A Rational Model of Dimension-reduced Human Categorization

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Abstract

Existing models in cognitive science typically assume human categorization as graded generalization behavior in a multidimensional psychological space. However, category representations in these models may suffer from the curse of dimensionality in a natural setting. People generally rely on a tractable yet sufficient set of features to understand the complex environment. We propose a rational model of categorization based on a hierarchical mixture of probabilistic principal components, that simultaneously learn category representations and an economical collection of features. The model captures dimensional biases in human categorization and supports zero-shot learning. We further exploit a generative process within a low-dimensional latent space to provide a better account of categorization with high-dimensional stimuli. We validate the model with simulation and behavioral experiments.

1 Introduction

Categorization refers to the behavior of grouping stimuli into discrete groups [1]. It requires abstraction of observations to form category representations, which may correspond to regions [2] or distributions [3] in a psychological space. The process of category learning reflects principles of human generalization, as objects within the same category are assumed to share similar features.

Human generalization is extremely flexible in that, people tend to generalize along a certain set of dimensions, which are termed *separable*, while performing isotropic generalization on *integral* dimensions. The dimensional biases match either rule-based or information-integration categorization behaviors [4]. The preference for grouping stimuli along certain axes develops with age [5], suggesting that the preferred directions are learned [6]. The dimensional biases are closely related to features learned during interaction with the environment [7, 8]. It is highly likely that these features are learned simultaneously with the categories.

One of the challenges for learning effective category representations is that natural stimuli can have numerous features. Uninformative noise dimensions hurt categorization performance [9] and need to be excluded. Humans can learn new categories with only a few samples [10, 11] and generalize flexibly. It was also found that low-dimensional and highly explainable representation is sufficient to

predict human similarity judgements [12] and categorization behavior [13]. However, to the best of our knowledge, dimension reduction techniques have not been widely incorporated into models of categorization, as they fail to produce the equivalence of separable dimensions [14].

In comparison, although state-of-the-art machine learning models can achieve accuracy comparable to humans, they show significantly different patterns of representation and generalization [15]. Human-like generalization patterns are desirable for their robustness against distributional shift [15], and can withstand the curse of dimensionality and data sparsity. Therefore, we would like to investigate: how do human beings spontaneously learn tractable yet sufficient features along with categories, which meanwhile support few-shot categorization and generalization?

We propose a rational model of categorization based on the hierarchical mixture of probabilistic principal component analyzers (PPCA) [16, 17], with nonparametric Bayesian priors to share feature components among clusters. The model is capable of learning both categories and principal components from scratch, and exhibit dimensional biases that reflect natural variation within categories. The principal components formed during category learning constitute an expressive feature set, which not only supports a generative approach to categorization and mitigates the curse of dimensionality, but also allows zero-shot learning within or among categories.

The outline of this paper is as follows. Section 2 summarizes the related cognitive models of human categorization and generalization. Section 3 introduces the rational model of categorization based on the hierarchical mixture of PPCA. Section 4 shows the results of simulations and how well the proposed model explains human behavior in experiments. Sections 5, 6 and 7 present the discussion, limitation and conclusion of this study.

2 Related work

Models of human category learning Cognitive models of category learning make various assumptions about *category representations*. Among the family resemblance models, prototype models [18] assume that a category can be represented as a single prototype, while exemplar models [19, 20] represent a category by all known members of the category. Rational model of categorization (RMC) [3] assumes that human categorization reflects adaption towards optimal prediction of features, representing categories as distributions, and casts categorization as Bayesian density estimation. In fact, density estimation, especially the estimated covariance structure, is closely related to the concept of gradient of generalization [2, 6]. Moreover, hierarchical Dirichlet process (HDP) is proposed as a unifying model for hierarchical categorization [21]. Clusters can be learned with supervision and shared among categories under the framework of HDP.

Learning features and generalization metrics Formal cognitive models introduce different distance metrics to describe different patterns of generalization [2, 19], but cannot explain the formation of dimensional biases. Hierarchical rational models exploit the hierarchical prior to [6, 22] learn proper rotations of the psychological space in addition to the dimensional weights. The higher-level prior is able to transfer learned metrics to novel stimuli. Despite their amazing ability of learning dimensional biases, such models retain all input dimensions in analysis, which is costly in high-dimensional tasks. Nonparametric Bayesian models (e.g., the Indian buffet process, [7, 8]) allow the learning of infinitely many features with the preference for simpler posteriors. Our proposed categorization model can also be extended to incorporate IBP priors for nonparametric feature selection.

3 A rational model of dimension-reduced categorization

3.1 The framework of rational analysis

The rational model of categorization (RMC) assumes that human behavior reflects optimal adaptation to the environment [3]. Specifically, RMC casts categorization as a problem of density estimation [21] through Bayesian analysis:

$$P(x_n|\mathbf{x}_{n-1},\mathbf{c}_{n-1}) = \sum_{k=1}^{K} P(c_n = k|\mathbf{c}_{n-1})P(x_n|c_n = k,\mathbf{c}_{n-1},\mathbf{x}_{n-1}),$$
(1)

where x_n is the new stimulus, c_n being its category assignment. The formulation decomposes the prediction task into two parts: the prior encodes how often a certain category has been observed, and the likelihood evaluates how typical a stimulus is given the category. The prior can take the form of a Chinese restaurant process (CRP) [23]. In a CRP, the probability of assigning a new sample to an existing component is proportional to the the number of existing samples assigned to it, and the probability of sampling a new component is proportional to a scalar concentration parameter:

$$P(c_n = k | \mathbf{c}_{n-1}) \propto \begin{cases} M_k & \text{if } M_k > 0 \ (k \text{ is old}) \\ \gamma & \text{if } M_k = 0 \ (k \text{ is new}) \end{cases}$$
 (2)

Starting from the original formulation of RMC, Rational Exclusive Family Resemblance Hierarchy (REFRESH) and other Bayesian hierarchical models are proposed, to capture more fine-grained covariance structure by replacing the prior of covariance with a mixture distribution [6, 22]. Specifically,

$$p(\Sigma_k|u_{k-1}, \Phi) = \sum_{j=1}^{J} p(u_k = j|u_{k-1}) p(\Sigma_k|\Phi_j, u_k = j),$$
(3)

where Φ_j is a component. As the hierarchical prior is shared across contexts, flexible generalization along an invariant set of multiple dimensions can be realized. REFRESH produces fascinating learning trajectories and generalization patterns. These models can learn the dimensional bias, along with proper rotation of the representation space. However, modeling covariance as a mixture of Inv-Wishart components does support explicit dimension reduction, as covariance matrices are full-rank. Besides, these models require each category to be related to a rotation of the original space, with little shared statistical strength between these category-specific dimensional biases. Therefore, we propose a new model of the hierarchical Bayesian type using a hierarchical mixture of probabilistic principal component analyzers (PPCA). Dependencies between clusters with shared dimensional biases are captured through the hierarchical Dirichlet process (HDP).

3.2 Explicit modeling of dimensional biases and dimension reduction

Covariance structure of mental representation mirrors the patterns of probabilistic generalization. Instead of using the scale matrices of Inv-Wishart priors, we base our model on probabilistic principal component analysis (PPCA) [17]. PPCA explicitly decomposes the covariance into variation along principal components and isotropic noise, with the following generative process

$$t_n = Wx_n + \mu + \epsilon_n,\tag{4}$$

where the columns of W stand for principal components. The low-dimensional latent variable x_n explicitly demonstrates the score on the corresponding direction. Within the probabilistic framework, PPCA can be extended to a mixture model[17], with additional assignment variable c_n . The marginal distribution of observed data t_n is

$$t_n \sim N(\mu_{c_n}, W_{c_n} W_{c_n}^T + \sigma^2 I_d) \tag{5}$$

given normal priors $x_n \sim N(0, I_q)$ and $\epsilon_n \sim N(0, \sigma^2 I_d)$. In this way, the model performs dimension analysis to select a subspace (spanned by columns of W) that efficiently explains the variation within the clusters.

Compared with previous rational models of categorization that attempts to explain the emergence of dimensional bias, PPCA-mixture-based models are at least as powerful in capturing dimensional biases. To be more specific, we can decompose any covariance prior component, which is a symmetric, positive definite matrix, into low rank approximations. As shown in the appendix, for any $\Phi \in \mathbb{R}^{d \times d}$ that is symmetric, we can find $\sigma^2 \in \mathbb{R}, W \in \mathbb{R}^{d \times q}, q \leq d-1$, such that

$$\Phi = WW^T + \sigma^2 I \tag{6}$$

PPCA-mixture-based model also facilitates dimension reduction within each category. Under the PPCA framework, we consider raw stimuli in the continuous psychological space as transformed mixtures of low dimensional latent variables. The latent variable can be inferred in a Bayesian manner (7), along with a low-rank reconstruction of the raw input (8).

$$x_n|t_n, c_n \sim N((W_{c_n}^T W_{c_n} + \sigma^2 I)^{-1} W_{c_n}^T (t_n - \mu_{c_n}), \sigma^2 (W_{c_n}^T W_{c_n} + \sigma^2 I)^{-1})$$
(7)

$$\hat{t}_n = W_{\hat{c}_n} (W_{\hat{c}_n}^T W_{\hat{c}_n})^{-1} W_{\hat{c}_n}^T (t_n - \mu_{\hat{c}_n}) + \mu_{\hat{c}_n}.$$
(8)

3.3 hierarchical Dirichlet process mixture of PPCA

Here, we describe our hierarchical Dirichlet process mixture of PPCA (HDP-PPCA) model. We introduce dependencies between clusters based on the assumption that humans utilize a central repository of features [8]. The cluster-specific principal component w_c is sampled from a Dirichlet process with Normal base measure. There is no limit on the number of features, but only finitely many will be learned. These atoms are shared across clusters. Now there are two infinite mixtures: a higher-level mixture assigning clusters to features, and a lower-level mixture assigning samples to clusters. In this paper, we assume that each category possesses only one principal component, which is equivalent to learning the basic-level categories in a hierarchy, along with a preferred axis.

HDP-PPCA can be thought of as an extended variant of previous feature learning methods [7, 8, 24], with an extra multiplicative latent variable x_n on feature ownership. As a result, HDP-PPCA enjoys the flexibility to learn continuous features from raw sensory representations. In our model, the feature ownership matrix constrains each row sum equal to one, and features images (principal components) are from an isotropic normal prior. The posterior of learned principal components forms a parsimonious set of features that can explain the majority variations within categories. For a new sample to be categorized, the model tends to generalize along the learned directions of principal components.

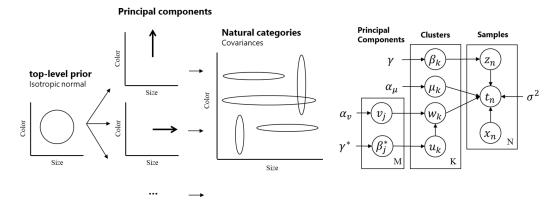


Figure 1: **Left.** Schemetic illustration of the hierarchical prior over principal components. Natural categories share these components, which explains common patterns of within-category variation. **Right.** Representation of HDP mixture of PPCA as a probabilistic graphical model.

3.4 Categorization with generative process

With the HDP-PPCA model, we propose a categorization framework based on a generative process. Category assignments can be inferred with generated exemplars beyond stored exemplars or prototypes. Specifically, we can express the likelihood term as an integral over the latent space, which can be approximated with samples.

$$p(c_n|t_n, \mathbf{c}_{n-1}, \mathbf{t}_{n-1}) \propto p(c_n|\mathbf{c}_{n-1})p(t_n|c_n, \mathbf{c}_{n-1}, \mathbf{t}_{n-1})$$

$$\tag{9}$$

$$\approx p(c_n|\mathbf{c}_{n-1}) \left[\frac{1}{N_g} \sum_{x_i \sim p(x)}^{N_g} p(x) N(\cdot|\hat{\mu}_{c_n} + \hat{w}_{c_n} x, \hat{\sigma}^2 I) \right]$$
(10)

The parameter N_g represents the number of *mental samples* [25]. An extreme case would be taking $N_g \to \infty$, which is equivalent to the conventional way, evaluating the marginal likelihood. Another

extreme would be generating only $N_g = 1$ exemplar, with maximum a posteriori (MAP) estimation of the latent variable in equation (7) and (8). Compared with exemplar models that store all historical observations, such an approach require less memory capacity. It also agrees with recent findings about how humans generalize beyond observed exemplars [26].

The generative approach we propose benefits from dimension reduction, and is in line with empirical findings about human similarity judgements [12]. All generated exemplars in our model lie on a low-dimensional subspace, and thus are immune to the curse of dimensionality. They constitute a more representative sample of the category than original observations. This makes the generative approach more effective than conventional approaches in few-shot learning settings.

Compositional few-shot generalization

Under the PPCA framework, the posterior of principal components in our model serves as a feature system that supports compositional generalization behavior. Compositionality resides in the additive nature of principal components when serving as category-specific features. The compositional generalization behavior explained with HDP-PPCA cannot be captured with classic models and previous rational models, as there are no explicit scores on the preferred dimensions.

We propose two possible manners of compositional generalization under our framework, either within or beyond an existing category. As the principal components can be considered as semantically meaningful, generalization with PCs and conditioning on latent variables require no observational samples, thus can be thought of as zero-shot learning of a new category. Consider the case of a cluster c, with its related principal component $w_{u_c}, u_c \in \{1, ..., k\}$. The central repository of principal components $\{w_k\}_{k=1}^K$ contains all existing posterior PCs.

Learning sub-categories First consider forming a sub-level category within cluster c, with a latent variable x_{sub} as description of the subcategory. PPCA allows conditioning on latent variables, with the conditional distribution

$$t_{sub}|x_{sub} \sim N(\mu_c + w_c x_{sub}, \sigma^2 I) \tag{11}$$

 $t_{sub}|x_{sub} \sim N(\mu_c + w_c x_{sub}, \sigma^2 I) \tag{11}$ The sub-category prototype is acquired in a additive manner, with isotropic generalization inherited from its parent class. Under a category learning framework, the emergence of categories may be explained as a result of coagulation along principal components. In other words, category prototypes serve as discrete proxies of key features in human psychological space.

Learning new categories The second way is to generalize on a direction that is not related with variation within the cluster, but learned with other (or other levels of) clusters. The new category learned in this manner will inherit generalization patterns from the existing category, i.e. transfer the principal component sample to the new cluster. Given a description on a feature w and score on that feature x, the new category is instructed as

$$t_{new}|w, x \sim N(\mu_c + wx, \sigma^2 I + w_c w_c^T)$$
(12)

4 Results

4.1 Learning the dimensional biases

With the principal components specifying preferred dimensions, HDP-PPCA can learn dimensional biases with reduced dimensionality. We run several simulations to illustrate the context-dependent learning of preferred dimensions. We first train our models with axes-aligned artificial clusters similar to those in previous experiments [6, 22] without any supervision. We set the concentration parameters in the Dirichlet processes γ^* and γ to 1, reflecting moderate preference for new components. We also set vague priors $\Gamma(1,1)$ on parameters α_u , α_v , as well as the precision $\tau=1/\sigma^2$.

We choose variational inference (VI) for posterior inference. VI approximates the intractable posterior with a family of tractable distributions, and performs optimization on the variational distributions with respect to the Evidence-Lower-Bound (ELBO) [27]. It is a common alternative to the Monte Carlo Markov Chain (MCMC) techniques [28, 29], and enjoys better scalability. In recent years, generalpurpose variational inference algorithms have been proposed to support probabilistic model fitting [30, 31], with no conjugacy assumptions required. We adopted pyro [32], an expressive, scalable and flexible Probabilistic Program Languages, to construct our model and perform variational inference. **Gradient of generalization** The generalization patterns of our model is shown in Figure. 2. Generalization tend to be aligned with previously encountered principal components. Although PPCA models do not add hard constraints on the components *a priori*, it is unlikely that posteriors will contain highly correlated components (unless necessary). Similar as they may seem, HDP-PPCA learn dimensional biases, or generalization metrics, based on lower dimensionality, while scale matrices inferred by REFRESH are full-rank.

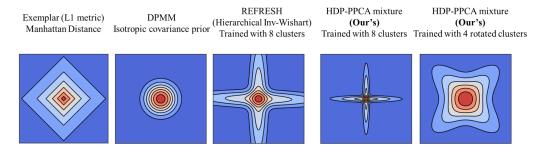


Figure 2: Generalization gradients of exemplar model (with L1 metric), DPMM, REFRESH and HDP-PPCA. The last two were trained on axes-aligned or rotated clusters without supervision.

4.2 Categorization with generated exemplars in low-dimensional subspace

The curse of dimensionality leads to sparse distributions in the high-dimensional space. A simulation study with dimension-varying stimuli is presented to illustrate whether PPCA-based generative models of categorization help tackle the curse of dimensionality. The categories were designed to be significantly different on the first dimension, and have major within-category variations on the second. Then we add noise dimensions to the categories.

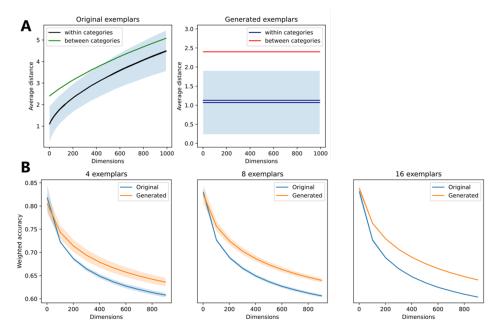


Figure 3: **A.** Average distances between and within categories. Notice that the gap (measured by standard deviation) of average distances between the original exemplars within and between categories decreases as dimensionality grows, but remains unchanged with the generated ones. **B.** Accuracy weighted by confidence shows that generated exemplars perform better than original ones against the curse of dimensionality. (Shadow areas correspond to 95% confidence interval)

In the first part of the simulation study, we calculate the average distance within and between samples from categories, and record how they change when more noise dimensions are added. As the number

of dimensions goes up, the average distance between samples from different categories gets closer to average within-category sample distances (measured by the standard deviation). This suggests the category distributions have 'mixed up' in the high-dimensional space. By contrast, generated exemplars rest on a low-dimensional subspace, and are not affected by the increasing dimensionality.

We base our second simulation study on the intuition that exemplars in a high-dimensional space may not be representative enough. By contrast, generated exemplars capture the major variation within the category, and can be more representative as they are sampled from a low-dimensional manifold. In the simulation study, we run multiple runs with different numbers of randomly sampled exemplar. Then, we test simple exemplar-based categorization with high-dimensional test data, with statistics (point and interval estimation) provided in Figure 3. It is clear that generated low-dimensional exemplars can support significantly better categorization performance in a high-dimensional environment.

4.3 Few-shot generalization

We carried out two experiments to study human few-shot generalization behavior, examining how different models account for human generalization. We make the assumption that humans have already learned a set of principal components, which correspond to separable dimensions in previous empirical findings. Human generalization are tested within or beyond artificial categories that are designed to align with these dimensions. Given our hypothesis that humans use scores on these dimensions to generalize, semantic description of size or color will be considered equivalent to providing a latent variable.

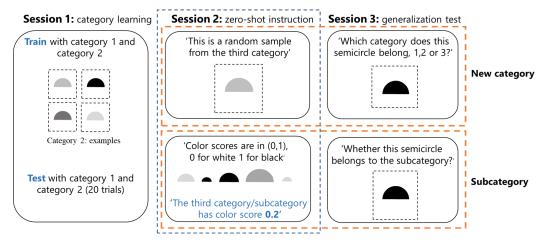


Figure 4: Procedure of zero-shot generalization experiment. After training and testing on the artificial categories in session 1, session 2 provides either zero-shot or one-shot instruction. Generalization patterns are tested in session 3.

The two experiments test the two hypothesized generalization patterns in the previous section. In the first experiment, semantic description of a subcategory is provided. In the second experiment, we try out an alternative approach by providing one visual example of a new category. The experiment stimuli we choose are semicircles with varying color and size, which are two commonly used separable dimensions [5, 6]. We first collected dimension rating data to scale the parameters of the generated stimuli. Experiment data is collected from the online platform Credamo, with reward for participants who respond. For each experiment setting (subcategory and new category), we recruited 200 participants, among which 172 and 186 respectively provided valid response. Participants were first familiarized with the artificial categories, with a training session and the first test session in session 1. After that, we provide the semantic description or provide a single example of the new category. Then the participants were tested on the instructed category structure. In the first experiment, the second test session includes samples from one of the clusters, some of which come from the semantically-informed subcategory. The participants need to judge whether the test stimuli come from the subcategory or not. In the second experiment, participants need to classify the samples into one of the three categories, where the third category was only illustrated with a single example.

We compare our model with the Exemplar model with and without dimensional weights [19]. We choose exemplar model because other models like the prototype model [18] cannot estimate the covariance with a single sample. We obtain maximum likelihood estimation (MLE) of model parameters using either only the training stimuli or stimuli that participants assign to a certain category. Average accuracy on predicting human category assignments on the subcategory (in experiment 1) and new category (in experiment 2) was recorded in Table. 1. The results suggest that our model perform better on predicting human few-shot generalization behavior. Exemplar models, on the other hand, are biased towards categories with more observations. In the cases where data is scarce, exemplar models may fail to capture rapid learning of a new category.

Table 1: Results of few-shot generalization experiment

	subcategory		new category	
Model	train	test	train	test
Exemplar Exemplar+Attention HDP-PPCA (Ours)	0.260 0.260 0.740	0.260 0.260 0.740	0.378 0.378 0.713	0.334 0.336 0.714

5 Discussion

Cross-categorization and context-dependent behavior There are two interesting properties of PPCA that support the flexibility of our model. First, PPCA-based models do not impose any ordering among principal components, which can be expressed as parent-child relationships in the tree-structured models [33]. Therefore, HDP-PPCA mixture allows for context-dependent ordering of features and construction of hierarchies, as in the behavior of cross-categorization [34]. Second, PPCA does not assume orthogonality like the conventional principal component analysis. Despite the fact that orthogonal components will be more economical, HDP-PPCA mixture can learn correlated features when necessary. This resembles how human learns new *integral* dimensions through training. A typical example is the saturation and brightness of a color, which are separable for color experts but not for ordinary people [14]. This may also explain how humans learn a simple, possibly temporary rule for categorization when the categories are not axes-aligned.

Prototypes and principal components Our model contains both category prototypes (represented by the means of each category) and variability of features within and among categories (represented by the various principal components). However, the two ingredients of our model can be related to each other and pose the problem of un-identifiability. In the experiment of zero-shot generalization, we show that people can use principal components to locate the prototype of a subcategory. Similarly, people can learn directions of variation within all the stimuli, then coagulate in the latent space to form categories. Although we have only tested such human behavior with simple hypothetical categories, it is possible that all prototypes are represented in a semantically-meaningful latent space, spanned by a set of principal components. Although direct composition of prototypes is not supported in our current model, it may become possible in an extension with latent-space representations of prototypes. In fact, principal components can be thought of as continuous solution to clustering [35]. We find it reasonable to consider human mental representations as a combination of category prototypes and principal components. The discrete prototypes serve as proxies for rapid recognition of an object, while continuous latent variables help with more fine-grained description, thus reaching a balance between cognitive cost and reliability.

Structural organization of representations Humans often use structures (e.g., trees) to organize the hierarchy of categories [36]. Given any category-feature ownership matrix, which has each row sum equal to one in HDP-PPCA, a structure can be learned *a posteriori* in a Bayesian manner [37]. Previous tree-based models can be thought of as a structure-constrained version of the hierarchical mixture model. On the other hand, the HDP-PPCA model can be naturally regarded as a tree with two levels: the first-level nodes correspond to the principal components, and the second-level nodes represent (basic) categories sharing these components. This relationship also implies the possibility of few-shot generalization, which we have found with the behavioral experiment [10]; i.e., proper covariance structures are transformed with the inductive bias encoded in higher-level nodes in a tree.

We intend to further investigate structural learning with latent representations of categories, which may explain how humans represent their knowledge, and why.

Embracing naturalist settings Most of the experiments of human categorization were carried out in laboratories, with low-dimensional stimuli representations based on Multidimensional Scaling (MDS) [2, 19]. Meanwhile, cognitive scientists are embracing a recent trend, where large-scaled experiments with high-dimensional naturalist stimuli become more popular [20, 38]. We can explore human categorization and generalization behavior with naturalist settings, as the model and estimation algorithm are readily scalable to high-dimensional data.

6 Limitation

First, the assumption that all principal components come from sensory category learning is not realistic. In the real world, humans learn about categories and concepts through not only observing but also in other ways, like social learning. A child may learn about a polar bear with only pictures or descriptions of it. It's unlikely that all the principal components in human minds, if any, are learned through observation of natural stimuli. We make the assumption for simplification. Other aspects of human categorization learning are out of the scope of this paper.

Second, we perform estimation by variational inference in this study. However, unlike Gibbs sampling or particle filters [29], the psychological plausibility of variational inference is unclear. We use variational inference to provide a satisfactory approximation to the posterior, with no guarantee that it has converged to the global optimum or approached the true posterior. Whether variational inference is psychologically sensible for the rational model of categorization needs to be carefully investigated.

7 Conclusion

We propose a rational model of categorization that can simultaneously learn a set of clusters and a tractable yet expressive feature representation. It is able to produce dimensional biases similar to human behavior. It inspires a generative framework for categorization, which improves accuracy with a few samples in a high-dimensional setting. The model can also explain human zero-shot generalization behavior. Using this model based on hierarchical mixture of PPCA, we hope to provide a better account of human generalization in the natural world.

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8 Supplementary Material

Proof of equation (6) For any symmetric matrix Φ , we can perform diagonalization by

$$\Phi = R\Lambda R^T = \sum_{i=1}^d \lambda_i r_i r_i^T \tag{13}$$

where λ_i is the i^{th} diagonal element of the diagonal matrix Λ , and r_i is the i^{th} column of the orthogonal matrix R, satisfying $R^TR=I$. Given that Φ is full rank, for any vector $x\in\mathbb{R}^d$, we can express it using columns of R, $x=\sum_{i=1}^d y_i r_i$. It is obvious that $r_i^Tx=y_i$. Without loss of generality, let λ_1 be the largest diagonal element of Λ , and let $\sigma^2=\lambda_1$, we have

$$\Phi x = \sigma^2 x + \sum_{i=1}^d \lambda_i (r_i^T x) r_i - \sigma^2 y_i r_i$$
(14)

$$= \sigma^2 x + \sum_{i=1}^d (\lambda_i - \sigma^2) y_i r_i \tag{15}$$

Given that we've set $\sigma^2 = \lambda_1$, there are at most d-1 nonzero components in the second term. Therefore,

$$\Phi x = (\sigma^2 I + W W^T) x \tag{16}$$

stands true for any $x \in \mathbb{R}^d$.

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