c1} = \frac{1}{C_1 s}[I_1(s) - I_2(s)]$$

$$I_2(s) = \frac{1}{R_2}[U_{c1} - E_o(s)]$$

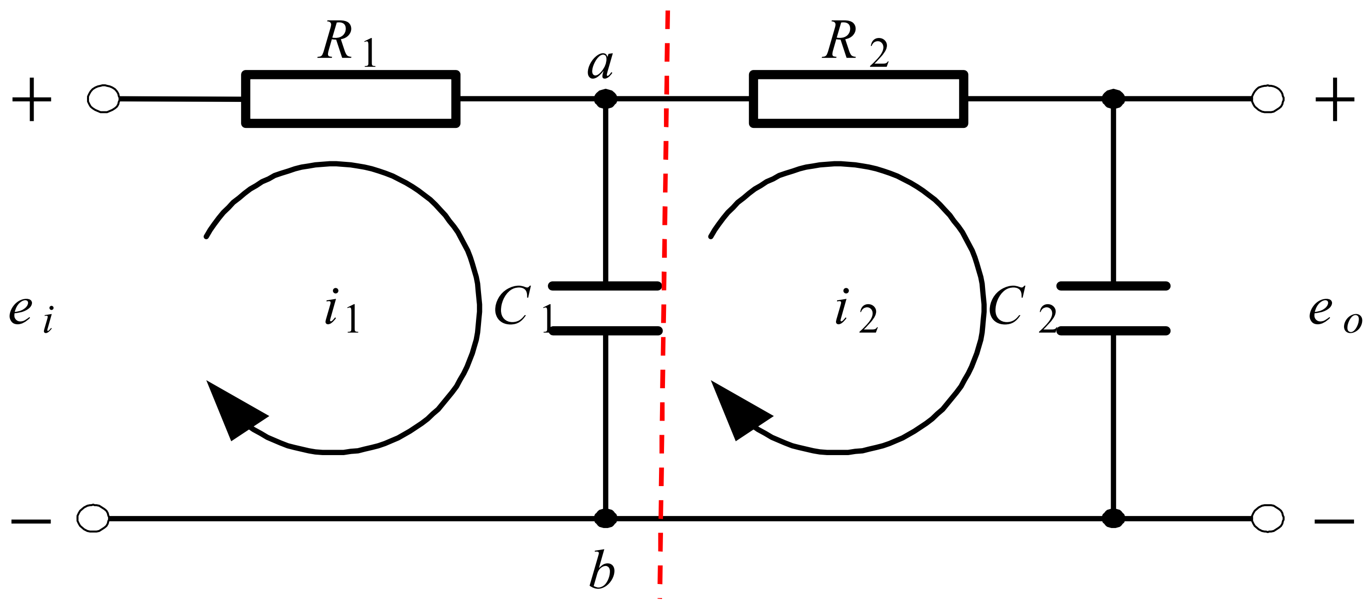
$$E_0(s) = \frac{1}{C_2 s} I_2(s)$$



from which we obtain

$$\frac{E_0(s)}{E_i(s)} = \frac{1}{R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1}$$

The above analysis shows that, if two RC circuits connected in cascade so that the output from the first circuit is the input to the second, the overall transfer function is not the product of $1/(R_1C_1s+1)$ and $1/(R_2C_2s+1)$ due to the ***loading effect*** (certain amount of power is withdrawn).



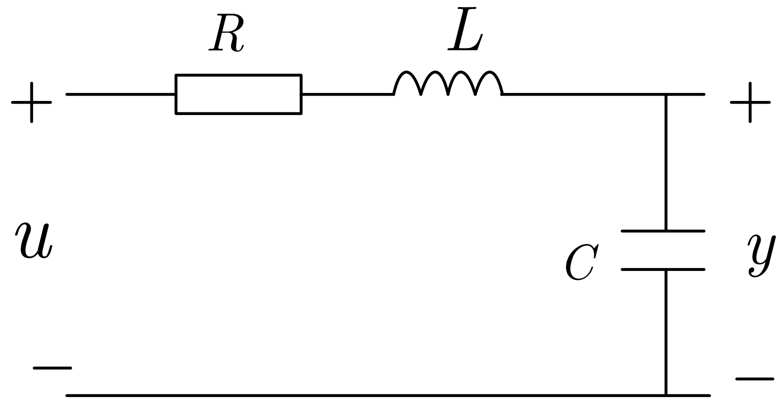
3. Complex impedances

Resistance R: R

Capacitance: $1/Cs$

Inductance: Ls

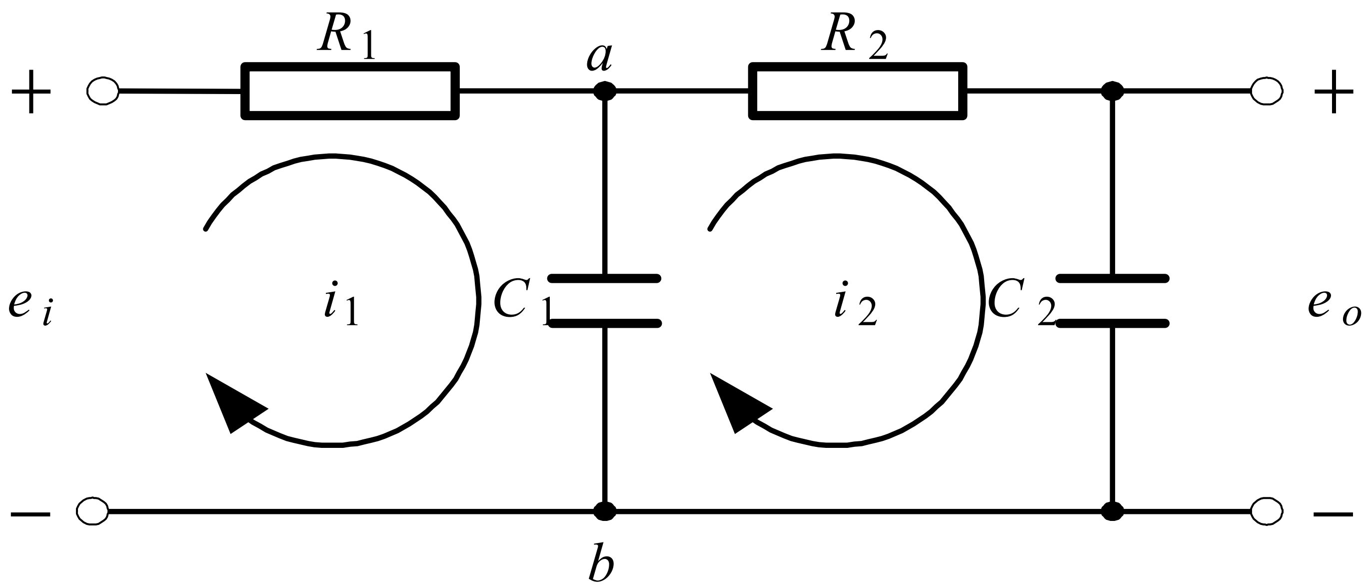
Example. Find $Y(s)/U(s)$



Solution:

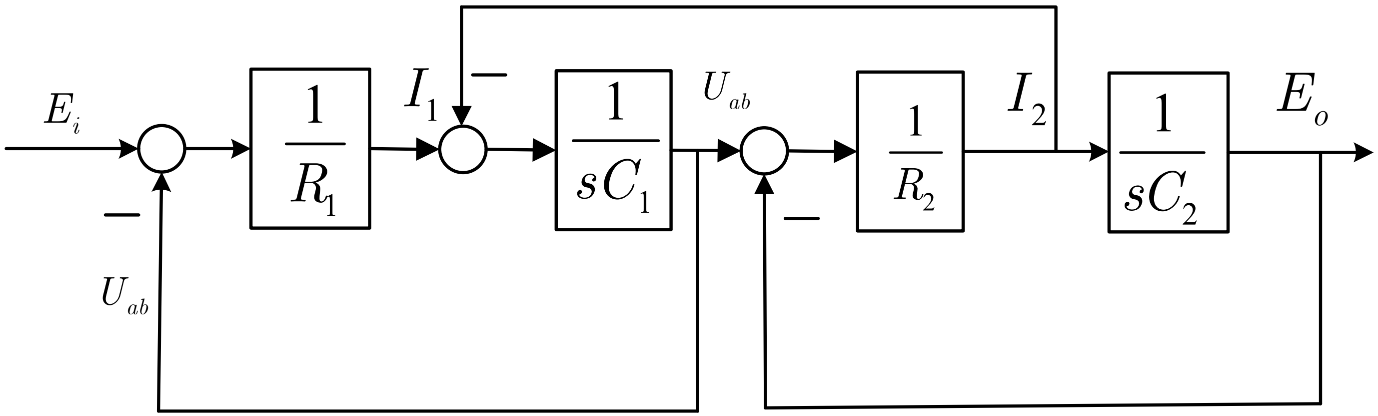
$$\frac{Y(s)}{U(s)} = \frac{1/Cs}{R + Ls + 1/Cs}$$

Example. Find $E_o(s)/E_i(s)$.



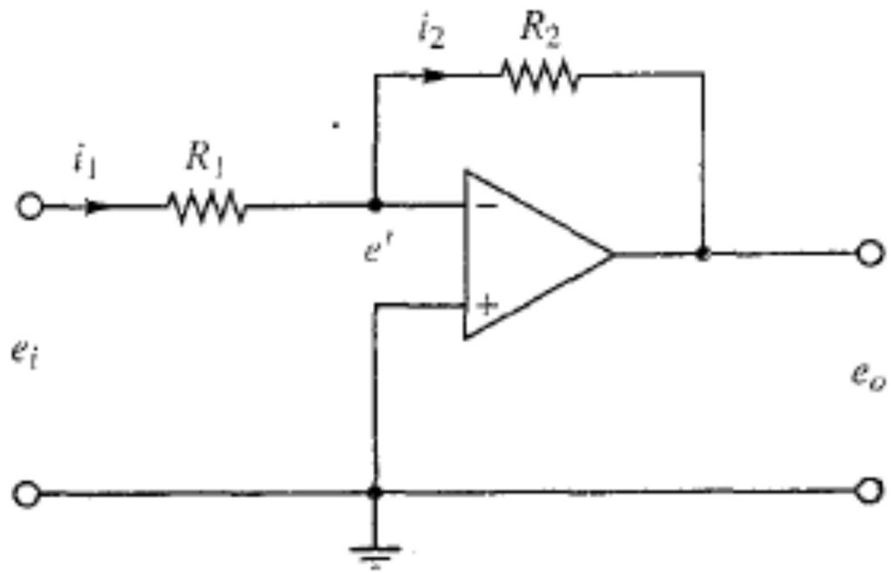
Solution: Utilizing complex impedance approach, we have (from left to right)

$$
\begin{cases}
I_1(s) = \dfrac{E_i(s) - U_{ab}(s)}{R_1} = \dfrac{1}{R_1}[E_i(s) - U_{ab}(s)] \\[2ex]
U_{ab}(s) = \dfrac{1}{sC_1}[I_1(s) - I_2(s)] \\[2ex]
I_2(s) = \dfrac{1}{R_2}[U_{ab}(s) - E_0(s)] \\[2ex]
E_0(s) = \dfrac{1}{sC_2} I_2(s)
\end{cases}
$$

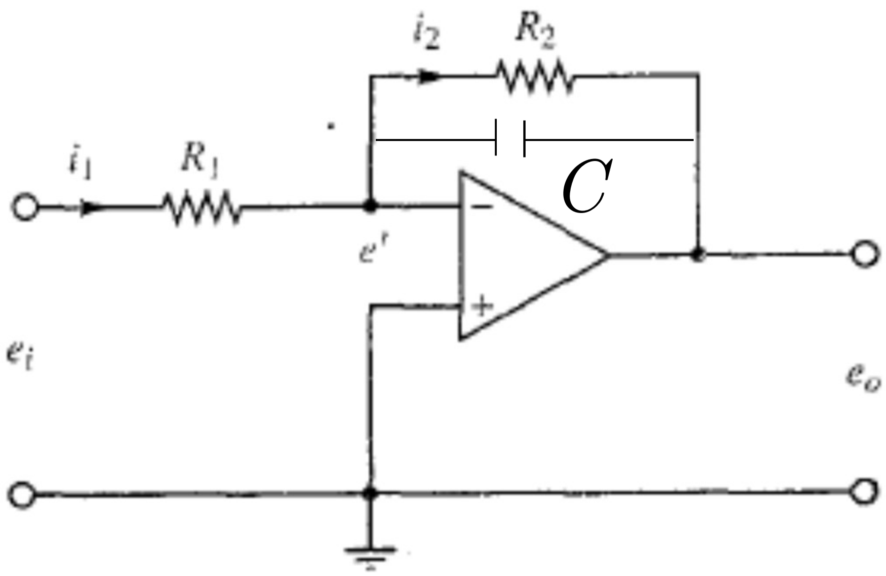


4. Operational amplifiers

In the ideal op amp, no current flows into the input terminals, and the output voltage is not affected by the load connected to the output terminal. In other words, the input impedance is infinite and the output impedance is zero.



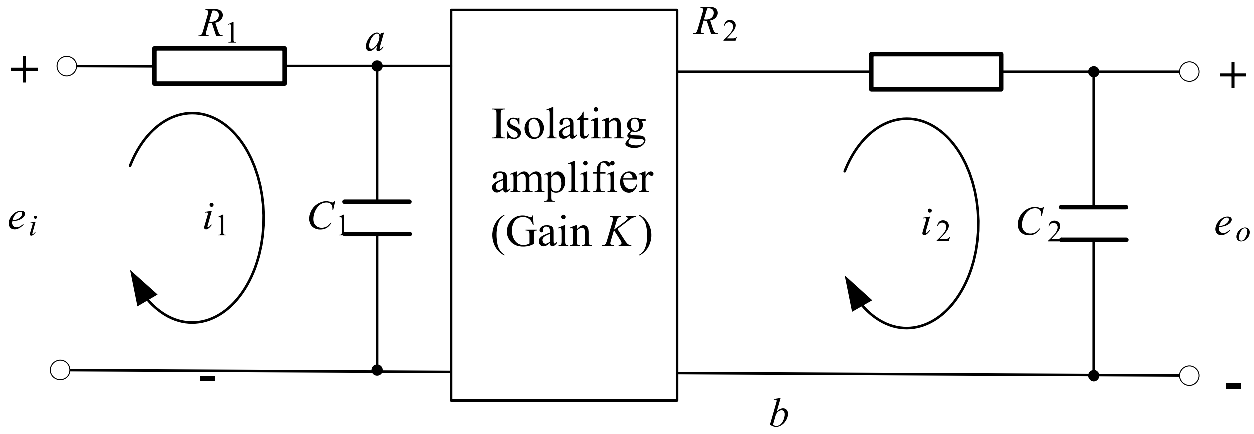
Example. Find $E_o(s)/E_i(s)$.



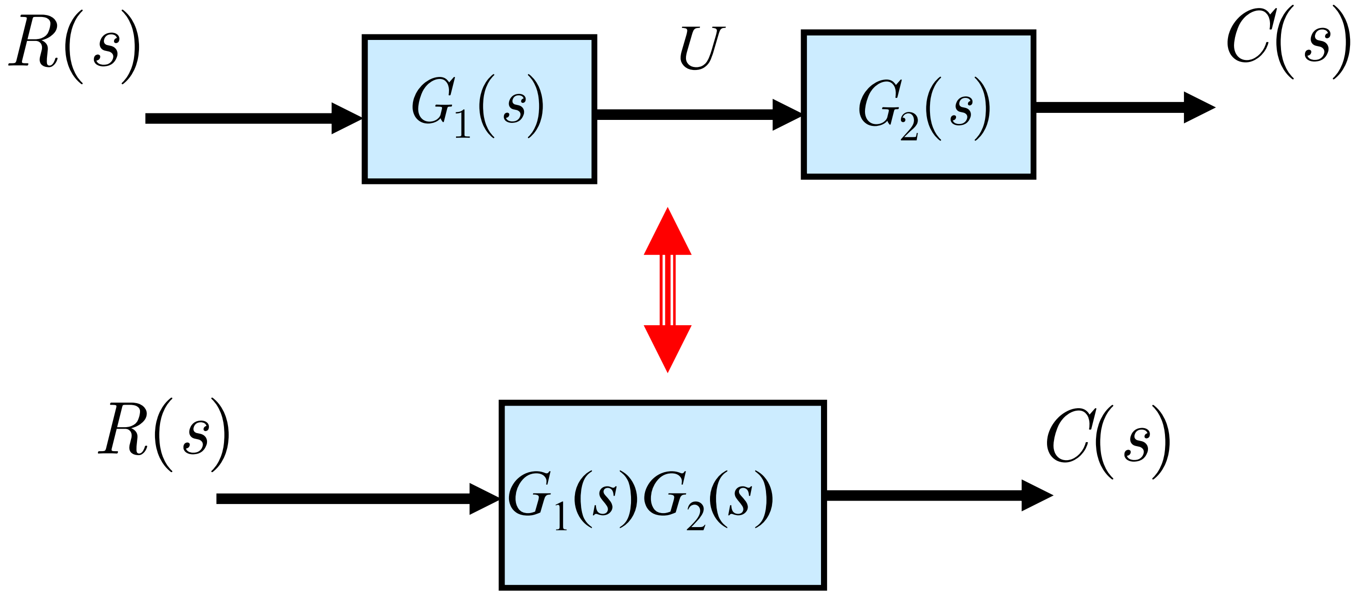
5. Transfer functions of non-loading cascaded elements

Again, consider the two simple RC circuits. Now, the circuits are isolated by an amplifier as shown below and therefore, have negligible loading effects, and the transfer function

$$\frac{E_0(s)}{E_i(s)} = \frac{K}{(R_1C_1s+1)(R_2C_2s+1)}$$



In general, the transfer function of a system consisting of two or more non-loading cascaded elements can be obtained by eliminating the intermediate inputs and outputs:



$$C(s) = G_2(s)U(s) = G_2(s)G_1(s)R(s)$$

6. Examples: DC motor and servo system

Example. Find the transfer function of a DC motor. Assume that the rotor has inertia J_m and viscous friction coefficient f_m. Determine $\Theta_m(s)/U_a(s)$, where θ_m is the rotational angle and u_a is the power supply voltage.

$$U_a(s) \longrightarrow \boxed{\begin{array}{c} DC \\ \text{motor} \end{array}} \longrightarrow \Theta_m(s)$$

Stator winding: 定子绕组;

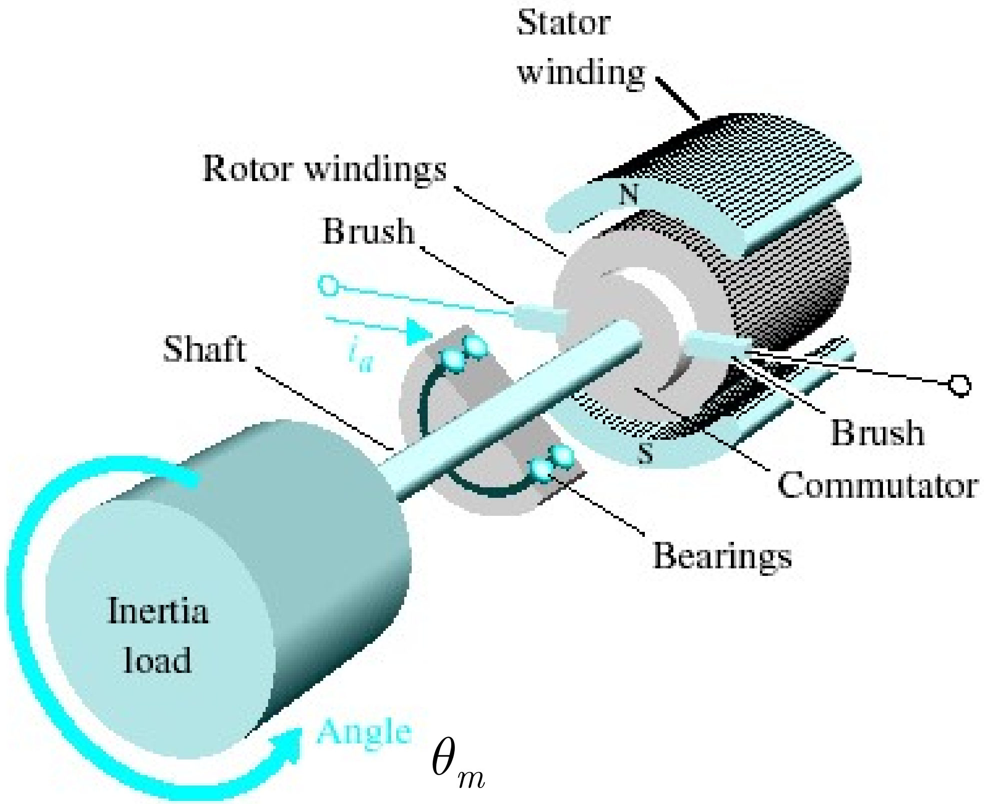
Rotor windings: 转子绕组;

Shaft: 电机轴;

Inertia load: 惯性负载;

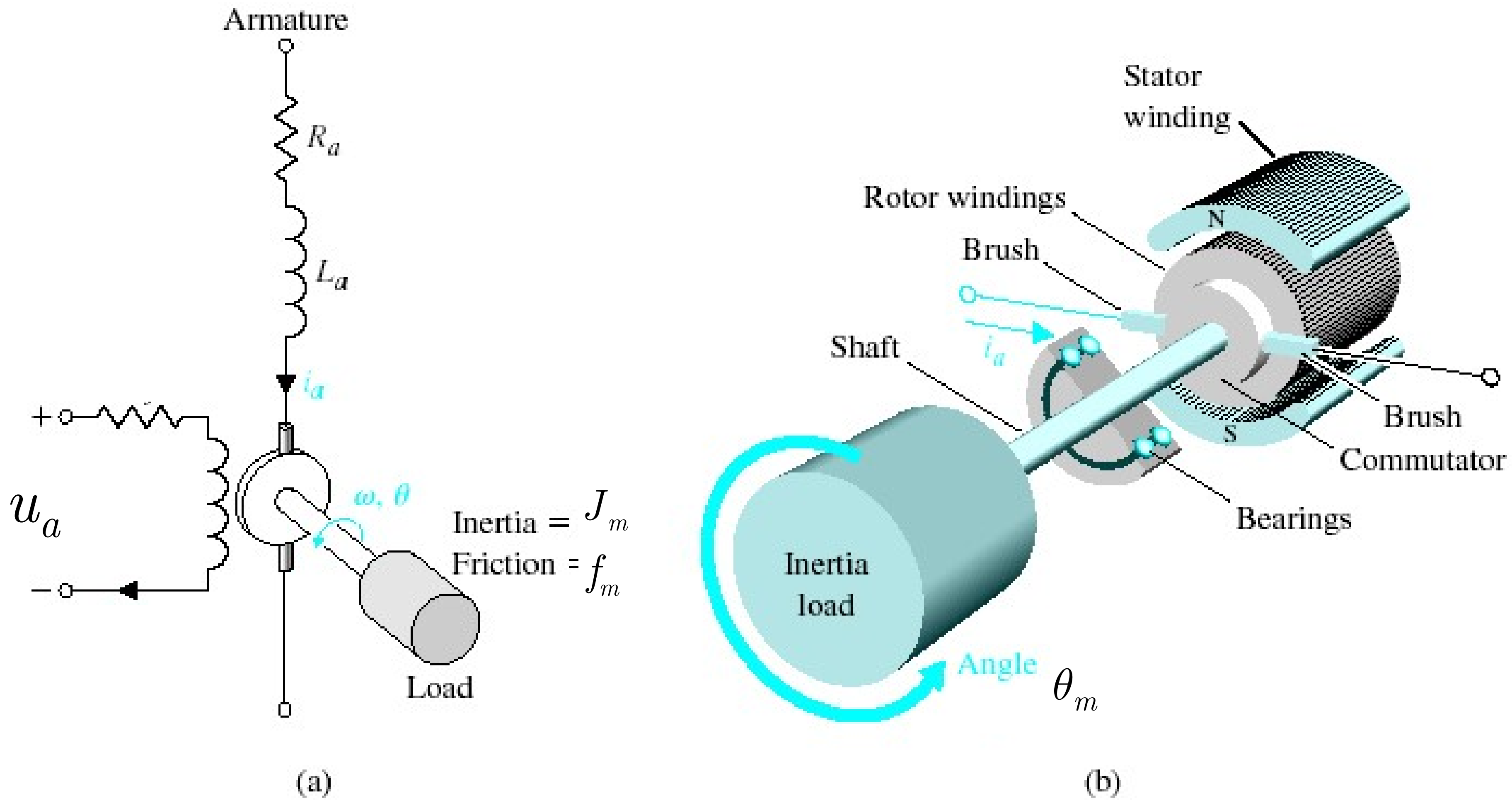
Bearings: 轴承

Commutator：换向器



(b)

Rotor: 转子；Torque: 电磁转矩；Back emf voltage: 反电势； Armature current:电枢电流;



θ_m: rotational angle; u_a: power supply voltage

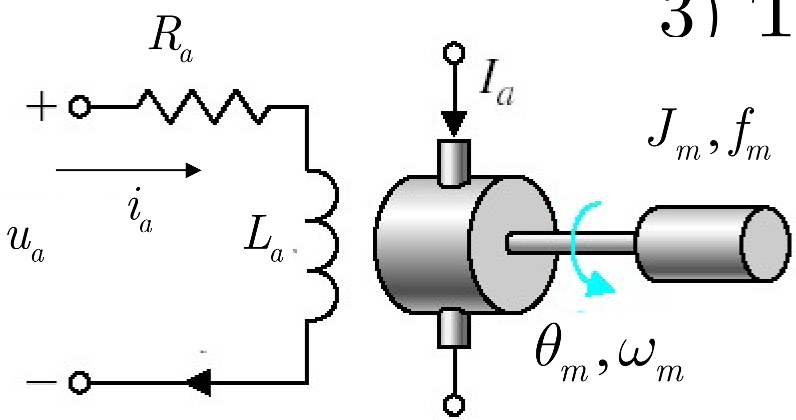
The motor equations:

1) The electromagnetic torque M_m on the rotor in terms of the armature current i_a (C_m: torque constant):

$$M_m = C_m i_a \quad (1)$$

2) The back emf voltage E_b in terms of the shaft's rotational velocity $d\theta_m/dt$ is (K_b : back emf constant):

$$E_b = K_b \dot{\theta}_m \quad (2)$$



3) The electrical equation:

$$u_a = i_a R_a + L_a \frac{di_a}{dt} + E_b \quad (3)$$

4) The torque equation: by using Newton's law,

$$J_m \ddot{\theta}_m = M_m - f_m \dot{\theta}_m \Rightarrow J_m \ddot{\theta}_m + f_m \dot{\theta}_m = M_m \quad (4)$$

Taking the Laplace transform of (3) and in view of (2) yields

$$U_a = I_a R_a + L_a s I_a + K_b s \Theta_m$$

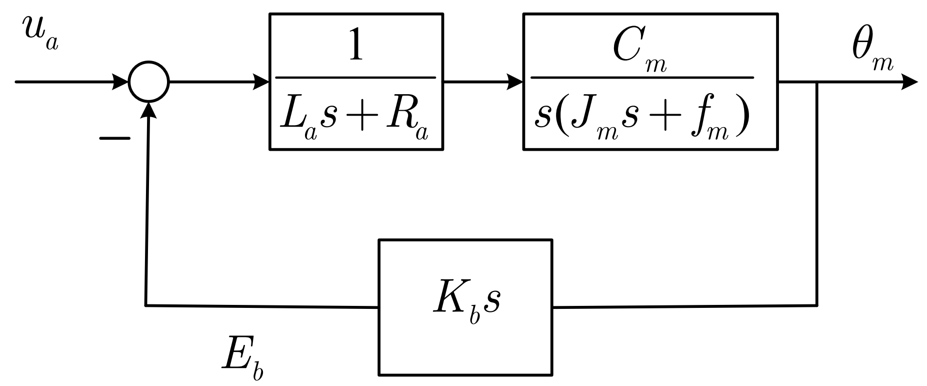
Hence,

$$I_a = \frac{1}{L_a s + R_a} [U_a - K_b s \Theta_m] \quad (5)$$

Taking the Laplace transform of (4) and in view of (1) yields

$$\Theta_m = \frac{1}{J_m s^2 + f_m s} C_m I_a \quad (6)$$

From (5) and (6), we have



$$\frac{\Theta_m}{U_a(s)} = \frac{C_m}{[(L_a s + R_a)(J_m s^2 + f_m s) + C_m K_b s]}$$

In particular, if $L_a \approx 0$, we can simplify the model as

$$\frac{\Theta_m}{U_a(s)} = \frac{K_m}{s(T_m s + 1)}$$

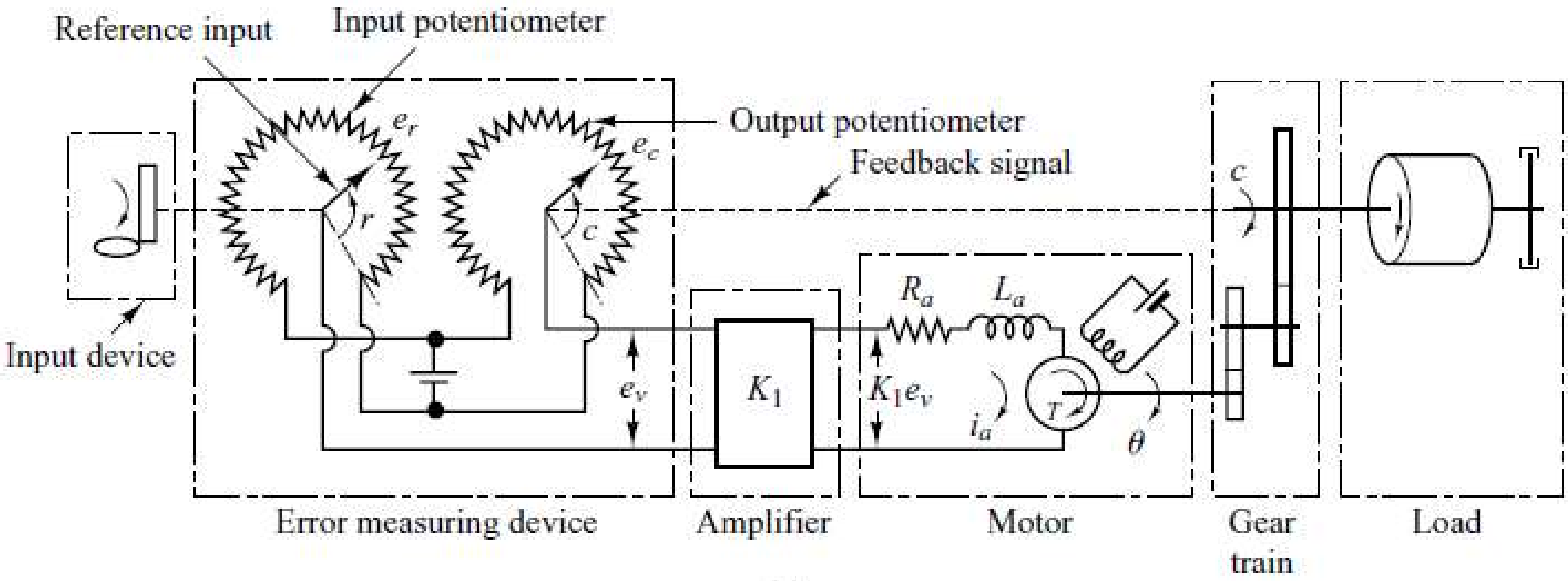
where

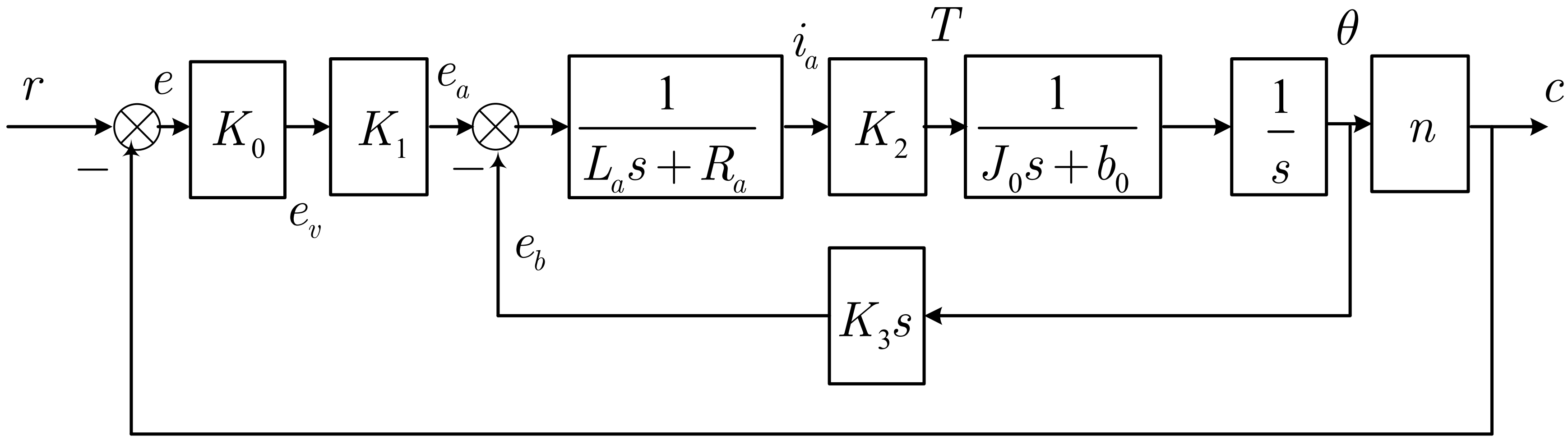
$$K_m = \frac{C_m}{R_a f_m + C_m K_b} \qquad T_m = \frac{J_m R_a}{R_a f_m + C_m K_b}$$

Example. A servo system is shown below, where r and c are the input and output (that are proportional to the angular positions of potentiometers),

$$e = r - c,\ (e_r - e_c) = K_0(r - c)$$

K_0: proportionality constant; K_1: amplifier gain; $e_a = K_1 e_v$: power supply voltage; n: gear ratio.





where in this example, for the DC motor

1) The electromagnetic torque (K_2: torque constant):

$$T = K_2 i_a \quad (1)$$

2) The back emf voltage (K_3 : back emf constant):

$$e_b = K_3 \dot{\theta} \quad (2)$$

where θ is the rotational angle;

3) The electrical equation:

$$e_a = i_a R_a + L_a \frac{di_a}{dt} + e_b \quad (3)$$

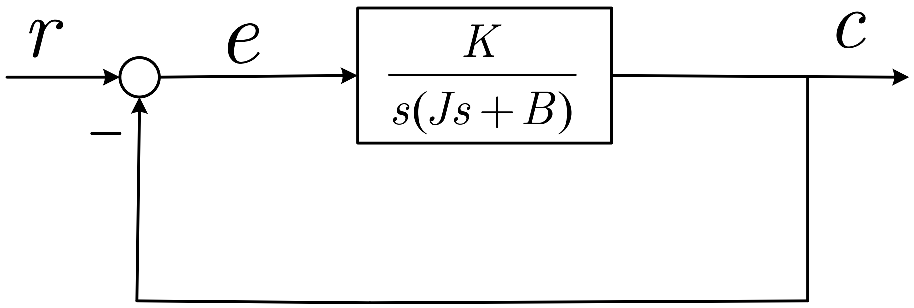
4) The torque equation:

$$J_0 \ddot{\theta} + b_0 \dot{\theta} = T = K_2 i_a \qquad (4)$$

where J_0 and b_0 denote the moment of inertia and friction constant, respectively.

Then, by neglecting L_a, the open-loop transfer function can be simplified as

$$\frac{C(s)}{E(s)} = \frac{K}{s(Js+\mathrm{B})}$$



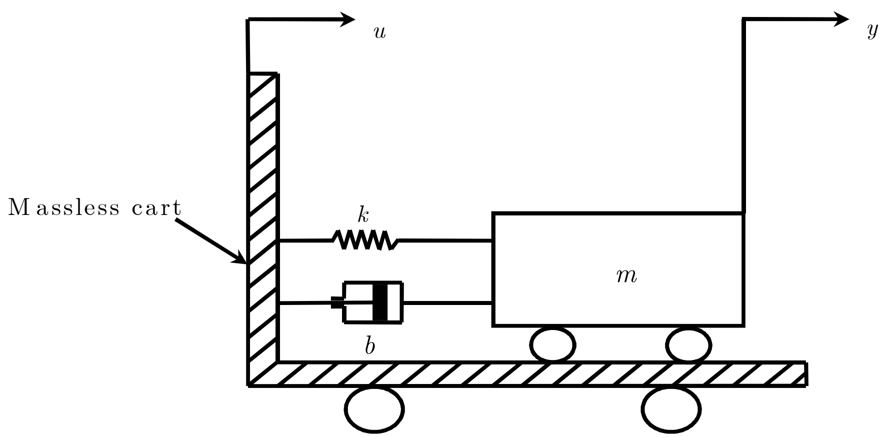
where

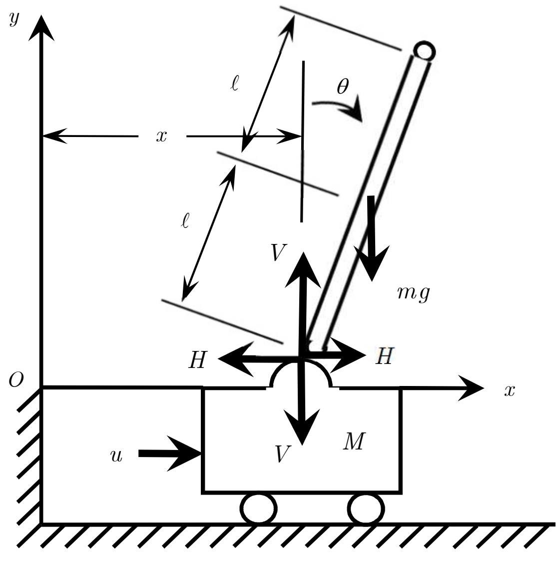
$$\begin{cases} J = J_0 / n^2 \\ B = [b_0 + (K_2 K_3 / R_a)] / n^2 \\ K = K_0 K_1 K_2 / R_a n \end{cases}$$

Summary of Chapter 3

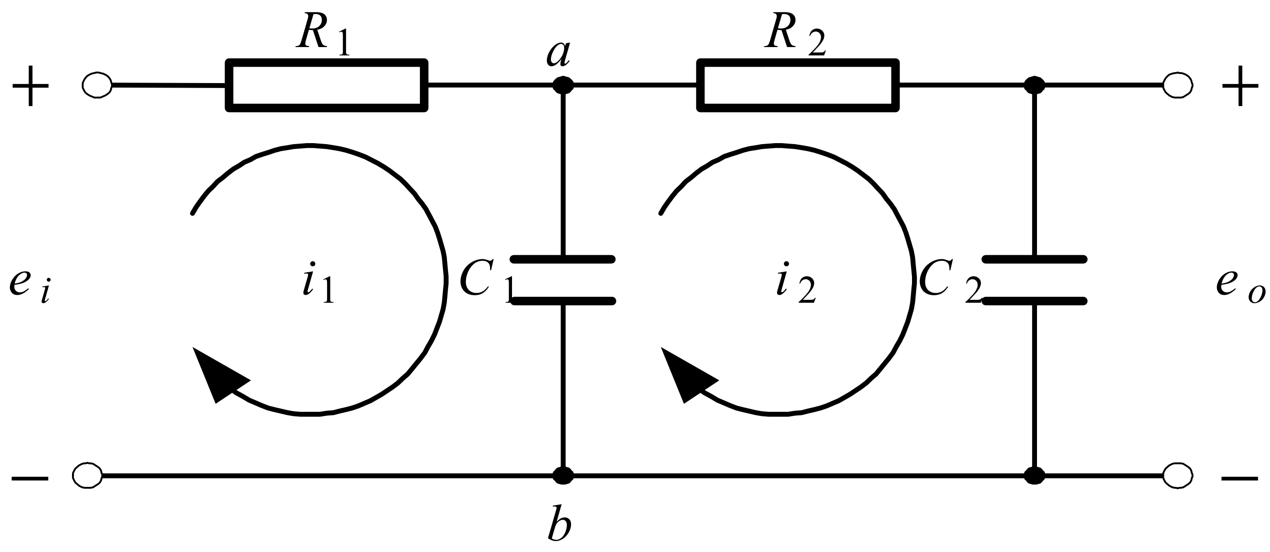
In this chapter, we studied the mathematical modeling of some simple mechanical and electrical systems.

1. For mechanical systems, Newton's second law is applied to establish the mathematical models of some spring-mass-damper systems and inverted pendulum systems.





2. For electrical systems, *Kirchhoff*'s laws are used. In particular, the concept of complex impedance is introduced, which provides a convenient way to obtain system transfer function.



3. The mathematical modeling of DC motor, which combines mechanical and electrical systems, is introduced and can be expressed as

$$\begin{cases} M_m = C_m i_a \\ E_b = K_b \dot{\theta}_m \\ u_a = i_a R_a + L_a \dfrac{di_a}{dt} + E_b \\ J_m \ddot{\theta}_m + f_m \dot{\theta}_m = M_m \end{cases} \Rightarrow \frac{\Theta_m}{U_a(s)} = \frac{C_m}{[(L_a s + R_a)(J_m s^2 + f_m s) + C_m K_b s]}$$