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ANALYTIC NATURAL DEDUCTION

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Introduction. We consider some natural deduction systems for quantification theory whose only quantificational rules involve elimination of quantifiers. By imposing certain restrictions on the rules, we obtain a system which we term *Analytic Natural Deduction*; it has the property that the only formulas used in the proof of a given formula X are either subformulas of X, or negations of subformulas of X. By imposing further restrictions, we obtain a canonical procedure which is bound to terminate, if the formula being tested is valid. The procedure (ultimately in the spirit of Herbrand [1]) can be thought of as a partial linearization of the method of semantical tableaux [2], [3]. A further linearization leads to a variant of Gentzen's system which we shall study in a sequel.

The completeness theorem for semantical tableaux rests essentially on König's lemma on infinite graphs [4]. Our completeness theorem for natural deduction uses as a counterpart to König's lemma, a lemma on infinite "nest structures", as they are to be defined. These structures can be looked at as the underlying combinatorial basis of a wide variety of natural deduction systems.

In § 1 we study these nest structures in complete abstraction from quantification theory; the results of this section are of a purely combinatorial nature. The applications to quantification theory are given in § 2.

§ 1. Nest structures.

We let π be a finite 1–1 sequence $a_1, a_2, \ldots, a_n$, or a denumerable 1–1 sequence $a_1, a_2, \ldots, a_n, \ldots$ of elements called *points*. For $i \leqq j$, by the *interval* $[a_i, a_j]$ we mean the sequence $(a_i, a_{i+1}, \ldots, a_j)$. We allow the unit interval $[a_i]$, but we exclude the empty interval. Now let Σ be a finite or denumerable collection of these intervals, and let N be the ordered pair $\langle\pi, \Sigma\rangle$. We call N a *nest structure* iff the following conditions holds.

(N_1): *If two distinct intervals overlap, then one of them wholly contains the other, and properly extends it at both ends.*

More precisely:

(a) If two distinct intervals overlap, then one of them wholly contains the other.

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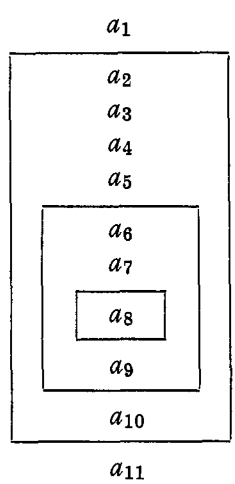
(b) No point begins two distinct intervals, nor ends two distinct intervals.

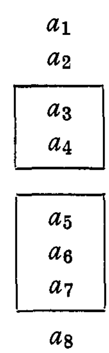
We call N a *finite* or *infinite* nest structure, depending respectively on whether the sequence π is finite or infinite. We refer to the number of terms of π as the *length* of the nest structure N.

We shall think of the sequence π as being written down vertically rather than horizontally, and we shall display nest structures by drawing *boxes* to represent the intervals. In the following examples, (a) and (b) are nest structures, but (c), (d) and (e) are not (each one violates condition (N_1) in one way or another)

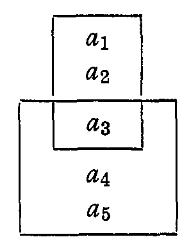
(a)



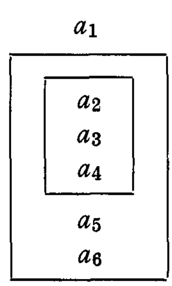
(b)



(c)

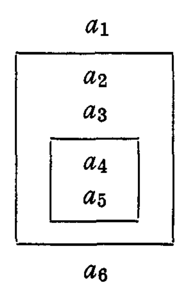


(d)



(a_2 begins 2 distinct intervals)

(e)



(a_5 ends 2 distinct intervals)

We shall call N a *regular* nest structure iff the following condition also holds.

(N_2): *For each positive integer i, if a_i ends an interval, then a_{i+1} does not begin an interval.*

In the above examples, (a) is a regular nest structure, but (b), though a nest structure, is not regular. Regular nest structures are the only ones which shall come up in our study of quantification theory (cf. § 2). The following is our fundamental lemma.

LEMMA. *For any infinite regular nest structure, infinitely many points lie outside all intervals.*

PROOF. Let N be infinite and regular. We call a point "free" if it lies outside all intervals. We shall show: (i) N contains at least one free point; (ii) for any free point, there is another free point under it.

(i) Consider the point a_1. If it is free, then there is nothing more to prove. Otherwise, it begins some interval $[a_1, a_i]$. We assert that a_{i+1} must then be free. For a_{i+1} cannot begin an interval, by the hypothesis of regularity. Hence, any interval about a_{i+1} would overlap $[a_1, a_i]$ in a manner violating condition N_1. Thus there exists at least one free point.

(ii) Suppose a_i is free. We must find a free point under a_i. If a_{i+1} is free, there is nothing more to prove. Otherwise a_{i+1} is contained in some interval; this interval cannot contain a_i (since a_i is free), so a_{i+1} must begin this interval. Let $[a_{i+1}, a_j]$ be this interval. We assert that a_{j+1} must be free. For suppose some interval I contains a_{j+1}. The first point of I cannot be a_{j+1} (since a_j ends an interval, and N is regular). The first point of I also cannot be any point in the interval $[a_{i+1}, a_j]$, or again condition N_1 would be violated. Therefore, the first point of I must be a_i, or some point above a_i, which is contrary to the hypothesis that a_i is free. This concludes the proof.

We shall subsequently employ the following terminology. Consider any nest structure N; let L be its length. Let $i+1 \leqq L$. Either a_{i+1} begins some interval or it does not. If the latter, and if a_i does not end any interval, then we call a_{i+1} the *sole successor* of a_i. If a_{i+1} *does* begin some interval $[a_{i+1}, a_j]$, then we call a_i a *junction point*, and we refer to a_{i+1} as the *first* successor of a_i, and a_{j+1} as the *second successor* of a_i (the latter, of course, only if a_{j+1} exists – i.e. only if $j+1 \leqq L$). Thinking pictorially in terms of boxes, a junction point is one which comes just before the top of some box. If we "cross" the top of the box, we hit the first successor of the point; if we "leap over" the box, we hit the second successor of the point. If point b is a successor of a (either sole, first or second) then we shall say that a is *connected* to b. We define a *path* as any finite or denumerable sequence of points, beginning with the first free point of N, and such that each term of the sequence is *connected* to the next. What properly results from the proof of our lemma is that any infinite regular nest structure contains an infinite path.[1] In terms of our notion of "successors", if a non-junction point

[1] This is reminiscent of the König lemma that every infinite tree in which each point branches to only finitely many points, must contain at least one infinite path. In a sequel to this paper, we shall show how our lemma on nest structures relates to the König lemma.

is free, so is its sole successor. If a junction point (of a *regular* nest structure) is free, so is its second successor. Thus, we can "take" the infinite path by starting with the first free point, and going down the sequence π, "leaping" over all boxes that come our way. We note that this is the *only* infinite path, for if we ever "enter" a box (i.e. take the *first* successor of a point), then we can never get out again, and all boxes are finite. We shall refer to this path as the principal path of N. It is also the path obtained by always taking the second successor, when we are at a junction point.

In preparation for the next section, we need a few more notions pertinent to nest structures. Let j be any positive integer less than or equal to the length of N, and let $i \leqq j$. We shall say that a_i is *covered* at stage j, if a_i is contained in at least one box $[a_k, a_l]$ which terminates before the point a_j — i.e., which is such that $l < j$. If a_i is not covered at stage j, then we say that a_i is *alive* or *free* at stage j. We note that any point a_i is automatically alive at stage i (even if the unit interval $[a_i]$ is present). Also any point a_i which is a free point of N is alive at stage j, for every $j \geqq i$ (providing, of course, that j is $\leqq$ to the length of N). Finally, we observe that the statement "a_i is connected to a_j" is equivalent to the statement" a_i is the last point before a_j which is alive at stage j."

In § 2, the "points" of our nest structures will be sentences (closed wffs) of quantification theory, or more precisely, *occurrences* of sentences on a line (not contained within an occurrence of a larger formula on that line). In this context, we shall indeed refer to our points as "lines".

§ 2. A system of quantification theory. We shall consider quantification theory as based on the primitives $\sim$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (material implication), $\forall$ (universal quantification), $\exists$ (existential quantification). For any formula F, we use F' synonymously with $\sim F$. We use individual variables $x_1, x_2, \ldots, x_n, \ldots$ and individual constants $a_1, a_2, \ldots, a_n, \ldots$ These constants (sometimes called "parameters") differ from the variables in that they are not to be quantified upon — i.e. $(\forall a_i)$ does not occur as a part of any well formed formula. By a *sentence* we mean a closed (well formed) formula; if no constants are present, then we call it a *pure* sentence. We use "F", "G" as metalinguistic variables ranging over all (well formed) formulas; "X", "Y", "Z" as metalinguistic variables ranging over all *sentences*; "x", "y", "z" as metalinguistic variables ranging over all individual variables $x_1, x_2, x_3, \ldots$ and "a", "b", "c" as metalinguistic variables ranging over our parameters $a_1, a_2, a_3, \ldots$ By F^x_a we mean the result of substituting a for every free occurrence of x in F. [In our system of quantification theory, we never need substitute a variable for a variable, but rather only a constant for a variable. We are thus not troubled with any possibility of collision of quantifiers]. For purely typographical reasons we sometimes abbreviate $F^x_{a_i}$ by F^x_i. The notion of *immediate subformula* is explicitly given by the rules: (i) F is an immediate subformula of $\sim F$; (ii) F, G are both immediate subformulas of $F \wedge G$, $F \vee G$, and $F \supset G$; (iii) F^x_a is an immediate subformula of $(\forall x)F$ and of $(\exists x)F$. We define G to be a *subformula* of F iff there exists a finite sequence of formulas, whose first term is F and whose last term is G and which is such that each term, other than the first, is an immediate subformula of the preceding term.

Natural deduction systems in general are characterized by the introduction of "premisses", which may subsequently be discharged. We shall display premisses by placing a *check mark* to the left of the sentence. To *introduce* X as a premiss (at a given stage n of the construction of a natural deduction) means to adjoin $\surd X$ as line $n+1$. To *discharge* a premiss (or more accurately, a line L_i which is a premiss) at a given stage n, means to draw a box about L_i to the last line L_n inclusive.

It is these "boxes" which make natural deductions special cases of nest structures. In the systems which we shall study, one only discharges the *last* premiss *alive* at a given stage. Also, when one discharges a premiss, one then subjoins a new line which is never itself a premiss. Hence, the nest structure obtained is always *regular*.

A typical discharge rule is the well known rule of *conditionalization* "for any deduction of length n, if $\surd X$ is the last premiss alive at stage n and Y is the n^{th} line of the deduction, the premiss $\surd X$ may be discharged and the sentence $X \supset Y$ subjoined". This rule is schematically displayed:

$$\begin{array}{c} \vdots \\ \boxed{\begin{array}{c} \surd X \\ \vdots \\ Y \end{array}} \\ X \supset Y \end{array}$$

[The idea behind the rule is, of course, that if Y is a logical consequence of X, together with other hypotheses $X_1, \ldots, X_m$, then $X \supset Y$ is a logical consequence of $X_1, \ldots, X_m$ alone]. As a matter of fact, we shall *not* use the rule of conditionalization, since it violates the subformula principle, and leads to a system of a radically different sort.

Aside from the rule of premiss introduction, our rules for constructing natural deductions fall into two categories: (i) direct rules; (ii) discharge rules.

The "direct" rules have the general format $\dfrac{A}{B}$, meaning "given a natural deduction of length n, if A is alive at stage n, then we may adjoin B as line

$n+1$". We also write $\begin{array}{c} A \\ \hline B \\ C \end{array}$ to mean "if A is alive at stage n, then we may adjoin either B or C (as line $n+1$)". This is really but a shorthand device for expressing both $\frac{A}{B}$ and $\frac{A}{C}$.

The "discharge" rules are of a different character. First we say that a finite natural deduction (say of length n) is in a *contradictory state* if there is a sentence Y such that Y, Y' are both alive (at stage n). Our discharge rules all have the following form:

$$\begin{array}{c} X \\ \vdots \\ \boxed{\begin{array}{l} \sqrt{Y} \\ \quad \vdots \\ \quad \textit{cont.} \end{array}} \\ Z \end{array}$$

In words, "if Y is the last premiss alive at stage n, and if X has an earlier occurrence, also alive at stage n, and if the deduction (as of stage n) is in a contradictory state, then we may discharge Y and subjoin Z.

A prototype of the discharge rule is the discharge rule for *disjunction*.

$$\begin{array}{c} X \vee Y \\ \vdots \\ \boxed{\begin{array}{l} \sqrt{X} \\ \quad \vdots \\ \quad \textit{cont.} \end{array}} \\ Y \end{array}$$

The idea behind the rule is that if $X \vee Y$ is true (under a set of hypotheses), but if X (together with these hypotheses) leads to a contradiction, then it must be Y that is true (under these hypotheses).

Now we state all our rules. First we have

(P) [Premiss Introduction] — At any stage we may introduce any premiss $\sqrt{X}$ as line n, providing it contains no *new* parameters — i.e., providing every parameter a which occurs in X occurs in at least one earlier line which is alive at stage n.

Next we consider the *direct* rules for the logical connectives.

$(\sim)$ $$\frac{\sim\sim X}{X}$$

$(\wedge 1)$ $$\begin{array}{c} X \wedge Y \\ \hline X \\ Y \end{array}$$

$(\vee 1)$ $$\begin{array}{c} \sim(X \vee Y) \\ \hline \sim X \\ \sim Y \end{array}$$

$(\supset 1)$ $$\begin{array}{c} \sim(X \supset Y) \\ \hline X \\ \sim Y \end{array}$$

Next the discharge rules

$(\vee 2)$ $$\begin{array}{c} X \vee Y \\ \vdots \\ \hline \boxed{\begin{array}{l} \surd X \\ \quad\vdots \\ \quad \textit{cont.} \end{array}} \\ Y \end{array}$$

$(\wedge 2)$ $$\begin{array}{c} \sim(X \wedge Y) \\ \vdots \\ \hline \boxed{\begin{array}{l} \surd \sim X \\ \quad\vdots \\ \quad \textit{cont.} \end{array}} \\ \sim Y \end{array}$$

$(\supset 2)$ $$\begin{array}{c} X \supset Y \\ \vdots \\ \hline \boxed{\begin{array}{l} \surd \sim X \\ \quad\vdots \\ \quad \textit{cont.} \end{array}} \\ Y \end{array}$$

Before considering the rules for the quantifiers, we remark that we could take fewer logical connectives as primitive and get along with fewer rules

— e.g., if we take just $\sim$, $\vee$ as primitive, we need use only rules (P), $(\sim)$, $(\vee_1)$, $(\vee_2)$.

The rules for the quantifiers are all direct. They are

$(\forall_1)$ $\quad \dfrac{(\forall x)F}{F_a^x}$ $\quad$ [a is *any* parameter]

$(\exists_1)$ $\quad \dfrac{\sim(\exists x)F}{\sim F_a^x}$ $\quad$ [a is *any* parameter]

$(\exists_2)$ $\quad \dfrac{(\exists x)F}{F_a^x}$, provided a is a *new* parameter (i.e. does not occur in any earlier line).

$(\forall_2)$ $\quad \dfrac{\sim(\forall x)F}{\sim F_a^x}$, provided a is a *new* parameter.

By a *natural deduction* we shall mean any nest structure — finite or infinite — such that each line and box (interval) has been introduced in accordance with the above rules. The "points" (lines) of our natural deductions are (occurrences) of *sentences* (with or without $\surd$). We note that (in our present system) the first line of a natural deduction can never be discharged; also no finite deduction has a box (interval) which includes the last line. Also the first line must be a premiss, and it cannot contain any parameters. By a *closed* natural deduction we shall mean a finite natural deduction which is in a contradictory state and which contains no premiss alive at the last stage except for the first line. By a *refutation* of a (pure) sentence X we shall mean a closed natural deduction whose first line is the premiss $\surd X$. By a *proof* of X we shall mean a refutation of X'.

Discussion. Rule $\forall_1$ is substantially Quine's rule *U.I.* (universal instantiation) — cf. [5]. Rule $\exists_1$ can be looked at as a trivial variant of *U.I.* [In fact, it could be replaced by the rule "to infer $(\forall x)\sim F$ from $\sim(\exists x)F$".] Rule $\exists_2$ is substantially Quine's rule *E.I.* (existential instantiation), and $\forall_2$ is a trivial variant of $\exists_2$. The intuitive idea behind the use of rule $\exists_2$ is this. If $(\exists x)F$ is true, then at least one element satisfies F — *let a be any such element.* If we subsequently derive another sentence $(\exists x)G$, we cannot legitimately say "let a be any element satisfying G", for we have already committed a to being the name of some element satisfying F, and we do not know that there is an element satisfying both F and G. This is the reason for the restrictive clause in Rule $\exists_2$; analogous considerations apply to Rule $\forall_2$. Actually we can liberalize these two rules by replacing the clause "providing a is new" by "providing a has not been previously introduced by Rule $\exists_2$ or Rule $\forall_2$", and does not occur in F". Under this liberalization, proofs can sometimes be shortened (cf. second example below).

It would appear that our present natural deduction system — call it "(N)" — is *sound* in the sense that every refutable sentence is unsatisfiable (and hence every provable sentence is valid). We discuss this further in § 3.

As an example of a natural deduction, the following is a proof of the sentence $(\forall x)[Px \supset Qx] \supset [(\forall x)Px \supset (\forall x)Qx]$

$$
\begin{array}{l}
\surd{\sim}[(\forall x)[Px \supset Qx] \supset [(\forall x)Px \supset (\forall x)Qx]] \\
\quad (\forall x)(Px \supset Qx) \\
\quad {\sim}[(\forall x)Px \supset (\forall x)Qx] \\
\quad (\forall x)Px \\
\quad {\sim}(\forall x)Qx \\
\quad {\sim}Qa_1 \\
\quad Pa_1 \\
\quad Pa_1 \supset Qa_1 \\
\boxed{\surd{\sim}Pa_1} \\
\quad Qa_1
\end{array}
$$

We wish to consider another example to illustrate a point. The following is a proof of the sentence $(\exists y)[Py \supset (\forall x)Px]$

$$
\begin{array}{ll}
\surd{\sim}[(\exists y)[Py \supset (\forall x)Px]] & \\
\quad {\sim}[Pa_1 \supset (\forall x)Px] & \\
\quad Pa_1 & \\
\quad {\sim}(\forall x)Px & \\
\quad {\sim}Pa_2 & \\
\quad {\sim}[Pa_2 \supset (\forall x)Px] & \text{(from line 1)} \\
\quad Pa_2 &
\end{array}
$$

If we liberalize Rule $\forall_2$ as indicated in our previous discussion, we can obtain the following shorter proof

$$
\begin{array}{l}
\surd{\sim}(\exists y)[Py \supset (\forall x)Px] \\
\quad {\sim}[Pa_1 \supset (\forall x)Px] \\
\quad Pa_1 \\
\quad {\sim}(\forall x)Px \\
\quad {\sim}Pa_1
\end{array}
$$

ANALYTIC NATURAL DEDUCTION. We shall call a natural deduction *analytic* if it is constructed in accordance with the following restriction (R) on premiss introduction.

(R) Aside from the first line, no premiss may be introduced except under the following conditions:

(a) If $X \vee Y$ is alive (at stage n, where n is the length of the deduction at hand), and if neither X nor Y is alive, then $\surd X$ may be introduced as a premiss (i.e. adjoined as line $n+1$).

(b) If $\sim(X \wedge Y)$ is alive, but neither X', Y' are alive, then $\surd X'$ may be introduced as a premiss.

(c) If $X \supset Y$ is alive, but neither X' nor Y are, then we may introduce $\surd X'$ as a premiss.

It is obvious that in an *analytic* deduction, every formula which appears must be a subformula or the negation of a subformula of the first line. Our aim is now to show that every valid pure sentence is provable by some analytic natural deduction.

A UNIFYING NOTATION. It will save us considerable repetition of essentially similar arguments in the metatheory, if we use the unified notation in [6].

We use "α" to denote any sentence of one of the forms $X \wedge Y$, $\sim(X \vee Y)$, $\sim(X \supset Y)$, $\sim\sim X$. By α_1 we respectively mean X, $\sim X$, X, X, and by α_2 we respectively mean Y, $\sim Y$, $\sim Y, X$. [E.g., if $\alpha = \sim(X \supset Y)$, then $\alpha_1 = X$ and $\alpha_2 = \sim Y$]. In each case, α is truth-functionally equivalent to the *conjunction* of α_1 and α_2. We might refer to sentences α as of *conjunctive type* (this includes as special cases sentences which are literally conjunctions). By a sentence β of *disjunctive type* we mean a sentence of one of the forms $X \vee Y$, $\sim(X \wedge Y)$, $X \supset Y$; by β_1 we respectively mean X, $\sim X$, $\sim X$ and by β_2 we respectively mean Y, $\sim Y$, Y. In each case β is truth-functionally equivalent to the *disjunction* of β_1, β_2. By a sentence of *universal type* — denoted by "γ" we mean a sentence of one of the two forms $(\forall x)F$, $\sim(\exists x)F$; for any parameter a, by $\gamma(a)$ we respectively mean F^x_a, $\sim F^x_a$. By a sentence of *existential type* — denoted by "δ", we mean a sentence of one of the two forms $(\exists x)F$, $\sim(\forall x)F$; by $\delta(a)$ we respectively mean F^x_a, $\sim F^x_a$.

To summarize, we are using "α", "β", "γ", "δ" as metalinguistic variables ranging over sentences which are respectively of conjunctive, disjunctive, universal, existential types. Every sentence, except for an atomic sentence or its negation, is uniquely of one of these four types.

Our rules for natural deduction — other than premiss introduction — can now be more succinctly formulated as follows

Rule A — [Direct sentential] — $\dfrac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}}$

Rule B — [Discharge]

$$\begin{array}{c} \beta \\ \vdots \\ \boxed{\begin{array}{c} \surd\beta_1 \\ \vdots \\ \textit{cont.} \end{array}} \\ \beta_2 \end{array}$$

Rule C — $\dfrac{\gamma}{\gamma(a)}$ *Rule D* — $\dfrac{\delta}{\delta(a)}$, providing a is new.

For *analytic natural deduction*, the three cases for premiss introduction (other than the first line) are subsumed under the rule.

(R) — If β is alive at stage n, but neither β_1, β_2 are so alive, then $\surd\beta_1$ may be taken as a premiss.

A CANONICAL PROOF PROCEDURE. We now wish to give a canonical proof procedure having the property that any analytic deduction, whose first line is $\surd{\sim}X$, and which is constructed in accordance with the procedure, is bound to terminate in a proof of X, providing X is valid. To technically facilitate the description of the procedure, we introduce the colon as a new formal symbol, and we define an *auxiliary sentence* to be an expression of the form $\gamma{:}a_n$. [We could interpret $(\forall x)F{:}a_n$ as saying that F holds for every x other than $a_1, \ldots, a_n$, and we could interpret ${\sim}(\exists x)F{:}a_n$ as saying that ${\sim}F$ holds for every x other than $a_1, \ldots, a_n$. Actually it is not really necessary to consider the question of the interpretation of these auxiliary sentences; they are basically but technical devices to force our deductions to have a purely syntactical closure property which we need]. We shall use "$\gamma{:}0$" synonymously with "γ", and "$\gamma{:}n$" for "$\gamma{:}a_n$", $n = 1, 2, \ldots$.

Let (A) be our system of analytic natural deduction. We let (A_1) be the system obtained from (A) by replacing Rule C by the following.

Rule C_1 — $\dfrac{\gamma{:}n}{\begin{array}{l}\gamma(n+1)\\ \gamma{:}n+1\end{array}}$ $(n = 0, 1, 2, \ldots)$

This is really a collection of four rules (where now n is a *positive* integer).

$$\dfrac{(\forall x)F}{\begin{array}{l}F(a_1)\\(\forall x)F{:}a_1\end{array}} \qquad \dfrac{{\sim}(\exists x)F}{\begin{array}{l}{\sim}F(a_1)\\{\sim}(\exists x)F{:}a_1\end{array}} \qquad \dfrac{(\forall x)F{:}a_n}{\begin{array}{l}F(a_{n+1})\\(\forall x)F{:}a_{n+1}\end{array}} \qquad \dfrac{{\sim}(\exists x)F{:}a_n}{\begin{array}{l}{\sim}F(a_{n+1})\\{\sim}(\exists x)F{:}a_{n+1}\end{array}}$$

Suppose D_1 is a natural deduction in the system (A_1). Some of the lines will, in general, not be sentences of quantification theory, but rather these *auxiliary* sentences. However, if we delete these auxiliary lines, the resulting nest structure D that remains is obviously a natural deduction in the system (A).

It is now important to note that if D is any deduction, either in the system (A) or the system (A_1), of finite length n, which is in a contradictory state, but not closed, then it is possible to use a discharge rule — moreover in such a manner that the subjoined line is not a repetition of an earlier line alive at stage $n+1$.[2] To see this, consider the last premiss alive at stage n; let k be the number of the line where it appears (clearly $1 < k \leqq n$). Now, by the restriction rule (R) on premiss introduction, the premiss in question must be of the form β_1, where β is alive at stage $k-1$, but β_2 is *not* alive at stage $k-1$. If we then discharge β_1 and subjoin β_2, we get no repetition of a line alive at stage $n+1$. [It is obvious that the only lines alive at stage $n+1$ are those alive at stage $k-1$ together with the $n+1^{th}$ line itself].

When we employ the premiss introduction rule (in accordance with restriction (R)) we say that we *use* β to *introduce* β_1. When we employ a discharge rule, then we say that we *use* β to *subjoin* β_2. When we employ a direct rule X/Y, we say that we *use* X to *adjoin* Y. We note that in our original system (A), a sentence γ could be used to adjoin infinitely many sentences. In our present system (A_1), γ can be used to adjoin at most two sentences. Thus, e.g., we cannot *use* $(\forall x)F$ to obtain $F(a_2)$; rather we *use* $(\forall x)F$ to obtain $(\forall x)F{:}a_1$, which in turn may be *used* to obtain $F(a_2)$. More generally, we can use a sentence of the form $\gamma{:}n$ in only two possible ways. We shall say that a line on a path P is *used on path* P iff it has been used to obtain some line (perforce below it) on path P. We say that a line on path P is *fulfilled* (on path P) iff one of the following conditions holds:

(i) it is a sentence of the form α, and both α_1, α_2 appear on P.

(ii) it is of the form β, and either β_1 or β_2 appears on P.

(iii) it is of the form γ, and both $\gamma(1)$ and $\gamma{:}1$ appears on P.

(iv) it is of the form $\gamma{:}n$, for n a *positive* integer, and either a_{n+1} does not appear in any line on path P, or else both $\gamma(n+1)$ and $\gamma{:}n+1$ appear on path P.

(v) it is of the form δ, and for some positive integer n, $\delta(n)$ appears on P.

Suppose a natural deduction D (in the system A_1) is infinite. Then, by our fundamental lemma on infinite regular nest structures, infinitely many lines lie outside all boxes; these lines are the elements of the so-called *principal path*. If a deduction D is finite, we also speak of the principal path of D; the elements of this path are those lines which are alive at the last stage of the deduction. Whether D is finite or infinite, we shall say that D is *fulfilled* iff the following conditions both hold: (1) Every line

[2] The additional fact about avoiding repetition is not needed in this paper, but will be used in a sequel.

on the principal path of D is fulfilled (on the principal path); (2) D is not closed.

If an infinite analytic deduction D in the system (A_1) is constructed at random, it is not at all necessary that D is fulfilled. The purpose of a canonical procedure is to guarantee that (i) every infinite deduction constructed according to the procedure is fulfilled; (ii) every finite deduction constructed according to the procedure which cannot be extended further without violating the procedure is either fulfilled or closed.

Many canonical procedures could be given. One such is as follows. Start the deduction with any premiss. Now suppose the deduction has been carried through to stage n. If, as of this stage, the deduction is either closed or fulfilled, then we stop. If the deduction is neither closed nor fulfilled, then our next act is determined by the following conditions.

(1) If the deduction is in a contradictory state, then we must use a discharge rule (which is indeed possible, as we have shown). And we use the earliest line which can be used to effect the discharge. [This line is perforce of type β and unfulfilled as of stage n, but fulfilled as of stage $n+1$].

(2) If the deduction is not in a contradictory state, then we pick the earliest unfulfilled line. If it is of the form α, then we adjoin the first of the pair α_1, α_2 which is not alive at stage n. If it is a sentence β, then we adjoin the premiss $\surd\beta_1$. If it is of the form $\gamma:n$, then we adjoin the first of the pair $\gamma(n)$, $\gamma:n$ which is not alive. If it is of the form δ, then we take the first integer m such that a_m occurs in no line alive at stage n, and we adjoin $\delta(m)$.

This concludes the description of the canonical procedure. It is a routine matter to check that this procedure does have the desired properties.

Suppose now that D is an infinite deduction constructed according to the canonical procedure. Let S^* be the set of all sentences which occur (with or without a check mark) on the principal path. Since D is fulfilled, then S^* must satisfy the following conditions.

M_0: No atomic sentence and its negation are both in S^*.

M_1: If $\alpha \epsilon S^*$, then α_1, α_2 are both in S^*.

M_2: If $\beta \epsilon S^*$, then either β_1 or β_2 is in S^*.

M_3: If $\delta \epsilon S^*$, then for at least one parameter a, $\delta(a) \epsilon S^*$.

M_4^*: For each positive integer n, if $\gamma:n \epsilon S^*$, then $\gamma(n+1)$ and $\gamma:n+1$ are both in S^*.

Let S be the set of all elements of S^* which are not auxiliary sentences. The set S obviously obeys conditions M_0—M_3 (reading "S" for "S^*"). And since S^* obeys condition M_4^*, then by an obvious induction argument, the set S obeys the condition:

M_4: If $\gamma \epsilon S$, then for *every* positive integer n, $\gamma(n) \epsilon S$.

Thus the set S of all non-auxiliary sentences on the principal path of D obeys conditions M_0—M_4. Such a set S is called a *model set* in Hintikka [3], and is denumerably satisfiable.[3] Thus if the canonical deduction, starting with a given premiss $\surd X$, runs on infinitely, then X is denumerably satisfiable.

Suppose that we run a canonical deduction starting with $\surd X$, and that we reach a stage in which every line on the principal path is fulfilled, but the deduction is not in a contradictory state. Let a_n be the last parameter which occurs on the principal path. Then the set S of all non-auxiliary sentences on the principal path satisfies conditions M_0—M_3, and in place of M_4, it satisfies the condition.

M'_4: If $\gamma \epsilon S$, then for every $i \leqq n$, $\gamma(i) \epsilon S$.

Such a set S is satisfiable in a finite domain of n elements (by an argument analogous to that in footnote 3).

We have thus shown that if X has no finite or denumerable model, then the canonical deduction starting with $\surd X$ must terminate in a refutation of X. Therefore, if a sentence X is valid (or even denumerably valid) then the canonical deduction starting with $\surd X'$ must terminate in a refutation of X'. Therefore, if a sentence X is valid (or even denumerably valid) then the canonical deduction starting with $\surd X'$ must terminate in a *proof* of X. So, of course, every valid (pure) sentence is provable by some analytic deduction, since the canonical deduction is analytic.

We have, of course, given a canonical procedure for the system (A_1), and have shown that every valid pure sentence is provable by an analytic deduction in (A_1). But this implies that every pure valid sentence is provable by an analytic deduction in the system (A).

REMARKS CONCERNING THE PROCEDURE. The above procedure, though theoretically adequate, is not a particularly good one from the point of view of obtaining short proofs. The following procedure, though a bit more difficult to justify, is considerably better as a working procedure.

Suppose at stage n, the deduction is not in a contradictory state. In the procedure we have given, we then always give priority to the earliest unfulfilled line. It is better rather to give first priority to lines of the form δ, second priority to lines of the form α, third priority to lines of the form $\gamma{:}n$. And within each of these groups, we give priority to the earliest unfulfilled line of that group.

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[3] Briefly, the argument is this. Take for the universe of discourse the set of parameters themselves. Assign to each n-ary predicate, the set of all n-tuples of parameters such that the predicate followed by the n-tuple lies in S. Under this interpretation, every sentence X of S is true, (by an induction argument on the number of occurrences of logical connectives and quantifiers in X).

A few working examples will convince the reader of the superiority of this procedure. Needles to say, it is still capable of many practical improvements. Such a study is a subject in itself, and we shall deal with this elsewhere.

§ 3. Concluding remarks. (1) We now see how our lemma on infinite nest structures can be used in place of the König graph lemma in proving the completeness of quantification theory. One can easily use the completeness of analytic natural deduction to prove the completeness of the better known Hilbert type formalizations, as well as the Gentzen formulations.

(2)** We wish to briefly discuss some related natural deduction systems. The system (N) which we first introduced is of little interest in itself. For suppose we introduce a premiss. Unless it be of the form $\sqrt{}\beta_1$, where β is already alive, it can never be discharged. Suppose it is of the form $\sqrt{}\beta_1$, with β already alive. If β_1 is already alive, the introduction of $\sqrt{}\beta_1$ is strategically pointless, and will only create a redundancy; if β_2 is alive, then $\sqrt{}\beta_1$ cannot be discharged without incurring a repetition of β_2. Thus any *closed* deduction in (N), which contains no redundancy, is automatically analytic.

Of more interest is the following extension $(N)^*$ of (N). The rules of $(N)^*$ are the rules P, A, B, C, D of (N) together with the following discharge rule

Rule E —

$$\begin{array}{c} \vdots \\ \hline \begin{array}{|l|} \sqrt{}{\sim}X \\ \qquad\vdots \\ \qquad \textit{cont.} \\ \hline \end{array} \\ X \end{array}$$

In the system $(N)^*$, the first line of a deduction is allowed to be discharged and the deduction then continued. By a *proof* of (a *pure* sentence) X in $(N)^*$ we shall mean a finite deduction in $(N)^*$ whose last line is X, and which contains no premisses alive at the last stage.

The system $(N)^*$ (which obviously does not obey the subformula principle — even in the weaker form allowing negations of subformulas) comes much closer to the usual natural deduction systems than does our analytic system (A). Indeed it is trivial to obtain Quine's rules of conditionalization and truth-functional implication as derived rules in $(N)^*$. Alternatively, we could start with the system — call it (M) — whose rules are Quine's rules of premiss introduction, truth-functional implication and premiss

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introduction, combined with our quantificational rules C, D, and easily establish the rules of $(N)^*$ as derived rules.[4]

If D is a proof of X in the system (A) (or indeed in (N)), then one application of Rule E converts D into a proof of X in the system $(N)^*$. So it is trivial that every sentence provable in (A) (or in (N)) is provable in $(N)^*$.

An argument justifying the *soundness* of the system $(N)^*$ (and hence also of the systems (N) and (A)) is briefly as follows. The crucial point to observe is that if S is a set of sentences which is (simultaneously) satisfiable, and if $\delta \epsilon S$ and if a is a parameter which occurs neither in δ nor any element of S, then $S \cup \{\delta(a)\}$ is again satisfiable. Therefore Rule D preserves *soundness* of natural deductions in the following sense: Call a finite deduction D *sound* if for every interpretation of all predicates alive in D (i.e., which occur in lines alive at the last stage of D) in a non-empty universe U, there exists an interpretation of the live parameters of D such that if all live premisses are true under the interpretation, so are all live lines of D. That the remaining rules P, A, B, C, E also preserve soundness is completely trivial. Therefore (by induction) every finite natural deduction (in $(N)^*$) is sound. This easily implies that every provable sentence is valid (since it contains no parameters, and no premisses are alive at the last stage of a proof).

The soundness of $(N)^*$, together with the completeness of (A), implies that every sentence provable in $(N)^*$ is also provable in (A). Of course, this proof is purely non-finitary. A finitary proof of the equivalence of the systems $(N)^*$ and (A) is virtually equivalent to a finitary proof of Gentzen's Haupsatz — indeed it could be obtained as a consequence of the Haupsatz, or constructed independently along the lines of Gentzen's proof.

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[4] Pedagogically we have found the systems M, $(N)^*$, to be particularly suitable as an introduction to Quantification Theory. We are not restricted to analytic proofs; indeed the systems are technically quite versatile. The restriction in Rule D (in all our systems) is of a relatively simple sort; there is no worry about alphabetic variance, flagging of variables, nor collision of quantifiers. The soundness is relatively easy to establish (since we do not have generalization rules working in conjunction with instantial rules, though it is not difficult to establish certain generalization rules as *derived* rules of the system).

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