Chapter 5

Transient and Steady-State Response Analysis (4)

5-7 The steady state error

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to ramp input.

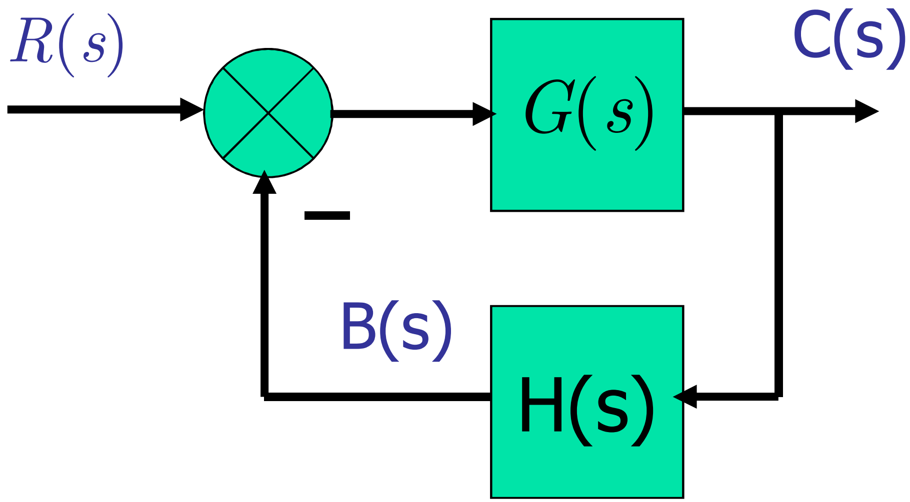
In this section, we shall introduce the definitions of system's error and the method to evaluate the steady-state error.

1. Two definitions of system error

$$E(s) = R(s) - C(s)$$

or

$$E(s) = R(s) - B(s)$$



The definitions reduce to the same if the system is unity-feedback.

The **steady state error** of the control system is then defined as

$$e_{ss} = \lim_{t \to \infty} e(t)$$

where $e(t) = \mathcal{L}^{-1}\{E(s)\}$.

2. Final Value Theorem and its applying condition

Theorem: Suppose $f(t)$ has the Laplace transform $F(s)$ and the limits $\lim_{t\to\infty} f(t)$ and $\lim_{s\to 0} sF(s)$ exist. Then, the final value is

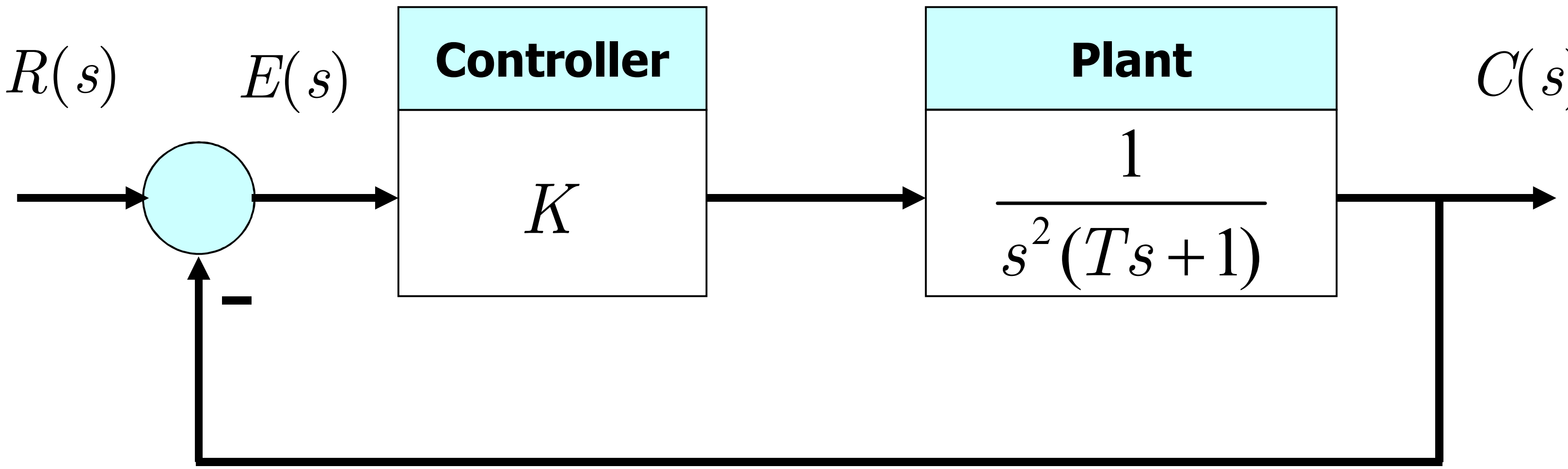
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

Let

$$E(s) = R(s) - B(s) = \frac{1}{1+G(s)H(s)} R(s)$$

and assume $E(s)$ is a rational proper function. Then, the Final Value Theorem can be applied provided **all the poles of $sE(s)$ lie in the left-half s-plane!**

Example. Consider the following control system:



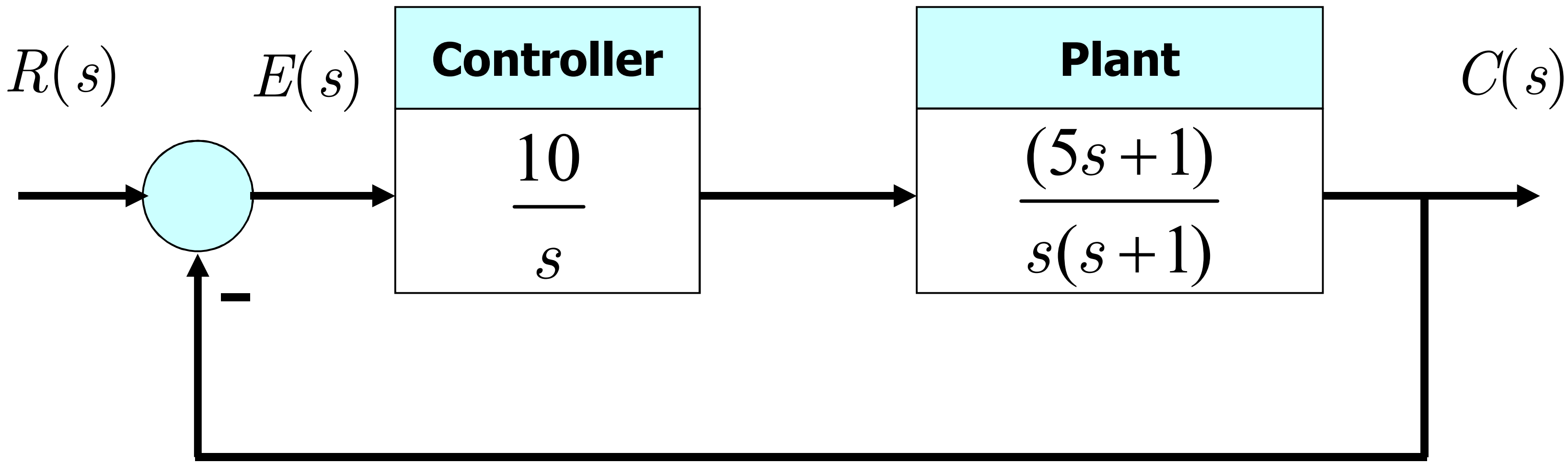
Let $r(t)=1(t)$. Find its steady-state error e_{ss}.

Solution: $sE(s) = s(R(s) - B(s)) = s\dfrac{1}{1+G(s)H(s)}\dfrac{1}{s}$

$$= \frac{1}{1+G(s)H(s)} = \frac{s^2(Ts+1)}{Ts^3+s^2+K}$$

The coefficient of the power s is zero, which implies that the system has poles in right-half s-plane.

Example. Consider the following control system:



Let $r(t)=\sin(\omega t)$. Find its steady-state error e_{ss}.

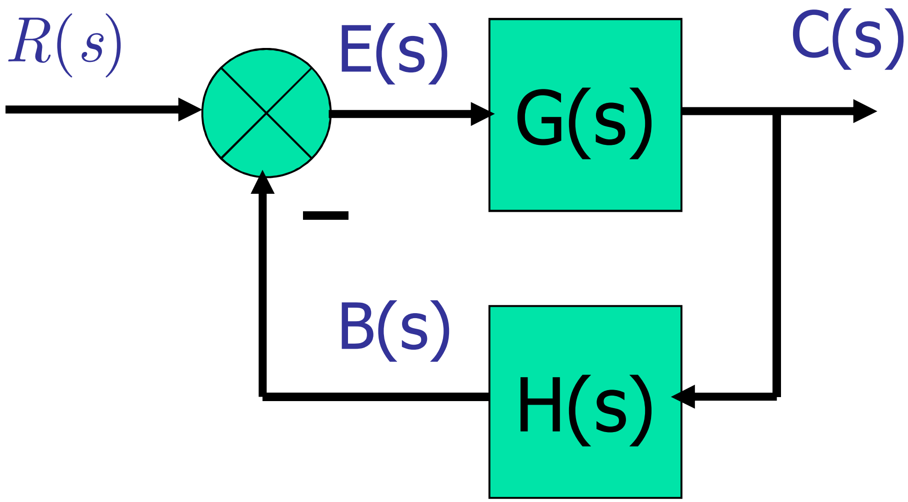
Solution: Since $\mathcal{L}(\sin\omega t)=\omega/(s^2+\omega^2)$,

$$sE(s)=s[R(s)-B(s)]=\frac{s}{1+G(s)H(s)}R(s)$$

It does not agree with the applying condition of the FV theorem since $sE(s)$ has two poles on the imaginary axis.

3. Classification of control systems

Define: $E(s) = R(s) - B(s)$



$$E(s) = \frac{1}{1+G(s)H(s)} R(s)$$

Let

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N (T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = \frac{KN_0(s)}{s^N D_0(s)}$$

Then

$$N = \begin{cases} 0, & \text{Type 0 system} \\ 1, & \text{Type 1 system} \\ 2, & \text{Type 2 system} \end{cases}$$

4. Static position error constant Kp (r=1(t))

Let

$$r(t) = 1(t)$$

Then

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} R(s)$$

$$= \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s} = \lim_{s\to 0} \frac{1}{1+\frac{KN_0(s)}{s^N D_0(s)}}$$

where

$$N_0(0) = 1 \qquad D_0(0) = 1$$

The static position error constant K_p is defined as

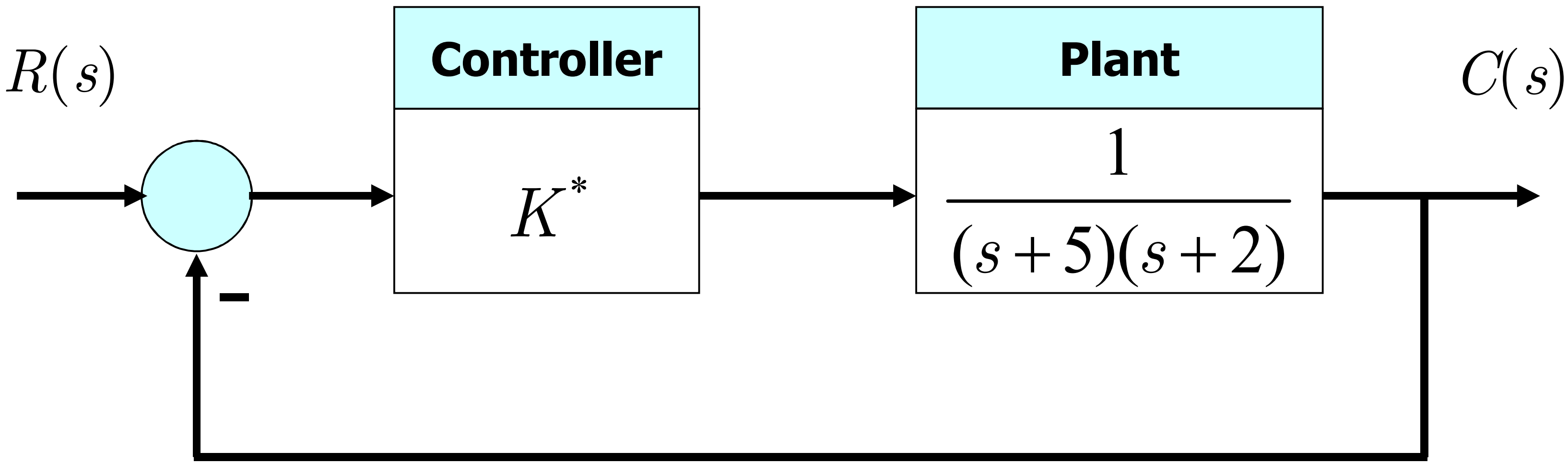
$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{KN_0(s)}{s^N D_0(s)} = \begin{cases} K, & \text{type} = 0 \\ \infty, & \text{type} \geq 1 \end{cases}$$

Therefore,

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} \frac{1}{s}$$

$$= \begin{cases} \dfrac{1}{1+K}, & \text{if } N = 0, \\ 0, & \text{if } N \geq 1 \end{cases}$$

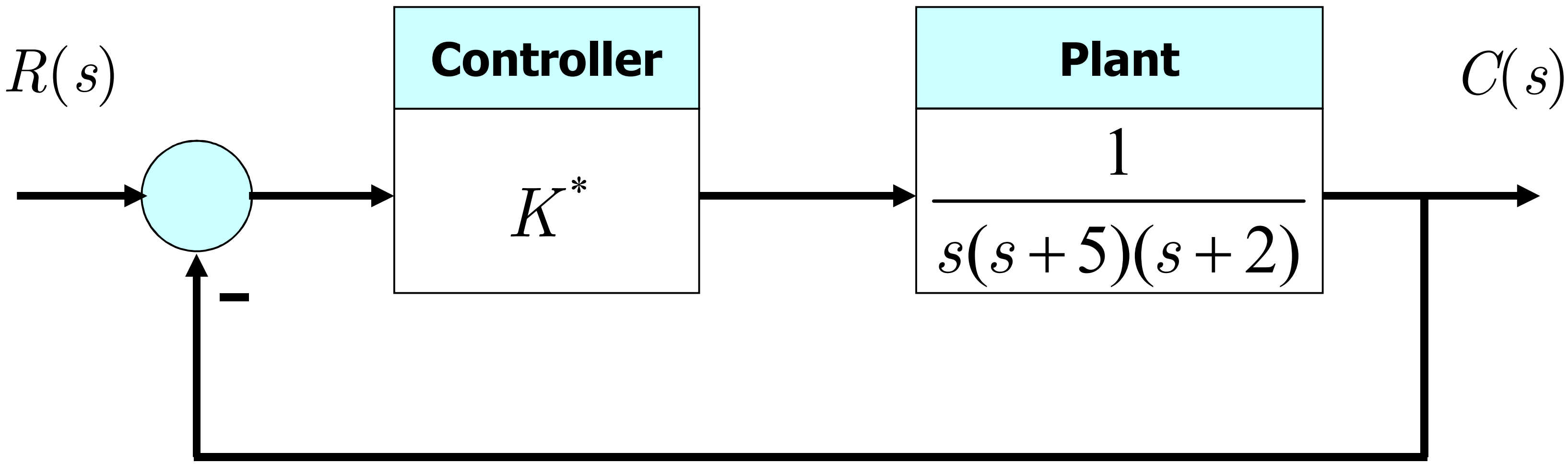
Example. Consider the following control system:



where $K^*>0$. Let $r(t)=1(t)$. Find its steady-state error e_{ss}.

Solution: Firstly, check whether the poles of $sE(s)$ lie in the left half s-plane. Then, calculate e_{ss}.

Example. Consider the following control system:

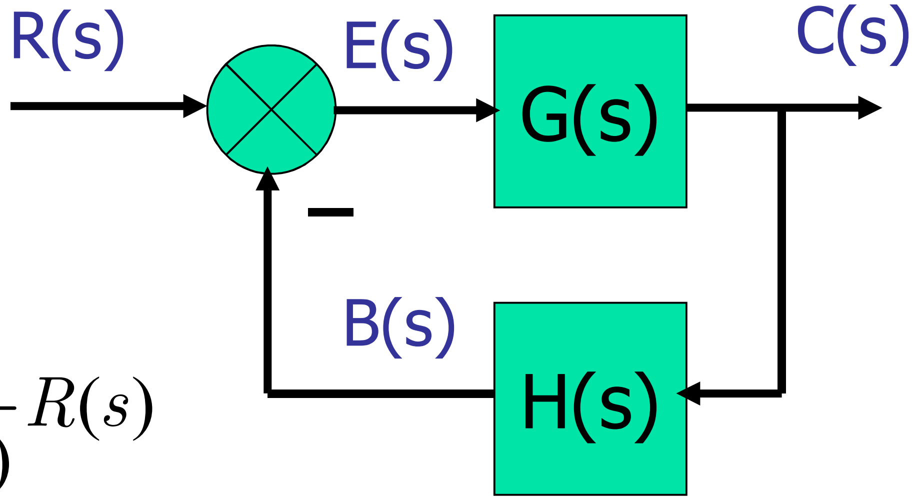


where $0 < K^* < 10$. Let $r(t) = 1(t)$. Find its steady-state error e_{ss}.

Solution: Firstly, check whether the poles of $sE(s)$ lie in the left half s-plane. Then, calculate e_{ss}.

5. Static velocity error constant Kv (r=t1(t))

Let $r(t) = t \cdot 1(t) \Rightarrow R(s) = \dfrac{1}{s^2}$



Then

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s}{1+G(s)H(s)} R(s)$$

$$= \lim_{s\to 0} \frac{1}{s + sG(s)H(s)} = \lim_{s\to 0} \frac{1}{s\dfrac{KN_0(s)}{s^N D_0(s)}}$$

where

$$N_0(0) = 1 \qquad D_0(0) = 1$$

Define the velocity error constant

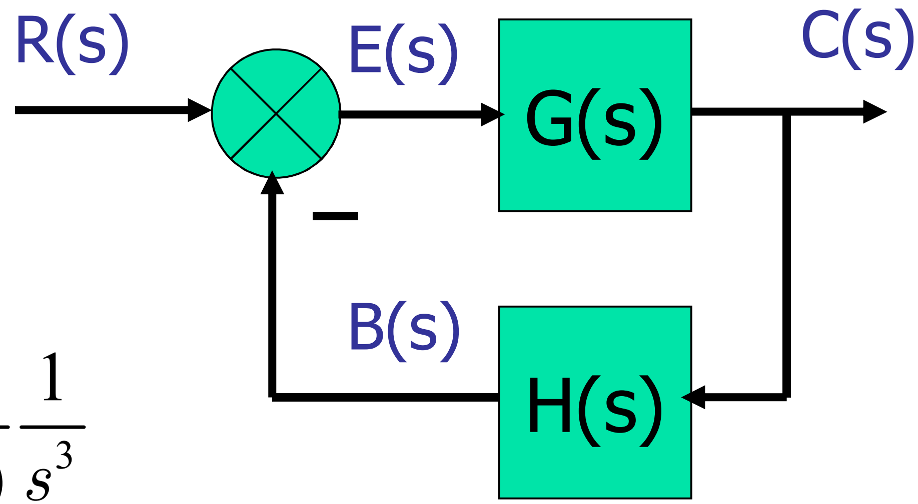
$$K_v = \lim_{s \to 0} sG(s)H(s) = \begin{cases} 0, & \text{Type 0 systems} \\ K, & \text{Type 1 systems} \\ \infty, & \text{Type 2 systems} \end{cases}$$

Then,

$$e_{ss} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)} = \begin{cases} \infty, & \text{Type 0 systems} \\ \dfrac{1}{K}, & \text{Type 1 systems} \\ 0, & \text{Type 2 or higher systems} \end{cases}$$

6. Static acceleration error constant Ka ($r=t^2/2$)

Let $r(t)=\frac{t^2}{2} \Rightarrow R(s)=\frac{1}{s^3}$



Then

$$e_{ss}=\lim_{t\to\infty} e(t)=\lim_{s\to 0}\frac{s}{1+G(s)H(s)}\frac{1}{s^3}$$

$$=\lim_{s\to 0}\frac{1}{s^2+s^2G(s)H(s)}=\lim_{s\to 0}\frac{1}{s^2\dfrac{KN_0(s)}{s^N D_0(s)}}$$

where

$$N_0(0)=1 \qquad D_0(0)=1$$

Define the acceleration error constant

$$K_a = \lim_{s \to 0} s^2 G(s) H(s) = \begin{cases} 0, & \text{Type 0 and 1 systems} \\ K, & \text{Type 2 system} \\ \infty, & \text{Type 3 or higher systems} \end{cases}$$

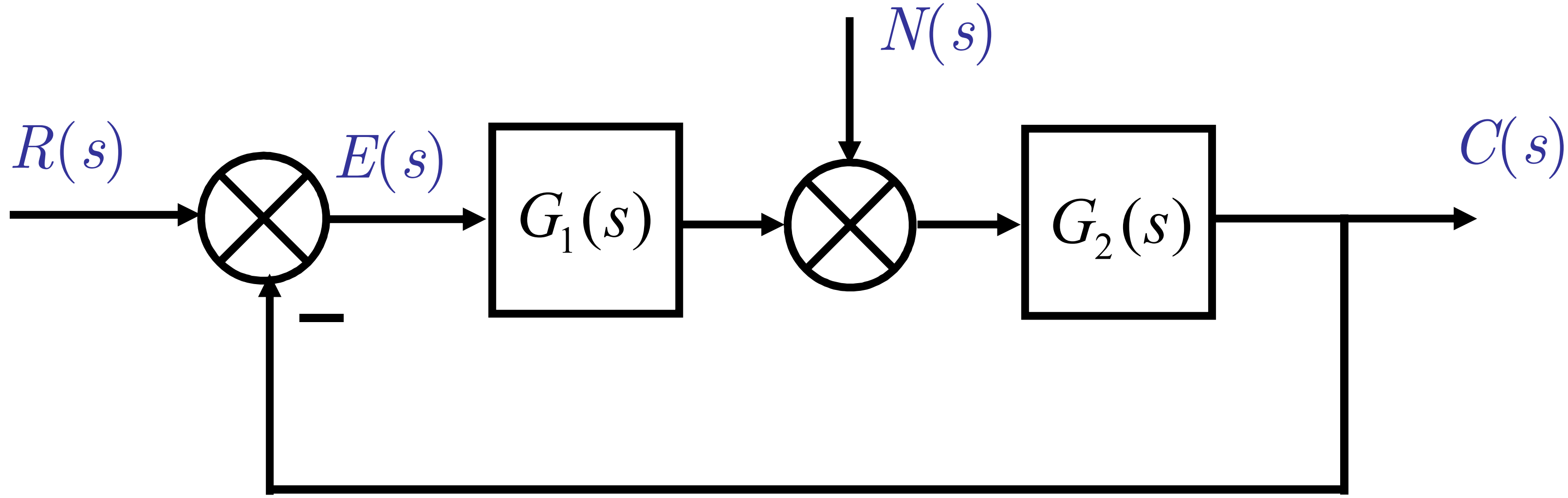
Then,

$$e_{ss} = \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s) H(s)}_v = \begin{cases} \infty, & \text{Type 0 and 1 systems} \\ \dfrac{1}{K}, & \text{Type 2 systems} \\ 0, & \text{Type 3 or higher systems} \end{cases}$$

The steady state error in terms of Gain K

Number of Integrations in $G(s)H(s)$	Input		
	$A \cdot 1(t)$	$A \cdot t1(t)$	$A \cdot t^2/2$
Type 0	$e_{ss} = \frac{A}{1+K}$	∞	∞
Type 1	$e_{ss} = 0$	$e_{ss} = \frac{A}{K}$	∞
Type 2	$e_{ss} = 0$	$e_{ss} = 0$	$e_{ss} = \frac{A}{K}$

7. The steady state error with disturbance n(t)



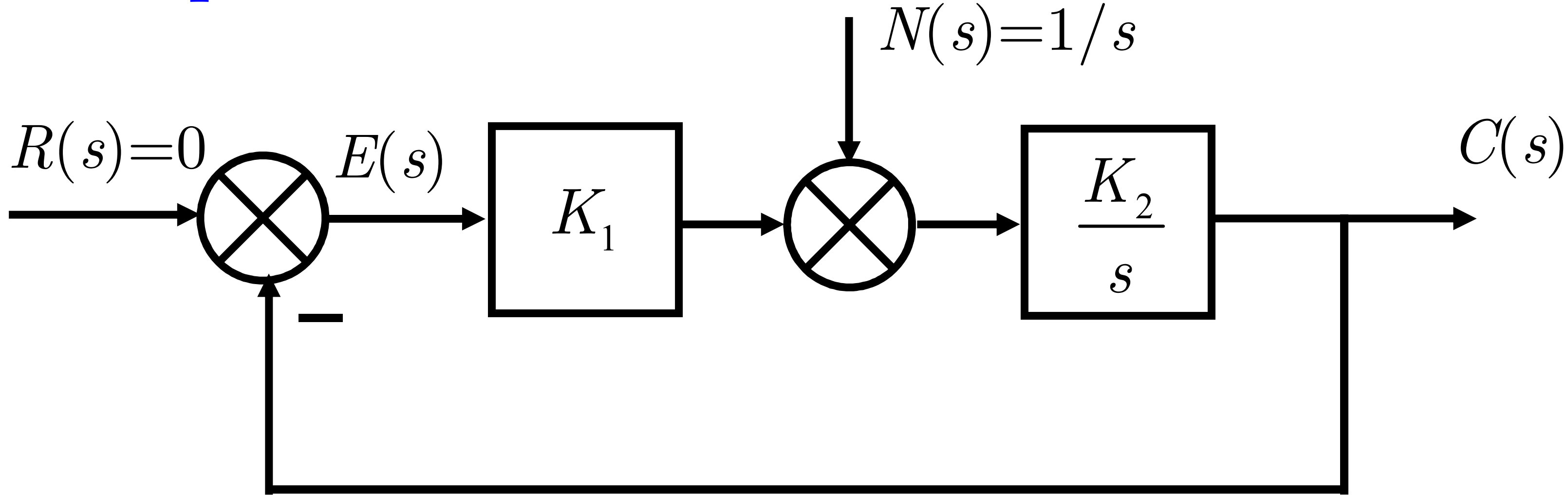
By the principle of superposition, we have

$$E(s)=\frac{1}{1+G_1(s)G_2(s)}R(s)-\frac{G_2(s)}{1+G_1(s)G_2(s)}N(s)$$

When $R(s)=0$

$$E(s)=-\frac{G_2(s)}{1+G_1(s)G_2(s)}N(s)$$

Example.

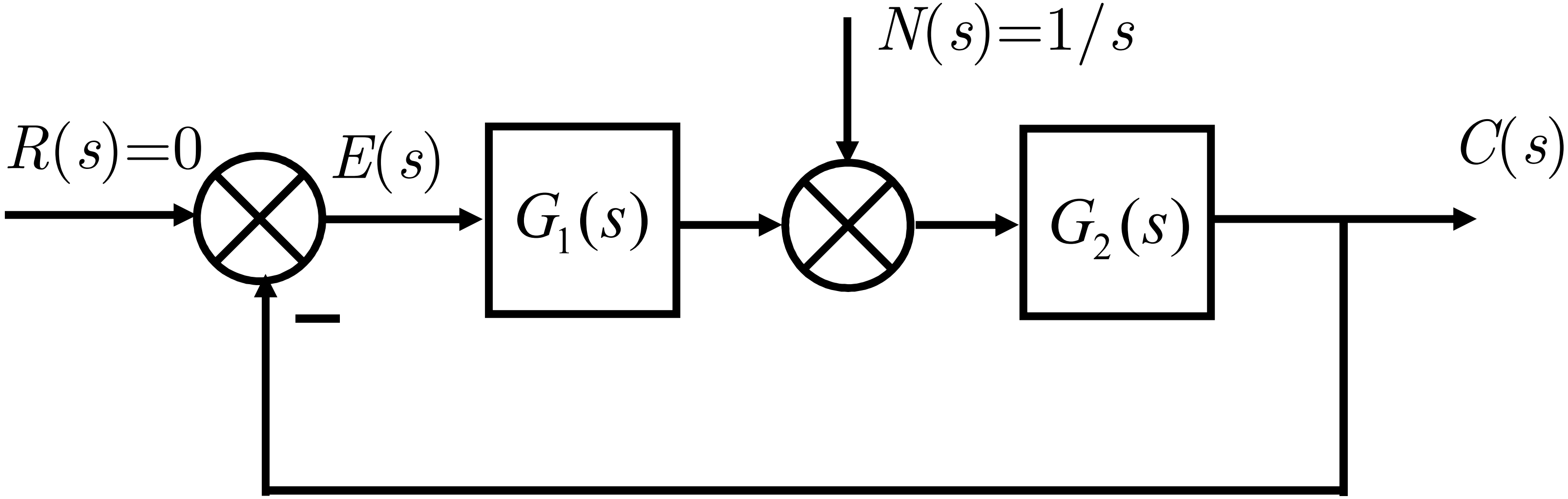


Determine its steady-state error.

Solution: Since $sE(s)$ has a stable pole, we have

$$e_{ss} = \lim_{s\to 0} sE(s) = -\lim_{s\to 0}\frac{sK_2}{s+K_1K_2}\cdot\frac{1}{s} = -\frac{1}{K_1}$$

Note that attention must be paid to the locations of the poles of $sE(s)$. For instance,



$$G_1(s) = \frac{K_1}{s} \qquad G_2(s) = \frac{K_2}{s} \qquad K_1 > 0, K_2 > 0$$

$$sE(s) = -s\frac{sK_2}{s^2 + K_1K_2} \cdot \frac{1}{s}$$

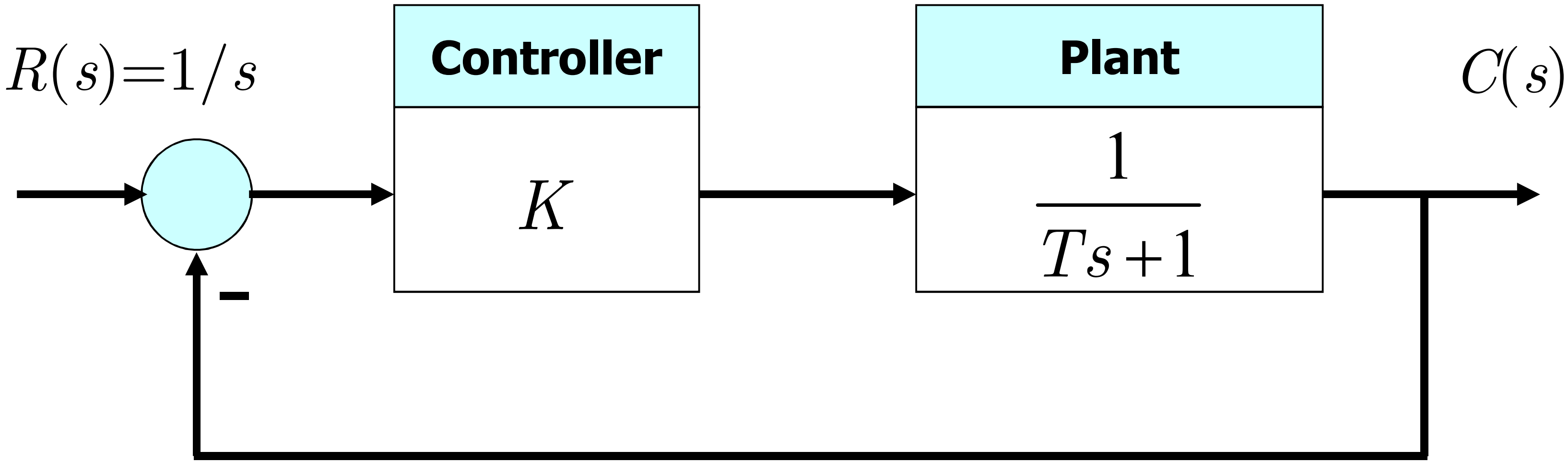
Since $sE(s)$ has imaginary poles, e_{ss} does not exist.

8. Effects of integral and derivative control actions on system performance

In the proportional control of a plant whose transfer function does not possess an integrator $1/s$, there is a steady-state error, or offset, in the response to a step input. Such an offset can be eliminated if an integral control action is included in the controller.

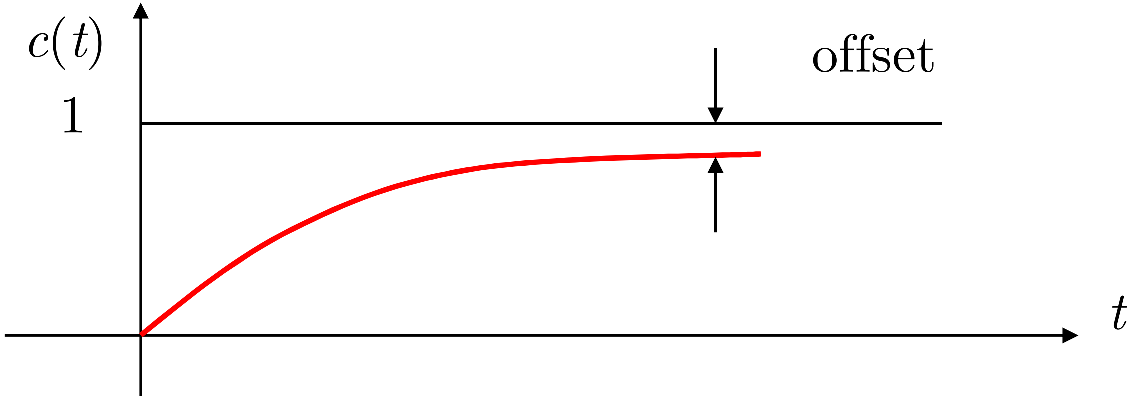
1. Proportional control of systems

Consider the following control system:

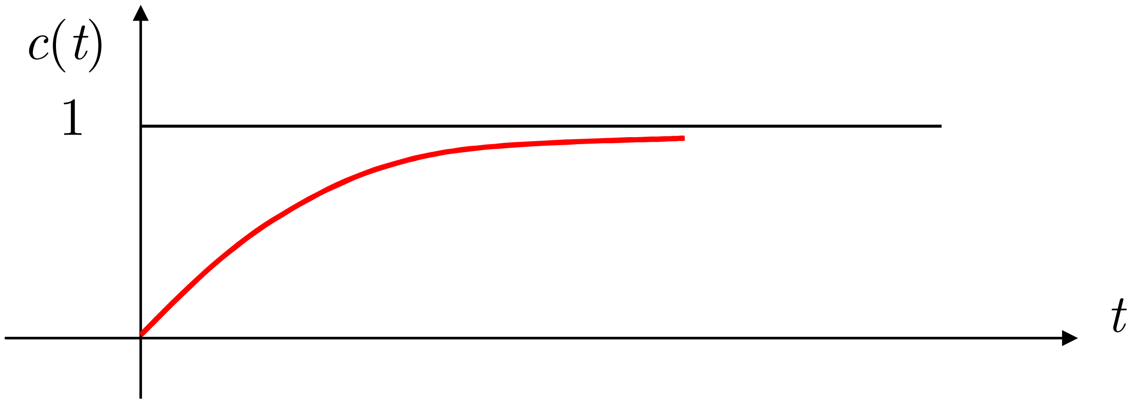


Since the system is type 0 system,

$$e_{ss} = \frac{1}{1+K}$$



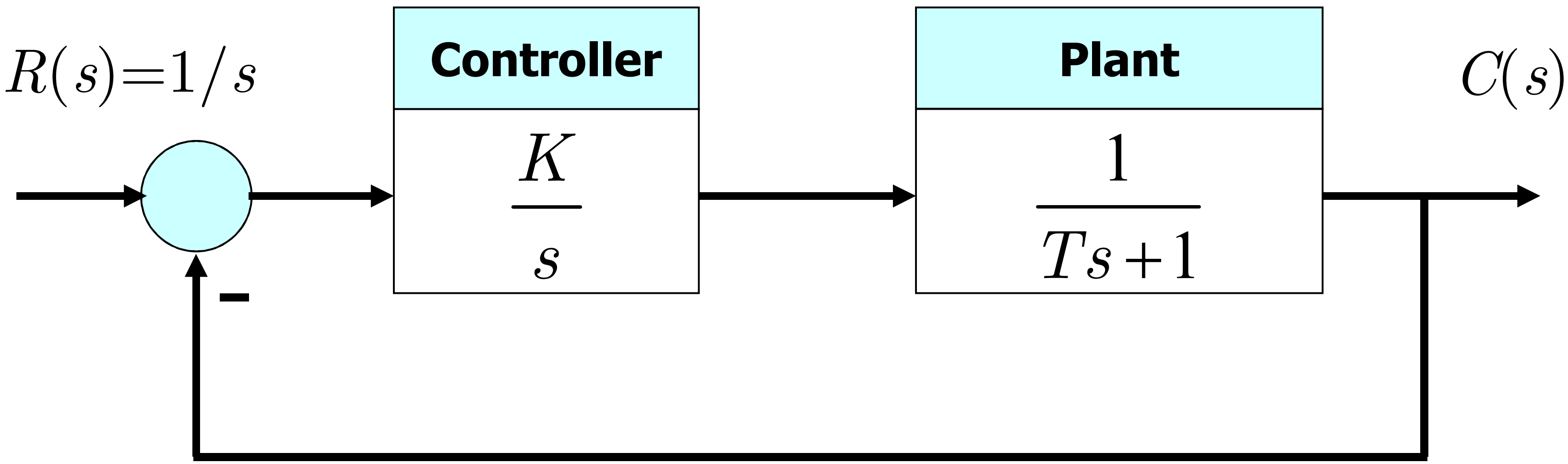
A higher K can improve the steady-state error, but may cause saturation of the amplifier. The derivation of an integral control action may eliminate the steady-state error.



A higher K leads to a smaller offset.

2. Integral control of systems

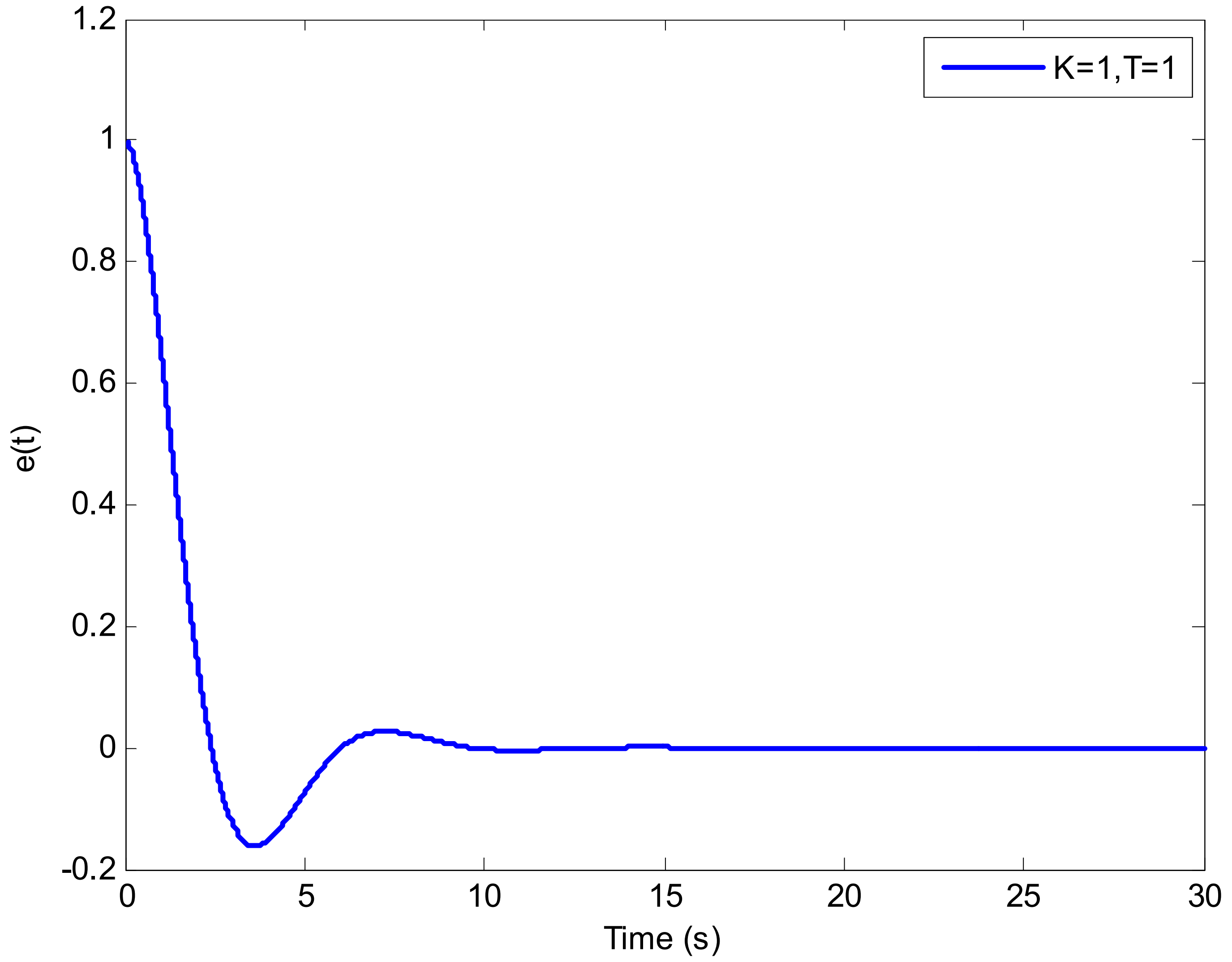
Consider the following control system:



Since the system is type 1 and yet stable,

$$e_{ss}=0$$

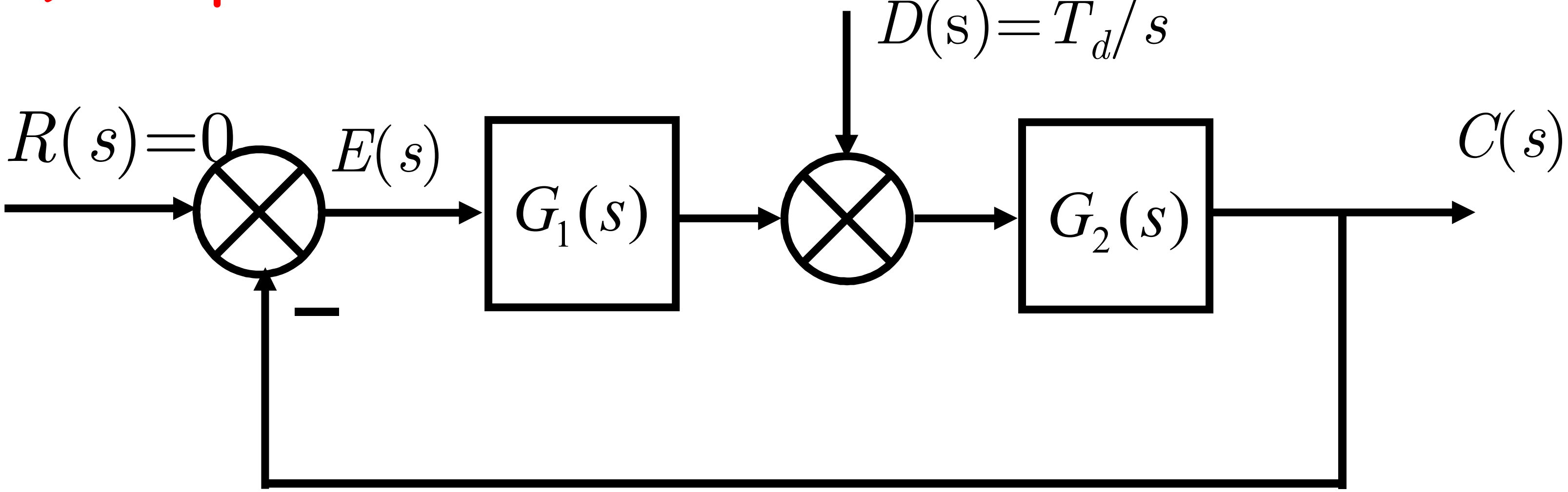
This is an important improvement over the proportional control alone, which gives offset.



Tracking error $e(t)$ of the integral control: e_{ss}=0

3. Application example: Response to torque disturbance

a) Proportional control



$$G_1(s)=K_p,\quad G_2(s)=\frac{1}{s(Js+b)},K_p>0,J>0,b>0$$

$$E(s)=-C(s)=-\frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)=-\frac{1}{Js^2+bs+K_p}\frac{T_d}{s}$$

$$sE(s) = -sC(s) = -s\frac{1}{Js^2 + bs + K_p}\frac{T_d}{s}$$

$sE(s)$ has two stable poles and therefore,

$$e_{ss} = \lim_{s\to 0} sE(s) = -\frac{T_d}{K_p}$$

b) Proportional + Integral (PI) control

Let $G_1(s)$ be replaced by a proportional+integral controller:

$$G_1(s) = K_p(1 + \frac{1}{T_i s}), \quad T_i > 0$$

Then

$$sE(s) = -sC(s) = -\frac{s^2}{Js^3 + bs^2 + K_p s + K_p / T_i}\frac{T_d}{s}$$

$sE(s)$ is stable if and only if

$$bT_i > J$$

On this premise, we have

$$e_{ss} = \lim_{s\to 0} sE(s)$$

$$= -\lim_{s\to 0} \frac{s^2}{Js^3 + bs^2 + K_p s + K_p / T_i} \frac{T_d}{s} = 0$$

Remark: It is important to point out that if the controller were an integral controller alone, the closed-loop characteristic equation would be

$$Js^3 + bs^2 + K = 0$$

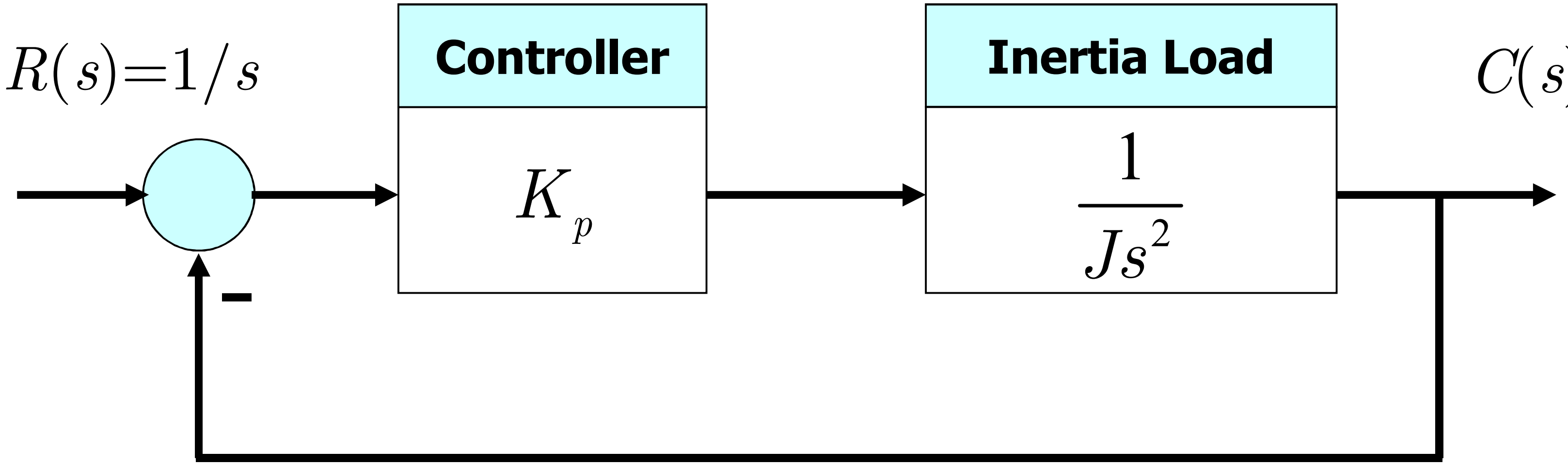
which is unstable.

4. Derivative control action

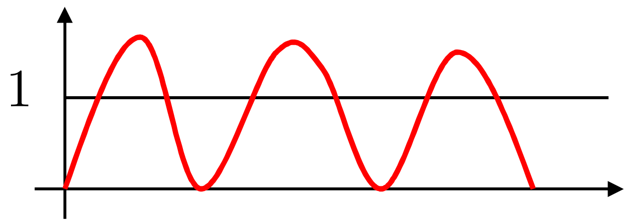
An advantage of using derivative control action is that it responds to the rate of change of the actuating error and can produce a significant correction before the magnitude of the actuating error becomes too large. Derivative control thus

anticipates the actuating error, initiates an early corrective action, and tends to increase the stability of the system.

Consider the following proportional control of an inertia load:



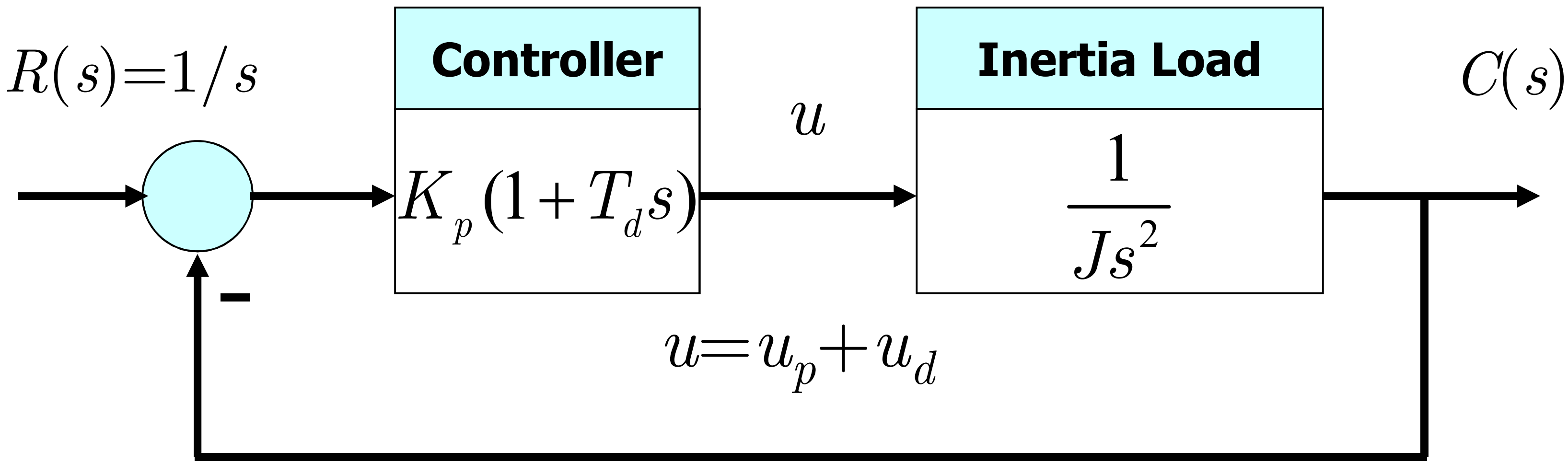
$$\frac{C(s)}{R(s)} = \frac{1}{Js^2 + K_p}$$



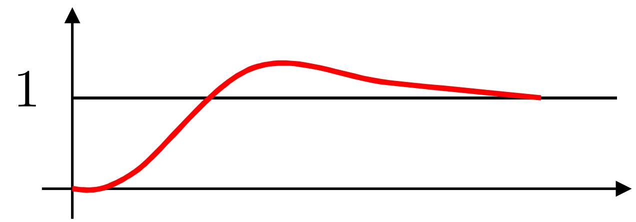
The response to a unit-step input oscillates indefinitely.

a) Proportional + Derivative (PD) control of a system with inertia load

Consider the following PD control of an inertia load:

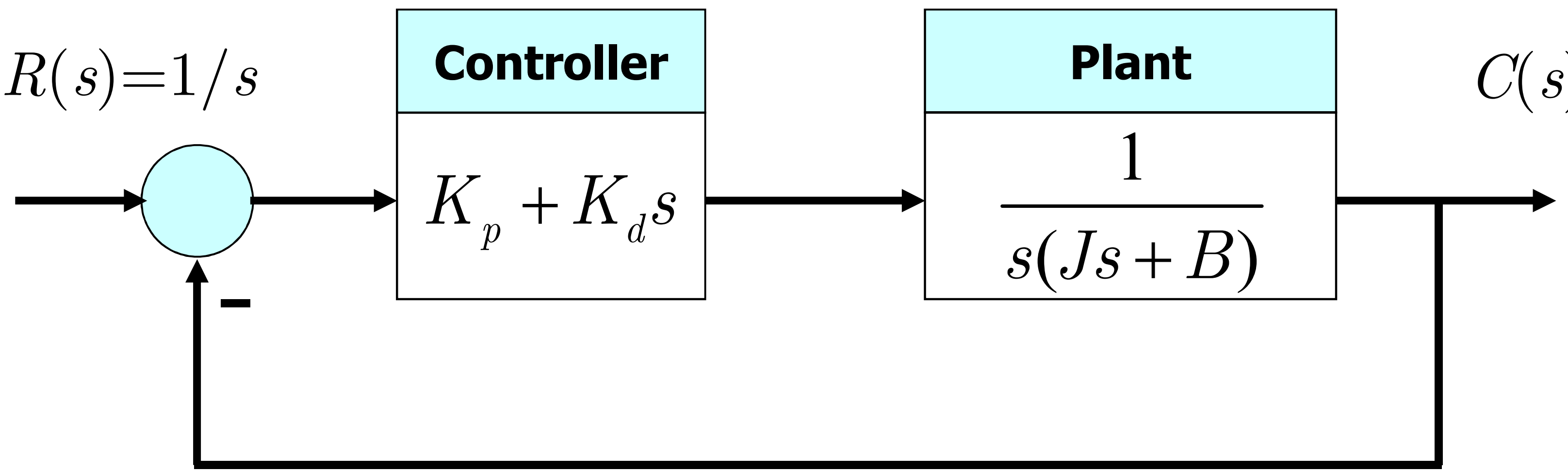


$$\frac{C(s)}{R(s)} = \frac{K_p(1+T_d s)}{Js^2 + K_p T_d s + K_p}$$



b) Proportional + Derivative (PD) control of Second-order systems

Consider the following proportional+derivative control of an inertia load:



$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (B + K_d)s + K_p}$$

The steady-state error for a unit-ramp input is

$$e_{ss} = \frac{B}{K_p}$$

The characteristic equation is

$$s^2 + [(B + K_d)/J]s + K_p / J = 0 \Rightarrow s^2 + 2\zeta_d \omega_n s + \omega_n^2 = 0$$

$$\omega_n = \sqrt{\frac{K_p}{J}}, \quad \zeta_d = \frac{B + K_d}{2\sqrt{K_p J}} > \zeta = \frac{B}{2\sqrt{K_p J}}$$

it is possible to make both the steady-state error e_{ss} for a ramp input and the maximum overshoot for a step input small by making B small, K_p large, and K_d large enough so that ζ_d is between 0.4 and 0.7.

Summary of Chapter 5

In this chapter, time domain analysis is given.

1. We studied the responses of the following systems:

- First-order system:

$$C(s)=\frac{1}{Ts+1}R(s)$$

$$\begin{cases} c(t)=1-e^{-t/T}, t\geq 0 & \text{unit-step} \\ c(t)=t-T+Te^{-\frac{t}{T}}, t\geq 0, & \text{unit-ramp} \\ c(t)=\dfrac{1}{T}e^{-t/T}, t\geq 0 & \text{unit-impulse} \end{cases}$$

Main properties of the unit-step response are:

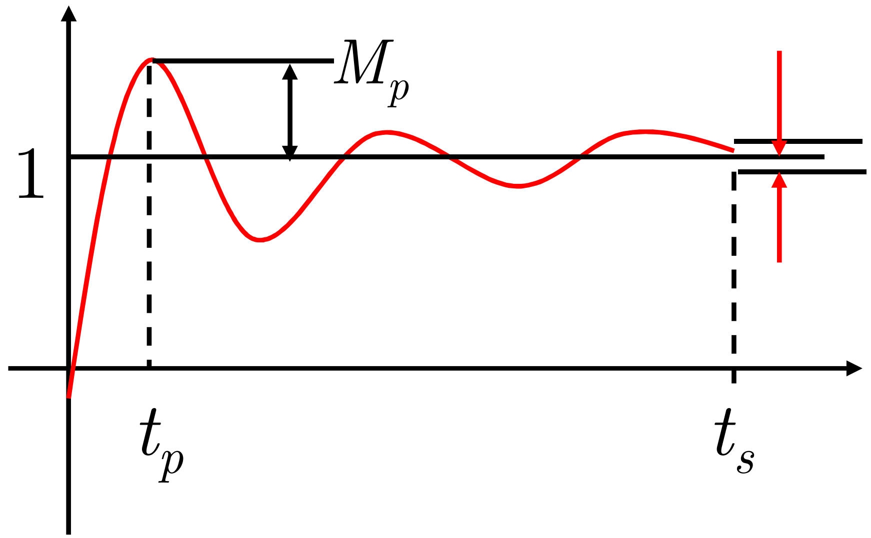
$$\begin{cases} t_s = 3T \\ e_{ss} = 0 \end{cases}$$

- Second-order system: $C(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s)$

$$\begin{cases} c(t) = 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \beta\right), t \geq 0, \quad 0 < \zeta < 1 \\ c(t) = 1 - \cos\omega_n t, \quad t \geq 0, \quad \zeta = 0 \\ c(t) = 1 - e^{-\omega_n t}(1 + \omega_n t), \quad t \geq 0, \quad \zeta = 1 \\ c(t) = 1 + \dfrac{1}{\dfrac{T_2}{T_1} - 1} e^{-\frac{1}{T_1}t} + \dfrac{1}{\dfrac{T_1}{T_2} - 1} e^{-\frac{1}{T_2}t}, t \geq 0, \quad \zeta > 1 \end{cases}$$

Performance specifications for unit-step response of a second-order system with $0<\zeta<1$:

$$\begin{cases} t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} \\ M_p = e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\% \\ t_s = \dfrac{3.5}{\zeta\omega_n} \\ e_{ss} = 0 \end{cases}$$



Performance specifications for unit-step response of a second-order system with $\zeta\geq 1$: No oscillation and

$$\begin{cases} t_s : \text{See the given table} \\ e_{ss} = 0 \end{cases}$$

- Unit-step response for higher-order systems:

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k (s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k^2}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

which is the sum of a series of first-order and second-order systems.

The performance analysis for higher-order systems is based on *dominant poles*, where to determine dominant poles, *dipole* must be excluded.

2. We studied the relationships among the impulse, the step and the ramp responses:

$$\frac{d}{dt}c_t(t) = c_1(t)$$

$$\frac{d}{dt}c_1(t) = c_\delta(t)$$

from which the following equations also hold:

$$\int_0^t c_\delta(\tau)d\tau = c_1(t)$$

$$\int_0^t c_1(\tau)d\tau = c_t(t)$$

The same conclusion can be applied to the *parabolic response*.

3. We studied the stability criterion, *Routh's stability criterion*. For a given *closed-loop characteristic equation*

$$D(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

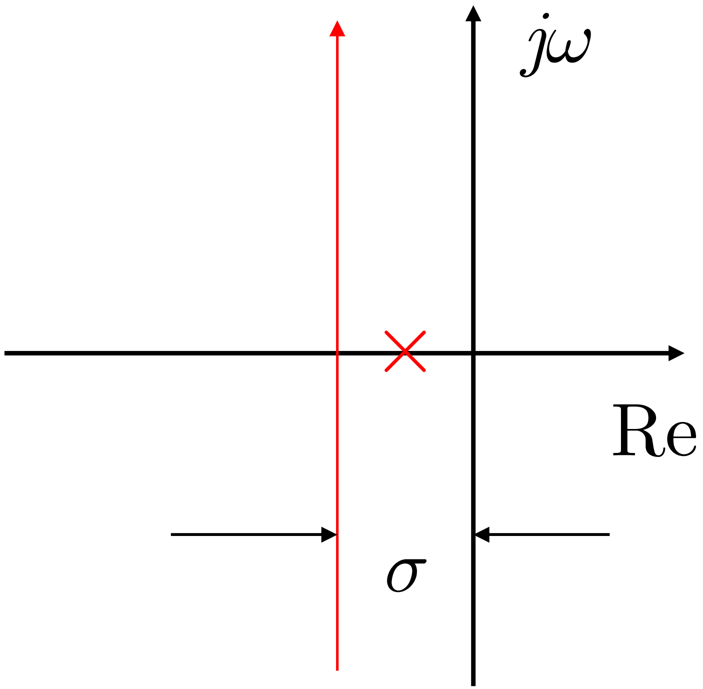
1) The system is stable if and only if

- All the coefficients of $D(s)$ are positive and
- All the terms in the first column of the array have positive signs.

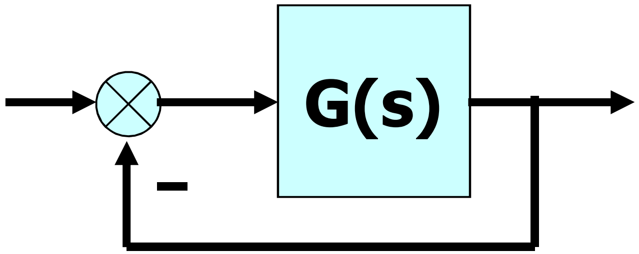
2) The number of roots of the D(s) with *positive* real parts is equal to the number of changes in sign of the coefficients of the first column of the array.

Some special cases are also discussed.

4. By utilizing Routh's stability criterion, we are able to address the relative stability and conditional stability problems:



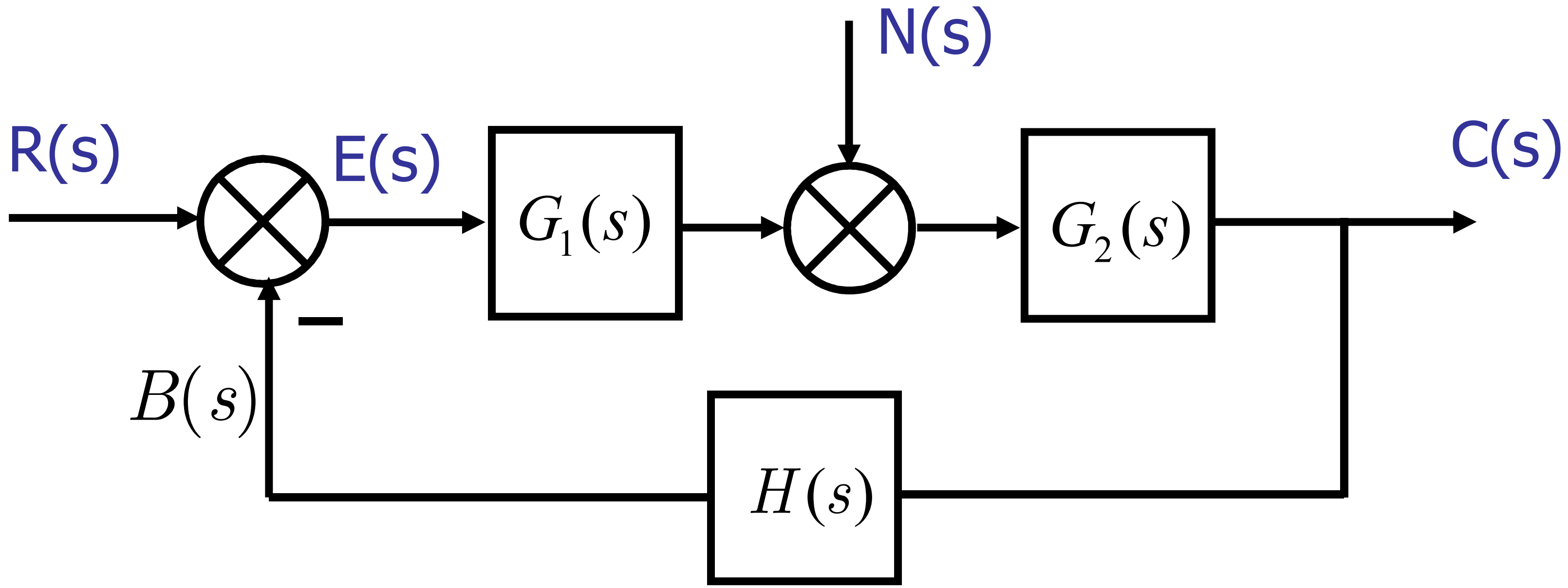
$$s = \hat{s} - \sigma \Rightarrow \hat{s} = s + \sigma$$



$$G(s) = \frac{K}{s(0.1s+1)(0.25s+1)}$$

$$0 < K < 14$$

5. We studied the steady-state error.



Let

$$E(s) := R(s) - B(s)$$

then

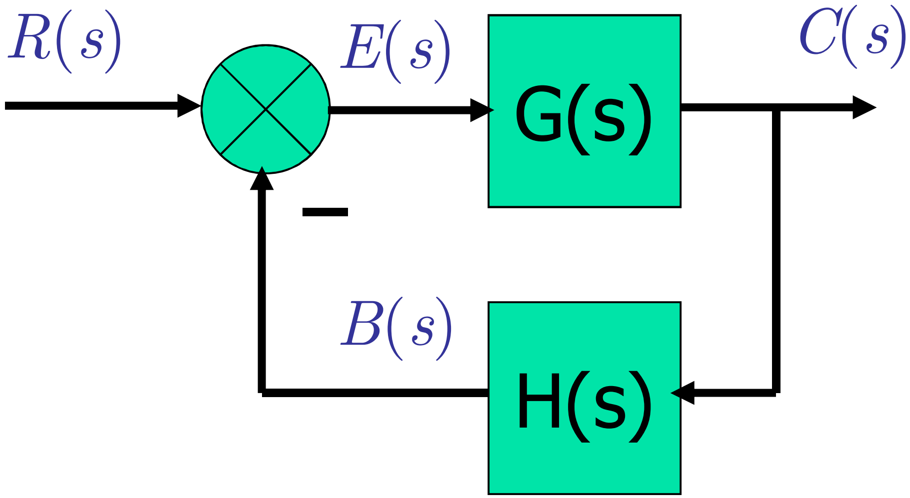
$$E(s) = E_R(s) + E_N(s)$$

To apply the Final Value Theorem, it is required that all the poles of both $sE_R(s)$ and $sE_N(s)$ lie in the left-half s-plane:

$$sE_R(s) = \frac{s}{1+G_1(s)G_2(s)H(s)}R(s)$$

$$sE_N(s) = -\frac{sG_2(s)H}{1+G_1(s)G_2(s)H(s)}N(s)$$

Further, in case of no disturbance, for typical input signals $1(t)$, $t{\cdot}1(t)$ and $t^2{\cdot}1(t)/2$, the system



type, the number of the integrations of $G(s)H(s)$, makes us determine e_{ss} conveniently. For

$$G(s)H(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)} = \frac{KN_0(s)}{s^N D_0(s)}$$

(1) When input signal $r = 1(t)$

$$e_{ss} = \begin{cases} \dfrac{1}{1+K}, & \text{Type } 0 \\ 0, & \text{Type } \geq 1 \end{cases}$$

where *static position error constant* K_p:

$$K_p = \lim_{s\to 0} G(s)H(s) = \begin{cases} K, \text{type} = 0 \\ \infty, \ \text{type} \geq 1 \end{cases}$$

(2) When input signal $r = t \cdot 1(t)$

$$e_{ss} = \begin{cases} \infty, & \text{Type } 0 \\ \dfrac{1}{K}, & \text{Type } 1 \\ 0, & \text{Type } \geq 2 \end{cases}$$

where *static velocity error constant* K_v:

$$K_v = \lim_{s \to 0} sG(s)H(s) = \begin{cases} 0, & \text{Type } 0 \\ K, & \text{Type } 1 \\ \infty, & \text{Type } 2 \end{cases}$$

(3) When input signal $r = t^2 \cdot 1(t)/2$

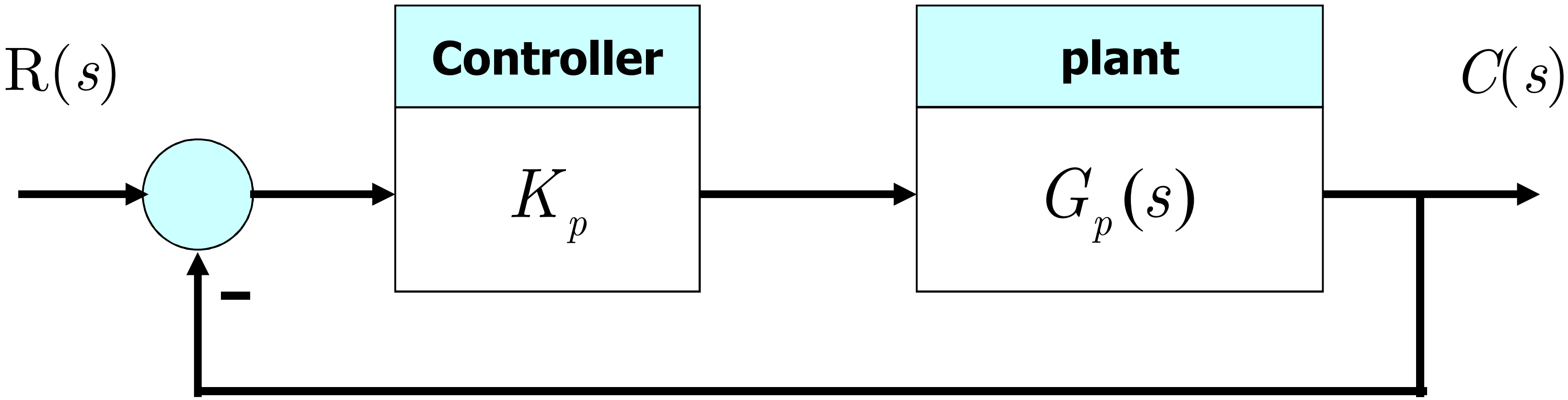
$$e_{ss} = \begin{cases} \infty, & \text{Type } 0,1 \\ \dfrac{1}{K}, & \text{Type } 2 \\ 0, & \text{Type } \geq 3 \end{cases}$$

where *static acceleration error constant* K_a:

$$K_a = \lim_{s \to 0} s^2 G(s)H(s)$$

6. We introduced some widely used controllers:

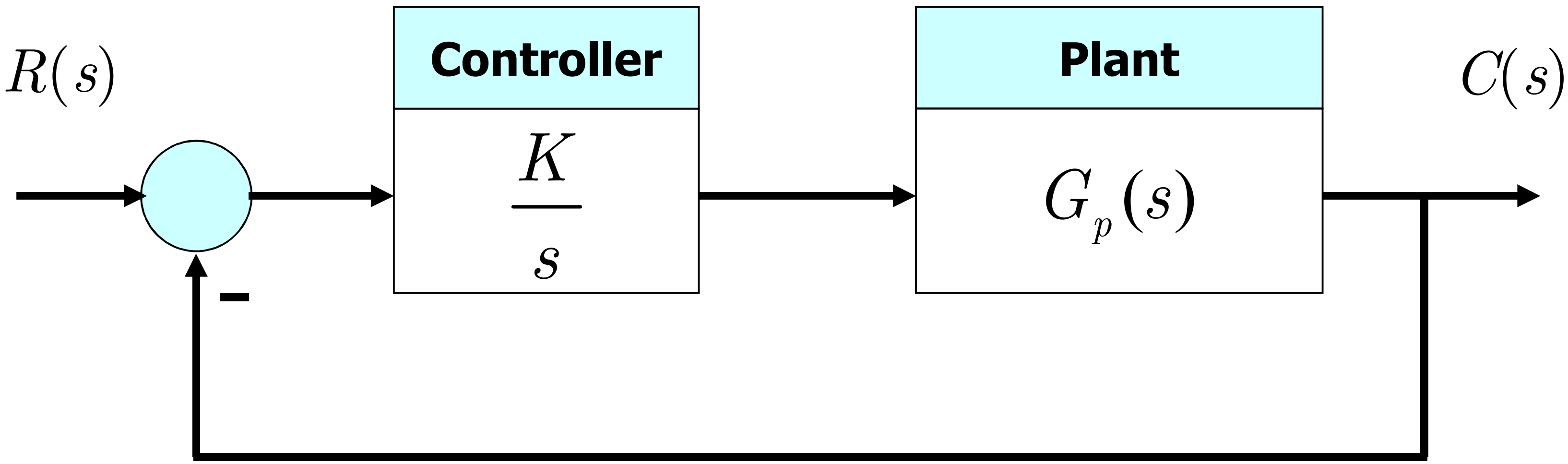
(1)Proportional controller



Advantage: In many cases, simply adjusting K_p is able to improve transient and steady-state performance;

Disadvantage: Large K may cause system to become unstable and cause saturation of the amplifier. Moreover, for type 0 $G_p(s)$, steady-state error is non-zero when $R(s)=1/s$.

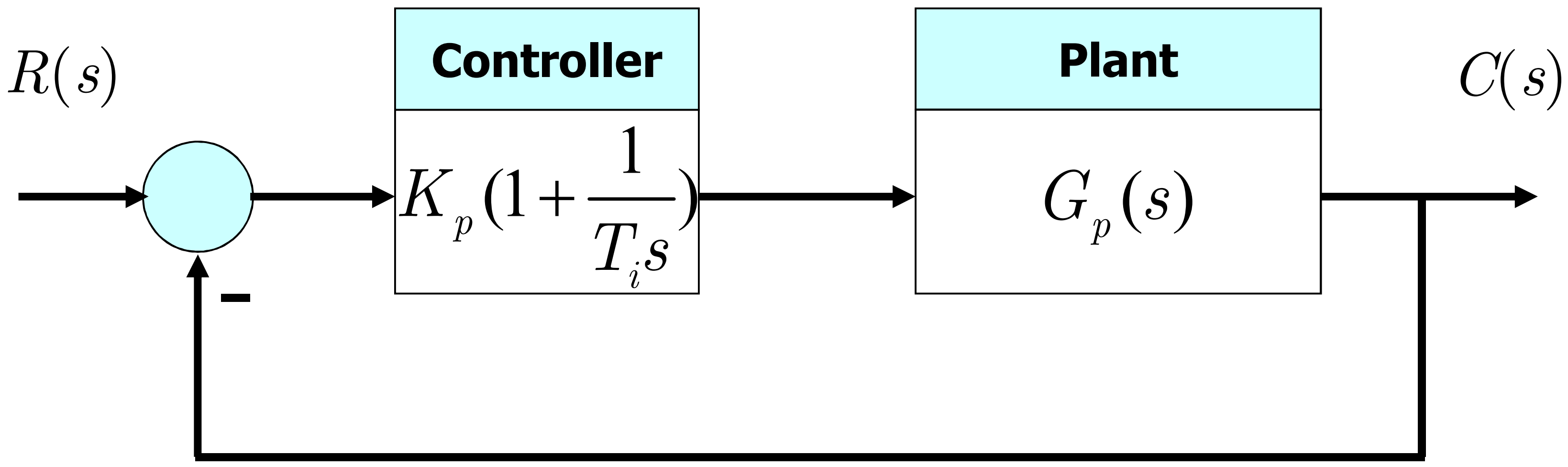
(2) Integral Controller



Advantage: is able to eliminate steady-state error.

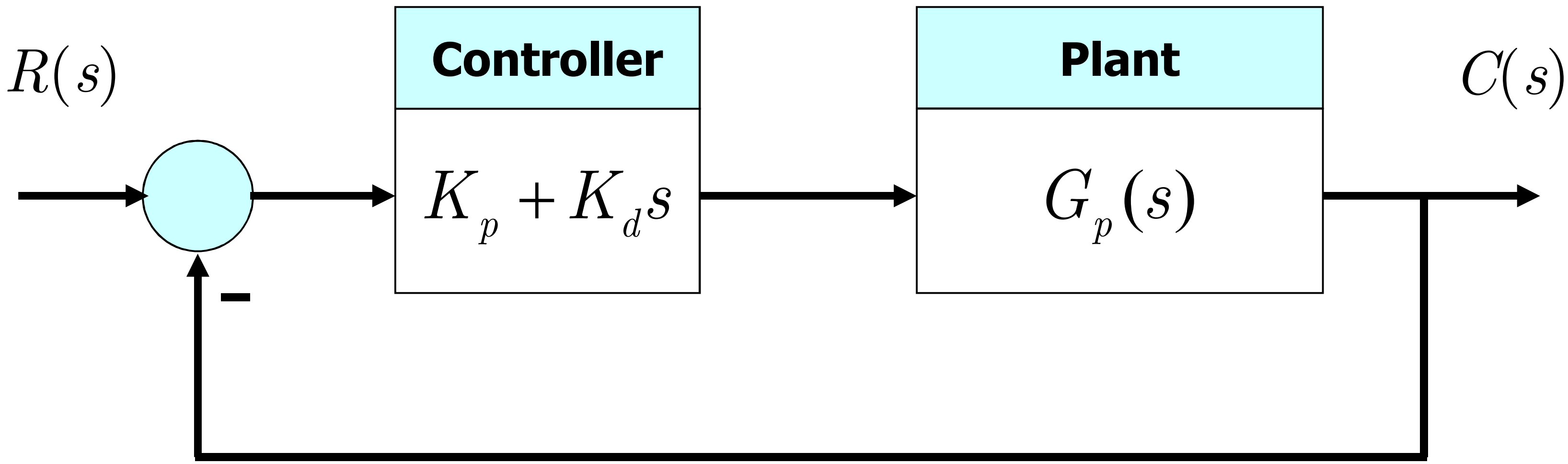
Disadvantage: may cause system to become unstable and therefore, is rarely used alone.

(4) Proportional +Integral (PI) controller



Advantage: is able to eliminate steady-state error while keeping system stability.

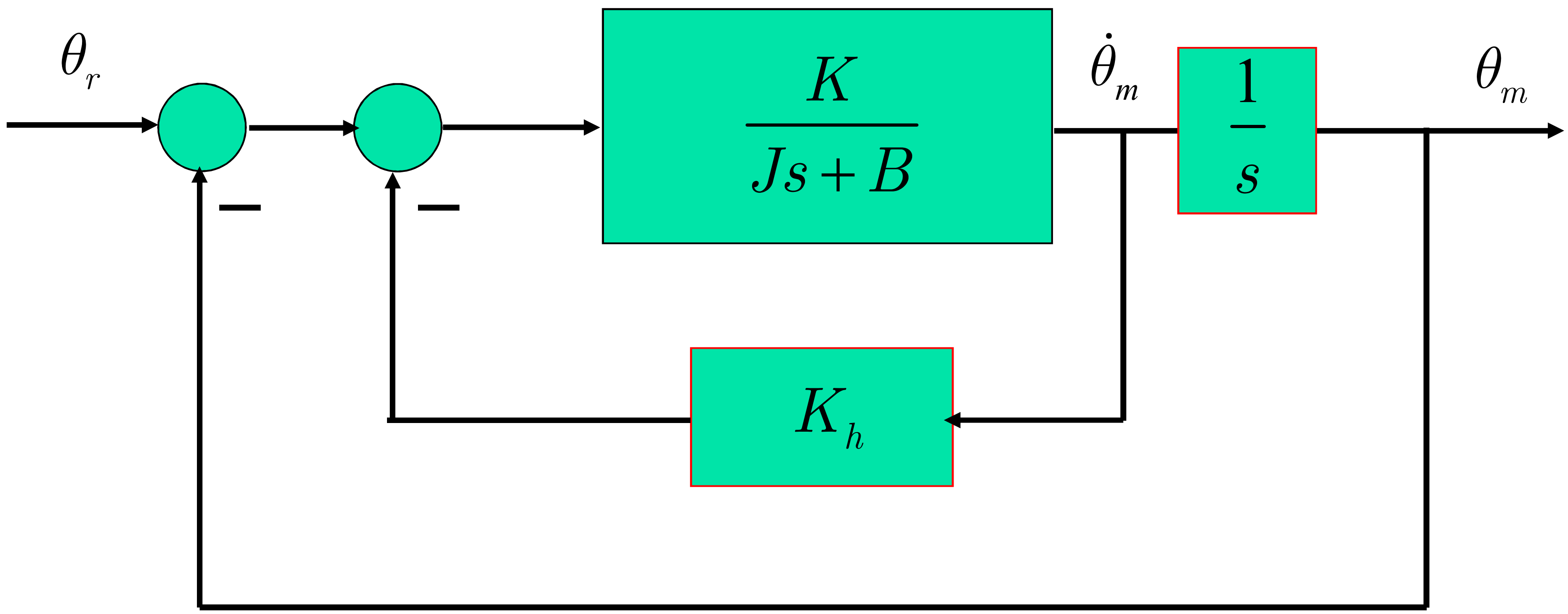
(5) Proportional + Derivative (PD) controller



Advantage: is able to improve system transient performance and stability;

Disadvantage: may amplify high frequency input disturbance.

(6) Servo system with velocity feedback



Advantage: is able to improve system transient performance while avoiding high frequency output disturbance.