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The assignment is an exercise in elementary mathematics which, however, has interesting connections with computing and information theory. This project will be suitable to COMP6730 students, or for Honours students studying Science and Mathematics. Background knowledge (or additional research) regarding the number representation in different bases will be needed, as well as elementary geometry of periodic lattices. The estimated size (including the code comments and docstrings) is approximately 250–300 lines of code. The time effort is about 15 hours.

The problem

Consider a rational number $Q = \frac{M}{N}$, $M < N$ which when expanded in a basis β , is described by a *periodic* sequence of digits. The digits, β_i are bounded by $0 \leq \beta_i < \beta$:

$$Q = \sum_{i=1}^{\infty} \frac{\beta_i}{\beta^i} = (0.\beta_1\beta_2\beta_3\dots)_{\beta}, \quad 0 \leq \beta_i < \beta, \quad \forall i,$$

(do not be confused with notations $(0.\beta_1\beta_2\dots)_{\beta}$ — it is a generalisation of the decimal expansion like $(0.123456\dots)_{10}$ where the base value 10 is usually omitted).

The sequence β_i is periodic: $\beta_{i+T} = \beta_i$ starting from some number $N : i \geq N$. For some rationals (like $\frac{1}{4}$ or $\frac{17}{150}$, which are expanded in base 10 as follows: 0.250000... and 0.113333333333...), the period $T=1$. These kind of rationals are not very interesting. Rationals with longer expansion period, for example, are

$$\frac{7}{11} = 0.6363636363\dots, \quad \text{period } T = 2,$$

or

$$\frac{1}{990} = 0.0010101010101010\dots, \quad \text{period } T = 2,$$

or

$$\frac{117}{151} = 0.77483443708609271523178\dots, \quad \text{period } T = 75,$$

One may consider a simple problem of finding the period of expansion of a rational number in the base β . You may take it as a warming up example. But more interesting numerical studies emerge when we consider a particular expansion base $\beta = 4$.

When the base of expansion is 4, the digits take 4 possible values 0, 1, 2, 3 **which can be considered as the directions of discrete step walks** – “East”, “North”, “West” and “South”. When we accept the interpretation, each rational number will be represented by a two-dimensional walk on a plane. Each step of the walk has a length of 1, and the direction of the n-th step corresponds to the n-th digit of the decimal in base 4. For example, the number $\frac{7}{11}$ has this expansion in the base 4:

$$\frac{7}{11} = 0.2202322023\dots, \quad \text{period } T = 5.$$

The 100 step walk which this number generates looks like this

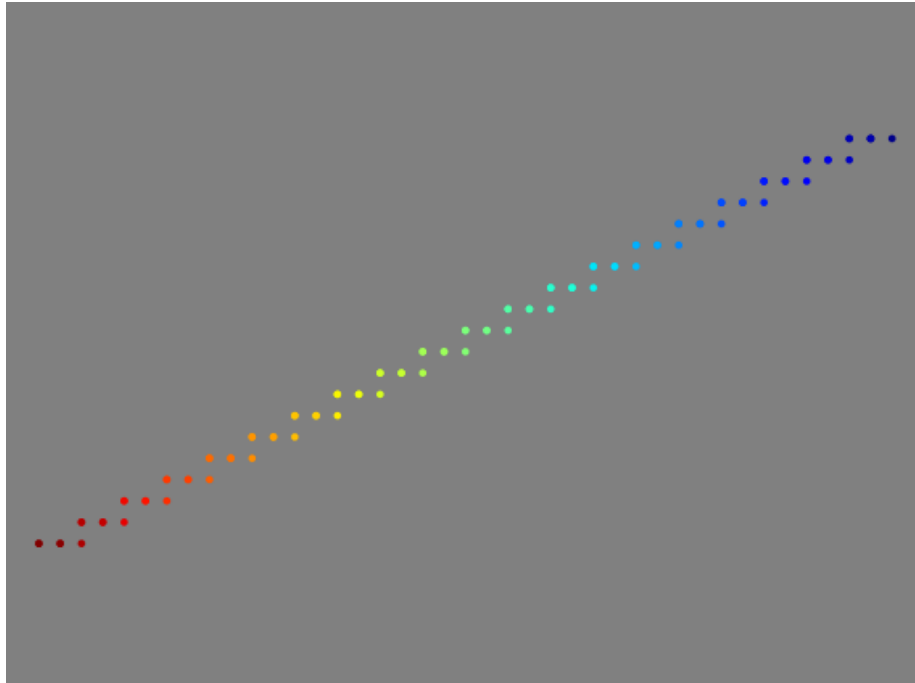


Figure 1: “Seven-11” walk

here the directions are chosen as {0: “East”, 1: “North”, 2: “West”, 3: “South”}, and the colour-coding is used to show the “time” of steps (the further the

time step from the start the redder is the colour — “the red shift”, like in Universe expansion).

Sometimes, walks look like random. For example the walks on Fig. 2 are created by two pairs of relatively small integers, the length is 250,000 in both cases:

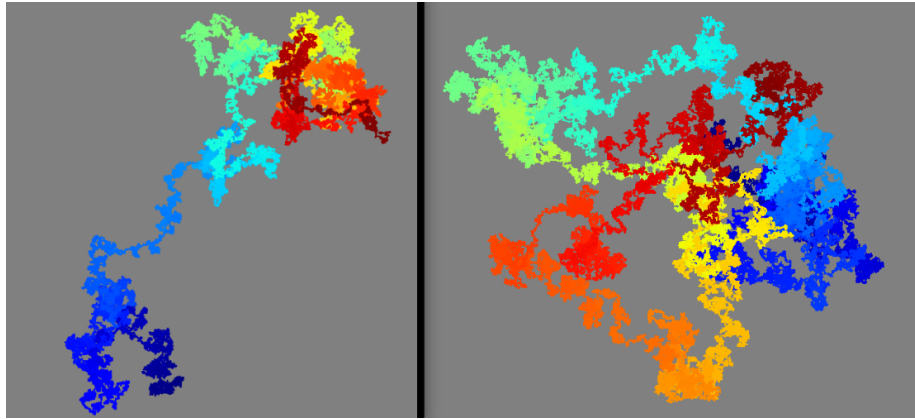


Figure 2: Pseudo-random walk on “small” numbers: $\frac{3624360069}{7000000001}$ and $\frac{23456789012}{1000000000061}$.

And sometimes, the walks show surprising regularity. For example, for two integers

M = 1049012271677499437486619280565448601617567358491560876166848380843144358447252875551
62924702775955557045371567931305878324772977202177081818796590637365767487981422801328592
0278610192581409571357487047122902674651513128059541953997504202061380373822338959713391954

N = 1612226962694290912940490066273549214229880755725468512353395718465191353017348814314
01750453996944547935301206438332726709700793305262920303509209736004509554561365966493250
7839146477284016238565137429529453089612268152748875615658076162410788075184599421938774835

the rational $Q = \frac{M}{N}$ generates a regular walk, which is also *periodic*, Fig. 3. (Do not confuse a periodic walk with a periodic expansion — the walk may not be periodic, but the expansion always is. For the walk to be periodic, it must return to the point of origin after the number of step which is integer factor of the expansion period.)



Figure 3: A 440 step base-4 walk on the number Q .

There are more examples of such remarkably regular number walks, and some mathematical theory behind this phenomenon. But we shall not pursue it further (if you like, one can learn about it in Ref. 1). Instead, we will build programming tools to explore these behaviours.

The choice of directions is somewhat arbitrary. For consistency, we shall assume that the East direction corresponds to the value $\beta_i = 0$, and that other directions are chosen in the anti-clockwise order ($\beta_i = 1$ corresponds to North, and so on).

It is interesting to note, that base-4 walks can be also generated using RNA/DNA sequences (which is a subject of study in [Project-3](#)). We cannot say whether such walks have any significance in the bioinformatics research, though.

Another consideration is about reversing the process of generating a walk from the number. Ask yourself — how were numbers like Q in the above example found? If we encode a trajectory which generates a target shape and use it as an expansion, we will be able to compute the rational number, given by the this expansion. In other words, if we know the shape (and its trajectory), we can calculate the rational number.

The programming tasks

This is what you will have to do in your program:

1. Define a function which accepts a 2-tuple comprising a numerator and a denominator, and a base of the numeral system in which to expand the rational number, and returns the period of the expansion sequence.
2. Define a function which accepts a 2-tuple of integers, a numerator and denominator representing a rational number, the base value and the positive integer $nsteps$, and returns the sequence of the expansion digits of this rational of the total length $nsteps$.
3. Using the plotting library `matplotlib`, create a walk which a rational (defined by a 2-tuple of integers) generates on a *square* lattice.
4. A sequence representing a rational number expansion in the base β can be written as a *prefix* sequence (in the above example $(\frac{17}{150})_{10}$ the prefix was 11) concatenated with a *period sequence* which is repeated infinitely (in the same example, it was a single element sequence 3). A prefix can be an empty sequence, a period sequence can consist of zeros (the example $\frac{1}{4}$ above).

Write a function that is passed a sequence *prefix*, a sequence *period* and a base b , which will return a 2-tuple of *mutually prime* integers to uniquely represent a rational. Again using the example of rational $\frac{17}{150}$, the asked function should return (17, 150) when the prefix (1, 1), the period sequence (3,) and the base 10 are passed for arguments.

5. Compute rationals which generate sample walks – a square, a circle (adopt a discrete approximation of your choice), a figure “8”, *Ying-Yang* or some other of your own choosing (for example, you can generate [Homer Simpson](#) which should give some idea that the topics we discuss in this assignment are related to the Fourier analysis).

References

1. “Walking on Real Numbers”, by Francisco J. Aragon Artacho, David H. Bailey, Jonathan M. Borwein, and Peter B. Borwein, *Mathematical Intelligencer*,

2012, <https://link.springer.com/article/10.1007/s00283-012-9340-x>, DOI 10.1007/s00283-012-9340-x

2. Most basic facts can be found in a Wikipedia article [Numeral system](#) on expressing numbers with symbols.
3. Same subject is discussed in a more general context in another Wikipedia article [Positional notation](#).