On the, not entirely trivial, issue of getting uniformly distributed random coordinates within a sphere.

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The rand-functions in c++ can give us random numbers uniformly distributed between 0 and 1. But we need somehow to convert these numbers into coordinates of uniformly distributed particles within a sphere of radius R_0.

The easy way to do this would be to get a uniform distribution of particles in a cube of volume $(2R_0)^3$, and then just throw away all the particles outside the sphere, but this is not very satisfying. A more elegant way to do this is to use spherical coordinates.

$$x = r \sin\theta \cos\phi \tag{1}$$

$$y = r \sin\theta \sin\phi \tag{2}$$

$$z = r \cos\theta \tag{3}$$

$$\theta \in [0, \pi], \phi \in [0, 2\pi], r \in [0, R_0].$$

The problem is, it is not obvious how to get a uniform density of particles. If we distribute the particles uniformly in r, the density will be much higher in the centre. And if we distribute the particles uniformly in θ they will clump towards the poles.

There is a easy trick to solve this problem, we can use the fact that the volume element should be the same for any coordinate system. If we introduce three new variables with a uniform distribution between 0 and 1 we can write the volume element in therms of these coordinates, and tie each coordinate to one of the spherical coordinates. We get

$$r^2 \sin\theta dr d\theta d\phi = A du dv dw,$$

where A is just a constant and $u, v, w \in [0, 1]$. If we separate the coordinates we get three equations

$$r^2 dr = a du, \; \sin\theta d\theta = b dv, \; d\phi = c dw,$$

where a, b, c are just constants obeying $A = abc$.

Lets start with the easy one and just integrate up,

$$\begin{aligned} d\phi &= c dw \Rightarrow & (4)\\ \phi &= c w \Rightarrow & (5)\\ 2\pi &= c, & (6) \end{aligned}$$

where we have used that $\phi = 2\pi$ corresponds to $w = 1$.

Lets go on using the same approach

$$\begin{aligned} r^2 dr &= a du \Rightarrow & (7)\\ \int_0^r r'^2 dr' &= a \int_0^u du' \Rightarrow & (8)\\ \frac{1}{3} r^3 &= a u \Rightarrow & (9)\\ \frac{1}{3} {R_0}^3 &= a, & (10)\\ r &= \sqrt[3]{3 \frac{1}{3} {R_0}^3 u} \Rightarrow & (11)\\ r &= R_0 \sqrt[3]{u}. & (12) \end{aligned}$$

Confident that we are on the right track we go on to the last case

$$\begin{aligned} \sin\theta d\theta &= b dv \Rightarrow & (13)\\ \int_0^\theta \sin\theta' d\theta' &= b \int_0^u dv' \Rightarrow & (14)\\ 1 - \cos\theta &= b v \Rightarrow & (15)\\ 2 &= b, & (16)\\ \theta &= \arccos(1 - 2v). & (17) \end{aligned}$$

Thus if we now let our random numbers (between 0 and 1) represent the value of the three coordinates u, v, w, we can get values for the spherical coordinates. We can then use these to get back to the cartesian coordinates we want.