Linearize 1D problem with nonlinear coefficient

$$((1+u^2)u')' = 1 \qquad x \in (0,1), \qquad u(0) = u(1) = 0 \tag{1}$$

(a) Picard Iteration

We can either solve the differential equation as it stands or integrate it to get an algebraic equation instead of a differential equation (which is possible in this special case). Picard iteration applied to the differential equation results in

$$((1+(u^-)^2)u')' = 1 \tag{2}$$

To turn the differential equation in this case into an algebraic equation, we integrate twice:

$$\begin{aligned}
(1+u^2)u' &= x \\
\int_0^x (1+u^2)du &= \int_0^x x\,\mathrm{d}x \\
u(1+\frac{1}{3}u^2) &= \frac{1}{2}x^2 \\
u &= \frac{1}{2}\frac{x^2}{1+\frac{1}{3}u^2} \\
u &\approx \frac{1}{2}\frac{x^2}{1+\frac{1}{3}(u^-)^2}
\end{aligned} \tag{3}$$

(b) Newton's Method

First we apply Newton's method to the differential equation by replacing u by $u^- + \delta u$ and then linearize terms in δu. Inserting $u^- + \delta u$ for u gives

$$((1+(u^- + \delta u)^2)(u^- + \delta u)')' = 1$$

Expanding the parenthesis,

$$((1+(u^-)^2 + 2u^-\delta u + (\delta u)^2)((u^-)' + \delta u'))' = 1$$

Since $(\delta u)^2 << \delta u$ we can neglect all products of δu with itself or it's derivative.

$$((1+(u^-)^2)\delta u')' + (1+(u^-)^2 + 2u^-\delta u)((u^-)')' = 1$$

This is the same equation as for Picard iteration, but with an extra term on the left-hand side. We can also apply Newton's method to the algebraic equation arising from this differential equation.

$$F(u) = u(1 + \frac{1}{3}u^2) - \frac{1}{2}x^2 = 0 \tag{4}$$

We use a Taylor expansion:

$$\hat{F}(u) = F(u^-) + F'(u^-)(u - u^-) = 0 \tag{5}$$

Solving this for u gives:

$$u = u^- - \frac{F(u^-)}{F'(u^-)} \tag{6}$$

We need an expression for $F'(u)$:

$$F'(u) = 1 + u^2 \tag{7}$$

We are now able to find u from u^-:

$$\begin{aligned} u &= u^- - \frac{u^-(1 + \frac{1}{3}(u^-)^2) - \frac{1}{2}x^2}{1 + (u^-)^2} \\ u &= \frac{\frac{2}{3}(u^-)^3 - \frac{1}{2}x^2}{1 + (u^-)^2} \end{aligned} \tag{8}$$

Finite difference and Picard

In the previous exercise we derived a finite difference scheme on the form $A(u)u = b(u)$. Where the entries in A and b was:

$$\begin{aligned} A_{i,i+1} &= \frac{1}{dx^2}(1 + u_{i+1}u_i) \\ A_{i,i} &= -\frac{1}{dx^2}\left(2 + u_{i+1}u_i + u_i u_{i-1}\right) \\ A_{i,i-1} &= \frac{1}{dx^2}(1 + u_i u_{i-1}) \\ b_i &= 1 \end{aligned} \tag{9}$$

In the Picard Iteration we linearize the equation be replacing u with a guess for u denoted u^- in A and b.

$$A(u^-)u = b(u^-) \tag{10}$$

We solve the equation with respect to u, and use this as the next value for u^-.

Finite difference and Newton

$$F(u) = A(u)u - b(u) = 0 \tag{11}$$

Equation number i looks as follows (when using a geometric mean for $(1 + u^2)_{i+\frac{1}{2}}$):

$$F_i = [(1 + u_{i+1}u_i)u_{i+1} - (u_{i+1}u_i + u_i u_{i-1})\, u_i + (1 + u_i u_{i-1})u_{i-1}] - dx^2 \tag{12}$$

In Newtons method we need the Jacobian $\dfrac{\partial F_i}{\partial u_j}$:

$$\begin{aligned}
J_{i,i} &= \frac{\partial F_i}{\partial u_i} = (u_{i+1})^2 - 2(u_{i+1} + u_{i-1})u_i + (u_{i-1})^2 \\
J_{i,i-1} &= \frac{\partial F_i}{\partial u_{i-1}} = -u_i^2 + 2u_i u_{i-1} \\
J_{i,i+1} &= \frac{\partial F_i}{\partial u_{i+1}} = 2u_i u_{i+1} - u_i^2 \\
J_{i,j} &= 0 \,\text{for}\, j \notin [i-1, i, i+1]
\end{aligned} \tag{13}$$

We start by guessing at a value for u denoted u^- and solve:

$$J\delta u = -F(u^-) \tag{14}$$

Our new guess for u is then $u^- + \delta u$ or $u^- + \omega\delta u$ is we use a relaxation parameter.