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On the D_ω-Classical Orthogonal Polynomials

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Dedicated to Pascal MARONI on occasion of his 90th birthday.

Abstract. We investigate the D_ω-classical orthogonal polynomials, where D_ω is the weighted difference operator. So, we address the problem of finding the sequence of orthogonal polynomials such that their D_ω-derivatives is also orthogonal polynomials. To solve this problem we adopt a different approach to those employed in this topic. We first begin by determining the coefficients involved in their recurrence relations, and then providing an exhaustive list of all solutions. When $\omega = 0$, we rediscover the classical orthogonal polynomials of Hermite, Laguerre, Bessel and Jacobi. For $\omega = 1$, we encounter the families of discrete classical orthogonal polynomials as particular cases.

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1. Introduction and Preliminary Results

The orthogonal polynomials are characterized by the fact that they satisfy a second-order recurrence relation. They said to be classical if their derivatives also form a sequence of orthogonal polynomials [11]. Hahn generalized the classical orthogonal polynomials by generalizing their characteristic properties (see [2,6] for more details). For this, he considered the linear operator [12]

$$(H_{q,\omega}f)(x) := \frac{f(qx+\omega) - f(x)}{(q-1)x+\omega}, \tag{1.1}$$

for all polynomial f, with q and ω are two fixed complex numbers.

Hahn showed that there is no loss of generality in assuming ω to be zero so that in what follows q may be thought of as 1 or different to 1. For $q \neq 1$

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and $\omega = 0$, we obtain the q-difference operator (also known as the Jackson's q-operator) which we write $(\mathscr{D}_q f)(x) := (H_{q,0} f)(x)$. When $q = 1$ with $\omega \neq 0$ we get the discrete operator $(D_\omega f)(x) := (H_{1,\omega} f)(x)$, that is,

$$(D_\omega f)(x) = \frac{f(x+\omega) - f(x)}{\omega}. \tag{1.2}$$

Unless otherwise stated we assume that $\omega \in \mathbb{C}$.

The limiting case $\omega \to 0$ (resp. $q \to 1$) of D_ω (resp. $\mathscr{D}_q$) gives rise to the derivative operator $D = d/dx$, giving $(Df)(x) := f'(x)$. Because it is always possible to take such a limit, this point is not really important at this time. It will be dealt with a little bit later when necessary. Obviously, if we take $\omega = 1$ in (1.2), we recover the finite (or forward) difference operator $\Delta f(x) = f(x+1) - f(x)$.

Motivated by the several properties common to all of the classical orthogonal polynomials, Hahn [12] posed and solved five (equivalent) problems that are related to the operator $\mathscr{D}_q$ and obtained that all possible solutions lead to the same orthogonal polynomial sequences (OPS) which are the so-called classical q-orthogonal polynomials. Later on, the study of such polynomials has known an increasing interest (see for instance [14] and the references therein).

The first problem studied by Hahn is the following:

Find all OPS $\{P_n\}_{n\geqslant 0}$ such that $\{\mathscr{D}_q P_n\}_{n\geqslant 0}$ is also an OPS.

For more details about solutions of these problems we refer the reader to [2,6,11,13].

In [8] Douak and Maroni considered the problem of finding all OPS such that their D-derivatives are also OPS. Instead of basing the study of this problem on the various properties of orthogonal polynomials, the authors have rather founded their exposures in a purely algebraic point of view, focusing primarily on the explicit calculation of recurrence coefficients. The identified polynomials are none other than the classical orthogonal polynomials of Hermite, Laguerre, Jacobi and Bessel with the usual restrictions on the parameters.

Referring back to the operator D_ω, we will pose the analogous problem:

(**P**) *Find all OPS* $\{P_n\}_{n\geqslant 0}$ *such that* $\{D_\omega P_n\}_{n\geqslant 0}$ *are also OPS.*

When D_ω is replaced by the operator Δ, the analogous problem was solved by Lancaster [15] by considering a second order difference equation of the Sturm–Liouville type (see [2] for more details). He obtained that the eigenfunctions of the related difference operator are the discrete orthogonal polynomials. Later Lesky in [16,17] stated the same conclusion.

Note that in this field a general method of studying the classical orthogonal polynomials of a discrete variable as solutions of a second-order difference equation of hypergeometric type was considered by Nikiforov et al. [23]. On the other hand, by introducing a new matrix approach using the basic algebraic properties of the matrices, Verde-Star [27] solved similar problems by looking for sequences of orthogonal polynomials possessing Hahn's property.

This work is mainly intended to constructing the D_ω-classical orthogonal polynomials by proceeding as in [8]. Such approach is rather new and, of course, different to those previously used in several studies dedicated to this topic (see for instance [1,10,11,23] and the references therein). After determining the recurrence coefficients explicitly, we proceed to the identification of the resulting polynomials. Under some restrictions on the parameters, we establish that these polynomials can be reduced to one of the well-known families of discrete classical orthogonal polynomials.

Similar method was also used within the d-orthogonality context ($d \geqslant 1$) to provide many extensions of the classical orthogonal polynomials (see, e.g., [7–9,18] and the references therein). The same approach used in the Dunkl context, namely, when the ordinary derivative D (or D_ω) is replaced by the Dunkl operator, also gives interesting results, as will be published elsewhere.

In an earlier survey, Abdelkarim and Maroni [1] investigated the problem (**P**) according to a functional approach. The authors had established various equivalent properties characterizing the resulting polynomials. Particularly, they showed that those polynomials satisfy the so-called functional Rodrigues's formula (1.14). Based on this last characterization, up to linear transformation of the variable, they found that there are four classes of D_ω-classical orthogonal sequences satisfying Rodrigues's formula, including the Charlier, Meixner, Krawchuk and Hahn polynomials as special cases of them.

Let $\mathscr{P}$ be the vector space of polynomials of one variable with complex coefficients and let $\mathscr{P}'$ be its algebraic dual. We denote by $\langle ., .\rangle$ the duality brackets between $\mathscr{P}'$ and $\mathscr{P}$. By $\{P_n\}_{n\geqslant 0}$ we denote a polynomials sequence (PS), $\deg P_n = n$, and $\{u_n\}_{n\geqslant 0}$ its associated dual sequence (basis) defined by $\langle u_n, P_m\rangle = \delta_{nm}$; $n, m \geqslant 0$, where δ_{nm} is the Kronecker's delta symbol.

The first element u_0 of the dual sequence is said to be the *canonical* form associated to the PS $\{P_n\}_{n\geqslant 0}$. Throughout this article, we will always consider the sequence of *monic* polynomials, i.e., the leading coefficient of each polynomial P_n is one ($P_n(x) = x^n + \cdots$).

Given a form $u \in \mathscr{P}'$. The sequence of complex numbers $(u)_n,\ n = 0, 1, 2, \ldots,$ defined by $(u)_n := \langle u, x^n\rangle$ denotes the moments of u with respect to the sequence $\{x^n\}_{n\geqslant 0}$. The form u is called *regular* (or *quasi-definite*) if we can associate with it a PS $\{P_n\}_{n\geqslant 0}$ such that

$$\langle u, P_n P_m\rangle = k_n \delta_{n,m},\ n, m \geqslant 0\ ;\ k_n \neq 0,\ n \geqslant 0.$$

In this case $\{P_n\}_{n\geqslant 0}$ is an orthogonal polynomials sequence (OPS) with respect to (w.r.t.) u. As an immediate consequence of the regularity of u, we have $(u)_0 \neq 0$ and $u = \lambda u_0$ with $\lambda \neq 0$. Furthermore, the elements of the dual sequence $\{u_n\}_{n\geqslant 0}$ are such that

$$u_n = \left(\langle u_0, P_n^2\rangle\right)^{-1} P_n u_0, \quad n = 0, 1, 2, \ldots. \tag{1.3}$$

So, in all what follows, we consider the orthogonality of any PS w.r.t. its canonical form u_0.

First, let us introduce the two operators h_a and τ_b defined for all $f \in \mathscr{P}$ by

$$(h_a f)(x) = f(ax) \ \text{ and } \ (\tau_b f)(x) = f(x-b), \quad a \in \mathbb{C}^* := \mathbb{C}\backslash\{0\},\ b \in \mathbb{C}.$$

(1.4)

On the other hand, for any functional u, we can write by transposition

$$\langle \tau_{-b} u, f(x) \rangle := \langle u, \tau_b f(x) \rangle = \langle u, f(x-b) \rangle, f \in \mathscr{P}, \tag{1.5}$$

$$\langle h_a u, f(x) \rangle := \langle u, h_a f(x) \rangle = \langle u, f(ax) \rangle, f \in \mathscr{P}. \tag{1.6}$$

For further formulas and other properties fulfilled by the operator D_ω see [1].

We now consider the sequence of monic polynomials $\{Q_n(x)\} n \geqslant 0$ defined by $\{Q_n(x) := (n+1)^{-1} D_\omega P_{n+1}(x)\}_{n \geqslant 0}$, with its associated dual sequence denoted by $\{v_n\}_{n \geqslant 0}$ and fulfilling

$$D_{-\omega}(v_n) = -(n+1) u_{n+1}, \ n \geqslant 0, \tag{1.7}$$

where by definition

$$\langle D_{-\omega} u\,,\, f \rangle = -\langle u\,,\, D_\omega f \rangle, \ u \in \mathscr{P}', \ f \in \mathscr{P}. \tag{1.8}$$

Next, in the light of the so-called Hahn property [11], we give the following definition.

Definition 1.1. The OPS $\{P_n\}_{n \geqslant 0}$ is called "D_ω-classical" if the sequence of its derivatives $\{Q_n\}_{n \geqslant 0}$ is also a OPS.

Thus, $\{P_n\}_{n \geqslant 0}$ is orthogonal w.r.t. u_0 and satisfies the second-order recurrence relation

$$P_{n+2}(x) = (x - \beta_{n+1}) P_{n+1}(x) - \gamma_{n+1} P_n(x), \ n \geqslant 0, \tag{1.9a}$$

$$P_1(x) = x - \beta_0, \ P_0(x) = 1, \tag{1.9b}$$

and $\{Q_n\}_{n \geqslant 0}$ is orthogonal w.r.t. v_0 and satisfies the second-order recurrence relation

$$Q_{n+2}(x) = (x - \tilde{\beta}_{n+1}) Q_{n+1}(x) - \tilde{\gamma}_{n+1} Q_n(x), \ n \geqslant 0, \tag{1.10a}$$

$$Q_1(x) = x - \tilde{\beta}_0, \ Q_0(x) = 1, \tag{1.10b}$$

with the regularity conditions $\gamma_n \neq 0$ and $\tilde{\gamma}_n \neq 0$ for every $n \geqslant 1$.

Finally, we will summarize in the following proposition the important properties characterizing the D_ω-classical orthogonal polynomials as stated in [1, Propositions 2.1–2.3].

Proposition 1.2. *For any* OPS $\{P_n\}_{n \geqslant 0}$, *the following are equivalent statements:*

(a) *The sequence* $\{P_n\}_{n \geqslant 0}$ *is* D_ω*-classical.*
(b) *The sequence* $\{Q_n\}_{n \geqslant 0}$ *is orthogonal.*
(c) *There exist two polynomials* Φ *monic* (*with* $\deg \Phi = t \leqslant 2$) *and* Ψ (*with* $\deg \Psi = 1$), *and a sequence* $\{\lambda_n\}_{n \geqslant 0}$, $\lambda_n \neq 0$ *for all* n, *such that*

$$\Phi(x) \left(D_\omega \circ D_{-\omega} P_{n+1} \right)(x) - \Psi(x) \left(D_{-\omega} P_{n+1} \right)(x) + \lambda_n P_{n+1}(x) = 0, \ n \geqslant 0. \tag{1.11}$$

(d) *The sequences* $\{Q_n\}_{n\geqslant 0}$ *and* $\{P_n\}_{n\geqslant 0}$ *are interlinked via the differentiation formula*

$$\Phi(x)Q_n(x) = \alpha^2_{n+2}P_{n+2}(x) + \alpha^1_{n+1}P_{n+1}(x) + \alpha^0_n P_n(x),\ n \geqslant 0, \quad (\alpha^0_n \neq 0). \tag{1.12}$$

This identity is referred to as the first structure relation of the OPS $\{P_n\}_{n\geqslant 0}$.

(e) *The form* u_0 *is* D_ω*-classical, say, it is regular and satisfies the functional equation*

$$D_{-\omega}(\Phi u_0) + \Psi u_0 = 0. \tag{1.13}$$

(f) *There exist a monic polynomial* Φ, $\deg\Phi \leqslant 2$, *and a sequence* $\{\lambda_n\}_{n\geqslant 0}$, $\lambda_n \neq 0$ *for all* n, *such that the canonical form* u_0 *satisfies the so-called functional Rodrigues formula*

$$P_n u_0 = \lambda_n D^n_{-\omega}\Big\{\Big(\prod_{\nu=0}^{n-1}\tau_{-\nu\omega}\Phi\Big)u_0\Big\},\ n \geqslant 0, \quad \textit{with} \prod_{\nu=0}^{-1} := 1. \tag{1.14}$$

It is worth noting that the structure relation (1.12) was also explained in [21, Proposition 4.5] in a more general setting, namely that of the D_ω-semi-classical case. Here we will add a new characterization to those established in the preceding proposition. This will be proved once we give the lemma below. To begin with, apply the operator D_ω to (1.9a)–(1.9b), taking into account (1.10a)–(1.10b), we obtain

$$P_{n+2}(x) = Q_{n+2}(x) + \tilde{\alpha}^1_{n+1}Q_{n+1}(x) + \tilde{\alpha}^0_n Q_n(x),\ n \geqslant 0, \tag{1.15a}$$

$$P_1(x) = Q_1(x) + \tilde{\alpha}^1_0,\ P_0(x) = Q_0(x) = 1. \tag{1.15b}$$

We refer to the above relation as the *second structure relation* of the polynomials P_n, $n \geqslant 0$. This will be instrumental in Sect. 2 to derive the system connecting the coefficients α^2_{n+2}, α^1_{n+1} and α^0_n, for $n \geqslant 0$, and the pair of recurrence coefficients (β_n, γ_n).

Combining (1.15a)–(1.15b) with (1.9a)–(1.9b) and then use (1.10a)–(1.10b), we infer that

$$\tilde{\alpha}^1_n = (n+1)(\beta_{n+1} - \tilde{\beta}_n - \omega) \text{ and } \tilde{\alpha}^0_n = (n+1)\gamma_{n+2} - (n+2)\tilde{\gamma}_{n+1}, n \geqslant 0. \tag{1.16}$$

Reminder that, for the classical orthogonal polynomials ($\omega = 0$), the first structure relation was given by Al Salam and Chihara [3], and the second one was established by Maroni [20].

When D_ω is replaced by the finite difference operator Δ, Garcia et al. [10] proved that the structure relations (1.12) and (1.15a)–(1.15b), as well as the functional Rodrigues formula (1.14) characterize the discrete classical polynomials of Charlier, Meixner, Krawchuk and Hahn.

We will now return to the operator D_ω and show that (1.15a)–(1.15b) also characterize the D_ω-classical orthogonal polynomials. To do so, we need the following lemma.

Lemma 1.3. ([19]) *Let $\{P_n\}_{n\geqslant 0}$ be a sequence of monic polynomials and let $\{u_n\}_{n\geqslant 0}$ be its associated dual sequence. For any linear functional u and integer $m \geqslant 1$, the following statements are equivalent:*

(i) $\langle u\,,\,P_{m-1}\rangle \neq 0\ ;\ \langle u\,,\,P_n\rangle = 0,\ n \geqslant m;$

(ii) $\exists\ \lambda_\nu \in \mathbb{C},\ 0 \leqslant \nu \leqslant m-1,\ \lambda_{m-1} \neq 0,$ *such that* $u = \sum_{\nu=0}^{m-1} \lambda_\nu u_\nu$.

Proposition 1.4. *Let $\{P_n\}_{n\geqslant 0}$ be an* OPS *satisfying* (1.9a)–(1.9b)*. The sequence $\{P_n\}_{n\geqslant 0}$ is D_ω-classical if and only if it fulfils* (1.15a)–(1.15b).

Proof. The proof is similar in spirit to that of [10, Proposition 2.10]. The necessary condition has already been shown. Conversely, suppose that the OPS $\{P_n\}_{n\geqslant 0}$ fulfils (1.15a) with (1.15b).

The action of the functional v_0 on both sides of the aforementioned identities gives rise to

$$\langle v_0\,,\,P_0\rangle = 1,\ \langle v_0\,,\,P_1\rangle = \tilde{\alpha}_0^1\ \ \langle v_0\,,\,P_2\rangle = \tilde{\alpha}_0^0 \ \text{ and } \ \langle v_0\,,\,P_n\rangle = 0, \ \text{ for } \ n \geqslant 3.$$

Application of Lemma 1.3 shows that

$$v_0 = \lambda_0 u_0 + \lambda_1 u_1 + \lambda_2 u_2, \tag{1.17}$$

with $\lambda_0 = 1$, $\lambda_1 = \tilde{\alpha}_0^1 = \beta_1 - \tilde{\beta}_0 - \omega$ and $\lambda_2 = \tilde{\alpha}_0^0 = \gamma_2 - 2\tilde{\gamma}_1$.

Now, the use of (1.3) enables us to write $u_1 = \left(\langle u_0, P_1^2\rangle\right)^{-1} P_1 u_0$ and $u_2 = \left(\langle u_0, P_2^2\rangle\right)^{-1} P_2 u_0$.

In (1.17) we replace u_1 and u_2 by their respective expressions given above, to deduce that there exists a polynomial Φ, with $\deg\Phi \leqslant 2$, such that $v_0 = \Phi u_0$.

On the other hand, setting $n = 0$ in (1.7), it follows immediately that

$$D_{-\omega}(v_0) = -u_1 = -\left(\langle u_0, P_1^2\rangle\right)^{-1} P_1 u_0 := -\Psi u_0.$$

Combining these last results, we deduce that

$$D_{-\omega}(\Phi u_0) + \Psi u_0 = 0, \quad \text{with} \quad \deg\Phi \leqslant 2 \quad \text{and} \quad \deg\Psi = 1.$$

By Proposition 1.2, we easily conclude that the orthogonal polynomials sequence $\{P_n\}_{n\geqslant 0}$ is D_ω-classical, and the proof is complete. $\square$

At the end of this section, let us remember the definition of the *shifted* polynomials denoted $\{\hat{P}_n\}_{n\geqslant 0}$ corresponding to the PS $\{P_n\}_{n\geqslant 0}$. For all $n, n = 0, 1, \ldots$, we have

$$\hat{P}_n(x) := \hat{a}^{-n} P_n(\hat{a}x + \hat{b}),\ \text{for } (\hat{a}; \hat{b}) \in \mathbb{C}^* \times \mathbb{C}. \tag{1.18}$$

Since the classical character of the considered polynomials is preserved by any linear change of the variable, for the OPS $\{P_n\}_{n\geqslant 0}$ satisfying (1.9a)–(1.9b), we obtain that the polynomials $\hat{P}_n, n = 0, 1, \ldots$, satisfy also the second-order recurrence relation

$$\hat{P}_{n+2}(x) = (x - \hat{\beta}_{n+1})\hat{P}_{n+1}(x) - \hat{\gamma}_{n+1}\hat{P}_n(x),\ n \geqslant 0, \tag{1.19a}$$

$$\hat{P}_1(x) = x - \hat{\beta}_0,\ \hat{P}_0(x) = 1, \tag{1.19b}$$

with

$$\hat{\beta}_n = \frac{\beta_n - \hat{b}}{\hat{a}},\ n \geqslant 0, \ \text{ and } \ \hat{\gamma}_{n+1} = \frac{\gamma_{n+1}}{\hat{a}^2},\ n \geqslant 0 \quad (\hat{a} \neq 0). \tag{1.20}$$

Furthermore, from (1.5)–(1.6) we readily see that (1.18) becomes $\hat{P}_n(x) = \hat{a}^{-n}\big(h_{\hat{a}} \circ \tau_{-\hat{b}} P_n\big)(x)$.

In addition, if u_0 is D_ω-classical, then Eq. (1.13) leads to

$$D_{-\frac{\omega}{\hat{a}}}\big(\hat{\Phi}\hat{u}_0\big) + \hat{\Psi}\hat{u}_0 = 0, \tag{1.21}$$

where $\hat{\Phi}(x) = \hat{a}^{-t}\Phi(\hat{a}x + \hat{b})$, $\hat{\Psi} = \hat{a}^{1-t}\Psi(\hat{a}x + \hat{b})$ and $\hat{u}_0 = (h_{\hat{a}^{-1}} \circ \tau_{-\hat{b}})u_0$.

The paper is organized as follows. In the next section we pose and solve two nonlinear systems. The first and most fundamental one relates the recurrence coefficients β_n, γ_{n+1} with $\tilde{\beta}_n$, $\tilde{\gamma}_{n+1}$. The second system combines the coefficients α_n^i, $i = 0, 1, 2$; $\tilde{\alpha}_n^j$, $j = 0, 1$, $\tilde{\beta}_n$ and $\tilde{\gamma}_{n+1}$ with those of the polynomial Φ (Proposition 2.2). This allows to express the coefficients α_n^i, in terms of β_n or γ_{n+1}. In Sect. 3, we investigate the four canonical families of D_ω-classical orthogonal polynomials which we identify after assigning particular values to the free parameters. The last section is devoted to the sequences of higher order derivatives. We give, principally, the explicit expressions of their recurrence coefficients in terms of the coefficients $\big(\beta_n, \gamma_{n+1}\big)_{n\in\mathbb{N}}$.

When $\omega = 0$, we rediscover the link between every higher order derivative sequences for the classical polynomials of Hermite, Laguerre, Bessel and Jacobi with each of these families.

2. Computation of the Related Coefficients

In order to compute the various related recurrence coefficients, the first step is to establish the main system connecting the recurrence coefficients $\{\beta_n\}_{n\geqslant 0}$, $\{\gamma_{n+1}\}_{n\geqslant 0}$ with $\{\tilde{\beta}_n\}_{n\geqslant 0}$, $\{\tilde{\gamma}_{n+1}\}_{n\geqslant 0}$. To do this, we proceed as follows. Substituting in (1.9a), P_{n+2}, P_{n+1} and P_n by their expressions provided in (1.15a) we derive a relation in terms of the polynomials Q_k for $k = n+2, \ldots, n-2$. Now, in this new relation we replace xQ_{n+1}, xQ_n and xQ_{n-1} by their respective expressions derived from the recurrence relation (1.10a) obtaining an expansion depending only on the polynomials $Q_{n+2}, \ldots, Q_{n-2}$.

By rearranging the terms in the resulting expansion and after a few further simplifications, the next system (valid for all $n \geqslant 1$) follows by identification

$$\begin{aligned}
&(n+2)\tilde{\beta}_n - n\tilde{\beta}_{n-1} = (n+1)\beta_{n+1} - (n-1)\beta_n - \omega;\\
&2\tilde{\beta}_0 = \beta_1 + \beta_0 - \omega,\\
&(n+3)\tilde{\gamma}_{n+1} - (n+1)\tilde{\gamma}_n\\
&\qquad = (n+1)\gamma_{n+2} - (n-1)\gamma_{n+1} + (n+1)\big(\beta_{n+1} - \tilde{\beta}_n\big)\big(\beta_{n+1} - \tilde{\beta}_n - \omega\big);\\
&3\tilde{\gamma}_1 = \gamma_2 + \gamma_1 + \big(\beta_1 - \tilde{\beta}_0\big)\big(\beta_1 - \tilde{\beta}_0 - \omega\big),\\
&(n+1)\tilde{\gamma}_n\big(2\beta_{n+1} - \tilde{\beta}_n - \tilde{\beta}_{n-1} - \omega\big) - n\gamma_{n+1}\big(\beta_{n+1} + \beta_n - 2\tilde{\beta}_{n-1} - \omega\big) = 0,\\
&(n+2)\tilde{\gamma}_n\tilde{\gamma}_{n+1} - 2(n+1)\tilde{\gamma}_n\gamma_{n+2} + n\gamma_{n+1}\gamma_{n+2} = 0.
\end{aligned}$$

We should mention here the important role played by the Hahn property (Definition 1.1) to establish such a system, since we just assume that these polynomials as well as their D_ω-derivatives are orthogonal w.r.t. regular forms. In other words, we only rely on the fact that such sequences satisfy a second-order recurrence relation, as it is introduced in Sect. 1.

When $\omega = 0$, we recover the system initiated and solved by Douak and Maroni [8] whose the solutions provide the classical OPS of Hermite, Laguerre, Jacobi and Bessel, after assigning particular values to the free parameters. We will encounter these families again in this paper.

To solve the above system, let us introduce the auxiliary coefficients δ_n and θ_n by writing

$$\tilde{\beta}_n = \beta_{n+1} + \delta_n\,,\ n \geqslant 0, \tag{2.1}$$

$$\tilde{\gamma}_n = \frac{n}{n+1}\gamma_{n+1}\theta_n\,,\ n \geqslant 1, \quad \big(\theta_n \neq 0\big). \tag{2.2}$$

With these considerations, the two equalities (1.16) take the form

$$\tilde{\alpha}_n^1 = -(n+1)(\delta_n + \omega),\ \ n \geqslant 0\,;\ \ \tilde{\alpha}_n^0 = (n+1)\gamma_{n+2}(1-\theta_{n+1}),\ \ n \geqslant 0. \tag{2.3}$$

2.1. The Coefficients of the Recurrence Relations

Our main objective here is to initially compute the auxiliary coefficients δ_n and θ_n, and then give the explicit expressions of the coefficients β_n and γ_n. This in turn allows to determine the coefficients $\tilde{\beta}_n$ and $\tilde{\gamma}_n$ and write significantly better each of the coefficients α_n^i and $\tilde{\alpha}_n^j$.

Under the formulas (2.1)–(2.2), it is easy to see that the above system can be transformed into

$$\beta_{n+1} - \beta_n = n\delta_{n-1} - (n+2)\delta_n - \omega,\ n \geqslant 0, \quad (\delta_{-1} = 0), \tag{2.4}$$

$$\big[(n+3)(\theta_{n+1}-1)+1\big]\frac{\gamma_{n+2}}{n+2} - \big[n(\theta_n - 1)+1\big]\frac{\gamma_{n+1}}{n+1} = \delta_n(\delta_n+\omega),\ n \geqslant 1, \tag{2.5}$$

$$\big(3\theta_1 - 2\big)\gamma_2 - 2\gamma_1 = 2\delta_0(\delta_0 + \omega), \tag{2.6}$$

$$\big[(n+3)(\theta_n-1)+1\big]\delta_n - \big[(n-1)(\theta_n-1)+1\big]\delta_{n-1} + 2(\theta_n-1)\omega = 0,\ n \geqslant 1, \tag{2.7}$$

$$(\theta_{n+1}-2)\,\theta_n + 1 = 0,\ n \geqslant 1. \tag{2.8}$$

To solve this system, we begin with the Riccati equation (2.8) whose solutions are

$$\textbf{A}.\quad \theta_n = 1, \quad n \geqslant 1,$$

$$\textbf{B}.\quad \theta_n = \frac{n+\theta+1}{n+\theta}, \quad n \geqslant 1, \quad \theta \neq -1, -2, \ldots.$$

Hence, the first three equations must be examined in the light of these solutions.

Case A. For $\theta_n = 1$, the above system reduces to

$$\beta_{n+1} - \beta_n = n\delta_{n-1} - (n+2)\delta_n - \omega,\ n \geqslant 0, \tag{2.9}$$

$$\frac{\gamma_{n+2}}{n+2} - \frac{\gamma_{n+1}}{n+1} = \delta_n(\delta_n + \omega),\ n \geqslant 0, \tag{2.10}$$

$$\delta_{n+1} - \delta_n = 0,\ n \geqslant 0. \tag{2.11}$$

Equation (2.11) clearly shows that $\delta_n = \delta_0,\ n \geqslant 0$, giving $\delta_n(\delta_n + \omega) = \delta_0(\delta_0 + \omega),\ n \geqslant 0$.

Thus it is quite natural to single out the two statements $\delta_0(\delta_0+\omega) = 0$ and $\delta_0(\delta_0+\omega) \neq 0$. But right now we go back to the two first equations from which we readily deduce that

$$\beta_n = \beta_0 - (2\delta_0 + \omega)n,\ n \geqslant 0, \tag{2.12}$$

$$\gamma_{n+1} = (n+1)\big(\delta_0(\delta_0+\omega)n + \gamma_1\big),\ n \geqslant 0. \tag{2.13}$$

If we take $\omega = 0$, we recover the recurrence coefficients of the Hermite or Laguerre polynomials as shown in [8]. This will be made more precise in the subcases $\mathbf{A}_1$ and $\mathbf{A}_2$ below.

Case B. For $\theta_n = (n+\theta+1)/(n+\theta)$, the system (2.4)–(2.8) becomes

$$\beta_{n+1} - \beta_n = n\delta_{n-1} - (n+2)\delta_n - \omega,\ n \geqslant 0, \tag{2.14}$$

$$\frac{(2n+\theta+4)}{(n+\theta+1)}\frac{\gamma_{n+2}}{n+2} - \frac{(2n+\theta)}{(n+\theta)}\frac{\gamma_{n+1}}{n+1} = \delta_n(\delta_n+\omega),\ n \geqslant 0, \tag{2.15}$$

$$\big(2n+\theta+3\big)\delta_n - \big(2n+\theta-1\big)\delta_{n-1} = -2\omega,\ n \geqslant 1, \tag{2.16}$$

unless, of course, θ happen to be zero for the index $n = 0$. So, we will first discuss the solution of the above system when $\theta \neq 0$. The case $\theta = 0$ is special, it will be considered separately.

When $\omega = 0$, with appropriate choices of the parameters, the only orthogonal polynomials obtained as solutions of this problem are those of Bessel and Jacobi (see [8] for more details).

We now return to seeking solutions for the equations (2.14)–(2.16). Observe first that the RHS of Eq. (2.15) vanishes if and only if $\delta_n = -\omega$ or $\delta_n = 0$. Each of these is possible.

It is easy to check that the former statement is dismissed, since it contradicts Equality (2.16). For the latter, if we replace $\delta_n = 0$ in (2.16), we immediately see that this leads to $\omega = 0$.

Straightforwardly from Eqs. (2.14) and (2.15), one has

$$\beta_n = \beta_0\ ;\ \gamma_{n+1} = \gamma_1\frac{(\theta+2)(n+1)(n+\theta)}{(2n+\theta+2)(2n+\theta)},\ n \geqslant 0. \tag{2.17}$$

If we set $\theta = 2\lambda$ and make a linear transformation with the choice $\hat{a}^2 = 2(\lambda+1)\gamma_1\ ;\ \ \hat{b} = \beta_0$, then

$$\hat{\beta}_n = 0\ ;\ \hat{\gamma}_{n+1} = \frac{(n+1)(n+2\lambda)}{(n+\lambda+1)(n+\lambda)},\ n \geqslant 0. \tag{2.18}$$

We thus meet the Gegenbauer polynomials which will reappear again in Subcase $\mathbf{B}_{21}$.

From now on we assume that $\delta_n(\delta_n+\omega) \neq 0,\ n \geqslant 0$. Starting from Eq. (2.16), multiply both sides by $2n+\theta+1$, after summation we get

$$\delta_n = \frac{\delta_0(\theta+3)(\theta+1) - 2\omega n(n+\theta+2)}{(2n+\theta+3)(2n+\theta+1)},\ n \geqslant 0. \tag{2.19}$$

Use a division to obtain

$$\delta_n = \frac{2\mu}{\big(2n+\theta+3\big)\big(2n+\theta+1\big)} - \frac{1}{2}\omega,\ n \geqslant 0, \tag{2.20a}$$

$$\delta_n = \mu\big(\vartheta_n - \vartheta_{n+1}\big) - \frac{1}{2}\omega,\ n \geqslant 0, \tag{2.20b}$$

where we have written $\mu := \frac{1}{4}(2\delta_0+\omega)(\theta+3)(\theta+1)$ and $\vartheta_n = (2n+\theta+1)^{-1}$, $n \geqslant 0$.

Thanks to the identity (2.20b), we can write

$$(2n+\theta+2)\,\delta_n(\delta_n+\omega) = \mu^2\big(\vartheta_n^2 - \vartheta_{n+1}^2\big) - \frac{1}{4}\omega^2(2n+\theta+2),\ \ n \geqslant 0. \tag{2.21}$$

The objective of course is to incorporate this new expression into (2.15) to derive the coefficients $\gamma_n,\ n \geqslant 1$, which will in fact be processed in a next step. We first calculate the coefficients β_n. For this, observe that the RHS of (2.14) may be rewritten using Eq. (2.16) in the form

$$n\delta_{n-1} - (n+2)\delta_n - \omega = \frac{1}{2}(\theta-1)(\delta_n - \delta_{n-1}),\ \ n \geqslant 1. \tag{2.22}$$

It is easily seen that, if θ assumes the value 1, Eq. (2.22) provides $\delta_n = -\frac{1}{2}\omega,\ n \geqslant 0$.

As a straightforward consequence of this last result, one sees immediately that (2.14), (2.15) respectively provides

$$\beta_n = \beta_0,\ \ n \geqslant 0, \tag{2.23}$$

$$\gamma_{n+1} = -\frac{1}{4}\frac{(n+1)^2\,\big(\omega^2 n(n+2) - 12\gamma_1\big)}{(2n+3)(2n+1)},\ \ n \geqslant 0. \tag{2.24}$$

When $\omega = 0$, under the transformation $\hat{a}^2 = 3\gamma_1, \hat{b} = \beta_0$, we meet the Legendre polynomials.

We now turn to the case $\theta \neq 1$. Using the identity (2.22), Eq. (2.14) gives rise to

$$\beta_{n+1} - \beta_n = \frac{1}{2}(\theta-1)(\delta_n - \delta_{n-1}),\ \ n \geqslant 1, \tag{2.25}$$

$$\beta_1 - \beta_0 = -(2\delta_0 + \omega), \tag{2.26}$$

with δ_n is given by (2.19). From this, it may be concluded that

$$\beta_n = \beta_0 - \frac{(2\delta_0+\omega)(\theta+3)n(n+\theta)}{(2n+\theta+1)(2n+\theta-1)},\ \ n \geqslant 0. \tag{2.27}$$

We can now proceed to compute the coefficients $\gamma_{n+1}, n \geqslant 0$. To do so, multiply both sides of Eq. (2.15) by $2n+\theta+2$ and set

$$\Theta_{n+1} = \frac{(2n+\theta+2)\,(2n+\theta)}{(n+\theta)}\,\frac{\gamma_{n+1}}{n+1},\ \ n \geqslant 0. \tag{2.28}$$

Then, taking into consideration (2.21), we easily check that (2.15) takes the form

$$\Theta_{n+2} - \Theta_{n+1} = \mu^2\big(\vartheta_n^2 - \vartheta_{n+1}^2\big) - \frac{1}{4}\omega^2(2n+\theta+2),\ n \geqslant 0.$$

By summation, we deduce that

$$\Theta_{n+1} = \Theta_1 + \mu^2\big(\vartheta_0^2 - \vartheta_n^2\big) - \frac{1}{4}\omega^2 n(n+\theta+1),\ n \geqslant 0. \tag{2.29}$$

Substituting (2.28) into (2.29) yields

$$\gamma_{n+1} = -\frac{(n+1)(n+\theta)\Big\{\big[\frac{1}{4}\omega^2 n\,(n+\theta+1) - \big(\mu^2\vartheta_0^2 + (\theta+2)\gamma_1\big)\big](2n+\theta+1)^2 + \mu^2\Big\}}{(2n+\theta+2)(2n+\theta+1)^2(2n+\theta)}. \tag{2.30}$$

It is possible to write the expression between braces in the numerator of (2.30) in the form

$$\big(\omega n(n+\theta+1) + \varrho n + \rho_1\big)\big(\omega n(n+\theta+1) - \varrho n + \rho_2\big),$$

where the three parameters ρ_1, ρ_2 and ϱ are such that

$$(\theta+1)\varrho^2 + (\rho_2 - \rho_1)\varrho = 0, \tag{2.31}$$

$$\varrho^2 - (\rho_2+\rho_1)\omega = \big((\theta+3)\delta_0 + (\theta+2)\omega\big)\big((\theta+3)\delta_0 + \omega\big) + 4(\theta+2)\gamma_1, \tag{2.32}$$

$$\rho_2\rho_1 = -(\theta+1)^2(\theta+2)\gamma_1. \tag{2.33}$$

The roots of the quadratic equation (2.31) are $\varrho = 0$ and $\varrho = (\rho_1 - \rho_2)/(\theta+1)$ for $\rho_2 \neq \rho_1$, with the root $\varrho = 0$ being double, if $\rho_2 = \rho_1$. The last equation clearly shows that $\rho_2\rho_1 \neq 0$.

All these parameters will be well specified when dealing with the canonical families. But, in any way, we have to consider the following two cases.

1. For $\varrho = 0$, we obtain

$$\gamma_{n+1} = -\frac{(n+1)(n+\theta)\big(\omega n(n+\theta+1) + \rho_1\big)\big(\omega n(n+\theta+1) + \rho_2\big)}{(2n+\theta+2)(2n+\theta+1)^2(2n+\theta)},\ n \geqslant 0. \tag{2.34}$$

In the particular case $\rho_2 = \rho_1 := \rho$, (2.34) simplifies to

$$\gamma_{n+1} = -\frac{(n+1)(n+\theta)\big(\omega n(n+\theta+1) + \rho\big)^2}{(2n+\theta+2)(2n+\theta+1)^2(2n+\theta)},\ n \geqslant 0. \tag{2.35}$$

2. For the general case $\varrho \neq 0$, we have

$$\gamma_{n+1} = -\frac{(n+1)(n+\theta)\big(\omega n(n+\theta+1) + \varrho n + \rho_1\big)\big(\omega n(n+\theta+1) - \varrho n + \rho_2\big)}{(2n+\theta+2)(2n+\theta+1)^2(2n+\theta)},\ n \geqslant 0. \tag{2.36}$$

We now turn to the special case $\theta = 0$. Substituting this in (2.4)–(2.7) yields

$$\begin{aligned}
&\beta_{n+1} - \beta_n = n\delta_{n-1} - (n+2)\delta_n - \omega,\ n \geqslant 0,\\
&\gamma_{n+2} - \gamma_{n+1} = \frac{1}{2}(n+1)\delta_n(\delta_n + \omega),\ n \geqslant 1,\\
&\gamma_2 - \frac{1}{2}\gamma_1 = \frac{1}{2}\delta_0(\delta_0 + \omega),\\
&(2n+3)\delta_n - (2n-1)\delta_{n-1} = -2\omega,\ n \geqslant 1.
\end{aligned}$$

The same reasoning applies to this case gives

$$\delta_n = \frac{3\delta_0 - 2\omega n(n+2)}{(2n+3)(2n+1)},\ n \geqslant 0,$$
$$\beta_n = \beta_0 - \frac{3(2\delta_0+\omega)n^2}{(2n+1)(2n-1)},\ n \geqslant 0,$$
$$\gamma_{n+1} = -\frac{\big(\omega n(n+1) + \tau n + \tau_1\big)\big(\omega n(n+1) - \tau n + \tau_2\big)}{4(2n+1)^2}, n \geqslant 1,$$

where τ_1, τ_2 and τ are such that

$$\tau^2 + (\tau_2 - \tau_1)\tau = 0,$$
$$\tau^2 - (\tau_2 + \tau_1)\omega = (3\delta_0 + 2w)(3\delta_0 + \omega) + 8\gamma_1,$$
$$\tau_2\tau_1 = -2\gamma_1.$$

When $\omega = 0$, if moreover $\delta_0 = 0$, which we may assume, it follows that

$$\delta_n = 0,\ n \geqslant 0,\ \beta_n = \beta_0,\ n \geqslant 0,\ \gamma_{n+1} = \frac{1}{2}\gamma_1, n \geqslant 1.$$

Thus, choosing $\beta_0 = 0$ and $\gamma_1 = \frac{1}{2}$, we meet the Tchebychev polynomials of the first kind.

After having finished solving the first system, we now proceed to the determination of the coefficients $\tilde{\alpha}_n^j$ and α_n^j in terms of β_n and γ_{n+1}, $n \geqslant 0$. This will be done in the next subsection.

2.2. The Coefficients of the Structure Relations

Proposition 2.1. *Let Φ be the polynomial arising in Proposition 1.2. We let the degree of Φ to be two 2 and write $\Phi(x) = a_2x^2 + a_1x + a_0$. Then the coefficients implicated in the two structure relations (1.12) and (1.15a)–(1.15b) are interlinked through the following system*

$$\alpha_{n+2}^2 = a_2,\ n \geqslant 0, \tag{2.37}$$
$$\alpha_{n+1}^1 + a_2\tilde{\alpha}_{n+1}^1 = a_2(\tilde{\beta}_{n+1} + \tilde{\beta}_n) + a_1,\ n \geqslant 0, \tag{2.38}$$
$$\alpha_{n+1}^1\tilde{\alpha}_n^1 + a_2\tilde{\alpha}_n^0 + \alpha_n^0 = a_2(\tilde{\gamma}_{n+1} + \tilde{\gamma}_n + \tilde{\beta}_n^2) + a_1\tilde{\beta}_n + a_0,\ n \geqslant 0, \tag{2.39}$$
$$\alpha_{n+1}^1\tilde{\alpha}_{n-1}^0 + \alpha_n^0\tilde{\alpha}_{n-1}^1 = a_2\tilde{\gamma}_n(\tilde{\beta}_n + \tilde{\beta}_{n-1}) + a_1\tilde{\gamma}_n,\ n \geqslant 1, \tag{2.40}$$
$$\alpha_{n+1}^0\tilde{\alpha}_{n-1}^0 = a_2\tilde{\gamma}_{n+1}\tilde{\gamma}_n,\ n \geqslant 1, \tag{2.41}$$

where we have adopted the convention that $\tilde{\gamma}_0 := 0$ so that (2.39) remains valid for $n = 0$.

Proof. As for the preceding system, we give only the main ideas of the proof. First, comparison of coefficients in (1.12) shows that $\alpha_{n+2}^2 = a_2$. Now, use the recurrence relation (1.10a) twice to write the product $\Phi(x)Q_n$, which is the LHS of (1.12), in terms of the polynomials $Q_{n+2}, \ldots, Q_{n-2}$. Then replace in the RHS of (1.12) P_{n+2}, P_{n+1} and P_n by their expressions provided in (1.15a) to obtain another expansion in terms of the polynomials $Q_{n+2}, \ldots, Q_{n-2}$. After some simplifications, and by identification the equations (2.38)–(2.41) follow. ∎

As far as the author knows, the technique used here to find explicit expressions for the coefficients involving in (1.15a) with (1.15b) is new, and the results obtained still unknown. So, the solution of the above system brings an answer to this question. But before doing so, recall that when the form u_0 is D_ω-classical, namely, it is regular and satisfies (1.13), the polynomials Φ and Ψ necessarily satisfy (see [1, p. 7] for further details):

$$\kappa\Phi(x) = (1-\theta_1)x^2 - \big((1-\theta_1)(\beta_1+\beta_0)+\delta_0+\omega\big)x + \big((1-\theta_1)\beta_1 + \delta_0+\omega\big)\beta_0 + \theta_1\gamma_1, \tag{2.42}$$

$$\kappa\Psi(x) = P_1(x) = x-\beta_0. \tag{2.43}$$

Note that the expression in the RHS of (2.42) is slightly transformed in accordance with the relations (2.1) and (2.2). The coefficient κ is to be chosen later so that $\Phi(x)$ is being monic.

Since we are proceeding following the two situations Case **A** and Case **B**, we see that the leading coefficient a_2 of $\Phi(x)$ assumes the value 0, when $\theta_1 = 1$, or is such that $\kappa a_2 = -(\theta+1)^{-1}$, when $\theta_1 = (\theta+2)/(\theta+1)$. To get $a_2 = 1$ in the latter case, we choose $\kappa = -(\theta+1)^{-1}$ which, in turn, allows to write the polynomial $\Phi(x)$ in one of the four standard forms

$$\Phi(x) = 1,\ \Phi(x) = x,\ \Phi(x) = x^2 \text{ and } \Phi(x) = (x+1)(x-c),\ c\in\mathbb{C}\backslash\{-1\}.$$

The classification achieved according to the degree of Φ as in [1] is of course exhaustive and is equivalent to that based on the values of θ_n. It is this second alternative that we will retain in the next section to go over the diverse families of D_ω-classical orthogonal polynomials or some relevant cases. But before doing so, let us discuss the solutions of the system (2.37)–(2.41) when a_2 takes one of the values 0 or 1, with use of (2.3).

I. For $a_2 = 0$, since $\alpha_n^0 \neq 0$ for each n, Eq. (2.41) readily gives $\tilde{\alpha}_n^0 = 0, n \geqslant 0$, so, due to (2.3), we necessarily have $\theta_n = 1$ for all $n \geqslant 1$. It turns out that $\delta_n = \delta_0, n\geqslant 0$, and so $\tilde{\alpha}_n^1 = -(n+1)(\delta_0+\omega), n \geqslant 0$. In this case, it is easily seen that the coefficients β_n and γ_{n+1} are given by (2.12)–(2.13).

This actually happens in the case **A** when the polynomial Φ takes the form

$$\kappa\Phi(x) = -(\delta_0+\omega)x + (\delta_0+\omega)\beta_0 + \gamma_1. \tag{2.44}$$

Using the fact that the polynomial $\Phi(x)$ is monic, we are brought to consider the following two situations according with the degree of this polynomial.

- **(i)** $\Phi(x)$ is constant. In this case we have $\delta_0+\omega = 0$ which leads to $a_2 = a_1 = 0$ and $a_0 = 1$. It follows that $\kappa = \gamma_1$, $\Phi(x) = 1$ and $\Psi(x) = \gamma_1^{-1}P_1(x)$. Therefore $\tilde{\alpha}_n^1 = 0,\ n \geqslant 0$, and the above system readily gives $\alpha_{n+2}^2 = \alpha_{n+1}^1 = 0\ ;\ \alpha_n^0 = 1, n\geqslant 0$. Thus, $Q_n = P_n$, $n \geqslant 0$, and so the two structure relations coincide.
- **(ii)** $\Phi(x)$ is linear. By setting $\kappa = -(\delta_0+\omega) \neq 0$ and $(\delta_0+\omega)\beta_0+\gamma_1 = 0$, we conclude that $a_2 = a_0 = 0$ and $a_1 = 1$ and so $\Phi(x) = x$ and $\Psi(x) = \gamma_1^{-1}\beta_0 P_1(x)$. We thus have

$$\alpha_{n+2}^2 = 0;\ \alpha_{n+1}^1 = 1 \text{ and } \alpha_n^0 = \beta_0 - \delta_0 n,\ n \geqslant 0.$$

Accordingly, the two structure relations may be written as follows

$$xQ_n(x) = P_{n+1}(x) + (\beta_0 - \delta_0 n)P_n(x),\ n \geqslant 0,$$
$$P_n(x) = Q_n(x) - (\delta_0 + \omega)nQ_{n-1}(x),\ n \geqslant 0,\ \ (Q_{-1} := 0).$$

For $n = 1$ in the first relation, we recover the equality $(\delta_0+\omega)\beta_0+\gamma_1 = 0$, that is, $\kappa\beta_0 = \gamma_1$. This interconnection between β_0 and γ_1 will prove useful in Subcase $\mathbf{A}_2$.

II. For $a_2 \neq 0$, $\Phi(x)$ is then quadratic and so $\kappa = 1 - \theta_1 = -(\theta+1)^{-1}$, since we take a_2 to be 1.

Changing n into $n+1$ in (2.41) yields $\alpha^0_{n+2}\tilde{\alpha}^0_n \neq 0$, $n \geqslant 0$, from which we see that the coefficients $\tilde{\alpha}^0_n$ are not identically 0 for all n, and hence, due to (2.3), we conclude that $\theta_n \neq 1,\ n \geqslant 1$. This in fact shows that the case **B** is the one to be naturally considered here. We thus have

$$\tilde{\alpha}^1_n = -(n+1)(\delta_n + \omega) \quad \text{and} \quad \tilde{\alpha}^0_n = -\frac{n+1}{n+\theta+1}\gamma_{n+2},\ n \geqslant 0.$$

Additionally, the coefficients β_n are given by (2.31), while the γ_n are generated either by (2.34) or by (2.36). Moreover, taking into account the position of the zeros of the polynomial Φ, we have to consider again two subcases .

(i) $\Phi(x) = x^2$. Since $a_2 = 1$ and $a_1 = a_0 = 0$, (2.42) leads to $\beta_1 + \beta_0 = (\theta+1)(\delta_0+\omega)$ and $\beta_0^2 = -(\theta+2)\gamma_1$. Analogously to the former case, taking into consideration (2.3), the system (2.37)–(2.41) readily provides

$$\alpha^2_{n+2} = 1\ ;\ \alpha^1_{n+1} = \beta_{n+1} + \beta_n + n(\delta_{n-1}+\omega) \quad \text{and}$$
$$\alpha^0_n = -\frac{n+\theta+1}{n+1}\gamma_{n+1},\ n \geqslant 0.$$

(ii) $\Phi(x) = (x+1)(x-c),\ c \neq -1$. We have $a_2 = 1$, $a_1 = 1-c$ and $a_0 = -c$. Therefore, (2.42) gives rise to $\beta_1 + \beta_0 = (\theta+1)(\delta_0+\omega) + c - 1$ and $(\beta_0+1)(\beta_0-c) = -(\theta+2)\gamma_1$. In the same manner we can see that

$$\alpha^2_{n+2} = 1\ ;\ \alpha^1_{n+1} = \beta_{n+1} + \beta_n + n(\delta_{n-1}+\omega) - c + 1 \ \text{ and}$$
$$\alpha^0_n = -\frac{n+\theta+1}{n+1}\gamma_{n+1},\ n \geqslant 0.$$

We could now proceed to present an exhaustive classification of the resulting polynomials. The sequences identified are, of course, consistent with those found in [1] and are pointed out with some of their special cases. They will be regarded as the *canonical* (or *representatives*) families. To provide these results, depending on certain parameters, we review all situations occurring in either of the cases **A** and **B** as we will see in the next section.

3. The Canonical Families of D_ω-Classical Polynomials

In the sequel we are only interested in regular OPS, the finite sequences are not considered here. Under some restrictions on the parameters, we rediscover the well-known families of discrete classical orthogonal polynomials or some particular cases. This is achieved on account of the specific conditions observed in Sect. 2.2. For each situation, we summarise the relevant properties of the corresponding family of polynomials. We often use the linear transformation (1.18) with (1.20) to provide the desired results.

▷ **Case A.** We first investigate the two main subcases, namely, $\delta_0+\omega=0$ and $\delta_0+\omega\neq 0$. Then, we consider the particular subcase when δ_0 assumes the value 0.

$\mathbf{A}_1$: $\delta_0+\omega=0$. From (2.12)–(2.13), taking into account (2.1)–(2.2), we immediately obtain $\tilde\beta_n=\beta_n=\beta_0+\omega n$; $\tilde\gamma_{n+1}=\gamma_{n+1}=\gamma_1(n+1),\ n\geqslant 0$. It follows that $Q_n=P_n$, $n\geqslant 0$, and hence the PS $\{P_n\}_{n\geqslant 0}$ belongs to the class of the so-called D_ω-Appell sequences.

$\mathbf{A_{1a}}$. With the choice $\hat a^2=2\gamma_1$ and $\hat b=\beta_0$, we easily get $\hat\beta_n=\frac{\omega}{\hat a}n$; $\hat\gamma_{n+1}=\frac{1}{2}(n+1),\ n\geqslant 0$. Now, replacing ω by $\hat a\omega$, we obtain $\hat\beta_n=\omega n$; $\hat\gamma_{n+1}=\frac{1}{2}(n+1),\ n\geqslant 0$.

When $\omega=0$, and so $\delta_0=0$, we meet the Hermite polynomials.

$\mathbf{A_{1b}}$. The choice $\hat a=\omega$ and $\hat b=\beta_0-\omega a$ with $\gamma_1=a\omega^2$ gives $\hat\beta_n=a+n$; $\hat\gamma_{n+1}=a(n+1),\ n\geqslant 0$. We thus encounter the Charlier polynomials [5].

$\mathbf{A_2}$: $\delta_0+\omega\neq 0$. We can assume without loss of generality that $\delta_0(\delta_0+\omega)=1\Leftrightarrow\omega=\delta_0^{-1}-\delta_0$. Use of (2.12)–(2.13) then shows that $\beta_n=\beta_0-(\delta_0^{-1}+\delta_0)n$; $\gamma_{n+1}=(n+1)\big(n+\gamma_1\big),\ n\geqslant 0$, where the parameters β_0 and γ_1 are related via $\beta_0=-\delta_0^{-1}\gamma_1$ as in the statement **I-(ii)** above.

By setting $\gamma_1:=\alpha+1$, we can write $\beta_0=-\delta_0^{-1}(\alpha+1)$.

If we take now $\omega=0$ which yields $\delta_0^2=1$, we recover the Laguerre polynomials for $\delta_0=-1$, and the shifted Laguerre polynomials for $\delta_0=1$.

For $\delta_0=-1$, we have $\beta_n=2n+\alpha+1$; $\gamma_{n+1}=(n+1)\big(n+\alpha+1\big),\ n\geqslant 0$.

From now on, we assume that $\delta_0\neq -1$ and set $\gamma_1:=\alpha+1$ again. We wish to examine two interesting situations already investigated in [1] with slight differences in notation.

The first one occurs for $\delta_0:=-e^{-\varphi},\varphi\neq 0$, so that $\omega=-2\sinh\varphi$. It follows that

$$\beta_n=e^{\varphi}(\alpha+1)+2n\cosh\varphi\ ;\ \gamma_{n+1}=(n+1)(n+\alpha+1),\ n\geqslant 0.$$

Afterwards, choose $\hat a=\omega$; $\hat b=0$ and put $c:=e^{2\varphi}$, to get

$$\hat\beta_n=\frac{c}{1-c}(\alpha+1)+\frac{1+c}{1-c}n\ ;\ \hat\gamma_{n+1}=\frac{c}{(1-c)^2}(n+1)(n+\alpha+1),\ n\geqslant 0.$$

When the parameter $c\in\mathbb{R}\backslash\{0,1\}$, we obtain the Meixner polynomials of the first kind [22]. The Krawtchouk polynomials are a special case of the Meixner polynomials of the first kind.

The second situation appears for $\delta_0:=e^{i\phi},\ 0<\phi<\pi$ by taking $2\lambda=\gamma_1:=\alpha+1$. After making a linear transformation via the changes $\hat a=i\omega$; $\hat b=-\lambda\omega$, we obtain that

$$\hat\beta_n=-(n+\lambda)\cot\phi\ ;\ \hat\gamma_{n+1}=\frac{1}{4}\frac{(n+1)(n+2\lambda)}{\sin^2\phi},\ n\geqslant 0.$$

From this, we conclude that the resulting polynomials are those of Meixner–Pollaczek [25].

$\mathbf{A_3}$: For $\delta_0 = 0$, one sees that (2.12)–(2.13) yields $\beta_n = \beta_0 - \omega n$; $\gamma_{n+1} = \gamma_1(n+1),\ n \geqslant 0$. From this, taking into consideration (2.1)–(2.2), we check at once that $\tilde{\beta}_n = \beta_n - \omega$ and $\tilde{\gamma}_{n+1} = \gamma_{n+1},\ n \geqslant 0$.

In accordance with (1.18), it is easily seen that $Q_n(x) = \tau_{-\omega} P_n(x) = P_n(x+\omega)$ for all $n \geqslant 0$.

It is worth pointing out the following two subcases for which we can proceed analogously to the generation of $\mathbf{A}_{1a}$ and $\mathbf{A}_{1b}$. This case could therefore be regarded as a subcase of $\mathbf{A}_1$.

$\mathbf{A_{3a}}$. Similarly to $\mathbf{A}_{1a}$, the choice $\hat{a}^2 = 2\gamma_1$; $\hat{b} = \beta_0$ yields $\hat{\beta}_n = -\frac{\omega}{\hat{a}} n$; $\hat{\gamma}_{n+1} = \frac{1}{2}(n+1),\ n \geqslant 0$. Replacing ω by $-\hat{a}\omega$, we get $\hat{\beta}_n = \omega n$; $\hat{\gamma}_{n+1} = \frac{1}{2}(n+1),\ n \geqslant 0$, which for $\omega = 0$ gives rise to the Hermite polynomials.

$\mathbf{A_{3b}}$. If we take $\hat{a} = -\omega$; $\hat{b} = \beta_0 + \omega a$ with $\gamma_1 = a\omega^2$, we get the Charlier polynomials again. Note that, with $\hat{a} = i\omega$; $\hat{b} = \beta_0 + i\omega b$ and $\gamma_1 = a\omega^2$, with $a \neq 0$ and b an arbitrary constant, a specific case relative to this situation have been mentioned in [1], where $\hat{\beta}_n = b + in$; $\hat{\gamma}_{n+1} = -a(n+1),\ n \geqslant 0$.

Note that the resulting polynomials encountered here are already identified in Subcases $\mathbf{A}_{1a}$ and $\mathbf{A}_{1b}$. All the polynomials obtained in this first case are summarized below in Table 1.

▷ **Case B.** Two main situations will be also investigated with some of their special subcases. The remarks referred to in the statements **II-(i)** and **II-(ii)** above must of course be taken into account to choose more precise certain parameters involved in the recurrence coefficients.

In what follows, unless otherwise stated, we assume that $\theta \neq 1$.

$\mathbf{B}_1$: $\theta = 2\alpha - 1$. From (2.27) and (2.30), if we choose $\beta_0 = \frac{1}{2}\alpha\omega - \mu_1$, we get

$$\beta_n = \frac{1}{2}\alpha\omega - \frac{\alpha(\alpha-1)\mu_1}{(n+\alpha)(n+\alpha-1)},\ n \geqslant 0, \tag{3.1}$$

$$\gamma_{n+1} = -\frac{(n+1)(n+2\alpha-1)\big(\omega(n+\alpha)^2 - 2\alpha\mu_1\big)^2}{(2n+2\alpha+1)(2n+2\alpha)^2(2n+2\alpha-1)},\ n \geqslant 0, \tag{3.2}$$

where we have set $\mu_1 := -\frac{1}{2}(\alpha+1)(2\delta_0 + \omega)$.

Note that (3.1) is valid for $n = 0$, except that it becomes worthless if concurrently $\alpha = 1$.

For $\alpha = \frac{1}{2}$, and so $\theta = 0$, the coefficients β_n and γ_{n+1} coincide with those previously established in Sect. 2.1.

$\mathbf{B}_{11}$. For $\mu_1 \neq 0$, the choice $\hat{a} = \alpha\mu_1 = 1$; $\hat{b} = 0$, leads to

$$\hat{\beta}_n = \frac{1}{2}\alpha\omega + \frac{1-\alpha}{(n+\alpha)(n+\alpha-1)},\ n \geqslant 0, \tag{3.3}$$

$$\hat{\gamma}_{n+1} = -\frac{(n+1)(n+2\alpha-1)\big(\omega(n+\alpha)^2 - 2\big)^2}{4(2n+2\alpha+1)(n+\alpha)^2(2n+2\alpha-1)},\ n \geqslant 0. \tag{3.4}$$

When $\omega = 0$, we clearly encounter the Bessel polynomials.

$\mathbf{B}_{12}$. For $\omega \neq 0$ with the choice $\hat{a} = i\omega$; $\hat{b} = \frac{1}{2}\alpha\omega$, another specific case was discovered in [1]. If on top of that we take $\mu_1 = 0$ and $\alpha > 0$, the

Table 1. Description of the OPS in Case A

Case A	$\hat{\beta}_n$	$\hat{\gamma}_{n+1}$	$\hat{P}_n$
$\mathbf{A}_1 : \delta_0 + \omega = 0$	$\beta_0 + \omega n$	$\gamma_1(n+1)$	D_ω-Appell
$\omega \to \hat{a}\omega,\ \omega \neq 0$			
$\hat{a}^2 = 2\gamma_1,\ \hat{b} = \beta_0$	ωn	$\frac{1}{2}(n+1)$	
$\hat{a} = \omega,\ \hat{b} = \beta_0 - \omega a,\ a \neq 0$	$a + n$	$a(n+1)$	Charlier
$\omega = 0$	0	$\frac{1}{2}(n+1)$	Hermite
$\mathbf{A}_2 : \delta_0(\delta_0 + \omega) = 1$	$\beta_0 - (\delta_0^{-1} + \delta_0)n$	$(n+1)\big(n+\gamma_1\big)$	
$\omega \neq 0, \delta_0 := -e^{-\varphi},\ \varphi \neq 0$	$e^{\varphi}(\alpha+1) + 2n\cosh\varphi$	$(n+1)\big(n+\alpha+1\big)$	
$c := e^{2\varphi} \in \mathbb{R} - \{0,1\}$			
$\hat{a} = \omega,\ \hat{b} = 0$	$\frac{c(\alpha+1)}{1-c} + \frac{1+c}{1-c}n$	$\frac{c(n+1)(n+\alpha+1)}{(1-c)^2}$	Meixner
$c := e^{i\phi},\ 0 < \phi < \pi$			
$\hat{a} = i\omega,\ \hat{b} = -\lambda\omega,\ 2\lambda = \alpha + 1$	$-(n+\lambda)\cot\phi$	$\frac{1}{4}\frac{(n+1)(n+2\lambda)}{\sin^2\phi}$	Meixner–Pollaczek
$\omega = 0, \delta_0 = -1$	$2n + \alpha + 1$	$(n+1)\big(n+\alpha+1\big)$	Laguerre
$\mathbf{A}_3 : \omega \neq 0, \delta_0 = 0$	$\beta_0 - \omega n$	$\gamma_1(n+1)$	
$\omega \to -\hat{a}\omega$			
$\hat{a}^2 = 2\gamma_1,\ \hat{b} = \beta_0$	ωn	$\frac{1}{2}(n+1)$	
$\hat{a} = -\omega,\ \hat{b} = \beta_0 + \omega a,\ \gamma_1 = a\omega^2$	$a + n$	$a(n+1)$	Charlier
$\hat{a} = i\omega,\ \hat{b} = \beta_0 + i\omega b,\ \gamma_1 = a\omega^2$	$b + in$	$-a(n+1)$	

corresponding orthogonal polynomials are symmetric and their associated form is positive definite.

$\mathbf{B}_2 : \theta = \nu - 1$. Let $c \in \mathbb{C} - \{-1\}$, thus from and (2.36), we obtain

$$\beta_n = \frac{1}{4}\nu\omega - \frac{1}{2}(1-c) + \frac{\nu(\nu-2)\mu_2}{(2n+\nu)(2n+\nu-2)}, \ n \geqslant 0, \tag{3.5}$$

$$\gamma_{n+1} = -\frac{(n+1)(n+\nu-1)}{(2n+2\nu+1)(2n+2\nu)^2(2n+2\nu-1)} \times \big[\omega n(n+\nu) + (1+c)n + \nu(\beta_0+1)\big]\big[\omega n(n+\nu) - (1+c)n + \nu(\beta_0 - c)\big], \ n \geqslant 0, \tag{3.6}$$

where we have put $\mu_2 := \frac{1}{4}(\nu+2)(2\delta_0+\omega)$ and chosen $\beta_0 := \frac{1}{4}\nu\omega - \frac{1}{2}(1-c) + \mu_2$.

$\mathbf{B}_{21}$. For $\omega = 0$ and $c = 1$, we may write $\nu(1+\beta_0) = 2(\alpha+1)$ and $\nu(1-\beta_0) = 2(\beta+1)$, so that

$$\nu = \alpha + \beta + 2 \ \text{ and } \ \beta_0 = \frac{\alpha - \beta}{\alpha + \beta + 2}.$$

Hence, we rediscover the Jacobi polynomials whose coefficients are

$$\beta_n = \frac{\alpha^2 - \beta^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}, \ n \geqslant 0,$$

$$\gamma_{n+1} = 4\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+1)}, \ n \geqslant 0.$$

For $\alpha = \beta = \lambda - \frac{1}{2}$, we meet again the Gegenbauer polynomials.

$\mathbf{B}_{22}$. When $\omega \neq 0$, we first write the two expressions between square brackets in (3.6) as follows

$$\omega n(n+\nu) + (1+c)n + \nu(\beta_0+1) = \big[\omega(n+\beta+1) + (1+c)\big](n+\alpha+1),$$
$$\omega n(n+\nu) - (1+c)n + \nu(\beta_0 - c) = \big[\omega(n+\alpha+1) - (1+c)\big](n+\beta+1).$$

If we set $\eta := \alpha - (1+c)/\omega$ and apply a linear transformation with $\hat{a} = i\omega$; $\hat{b} = \frac{1}{4}\nu\omega - \frac{1}{2}(1-c)$, we may rewrite (3.5)–(3.6) as

$$\hat{\beta}_n = \frac{1}{2}i(\alpha^2 - \beta^2)\frac{\eta - \frac{1}{2}(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}, \ n \geqslant 0, \tag{3.7}$$

$$\hat{\gamma}_{n+1} = \frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+\beta+1-\eta)(n+\eta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+1)}, \ n \geqslant 0. \tag{3.8}$$

Observe that if $\alpha^2 - \beta^2 = 0$ or $\eta = \frac{1}{2}(\alpha+\beta)$, the corresponding polynomials are symmetric. In consequence, it is worth while to discuss the following subcases mentioned in [1].

$\mathbf{B}_{22a}$. For $\alpha - \beta = 0$, we obtain

$$\hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+2\alpha+1)(n+2\alpha+1-\eta)(n+\eta+1)}{(2n+2\alpha+3)(2n+2\alpha+1)}, \ n \geqslant 0. \tag{3.9}$$

When $-1 < \eta < 2\alpha+1$ or when $\alpha \in \mathbb{R}$ and $\eta + \bar{\eta} = 2\alpha, \ \alpha+1 > 0$, the obtained polynomials are orthogonal with respect to a positive definite form.

– For $\alpha = 0$, the resulting polynomials are related to Pasternak polynomials [24] having the coefficients

$$\hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)^2(n+\eta+1)(n-\eta+1)}{(2n+3)(2n+1)},\ n \geqslant 0. \tag{3.10}$$

.

– For $\eta = 0$, the resulting polynomials are the Touchard ones [26] which are themselves particular cases of the continuous Hahn polynomials [4]. In this case we have

$$\hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)^2(n+\alpha+1)(n-\alpha+1)}{(2n+3)(2n+1)},\ n \geqslant 0. \tag{3.11}$$

$\mathbf{B}_{22b}$. For $\alpha + \beta = 0$, we get

$$\hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+\alpha+1)(n-\alpha+1)(n+\eta+1)(n-\eta+1)}{(2n+3)(2n+1)},\ n \geqslant 0. \tag{3.12}$$

Again, observe that the associated form to these orthogonal polynomials is positive definite when $-1 < \eta < 2\alpha+1$ or when $\alpha \in \mathbb{R}$ and $\eta + \bar{\eta} = 2\alpha,\ \alpha + 1 > 0$.

– When $\alpha = 0$, this leads to the Pasternak polynomials.
– When $\eta = 0$, we meet again the Touchard polynomials.

$\mathbf{B}_{22c}$. For $\eta = \frac{1}{2}(\alpha+\beta)$, we get

$$\hat{\gamma}_{n+1} = \frac{1}{4}\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+1)},\ n \geqslant 0. \tag{3.13}$$

For $\alpha + 1 > 0$ and $\beta + 1 > 0$, we have $\hat{\gamma}_{n+1} > 0$, for all $n \geqslant 0$. The form associated to these polynomials is then positive definite.

On the other hand, if both α and β are assumed to be real numbers, and so $\eta + \bar{\eta} = \alpha + \beta$, it is possible to find two numbers a and b, such that

$$a + \bar{a} = \alpha + 1,\ b + \bar{b} = \beta + 1,\ a + \bar{b} = \eta + 1 \text{ and } \bar{a} + b = \alpha + \beta + 1 - \eta.$$

In addition, under the conditions $\Re a > 0$ and $\Re b > 0$, it was shown in [1, p.18] that the resulting polynomials are orthogonal w.r.t. a positive definite form and coincide with the continuous Hahn polynomials. For more details we refer the reader to the aforementioned paper. The description of the main results obtained in the second case are summarized in Table 2.

4. The Recurrence Coefficients of the Higher Order Derivatives

Let k be a positive integer and let $\{P_n\}_{n\geqslant 0}$ be a D_ω-classical OPS. The sequence of the normalized higher order D_ω-derivatives, denoted as $\{P_n^{[k]}\}_{n\geqslant 0}$, is recursively defined by

$$P_n^{[k]}(x) := \frac{1}{n+1} D_\omega P_{n+1}^{[k-1]}(x),\quad k \geqslant 1, \tag{4.1a}$$

Table 2. Description of the OPS in Case B

Case B	$\hat{\beta}_n$	$\hat{\gamma}_{n+1}$	$\hat{P}_n$
$\mathbf{B}_1: \theta = 2\alpha - 1,\ \alpha \neq 1$	$\frac{1}{2}\alpha\omega - \frac{\alpha(\alpha-1)\mu_1}{(n+\alpha)(n+\alpha-1)}$	$-\frac{(n+1)(n+2\alpha-1)(\omega(n+\alpha)^2-2\alpha\mu_1)^2}{4(2n+2\alpha+1)(n+\alpha)^2(2n+2\alpha-1)}$	
$\omega \neq 0, \mu_1 \neq 0$			
$\hat{a} = \alpha\mu_1 = 1,\ \hat{b} = 0$	$\frac{1}{2}\alpha\omega + \frac{1-\alpha}{(n+\alpha)(n+\alpha-1)}$	$-\frac{(n+1)(n+2\alpha-1)(\omega(n+\alpha)^2-2)^2}{4(2n+2\alpha+1)(n+\alpha)^2(2n+2\alpha-1)}$	
$\omega = 0$	$\frac{1-\alpha}{(n+\alpha)(n+\alpha-1)}$	$-\frac{(n+1)(n+2\alpha-1)}{(2n+2\alpha+1)(n+\alpha)^2(2n+2\alpha-1)}$	Bessel
$\mathbf{B}_2: \theta = \nu - 1,\ \nu \neq 2$	$\frac{1}{4}\nu\omega - \frac{1}{2}(1-c) + \frac{\nu(\nu-2)\mu_2}{(2n+\nu)(2n+\nu-2)}$	$-\frac{(n+1)(n+\nu-1)}{(2n+2\nu+1)(2n+2\nu)^2(2n+2\nu-1)} \times$	
	$c \neq -1$	$\{[\omega n(n+\nu) + (1+c)n + \nu(\beta_0+1)]$	
		$[\omega n(n+\nu) - (1+c)n + \nu(\beta_0 - c)]\}$	
$\hat{a} = i\omega,\ \hat{b} = \frac{1}{4}\nu\omega - \frac{1}{2}(1-c)$	$\frac{1}{2}i(\alpha^2-\beta^2)\frac{\eta-\frac{1}{2}(\alpha+\beta)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}$	$\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+\beta+1-\eta)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+1)} \times$	
$\omega \neq 0, \nu = \alpha+\beta+2$	$\alpha^2 - \beta^2 \neq 0$	$\{(n+\eta+1)(n+\alpha+1)(n+\beta+1)\}$	
$\alpha - \beta = 0$	0	$\frac{1}{4}\frac{(n+1)(n+2\alpha+1)(n+2\alpha+1-\eta)(n+\eta+1)}{(2n+2\alpha+3)(2n+2\alpha+1)}$	
$\alpha = \beta = 0$	0	$\frac{1}{4}\frac{(n+1)^2(n+\eta+1)(n-\eta+1)}{(2n+3)(2n+1)}$	Pasternak
$\eta = 0$	0	$\frac{1}{4}\frac{(n+1)^2(n+\alpha+1)(n-\alpha+1)}{(2n+3)(2n+1)}$	Touchard
$\alpha + \beta = 0$	0	$\frac{1}{4}\frac{(n+\alpha+1)(n-\alpha+1)(n+\eta+1)(n-\eta+1)}{(2n+3)(2n+1)}$	
$\eta = \frac{1}{2}(\alpha+\beta)$	0	$\frac{1}{4}\frac{(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+1)}$	
$\omega = 0, \nu = \alpha+\beta+2$	$\frac{\alpha^2-\beta^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta)}$	$\frac{4(n+1)(n+\alpha+\beta+1)(n+\alpha+1)(n+\beta+1)}{(2n+\alpha+\beta+3)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+1)}$	Jacobi
	$c = 1$		

Here $\mu_1 := -\frac{1}{2}(\alpha+1)(2\delta_0+\omega)$; $\mu_2 := \frac{1}{4}(\nu+2)(2\delta_0+\omega)$ and $\eta := \alpha - (1+c)/\omega$

or, equivalently,

$$P_n^{[k]}(x) := \frac{1}{(n+1)_k} D_\omega^k P_{n+k}(x), \quad k \geqslant 1, \tag{4.1b}$$

where $(\mu)_n = \mu(\mu+1)\cdots(\mu+n-1)$, $(\mu)_0 = 1$, $\mu \in \mathbb{C}, n \in \mathbb{N}$, is the Pochhammer symbol. Application of Definition 1.1, with the special notations $P_n^{[1]} := Q_n$ and $P_n^{[0]} := P_n$, allows one to write $\beta_n^{[1]} := \tilde{\beta}_n$, $\gamma_n^{[1]} := \tilde{\gamma}_n$ and $\beta_n^{[0]} := \beta_n$, $\gamma_n^{[0]} := \gamma_n$, respectively.

The following corollary, which is in fact an immediate consequence of Proposition 1.2, plays an important role in establishing our results.

Corollary 4.1. ([1]) *If the OPS $\{P_n\}_{n\geqslant 0}$ is D_ω-classical, then the sequence $\{P_n^{[k]}\}_{n\geqslant 0}$ is also D_ω-classical OPS for any $k \geqslant 1$.*

By an application of this corollary, if we denote by $\left(\beta_n^{[k]}, \gamma_{n+1}^{[k]}\right)_{n\in\mathbb{N}}$ the recurrence coefficients corresponding to the OPS $\{P_n^{[k]}\}_{n\geqslant 0}$, with $k \geqslant 1$, then

$$P_{n+2}^{[k]}(x) = (x - \beta_{n+1}^{[k]}) P_{n+1}^{[k]}(x) - \gamma_{n+1}^{[k]} P_n^{[k]}(x), \; n \geqslant 0, \tag{4.2a}$$

$$P_1^{[k]}(x) = x - \beta_0^{[k]}, \; P_0^{[k]}(x) = 1. \tag{4.2b}$$

Our objective here is to express the coefficients $\beta_n^{[k]}$ and $\gamma_{n+1}^{[k]}$ in terms of the corresponding coefficients of the OPS $\{P_n\}_{n\geqslant 0}$, namely, β_n and γ_{n+1} obtained either in Case **A** or in Case **B**.

This will be stated in the next proposition.

Proposition 4.2. *Let $\{P_n\}_{n\geqslant 0}$ be a D_ω-classical OPS. Then, for every $k \in \mathbb{N}$, we have*

$$\beta_n^{[k]} = \beta_{n+k} + k\delta_{n+k-1}, \quad n \geqslant 0, \tag{4.3}$$

$$\gamma_n^{[k]} = \frac{n}{n+k} \gamma_{n+k} \big(k(\theta_{n+k-1} - 1) + 1 \big), \quad n \geqslant 1, \tag{4.4}$$

where δ_n and θ_n are solutions for the Eqs. (2.7) and (2.8), respectively.

Proof. From Corollary 4.1 it follows that each sequence $\{P_n^{[k]}\}_{n\geqslant 0}$, $k \geqslant 1$, is also D_ω-classical. Accordingly, both of the OPS $\{P_n^{[k]}\}_{n\geqslant 0}$ and $\{P_n^{[k+1]}\}_{n\geqslant 0}$ are characterized by the fact that they satisfy a second structure relation of type (1.15a)–(1.15b). For $\{P_n^{[k]}\}_{n\geqslant 0}$, this is given by

$$P_{n+2}^{[k-1]} = P_{n+2}^{[k]} + \alpha_{n+1}^{k,1} P_{n+1}^{[k]} + \alpha_n^{k,0} P_n^{[k]}, \; n \geqslant 0, \tag{4.5a}$$

$$P_1^{[k-1]} = P_1^{[k]} + \alpha_0^{k,1}, \quad P_0^{[k-1]} = P_0^{[k]} = 1, \tag{4.5b}$$

where

$$\alpha_n^{k,1} = (n+1) \left(\beta_{n+1}^{[k-1]} - \beta_n^{[k]} - \omega \right) \quad \text{and}$$

$$\alpha_n^{k,0} = (n+1)\gamma_{n+2}^{[k-1]} - (n+2)\gamma_{n+1}^{[k]}. \tag{4.6}$$

Likewise, for $\{P_n^{[k+1]}\}_{n\geqslant 0}$, an equivalent relation may be written as

$$P_{n+2}^{[k]} = P_{n+2}^{[k+1]} + \alpha_{n+1}^{k+1,1} P_{n+1}^{[k+1]} + \alpha_n^{k+1,0} P_n^{[k+1]}, \; n \geqslant 0, \tag{4.7a}$$

$$P_1^{[k]} = P_1^{[k+1]} + \alpha_0^{k+1,1}, \quad P_0^{[k]} = P_0^{[k+1]} = 1, \tag{4.7b}$$

with

$$\alpha_n^{k+1,1} = (n+1)\left(\beta_{n+1}^{[k]} - \beta_n^{[k+1]} - \omega\right) \quad \text{and}$$
$$\alpha_n^{k+1,0} = (n+1)\gamma_{n+2}^{[k]} - (n+2)\gamma_{n+1}^{[k+1]}. \tag{4.8}$$

To prove the equalities (4.3) and (4.4), we can proceed by induction on k. For this, let $\mathrm{P}(k)$ be the proposition that these two equalities hold. Observe first that the assertion $\mathrm{P}(1)$ is trivial. Let us check that $\mathrm{P}(2)$ is true. Setting $k = 1$ in (4.5a)–(4.5b) we get

$$P_{n+2} = P_{n+2}^{[1]} + \alpha_{n+1}^{1,1} P_{n+1}^{[1]} + \alpha_n^{1,0} P_n^{[1]}, \ n \geqslant 0, \tag{4.9a}$$
$$P_1 = P_1^{[1]} + \alpha_0^{1,1}, \quad P_0 = P_0^{[1]} = 1, \tag{4.9b}$$

where $\alpha_n^{1,1} := \tilde{\alpha}_n^1$ and $\alpha_n^{1,0} := \tilde{\alpha}_n^0$, since $P_n^{[1]} := Q_n$ and $P_n^{[0]} := P_n$. Roughly speaking, in this special case, the formulas (4.9a)–(4.9b) and (1.15a)–(1.15b) coincide.

Similarly, if we take $k = 2$ in (4.5a)–(4.5b), we have

$$P_{n+2}^{[1]} = P_{n+2}^{[2]} + \alpha_{n+1}^{2,1} P_{n+1}^{[2]} + \alpha_n^{2,0} P_n^{[2]}, \ n \geqslant 0, \tag{4.10a}$$
$$P_1^{[1]} = P_1^{[2]} + \alpha_0^{2,1}, \quad P_0^{[1]} = P_0^{[2]} = 1. \tag{4.10b}$$

Replace n by $n+1$ in (4.9a) and then apply the operator D_ω yields

$$(n+3)P_{n+2}^{[1]} = (n+3)P_{n+2}^{[2]} + (n+2)\alpha_{n+2}^{1,1} P_{n+1}^{[2]} + (n+1)\alpha_{n+1}^{1,0} P_n^{[2]}, \ n \geqslant 0. \tag{4.11}$$

Multiply both sides of (4.10a) by $(n+3)$ and compare this with (4.11) readily gives

$$(n+3)\alpha_{n+1}^{2,1} = (n+2)\alpha_{n+2}^{1,1}, \tag{4.12a}$$
$$(n+3)\alpha_n^{2,0} = (n+1)\alpha_{n+1}^{1,0}. \tag{4.12b}$$

By (4.6), for $k = 2$ and $k = 1$, it is easy to check that (4.12a) and (4.12b), respectively, lead to

$$(n+3)(n+2)\big[\beta_{n+2}^{[1]} - \beta_{n+1}^{[2]} - \omega\big] = (n+3)(n+2)\big[\beta_{n+3} - \beta_{n+2}^{[1]} - \omega\big],$$
$$(n+3)\big[(n+1)\gamma_{n+2}^{[1]} - (n+2)\gamma_{n+1}^{[2]}\big] = (n+1)\big[(n+2)\gamma_{n+3} - (n+3)\gamma_{n+2}^{[1]}\big].$$

Thanks to (2.1)–(2.2), we, respectively, deduce that

$$\beta_n^{[2]} = \beta_{n+2} + 2\delta_{n+1}, \quad n \geqslant 0,$$
$$\gamma_n^{[2]} = \frac{n}{n+2}\gamma_{n+2}\big(2\theta_{n+1} - 1\big), \quad n \geqslant 1,$$

which is precisely the desired conclusion, that is, $\mathrm{P}(2)$ is true.

Now, we must show that the conditional statement $\mathrm{P}(k) \to \mathrm{P}(k+1)$ is true for all positive integers k. We can proceed analogously to the proof of $\mathrm{P}(2)$. Starting from (4.5a), replacing n by $n+1$ and then apply the operator D_ω we find

$$(n+3)P_{n+2}^{[k]}=(n+3)P_{n+2}^{[k+1]}+(n+2)\alpha_{n+2}^{k,1}P_{n+1}^{[k+1]}+(n+1)\alpha_{n+1}^{k,0}P_{n}^{[k+1]},\ n\geqslant 0. \tag{4.13}$$

Multiply both sides of (4.7a) by $(n+3)$ and compare this with (4.13) readily gives

$$(n+3)\alpha_{n+1}^{k+1,1}=(n+2)\alpha_{n+2}^{k,1}, \tag{4.14a}$$

$$(n+3)\alpha_{n}^{k+1,0}=(n+1)\alpha_{n+1}^{k,0}. \tag{4.14b}$$

On account of (4.6)–(4.8), we have

$$(n+3)(n+2)\big[\beta_{n+2}^{[k]}-\beta_{n+1}^{[k+1]}-\omega\big]=(n+3)(n+2)\big[\beta_{n+3}^{[k-1]}-\beta_{n+2}^{[k]}-\omega\big],$$

$$(n+3)\big[(n+1)\gamma_{n+2}^{[k]}-(n+2)\gamma_{n+1}^{[k+1]}\big]=(n+1)\big[(n+2)\gamma_{n+3}^{[k-1]}-(n+3)\gamma_{n+2}^{[k]}\big].$$

From this, we deduce that

$$\beta_{n+1}^{[k+1]}=2\beta_{n+2}^{[k]}-\beta_{n+3}^{[k-1]}, \tag{4.15}$$

$$\gamma_{n+1}^{[k+1]}=2\frac{n+1}{n+2}\gamma_{n+2}^{[k]}-\frac{n+1}{n+3}\gamma_{n+3}^{[k-1]}. \tag{4.16}$$

On the other hand, according to the induction hypothesis we may write

$$\beta_{n+2}^{[k]}=\beta_{n+k+2}+k\delta_{n+k+1},\quad \beta_{n+3}^{[k-1]}=\beta_{n+k+2}+(k-1)\delta_{n+k+1},$$

$$\gamma_{n+2}^{[k]}=\frac{n+2}{n+k+2}\gamma_{n+k+2}\big(k(\theta_{n+k+1}-1)+1\big),$$

$$\gamma_{n+3}^{[k-1]}=\frac{n+3}{n+k+2}\gamma_{n+k+2}\big((k-1)(\theta_{n+k+1}-1)+1\big).$$

Substituting these into (4.15)–(4.16), and then changing n into $n-1$, it follows that

$$\beta_{n}^{[k+1]}=\beta_{n+k+1}+(k+1)\delta_{n+k},\quad n\geqslant 0,$$

$$\gamma_{n}^{[k+1]}=\frac{n}{n+k+1}\gamma_{n+k+1}\big((k+1)(\theta_{n+k}-1)+1\big),\quad n\geqslant 1.$$

These last equalities show that $\mathrm{P}(k+1)$ is also true, which completes the proof. □

Remark. Due to (4.3) and (4.4), from (4.6), it is seen that the two coefficients $\alpha_n^{k,1}$ and $\alpha_n^{k,0}$ involving in the structure relation (4.5a)–(4.5b) can be rewritten as

$$\alpha_n^{k,1}=-(n+1)\left(\delta_{n+k-1}+\omega\right)\ ;\ \alpha_n^{k,0}=\frac{(n+1)(n+2)}{n+k+1}\gamma_{n+k+1}\left(1-\theta_{n+k}\right),\ n\geqslant 0. \tag{4.17}$$

For $k=1$, this reduces to (2.3) providing the coefficients of (1.15a)–(1.15b).

Application.
When $\omega=0$, the identities (4.5a)–(4.5b) consist of the structure relation characterizing the higher order derivatives sequence of the classical orthogonal polynomials. In this case, a direct application of the preceding proposition

enables us to express the recurrence coefficients of the sequence $\{P_n^{[k]}\}_{n\geqslant 0}$ in terms of the recurrence coefficients for each of the four classical families.

To this purpose, if we denote by $\{\hat{H}_n\}_{n\geqslant 0}$, $\{\hat{L}_n(.;\alpha)\}_{n\geqslant 0}$, $\{\hat{B}_n(.;\alpha)\}_{n\geqslant 0}$ and $\{\hat{J}_n(.;\alpha,\beta)\}_{n\geqslant 0}$, respectively, the (monic) Hermite, Laguerre, Bessel and Jacobi polynomials, and by $\hat{H}_n^{[k]}(x)$, $\hat{L}_n^{[k]}(x;\alpha)$, $\hat{B}_n^{[k]}(x;\alpha)$ and $\hat{J}_n^{[k]}(x;\alpha,\beta)$ their corresponding sequences of derivatives of order k, then application of Formulas (4.3) and (4.4) successively give:

Case A. For $\theta_n = 1$, $n \geqslant 1$, and $\delta_n = \delta_0$, $n \geqslant 1$, we have

$$\hat{\beta}_n^{[k]} = \hat{\beta}_{n+k} + k\delta_0, \quad n \geqslant 0, \tag{4.18}$$

$$\hat{\gamma}_n^{[k]} = \frac{n}{n+k}\hat{\gamma}_{n+k}, \quad n \geqslant 1. \tag{4.19}$$

Hermite case: When $\delta_0 = 0$, $\hat{\gamma}_1 = \frac{1}{2}$, this yields

$$\hat{\beta}_n^{[k]} = 0, n \geqslant 0, \quad \text{and} \quad \hat{\gamma}_{n+1}^{[k]} = \frac{1}{2}(n+1), n \geqslant 0.$$

Laguerre case: When $\delta_0 = -1$, $\hat{\gamma}_1 = \hat{\beta}_0 = \alpha + 1$, we get

$$\hat{\beta}_n^{[k]} = 2n + \alpha + k + 1, n \geqslant 0, \quad \text{and} \quad \hat{\gamma}_{n+1}^{[k]} = (n+1)(n+\alpha+k+1), n \geqslant 0.$$

Case B. For $\theta_n = \frac{n+\theta+1}{n+\theta}$, $n \geqslant 1$, and $\delta_n = \frac{\delta_0(\theta+3)(\theta+1)}{\big(2n+\theta+3\big)\big(2n+\theta+1\big)}$, $n \geqslant 0$, we deduce that

$$\hat{\beta}_n^{[k]} = \hat{\beta}_{n+k} + \frac{k\delta_0(\theta+3)(\theta+1)}{\big(2(n+k)+\theta+1\big)\big(2(n+k)+\theta-1\big)}, \quad n \geqslant 0, \tag{4.20}$$

$$\hat{\gamma}_n^{[k]} = \frac{n(n+\theta+2k-1)}{(n+k)(n+\theta+k-1)}\hat{\gamma}_{n+k}, \quad n \geqslant 1. \tag{4.21}$$

Bessel case: When $\theta = 2\alpha - 1$, $\delta_0 = -1/(\alpha+1)\alpha$, we obtain

$$\hat{\beta}_n^{[k]} = \frac{1-(\alpha+k)}{(n+\alpha+k)(n+\alpha+k-1)}, \; n \geqslant 0,$$

$$\hat{\gamma}_{n+1}^{[k]} = -\frac{(n+1)(n+2(\alpha+k)-1)}{(2n+2(\alpha+k)+1)(n+\alpha+k)^2(2n+2(\alpha+k)-1)}, \; n \geqslant 0.$$

Jacobi case: When $\theta = \alpha+\beta+1$, $\delta_0 = 2(\alpha-\beta)/(\alpha+\beta+4)(\alpha+\beta+2)$, we get

$$\hat{\beta}_n^{[k]} = \frac{(\alpha-\beta)(\alpha+\beta+2k)}{(2n+\alpha+\beta+2k+2)(2n+\alpha+\beta+2k)}, \; n \geqslant 0,$$

$$\hat{\gamma}_{n+1}^{[k]} = 4\frac{(n+1)(n+\alpha+\beta+2k+1)(n+\alpha+k+1)(n+\beta+k+1)}{(2n+\alpha+\beta+2k+3)(2n+\alpha+\beta+2k+2)^2(2n+\alpha+\beta+2k+1)}, \; n \geqslant 0.$$

We thus rediscover the well known relations $\hat{H}_n^{[k]}(x) = \hat{H}_n(x)$, $\hat{L}_n^{[k]}(x;\alpha) = \hat{L}_n(x;\alpha+k)$, $\hat{B}_n^{[k]}(x;\alpha) = \hat{B}_n(x;\alpha+k)$ and $\hat{J}_n^{[k]}(x;\alpha,\beta) = \hat{J}_n(x;\alpha+k,\beta+k)$.

The results presented above are of course known and clearly show that the sequences of higher order derivative for the classical orthogonal polynomials belonging to the same class, provided that the parameters α and β take values in the range of regularity.

Corollary 4.1 asserts that the D_ω-classical character remain invariants under a differentiation of any D_ω-classical polynomials. However, except for the case $\omega = 0$, application of the kth order operator D_ω^k does not lead to a shift on the parameters of the original sequence.

Conclusion

We studied the D_ω-classical orthogonal polynomials using a new method in this domain. The results obtained in Sect. 3 are expected, where four representatives families are pointed out with some of their special cases. The recurrence coefficients of the resulting orthogonal polynomials are explicitly determined. Proposition 1.4 established a new characterization of these polynomials via a structure relation, and Proposition 4.2 provided relations connecting the recurrence coefficients of each sequence of polynomials with those of its higher order derivatives. For $\omega = 0$, the classical orthogonal polynomials are rediscovered.

We conclude, of course, by asking whether our approach provides a rather easy answer to the first Hahn's problem, that is, when D_ω is replaced by $\mathscr{D}_q$?

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Declarations

Conflict of interest The author declares no conflict of interest.

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