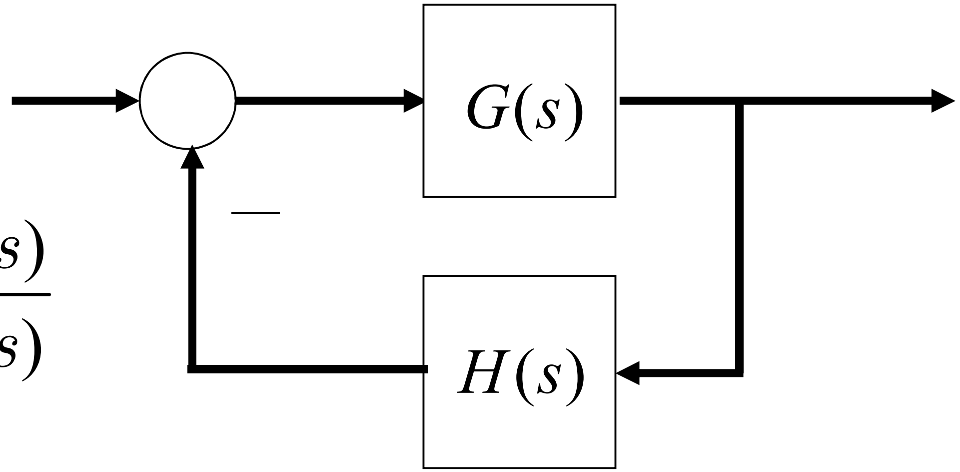
7-4 Nyquist Stability Criterion

1. Introduction to Nyquist stability criterion

The Nyquist stability criterion determines the stability of a closed-loop system from its **open-loop frequency response and open-loop poles**, and there is no need for actually determining the closed-loop poles.

To determine the stability of a closed-loop system, one must investigate the closed-loop characteristic equation of the system.

We consider the following typical closed-loop system:



where

$$G(s) := \frac{N_1(s)}{D_1(s)}, \ H(s) := \frac{N_2(s)}{D_2(s)}$$

From the above diagram, we have

$$\Phi(s) = \frac{G(s)}{1+G(s)H(s)}$$

We define

$$F(s) = 1 + G(s)H(s) = 0$$

which is also called the **characteristic equation** of the closed-loop system and will play an important role in stability analysis. In fact, we have

$$\boxed{F(s)=\frac{D(s)+N(s)}{D(s)}, \quad D=D_1D_2,\, N=N_1N_2}$$

Further, we assume that $\deg(N)<\deg(D)$, which implies that both the denominator and numerator polynomials of $F(s)$ have the same degree.

2. Preliminary Study

Consider, for example, the following open-loop transfer function:

$$G(s)H(s) = \frac{2}{s-1}$$

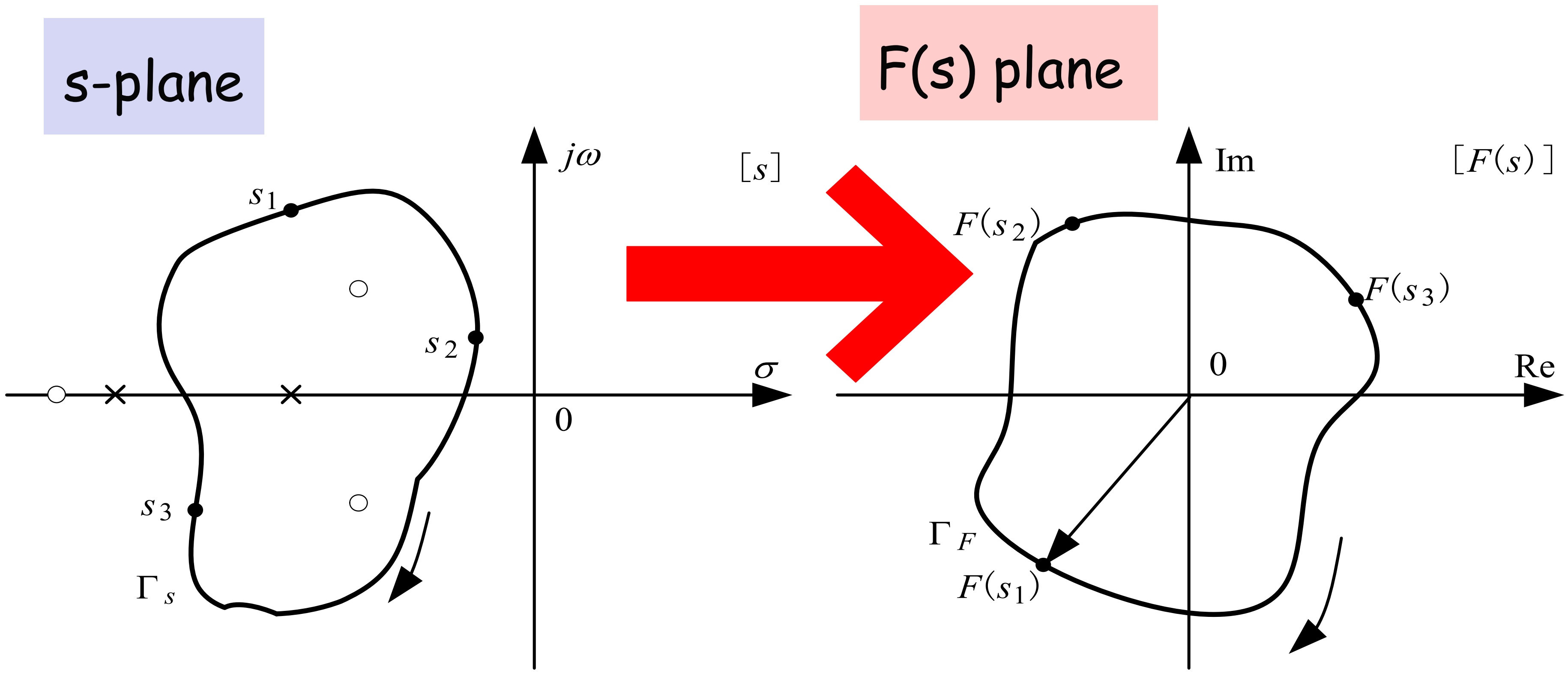
whose closed-loop **characteristic equation** is

$$F(s) = 1 + G(s)H(s) = \frac{s+1}{s-1} = 0$$

which is analytic everywhere except at its singular points. For each point analytic in s plane, $F(s)$ maps the point into $F(s)$ plane. For example, $s=2+j: \longrightarrow F(s)=2-j$.

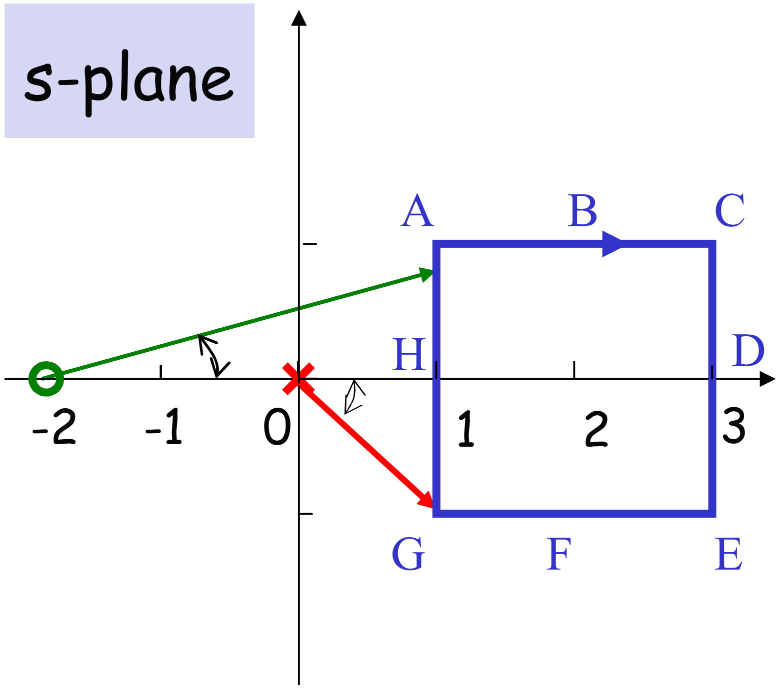
Further, we have the following conclusion:

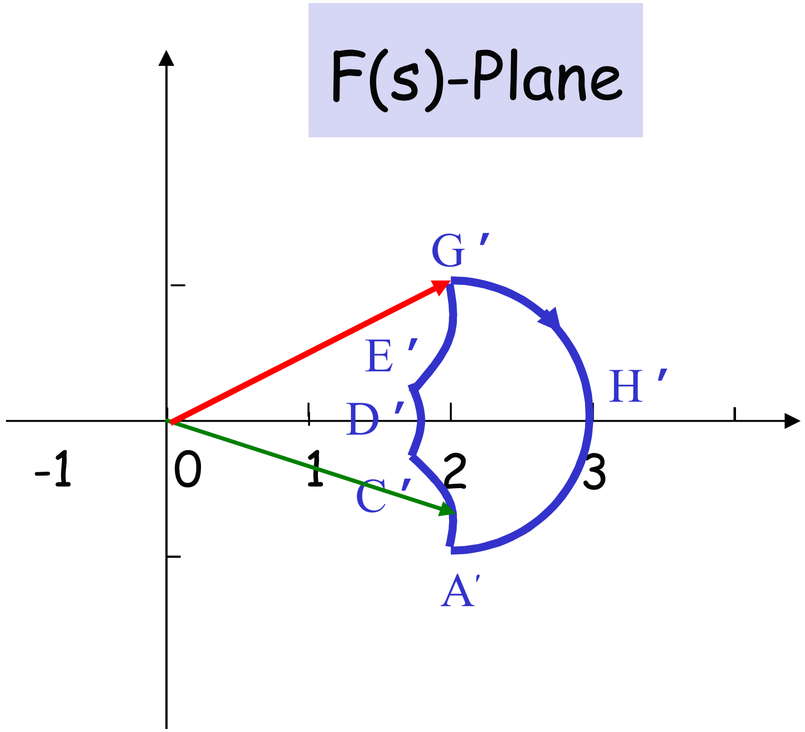
Conformal mapping: For any given continuous closed path in the s plane that does not go through any singular points, there corresponds a closed curve in the $F(s)$ plane.



Example. Consider $F(s) = \dfrac{s+2}{s}$

Case 1:





Γ_s: clockwise

Γ_F: clockwise

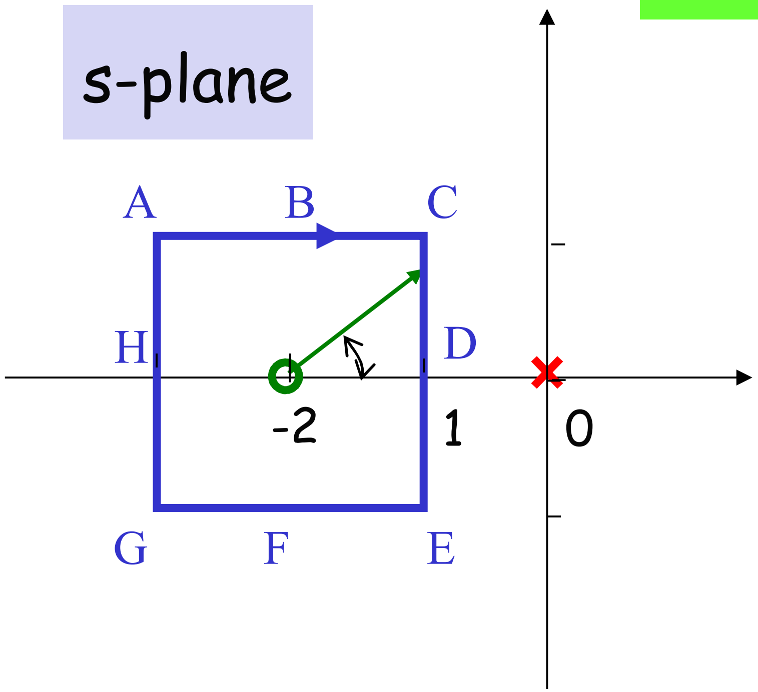
Γ_F encircles the origin:

$$\Delta\angle(s+2) = 0^\circ$$
$$\Delta\angle(s+0) = 0^\circ$$

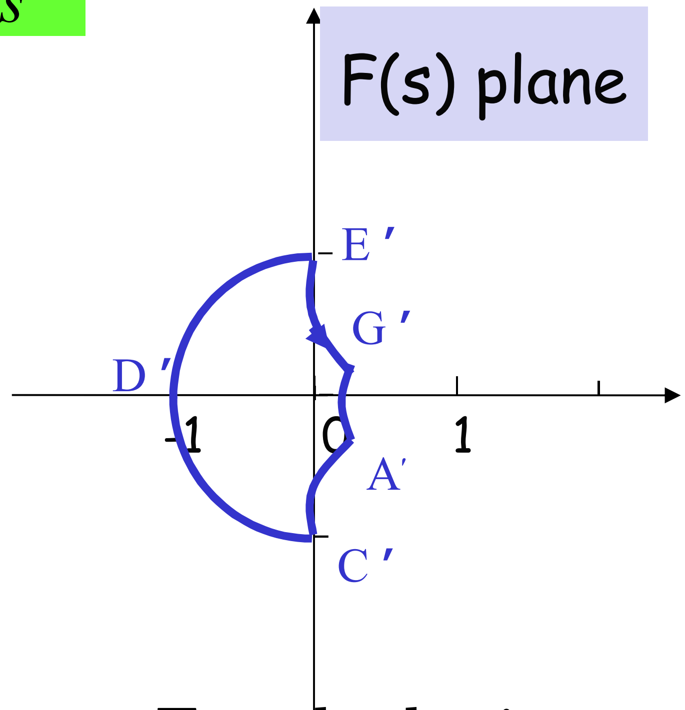
$$\Delta\angle F(s) = \Delta\angle(s+2) - \Delta\angle(s+0)$$
$$= 0^\circ$$

Case 2:

$$F(s) = \frac{s+2}{s}$$



Γ_s: clockwise



Γ_F: clockwise

$$\Delta\angle(s+2) = -360^\circ$$

$$\Delta\angle(s+0) = 0^\circ$$

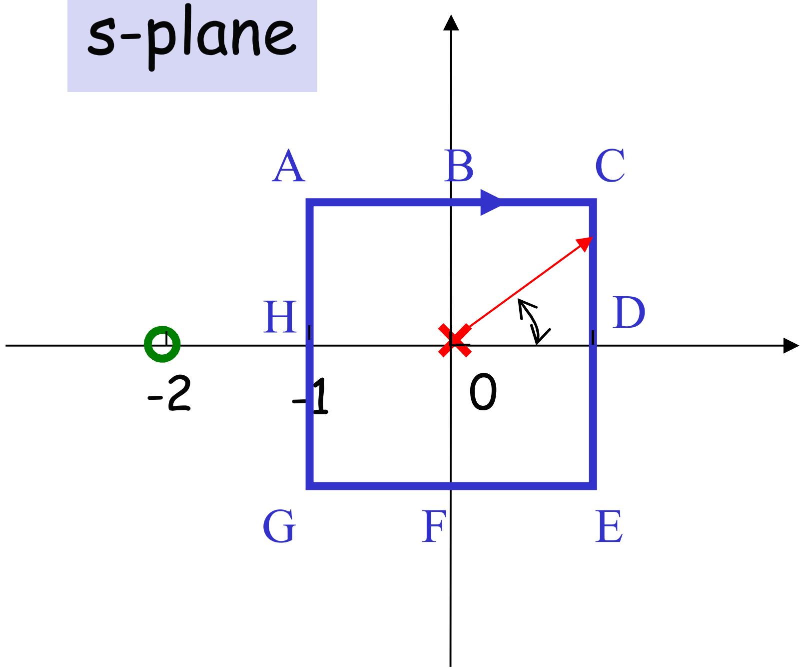
Γ_F encircles the origin:

$$\Delta\angle F(s) = \Delta\angle(s+2) - \Delta\angle(s+0)$$
$$= -360^\circ$$

Case 3:

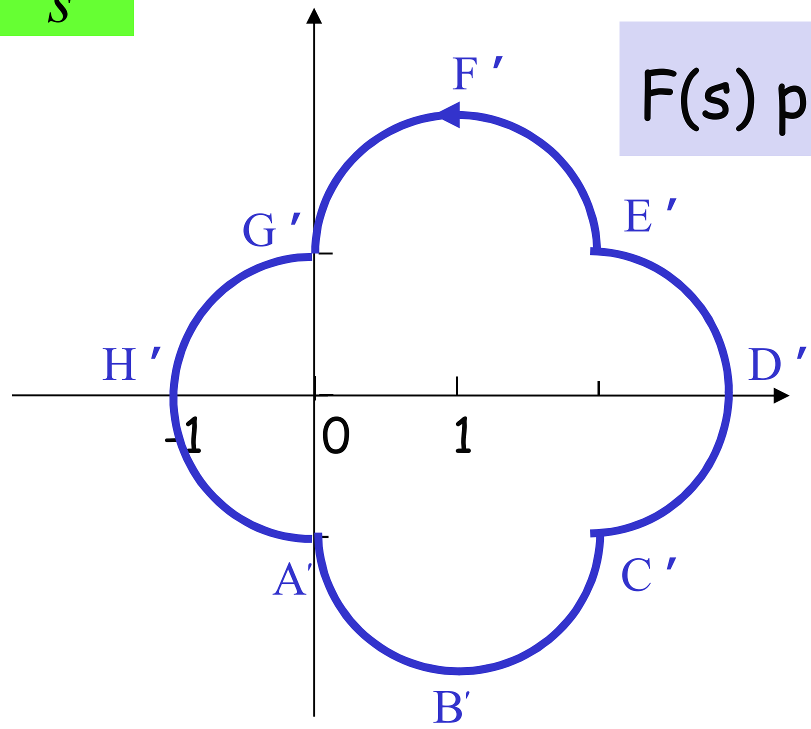
$$F(s) = \frac{s+2}{s}$$

s-plane



Γ_s: clockwise

F(s) plane



Γ_F: counterclockwise

$$\Delta\angle(s+2) = 0^0$$

$$\Delta\angle(s+0) = -360^\circ$$

Γ_F encircles the origin:

$$\Delta\angle F(s) = \Delta\angle(s+2) - \Delta\angle(s+0)$$

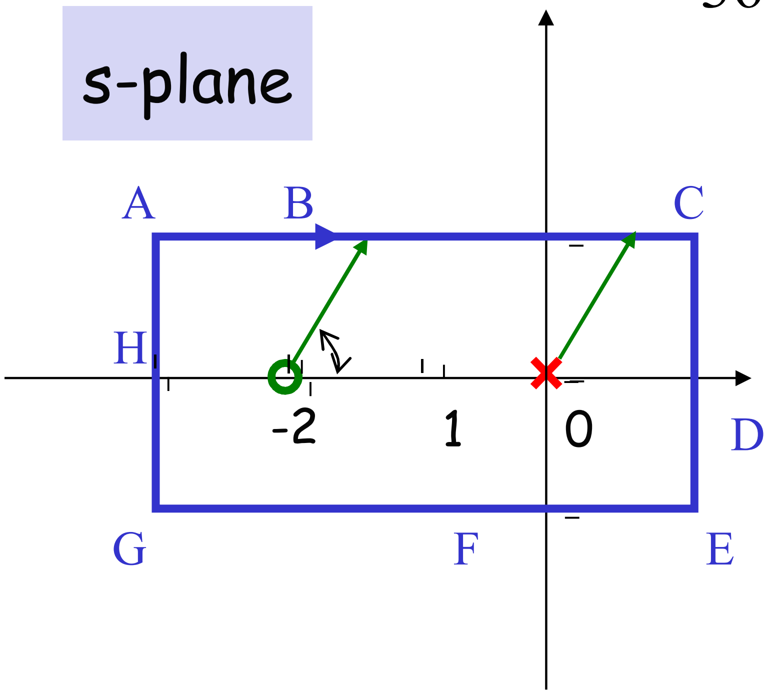
$$= 360^\circ$$

Case 4:

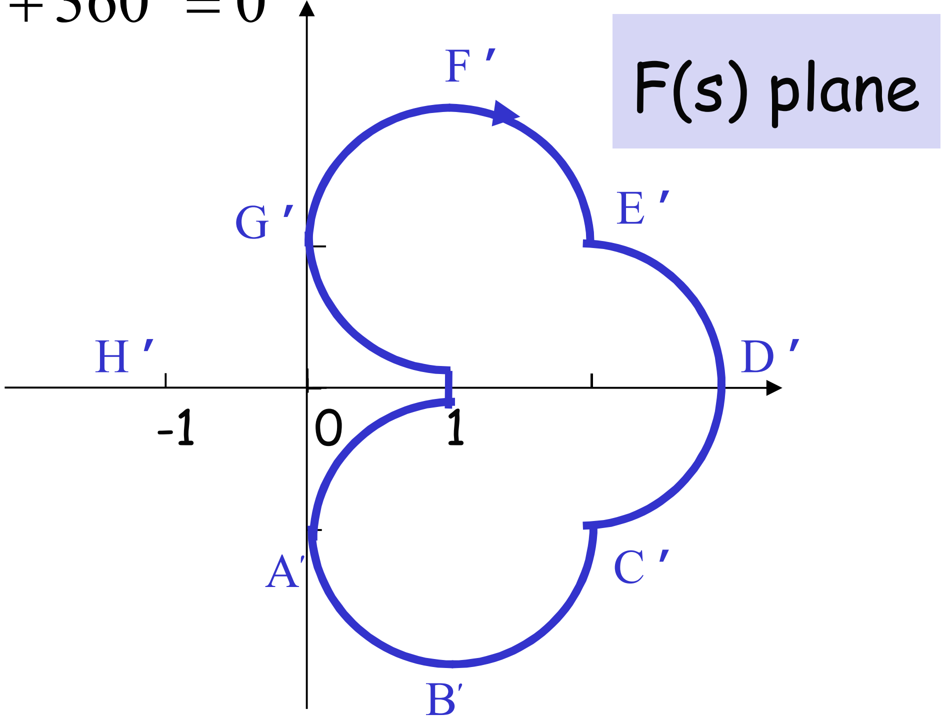
$$F(s)=\frac{s+2}{s}$$

In this case, the number of the Γ_F encircles the origin is $-1+1=0$ or in other words,

$$\Delta\angle F(s)=\Delta\angle(s+2)-\Delta\angle(s+0)$$
$$=-360^\circ+360^\circ=0^0$$



Γ_s: clockwise



Γ_F: clockwise

Mapping Theorem (The Principle of Argument)

Let $F(s)$ be a ratio of two polynomials in s.

$$F(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)}$$

Given a closed contour Γ_s which does not pass through **any poles and zeros of $F(s)$**. Then Γ_s in the s-plane is mapped into $F(s)$ plane as a closed curve, Γ_F.

Let the phase angle change of $F(s)$ be

$$\Delta\angle F(s) = \Delta\angle(s - z_1) + \Delta\angle(s - z_2) + \cdots + \Delta\angle(s - z_m) \\ - \Delta\angle(s - p_1) - \Delta\angle(s - p_2) - \cdots - \Delta\angle(s - p_n)$$

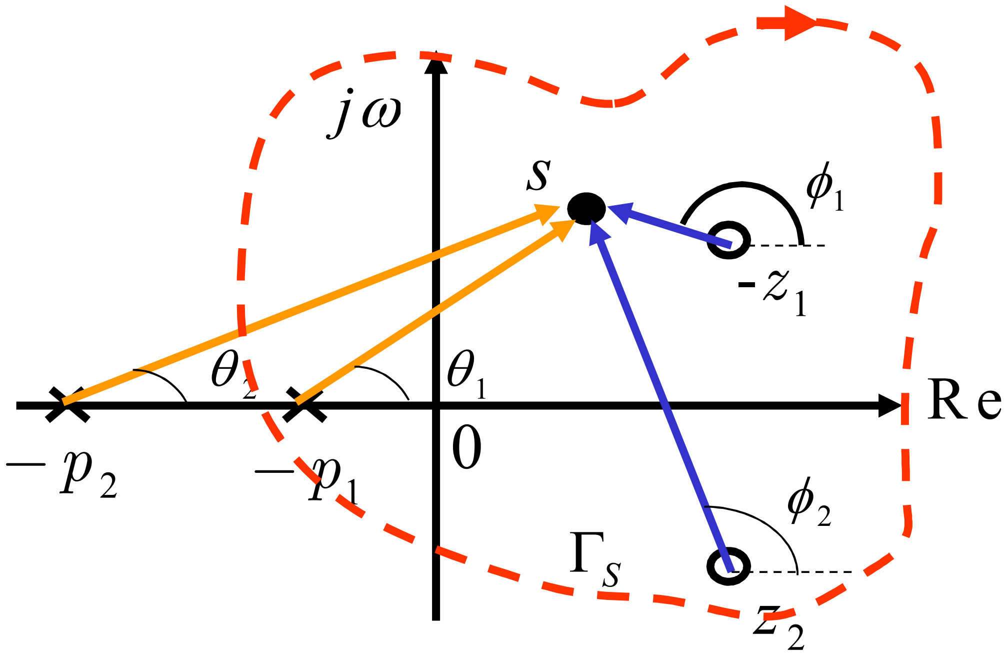
as s traces out Γ_s in the **clockwise** direction.

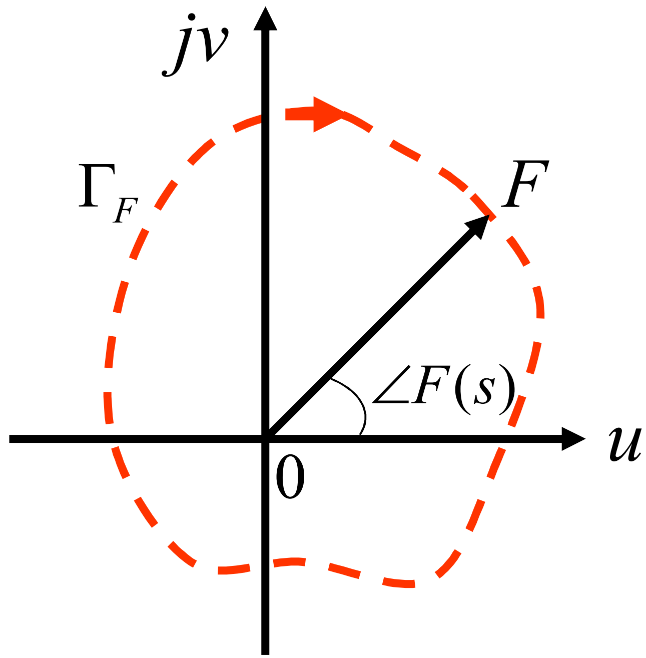
P**:** The number of **poles** of $F(s)$ that lie inside Γ_s

Z**:** The number of **zeros** of $F(s)$ that lie inside Γ_s

N**:** The total number of the **counterclockwise** encirclements of the origin in $F(s)$ plane, as s traces out Γ_s in the **clockwise** direction.

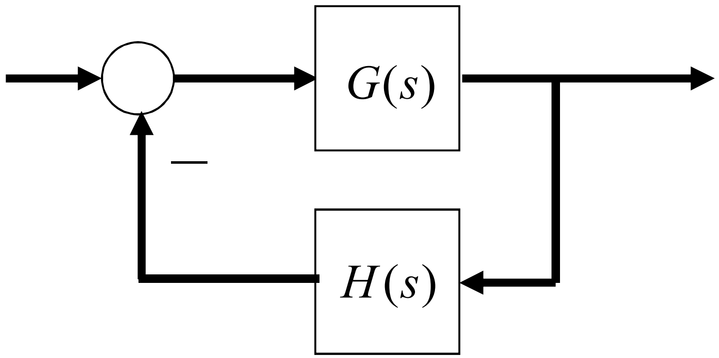
$$\boldsymbol{N = P - Z}$$





If, for example, $F(s)$ has two zeros and one pole are enclosed by Γ_s, it can be deduced from the above analysis that Γ_F encircles the origin of F plane one time in the clockwise direction as s traces out Γ_s.

3. Nyquist Stability Criterion



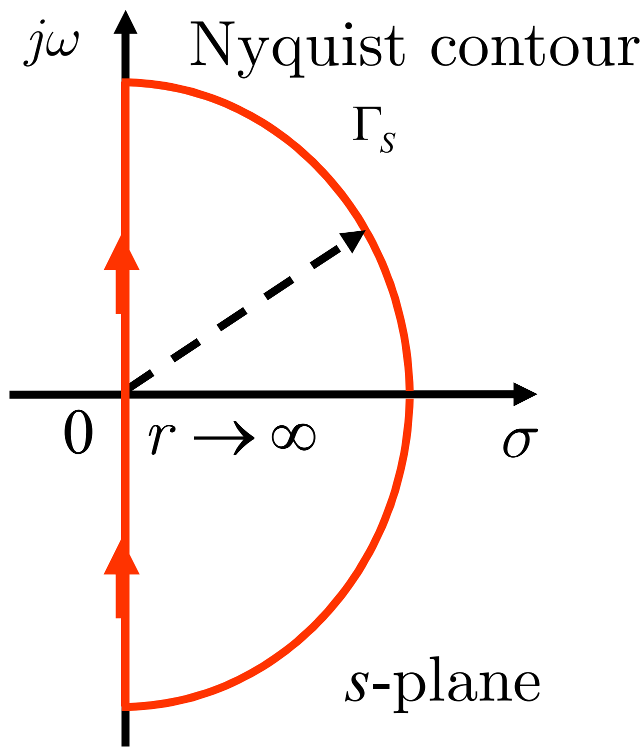
$$F(s)=\frac{D(s)+N(s)}{D(s)}$$

$$=\frac{\text{closed-loop characteristic equation}}{\text{open-loop characteristic equation}}$$

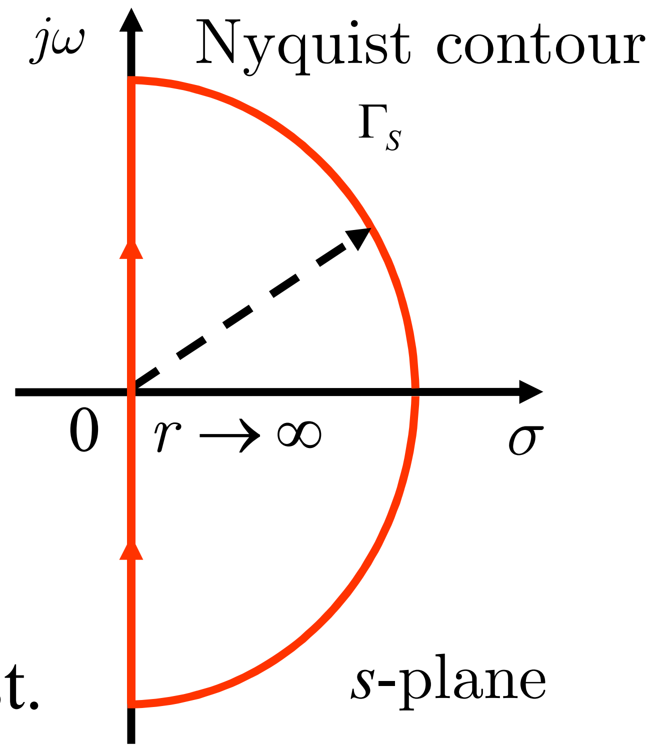
For a stable system, all the **zeros** of $F(\mathrm{s})$ must lie in the left-half s-plane.

Therefore, we choose a contour Γ_s, say, *Nyquist path*, in the s-plane that encloses the entire right-half s-plane. The contour consists of the entire $j\omega$ axis (ω varies from $-\infty$ to $+\infty$) and a semicircular path of infinite radius in the right-half s plane.

As a result, the Nyquist path encloses all the zeros and poles of $F(s)=1+G(s)H(s)$ having positive real parts.



Question: How to obtain the curve of $F(s)$ as s traces out Γ_s (*Nyquist* path) in the clockwise direction?



By assumption,

$$\lim_{s\to\infty} F(s)$$

$$= \lim_{s\to\infty}[1 + G(s)H(s)] = \text{const.}$$

$F(s)$ remains a constant when s traverses the semicircle of infinite radius → the number of encirclements of the origin of $F(s)$-plane is only determined by $F(j\omega)$ as ω varies from $-\infty$ to $+\infty$ provided that **no zeros or poles lie on the $j\omega$ axis**.

Application of the Mapping Theorem

***P*:** The number of **poles** of $F(s)=1+G(s)H(s)$ that lie inside Γ_s → The number of unstable **open-loop** poles.

***Z*:** The number of **zeros** of $F(s)=1+G(s)H(s)$ that lie inside Γ_s → The number of unstable **closed-loop** poles.

***N*:** The number of encirclements of the origin in F plane by $F(j\omega)$ as ω varies from $-\infty$ to $+\infty$ in the **counterclockwise** direction.

$$\boldsymbol{N = P - Z}$$

Known by drawing the curve

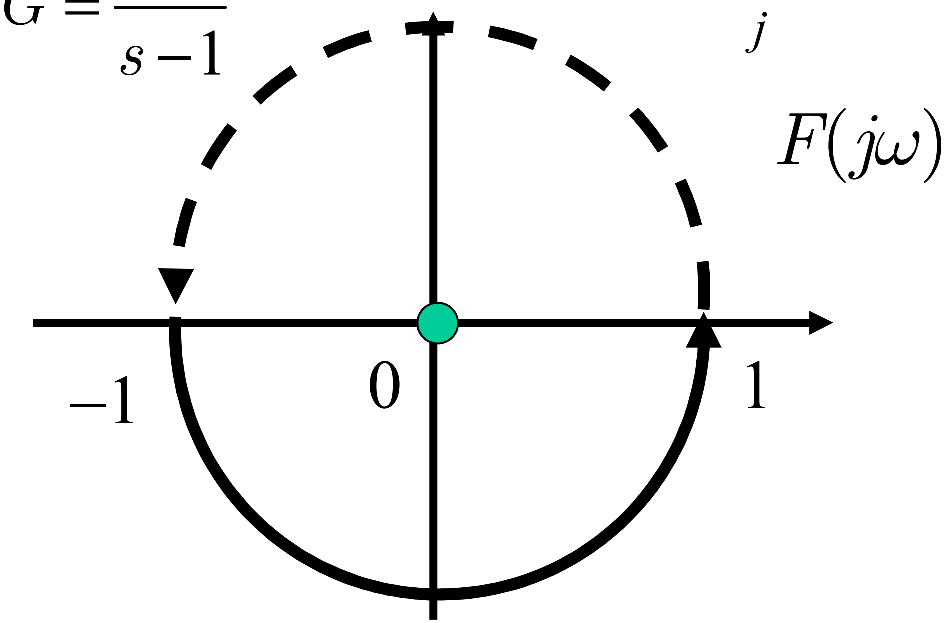
known

$Z = 0$ → The system is stable.
$Z > 0$ → The system is unstable.

Example.

Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s) = \frac{2}{s-1} \Rightarrow F(s) = 1 + G = \frac{s+1}{s-1}$$



$P = 1$ $N = 1$

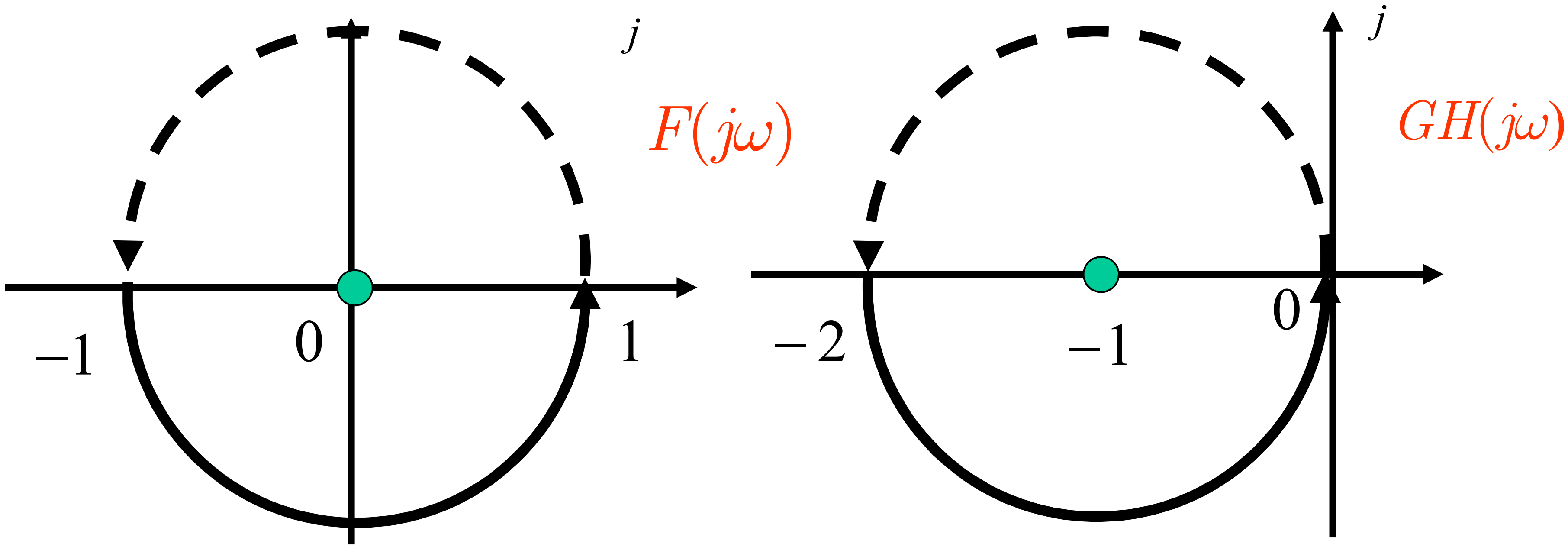
$Z = P - N = 0$

The system is stable.

Further, since

$$F(s) = 1 + GH$$

compare the two curves below:

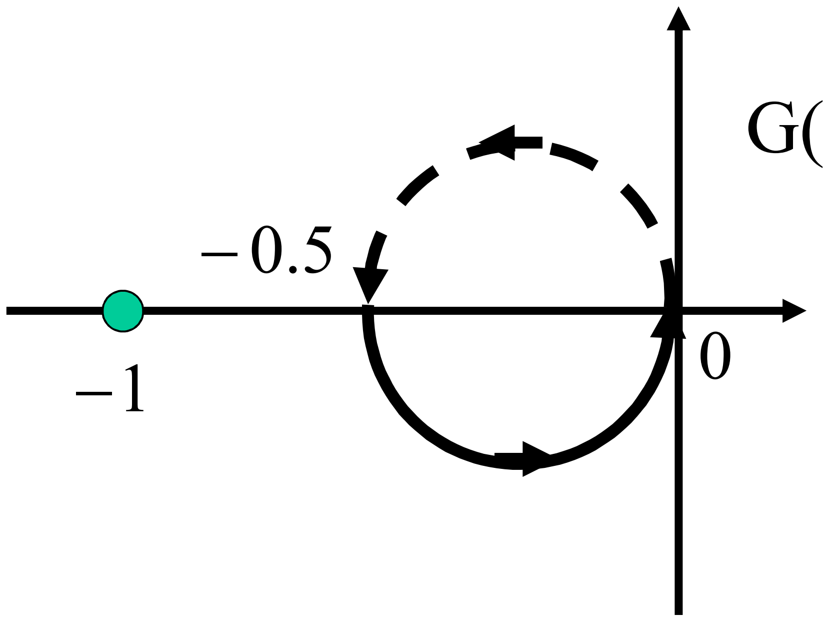


Clearly, the encirclement of the origin in F plane is equivalent to the encirclement of -1 point in GH plane!

Example.

Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s) = \frac{0.5}{s-1}$$



$P = 1 \qquad N = 0$

$Z = P - N = 1$

The system is unstable.

Example.

Consider a single-loop control system, where

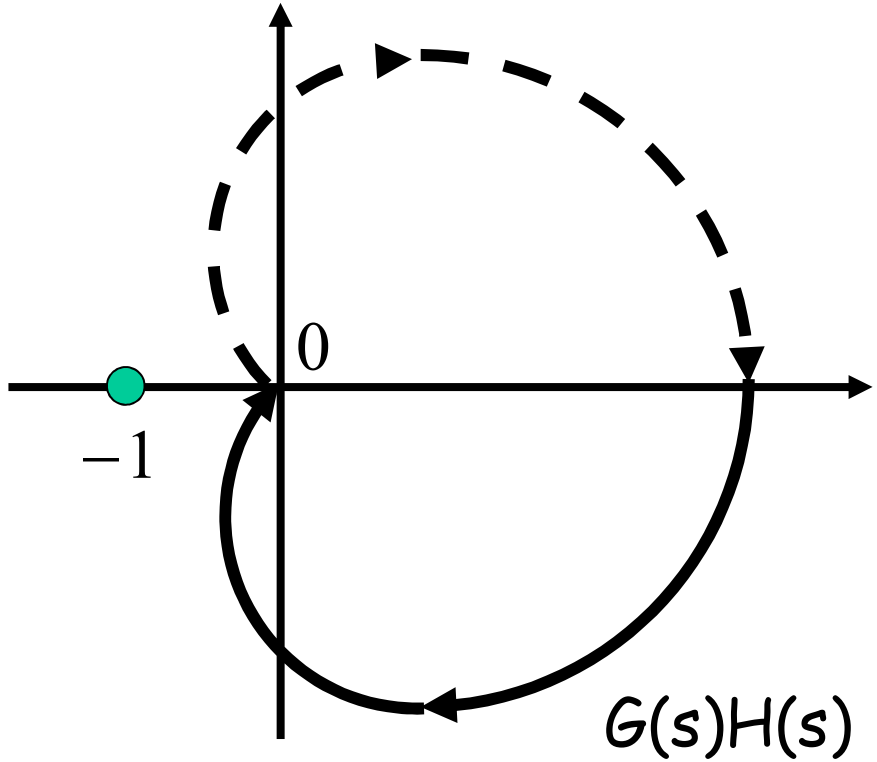
$$G(s)H(s) = \frac{100}{(s+1)(0.1s+1)}$$

Determine its stability.

$$P = 0, N = 0$$

$$Z = P - N = 0$$

The system is stable.



Nyquist stability criterion

A system is stable if and only if from

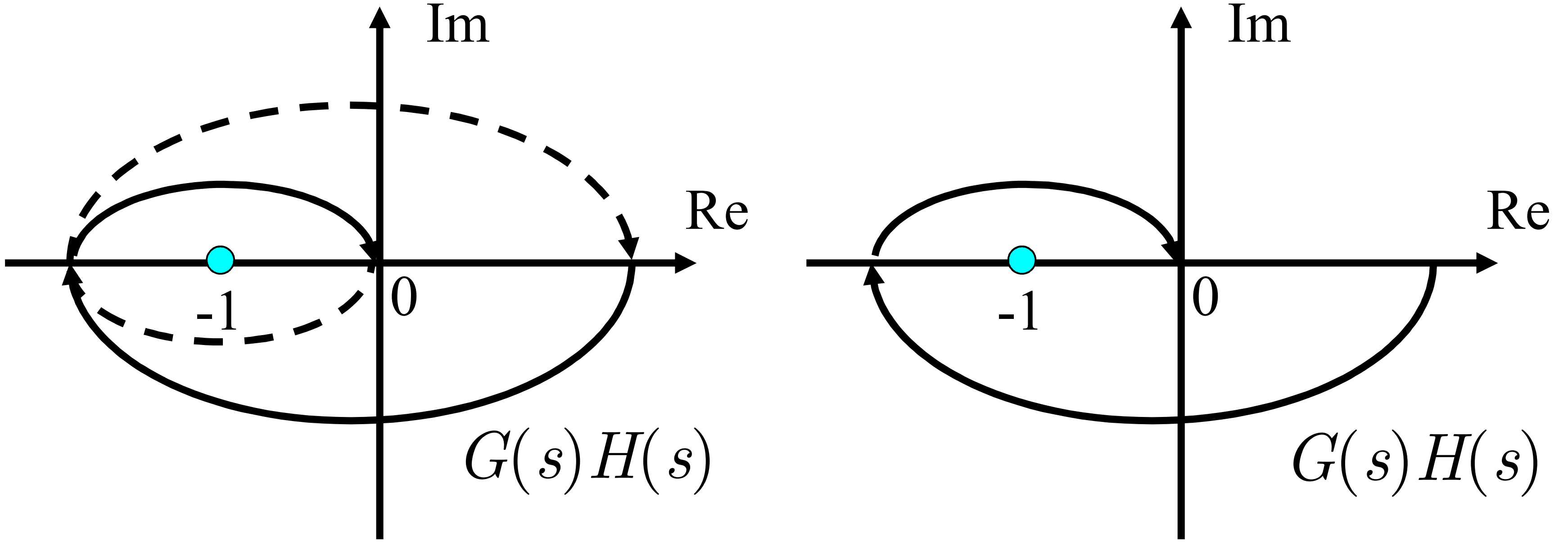
$$N = P - Z$$

we can obtain that $Z\!=\!0$, where P is the number of the open-loop poles in the right-hand half s-plane and N is the number of the encirclements of the point $(-1,\, j0)$ of $G(j\omega)H(j\omega)$ in the **counterclockwise** direction.

Since the plot of $G(j\omega)H(j\omega)$ and the plot of $G(-j\omega)H(-j\omega)$ are symmetrical with each other about the real axis, we can modify the stability criterion as

$$N = (P - Z)/2$$

where only the polar plot of $G(j\omega)H(j\omega)$ with ω varying from 0 to $+\infty$ is necessary.



Example.

Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s)=\frac{2}{s-1}$$

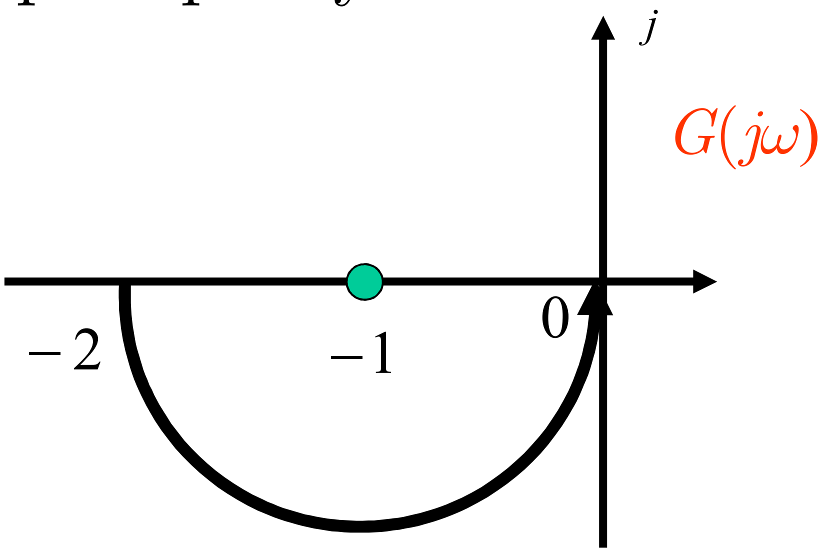
Determine its stability by using Nyquist criterion.

Solution: Obviously, P=1 (nonminimum phase plant). Now, drawing its Nyquist plot yields:

$$P=1 \qquad N=\frac{1}{2}$$

$$N=\left(P-Z\right)/2$$

$$\Rightarrow Z=P-2N=0$$



The system is stable.

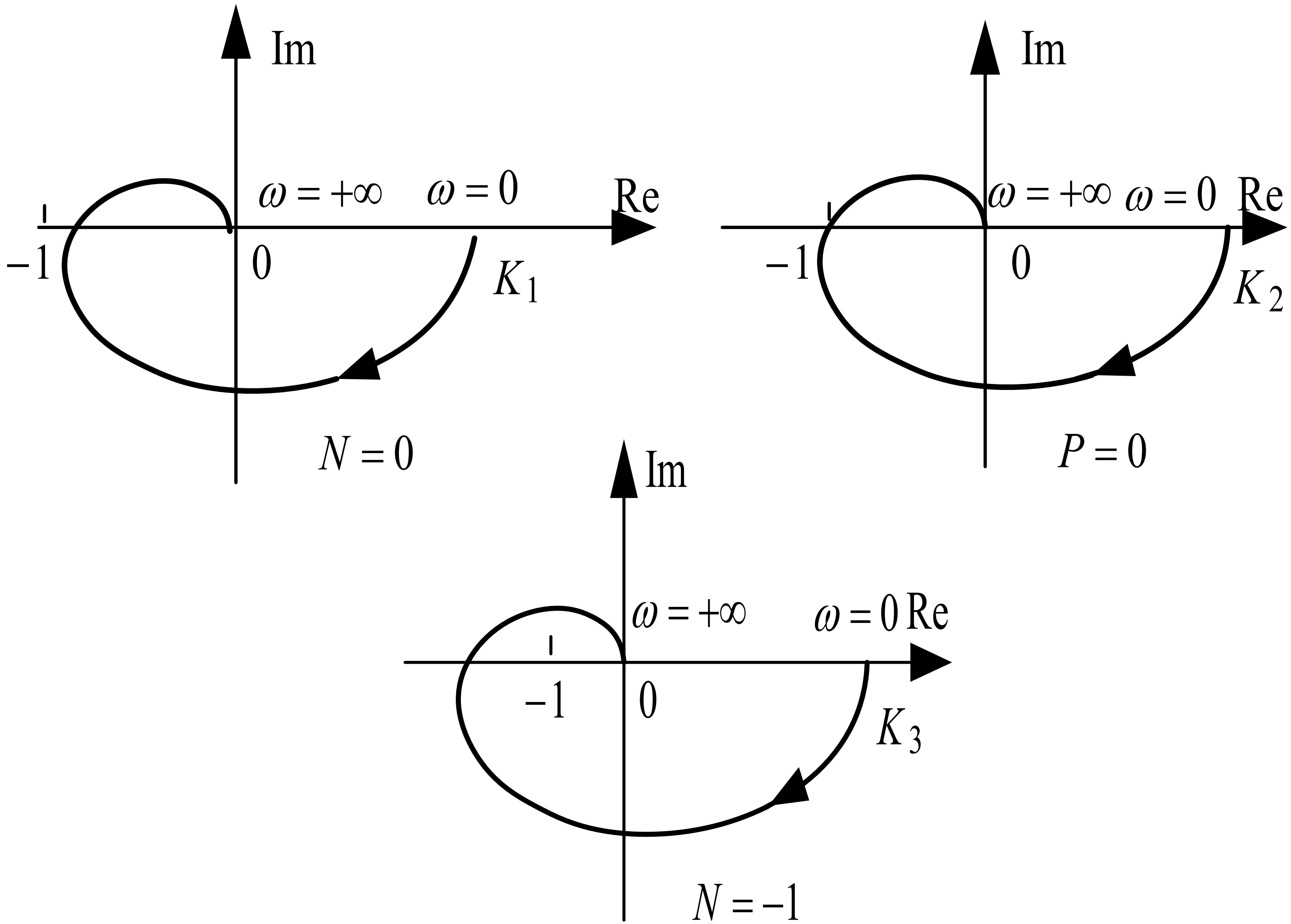
Example.

Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)(T_3 s + 1)}, \quad T_i > 0, K > 0$$

Using Nyquist plot, determine its stability.

Solution: Obviously, P=0. Now, drawing its Nyquist plot yields:



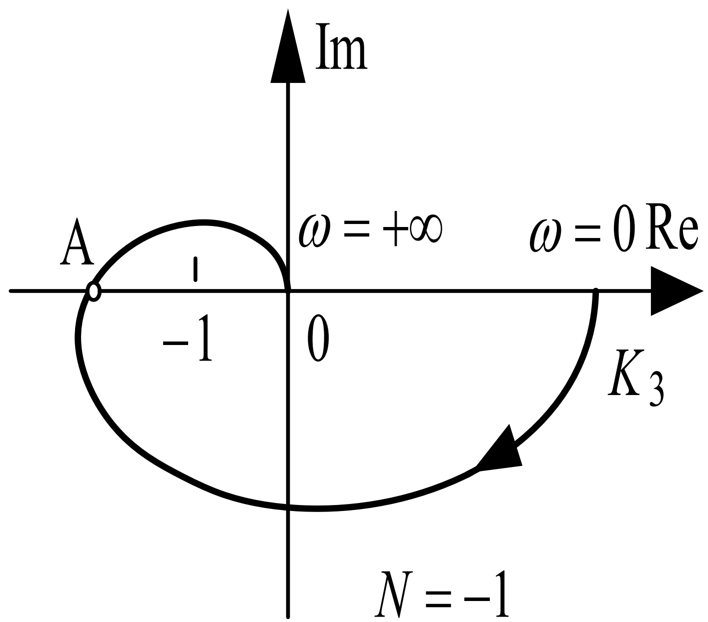
Im
Re
$\omega=+\infty$
$\omega=0$
-1
0
K_1
$N=0$
Im
Re
$\omega=+\infty$
$\omega=0$
-1
0
K_2
$P=0$
Im
Re
$\omega=+\infty$
$\omega=0$
-1
0
K_3
$N=-1$

4. Nyquist Criterion Based on Bode diagram

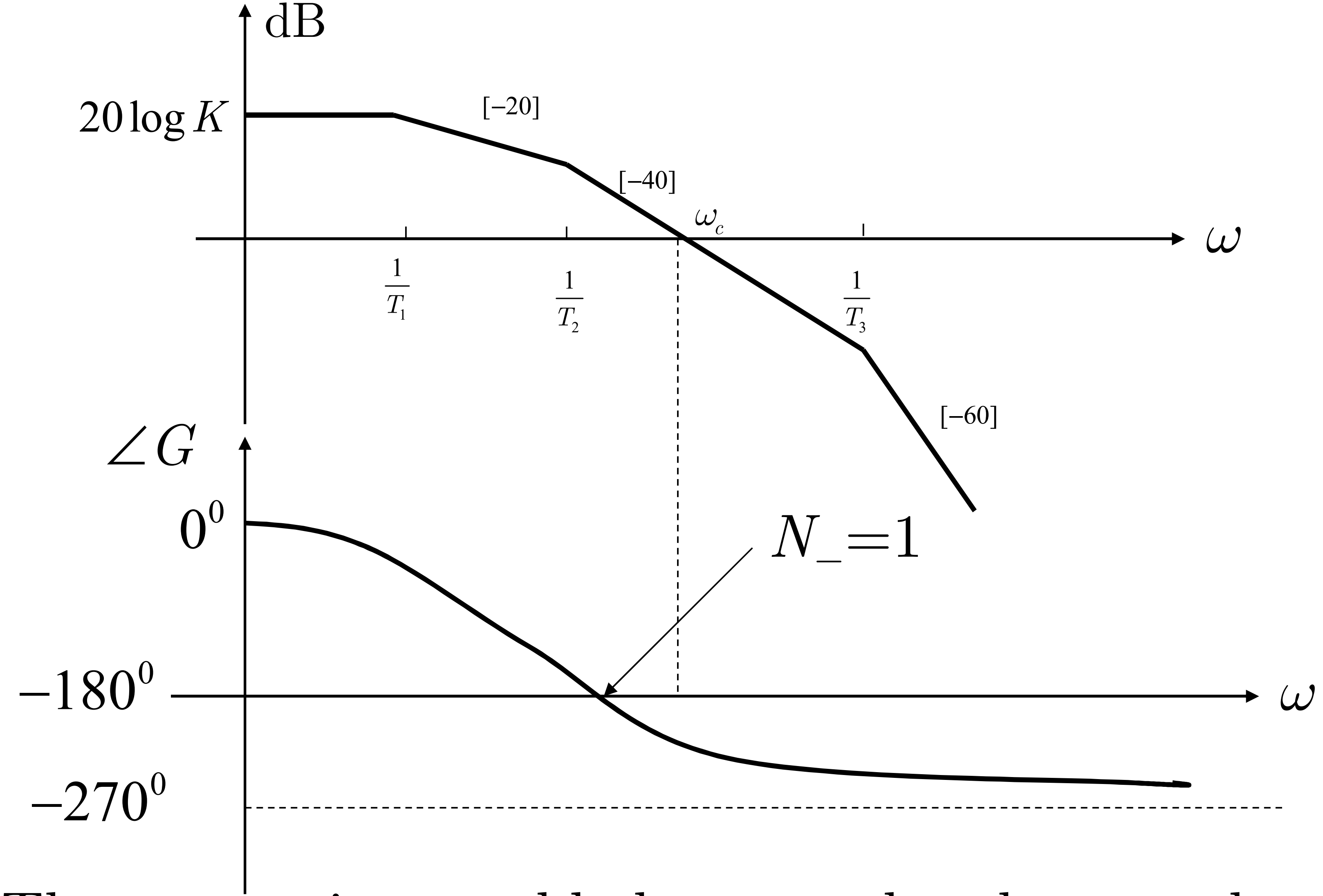
Again, we consider the open-loop transfer function with Nyquist curve below:

$$G(s)=\frac{K}{(T_1 s+1)(T_2 s+1)(T_3 s+1)}, \quad T_i>0, K>0$$

where P=0, N=−1 and the system is unstable. The key is the point that the Nyquist curve crosses the negative real axis is less than −1 (abs(A)>1).



The Bode diagram is:



The system is unstable because the phase angle plot crosses the -180^0 line at which $20\log|G(j\omega)|>0$.

- N_{-}: The number of crossings of -180^0 line at which $20\log|G(j\omega_i)|>0$ and each crossing makes the phase angle less than -180^0.
- N_{+}: The number of crossings of -180^0 line at which $20\log|G(j\omega_i)|>0$ and each crossing makes the phase angle larger than -180^0.

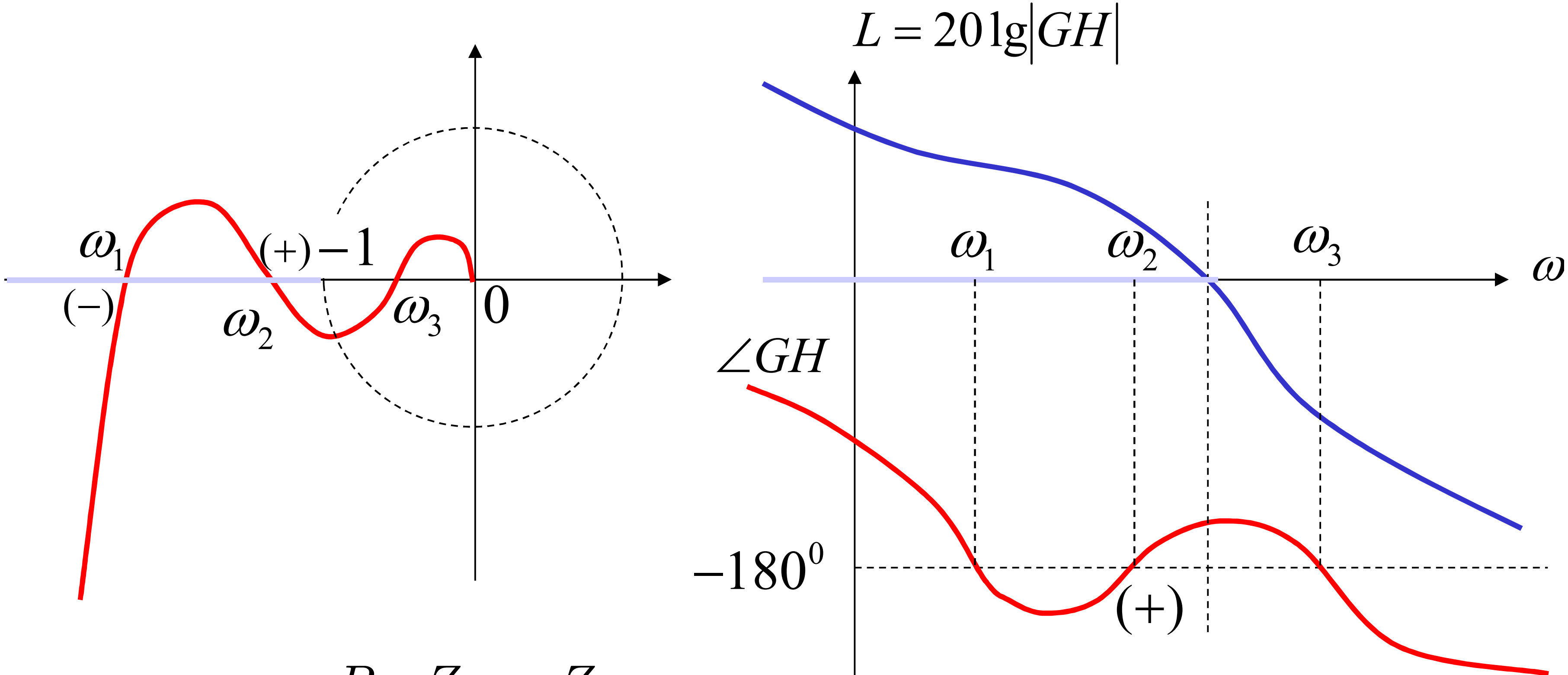
The Nyquist Theorem can thus be interpreted as:

Theorem: The system is stable if and only if from

$$N_{+} - N_{-} = (P - Z)/2$$

one can obtain that $Z=0$.

Example. Both the Nyquist plot and Bode diagram of an open-loop minimum phase transfer function are shown below. Determine its stability.



$$N_+ - N_- = \frac{P-Z}{2} = \frac{-Z}{2} = 0 \Rightarrow Z = 0$$

5. Special case when G(s)H(s) involves Poles and Zeros on the jω Axis

Consider the following open-loop transfer function:

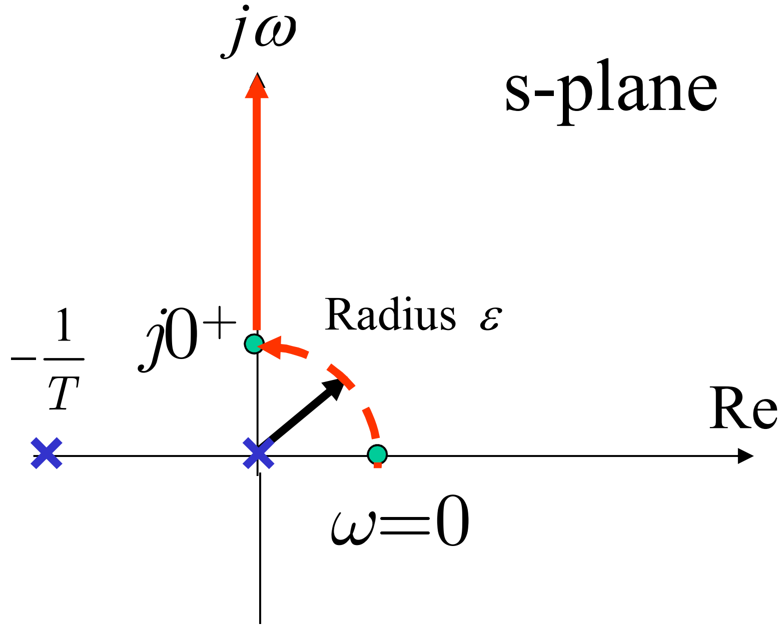
$$G(s)H(s) = \frac{K}{s(Ts+1)}$$

- s is a singular point.
- The Nyquist contour Γ_s must be modified.
- A way of modifying the contour near the origin is to use a quarter-circle with an infinitesimal radius $\varepsilon(>0)$:

$$s = \varepsilon \cdot e^{j\theta}$$

where θ varies from 0^0 to 90^0 as ω varies from 0 to 0^+.

A modified contour Γ_s :

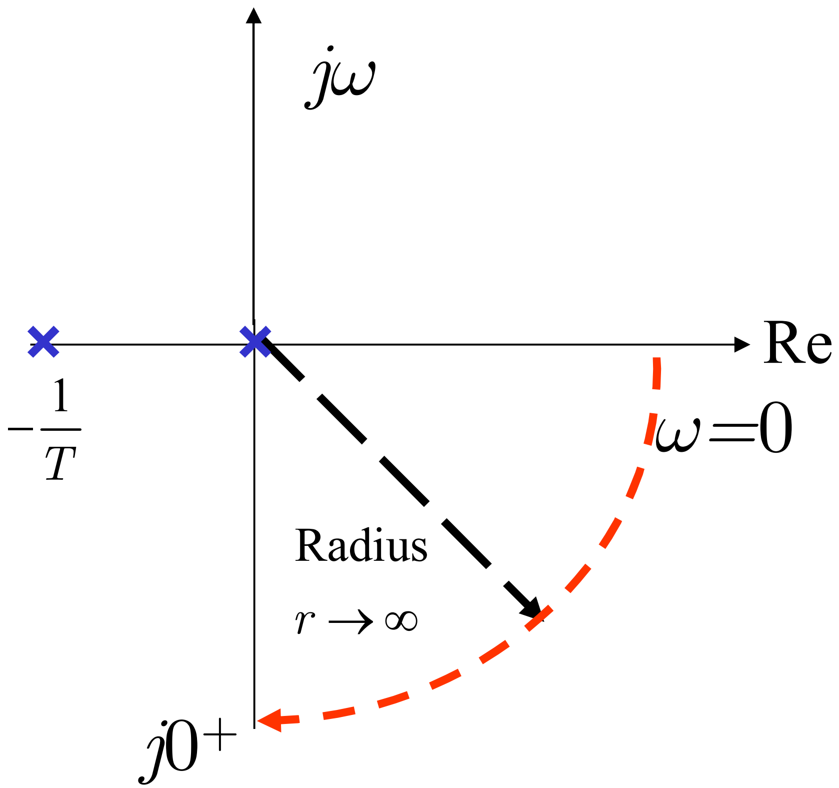


We regard $1/s$ as a stable pole.

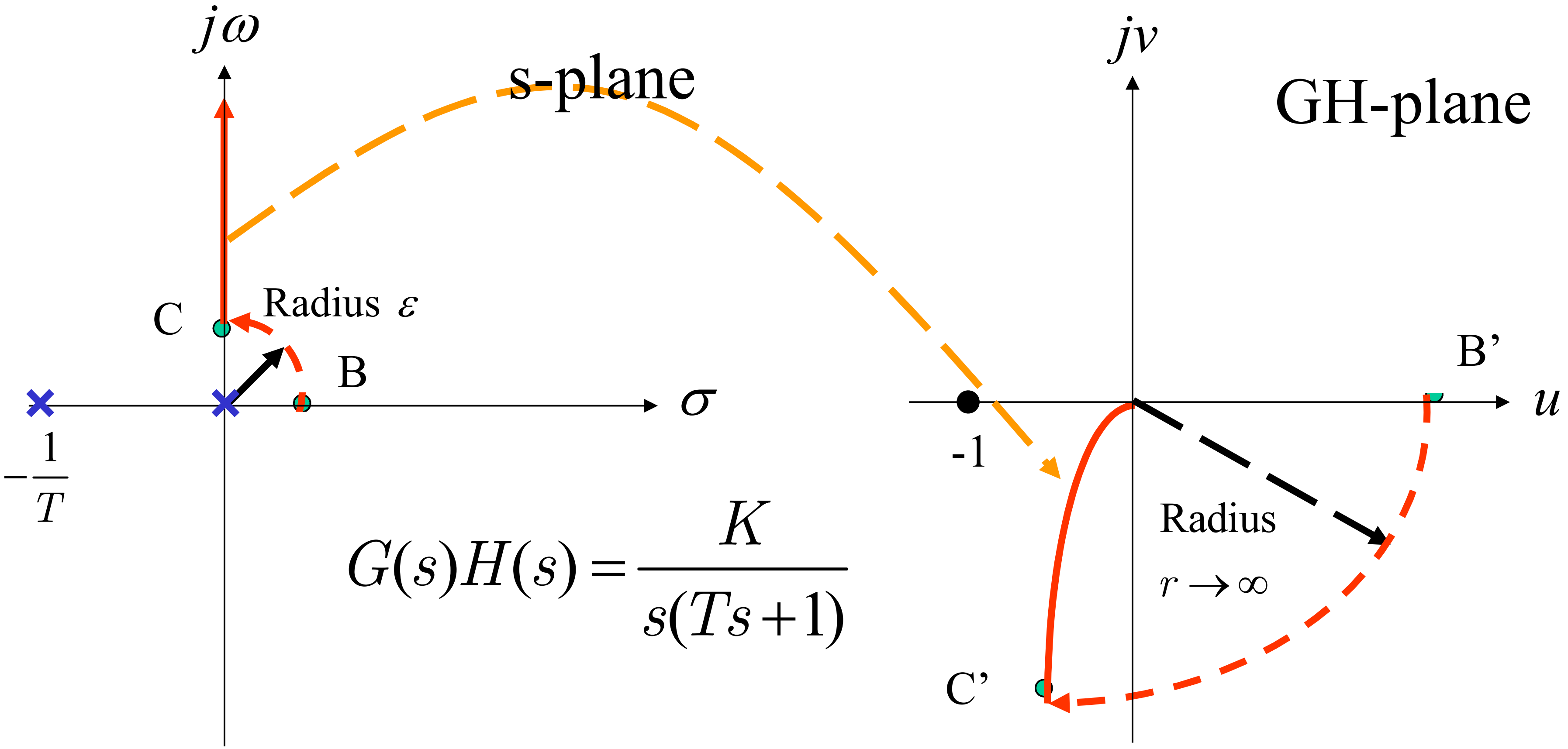
Then, for the controlled plant with integral factor $1/s$, we have

$$\frac{1}{s}=\frac{1}{\varepsilon}\cdot e^{-j\theta}, \quad -\theta : 0^0 \rightarrow -90^0$$

where $1/s$ rotates from 0^0 to -90^0 with an infinite radius as ω varies from 0 to 0^+.

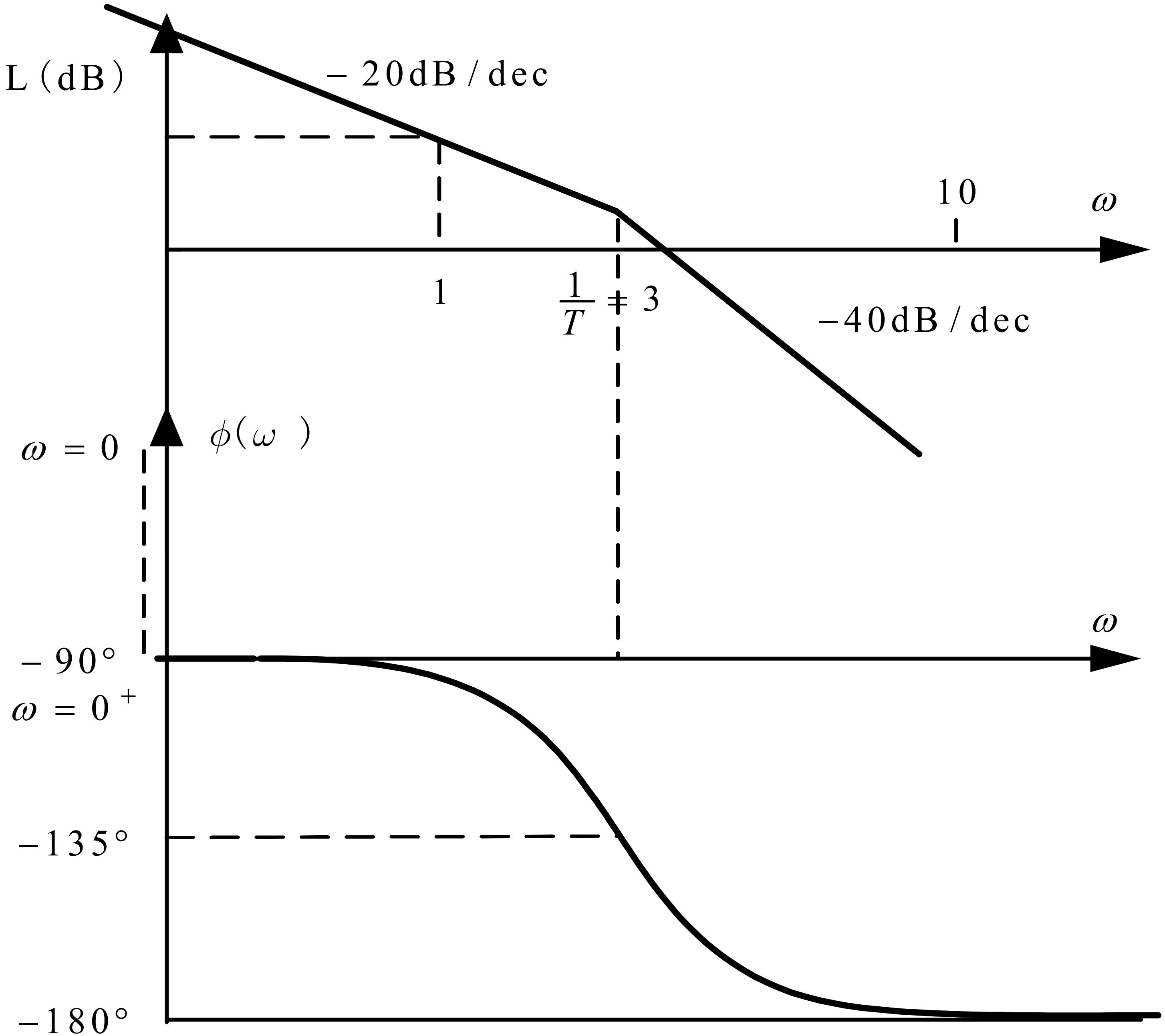


The integral factor $1/s$ rotates from 0^0 to -90^0 with an infinite radius as ω varies from 0 to 0^+.



The Nyquist curve starts from real axis when ω=0 and rotates -90^0 as ω=0^+.

$N = (P - Z)/2 \rightarrow Z$=0, **the system is stable**.


L（dB）
− 20dB / dec
10
ω
1
1/T = 3
−40dB / dec
ω = 0
φ(ω)
ω
− 90°
ω = 0 +
−135°
−180°

Example. Let the open-loop transfer function be

$$G(s)H(s) = \frac{K}{s^2(Ts+1)}, \quad T > 0, \quad K > 0$$

Determine its stability by using Nyquist Theorem.

Solution: When ω=0,

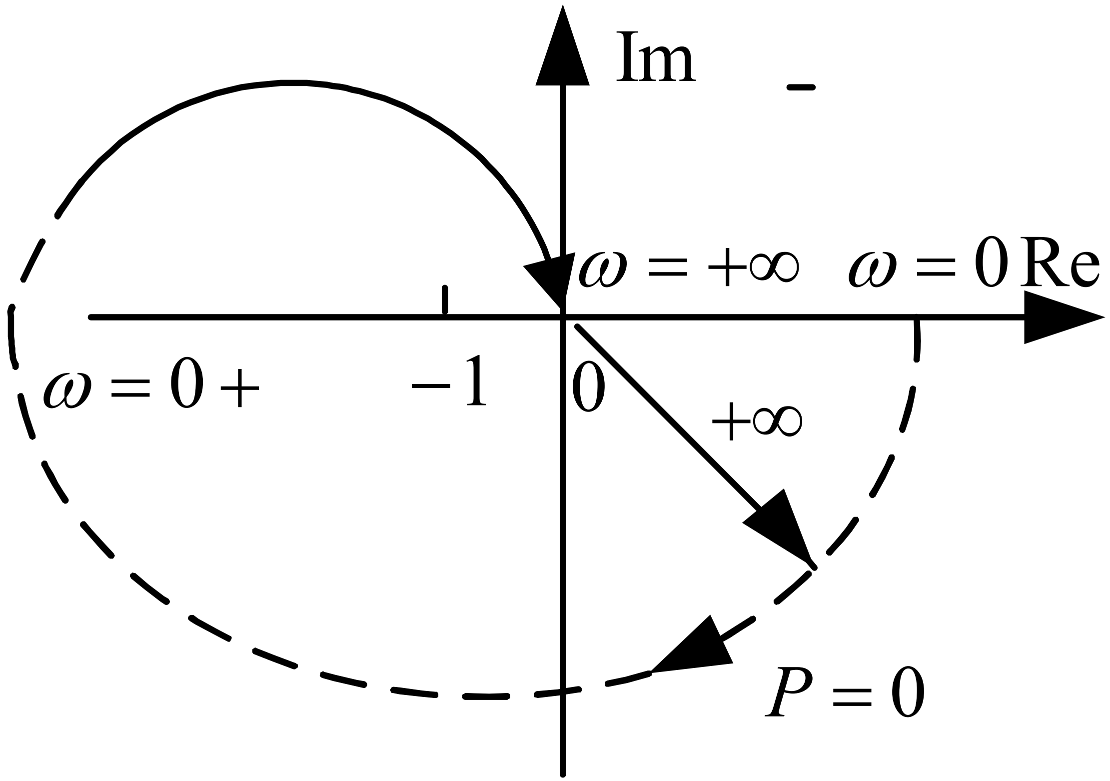
$$G(s)H(s) = \frac{K}{s^2(Ts+1)} \approx \frac{K}{\varepsilon^2} e^{-j2\theta}, \omega = 0, \theta = 0^0;$$

When ω=0⁺,

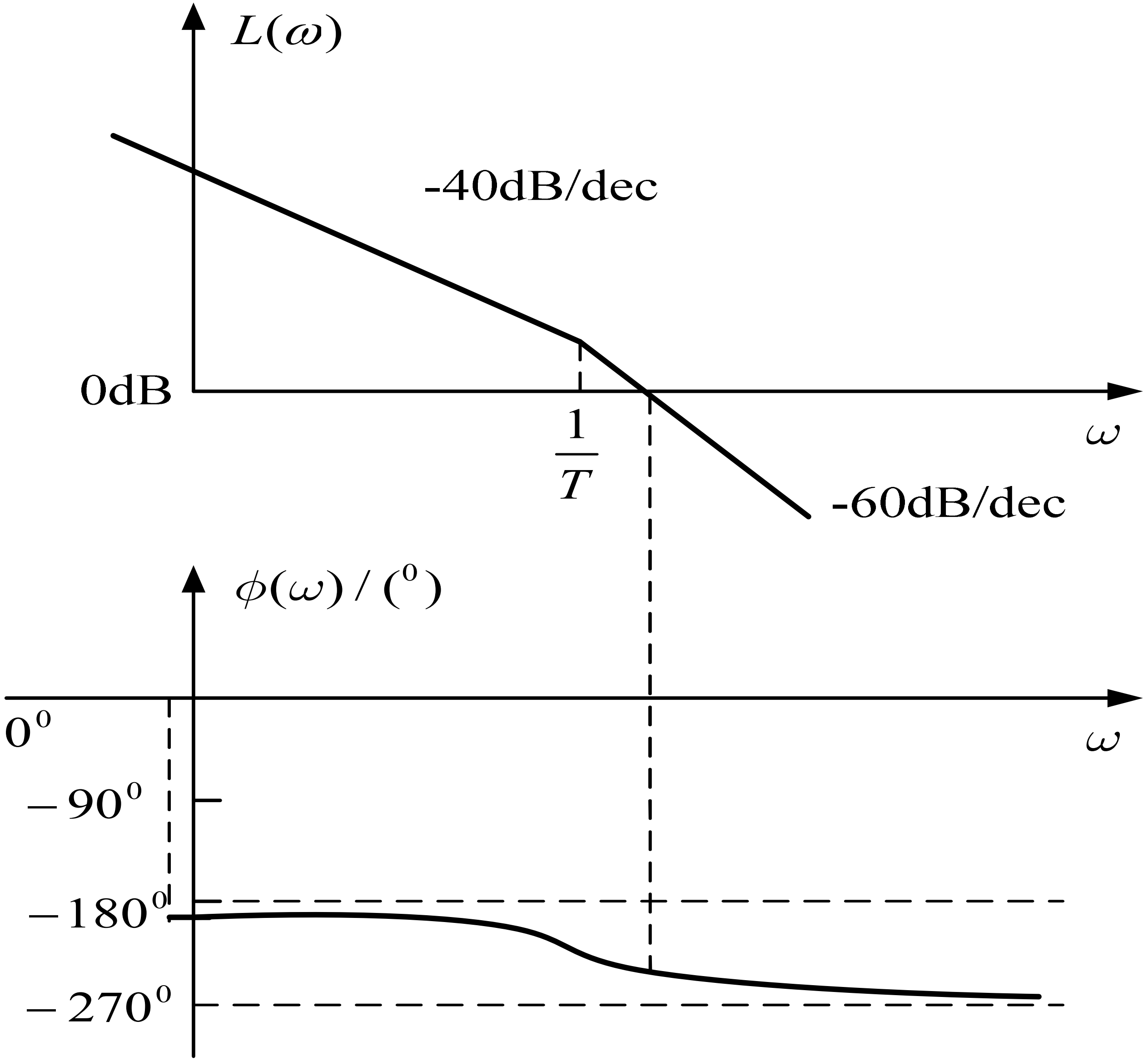
$$-2\theta = -180^0$$

Hence, when ω>0⁺,

$$\angle G(j\omega)H(j\omega) = -180^0 - \angle \tan^{-1} T\omega$$



$P{=}0,\ \ N{=}{-1}\ \Rightarrow\ N{=}(P\ -Z)/2\ \Rightarrow\ Z{=}2.$ **The system is unstable.**



P=0, N_-=1, N_+=0 $\Rightarrow$ $N_+ - N_- = (P - Z)/2 \Rightarrow$ Z=2. The system is unstable.

6. Stability Analysis

Example. Let the open-loop transfer function be

$$G(s)H(s)=\frac{K(T_2s+1)}{s^2(T_1s+1)},\quad T_1>0,\quad T_2>0,\quad K>0$$

Determine its stability by using Nyquist Theorem.

Solution: When ω=0,

$$G(s)H(s)=\frac{K(T_2s+1)}{s^2(T_1s+1)}\approx\frac{K}{\varepsilon^2}e^{-j2\theta},\omega=0,\theta=0^0;$$

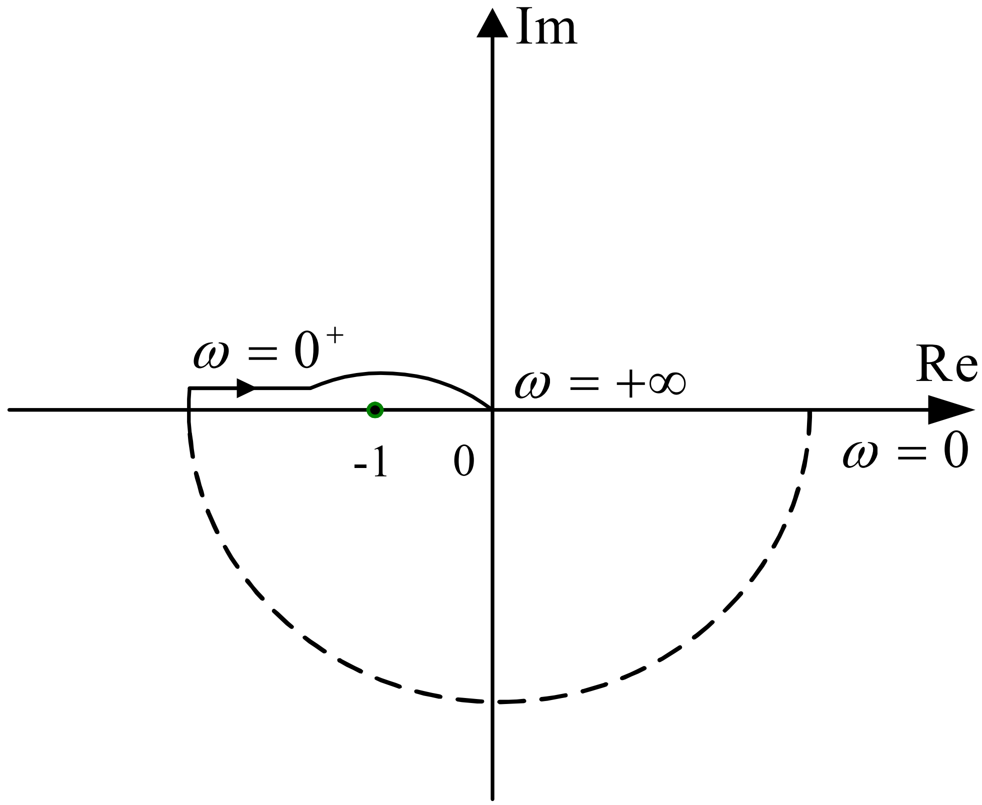
When $\omega=0^+$

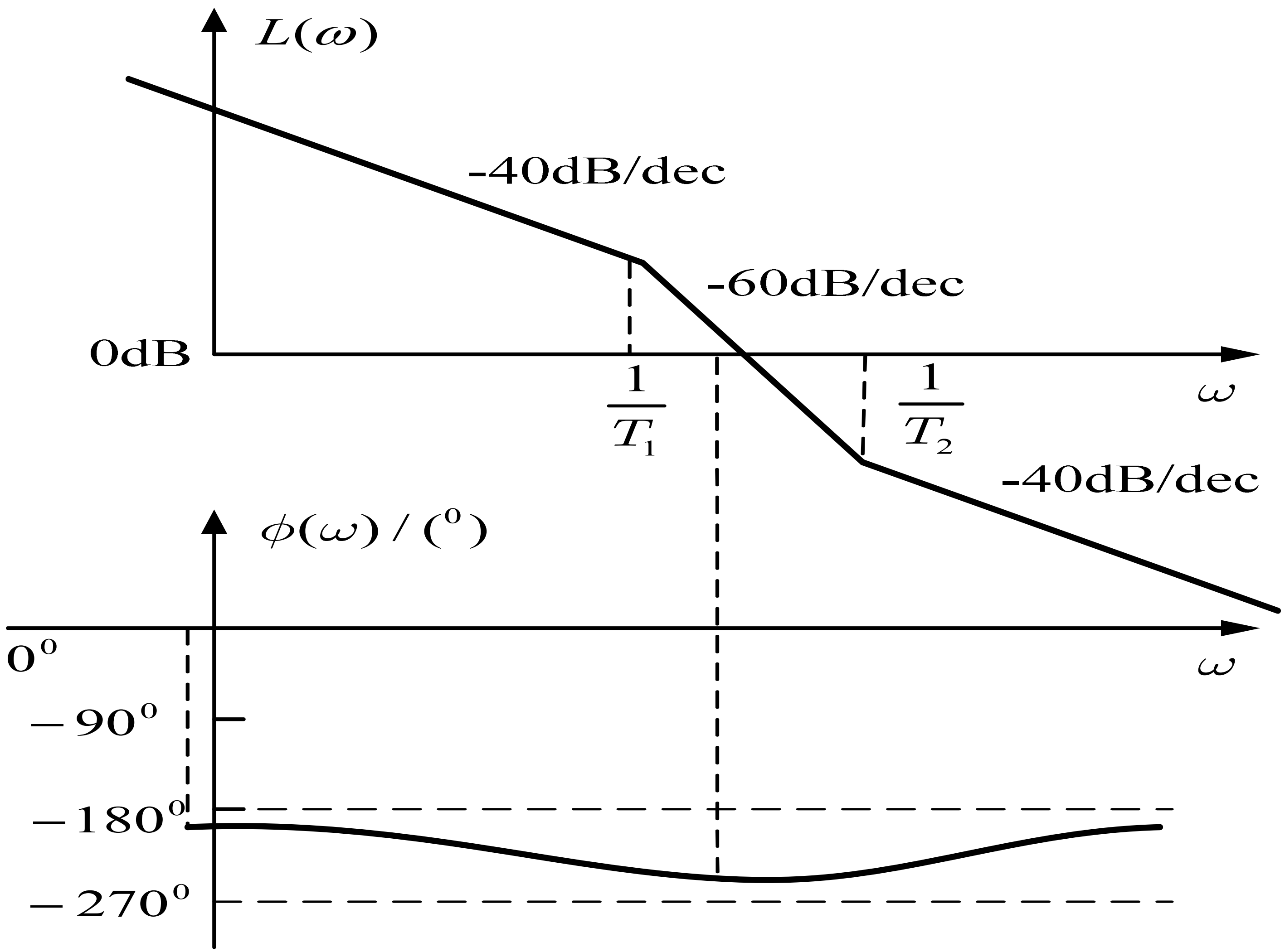
$$\omega=0^+,-2\theta=-180^0$$

Hence, when $\omega>0^+$,

$$\angle G(j\omega)H(j\omega)=-180^0-\angle\tan^{-1}T_1\omega+\angle\tan^{-1}T_2\omega$$

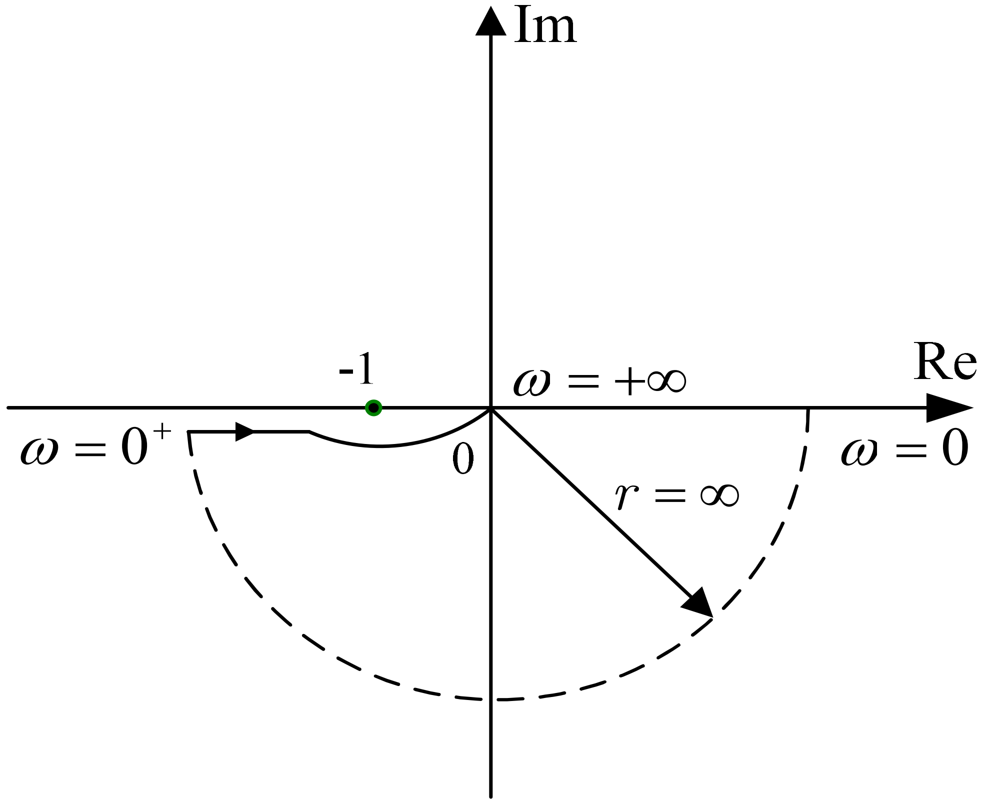
If $T_1 > T_2$, the Nyquist curve is shown below. Since $P=0$, $N=-1 \Rightarrow N=(P-Z)/2 \Rightarrow Z=2$. The system is unstable.

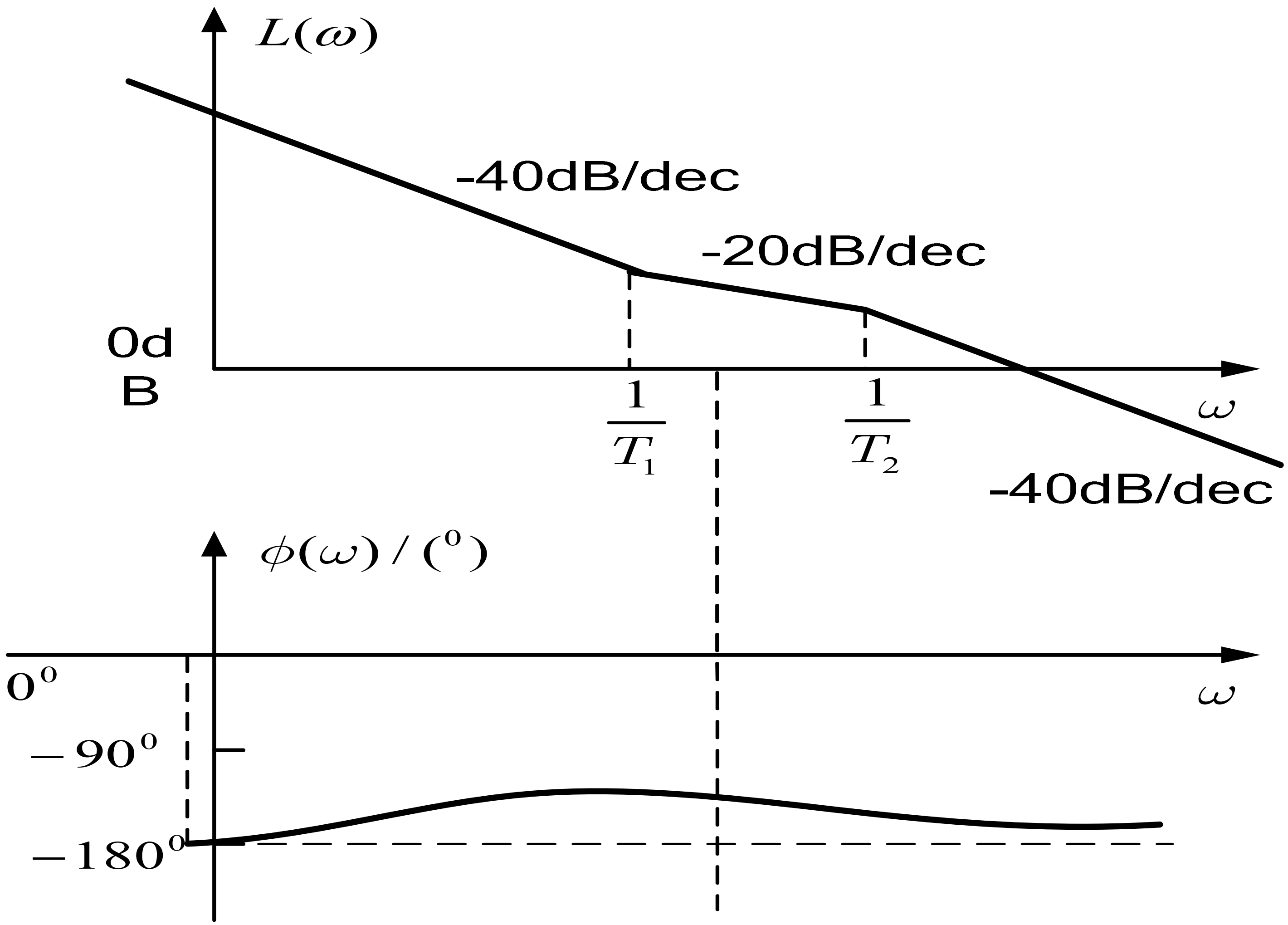




L(ω)
-40dB/dec
-60dB/dec
0dB
1/T1
1/T2
ω
-40dB/dec
φ(ω)/(°)
0°
ω
−90°
−180°
−270°

If $T_2 > T_1$, the Nyquist curve is shown below. Since $P=0$, $N=0 \Rightarrow N=(P-Z)/2 \Rightarrow Z=0$. The system is stable.





P=0, N_-=1/2, N_+=1/2 $\Rightarrow$ $N_+ - N_- = (P - Z)/2$
$\Rightarrow$ Z=0. The system is stable.

Example. Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s)=\frac{K}{s(Ts-1)}$$

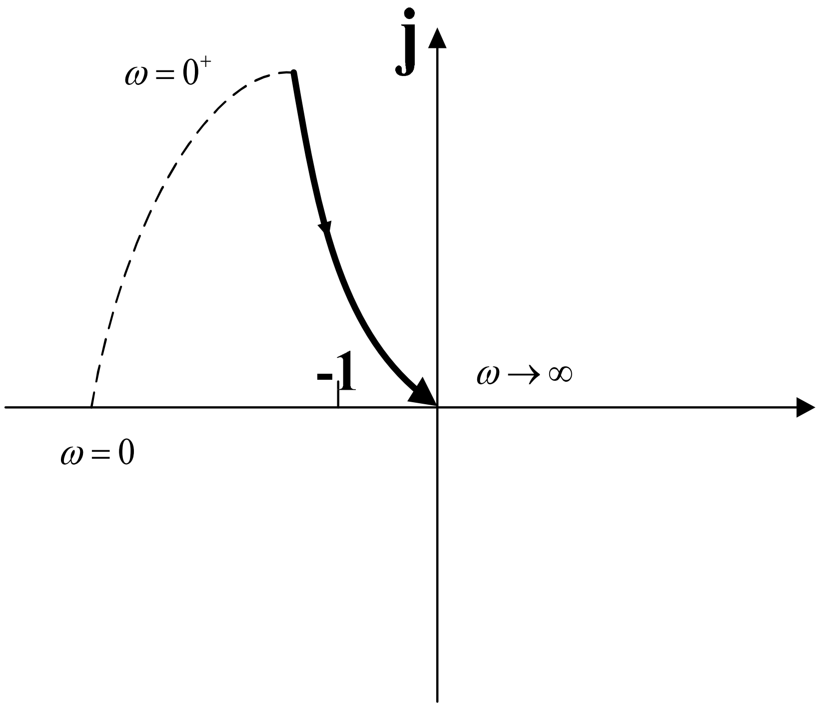
Investigate its stability by using Nyquist criterion.

Solution: P=1 (nonminimum phase plant).

Nyquist plot:

$$N=-\frac{1}{2}=\frac{(P-Z)}{2}$$

$$=\frac{(1-Z)}{2}\Rightarrow Z=2$$



The system is unstable.

Example. Consider a unity feedback system whose open-loop transfer function is as follows

$$G(s) = \frac{300}{s(s^2 + 2s + 100)}$$

Investigate its stability by using Nyquist criterion based on Bode diagram.

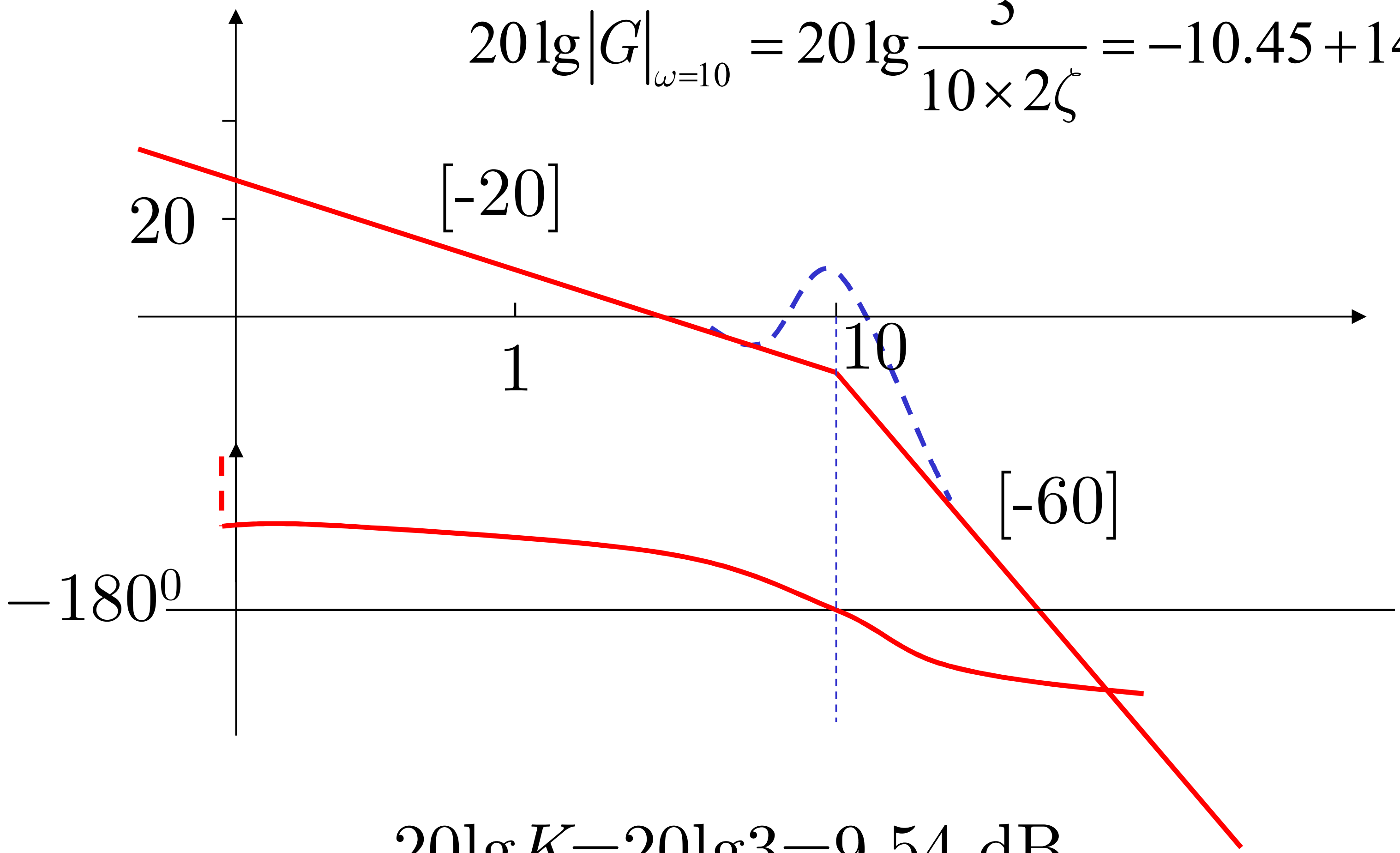
Solution: Write the transfer function in normalized form:

$$G(s) = \frac{3}{s(\frac{s^2}{10^2} + \frac{1}{50}s + 1)}$$

Note that for the second order factor, ω_n=10 rad/s, and ζ=0.1, which implies that its resonant peak value is large.

Without considering the effect of the resonant peak, the system seems stable. However, if we take the resonant peak into account, the system is unstable since at the point the phase angle crosses -180^0 line,

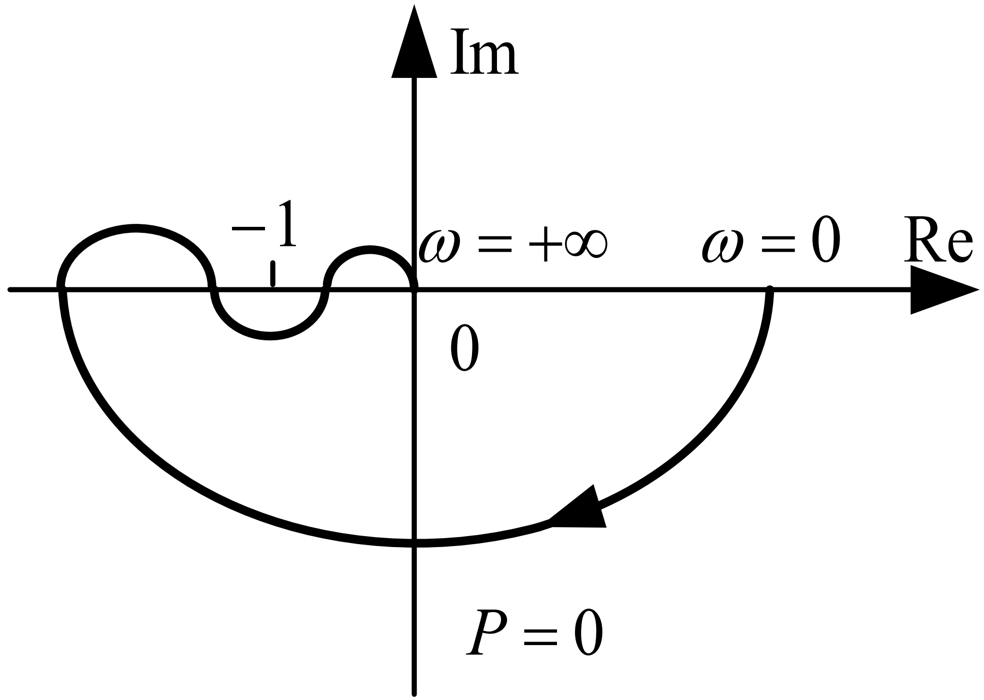
$$20\lg|G|_{\omega=10} = 20\lg\frac{3}{10\times 2\zeta} = -10.45 + 14 > 0$$



$$20\lg K = 20\lg 3 = 9.54\ \text{dB}$$

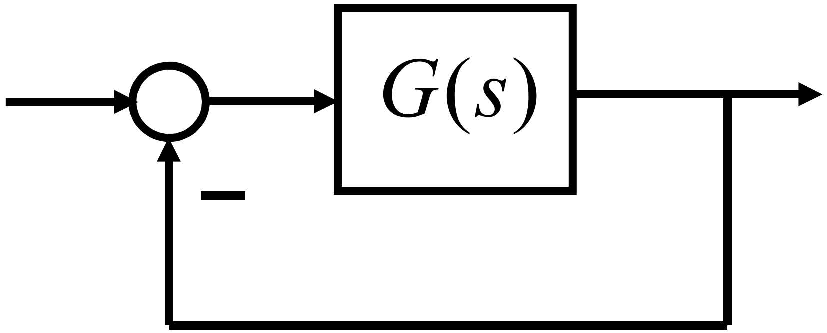
7. Conditionally Stable Systems

A conditionally stable system is stable for the value of the open-loop gain lying between critical values, but it is unstable if the open-loop gain is either increased or decreased sufficiently.

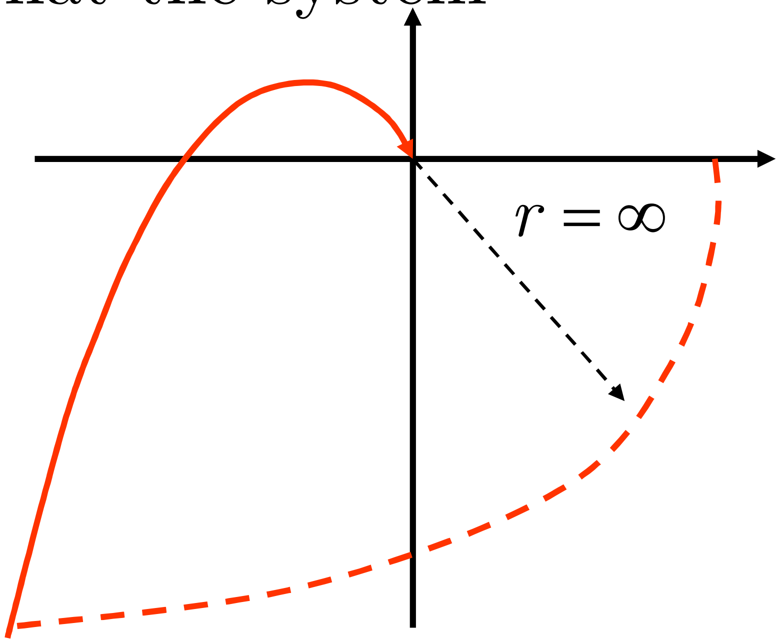


Example. Consider a single-loop control system, where its open-loop transfer function is

$$G(s) = \frac{K}{s(s^2+2s+4)}$$



Determine the range of K so that the system keeps stable.



$$\omega_n = 2$$

$$\angle G(j\omega_n) = -90^0 - 90^0 = -180^0$$

$$|G(j2)| = \frac{K}{2\times 4} < 1$$

$$0 < K < 8$$

Example. Consider a single-loop control system, where its open-loop transfer function is

$$G(s) = \frac{K(\tau_1 s + 1)(\tau_2 s + 1)}{s^3}$$

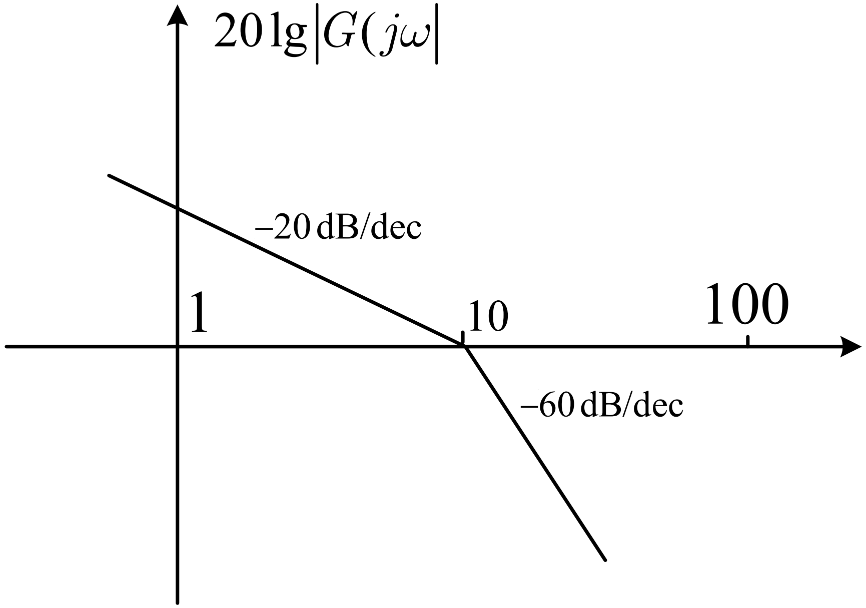
Discuss its stability by using Nyquist stability criterion.

Example. Consider a single-loop control system, where its open-loop transfer function is

$$G(s) = \frac{K(\tau s + 1)}{s(s - 1)}$$

Discuss its stability by using Nyquist stability criterion.

Example. Bode asymptotic magnitude curve of a minimum system is shown below



Determine its transfer function. Is the system stable?