作业2.1-2.2 习题答案

1. 已知空间直角坐标系中的四点$A(1,1,1),B(1,2,3),C(1,4,9),D(1,8,27)$,判断这四个点是否在同一平面上.

解: $\vec{AB}=(0,1,2),\vec{AC}=(0,3,8),\vec{AD}=(0,7,26)$

线性方程组$x\vec{AB}+y\vec{AC}+z\vec{AD}=0$即为$\begin{cases} x+3y+7z=0 \\ 2x+8y+26z=0 \end{cases}$ 未知量个数大于方程个数,固有非零解,所以四点在同一平面上.

2. 设$\alpha=(3,-2,-1,1),\beta=(-3,1,-2,1)$. 求向量$\gamma=(c_1,c_2,c_3,c_4)$,使$2\alpha+3\gamma=\beta$.

解: 将α,β,γ代入$2\alpha+3\gamma=\beta$,即

$$2(3,-2,-1,1)+3(c_1,c_2,c_3,c_4)=(-3,1,-2,1).$$

由此得

$$6+3c_1=-3,\quad -4+3c_2=1,$$
$$-2+3c_3=-2,\quad 2+3c_4=1.$$

于是$c_1=-3,c_2=\frac{5}{3},c_3=0,c_4=-\frac{1}{3}$. 即所求的向量$\gamma=(-3,\frac{5}{3},0,-\frac{1}{3})$.

3. 设$\alpha=(2,1,-2),\beta=(-4,2,3),\gamma=(-8,8,5)$.证明:存在数$k$使$2\alpha+k\beta=\gamma$.

证: 将α,β,γ代入$2\alpha+k\gamma=\beta$,得

$$2(2,1,-2)+k(-4,2,3)=(-8,8,5).$$

由此得

$$4-4k=-8,\quad 2+2k=8,\quad -4+3k=5.$$

他们有惟一解$k=3$.

4. 设$\alpha_1=(2,2,2),\alpha_2=(-1,3,0),\alpha_3=(2,0,3),\beta=(1,3,0)$. 问:向量$\beta$是否为向量组$\alpha_1,\alpha_2,\alpha_3$的线性组合?如果是,求一组组合系数.

解: 设$\beta = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$ 这就是要问如下线性方程组:
$\begin{cases} 2x_1 - x_2 + 2x_3 = 1 \\ 2x_1 + 3x_2 \qquad = 3 \\ 2x_1 + \qquad 3x_3 = 0 \end{cases}$ 是否有解,若有解其解就是组合系数.

因为$\begin{pmatrix} 2 & -1 & 2 & 1 \\ 2 & 3 & 0 & 3 \\ 2 & 0 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 2 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$

所以方程组有惟一解$x_1 = \frac{3}{2}, x_2 = 0, x_3 = -1$. 即$\beta = \frac{3}{2}\alpha_1 - \alpha_3$.

5.已知空间直角坐标系中的三点$A(1,1,1), B(1,2,4), C(1,3,9)$,是否能将空间任意一个向量$\vec{OD}$写成$\vec{OA}, \vec{OB}, \vec{OC}$的线性组合?以$OA, OB, OC$为棱的平行六面体的体积是否等于0?

解: $\vec{OA} = (1,1,1), \vec{OB} = (1,2,4), \vec{OC} = (1,3,9)$

因为$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$ 看出$\vec{OA}, \vec{OB}, \vec{OC}$线性无关,空间中任意向量$\vec{OD}$可以表示成$\vec{OA}, \vec{OB}, \vec{OC}$的线性组合.又

$$(\vec{OA} \times \vec{OB}) \cdot \vec{OC} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 2$$

进而以OA, OB, OC为棱的平行六面体的体积不为零.

6. 判断以下各向量组是否线性相关:

(1) $\alpha_1 = (2,1), \alpha_2 = (-1,4), \alpha_3 = (2,-3)$;

(2) $\alpha_1 = (2,1,1), \alpha_2 = (1,2,-1), \alpha_3 = (-2,3,0)$

解: (1)设有数x_1, x_2, x_3,使$x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = 0$,即

$$x_1(2,1) + x_2(-1,4) + x_3(2,-3) = 0$$

由此得方程组

$$\begin{cases} 2x_1 - x_2 + 2x_3 = 0 \\ x_1 + 4x_2 - 3x_3 = 0 \end{cases}$$

此方程组有无穷多解,故有非零解,所以$\alpha_1,\alpha_2,\alpha_3$线性相关.

(2)设$x_1\alpha_1+x_2\alpha_2+x_3\alpha_3=0$,得方程组

$$\begin{cases} 2x_1+x_2-2x_3=0 \\ x_1+2x_2+3x_3=0 \\ x_1-x_2=0 \end{cases}$$

由于$\begin{pmatrix} 2 & 1 & -2 & 0 \\ 1 & 2 & 3 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & -2 & 0 \end{pmatrix}$

故方程组只有零解:$x_1=x_2=x_3=0$.所以$\alpha_1,\alpha_2,\alpha_3$线性无关.

7. 证明:如果向量组$\alpha_1,\alpha_2,\cdots,\alpha_r$线性无关,而$\alpha_1,\alpha_2,\cdots,\alpha_r,\beta$ 线性相关,则向量β可以表示成$\alpha_1,\alpha_2,\cdots,\alpha_r$的线性组合.

证: 由题设有不全为零的数$l,l_1,\cdots,l_r$使$l\beta+l_1\alpha_1+\cdots+l_r\alpha_r=0$.将$l\beta$移项,得$(-l)\beta=l_1\alpha_1+\cdots+l_r\alpha_r$.现来证$l\neq 0$.用反证法,若$l=0$,由$l,l_1,\cdots,l_r$不全为零,故$l_1,\cdots,l_r$ 不全为零,且有$0=l_1\alpha_1+\cdots+l_r\alpha_r$.与题设$\alpha_1,\alpha_2,\cdots,\alpha_r$线性无关矛盾.故$l\neq 0$.将$(-l)\beta=l_1\alpha_1+\cdots+l_r\alpha_r$两端同乘以$\frac{1}{(-l)}$,则得

$$\beta=\frac{l_1}{(-l)}\alpha_1+\frac{l_2}{(-l)}\alpha_2+\cdots+\frac{l_r}{(-l)}\alpha_r$$

即向量β可以表示成$\alpha_1,\alpha_2,\cdots,\alpha_r$的线性组合.

8. 设有三维向量$\alpha_1=\big(1+\lambda,1,1\big),\alpha_2=\big(1,1+\lambda,1\big),\alpha_3=\big(1,1,1+\lambda\big),\beta=\big(0,\lambda,\lambda^2\big)$.问$\lambda$为何值时$\beta$可写成$\alpha_1,\alpha_2,\alpha_3$的线性组合,且表达式唯一.

解: 设$\beta=x_1\alpha_1+x_2\alpha_2+x_3\alpha_3$,$\beta$ 可否写成$\alpha_1,\alpha_2,\alpha_3$的线性组合即问如下方程组是否有解问题

$$\begin{cases} (1+\lambda)x_1+x_2+x_3=0 \\ x_1+(1+\lambda)x_2+x_3=\lambda \\ x_1+x_2+(1+\lambda)x_3=\lambda^2 \end{cases}$$

因为$\begin{pmatrix} 1+\lambda & 1 & 1 & 0 \\ 1 & 1+\lambda & 1 & \lambda \\ 1 & 1 & 1+\lambda & \lambda^2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1+\lambda & \lambda^2 \\ 0 & -\lambda & -\lambda(\lambda+2) & -\lambda^2(1+\lambda) \\ 0 & 0 & -\lambda(\lambda+3) & -\lambda(\lambda^2+2\lambda-1) \end{pmatrix}$

所以当$\lambda\neq 0$且$\lambda\neq -3$时,β可写成$\alpha_1,\alpha_2,\alpha_3$的线性组合,且表达式唯一;

当$\lambda = 0$时,β可写成$\alpha_1, \alpha_2, \alpha_3$的线性组合,但表达式不唯一;
当$\lambda = -3$时,β不能由$\alpha_1, \alpha_2, \alpha_3$线性表出.

9. 设向量β可由向量组$\alpha_1, \alpha_2, \cdots, \alpha_r$线性表示,但不能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示.证明:

(1) α_r不能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示;

(2) α_r能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}, \beta$线性表示.

证: (1)若不然,设α_r可由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示,且设

$$\alpha_r = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_{r-1}\alpha_{r-1}. \qquad (1)$$

由假设β可由$\alpha_1, \alpha_2, \cdots, \alpha_r$线性表示,可设

$$\beta = l_1\alpha_1 + l_2\alpha_2 + \cdots + l_{r-1}\alpha_{r-1} + l_r\alpha_r. \qquad (2)$$

将(1)式代入(2)式,并整理得

$$\beta = (l_1 + l_r k_1)\alpha_1 + \cdots + (l_{r-1} + l_r k_{r-1})\alpha_{r-1}$$

即β可由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示,与题设不符.
故α_r不能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示.

(2)由上面(2)式知,由于β不能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}$线性表示,故必$l_r \neq 0$.于是由(2)式可得

$$\alpha_r = \frac{1}{l_r}\beta - \frac{l_1}{l_r}\alpha_1 - \cdots - \frac{l_{r-1}}{l_r}\alpha_{r-1}$$

即α_r能由向量组$\alpha_1, \alpha_2, \cdots, \alpha_{r-1}, \beta$线性表示.

10. 设复数域上的向量$\alpha_1, \ldots, \alpha_n$ 线性无关. 对复数λ的不同值，判断向量组$S = \{\alpha_1 + \lambda\alpha_2, \ldots, \alpha_{n-1} + \lambda\alpha_n, \alpha_n + \lambda\alpha_1\}$ 是否线性无关，并求出S的秩.

解: 令$k_1(\alpha_1 + \lambda\alpha_2) + k_2(\alpha_2 + \lambda\alpha_3) + \cdots + k_n(\alpha_n + \lambda\alpha_1) = 0$
即$(k_1 + k_n\lambda)\alpha_1 + (k_2 + k_1\lambda)\alpha_2 + \cdots + (k_n + k_{n-1}\lambda)\alpha_n = 0$因为$\alpha_1, \ldots, \alpha_n$ 线性无关,所以有$\begin{cases} k_1 + k_n\lambda = 0 \\ k_2 + k_1\lambda = 0 \\ \cdots\cdots \\ k_n + k_{n-1}\lambda = 0 \end{cases}$ 即$\begin{pmatrix} 1 & 0 & 0 & \cdots & \lambda \\ \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \\ \vdots \\ k_n \end{pmatrix} = 0$

系数行列式为：$\lambda^n+1=0$
若$n=4m+2$时,$\lambda=i,-i$时上述系数行列式为零,从而线性方程组有非零解,此时向量组S线性相关,且秩为$n-1$,λ为其他值时都线性无关;
若$n=2m+1$,$\lambda=-1$时上述系数行列式为零,有非零解,此时向量组S线性相关,且秩为$n-1$,λ为其他值时都线性无关;
若n为其他情形,对λ的任何取值向量组S都线性无关.

11. 设$\alpha_1=(0,1,2,3),\alpha_2=(1,2,3,4),\alpha_3=(3,4,5,6),\alpha_4=(4,3,2,1),\alpha_5=(6,5,4,3)$; 将$\alpha_1,\alpha_2$扩充成$\{\alpha_1,\cdots,\alpha_5\}$ 的一个极大线性无关组.

解: $$(\alpha_1^T,\cdots,\alpha_5^T)=\begin{pmatrix}0&1&3&4&6\\1&2&4&3&5\\2&3&5&2&4\\3&4&6&1&3\end{pmatrix}\to\begin{pmatrix}1&0&-2&-5&-7\\0&1&3&4&6\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}$$
α_1,α_2本身就是一组极大线性无关组.

12. 设$\beta_1=\alpha_2+\alpha_3+\cdots+\alpha_r$, $\beta_2=\alpha_1+\alpha_3+\cdots+\alpha_r$, $\cdots$, $\beta_r=\alpha_1+\alpha_2+\cdots+\alpha_{r-1}$. 证明:$\beta_1,\beta_2,\cdots,\beta_r$ 与$\alpha_1,\alpha_2,\cdots,\alpha_r$ 有相同的秩.

证: 由题设$\beta_1,\beta_2,\cdots,\beta_r$可经$\alpha_1,\alpha_2,\cdots,\alpha_r$线性表出,所以$\beta_1,\beta_2,\cdots,\beta_r$的秩$\leq$ $\alpha_1,\alpha_2,\cdots,\alpha_r$的秩.又易计算得$\alpha_i=\frac{1}{r-1}(\beta_1+\beta_2+\cdots+\beta_r)-\beta_i,i=1,2,\cdots,r$. $\alpha_1,\alpha_2,\cdots,\alpha_r$ 可经$\beta_1,\beta_2,\cdots,\beta_r$线性表出,所以$\alpha_1,\alpha_2,\cdots,\alpha_r$ 的秩$\leq$ $\beta_1,\beta_2,\cdots,\beta_r$ 的秩.故它们的秩相同.