Classify Equations

Definition of a linear function

$$f(ax_1 + bx_2) = af(x_1) + bf(x_2) \tag{1}$$

Equation 1

$$b^2 = 1 \tag{2}$$

All terms are linear

Equation 2

$$a + b = 1, \qquad a - 2b = 0 \tag{3}$$

All terms are linear

Equation 3

$$mu'' + \beta|u'|u' + cu = \mathtt{F}(t) \tag{4}$$

All terms except $\beta|u'|u'$ are linear.

$$|(au_1 + bu_2)'|(au_1 + bu_2)' \neq a|u_1'|u_1 + b|u_2'|u_2 \tag{5}$$

Equation 4

$$u_t = \alpha u_{xx} \tag{6}$$

All terms are linear

Equation 5

$$u_{tt} = c^2 \nabla^2 u \tag{7}$$

All terms are linear

Equation 6

$$u_t = \nabla \cdot (\alpha(u) \nabla u) + f(x, y) \tag{8}$$

If $\alpha(u) \neq \mathtt{constant}$ then the term $\nabla \cdot (\alpha(u)\nabla u)$ is nonlinear. The rest of the terms are linear. One could also write the nonlinear term as:

$$\nabla \alpha(u) \cdot \nabla u + \alpha(u) \nabla^2 u \tag{9}$$

In order for the first term to be nonlinear $\alpha(u)$ can not be constant nor linear, while the second term is nonlinear as long as $\alpha(u)$ is not constant.

Equation 7

$$u_t + f(u)_x = 0 \tag{10}$$

The term $f(u)_x$ is nonlinear if $f(u)$ is not constant, linear or quadratic.

Equation 8

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + r \nabla^2 \mathbf{u}, \qquad \nabla \cdot \mathbf{u} = 0 \tag{11}$$

The term $\mathbf{u} \cdot \nabla \mathbf{u}$ is nonlinear.

Equation 9

$$u' = f(u, t) \tag{12}$$

The term $f(u,t)$ is linear if $f(au_1 + bu_2, t) = af(u_1, t) + bf(u_2, t)$, otherwise it's nonlinear.

Equation 10

$$\nabla^2 u = \lambda e^u \tag{13}$$

The term e^u is nonlinear since $e^{au_1 + bu_2} \neq ae^{u_1} + be^{u_2}$.