Mediterr. J. Math. (2024) 21:101
https://doi.org/10.1007/s00009-024-02647-8

Mediterranean Journal of Mathematics

Isomorphisms Between the Multiplier Algebras of Certain Topological Algebras

Marina Haralampidou, Lourdes Palacios and Carlos Signoret

Abstract. We study the topological algebra identification of the multiplier algebra of a certain algebra E and that of a closed left ideal in E. The case when one of the algebras is a Segal topological algebra in the other is considered. We also study this problem in the context of locally C^*-algebras.

Mathematics Subject Classification. Primary 46H05, 46H10; Secondary 46H15, 46H20, 46K05.

Keywords. Left (right) multiplier, multiplier algebra, Segal topological algebra, preannihilator algebra, left regular representation, locally C^*-algebra.

1. Introduction

In 1956, S. Helgason dealt with representation theory for commutative Banach algebras with special reference to harmonic analysis [11]. He approached this problem by defining the algebra of multipliers. This algebra consists of certain linear maps, called multipliers which are well behaved. Since after, a considerable number of papers referred to multipliers (see e.g., [11,15,17]). Some authors use the term “centralizer” in place of “multiplier” (see for instance [14], among many others). The general theory of multipliers can be found in [18].

In [25], B.J. Tomiuk dealt mainly with the identification up to a topological algebra isomorphism of the left multiplier algebras of two certain Banach algebras. The purpose of this paper is to present non-normed extensions of some results of Tomiuk stated in [ibid.]. We refer to generalizations as much as possible. In [9], a study of the left regular representation of certain topological algebras is given along with the properties of their closed ideals.

All algebras, employed below, are taken over the field $\mathbb{C}$ of complexes. A *topological algebra* E is an associative algebra which is a topological vector space and the ring multiplication is separately continuous (see e.g., [20]). We

Published online: 25 April 2024

refer to a *locally m-convex algebra* E, if the topology of E is defined by a family of submultiplicative seminorms [ibid.].

A *locally C^*-algebra* is a complete involutive locally convex algebra E whose topology is given by a (saturated) family of *C^*-seminorms* $(q_i)_{i\in I}$. This means that these seminorms satisfy the *C^*-property*, namely for each $i \in I$, $q_i(x)^2 = q_i(x^* x)$ for every $x \in E$ (see [7, Chapter II, p. 101] and [13]). It is known that C^*-seminorms are automatically $*$-preserving and submultiplicative [ibid.]

Throughout, we employ the symbol (0) to denote either the set containing the zero element or the ideal containing only the zero element. The symbol $\overline{S}^E$ stands for the topological closure of a subset S in a topological algebra E.

2. Preliminaries

In this paper, we consider mostly objects and notions on the left side, but most of all our work could be carried out on the right side and sometimes even on the bilateral setting. If needed, we will make declaratory comments in the appropriate places.

Let E be an algebra. If $S \subseteq E$, then

$$\mathcal{A}_l(S) = \{x \in E : x\, s = 0 \text{ for all } s \in S\}$$

denotes the *left annihilator* of S (with respect to E). A similar definition can be made for $\mathcal{A}_r(S)$, the *right annihilator* of S. $\mathcal{A}_l(S)$ (resp. $\mathcal{A}_r(S)$) is a left (right) ideal of E, which in particular, is two-sided if S is a left or right ideal. For a topological algebra (separately continuous multiplication), the previous ideals are closed.

An algebra E is called *left* (resp. *right) preannihilator* if $\mathcal{A}_l(E) = (0)$ (resp. $\mathcal{A}_r(E) = (0)$). If $\mathcal{A}_l(E) = (0) = \mathcal{A}_r(E)$, then E is named *preannihilator*. For more on preannihilator algebras, see [8].

Definition 2.1. An *approximate identity* in a topological algebra E is a net $\{e_\delta\}_{\delta\in\Delta}$ in E such that for each $x \in E$,

$$x - x\, e_\delta \underset{\delta}{\longrightarrow} 0 \text{ and } x - e_\delta\, x \underset{\delta}{\longrightarrow} 0.$$

An approximate identity is called *bounded* if it is a bounded subset of E, and it is called *uniformly bounded* if the set $\{(k^{-1}\, e_\delta)^n, \delta \in \Delta, n \in \mathbb{N}\}$ with k a positive constant, is bounded in E.

Any algebra with a unit element or yet a topological algebra with an orthogonal basis, or with an approximate identity, is preannihilator. Thus, a locally C^*-algebra, in particular a C^*-algebra, is preannihilator (see [13, p. 208, Theorem 2.6]). Further, any topologically semisimple topological algebra is semiprime and thus preannihilator (see [8, Lemma 2.3]).

In what follows, $L(E)$ denotes the algebra of all linear operators on an algebra E.

Definition 2.2. An element T in $L(E)$ is called a *left (right) multiplier* on E if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in E$. It is called a *two-sided multiplier* on E if it is both a left and a right multiplier.

We use the following notation: $M_l(E)$ (resp. $M_r(E)$) denotes the set of all left (resp. right) multipliers on E and $M(E)$ that of all two-sided multipliers on E. Clearly $M_l(E), M_r(E)$ and $M(E)$ are subalgebras of $L(E)$. Now, for $x \in E$, the operator L_x on E given by $L_x(y) = xy$ $(y \in E)$ is a left multiplier, due to the associativity of the multiplication on E. Similarly, the right multiplier with respect to $x \in E$, is defined and it is denoted by R_x. For the next, see [10, Proposition 2.2].

Proposition 2.3. *Let E be a preannihilator algebra. Then the following hold:*

(i) The mapping

$$L : E \longrightarrow M_l(E) \text{ given by } x \mapsto L_x \tag{1}$$

defines an algebra monomorphism which identifies E with a subalgebra of $M_l(E)$.

(ii) E is a left ideal of the algebra $M_l(E)$.

(iii) If E has a left identity, then L is onto.

Similar statements hold, if we consider right or two-sided multipliers. We just note that in the right case, the respective mapping R is an *algebra antimonomorphism*. If E is a topological algebra, then the linear operators L_x and R_x, for all $x \in E$, are continuous.

3. Segal Algebras

In harmonic analysis, there are examples of Banach algebras being ideals in another Banach algebra. Under certain conditions, this type of topological algebras is known as *abstract Segal algebras*. Segal algebras were introduced by Reiter in 1968, as subalgebras in the context of group algebras ([22] and [23, p. 173, Definition 6.2.1]). The notion of an abstract Segal algebra is due to J.T. Burnham ([4, p. 551, Definition 1.1]) and it is defined as a Banach algebra $(B, \|\cdot\|_B)$ which is continuously embedded as a dense ideal to another Banach algebra $(A, \|\cdot\|_A)$ satisfying also a norm-condition with respect to both norms. In particular, a C^*-*Segal algebra* is a Segal algebra in a C^*-algebra. For the Banach case, see also [18, p. 297, Definition].

In this section, we generalize the notion of an abstract Segal algebra from the context of Banach algebras to that of locally m-convex ones (see Definition 3.1). For the Banach case, see [19, p. 297, Definition]. The same notion, for Banach algebras, is used by B.J. Tomiuk in [25], some results of which we generalize here.

In [1], Mart Abel introduced Segal topological algebras (not necessarily Banach, nor locally m-convex, complete or not, nor even locally m-convex Fréchet algebras). In this framework, he generalizes some results from [2, 16, 21].

Definition 3.1. A left ideal S of a locally m-convex algebra $(E,(p_\lambda)_{\lambda\in\Lambda})$ is called a *left Segal locally m-convex algebra* in E, if the following hold true.

(i) S is dense in E,

(ii) S is a locally m-convex algebra with respect to some (saturated) family $(q_i)_{i\in I}$ of submultiplicative seminorms, and

(iii) for each $\lambda\in\Lambda$, there exists $i\in I$ and $k>0$ such that

$$p_\lambda(x)\leq k\,q_i(x) \text{ for every } x\in S. \tag{2}$$

Remark 3.2. Since the family $(q_i)_{i\in I}$ is saturated, by condition (iii), we get that the topology defined on S by the previous family is finer than the (subspace) locally m-convex topology defined on S by the family $\{p_\lambda\}_{\lambda\in\Lambda}$. This means that the identical bijection $x\mapsto x$ from $(S,(q_i)_{i\in I})$ onto $(S,(p_\lambda)_{\lambda\in\Lambda})$ is continuous (see e.g., [12, p. 98]).

Examples.

1. The case in which $(S,\|\cdot\|_S)$, $(E,\|\cdot\|_E)$ are Banach algebras and S is a classical Segal algebra in E (see e.g., [19, p. 297, Definition]) is clearly an example for Definition 3.1.
2. Let (A,τ_A) be a locally m-convex algebra and I a dense left (right or two-sided) ideal of A, equipped with the subspace topology. Then I is a left (respectively, right or two-sided) Segal locally m-convex algebra in A ([1, Example 2.3]).

For the topological algebra E, let us denote by $\mathcal{L}(E)$ the algebra of all continuous linear operators on E. Now we need the notion of the left regular representation L_E of E in the locally m-convex case (see also [9]).

Let E be a preannihilator locally m-convex algebra. Denote by $(p_\alpha)_{\alpha\in\Lambda}$ the set of all continuous seminorms defining the topology on E. We endow $\mathcal{L}(E)$ with the locally convex topology induced by the family of seminorms $(\tilde{p_\alpha})_{\alpha\in\Lambda}$, where, for $T\in\mathcal{L}(E)$,

$$\tilde{p_\alpha}(T)=\sup\{p_\alpha(T(x)),\ x\in E \text{ and } p_\alpha(x)\leq 1\}. \tag{3}$$

Under the assumption that each left multiplier on the topological algebra E is continuous, (this is the case, for instance, for a locally C^*-algebra, see [15, p. 75, Proposition 3.4]), then the algebra $M_l(E)$ of left multipliers with the composition of operators as ring multiplication, is a subalgebra of $\mathcal{L}(E)$ and becomes a complete locally m-convex algebra with this family of seminorms [ibid, p. 75, Theorem 3.5, (1)]. Moreover, the map L defined in Proposition 2.3 is a *seminorm-decreasing* homeomorphism.

Consider the set $\{L_x : x\in E\}$ in $M_l(E)$. Put

$$L_E=\overline{\{L_x : x\in E\}}^{M_l(E)}.$$

We call L_E the *left regular representation* of E. It is a closed left ideal in $M_l(E)$. Being a closed subspace of a complete space, L_E is itself complete.

Since E is a preannihilator algebra, the identification $x\mapsto L_x$ of E as a subalgebra of $M_l(E)$ allows us to identify E with a left ideal of L_E. In order to see this, take $T\in L_E$ and $x\in E$. Then, for each $y\in E$,

$$(T\cdot L_x)(y)=T(L_x(y))=T(x\cdot y)=T(x)\cdot y=L_{T(x)}(y).$$

So, $T \cdot L_x = L_{T(x)}$. Since $T \in L_E \subseteq M_l(E)$, then $T(x) \in E$, that is, $L_{T(x)} \in L_E$.

Moreover, for a preannihilator algebra E in which every left multiplier is continuous, we can identify E as a *dense* left ideal in L_E. For, just note that $E = \{L_x : x \in E\}$ in $M_l(E)$ implies $\overline{E}^{L_E} = L_E \cap \overline{E}^{M_l(E)} = L_E \cap \overline{\{L_x : x \in E\}}^{M_l(E)} = L_E$.

Proposition 3.3. *Let $(S, (q_i)_{i\in I})$ be a preannihilator locally m-convex algebra. Suppose that every left multiplier in S is continuous. Then S is a left Segal algebra in $(L_S, (\tilde{q}_i)_{i\in I})$.*

Proof. We have already pointed out that, since every left multiplier is continuous, S is a dense left ideal in L_S.

Now take $i \in I$ and $s \in S$. Then

$$\begin{aligned}\tilde{q}_i(L_s) &= \sup\{q_i(L_s(x) : x \in S,\ q_i(x) \leq 1\}\\ &\leq \sup\{q_i(s \cdot x) : x \in S,\ q_i(x) \leq 1\}\\ &\leq \sup\{q_i(s) \cdot q_i(x) : x \in S,\ q_i(x) \leq 1\}\\ &\leq k_i \cdot q_i(s),\end{aligned}$$

where $k_i = \sup\{q_i(x) : x \in S,\ q_i(x) \leq 1\}$, which exists because q_i is continuous in S. This completes the proof.

Based on the comment after relation (3), we directly get the next.

Corollary 3.4. *Let $(S, (q_i)_{i\in I})$ be a locally C^*-algebra. Then S is a left Segal algebra in $(L_S, (\tilde{q}_i)_{i\in I})$.*

4. Isomorphisms

In this section, the fact that E is preannihilator plays a central role, due to Proposition 2.3. The next lemma and statements concern the left regular representation L_S of a locally m-convex algebra S.

Lemma 4.1. *Let $(S, (q_i)_{i\in I})$ be a locally m-convex algebra which is a dense left ideal of a preannihilator complete locally m-convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$. Suppose that for each $\lambda \in \Lambda$, there exist $i \in I$ and $k > 0$ such that*

$$p_\lambda(s) \leq k \cdot q_i(s) \tag{4}$$

for every $s \in S$. Then, for each $i \in I$, there exist $\lambda \in \Lambda$ and $C > 0$ such that

$$q_i(xs) \leq C \cdot p_\lambda(x) \cdot q_i(s) \tag{5}$$

for every $s \in S$ and $x \in E$.

Proof. Take $s \in S$ and define the map $R_s : E \to S$ as $R_s(x) = xs$. We claim that R_s is continuous. Indeed, for $\lambda \in \Lambda$, take $i \in I$ that satisfies (4). For $\{x_n\} \subseteq E$, $t \in S$ and $\{p_\lambda(x_n)\} \to 0$, $q_i(R_s(x_n) - t) \to 0$, we get $q_i(x_n s - t) \to 0$. Since q_i dominates p_λ, we also get $p_\lambda(x_n s - t) \to 0$. But $p_\lambda(x_n s) \to 0$ too, therefore $t = 0$.

Take $i \in I$. Now, since R_s is continuous, for each $s \in S$, there exists $M_{\lambda,s} > 0$ such that

$$q_i(xs) \le M_{\lambda,s} \cdot p_\lambda(x) \tag{6}$$

for $x \in E$.

Take $x \in E$ and note that the linear map $L_x : S \to S$, $s \mapsto xs$ is also continuous.

Put $|L_x|_i = \sup\{q_i(L_x(s)) : q_i(s) \le 1\}$ and $\mathcal{E}_\lambda = \{L_x : x \in E,\ p_\lambda(x) \le 1\}$. Then

$$q_i(L_x(s)) \le M_{\lambda,s} \tag{7}$$

for each $s \in S$ and $L_x \in \mathcal{E}_\lambda$. Since E is complete, we can apply the Uniform Boundedness Theorem (see for instance [24, p. 43, Theorem 2.5]), and we get that there exists $C > 0$ such that, for $\lambda \in \Lambda$ and $L_x \in \mathcal{E}_\lambda$,

$$|L_x|_i \le C.$$

From this, we get that, if $s \in S$, $q_i(s) \neq 0$ and $x \in E$ with $p_\lambda(x) \le 1$,

$$q_i\left(\frac{s}{q_i(s)} x\right) \le C$$

and therefore

$$q_i(sx) \le C \cdot q_i(s), \tag{8}$$

which also holds if $q_i(s) = 0$, due to the continuity of L_x. Now, take an arbitrary $x \in E$. We can conclude that, if $p_\lambda(x) \neq 0$, then

$$q_i\left(s\, \frac{x}{p_\lambda(x)}\right) \le C \cdot q_i(s)$$

which implies

$$q_i(sx) \le C \cdot q_i(s) \cdot p_\alpha(x),$$

and this inequality also holds if $p_\lambda(x) = 0$, due to relation (6).

Remark 4.2. Note that if $(S, (q_i)_{i\in I})$ is a locally m-convex left Segal algebra in a locally m-convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$, then condition (4) is automatically fulfilled, by the very definition.

Theorem 4.3. *Let $(S, (q_i)_{i\in I})$ be a preannihilator complete locally convex algebra which is a left Segal algebra in a preannihilator complete locally m-convex algebra $(E, (p_\lambda)_{\lambda\in\Lambda})$. Then there exists a topological algebra isomorphism of E onto L_S which maps x into L_x, for all $x \in S$.*

Proof. Since S is preannihilator, the mapping $\Phi : S \to L_S$, $s \mapsto L_s$ is an algebraic monomorphism. Now, due to Lemma 4.1 and Remark 4.2, for each $i \in I$, there exist $\lambda \in \Lambda$ and $C > 0$ such that

$$q_i(xs) \le C \cdot p_\lambda(x) \cdot q_i(s) \tag{9}$$

for every $s \in S$ and $x \in E$. In particular, this also holds if $x \in S$.

This shows that

$$\tilde{q}_i(L_x) \le C \cdot p_\lambda(x) \tag{10}$$

for $x \in S$.

Then, due to relation (10), the map Φ is also continuous when S is endowed with the subspace topology of E. Since S is dense in E, Φ extends to a continuous map $\Psi : E \to L_S$, $x \mapsto L_x$.

Let K be the kernel of Ψ. Then K is a closed ideal in E and $K \cap S = \{0\}$. But $KS \subseteq K \cap S$ and so $KS = \{0\}$; therefore $KE = \{0\}$. Since E is preannihilator, this implies that $K = \{0\}$ and so, Ψ is also a monomorphism.

Next, we claim that $Im(\Psi)$ is closed. So, take $(T_\delta)_{\delta \in \Delta} \subseteq Im(\Psi)$. Then $T_\delta = L_{s_\delta}$ for some $s_\delta \in S$. Take $i \in I$. Applying relation (10), there exist $i \in I$ and $C > 0$ such that

$$\tilde{q}_i(T_\delta - T_\gamma) = \tilde{q}_i(L_{s_\delta} - L_{s_\gamma}) = \tilde{q}_i(L_{s_\delta - s_\gamma}) \leq C \cdot p_\lambda(s_\delta - s_\gamma),$$

so $(T_\delta)_{\delta \in \mathbf{N}}$ is a Cauchy net in L_S, which is complete. Then there exists $U \in L_S$ such that $\Psi(L_{T_\delta}) \longrightarrow U$. By the continuity of Ψ, $U = L_{\lim(s_\delta)}$, so $U \in Im(\Psi)$.

Finally, since $Im(\Psi)$ is closed and dense in L_S, the map Ψ is onto, and therefore bicontinuous.

Proposition 4.4. *Let $(S, (q_i)_{i \in I})$ be a preannihilator locally m-convex algebra. Then every continuous left multiplier U on S has a unique extension to a continuous left multiplier T on L_S and, for each $i \in I$, there exist a constant k_i such that*[1]

$$\tilde{q}_i(T) \leq k_i \cdot \tilde{q}_i(U). \tag{11}$$

Proof. For each $s \in S$ and $i \in I$, as in the proof of Proposition 3.3, there exists a constant k_i such that $\tilde{q}_i(L_s) \leq k_i \cdot q_i(s)$. Therefore,

$$\tilde{q}_i(L_{U(s)}) = \tilde{q}_i(U \circ L_s) \leq \tilde{q}_i(U) \cdot \tilde{q}_i(L_s) \leq \tilde{q}_i(U) \cdot k_i \cdot q_i(s).$$

Then U is continuous considered as a map from S to L_S. Since S is dense in L_S, there is a unique continuous map $T : L_S \to L_S$ that extends U and satisfies (11). Clearly $T \in M_l(L_S)$ and $T|_S = U$.

Remark 4.5. Note that it is also true that if $T \in M_l(L_S)$ is continuous, then $U = T|_S$ belongs to $M_l(S)$ and is continuous. This fact, in addition to Proposition 4.4, implies that, for a preannihilator locally m-convex algebra S, every left multiplier in S is continuous if and only if every left multiplier in L_S is continuous.

An approximate identity $\{e_\delta\}_\delta$ in a topological algebra E is said to be *quasi-bounded* if the set $\{L_{e_\delta}\}_\delta$ is bounded in L_E. It is easy to see that, if $\{e_\delta\}_\delta$ is a quasi-bounded left (right) approximate identity in E, then $\{L_{e_\delta}\}_\delta$ is a bounded left (right) approximate identity in L_E.

Moreover, if $\{e_\delta\}_\delta$ is a bounded approximate identity in a locally m-convex algebra E, and if q is a continuous seminorm in E, then $\tilde{q}(L_{e_\delta}) \leq q(e_\delta)$ and therefore the set $\{L_{e_\delta}\}_\delta$ is also bounded (in L_E), that is, $\{e_\delta\}_\delta$ is a (left) quasi-bounded approximate identity in E. In particular, every locally C^*-algebra has a quasi-bounded approximate identity.

[1] We are using the same notation $\tilde{q}_i$ for the uniform seminorm in $\mathcal{L}(S)$ and in L_S.

Definition 4.6. A topological algebra A is called *left regular factorable*, if

$$A = L_A \cdot A = \{xy : x \in L_A, y \in A\}.$$

Among left regular factorable topological algebras we can find locally bounded algebras (see [26]), locally convex F-algebras (see [5]) and fundamental F-algebras (see [3]), when there is a uniformly bounded approximate identity in each of them. For more information about this topic, the reader is also referred to [6,8].

Theorem 4.7. *Let $(S, (q_i)_{i\in I})$ be a preannihilator left regular factorable complete locally m-convex algebra with a quasi-bounded approximate identity. Suppose that every multiplier in S is continuous. Then $M_l(S) = M_l(L_S)$ within a topological algebra isomorphism.*

Proof. Take $T \in M_l(L_S)$. Since S is left regular factorable, $S = L_S \cdot S$. If $T' = T|_S$, then $T'(S) = T(S) = T(L_S \cdot S) = T(L_S) \cdot S \subseteq L_S \cdot S = S$ and hence T' maps S into S, that is, $T' \in M_l(S)$.

Let $\{e_\delta\}_\delta$ be a quasi-bounded left approximate identity in S. Due to Remark 4.5, T' is continuous and therefore the set $\{T'(e_\delta)\}_\delta$ is bounded in L_S. So, for each $i \in I$, there is a constant $D_i > 0$ such that, $\tilde{q}_i(T'(e_\delta)) \leq D_i$ for every δ.

Due to Proposition 3.3 and Remark 4.2, we can apply Lemma 4.1. Therefore, for each $i \in I$ there is $j \in I$ and $C > 0$ such that $q_i(b\, s) \leq C\, \tilde{q}_i(b)\, q_j(s)$ for every $b \in L_S$ and $s \in S$. So, we have that for each $s \in S$, given $i \in I$, there exist $j \in I$ and $C > 0$ such that

$$\begin{aligned} q_i(T'(s)) &= q_i(T'(\lim_\delta(e_\delta \cdot s))) = \lim_\delta(q_i(T'(e_\delta \cdot s))) = \lim_\delta(q_i(T'(e_\delta) \cdot s)) \\ &\leq \lim_\delta(C\, \tilde{q}_i(T(e_\delta))q_j(s)) \leq C \sup_\delta(\tilde{q}_i(T(e_\delta))q_j(s)) \leq C\, D_i\, q_j(s). \end{aligned}$$

Whence the map $T \mapsto T'$ is continuous.

Now, by Proposition 4.4, every $U \in M_l(S)$ has a unique extension T to L_S, $T \in M_l(L_S)$ and $\tilde{q}_i(T) \leq k_i \cdot \tilde{q}_i(U)$ (for each $i \in I$ and a constant k_i that depends on i). Hence, the mapping $T \to T'$ is an algebraic and topological isomorphism from $M_l(L_S)$ onto $M_l(S)$.

Corollary 4.8. *Let $(S, (q_i)_{i\in I})$ be a left regular factorable locally C^*-algebra. Then $M_l(S) = M_l(L_S)$ within a topological algebra isomorphism.*

Proof. We have already remarked that left multipliers in locally C^*-algebras are continuous. Also, every locally C^*-algebra is preannihilator and has a left quasi-bounded approximate identity. Then apply Theorem 4.7.

Corollary 4.9. *Let $(S, (q_i)_{i\in I})$ be a preannihilator left regular factorable Segal locally m-convex algebra in a preannihilator complete locally m-convex algebra E. Suppose that every multiplier in S is continuous and that E has a bounded left approximate identity contained in S. Then $M_l(S) = M_l(E)$ within a topological algebra isomorphism.*

Proof. First, due to Theorem 4.3, E and L_S are isomorphic as topological algebras. If $\{e_\delta\}_\delta$ is a bounded left approximate identity in E that lies in S, then it is also a left approximate identity in S. Therefore $\{e_\delta\}_\delta$ is a quasi-bounded left approximate identity in S. The conclusion follows from Theorem 4.7.

Author contributions All authors contributed equally to this work.

Data Availability Statement No datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare no competing interests.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Abel, M.: Generalization of Segal algebras for arbitrary topological algebras. Period. Math. Hung. **77**(1), 58–68 (2018)

[2] F. Abtahi, S. Rahnama, A. Rejali, *Segal Fréchet algebras* (2015). arXiv:1507.06577v1

[3] Ansari-Piri, E.: A class of factorable topological algebras. Proc. Edinb. Math. Soc. **33**, 53–59 (1990)

[4] Burnham, J.T.: Closed ideals in subalgebras of Banach algebras I. Proc. Am. Math. Soc. **32**(2), 551–555 (1972)

[5] Dixon, P.G.: Automatic continuity of positive functionals on topological involution algebras. Bull. Austral. Math. Soc. **23**, 265–281 (1981)

[6] Doran, R.S., Wichmann, J.: Approximate Identities and Factorization in Banach Modules. Lecture Notes in Mathematics, vol. 768. Springer, Berlin (1979)

[7] Fragoulopoulou, M.: Topological Algebras with Involution. North Holland Mathematics Studies, vol. 200. Elsevier, Amsterdam (2005)

[8] Haralampidou, M.: Annihilator topological algebras. Port. Math. **51**, 147–162 (1994)

[9] Haralampidou, M.: On the left regular representation of complemented topological algebras. Manuscript

[10] Haralampidou, M., Palacios, L., Signoret, C.: Multipliers in locally convex *-algebras. Rocky Mt. J. Math. **43**(6), 1931–1940 (2013)

[11] Helgason, S.: Multipliers of Banach algebras. Ann. Math. (2) **64**, 240–254 (1956)

[12] Horváth, J.: Topological Vector Spaces and Distributions, vol. I. Addison-Wesley Publ. Co., Reading (1966)

[13] Inoue, A.: Locally C^*-algebras. Mem. Fac. Sci. Kyushu Univ. (Ser. A) **25**, 197–235 (1971)

[14] Johnson, B.E.: An introduction to the theory of centralizers. Proc. Lond. Math. Soc. **14**, 299–320 (1964)

[15] Joiţa, M.: On bounded module maps between Hilbert modules over locally C^*-algebras. Acta Math. Univ. Comeneanae **74**(1), 71–78 (2005)

[16] Kauppi, J., Mathieu, M.: C^*-Segal algebras with order unit. J. Math. Anal. Appl. **392**(2), 785–797 (2013)

[17] Larsen, R.: The Multiplier Problem. Lectures Notes in Mathematics, vol. 105. Springer, Berlin (1969)

[18] Larsen, R.: Introduction to the Theory of Multipliers. Grundlehren Math. Wiss., vol. 175. Springer, Berlin (1971)

[19] Leinert, M.: A contribution to Segal algebras. Manuscr. Math. **10**, 297–306 (1973)

[20] Mallios, A.: Topological Algebras. Selected Topics, North-Holland, Amsterdam (1986)

[21] Mattas, J.: Segal algebras, approximate identities and norm irregularity in $C_0(X, A)$. Stud. Math. **215**(2), 99–112 (2013)

[22] Reiter, H.: Sur certains idéaux dans $L^1(G)$. C. R. Math. Acad. Sci. Paris **267**, 882–885 (1968)

[23] Reiter, H., Stegeman, J.D.: Classical Harmonic analysis and locally compact groups, 2nd edn. Oxford University Press, Clarendon Press, Oxford (2000)

[24] Rudin, W.: Functional Analysis. International Series in Pure and Applied Mathematics, vol. 8, 2nd edn. McGraw-Hill Science/Engineering/Math, New York (1991)

[25] Tomiuk, B.J.: Isomorhisms of multiplier algebras. Glasg. Math. J. **28**, 73–77 (1986)

[26] Żelazko, W.: Banach Algebras. PWN-Polish Scientific Publishers, Warszawa (1973)

Marina Haralampidou
National and Kapodistrian University of Athens
Panepistimioupolis
Athens
Greece
e-mail: mharalam@math.uoa.gr

Lourdes Palacios and Carlos Signoret
Universidad Autónoma Metropolitana Iztapalapa
Av. San Rafael Atlixco 186, Col. Leyes de Reforma, Iztapalapa
Mexico City 09140
Mexico
e-mail: pafa@xanum.uam.mx

Carlos Signoret
e-mail: casi@xanum.uam.mx

Received: January 26, 2024.

Revised: March 29, 2024.
Accepted: April 3, 2024.