Chapter 6

Root Loci Analysis (3):

Root Locus Approach to Control System Design

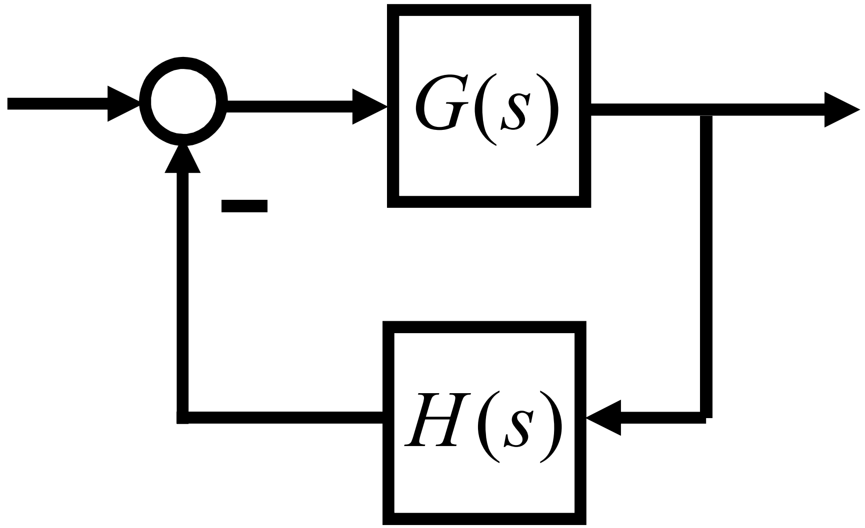
6-5 Root Locus Approach to Control System Design

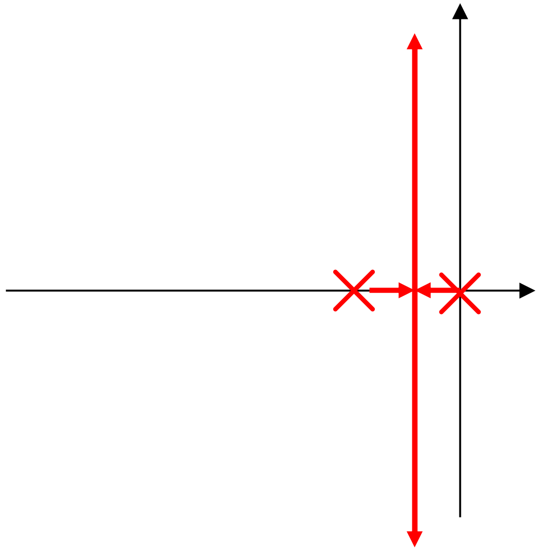
The primary objective of this section is to present procedures for the design and compensation of LTI control systems based on root-locus method.

1. Performance Specifications

Control systems are deigned to perform specific tasks, whose requirements on the systems are usually given by t_s, M_p and e_{ss} in step response (dominant poles can therefore be determined).

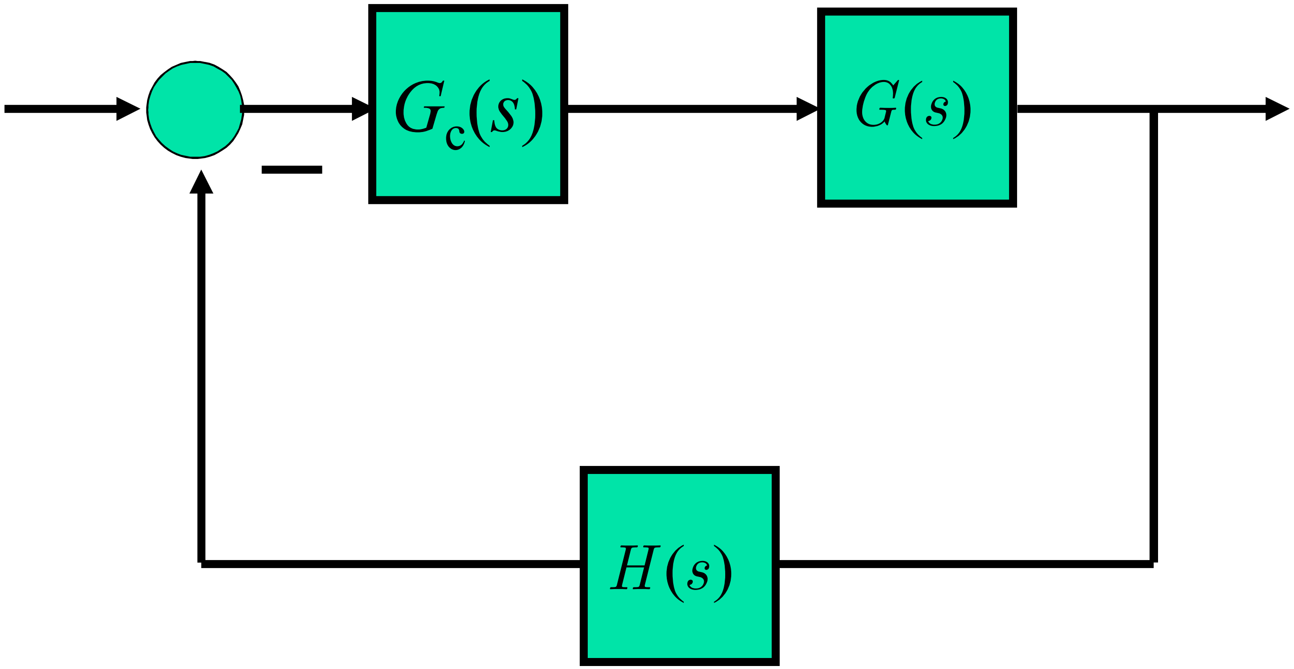
2. System Compensation





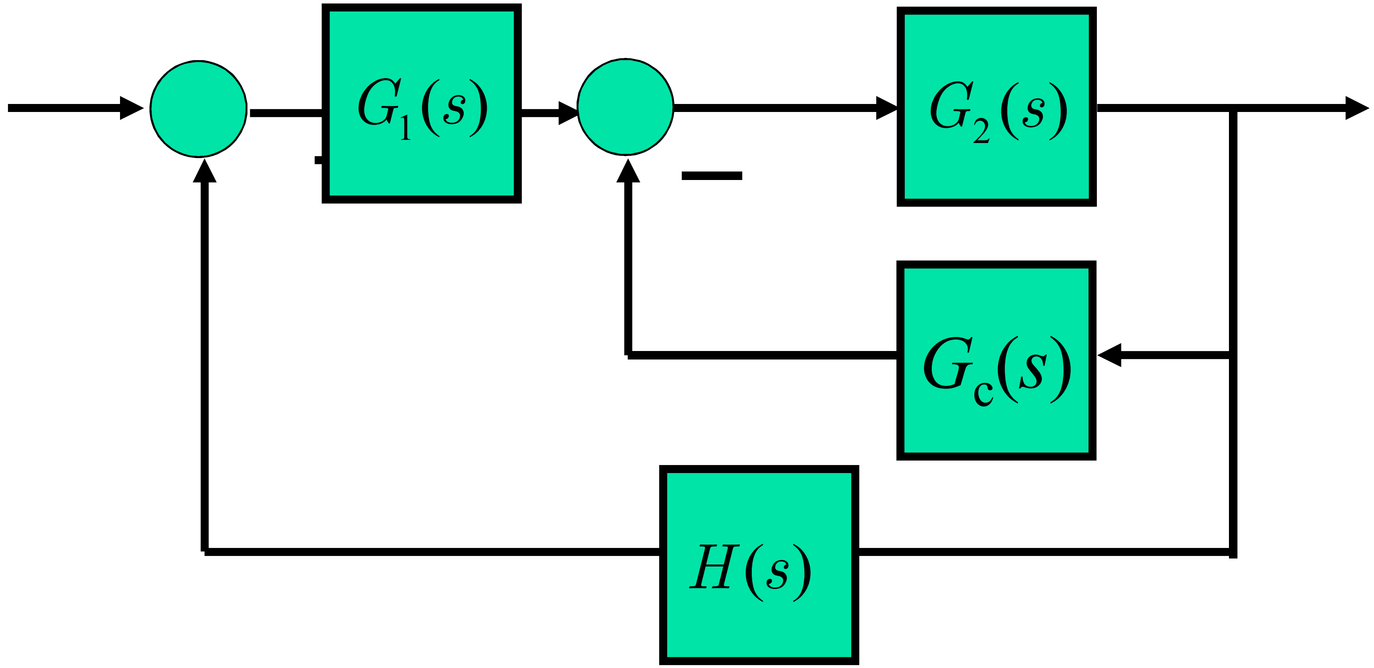
In many practical cases, the adjustment of the gain alone may not provide sufficient alteration of the system behavior to meet the given specifications. In that case, a compensator is necessary.

1) Series compensation



The block diagram shows the configuration where the compensator G_c(s) is placed in series with the plant. This scheme is called series compensation.

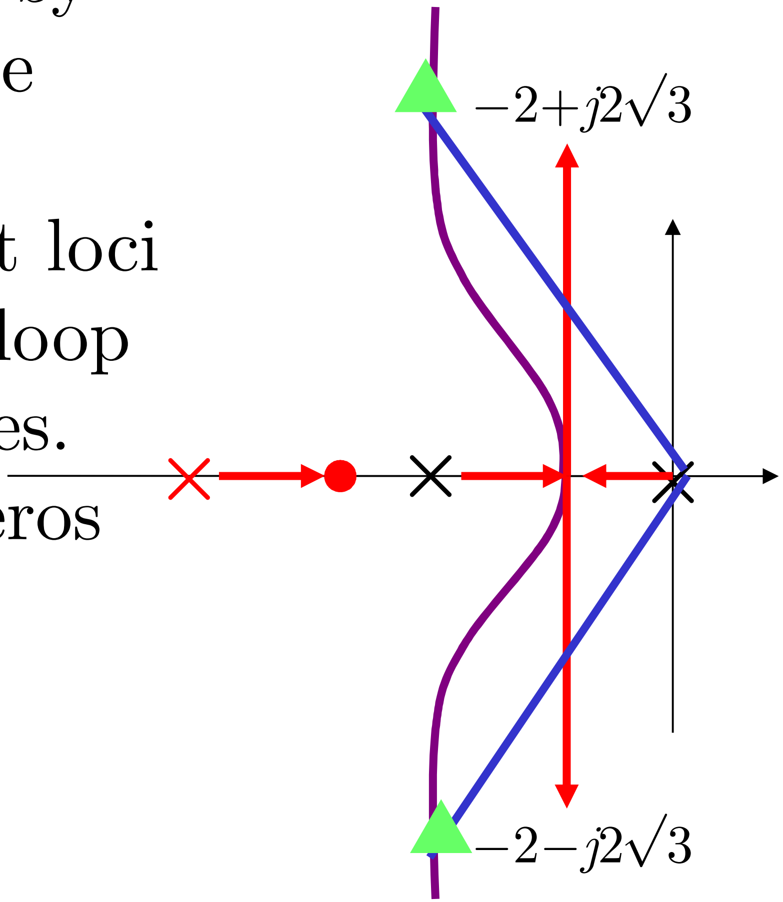
2) Feedback compensation



An alternative to series compensation is to feed back the signal(s) from some element(s) and place a compensator in the resulting inner feedback path, as shown in the block diagram. Such a compensation is called feedback compensation.

3. Design based on Root-Locus method

The design is based on reshaping the root-locus of the system by adding poles and zeros to the system's **open-loop** transfer function and forcing the root loci pass through desired closed-loop poles, usually, dominant poles. The addition of poles and zeros forms a compensator. For instance,



4. Commonly used compensators

Among the many kinds of compensators, widely employed compensators are

1) Lead compensator;

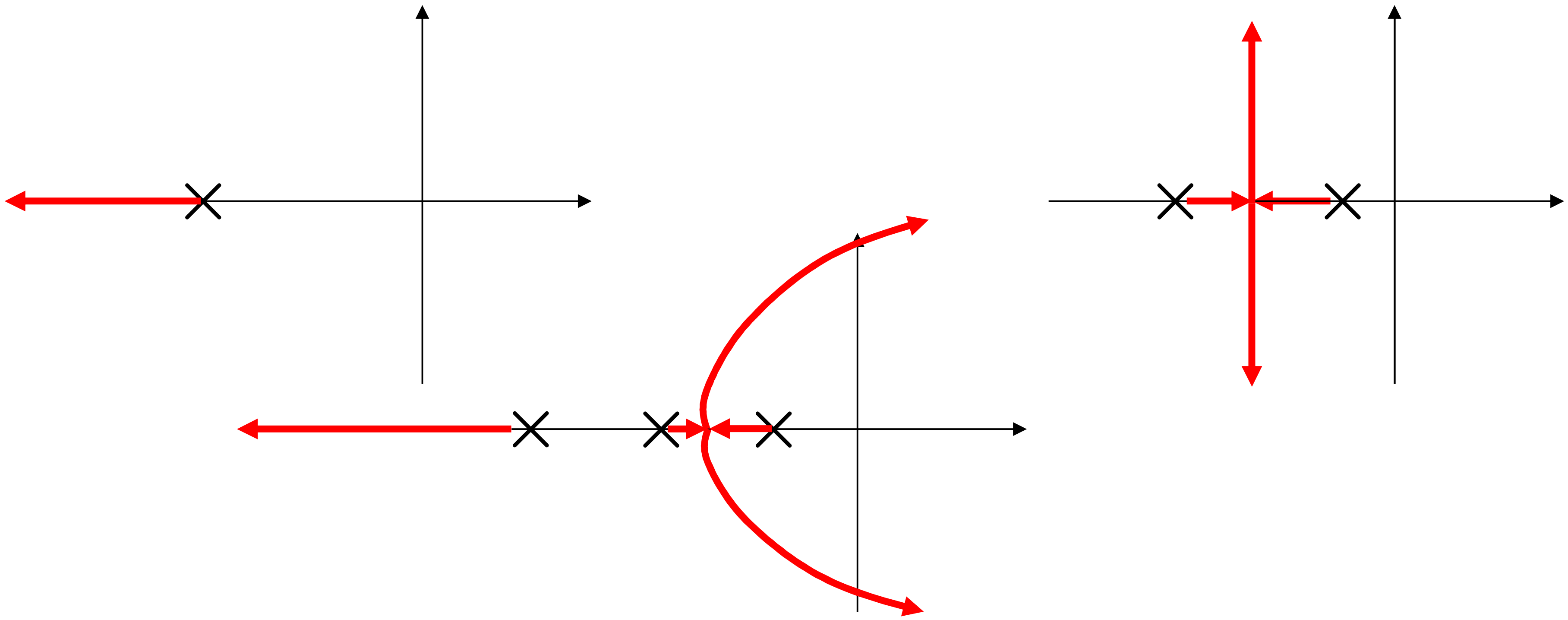
2) Lag compensator;

3) Lag-lead compensator;

4) Velocity-feedback (tachometer) compensator.

The names of lead, lag or lag-lead compensators will become clear in the frequency domain analysis.

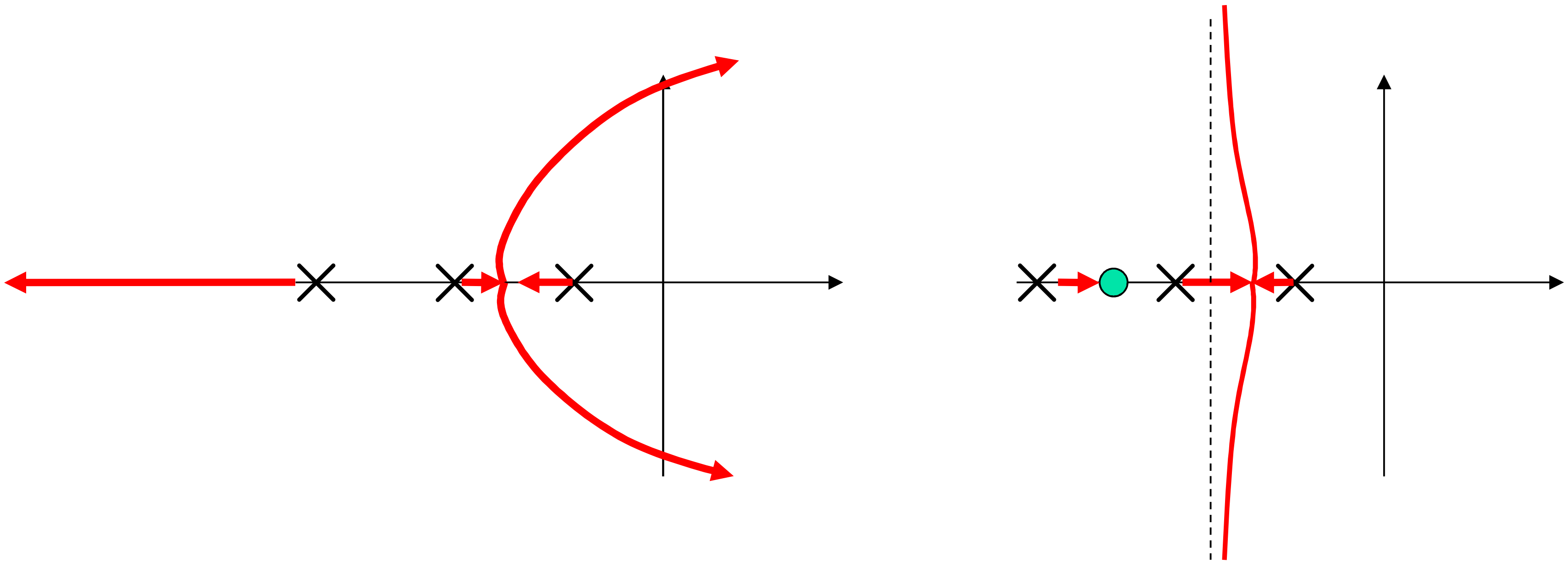
5. Effects of the addition of poles



The addition of a pole to the open-loop transfer function has the effect of pulling the root-locus to the right, tending to lower the system's relative stability and to increase the settling time of the response.

6. Effects of the addition of zeros

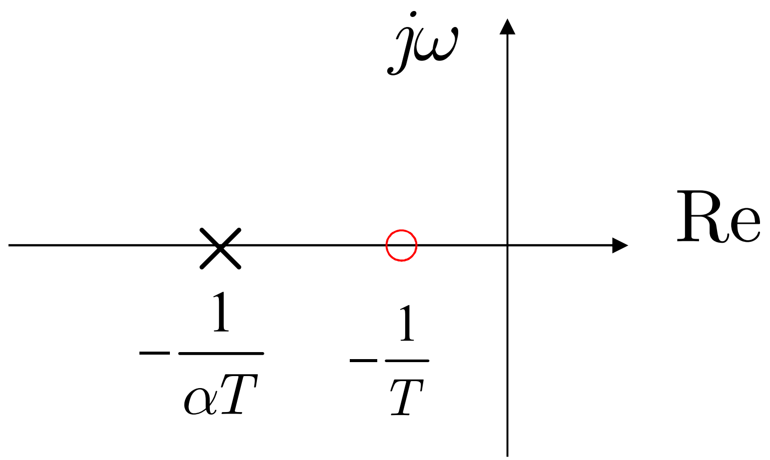
The addition of a zero to the open-loop transfer function has the effect of pulling the root locus to the left, tending to make the system more stable and to decrease the settling time of the response.



6-6 Lead Compensation

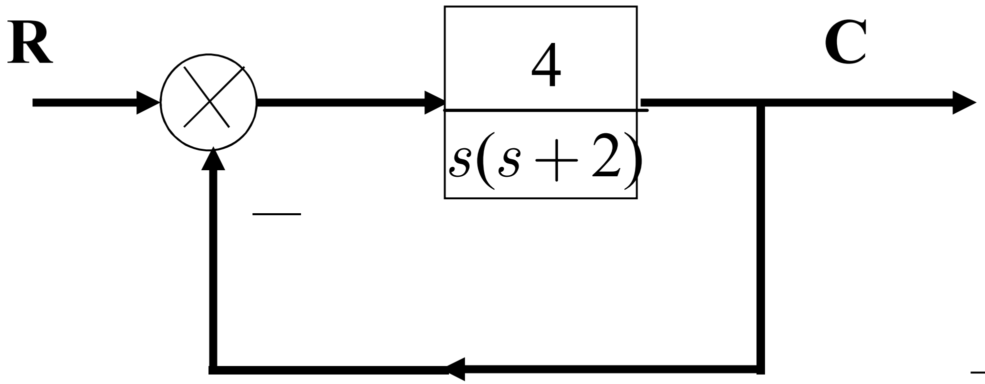
1. Mathematical model:

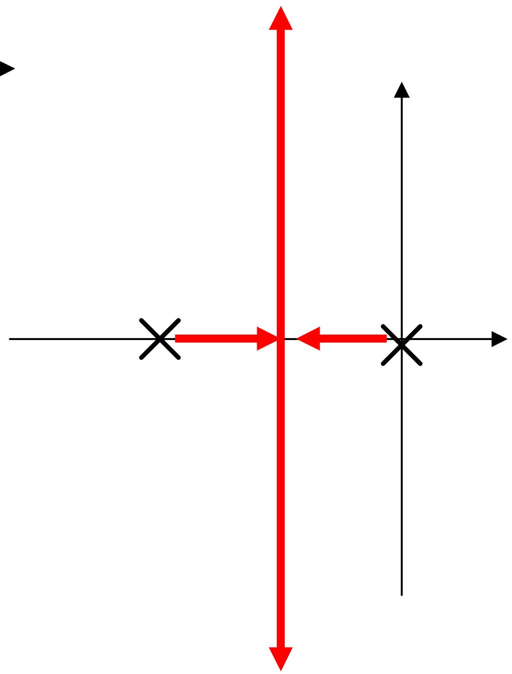
$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\alpha T}}, \quad 0<\alpha<1$$



2. Design Procedure: is illustrated via the following example.

Example.





$$\frac{C(s)}{R(s)}=\frac{4}{s^2+2s+4}\Rightarrow\begin{cases}s_{1,2}=-1\pm j\sqrt{3}\\ \zeta=0.5,\ \omega_n=2\text{rad/s}\end{cases}$$

from which we obtain that $t_s=3.5s,\ M_p=16.3\%.$

Step 1: Performance specifications: To determine the *desired locations* for the dominant closed-loop poles.

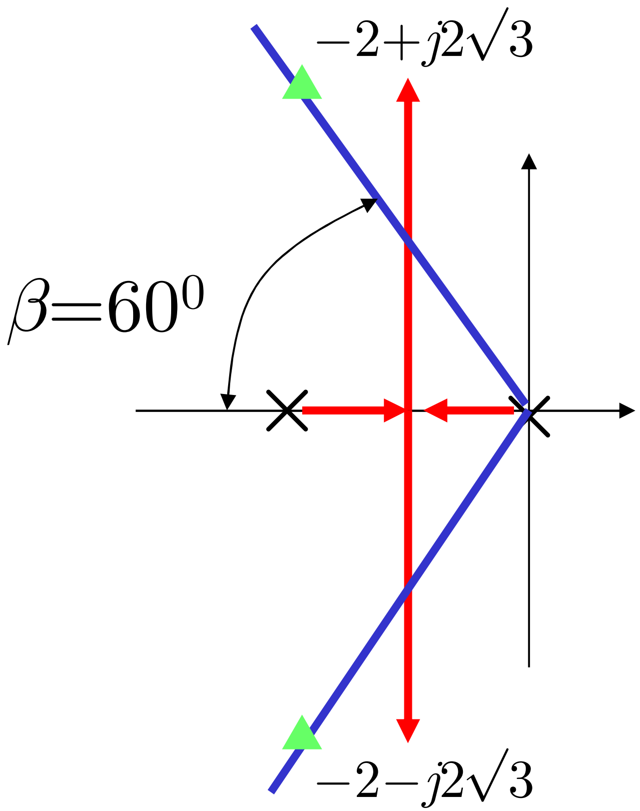
In this example, it is required to modify the closed-loop poles so that t_s=1.75s, M_p=16.3%, which can be converted into

$$\zeta=0.5\ (\beta=60^0),\ \omega_n=4\ \text{rad/s}$$

Hence, the **desired closed-loop poles** are

$$s=-2\pm j2\sqrt{3}$$

Step 2: By drawing the root-locus of the original system to ascertain whether or not the K^* alone can yield the desired closed-loop poles.



Step 3: Obviously, a lead compensator is needed for reshaping the root loci as desired locations.

$$G_c(s) = K_c \alpha \frac{Ts+1}{\alpha Ts+1} = K_c \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}},$$

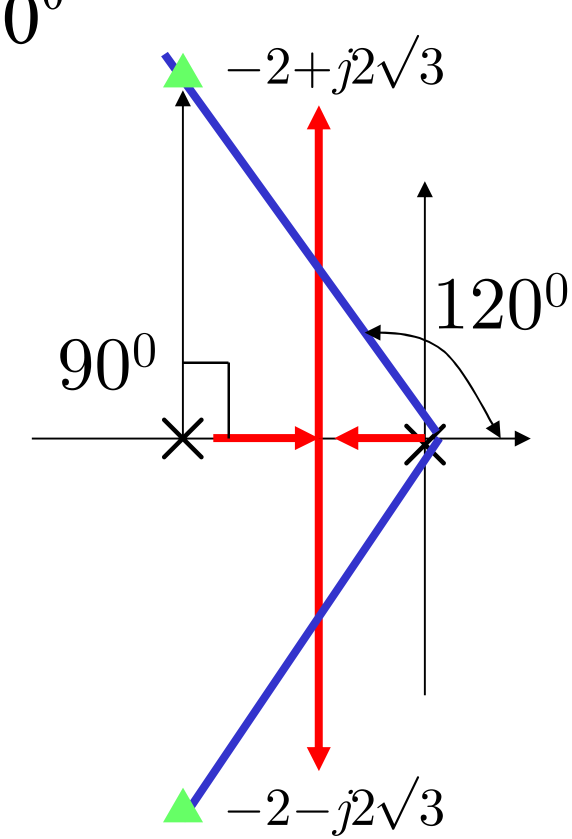
$$0 < \alpha < 1$$

Step 4: Determine the locations of the zero and pole of the lead compensator:

- Determine the angle deficiency ϕ so that the total sum of angles is $\pm 180^0(2k+1)$. In this example, since

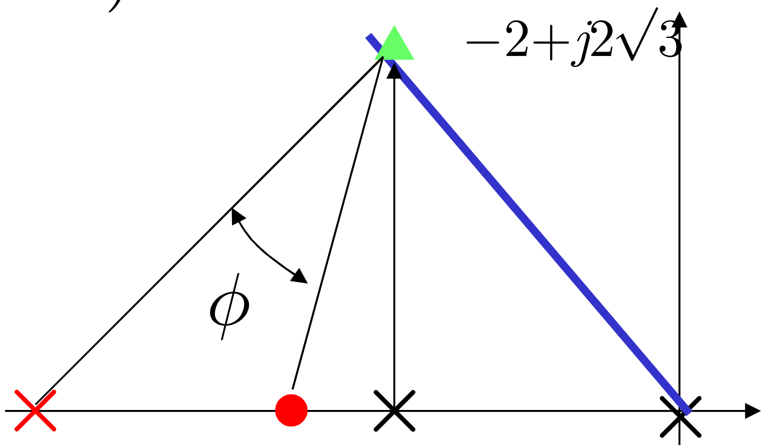
$$\angle \frac{4}{s(s+2)}\bigg|_{s=-2\pm j2\sqrt{3}} = -210^0$$

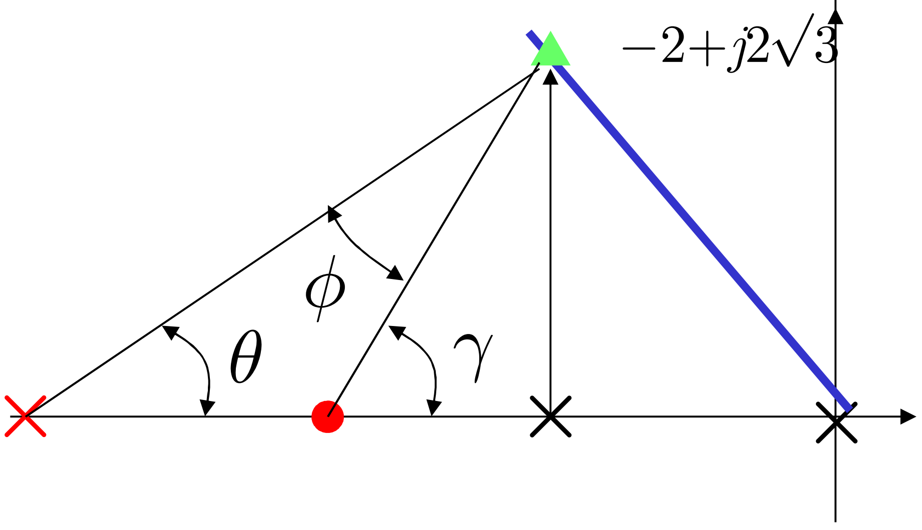
the lead compensator must contribute $\phi = 30^0$ at $s = -2 \pm j2\sqrt{3}$.

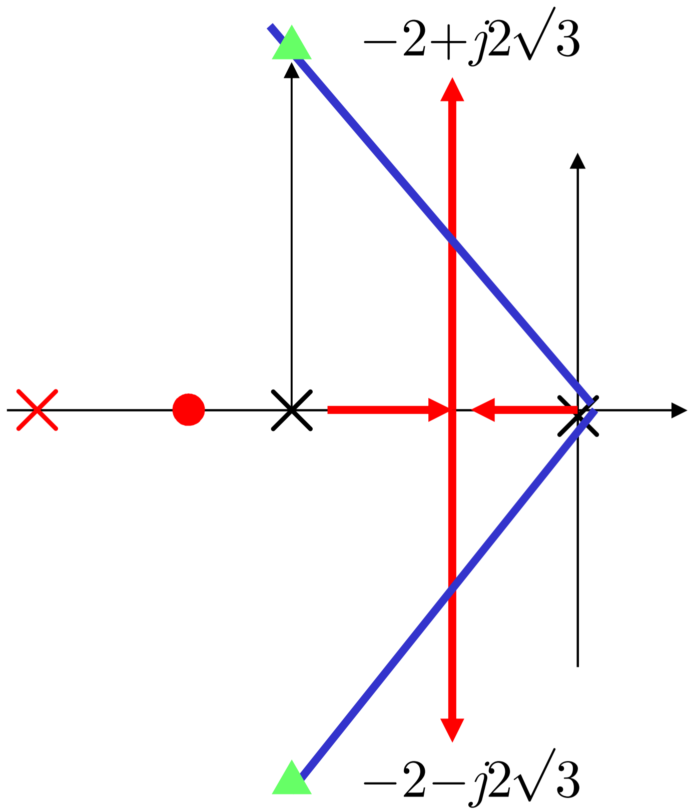


- Determine the zero and pole locations of the compensator.

Note that there are infinitely many choices as long as the angle between the two vectors is $\phi(=30^0)$.





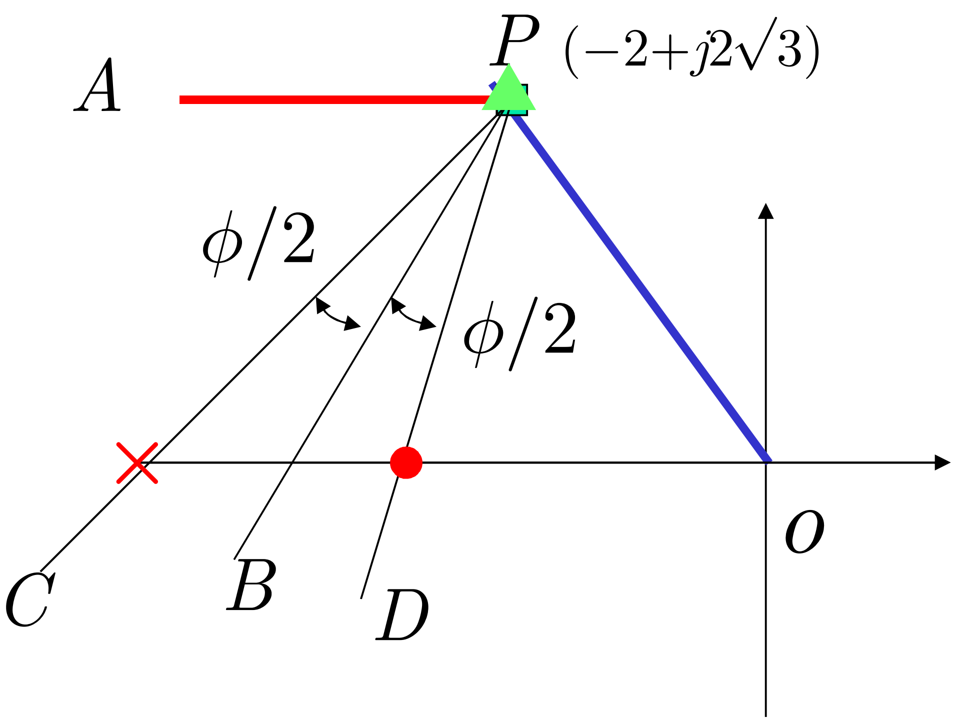


$$\phi=\gamma-\theta(=30^0)$$

- Draw horizontal line PA and the line PO;

- Bisect the angle $\angle APO$

to obtain line PB;



- Draw two lines PC and PD that make angles $\pm\phi/2$. The intersections of PC and PD with x-axis give the locations of the pole and zero of the compensator.

- Determine the values of α and T.
  In this example, by measuring the intersection points yields
  zero at $s=-2.9$;
  pole at $s=-5.4 \Rightarrow T=1/2.9=0.345$,
  $\alpha T=1/5.4=0.185 \Rightarrow \alpha=0.573$.

Step 5: Determine K_c of the compensator.

$$G(s)G_c(s) = K_c \frac{s+2.9}{s+5.4} \frac{4}{s(s+2)} = \frac{K^*(s+2.9)}{s(s+2)(s+5.4)}$$

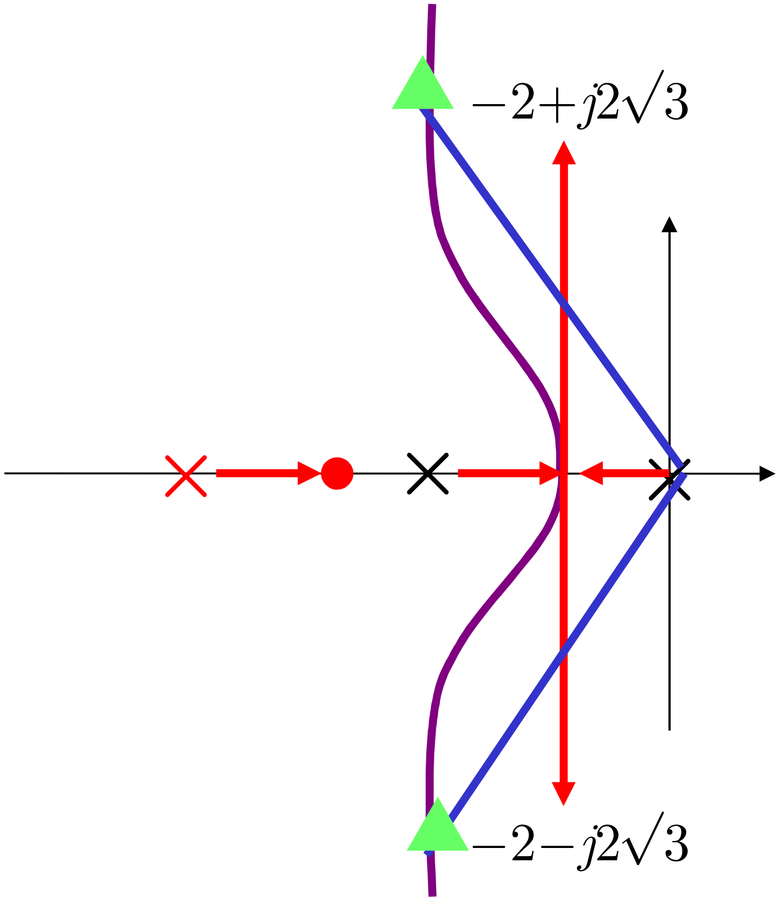
where $K^*=4K_c$, which can be evaluated by

$$\left| \frac{K^*(s+2.9)}{s(s+2)(s+5.4)} \right|_{s=-2+j2\sqrt{3}} = 1 \Rightarrow K^* = 18.7$$

$$\Rightarrow K_c = 18.7/4 = 4.68$$

Hence,

$$G_c(s) = 4.68\frac{(s+2.9)}{(s+5.4)}$$

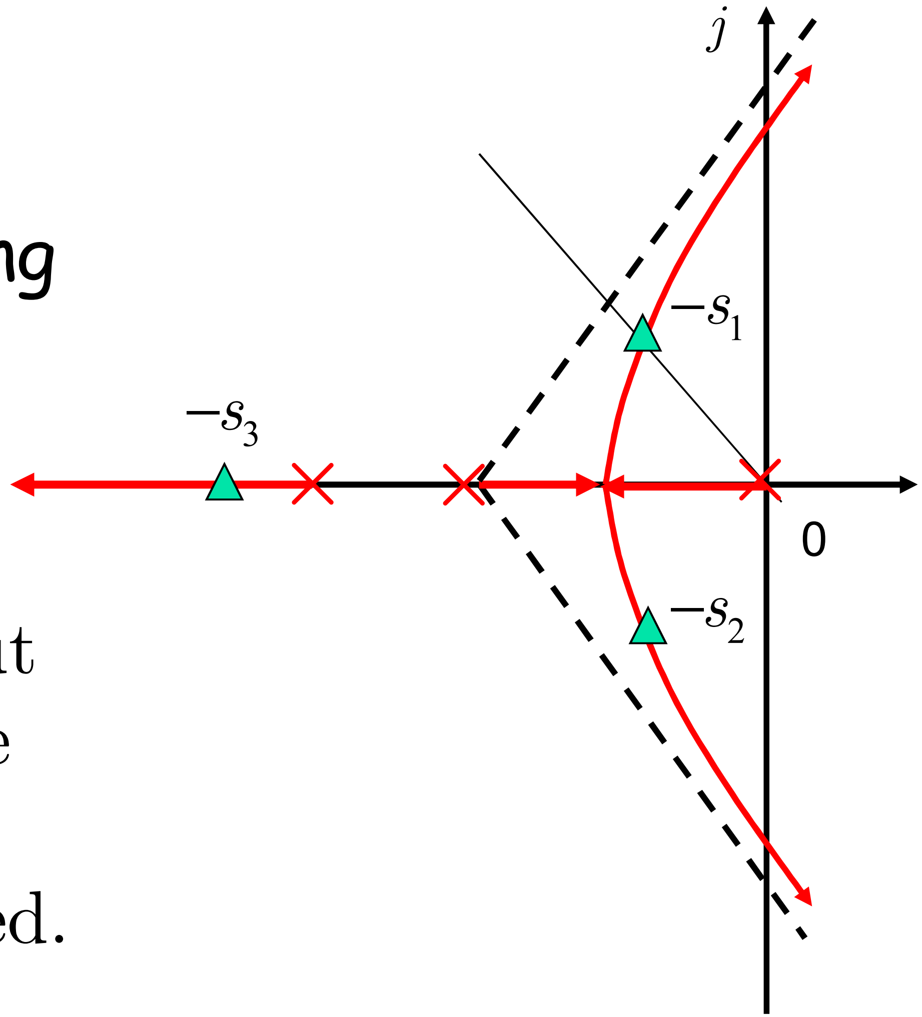


This completes our lead compensator design.

6-7 Lag Compensation

Motivation for introducing lag compensator:

If a system exhibits satisfactory transient response characteristics but unsatisfactory steady-state characteristics, a lag compensator can be applied.



For example, the system with two dominant poles has satisfactory transient response but its static velocity error does not meet the requirement.

Basic requirements

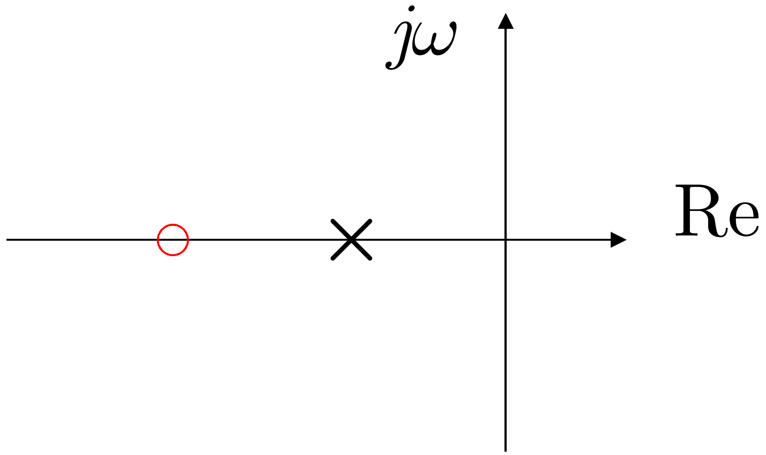
◆ Do not appreciably change the dominant closed-loop poles of the original system;

◆ The open-loop gain should be increased as much as needed.

This can be accomplished if a lag compensator is put in cascade with the given feedforward transfer function.

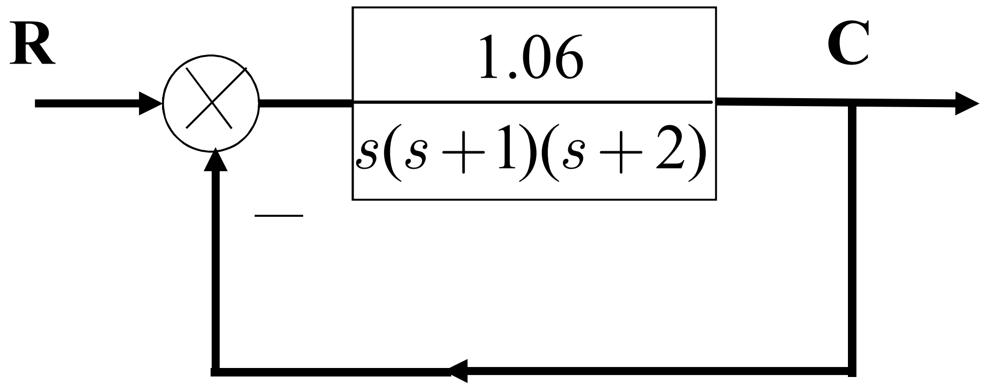
1. Mathematical model :

$$G_c(s) = \hat{K}_c \hat{\beta} \frac{Ts+1}{\hat{\beta}Ts+1} = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\hat{\beta}T}}, \quad \hat{\beta} > 1$$



2. Design Procedure: is illustrated via the following example.

Example.



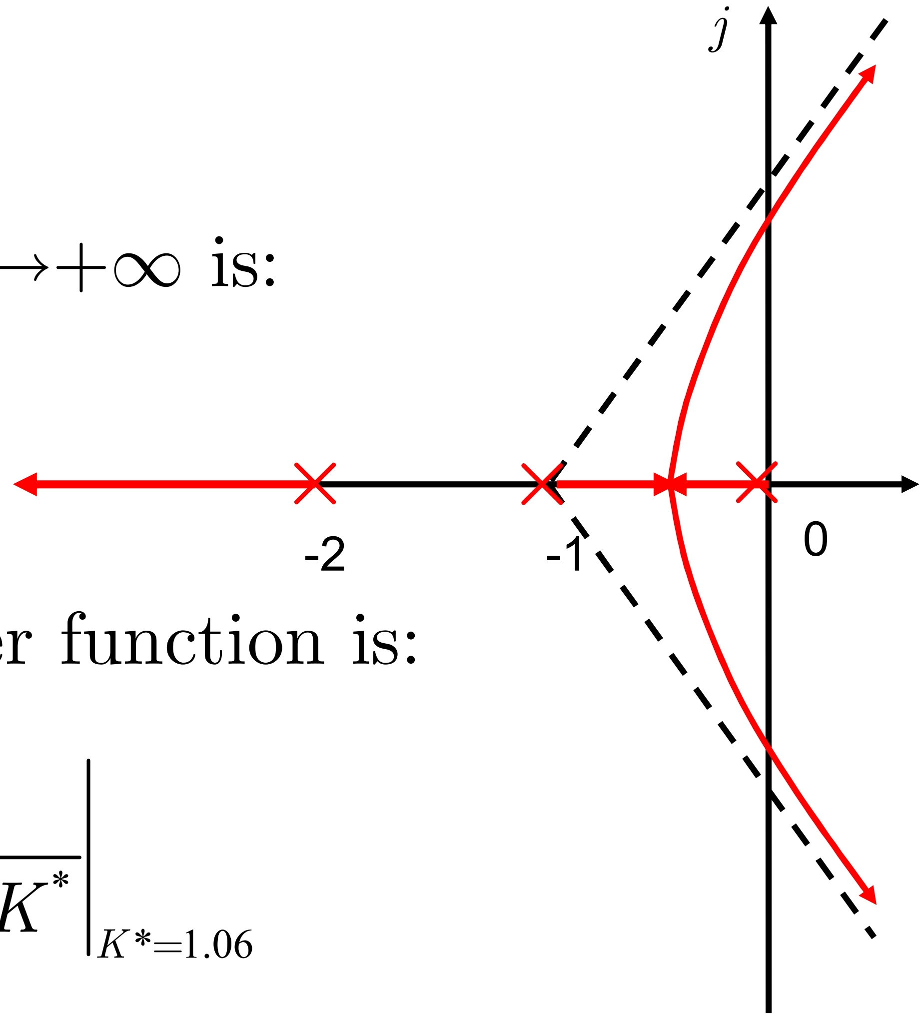
Performance specifications:

1) Transient performance specification: ζ=0.5 ($\Leftrightarrow\beta$=60^0);

2) Steady-state performance specification: Static velocity error e_{ss}=0.2 ($\Leftrightarrow K_v$=5).

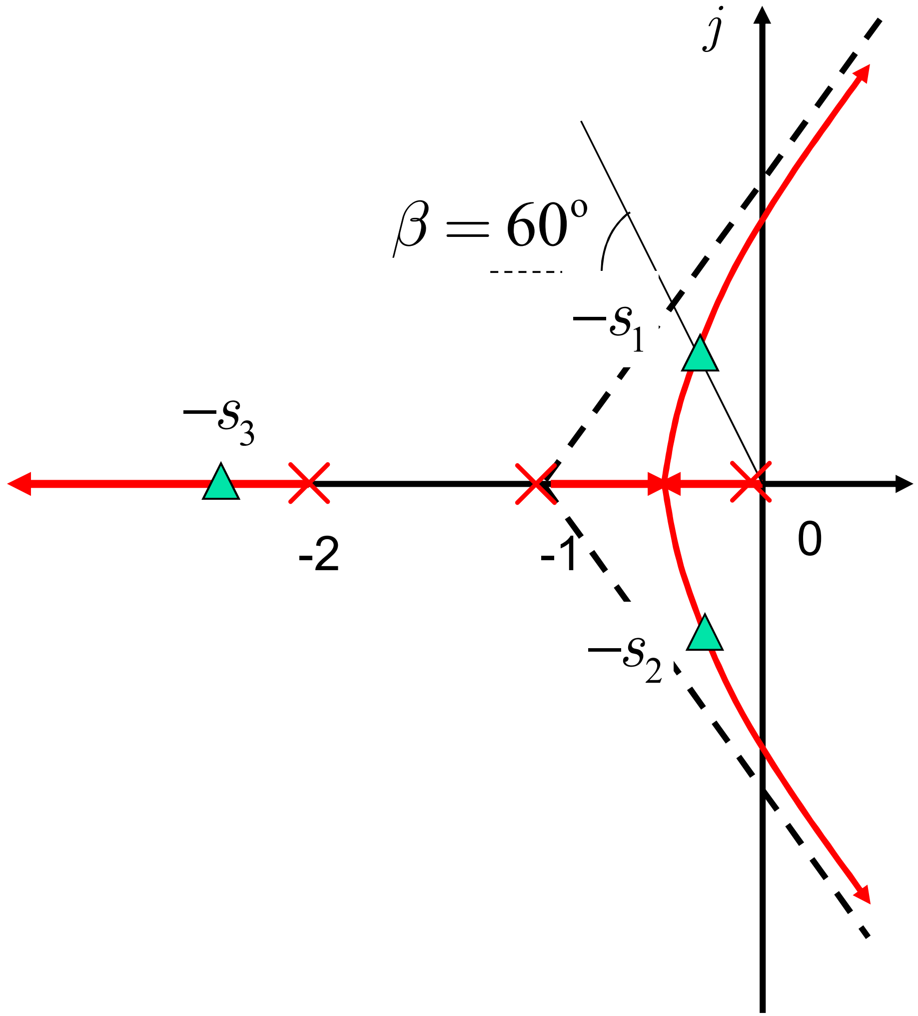
Step 1: Based on the transient-response specifications, locate the dominant closed-loop poles on the root locus by drawing the root-loci of **the original system**.

The root loci for K^*: $0 \rightarrow +\infty$ is:



The closed-loop transfer function is:

$$\frac{C(s)}{R(s)} = \frac{K^*}{s(s+1)(s+2)+K^*}\Bigg|_{K^*=1.06}$$



The dominant closed-loop poles for this example are:

$s_{1,2} = -0.33 \pm j0.59$,

and

$s_3 = -2.34$, where $s_{1,2}$ exactly correspond to 60^0 line ($\zeta = 0.5$),

$K_v = 0.53$,

$K^* = 1.06$.

Therefore, the original system

$$\frac{C(s)}{R(s)} = \frac{1.06}{s(s+1)(s+2)+1.06}$$

$$= \frac{1.06}{(s+0.3307-j0.5864)(s+0.3307+j0.5864)(s+2.3386)}$$

satisfies $\zeta=0.5$ ($\beta=60^0$) but does not satisfy $e_{ss}=0.2$ ($K_v=5$).

Step 2: Obviously, from Step 1, a lag compensator is needed and is of the form:

$$G_c(s) = \hat{K}_c \hat{\beta} \frac{Ts+1}{\hat{\beta}Ts+1} = \hat{K}_c \frac{s+\dfrac{1}{T}}{s+\dfrac{1}{\hat{\beta}T}}, \quad \hat{\beta} > 1$$

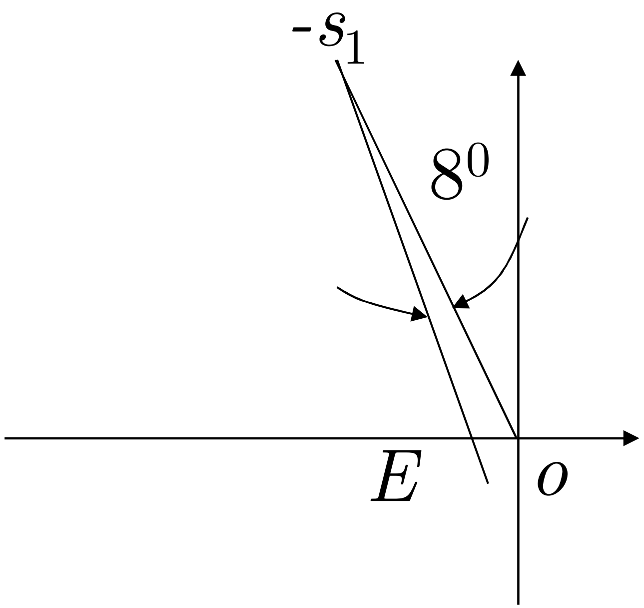
Step 3: Evaluate the particular static error constant specified in the problem.

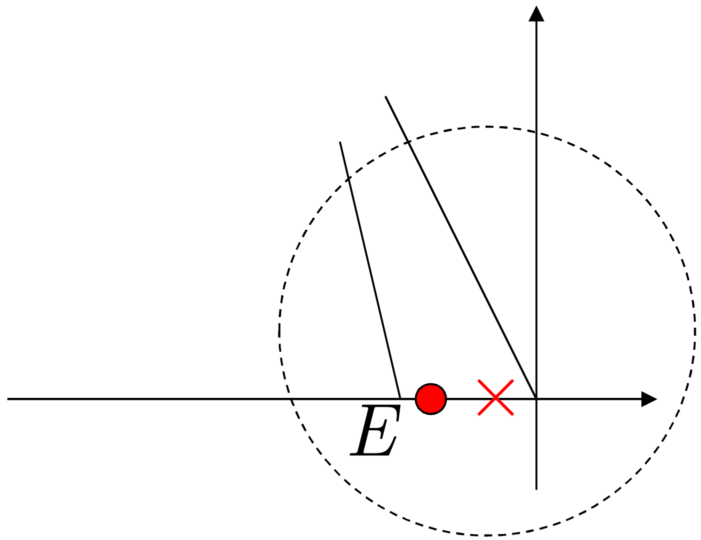
In this example, by requirement, K_v=5 (about **ten times** of the original K_v =0.53).

Step 4: Determine the amount of increase in the static error constant necessary to satisfy the specifications.

In this example, to meet K_v=5, we must increase the static velocity error constant by a factor of 10, that is, $\hat{\beta} = 10$, which will become clear later.

Step 5: Determine the pole and zero of the compensator as shown in the Figure below.

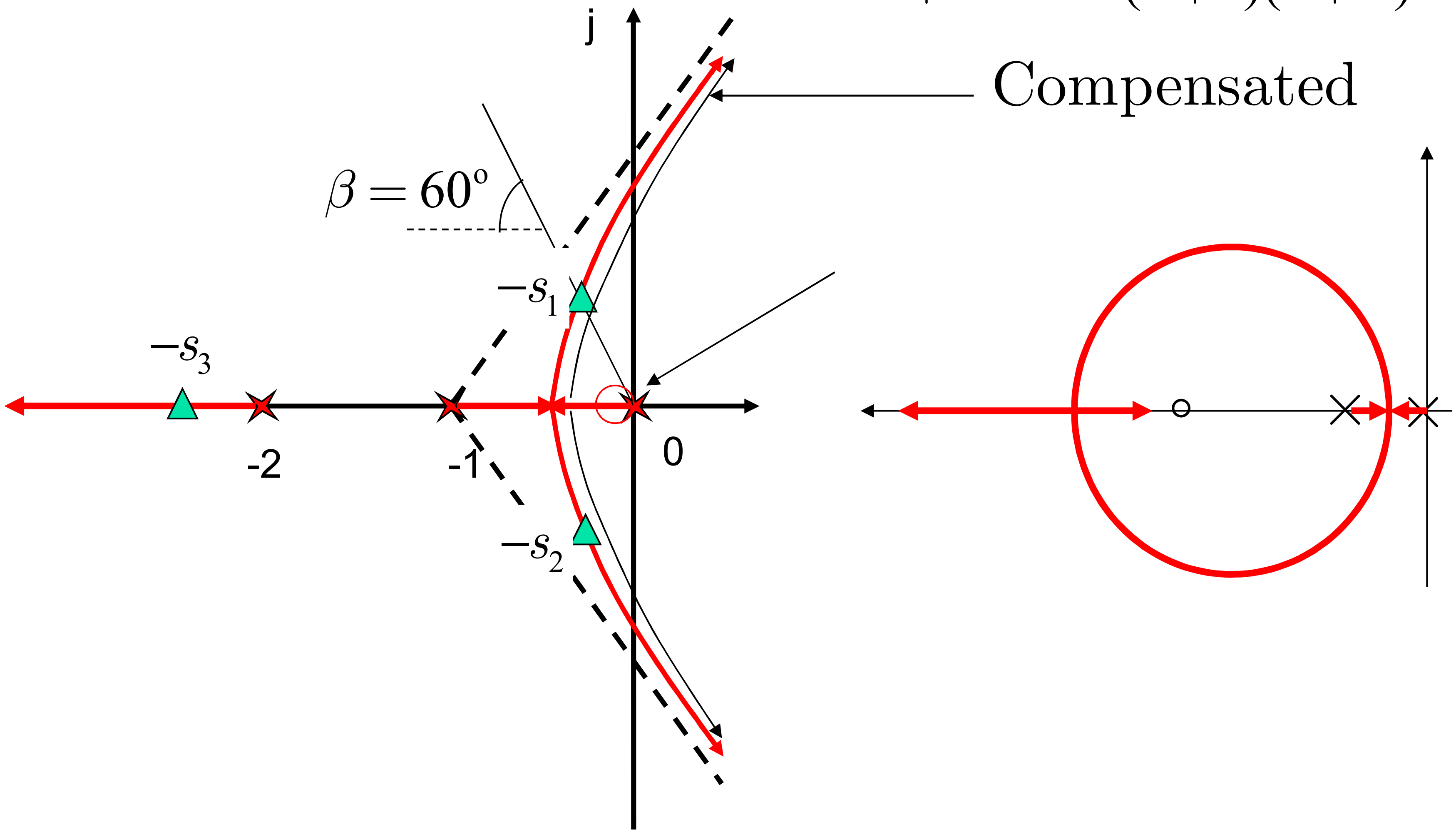




Draw a line from the dominant pole with an angle 6^0-10^0 that intersects with x-axis at E. Then, choose a point to the right of E to locate the zero of the compensator: zero$=-0.05$; hence, pole$=-0.005$.

Step 6: Draw the new root locus plot for the compensated system and *locate the desired dominant closed-loop poles.*

$$G(s)G_c(s) = \hat{K}_c \frac{s+0.05}{s+0.005} \frac{1.06}{s(s+1)(s+2)}$$



In this example, we assume the damping ratio of the new dominant closed-loop poles is kept the same (ζ=0.5 (β=60^0), then the poles obtained from the new root-locus plot are

$$s_{1,2} = -0.31 \pm j0.55$$

Step 7: Adjust gain $\hat{K}_c$ of the compensator from the magnitude condition so that the dominant closed-loop poles lie at the desired locations.

First, K^* can be determined by magnitude condition:

$$\left|G(s)G_c(s)\right| = \left|K^* \frac{s+0.05}{s+0.005} \frac{1}{s(s+1)(s+2)}\right|_{s=-0.31+j0.55} = 1$$

That is

$$K^* = \left| \frac{s(s+0.005)(s+1)(s+2)}{s+0.05} \right|_{s=-0.31+j0.55} = 1.0235$$

Since

$$K^* = \hat{K}_c \times 1.06$$

we have,

$$\hat{K}_c = \frac{1.06}{K^*} = \frac{1.06}{1.0235} = 0.9656$$

The transfer function of the lag compensator is thus obtained as

$$G_c(s) = \hat{K}_c \hat{\beta} \frac{Ts+1}{\hat{\beta}Ts+1} = 0.9656 \frac{s+0.05}{s+0.005} = 9.656 \frac{20s+1}{200s+1}$$

The compensated system has the following open-loop transfer function:

$$G(s)G_c(s) = \frac{1.0235(s+0.05)}{s(s+0.005)(s+1)(s+2)}$$

$$= \frac{5.12(20s+1)}{s(200s+1)(s+1)(0.5s+1)}$$

The static velocity error constant K_v is

$$K_v = \lim_{s\to 0} sG(s)G_c(s) = 5.12$$

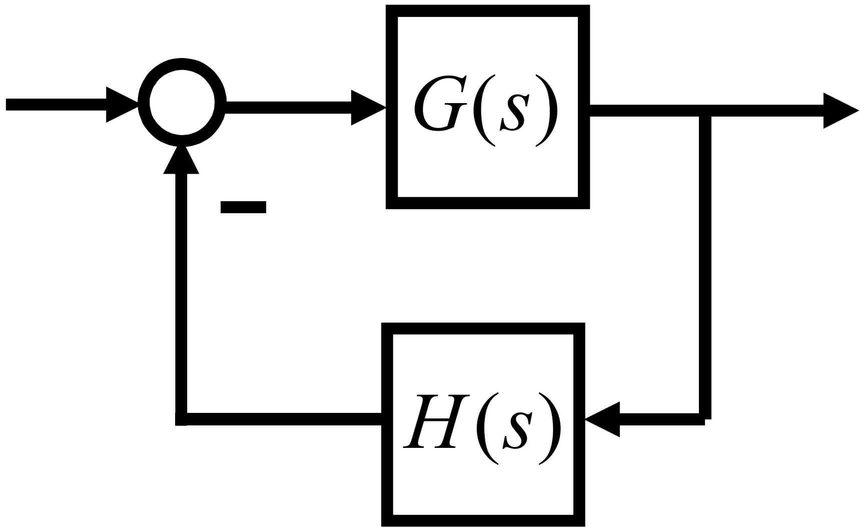
which is almost ten times of the original one (K_v=0.53). This completes the design.

6-8 Lag-Lead Compensation

- Lead compensation basically speeds up the response and increases the stability of the system.
- Lag compensation improves the steady-state accuracy of the system, but reduces the speed of the response.
- If improvements in both transient response and steady-state response are desired, usually we use a single lag-lead compensator.

Summary

1. Purpose of root locus method: To investigate the closed-loop stability and the system compensator design through *the open-loop transfer function* with variation of a certain system parameter, commonly, but not limited to, the open-loop gain.



2. Root loci construction rules

- The number of root locus branches is equal to the order of the characteristic equation.
- The loci are symmetrical about the real axis.

Negative feedback	Positive feedback
Rule 1. The root locus branches start from open-loop poles and end at open-loop zeros or zeros at infinity.	**Rule 1.** The same.

Rule 2. If the total number of real poles and real zeros to the right of a test point on the real axis is odd, then the test point lies on a root locus.	**Rule 2.** If the total number of real poles and real zeros to the right of a test point on the real axis is even, then the test point lies on a root locus.
Rule 3. $$\sigma_a = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m}$$ $$\phi_a = 180^0 \times \frac{(2k+1)}{n-m}$$	**Rule 3.** $$\sigma_a = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m}$$ $$\phi_a = 360^0 \times \frac{k}{n-m}$$

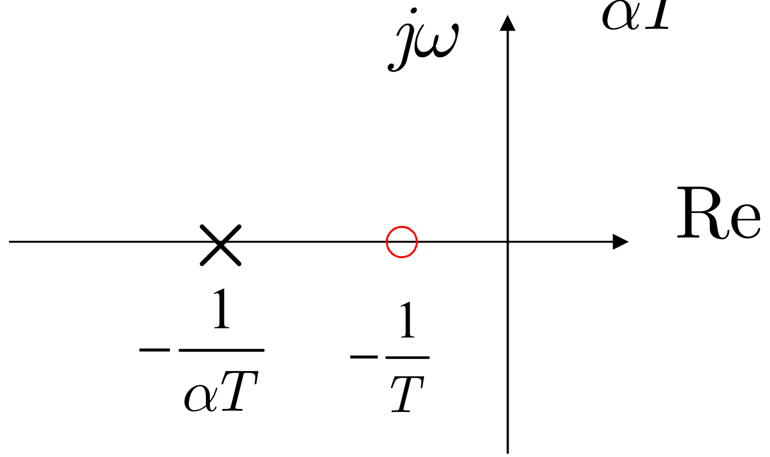
Rule 4. Breakaway point: $\sum_{j=1}^{m} \frac{1}{d+z_j} = \sum_{i=1}^{n} \frac{1}{d+p_i}$	**Rule 4.** The same.
Rule 5. $\theta_{p_i} = 180^0 + \sum_{j=1}^{m} \angle(-p_i + z_j) - \sum_{\substack{k=1 \\ k \neq i}}^{n} \angle(-p_i + p_k)$ $\phi_{z_i} = 180^0 - \sum_{\substack{j=1 \\ j \neq i}}^{m} \angle(-z_i + z_j) + \sum_{j=1}^{n} \angle(-z_i + p_j)$	**Rule 5.** $\theta_{p_i} = \sum_{j=1}^{m} \angle(-p_i + z_j) - \sum_{\substack{k=1 \\ k \neq i}}^{n} \angle(-p_i + p_k)$ $\phi_{z_i} = \sum_{\substack{j=1 \\ j \neq i}}^{m} \angle(-z_i + z_j) + \sum_{j=1}^{n} \angle(-z_i + p_j$

Rule 6. Intersection of the root loci with the imaginary axis. $$\left[\prod_{j=1}^{n}(s+p_j)+K^*\prod_{i=1}^{m}(s+z_i)\right]_{s=j\omega}=0$$	**Rule 6.** $$\left[\prod_{j=1}^{n}(s+p_j)-K^*\prod_{i=1}^{m}(s+z_i)\right]_{s=j\omega}=0$$
Rule 7. If $n \geq m+2$, the sum of poles remains unchanged as K^* varies form zero to infinity	**Rule 7.** The same.

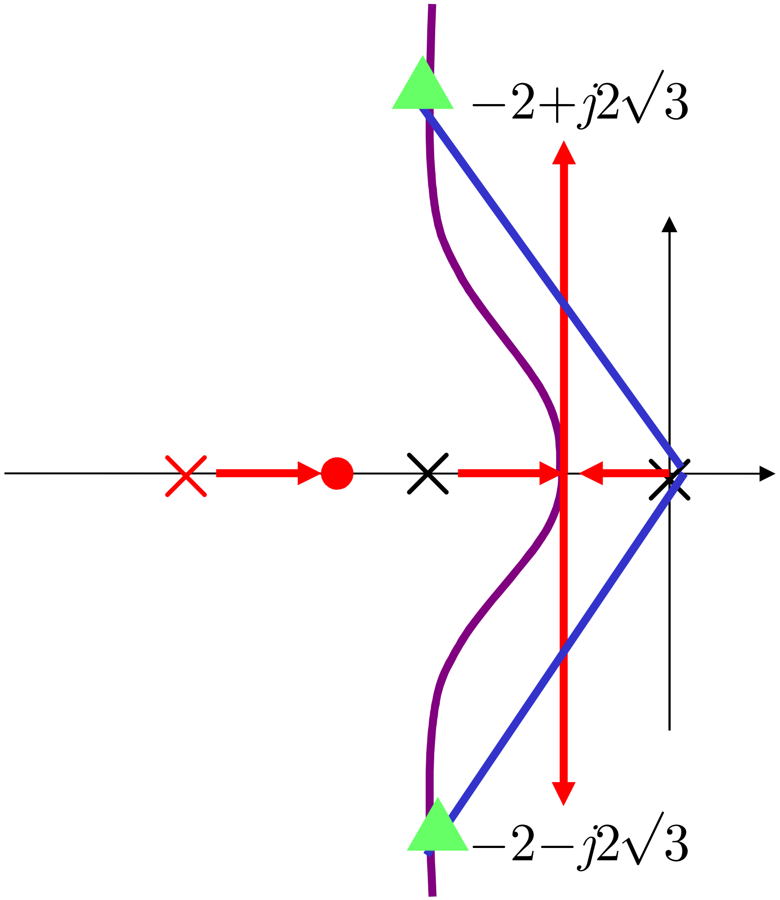
3. Compensator Design Base on Root Locus Method

1. Lead Compensation:

$$G_c(s) = K_c\alpha\frac{Ts+1}{\alpha Ts+1} = K_c\frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}}, \quad 0<\alpha<1$$

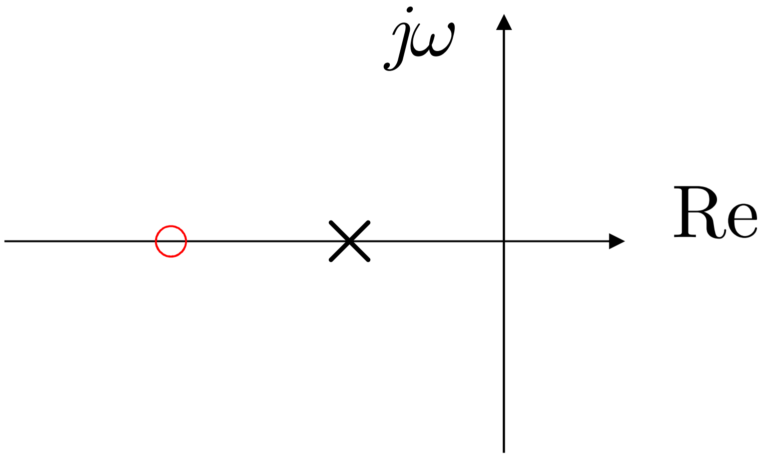


Zero is closer to the imaginary axis implies that the compensator dominates by PD control action which is able to make compensated system more stable and improve system transient performance.



2. Lag Compensation:

$$G_c(s) = \hat{K}_c \hat{\beta} \frac{Ts+1}{\hat{\beta}Ts+1} = \hat{K}_c \frac{s+\frac{1}{T}}{s+\frac{1}{\hat{\beta}T}}, \quad \hat{\beta} > 1$$



Pole is closer to the imaginary axis implies that the compensator dominates by PI control action which is able to improve system's steady state performance while keeping a satisfactory transient response.