Exercise 14.

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We will here study the following multi-dimensional PDE:

$$\varrho c(T)T_t = \nabla \cdot (k(T)\nabla T)$$

in a spatial domain Ω, with a nonlinear Robin boundary condition:

$$-k(T)\frac{\partial T}{\partial n} = h(T)(T - T_s(t))$$

Using a BE we get the following:

$$\varrho c(T^n)T^n - \Delta t\nabla \cdot (k(T^n)\nabla T) = \varrho c(T^n)T^{n-1}$$

$$T^n - \frac{\Delta t\nabla \cdot (k(T^n)\nabla T)}{\varrho c(T^n)} = T^{n-1}$$

Introducing new notation T for T^n and $T^{(1)}$ for T^{n-1}

$$T - \frac{\Delta t\nabla \cdot (k(T)\nabla T)}{\varrho c(T)} = T^{(1)}$$

The variational formula to find $u \in V$ such that:

$$\int_\Omega (Tv - \frac{\Delta t(k(T)\nabla T)}{\varrho c(T)} \cdot \nabla v - T^{(1)}v)\mathrm{d}x$$

for $\forall v \in V$ We can have the nonlinear algebraic equations to follow from setting $\psi_i = v$ and using the representation $T = \sum_k c_k\psi_k$ which gives us:

$$F_i = \int_\Omega (T\psi_i - \frac{\Delta t(k(T)\nabla T)}{\varrho c(T)} \cdot \nabla \psi_i - T^{(1)}\psi_i)\mathrm{d}x \tag{1}$$

For the Picard iteration we can use the most recent approximation as previous setting for k and c thus giving us:

$$F_i \approx \hat{F}_i = \int_\Omega (T\psi_i - \frac{\Delta t(k(T^-)\nabla T)}{\varrho c(T^-)} \cdot \nabla \psi_i - T^{(1)}\psi_i)\mathrm{d}x$$

We can then solve it by $\sum_{j\in I_s} A_{i,j}c_j = b_i, i \in I_s$ to find the unknown $\{c_i\}_{i\in I_s}$ by inserting $T = \sum_j c_j\varphi_j$. Giving us:

$$A_{i,j} = \int_\Omega (\varphi_j \psi_i - \frac{\Delta t(k(T^-)\nabla\varphi_j)}{\varrho c(T^-)} \cdot \nabla\psi_i)\mathrm{d}x$$

$$b_i = \int_\Omega T^{(1)}\psi_i \mathrm{d}x$$

As for the Newton's method we will go on to evaluate F_i with the known value u^- for u

$$F_i \approx \hat{F}_i = \int_\Omega (T^-\psi_i - \frac{\Delta t(k(T^-)\nabla T^-)}{\varrho c(T^-)} \cdot \nabla\psi_i - T^{(1)}\psi_i)\mathrm{d}x$$

and also note:

$$\frac{\partial T}{\partial c_j} = \sum_k \frac{\partial}{\partial c_j} c_k \psi_k = \psi_j$$

$$\frac{\partial \nabla T}{\partial c_j} = \sum_k \frac{\partial}{\partial c_j} c_k \nabla\psi_k = \nabla\psi_j$$

Now differentiating our equation (1) we get (writing the final result only):

$$J_{i,j} = \frac{\partial F_i}{\partial c_j} = \int_\Omega \left(\psi_j\psi_i - \frac{(\Delta t k'(T)\psi_j \nabla T \cdot \nabla\psi_i + \Delta t k(T)\nabla\psi_j \cdot \nabla\psi_i)\varrho c(T) + \varrho c'(T)\psi_j(\Delta t k(T)\nabla T \cdot \nabla\psi_i)}{(\varrho c(T))^2} \right) \mathrm{d}x$$

which is the Jacobian which we can further evaluate by inserting our Newton notation T^- and T giving us:

$$J_{i,j} = \int_\Omega \left(\psi_j\psi_i - \frac{(\Delta t k'(T^-)\psi_j \nabla T^- \cdot \nabla\psi_i + \Delta t k(T^-)\nabla\psi_j \cdot \nabla\psi_i)\varrho c(T^-) + \varrho c'(T^-)\psi_j(\Delta t k(T^-)\nabla T^- \cdot \nabla\psi_i)}{(\varrho c(T^-))^2} \right) \mathrm{d}x$$