Nonlinear Vibration ODE

$$mu'' + b|u'|u' + s(u) = \mathtt{F}(t) \tag{1}$$

where both m and b are positive constants. $s(u)$ is possibly a nonlinear function of u, and $\mathtt{F}(t)$ is a given function.

(a) System of two first order ODEs

We introduce $v = u'$:

$$\begin{aligned} v' &= \frac{1}{m}\left(F(t) - bv|v| - s(u)\right) \\ u' &= v \end{aligned} \tag{2}$$

We discretize the system with Crank-Nicolson and use arithmetic mean on linear terms and a geometric mean on the quadratic term:

$$\begin{aligned} \frac{v^{n+1} - v^n}{\Delta t} &= \left[\tfrac{1}{m}(F^n - b|v|v - s(u))\right]^{n+\frac{1}{2}} \\ &= \tfrac{1}{m}(F^{n+\frac{1}{2}} - b|v^{n+\frac{1}{2}}|v^{n+\frac{1}{2}} - s(u)^{n+\frac{1}{2}}) \\ &\approx \tfrac{1}{m}\big(\tfrac{1}{2}(F^{n+1} + F^n) - b|v^{n+\frac{1}{2}}|v^{n+\frac{1}{2}} - \tfrac{1}{2}(s(u^{n+1}) + s(u^n))\big) \\ &\approx \tfrac{1}{m}\big(\tfrac{1}{2}(F^{n+1} + F^n) - bv^{n+1}|v^n| - \tfrac{1}{2}(s(u^{n+1}) + s(u^n))\big) \end{aligned} \tag{3}$$

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}(v^{n+1} + v^n) \tag{4}$$

(b) Picard Iteration

We start by denoting $v^n = v_1$, $u^n = u_1$, $u^{n+1} = u$, $v^{n+1} = v$ and the unknown in the nonlinear terms u^-. We insert this into (3) and (4) and solv with respect to the unknowns v and u

$$\begin{aligned} v &= v_1 + \tfrac{\Delta t}{m}\big(\tfrac{1}{2}(F^{n+1} + F^n) - bv|v_1| - \tfrac{1}{2}(s(u_1) + s(u^-))\big) \\ &= v_1 + \left(\frac{\Delta t}{2m}\right)\frac{F^{n+1} + F^n - s(u_1) - s(u^-))}{1 + \frac{dt}{m}b|v_1|} \end{aligned} \tag{5}$$

$$u = u_1 + \tfrac{\Delta t}{2}(v + v_1) \tag{6}$$

We start our iteration by guessing that $v^- = v_1$ and $u^- = u_1$. We solve the equations for v and u. We then set $v^- = v$ and $u^- = u$, and solve for v and u. The iteration continues until the desired precision or max number of iterations is reached.

Newtons Method

We denote our unknown values v^{n+1} and u^{n+1} as v and u. The values at the previus timestep is written as v_1 and u_1.

$$G_1(v,u) = v - v_1 - \frac{dt}{m}\left(\frac{1}{2}(F^{n+1} + F^n) - \frac{1}{2}\left(s(u) + s(u_1)\right) - v|v_1|\right) = 0 \quad (7)$$

$$G_2(v,u) = u - u^1 - \frac{dt}{2}(v + v_1) = 0 \quad (8)$$

We denote $\mathbf{G}(\mathbf{u}) = \begin{pmatrix} G_1(v,u) \\ G_2(v,u) \end{pmatrix}$. We linearize $\mathbf{G}$ by a Taylor expansion around a known value $\mathbf{u}^-$.

$$\hat{\mathbf{G}}(\mathbf{u}) = \mathbf{G}(\mathbf{u}^-) + J(\mathbf{u}^-)(\mathbf{u} - \mathbf{u}^-) = \mathbf{G}(\mathbf{u}^-) + J(\mathbf{u}^-)\delta\mathbf{u} = 0 \quad (9)$$

Where the Jacobian, J, is a 2×2 matrix with entries:

$$J_{i,j} = \frac{\partial G_i}{\partial x_j} \quad (10)$$

where $x_1 = v$ and $x_2 = u$.

$$\begin{aligned} J_{1,1} &= \frac{\partial G_1}{\partial v} = 1 - \frac{dt}{m}v^1 \\ J_{1,2} &= \frac{\partial G_1}{\partial u} = \frac{dt}{2m}s'(u) \\ J_{2,1} &= \frac{\partial G_2}{\partial v} = -\frac{dt}{2} \\ J_{2,2} &= \frac{\partial G_2}{\partial u} = 1 \end{aligned} \quad (11)$$

We can now make an initial guess for $\mathbf{u}^-$ and iterate until we reach desired precision:

solve $J\delta\mathbf{u} = -\mathbf{G}(\mathbf{u}^-)$ wrt $\delta\mathbf{u}$

update $\mathbf{u} = \mathbf{u}^{-} + \delta\mathbf{u}$

new guess $\mathbf{u}^{-} \leftarrow \mathbf{u}$

check precision How close is $\mathbf{G}(\mathbf{u})$ to 0 or how large is $\delta\mathbf{u}$