Peg Solitaire

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introduction

Let b be a board of size n, this can be represented as a n bit binary number. A 1 in the i'th position in b represents a peg in the i'th board position, a 0 represents a cavity.

$$b = b_0 b_1 ... b_{n-2} b_{n-1},\ b_i \in \{0, 1\}$$

definitions

Let $p(b) = x$ where x is the the number of ones in b, aka. the pop count.

Let $a(b, i) = b'$ be the funtion negating the i'th bit and its neighbours. $a(b, i)$ is only defined when $b_i = 1$ and $b_{i-1} \neq b_{i+1}$. Applying a equates to a move and removes a peg from the board $p(b') = p(b) - 1$.

We can apply a repeatedly to b until it is no longer defined for any i. The board, s, we get at the end we will call a solution. If we do this in every posible way we get a set $S(b)$ of all solutions for b. The set of optimal solutions is $O(b) = \{s \in S(b) \mid \forall_{s' \in S(b)}\ p(s) \leq p(s')\}$

Let $f(b, m) = b'$ be the function applying $a(b, i_j)$ m times, with each application taking b closer to an optimal solution. If a can not be applied m times before becoming undefined then $f(b, m) = b',\ b' \in O(b)$.

Let $\cdot$ be the board concatenation operator. $b = x \cdot y = x_0 x_1 .. x_{n-1} y_0 y_1 .. y_{n-1}$.

Let $b^x = b^{x-1} \cdot b$, where $b^0 = id$ is the board such that $b = b \cdot id = id \cdot b$.

Let $|b|$ be the board length operator. $|b| = n$.

Let $o(b) = f(b, |b|) \in O(b)$

Let x_i be the position of the i'th 1 in b. Let $w(b) = x_{p(b)} - x_1$

Let $r(b)$ be the function that reverses the board. $r(x \cdot y) = r(y) \cdot r(x),\ r(1) = 1,\ r(0) = 0$

Conjectures

1. $|b| \geq p(b)$
2. $p(b) \geq p \circ f(b, m)$
3. $p \circ f(b, m) \geq p(b) - m$
4. $p(b) \leq 1 \Rightarrow f(b, m) = b$
5. $p(b) = |b| \Rightarrow f(b, m) = b$
6. $p(b) \geq 0$
7. $w(b) \leq |b|$
8. $p(b) = p \circ r(b)$
9. $w(b) = w \circ r(b)$
10. $p \circ o(b) = p \circ o \circ r(b)$
11. $b = b' \cdot 0^3 \cdot b'', \quad p \circ o(b) = p \circ o(b') + p \circ o(b'')$
12. $b = b' \cdot b'', \quad p \circ o(b) \leq p \circ o(b') + p \circ o(b'')$
13. $w \circ f(b, x) \leq w(b) + 2$
14. $b = 0 \cdot 1^n \cdot 0,\ p \circ o(b) = \frac{n+1}{2}$

Proofs

1 $|b| \geq p(b)$

There can never be more ones in b than there are bits.

2 $p(b) \geq p \circ f(b, m)$

Each application of a removes a one. Thus f can never add ones.

3 $p \circ f(b, m) \geq p(b) - m$

Each application of a removes a one. $f(b, m)$ applies a m or less times. Thus $f(b, m)$ can never remove more than m ones.

4 $p(b) \leq 1 \Rightarrow f(b,m) = b$

If $p(b) \leq 0$ then there are not enough ones to apply a. Therefore f will apply a zero times, thus making no changes.

5 $p(b) = |b| \Rightarrow f(b,m) = b$

If $p(b) = |b|$ then there are not enough zeroes to apply a. Therefore f will apply a zero times, thus making no changes.

6 $p(b) \geq 0$

We can never have a negative number of ones on a board.

7 $w(b) \leq |b|$

Since $w(b) = x_{p(b)} - x_1$ where $0 \leq x_1 \leq x_{p(b)} \leq |b|$ then w is maximized when $x_1 = 0$. Then we have $w(b) = x_{p(b)} \leq |b|$.

8 $p(b) = p \circ r(b)$

Changing the order of the bits does not change the number of ones.

9 $w(b) = w \circ r(b)$

$w(b) = x_{p(b)} - x_1 = (|b| - x_1) - (|b| - x_{p(b)}) = w \circ r(b)$

10 $p \circ o(b) = p \circ o \circ r(b)$

If we can apply $a(b,i)$ then we can also apply $a(r(b), |b| - i - 1)$. So $f(b,m) = r \circ f(r(b), m)$ and then $o(b) = r \circ o \circ r(b)$ and by 8 we have $p \circ o(b) = p \circ r \circ o \circ r(b) = p \circ o \circ r(b)$.