(4.3) (1). $\dot{x}_1 = -x_1 + x_1 x_2 \qquad \dot{x}_2 = -x_2.$

显然原点是平衡点. 取 $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

$V'(x) = x_1 x_1' + x_2 x_2' = x_1 \cdot (-x_1 + x_1 x_2) + x_2 \cdot (-x_2)$

$\therefore V'(x) = -x_1^2 + x_1^2 x_2 - x_2^2.$

设 D 为包含原点. 半径为 r 的圆. $\sqrt{(x_1^2+x_2^2)} \le r$, $|x_1| \le r$, $|x_2| \le r$

$\therefore V'(x) = -x_1^2 - x_2^2 + x_1 \cdot x_1 \cdot x_2 \le -(x_1^2 + x_2^2) + r \cdot x_1 x_2$

当 $r < 2$ 时. $V'(x) < -x_1^2 - x_2^2 + 2x_1 x_2 = -(x_1 - x_2)^2 \le 0$

~~原点是稳定的~~. $\therefore V'(x) < 0$, 为负定, 当 $r < 2$.

$\therefore$ 原点是渐近稳定的.

(2). $\dot{x}_1 = -x_2 - x_1(1 - x_1^2 - x_2^2) \qquad \dot{x}_2 = x_1 - x_2(1 - x_1^2 - x_2^2)$

显然原点是平衡点. 取 $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$V'(x) = x_1 x_1' + x_2 x_2' = -x_1 x_2 - x_1^2(1 - x_1^2 - x_2^2) + x_1 x_2 - x_2^2(1 - x_1^2 - x_2^2)$

$2V - 1 < 0$

$2V < 1$

$V < \frac{1}{2}$

$= (x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) = 2V(x) \cdot [2V(x) - 1]$

令 D 为 $x_1^2 + x_2^2 < \frac{1}{2}$. 此时 $V'(x) < 0$

$\therefore$ 原点是渐近稳定的.

(4.4.)

$J_1 \dot{\omega}_1 = (J_2 - J_3)\omega_2 \omega_3 + u_1$

$J_2 \dot{\omega}_2 = (J_3 - J_1)\omega_3 \omega_1 + u_2$

$J_3 \dot{\omega}_3 = (J_1 - J_2)\omega_1 \omega_2 + u_3.$

(a). 令 $V(\omega) = \frac{1}{2}(J_1 \omega_1^2 + J_2 \omega_2^2 + J_3 \omega_3^2)$, $u_i = 0$

$V'(\omega) = J\omega_1 \dot{\omega}_1 + J\omega_2 \dot{\omega}_2 + J\omega_3 \dot{\omega}_3 = [(J_2 - J_3) + (J_3 - J_1) + (J_1 - J_2)]\omega_1 \omega_2 \omega_3$

$= 0$

$\therefore$ 是稳定的. 对于任意 ω, $V(\omega)$ 恒为 0, 则必不渐近稳定

(b). $u_i = -k_i \omega_i$. $V'(\omega) = -k_1 \omega_1^2 - k_2 \omega_2^2 - k_3 \omega_3^2 = -(k_1 \omega_1^2 + k_2 \omega_2^2 + k_3 \omega_3^2)$

显然 $V'(\omega) \le 0$. 且对于任意 $\omega(t_0) \ne 0$, 除原点外, $V'(x) < 0$

$\therefore$ 为渐近稳定.

4.13

(1). $\dot{x}_1 = x_1^3 + x_1^2 x_2$ $\dot{x}_2 = -x_2 + x_2^2 + x_1 x_2 - x_1^3$.

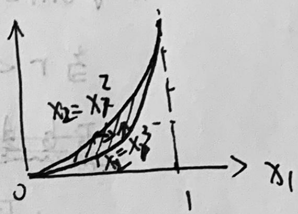
令 $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$，在 $x_2=0$ 直线上，任意接近原点的点有 $V(x)>0$.

$V'(x) = x_1\dot{x}_1 - x_2\dot{x}_2 = (x_1^2 + x_1x_2)^2 + x_2^2(1 - x_2 - x_1 - x_1^2)$.

易知，显然存在一定区域使 $1 - x_1 - x_1^2 > 0$ 即 $x^2 + x_1 < 1$ 即 D

在D内 $V'(x)$ 显然正定，则原点不稳定.

(2). $\dot{x}_1 = -x_1^3 + x_2$ $\dot{x}_2 = x^6 - x_2^3$. $\Gamma = \{0 \le x_1 \le 1\} \cap \{x_2 \ge x_1^3\} \cap \{x_2 \le x_1^2\}$



在边界 $x_2 = x_1^2$ 内，$\dot{x}_2 = 0$，$\dot{x}_1 > 0$，系统向 x_1 正方向移动；在边界 $x_2 = x_1^3$ 内，$\dot{x}_1 = 0$，$\dot{x}_2 > 0$ 系统向 x_2 正方向移动，则 Γ 为正不变集.

当在 Γ 内，即 ~~$x_2 < x_1^2$~~ $x_1^3 \le x_2 \le x_1^2$；$\dot{x}_1 > 0$，$\dot{x}_2 > 0$.

∴系统向 (1, 1) 移动，则系统原点显然不稳定.

4.14

$\dot{x}_1 = x_2$ $\dot{x}_2 = -g(x_1)(x_1 + x_2)$.

$V(x) = \int_0^{x_1} y\cdot g(y)\,dy + x_1x_2 + x_2^2$. $g(y) \ge 1$. 则 $\int_0^{x_1} y\cdot g(y)\,dy \ge \int_0^{x_1} y\,dy = \frac{1}{2}x_1^2$.

∴ $V(x) \ge \frac{1}{2}x_1^2 + x_1x_2 + x_2^2 = [x_1, x_2]\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

∴ $V(x)$ 正定，且 $x\to\infty$ 时，$V(x)$ 显然 $\to\infty$，则径向无界.

$V'(x) = x_1 \cdot g(x_1)\cdot\dot{x}_1 + x_2\dot{x}_1 + x_1\dot{x}_2 + 2x_2\dot{x}_2$

$= -g(x_1)(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 = -g(x_1)[x_1, x_2]\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + x_2^2$.

$1>0$，$\begin{vmatrix}1 & 1\\ 1 & 2\end{vmatrix} > 0$，则 $x^T\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}x$ 正定，则 $-g(x_1)x^T\begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}x$ 负定

∴ $V'(x) \le -(x_1^2 + 2x_1x_2 + 2x_2^2) + x_2^2 = -(x_1 + x_2)^2 \le 0$.

对于任意 $x(t) \ne 0$，除原点外，$V'(x) < 0$，

∴原点是全局渐近稳定的.

4.16

$\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1^3 - x_2^3$.

令 $V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$，$V'(x) = x_1^3x_2 - x_1^3x_2 - x_2^4 = -x_2^4 \le 0$. $x \in R^n$.

~~当 $x_2 = 0$ 时，$V'(x) = 0$，$x_1^3 = x_2^3 = 0$~~ 若X 设 $S = \{x \in R^n \mid V'(x) = 0\}$.

当 $x_1 \ne 0$ 时，$\dot{x}_2 \ne 0$ 矛盾，则除 $x(t) \equiv 0$ 外，无其他解

∴原点是全局渐近稳定的.