Problem 0:

2 has probability of $\frac{5}{10} = \frac{1}{2}$

unfair coin : $P(\text{Tails}) = 30\% = 0.3$

X	$P(X)$
1	$\frac{2}{10}$
2	$\frac{1}{2}$
4	$\frac{3}{10}$

$\operatorname{var}(X+Y) = \operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X,Y)$

$\operatorname{cov}(X,Y) = E[XY] - E[X]E[Y]$ -0.08

1 = Tails 0 = heads

Y	$P(Y \mid \text{fair})$
0	0.5
1	0.5

Y	$P(Y \mid \text{unfair})$
0	0.7
1	0.3

$E(X) = \sum_{c \in \{1,2,4\}} c P(X=c) = 1 \cdot \frac{2}{10} + 2 \cdot \frac{1}{2} + 4 \cdot \frac{3}{10}$

$= \frac{12}{5}$

$E[Y] = \sum_{c \in \{0,1\}} c \cdot P(Y=c) = 0 \cdot P(Y=0) + 1 \cdot P(Y=1)$

$= 0 + 0.4 = 0.4$

1 = Tails

0 = heads

$P(\text{fair coin}) = \frac{1}{2}$

$P(\text{unfair coin}) = \frac{2}{10} + \frac{3}{10} = \frac{5}{10} = \frac{1}{2}$

$P\{Y=0\} = 0.5 \cdot \frac{1}{2} + 0.7 \cdot \frac{1}{2}$
$= 0.6$

$P(Y=1) = 0.5 \cdot \frac{1}{2} + 0.3 \cdot \frac{1}{2} = 0.4$

x_i	y_i	$W = XY$	$P(X=x_i \text{ and } Y=y_i)$
1	0	0	$P(X=1 \text{ and } Y=0) = P(Y=0 \mid X=1)P(X=1) = 0.7 \cdot \frac{2}{10} = 0.14$
1	1	1	$P(X=1 \text{ and } Y=1) = P(Y=1 \mid X=1)P(X=1) = 0.3 \cdot \frac{2}{10} = 0.06$
2	0	0	$P(X=2 \text{ and } Y=0) = P(Y=0 \mid X=2)P(X=2) = 0.5 \cdot 0.5 = 0.25$
2	1	2	$P(X=2 \text{ and } Y=1) = P(Y=1 \mid X=2)P(X=2) = 0.5 \cdot 0.5 = 0.25$
4	0	0	$P(X=4 \text{ and } Y=0) = 0.7 \cdot \frac{3}{10} = 0.21$
4	1	4	$P(X=4 \text{ and } Y=1) = 0.3 \cdot \frac{3}{10} = 0.09$

$$E[XY] = E[W] = \sum_{w \in \{0,1,2,4\}} w\, P(W=w)$$

$P(W=0) = 0.14 + 0.25 + 0.21 = 0.6$ $\quad P(W=4) = 0.09$

$P(W=1) = 0.06$, $P(W=2) = 0.25$

$$\therefore E[XY] = E[W] = 0 \cdot 0.6 + 1 \cdot 0.06 + 2 \cdot 0.25 + 4 \cdot 0.09$$
$$= 0 + 0.06 + 0.5 + 0.36 = 0.92$$

$$\therefore Cov(X,Y) = E[XY] - E[X]E[Y] = 0.92 - (0.4 \cdot \tfrac{12}{5}) = 0.92 - 0.96$$
$$= -0.04$$

$2Cov(X,Y) = -0.08$

$\therefore Var(X+Y) = Var(X) + Var(Y) - 0.08$

$(E[X])^2 = \frac{144}{25}$, $(E[Y])^2 = (0.4)^2 = 0.16$

$$\therefore Var[X] = E[X^2] - (E[X])^2$$
$$= 7 - \frac{144}{25} = \frac{31}{25}$$

$$\therefore Var[Y] = E[Y^2] - (E[Y])^2$$
$$= 0.4 - 0.16 = 0.24$$

$$\therefore Var(X+Y) = \frac{31}{25} + 0.24 - 0.08 = \frac{7}{5}$$

$$E[X^2] = \sum_{c \in \{1,2,4\}} c^2 P(X=c)$$
$$= 1^2 \cdot \frac{2}{10} + 2^2 \cdot \frac{1}{2} + 4^2 \cdot \frac{3}{10}$$
$$= \frac{2}{10} + 2 + \frac{24}{5} = 7$$

$$E[Y^2] = \sum_{c \in \{0,1\}} c^2 P(Y=c)$$
$$= 0 + 1^2 \cdot P(Y=1) = 0.4$$

Problem 1:

L_i = # of people on the bus when drives away from bus stop i.

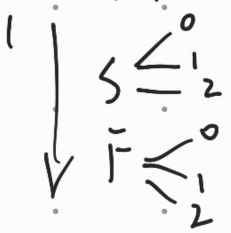
stop 1	$L_1=1$	prob
0	0	0.5
1	1	0.4
2	2	0.1

$L_2=1$

stop 2 ↓

L_1	L_2	prob	L_1-L_2	$(L_1-L_2)^2$
0	1	$(.8)(.5)(.4)$	0	0
1	2	$(.8)(.4)(.4)$	-1	1
2	3	$(.8)(.1)(.4)$	-2	4
0	0	$(.2)(.5)(.4)$	1	1
1	1	$(.2)(.4)(.4)$	0	0
2	2	$(0.2)(.1)(.4)$	-1	1

L_1 stop 2



$$E(L_1-L_2) = -1(.8)(.4)^2 - 2(.8)(.1)(.4) + 1(.2)(.5)(.4) - (.2)(.1)(.4) = -0.16$$

$$E[(L_1-L_2)^2] = 0.304$$

$$Var(L_1-L_2) = E(\{(L_1-L_2)-E[L_1-L_2]\})^2$$
$$= E[(L_1-L_2)^2] - (E(L_1-L_2))^2$$
$$= 0.304 - 0.0256 = 0.2784$$

$$E(L_1) = 1(.8)(.4)(.4) + 2(0.8)(0.1)(0.4) + (.2)(.4)(.4) + 2(0.2)(.1)(.4) = 0.24$$

$$E(L_1^2) = 0.32$$

$$Var(L_1) = E(L_1^2) - (E(L_1))^2 = 0.32 - 0.0576 = 0.2624$$

$$E(L_2) = (.8)(.5)(.4) + 2(.8\cdot.4\cdot.4) + 3(.8\cdot.1\cdot.4) + (.2\cdot.4\cdot.4) + 2(0.2\cdot0.1\cdot0.4)$$
$$= 0.56$$

$$E(L_2^2) = 1.024$$

$$\therefore Var(L_2) = E(L_2^2) - (E(L_2))^2$$
$$= 1.024 - 0.3136 = 0.7104$$

$$\therefore Var(L_1) + Var(L_2) = 0.9728$$

$$\therefore Var(L_1-L_2) \neq Var(L_1) + Var(L_2)$$

Problem 2:

$r = 4$, $s = 7$

Let the coin is fair coin , $P(\text{win/Loss}) = \frac{1}{2}$

			prob
min. stop	HHHH	4	$(\frac{1}{2})^4$
	THHHH	5	$(\frac{1}{2})^5$
	TTHHHH	6	$(\frac{1}{2})^6$
	HTHHHH	6	$(\frac{1}{2})^6$
		7	$1-(\frac{1}{2})^4-(\frac{1}{2})^5-(\frac{1}{2})^5$

$E[X] = 4\cdot(\frac{1}{2})^4 + 5(\frac{1}{2})^5 + 6(\frac{1}{2})^5 + 7(1-(\frac{1}{2})^4-(\frac{1}{2})^5-(\frac{1}{2})^5)$

$= 6.71875 = \frac{215}{32}$

Minimum Fee : $6.71875 \rightarrow 7$

Problem 3:

$p = P(X=1) \quad q = P(Y=1) \quad r = P(X=Y=1)$

$Var(3X-2Y) = E((3X-2Y)^2) - (E(3X-2Y))^2$

$\therefore E[(3X-2Y)^2] = 9E[X] - 12E[XY] + 4E[Y]$

$E[XY] = E[X]E[Y] = pq = P(X=Y=1) = r$

$\therefore E[(3X-2Y)^2] = 9p - 12r + 4q$

$[E(3X-2Y)]^2 = 9p^2 - 12pq + 4q^2$

$\therefore Var(3X-2Y) = 9p - 12r + 4q - 9p^2 + 12pq - 4q^2$

$= 9p - 9p^2 + 4q - 4q^2 - 12r + 12pq$

$= 9p(1-p) + 4q(1-q) - 12r$

Problem 4:

1. $\because$ dice 1 : $\{1 \dots 8\}$

dice 2 : $\{1 \dots 8\}$

$\therefore$ min. value : $1+1=2$

max. value : $8+8=16$

$\therefore$ X values take : $\{2, 3, 4, 5, \cdots, 15, 16\}$

2.

$8^2 = 64$

$P(X=2) = P(X=16) = \frac{1}{64}$

$P(X=3) = P(X=15) = \frac{2}{64}$

$P(X=4) = P(X=14) = \frac{3}{64}$

$P(X=5) = P(X=13) = \frac{4}{64}$

$P(X=6) = P(X=12) = \frac{5}{64}$

$\vdots$

$P(X=9) = \frac{8}{64}$

3.

$$E(X) = 2 \cdot \frac{1}{64} + 3 \cdot \frac{2}{64} + 4 \cdot \frac{3}{64} + \cdots + 9 \cdot \frac{8}{64} + 10 \cdot \frac{7}{64} + \cdots + 16 \cdot \frac{1}{64}$$

$$= \frac{576}{64} = 9$$

4.

$$Var(X) = E(X^2) - (E(X))^2$$

$$= E(X^2) - 81$$

$$= 10.5$$