Homework 0 - Vector Spaces, Linear Transformations and Matrices

Handout date:	October 3, 2024.
Submission deadline:	October 15, 2024, 11:59pm.

1 Vector Spaces

Problem 1 (10 pts). Determine the dimensions of the following spaces:

a) range $\left(\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\right)$

b) range $\left(\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 1\\ 0 & 1 & 1\end{bmatrix}\right)$

c) span $((1,-1,1),(-1,1,1),(1,1,-1),(1,1,1))$

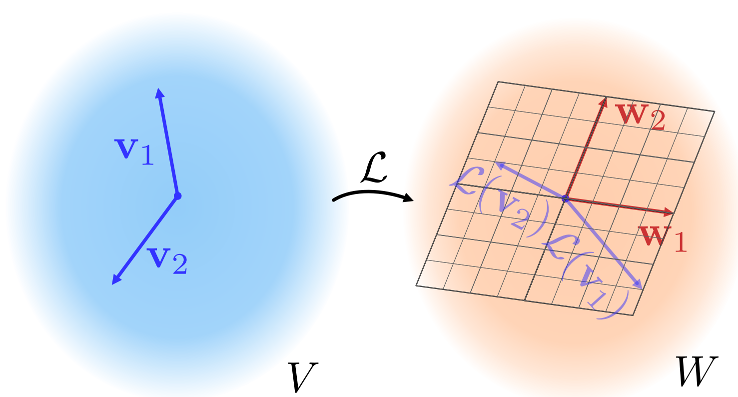
d) span $((1,2,3),(2,4,6),(3,6,9))$

e) P_3, the space of polynomials of degree less than or equal to 3

Problem 2 (5 pts). Suppose that S_1 and S_2 are subspaces of a vector space $(V,\mathbb{F})$. Show that their intersection $S_1 \cap S_2$ is also a subspace of $(V,\mathbb{F})$. Is their union $S_1 \cup S_2$ always a subspace?

2 Linear Transformations and Rank

Problem 3 (5 pts). Express the following linear transformation as a matrix relative to the bases $\{\mathbf{v}_1, \mathbf{v}_2\}$ and $\{\mathbf{w}_1, \mathbf{w}_2\}$. Please give a numerical answer.



Problem 4 (10 pts). For each function below, state whether it is linear and explain why.

a) $f(x, y, z) = 0$
b) $f(x, y, z) = 1$
c) $f(x, y, z) = (y + 2, z)$
d) $f(x, y, z) = (2z, y, 0)$
e) $f(x, y, z) = (x + y + z, x + 2y + 3z, x)$

Problem 5 (10 pts). We saw in class that differentiation is a linear transformation from P_n to P_{n-1}, where P_n denotes the space of polynomials of degree not exceeding n. We can represent this transformation as a matrix A using the monomial basis $\{1, x, x^2, \cdots\}$ for its input and output spaces. Write the matrix for $n = 4$.

Problem 6 (15 pts). The *outer product* of two column vectors $\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^n$, is the $m \times n$ matrix $\mathbf{x} \otimes \mathbf{y} := \mathbf{x}\,\mathbf{y}^\top$ (the product of a $m \times 1$ matrix with a $1 \times n$ matrix). This is called *a rank-one matrix* for the following reason: the rank of a matrix $A \in \mathbb{R}^{m \times n}$ equals 1 if and only if it can be written in the form $\mathbf{x}\,\mathbf{y}^\top$ for nonzero $\mathbf{x}$ and $\mathbf{y}$. Prove this fact.
Hint: think about what each column of $\mathbf{x}\,\mathbf{y}^\top$ looks like in terms of $\mathbf{x}$ and entries of $\mathbf{y}$.
Warning: there are two directions to prove in an "if and only if" statement!

3 Matrix Multiplication and Inversion

Problem 7 (5 pts). You are given a "black-box" function `multiply_A(x)` that computes $A\mathbf{x}$, the result of multiplying a matrix $A \in \mathbb{R}^{m \times n}$ by a column vector $\mathbf{x} \in \mathbb{R}^n$ that you pass in. What is the smallest number of function calls you need to make, and what $\mathbf{x}$ can you pass, to determine:

a) the j^{th} column of A;
b) the sum of the columns of A (if A has column vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots$, I want the answer $\mathbf{a}_1 + \mathbf{a}_2 + \cdots$);
c) the j^{th} *row* of A?

Problem 8 (15 pts). An upper triangular matrix A is one whose entries a_{ij} are zero if $i > j$. Prove that the product $C = AB$ of two upper triangular matrices is upper triangular.
Hint: there is a simple argument based on the interpretation of matrix multiplication AB as forming linear combinations of the columns of A.

Problem 9 (5 pts). Using the definition of an inverse, show that matrix inversion obeys:

a) $\left(A^{-1}\right)^\top = \left(A^\top\right)^{-1}$ (assuming A^{-1} exists)
b) $(AB)^{-1} = B^{-1}A^{-1}$ (assuming A^{-1} and B^{-1} exist)

You can assume the transposition rule for matrix multiplication: $(AB)^\top = B^\top A^\top$ (which is easy to prove from the defining formulas for matrix multiplication and transposition).

Problem 10 (20 pts). A matrix $A \in \mathbb{R}^{n \times n}$ is called *idempotent* if it satisfies $A^2 = A$.

a) Suppose the product $B^\top B$ is invertible for some matrix $B \in \mathbb{R}^{m \times k}$. Show that $B\left(B^\top B\right)^{-1} B^\top$ is idempotent.
b) If A is idempotent, show that $I_{n \times n} - A$ is also.
c) If A is idempotent, show that $\frac{1}{2} I_{n \times n} - A$ is invertible by giving an explicit formula for its inverse.
d) Suppose that A is idempotent and that we are given $\mathbf{x} \neq 0$ and λ satisfying $A\mathbf{x} = \lambda \mathbf{x}$. Show that $\lambda \in \{0, 1\}$.

4 What to Turn In

Please submit a *single PDF* to Canvas containing your solutions. Your submission can be handwritten, but to obtain credit it must be *easily readable*. Typesetting is preferred, but not required.

5 Late Work and Collaboration Policy

Late work will be accepted with a deduction of 10pts per day up until graded homework is returned. The deadline is strict: homework submitted after 11:59pm will be considered one day late.

Discussing the problems with classmates is allowed, but you may not collaborate as you write up your solutions, and you must not at any point look at another student's written solutions. You also must not search for solutions online: if you are stuck on a problem, please message me or post a question to the Canvas discussion board.