1D problem with nonlinear coefficient

$$((1+u^2)u')' = 1 \qquad x \in (0,1), \qquad u(0) = u(1) = 0 \tag{1}$$

(a) Finite Difference

We use a centered difference:

$$\begin{aligned} D_x(\alpha(u)D_x u) &= 1 \\ \frac{1}{dx^2}(\alpha(u)_{i+\frac{1}{2}}(u_{i+1}-u_i) - \alpha(u)_{i-\frac{1}{2}}(u_i-u_{i-1}) &= 1 \\ \frac{1}{dx^2}\left[\alpha(u)_{i+\frac{1}{2}}u_{i+1} - \left(\alpha(u)_{i+\frac{1}{2}} + \alpha(u)_{i-\frac{1}{2}}\right)u_i + \alpha(u)_{i-\frac{1}{2}}u_{i-1}\right] &= 1 \end{aligned} \tag{2}$$

A geometric mean inside α gives:

$$\alpha(u)_{i+\frac{1}{2}} = 1 + u_{i+1}u_i \tag{3}$$

Note that there are several other ways of taking the mean value. The scheme can then be written:

$$\frac{1}{dx^2}\left[(1+u_{i+1}u_i)u_{i+1} - (u_{i+1}u_i + u_iu_{i-1})\,u_i + (1+u_iu_{i-1})u_{i-1}\right] = 1 \tag{4}$$

We want this on the form $A(u)u = b(u)$. b is simply 1, while the entries in A is given as:

$$\begin{aligned} A_{i,i+1} &= \frac{1}{dx^2}(1+u_{i+1}u_i) \\ A_{i,i} &= -\frac{1}{dx^2}\left(2 + u_{i+1}u_i + u_iu_{i-1}\right) \\ A_{i,i-1} &= \frac{1}{dx^2}(1+u_iu_{i-1}) \end{aligned} \tag{5}$$

(b) Finite Element

The residual reads:

$$R = (\alpha(u)u')' - 1 \tag{6}$$

We use the Galerkin method to derive the variational form::

$$(R, v) = 0 \tag{7}$$

$$\begin{aligned} (\alpha(u)u')' - 1, v) &= 0 \\ ((\alpha(u)u')', v) &= (1, v) \\ (\alpha(u)u', v') &= -(1, v) \end{aligned} \tag{8}$$

In the integration by parts we have used that v is zero on the boundary. We use that $u = \sum_k c_k \phi_k$ and that $v \in \mathrm{span}\{\phi_i\}$. We integrate over the reference cell using the trapeziodal rule, where $\tilde{\phi}_k$ is k-th basisfunction mapped to a reference cell.

$$\int_{-1}^{1} \alpha(\sum \tilde{u}_k \tilde{\phi}_k) \tilde{\phi}'_r \tilde{\phi}'_s \frac{h}{2} \,\mathrm{d}X = \frac{2}{h} \int_{-1}^{1} \alpha(\sum \tilde{u}_k \tilde{\phi}_k) \tilde{\phi}_{r,X} \tilde{\phi}_{s,X} \,\mathrm{d}X \tag{9}$$

Since we have P1 elements $\tilde{\phi}_{s,X} \tilde{\phi}_{r,X} = \begin{cases} -\frac{1}{4} & s \neq r \\ \frac{1}{4} & s = r \end{cases}$

$$\begin{aligned} \frac{1}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_{-1}^{1} \alpha(\sum \tilde{u}_k \tilde{\phi}_k) \,\mathrm{d}X &= \frac{1}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Big(\alpha(\sum \tilde{u}_k \tilde{\phi}_k(-1)) + \alpha(\sum \tilde{u}_k \tilde{\phi}_k(1)) \Big) \\ &= \frac{1}{2h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} (\alpha(\tilde{u}_0) + \alpha(\tilde{u}_1)) \end{aligned}$$

Note that $\frac{1}{2}(\alpha(\tilde{u}_0) + \alpha(\tilde{u}_1))$ is the arithmetic mean of α over the element. If we write the system on the form $A(c)c = b$ (since we only use values at the nodes we can use that $u_i = c_i$) the entries in A are:

$$\begin{aligned} A_{i,-i} &= -\frac{1}{2h}(\alpha(u_i) + \alpha(u_{i-1})) = -\frac{1}{2h}(2 + u_i^2 + u_{i-1}^2) \\ A_{i,i} &= \frac{1}{2h}(\alpha(u_{i+1}) + 2\alpha(u_i) + \alpha(u_{i-1})) = \frac{1}{2h}(4 + u_{i+1}^2 + 2u_i^2 + u_{i-1}^2) \\ A_{i,i} &= -\frac{1}{2h}(\alpha(u_{i+1}) + \alpha(u_i)) = -\frac{1}{2h}(2 + u_{i+1}^2 + u_i^2) \end{aligned} \tag{10}$$

The entries in b are:

$$b_i = -\frac{h}{2} \begin{bmatrix} 1 \\ 2 \\ \vdots \\ 2 \\ 1 \end{bmatrix} \tag{11}$$

We see that if we use the arithmetic mean in the finite difference approach, the two methods would give the same scheme.