Machine Learning

Part 1: Mathematical Foundation of Machine Learning

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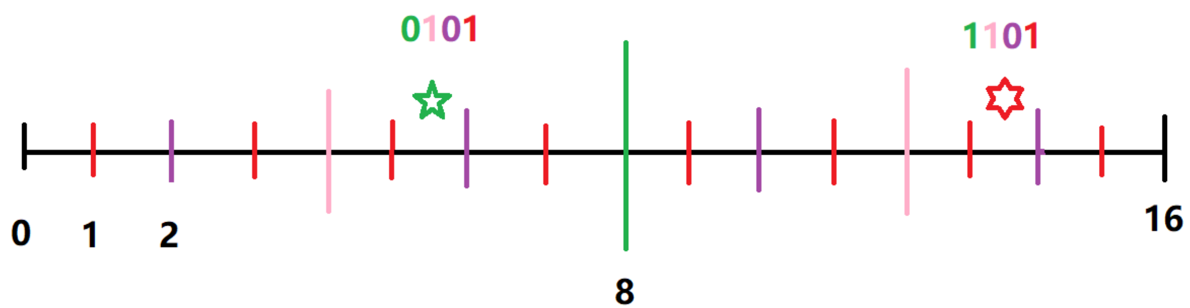
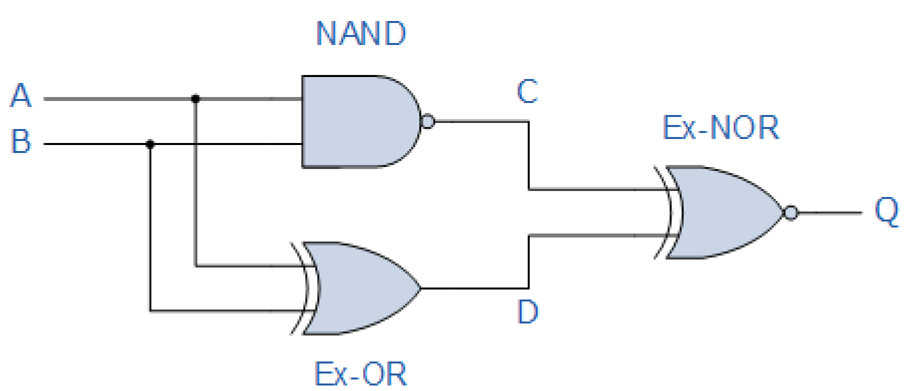
Function and Data Generalization

Numbers and Logic

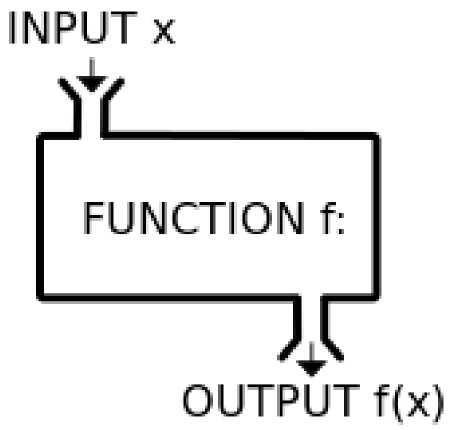
There are 10 kinds of people in the world, the ones who understand binary and those who do not.

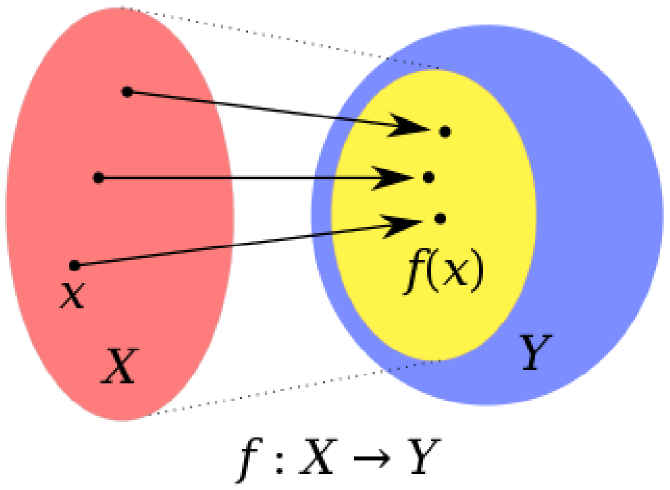




Functions

In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output.

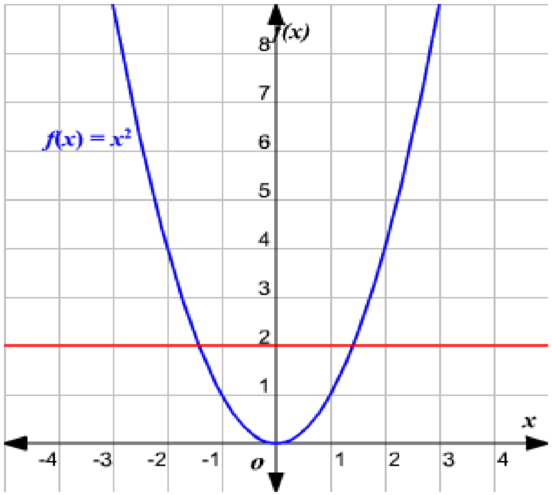


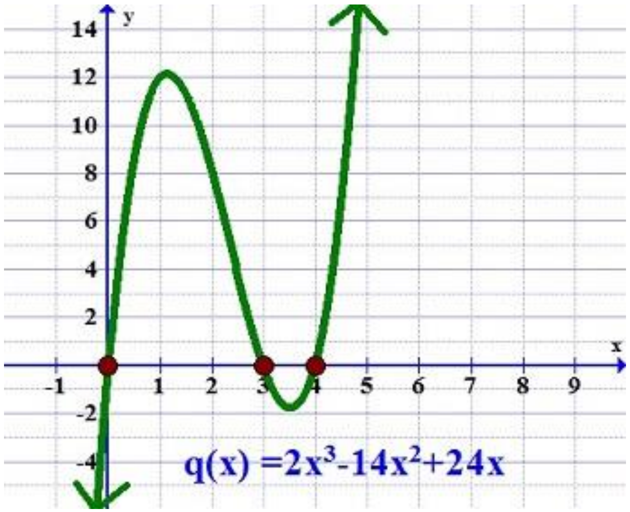


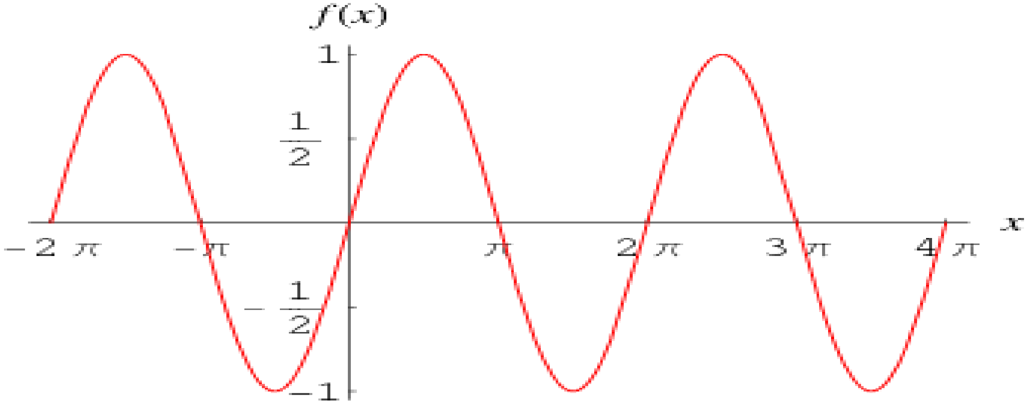
A sample function: f(x) = 2x+3

Functions

We have learned different types of functions.

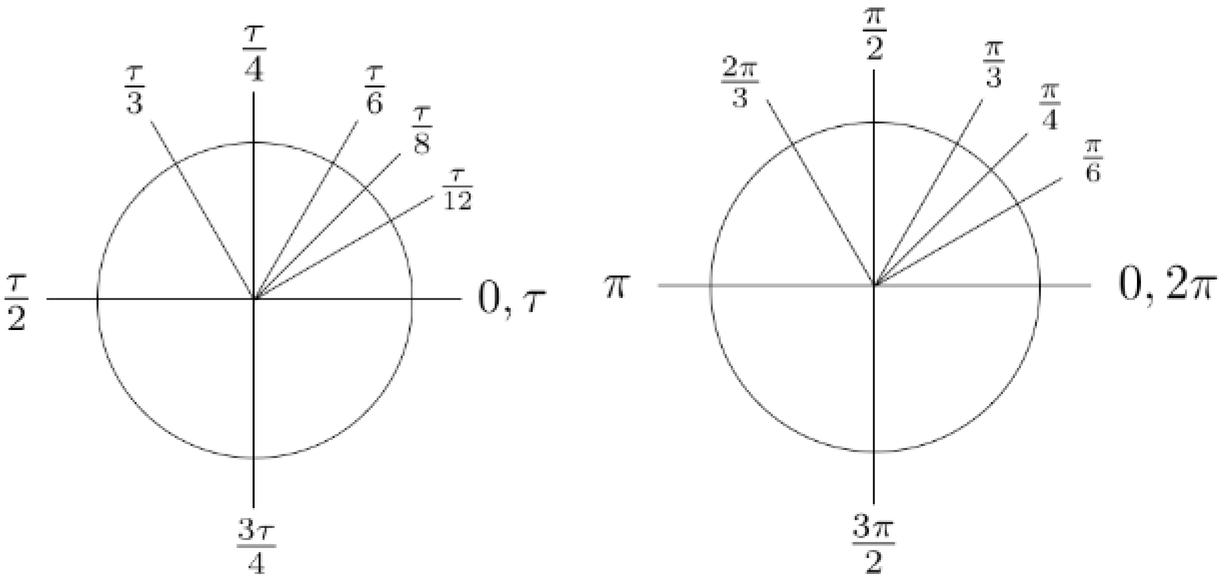






Tau or Pie?

Constant are somehow arbitrary, even the most important one!



https://www.theverge.com/tldr/2018/3/14/17119388/pi-day-pie-math-tau-circle-constant-mathematics-circumference-diameter-radius-holiday-truth

Functions

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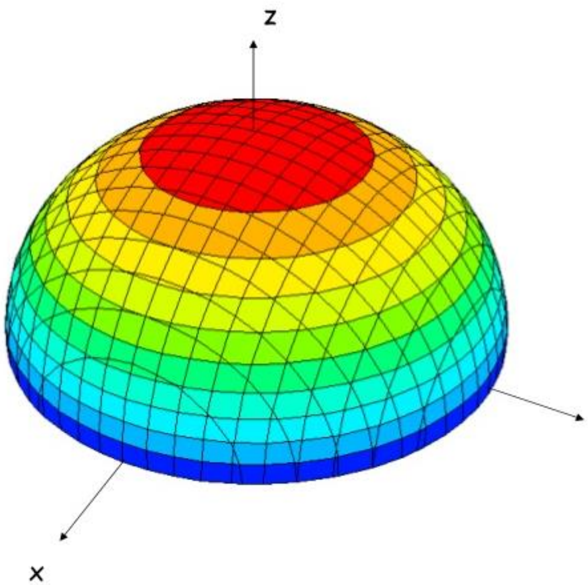
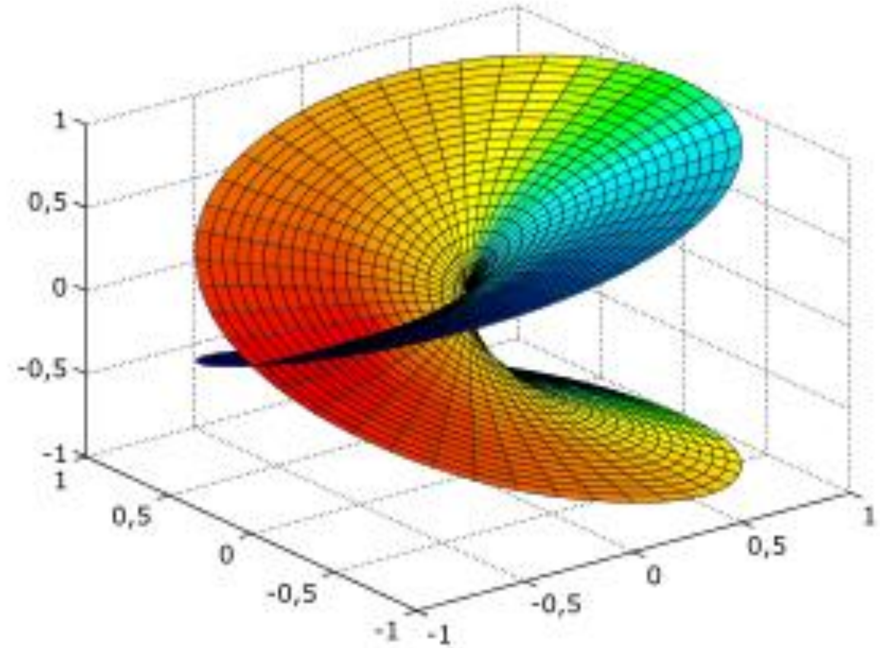
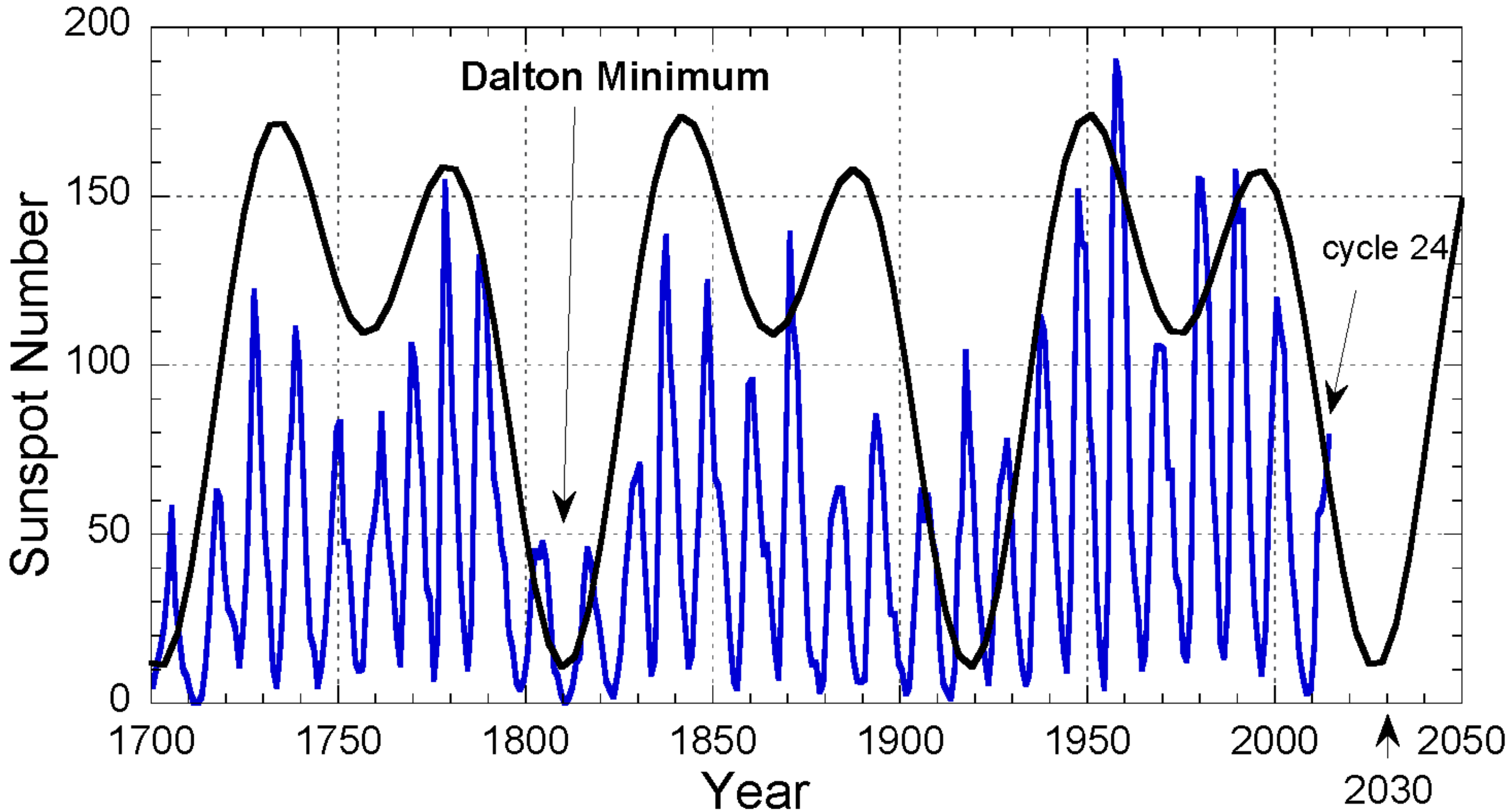


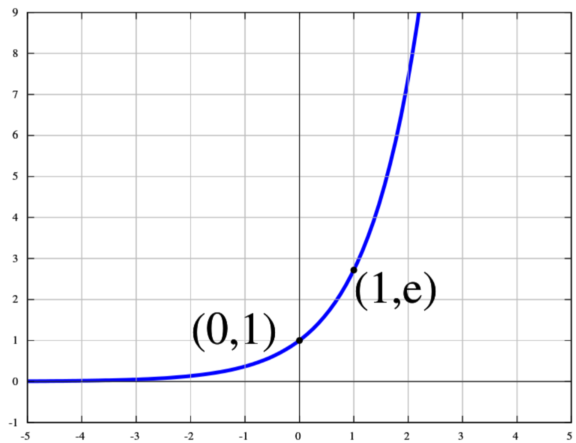


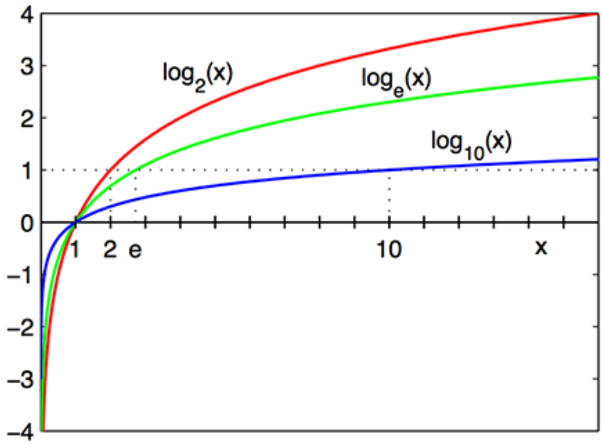
The Real-World Data

In the real-world, when we are investigating relations, we may find the following:

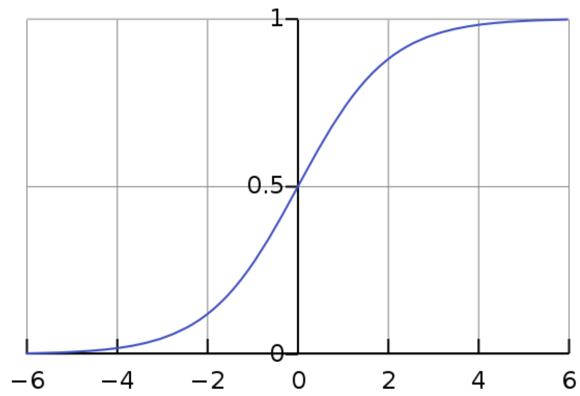


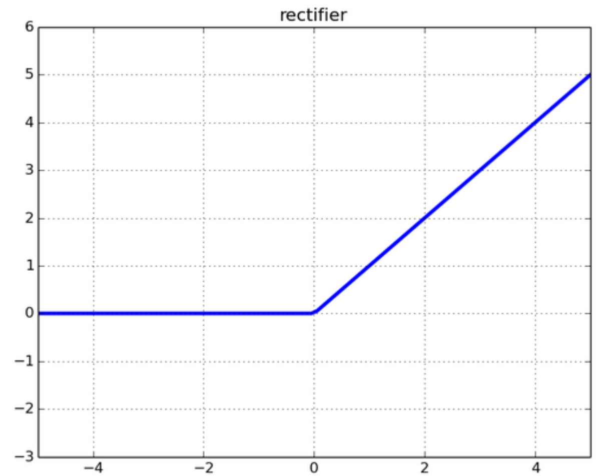
Some Functions





$y = e^x$ http://setosa.io/ev/exponentiation/





$$S(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1}$$

$$f(x) = x^+ = \max(0, x)$$

Function Decomposition

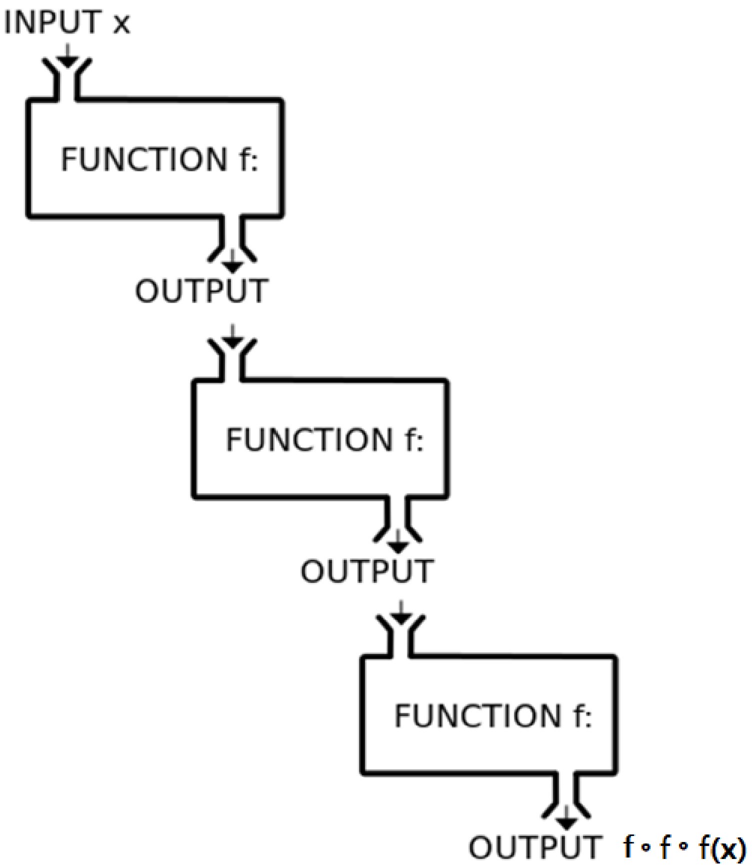
"Function Composition" is applying one function to the results of another:
The result of f() is sent through g()

It is written: $(g \circ f)(x)$
Which means: $g(f(x))$

$f(x) = 2x+3$

$f \circ f(x) = ?$
$f \circ f \circ f\ (x) = ?$

$f \circ f \circ f\ (2) =$

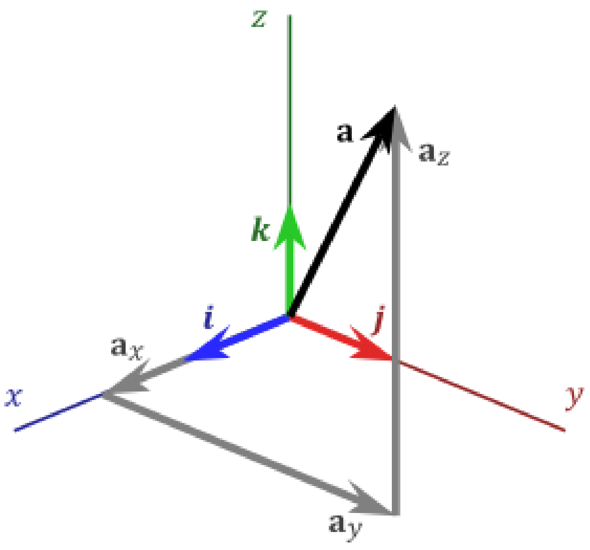


Linear Algebra

Vector

A vector space V is a set (the elements of which are called vectors) on which two operations are defined: vectors can be added together, and vectors can be multiplied by real numbers called scalars.

Can be written in column form or row form – Column form is conventional!



$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ \vdots \\ x_n+y_n \end{bmatrix}, \qquad \alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}$$

Vector Space

- Euclidean space is used to mathematically represent physical space, with notions such as distance, length, and angles.
- Although it becomes hard to visualize for n > 3, these concepts generalize mathematically in obvious ways.
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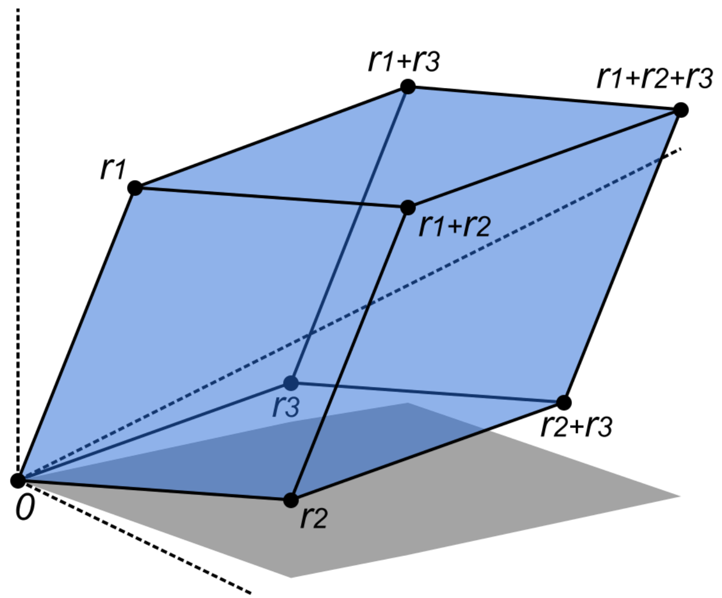
- Linear relations hold in high dimensional space.



Norm of Vectors

A **norm** on a real vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies

(i) $\|\mathbf{x}\| \geq 0$, with equality if and only if $\mathbf{x} = \mathbf{0}$

(ii) $\|\alpha\mathbf{x}\| = |\alpha|\|\mathbf{x}\|$

(iii) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ (the **triangle inequality** again)

We will typically only be concerned with a few specific norms on $\mathbb{R}^n$:

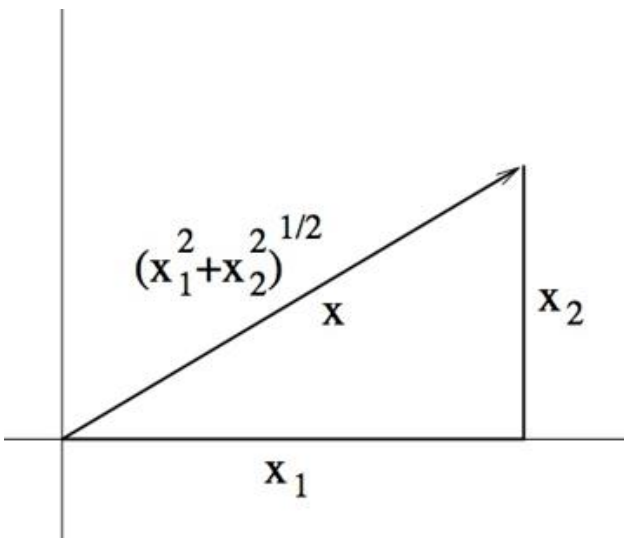
$$\|\mathbf{x}\|_1 = \sum_{i=1}^{n} |x_i| \qquad \|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \qquad (p \geq 1)$$

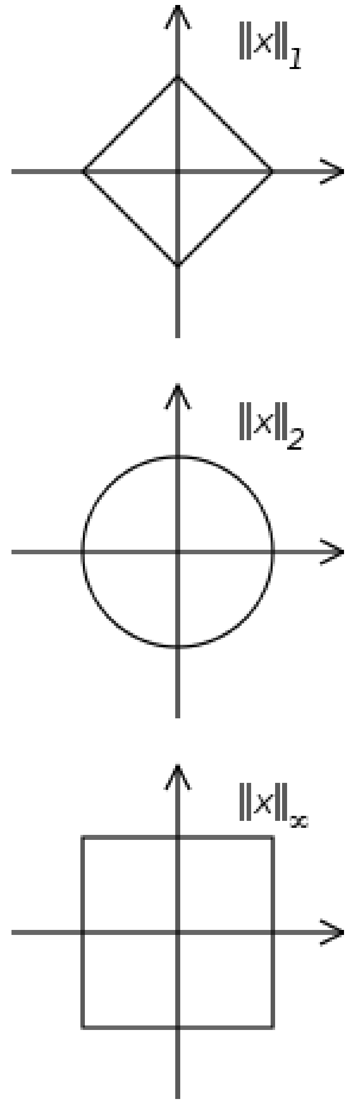
$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2} \qquad \|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i|$$

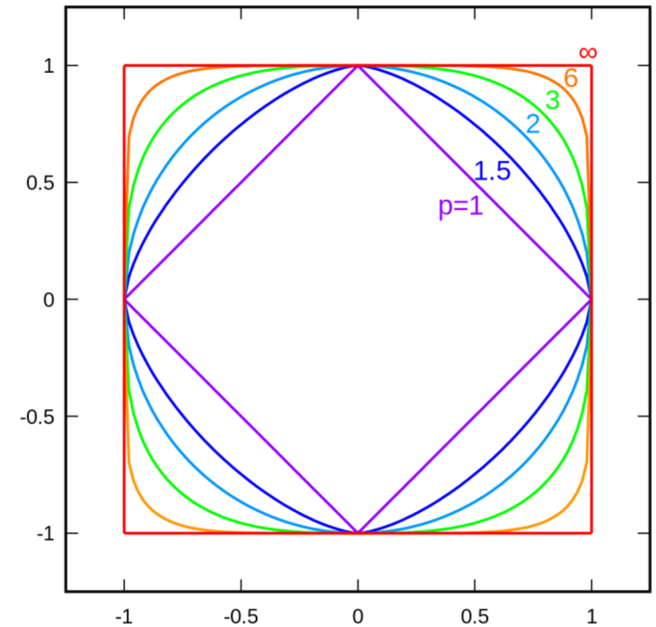
L-0 to L-infinity Norms

a **norm** is a function that assigns a strictly *positive length* to a vector。

A simple example is two dimensional Euclidean space R2 equipped with the "Euclidean norm"







$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$$

Matrix

A vector can be regarded as special case of a matrix, where one of matrix dimensions = 1.

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

Matrix transpose (denoted T)

$$\mathbf{A} = \begin{pmatrix} 2 & 7 & -1 & 0 & 3 \\ 4 & 6 & -3 & 1 & 8 \end{pmatrix} \qquad \mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 2 & 4 \\ 7 & 6 \\ -1 & -3 \\ 0 & 1 \\ 3 & 8 \end{pmatrix}$$

$$C = AB \qquad \Leftrightarrow \qquad c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

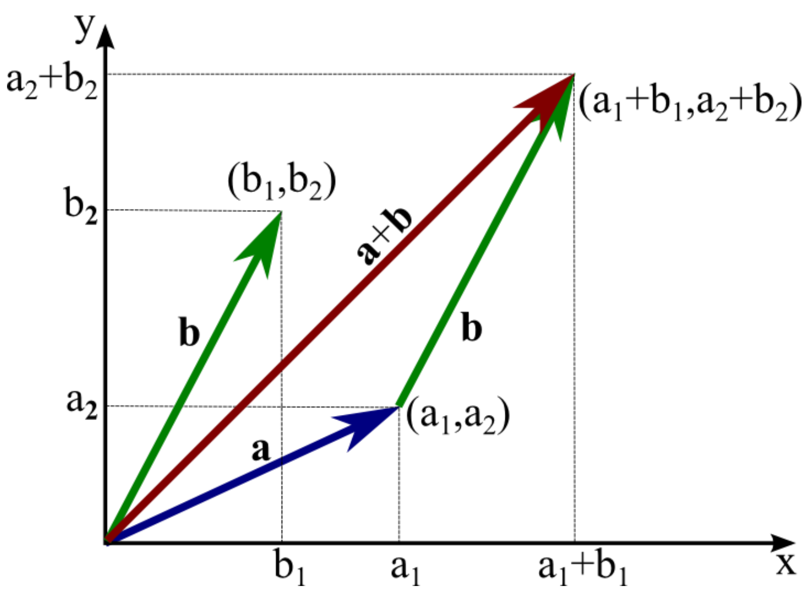
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

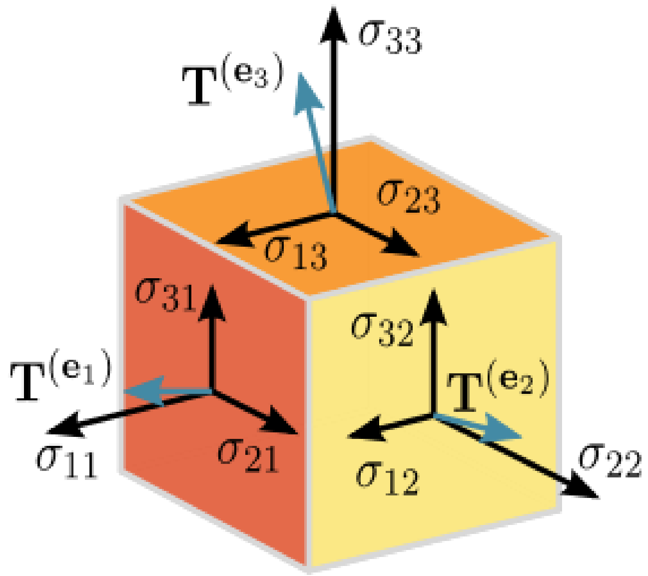
Vector to Tensor

https://www.quora.com/What-is-a-tensor

Columns are the stresses (forces per unit area) acting on the $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ faces of the cube.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \equiv \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$



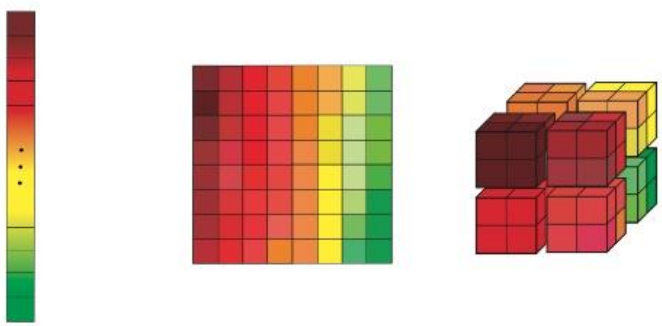


https://www.wukong.com/question/6531498435785261325/
https://www.bilibili.com/video/av10852829/

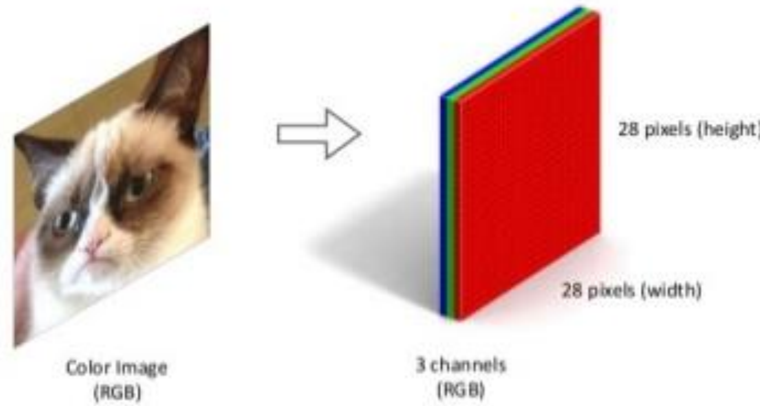
Tensor

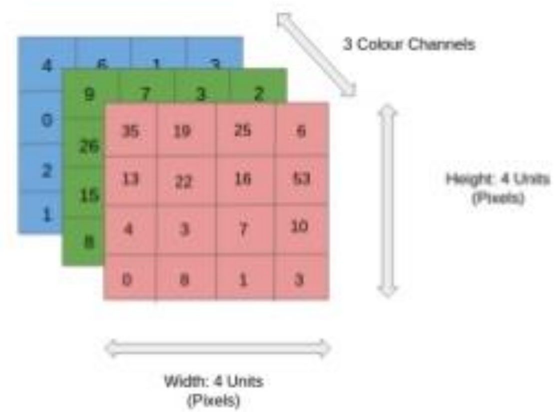
tensor = multidimensional array

vector matrix tensor



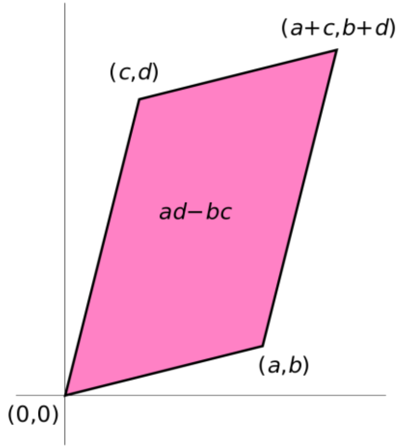
$\mathbf{v} \in \mathbb{R}^{64}$ $\quad X \in \mathbb{R}^{8x8}$ $\quad \boldsymbol{\mathcal{X}} \in \mathbb{R}^{4x4x4}$



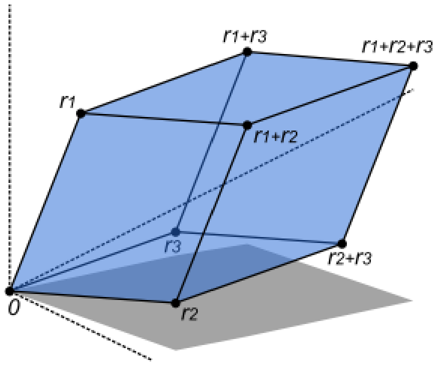


Determinant

In linear algebra, the determinant is a useful value that can be computed from the elements of a square matrix. The determinant of a matrix A is denoted det(A), det A, or |A|. It can be viewed as the scaling factor of the transformation described by the matrix.



$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$



$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$
$$= aei + bfg + cdh - ceg - bdi - afh.$$

Eigenvector and Eigenvalue

For a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$, there may be vectors which, when $\mathbf{A}$ is applied to them, are simply scaled by some constant. We say that a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ is an **eigenvector** of $\mathbf{A}$ corresponding to **eigenvalue** λ if

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

The zero vector is excluded from this definition because $\mathbf{A0} = \mathbf{0} = \lambda\mathbf{0}$ for every λ.

We now give some useful results about how eigenvalues change after various manipulations.

The **trace** of a square matrix is the sum of its diagonal entries:

$$\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} A_{ii}$$

http://setosa.io/ev/eigenvectors-and-eigenvalues/

Eigen Decomposition

Assume square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with n linearly independent eigenvectors $\mathbf{q}_i, i = 1, \ldots, n$ and n eigenvalues $\lambda_1, \ldots, \lambda_n$. Then $\mathbf{A}$ can be factorised as

$$\mathbf{A} = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}$$

where $\mathbf{Q} = [\mathbf{q}_1 \ldots \mathbf{q}_n]$ and $\mathbf{\Lambda}$ is a diagonal matrix whose diagonal elements are the corresponding eigenvalues, i.e. $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$.
Using the eigen-decomposition we can compute various powers of $\mathbf{A}$ as

$$\mathbf{A}^k = \mathbf{Q}\mathbf{\Lambda}^k\mathbf{Q}^{-1}.$$

We can easily verify the above for $k = 2$ as $\mathbf{A}^2 = \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^{-1} = \mathbf{Q}\mathbf{\Lambda}^2\mathbf{Q}^{-1}$. Then, we can easily prove the general case using induction.
In case $k = -1$ we can compute the inverse as

$$\mathbf{A}^{-1} = \mathbf{Q}\mathbf{\Lambda}^{-1}\mathbf{Q}^{-1}.$$

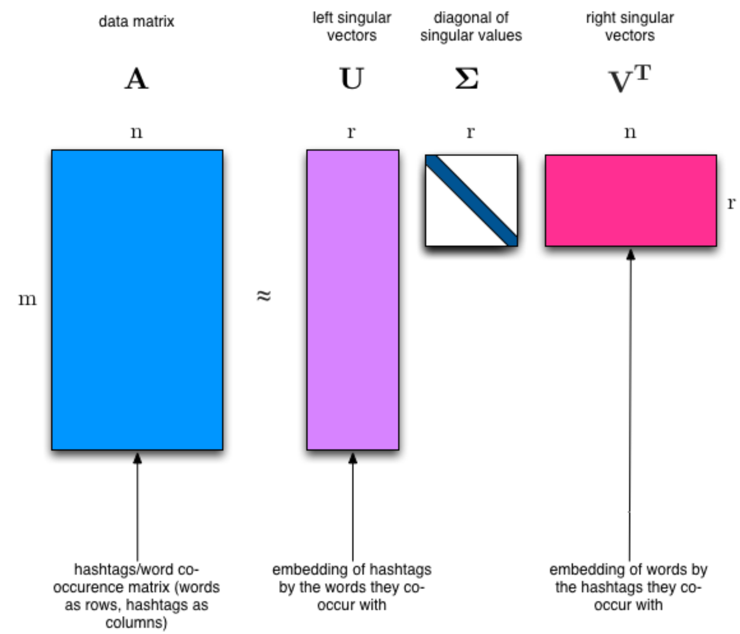
Singular Value Decomposition

Singular Value Decomposition:

Formally, the SVD of a real $m \times n$ matrix A is a factorization of the form $A = U \Sigma V^T$, where U is an $m \times m$ orthogonal matrix of left singular vectors, Σ is an $m \times n$ diagonal matrix of singular values, and V^T is an $n \times n$ orthogonal matrix of right singular vectors.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{V}^* = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$



Jacobian and Hessian Matrices

The **Jacobian** of $f : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix of first-order partial derivatives:

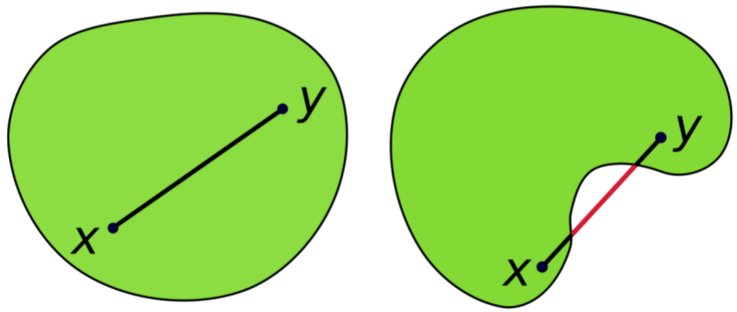
$$\mathbf{J}_f = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \quad \text{i.e.} \quad [\mathbf{J}_f]_{ij} = \frac{\partial f_i}{\partial x_j}$$

Note the special case $m = 1$, where $\nabla f = \mathbf{J}_f^\top$.

The **Hessian** matrix of $f : \mathbb{R}^d \to \mathbb{R}$ is a matrix of second-order partial derivatives:

$$\nabla^2 f = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} \quad \text{i.e.} \quad [\nabla^2 f]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Convex Set and Function



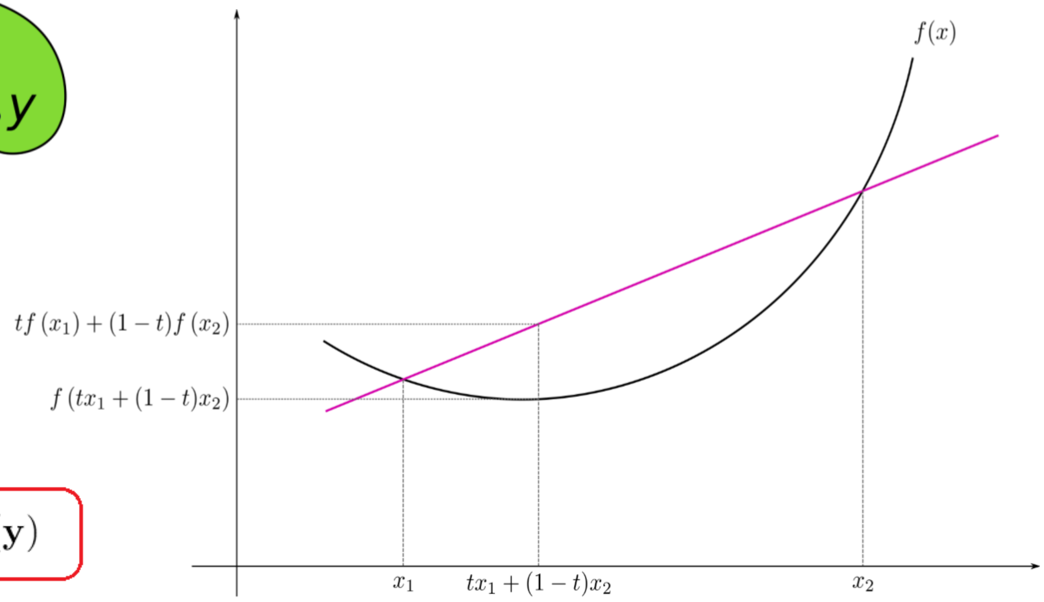


Figure 2: What convex functions look like

A function f is **convex** if

$$\boxed{f(t\mathbf{x} + (1-t)\mathbf{y}) \leq tf(\mathbf{x}) + (1-t)f(\mathbf{y})}$$

for all $\mathbf{x}, \mathbf{y} \in \operatorname{dom} f$ and all $t \in [0, 1]$.

References

This slide of this class is modified from Lecture Notes of Dimitri P. Bertsekas and John N. Tsitsiklis – Introduction to Probability, MIT, 2000. & Wikipedia.
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