MATHS260: Differential Equations

2017 Semester 2 Exam

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1. (13 marks) Consider the following differential equation:

$$\frac{dy}{dt} = \sqrt{y}\cos(t)$$

   (a) Find a family of solutions to this differential equation.

   (b) Consider this differential equation with the initial condition $y(0) = y_0$.

      i. Give a value of y_0 for which this initial value problem has a unique solution. Explain your answer.

      ii. Give a value of y_0 for which this initial value problem does not satisfy the conditions of the uniqueness theorem. Explain your answer.

      iii. Using the value of y_0 that you gave in part (ii), find a solution to the initial value problem.

      iv. Using the value of y_0 that you gave in part (ii), find a second solution to the initial value problem (that is not the same as the one you found in part (iii))

   (c) If an initial value problem does not satisfy the conditions of the uniqueness theorem, what does the uniqueness theorem tell you about solutions to the initial value problem?

Solution:

(a) Using separation of variables, as long as $\sqrt{y} \neq 0$.

$$\int \frac{1}{\sqrt{y}}\ dy = \int \cos(t)\ dt$$

$$2\sqrt{y} = \sin(t) + k$$

$$y = \frac{1}{4}(\sin(t) + k)^2$$

(b) i. Let $f(t,\ y) = \sqrt{y}\cos(t)$. $f(t,\ y)$ is continuous for all t and $y > 0$.

$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y}}\cos(t)$$

$\frac{\partial f}{\partial y}$ is continuous for all t and $y > 0$.
By the uniqueness theorem, as long as we have $y > 0$, both $f(t,\ y)$ and $\frac{\partial f}{\partial y}$ are continuous, so the IVP will have a unique solution.
Example: $y_0 = 1$

ii. $y_0 = 0$, $\frac{\partial f}{\partial y}$ is discontinuous, thus the uniqueness theorem conditions are not satisfied.

iii.

$$0 = \frac{1}{4}(\sin(0) + k)^2 \implies k = 0$$

$$y = \frac{1}{4}(\sin(t))^2$$

iv. $y = 0$ (the missing solution) is also a solution to the IVP.

(c) The existence and uniqueness theorem only give sufficient conditions, not necessary. i.e: we have no information when the existence and uniqueness theorems are not satisfied.

2. (15 marks) Consider the following second-order differential equation:

$$2\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = f(t) \tag{1}$$

   (a) Find the general solution to this equation when $f(t) = 0$.

   (b) Find a particular solution when $f(t) = 2t + 3$, and use it to find the general solution in this case.

   (c) Returning to the case where f(t) = 0, write equation (1) in the form

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

   where A is a 2×2 matrix.

   (d) Write down an initial condition for the system of equations that you found in part (c).

   (e) Write down an initial condition for equation (1) that corresponds to the linear system initial condition that you gave in part (d).

Solution:

(a) Solving the characteristic polynomial;

$$2\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = -\frac{1}{2} \pm \frac{1}{2}i$$

Since we have complex roots, we must find Y_R and Y_I.

$$e^{\lambda t} = e^{(-\frac{1}{2}\pm\frac{1}{2}i)t} = e^{-\frac{1}{2}t}\left[\cos\left(\frac{1}{2}t\right) + i\sin\left(\frac{1}{2}t\right)\right] = e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + ie^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right) = Y_R + iY_I$$

Let the general solution to the homogeneous equation be Y_H.

$$Y_H = k_1 Y_R + k_2 Y_I = k_1 e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + k_2 e^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right)$$

(b) UC set $= \{t, 1\}$
Guess that $y = at + b$ is a particular solution. $y' = a$ and $y'' = 0$.
LHS $= 2y'' + 2y' + y = (2a + b) + at$
RHS $= 2t + 3$
LHS = RHS when $a = 2$ and $b = -1$. Thus, our particular solution is:

$$y = 2t - 1$$

Let $Y(t)$ denote the general solution,

$$Y(t) = Y_H + y = k_1 e^{-\frac{1}{2}t}\cos\left(\frac{1}{2}t\right) + k_2 e^{-\frac{1}{2}t}\sin\left(\frac{1}{2}t\right) + 2t - 1$$

(c) Let $u = \frac{dy}{dt}$, thus $\frac{du}{dt} = \frac{d^2y}{dt^2}$.

$$2\frac{du}{dt} + 2u + y = 0 \implies \frac{du}{dt} = -u - \frac{1}{2}y$$

Thus we have the following system of differential equations:

$$\begin{cases} \frac{dy}{dt} = u \\ \frac{du}{dt} = -u - \frac{1}{2}y \end{cases}$$

$$\begin{pmatrix} \dot{y} \\ \dot{u} \end{pmatrix} = \begin{bmatrix} u \\ -u - \frac{1}{2}y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{pmatrix} y \\ u \end{pmatrix}$$

(d) At $t = 0$, set $u = 0$ and $y = 0$ as our initial condition.

(e) $y(0) = 0 \implies u(0) = y'(0) = 0$

3. (20 marks) Consider the following linear system of differential equations:

$$\frac{dx}{dt} = ax - 5y$$

$$\frac{dy}{dt} = x + y$$

(a) Suppose $a = 5$.
   i. Find the general solution to the linear system in terms of real-valued functions.
   ii. Sketch the phase plane.
   iii. Find a solution to the linear system which satisfies $x(0) = 9$, $y(0) = 4$.

(b) Find a value of a such that this system has more than one equilibrium solution. Find all equilibrium solutions in this case, and sketch the phase plane.

(c) Find a value of a for which this system has no nonzero solutions that tend towards the origin as $t \to \infty$ and no nonzero solutions that tend towards the origin as $t \to -\infty$ Sketch the phase plane for this value of a.

Solution:

(a) i.

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{bmatrix} 5 & -5 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Our characteristic polynomial is:

$$\lambda^2 - 6\lambda + 10 = 0$$

$$\lambda = 3 \pm i$$

We need to find the eigenvector(s) for when $\lambda = 3 + i$, we get:

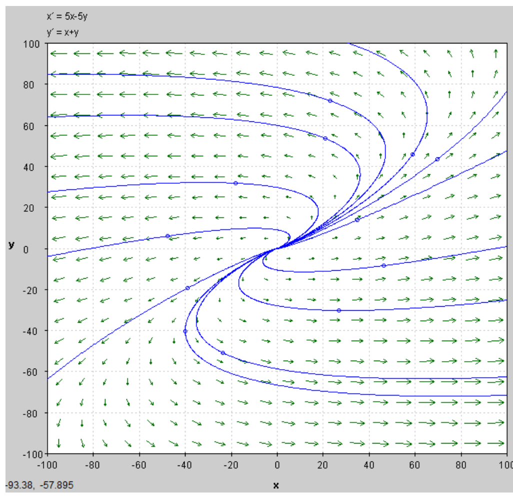
$$v = \begin{pmatrix} 1 \\ \frac{-2+i}{5} \end{pmatrix}$$

$$\begin{aligned} e^{\lambda t}v &= e^{(3+i)t}v = e^{3t}\left(\cos(t) + i\sin(t)\right)v \\ &= e^{3t}\begin{pmatrix} \cos(t) \\ -\frac{2}{5}\cos(t) - \frac{1}{5}\sin(t) \end{pmatrix} + ie^{3t}\begin{pmatrix} \sin(t) \\ \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) \end{pmatrix} \\ &= Y_R + iY_I \end{aligned}$$

Denote the general solution as $Y(t)$,

$$Y(t) = k_1Y_R + k_2Y_I = k_1e^{3t}\begin{pmatrix} \cos(t) \\ -\frac{2}{5}\cos(t) - \frac{1}{5}\sin(t) \end{pmatrix} + k_2e^{3t}\begin{pmatrix} \sin(t) \\ \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) \end{pmatrix}$$

ii. (0, 0) is a spiral source.



iii.

$$x(0) = k_1e^0\cos(0) + k_2e^0\sin(0) = 9 \implies k_1 = 9$$

$$y(0) = \frac{-18}{5} + \frac{1}{5}k_2 = 4 \implies k_2 = 38$$

$$Y(t) = 9e^{3t}\begin{pmatrix} \cos(t) \\ -\frac{2}{5}\cos(t) - \frac{1}{5}\sin(t) \end{pmatrix} + 38e^{3t}\begin{pmatrix} \sin(t) \\ \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) \end{pmatrix}$$

(b) We want a line of equilibria, this occurs when $\lambda = 0$. Thus we must have det(A)=0.

$$\det(A) = a + 5 = 0 \implies a = -5$$

When $a = -5$, the eigenvalues of A are $\lambda = 0$ and $\lambda = -4$.

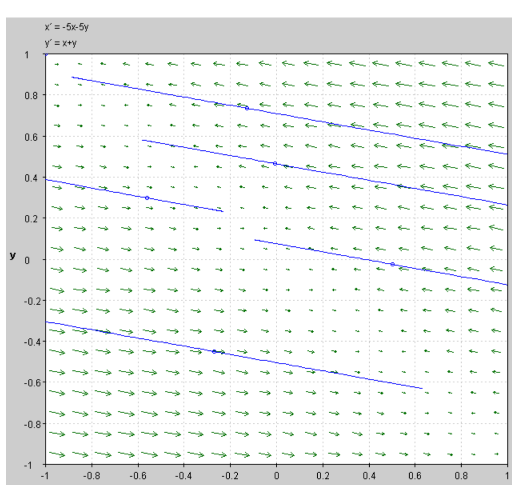
The eigenvector corresponding to the eigenvalue $\lambda = 0$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ Thus,

$$Y_1(t) = e^{0t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The line span $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ is the equilibria.

When $\lambda = 4$, the corresponding eigenvector is: $\begin{pmatrix} -5 \\ 1 \end{pmatrix}$.

Thus, the equilibria on the line span $\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}$ are sinks.



(c) The description in the question indicates that we want the origin to be a saddle. A saddle occurs when det(A)<0.

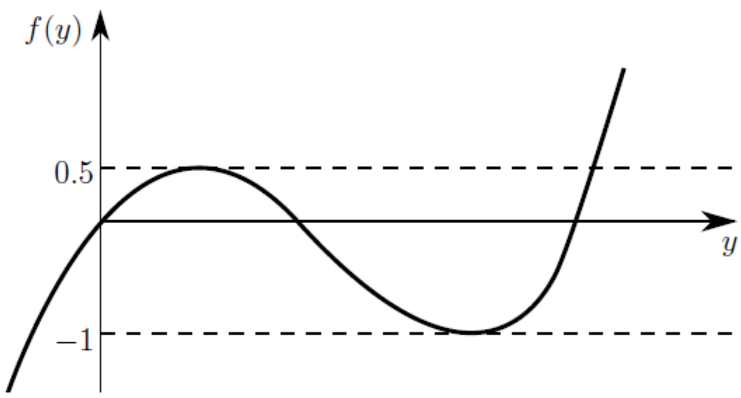
$$\det(A) = a + 5 < 0 \Longrightarrow a < -5$$

Example: Take $a = -6$, draw an arbitrary saddle.

4. (12 marks) Suppose that a differential equation takes the form

$$\frac{dy}{dt} = kf(y) - 1$$

where $k > 0$ is a parameter, and $f(y)$ has the following graph:



(a) Sketch the phase line when $k = 1$.
(b) Sketch the phase line when $k = 2$.
(c) Sketch the phase line when $k = 3$.
(d) At what value(s) of $k > 0$ will a bifurcation occur?
(e) Sketch a bifurcation diagram for this system.

Solution:

(a)

$$\frac{dy}{dt} = f(y) - 1$$

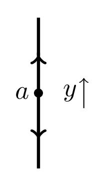
Equilibria occurs when $f(y) = 1$, which appears to happen for larger y. Denote that this happens when $y = a$.
For $a < 1$,

$$f(y) < 1 \Longrightarrow \frac{dy}{dt} < 0$$

For $a > 1$,

$$f(y) > 1 \Longrightarrow \frac{dy}{dt} > 0$$

$y = a$ is a source.



(b)

$$\frac{dy}{dt} = 2f(y) - 1$$

Equilibria occurs when $f(y) = 0.5$. On the graph of $f(y)$ provided, we can see that $f(y) = 0.5$ at two locations. Let these locations be a and b, where $a < b$.
For $y < a$,

$$f(y) < 0.5 \Longrightarrow \frac{dy}{dt} < 0.$$

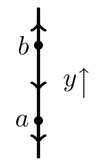
For $a < y < b$,

$$f(y) < 0.5 \Longrightarrow \frac{dy}{dt} < 0.$$

For $y > b$,

$$f(y) > 0.5 \Longrightarrow \frac{dy}{dt} > 0.$$

Thus, $y = a$ is a node and $y = b$ is a source.



(c)

$$\frac{dy}{dt} = 3f(y) - 1$$

Equilibria occurs when $f(y) = \frac{1}{3}$. On the graph of $f(y)$ provided, we can see that $f(y) = \frac{1}{3}$ at three locations. Let these locations be a, b and c, where $a < b < c$.
For $y < a$,

$$f(y) < \frac{1}{3} \implies \frac{dy}{dt} < 0.$$

For $a < y < b$,

$$f(y) > \frac{1}{3} \implies \frac{dy}{dt} > 0.$$

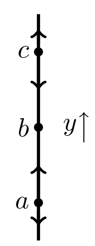
For $b < y < c$,

$$f(y) < \frac{1}{3} \implies \frac{dy}{dt} < 0.$$

For $y > c$,

$$f(y) > \frac{1}{3} \implies \frac{dy}{dt} > 0.$$

Thus, $y = a$ is a source, $y = b$ is a sink and $y = c$ is a source.

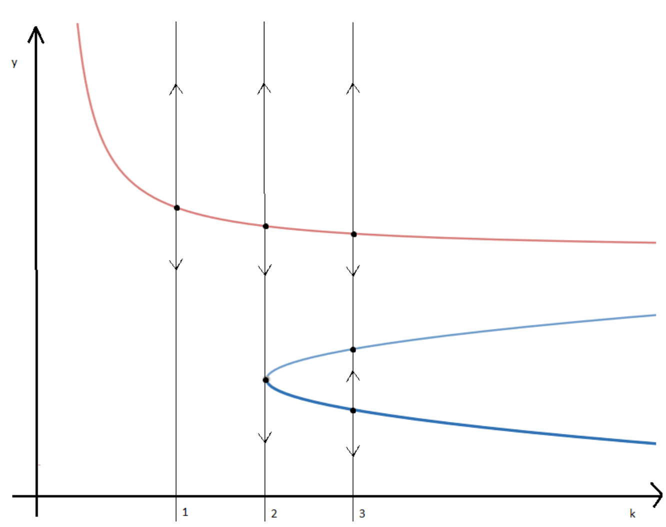


(d) Since $k > 0$, equilibria can only occur when $f(y) > 0$.

$$\frac{dy}{dt} = kf(y) - 1 = 0 \implies f(y) = \frac{1}{k}$$

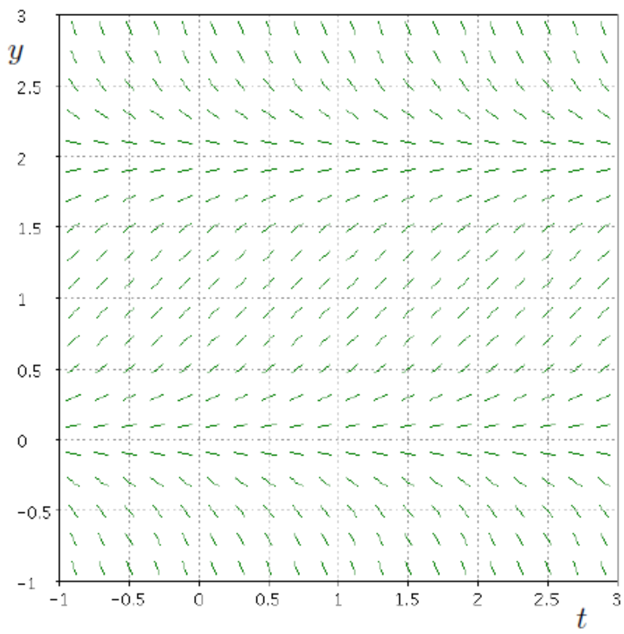
Notice that this is a horizontal line.
For $f(y) > \frac{1}{2}$, i.e: $k > 2$, there is only one such equilibria (where y is large).
For $0 < f(y) < \frac{1}{2}$, the horizontal line would intersect at 3 locations, thus there would be 3 equilibria. i.e: when $k > 2$.
When $k = 2$, we have already covered in (b), there are two such equilibria.
Hence, the bifurcation occurs at $k = 2$.

(e) Use your answers from (a), (b) and (c) to guide you;

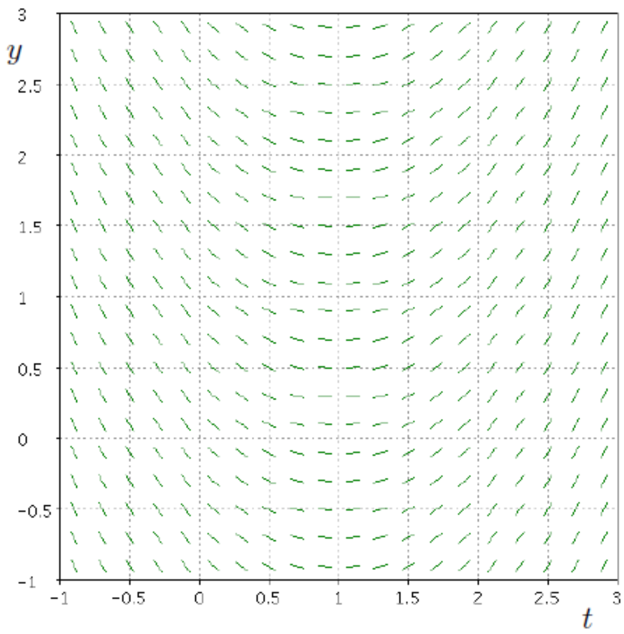


5. (8 marks) For each of the following direction field plots, write down a function $f(t,\ y)$ such that the differential equation $\frac{dy}{dt} = f(t,\ y)$ could have this direction field.

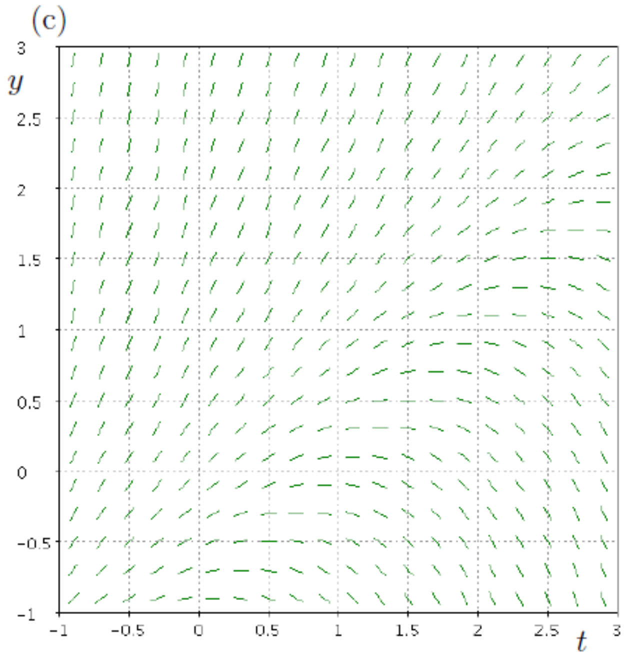
(a)



(b)



(c)



Solution:

(a) The top left direction field belongs to an autonomous differential equation, with equilibria at $y = 0$ and $y = 2$. From the direction field, for $0 < y < 2$, $\frac{dy}{dt} > 0$ and for $y < 0$ and $y > 2$ we have $\frac{dy}{dt} < 0$. This leads us to the following differential equation:

$$\frac{dy}{dt} = y(2 - y)$$

(b) $\frac{dy}{dt}$ does not depend on y explicitly, i.e: $\frac{dy}{dt} = f(t)$. Notice that $f(1) = 0$ and that for $t > 1$ we have $f(t) > 0$. For $t < 1$, $f(t) < 0$ and $f(t)$ increases as t increases. This leads us to the following differential equation:

$$\frac{dy}{dt} = t - 1$$

Notice that if we were to integrate this, we would get a quadratic which matches the direction field. Alternatively, we could try to find the quadratic from the direction field and then differentiate to get $\frac{dy}{dt}$.

(c) $\frac{dy}{dt}$ depends on both t and y explicitly. Notice that $\frac{dy}{dt} = 0$ when $y = t - 1$, for $y > t - 1$ we have $\frac{dy}{dt} > 0$ and for $y < t - 1$ we have $\frac{dy}{dt} < 0$. For fixed y, if t increases, $\frac{dy}{dt}$ decreases. This leads us to the following differential equation:

$$\frac{dy}{dt} = y - t + 1$$

6. (12 marks) In a large reserve in western Canada, a bear population eats salmon from the rivers in the reserve. The populations of salmon and bears can be described with the following model:

$$\frac{dS}{dt} = aS\left(1 - \frac{S}{K}\right) - bSB$$
$$\frac{dB}{dt} = -cB + dSB$$

where S is the population of salmon, B is the population of bears, and a, b, K, c, and d are positive constants. The time t is measured in years.

(a) Describe the physical significance of each term in the system.

(b) What is the long-term behaviour of the salmon population on the reserve if the initial bear population is zero?

(c) Suppose that fishing is introduced to the reserve, resulting in N salmon being harvested per year. How might you alter the model to reflect this?

(d) Briefly describe (i.e., in one or two paragraphs) some methods you could use to analyse these equations to get information about solutions to the model.

Solution:

(a) $\frac{dS}{dt}$ and $\frac{dB}{dt}$ are the rate of change of salmon and bear populations with tine t respectively.
$aS(1 - \frac{S}{K}$ denotes the logistic growth of salmon population in the absence of bears.
$-cB$ denotes the exponential decay in bear population when there are no salmon.
$-bSB$ and dSB denotes the interaction between the salmon and bear populations, with each interaction resulting in a loss of salmon numbers and an increase in bear population.

(b) The salmon population reaches the carrying capacity of K.

(c) Alter $\frac{dS}{dt}$ and keep $\frac{dB}{dt}$ the same.

$$\frac{dS}{dt} = aS\left(1 - \frac{S}{K}\right) - bSB - N$$

(d) We would need some sort of estimate on the parameters. Using the estimates, we could draw a phase portrait to visually see the solutions. It is best to use pplane to draw the phase portrait as we could easily adjust the parameters.

7. (20 marks) Consider the following system of equations:

$$\dot{x} = x(2 - x)$$
$$\dot{y} = 2 - x - y$$

A grid is provided on the next page. Use a copy of the grid for your answers to parts (b) and (c) of this question.

(a) Find all equilibrium solutions and determine their type (e.g., spiral source, saddle). For each equilibrium you find, draw a phase portrait showing the behaviour of solutions near that equilibrium.

(b) Find the nullclines for the system and sketch them on the grid provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait of the system on the grid provided. Your phase portrait should show the behaviour of solutions near the equilibria, and should show a representative selection of solution curves (i.e., some solutions in each different part of the phase plane).

(d) Describe the long term behaviour of the solutions passing through the following initial conditions:

i. $(x(0); y(0)) = (1, 1)$;
ii. $(x(0); y(0)) = (0, -1)$.

Solution:

(a) For equilibrium solutions, solve the system of equations:

$$\dot{x} = x(2 - x) = 0$$
$$\dot{y} = 2 - x - y = 0$$

which yields two equilibrium: (0, 2) and (2, 0).

$$J(x, y) = \begin{bmatrix} 2 - 2x & 0 \\ -1 & -1 \end{bmatrix}$$
$$J(0, 2) = \begin{bmatrix} 2 & 0 \\ -1 & -1 \end{bmatrix}$$

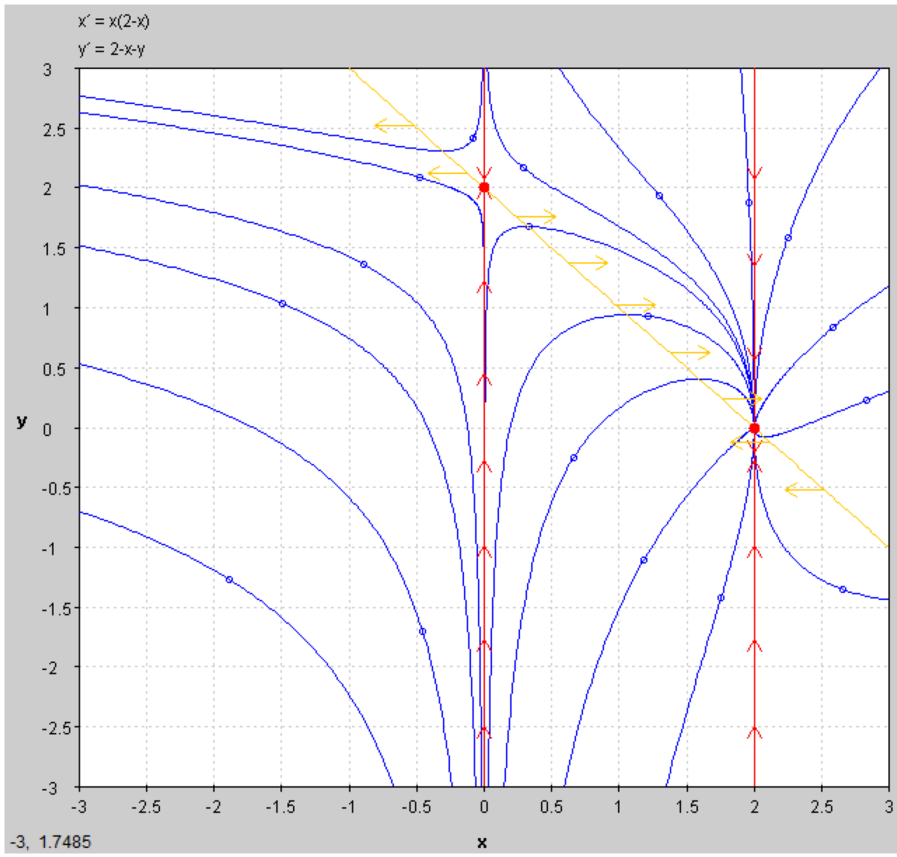
for which has eigenvalues -1 and 2, thus (0, 2) is a saddle.

$$J(2, 0) = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix}$$

for which has eigenvalues -1 and -2, thus (0, 2) is a nodal sink.

(b) The x-nullclines are $x = 0$ and $x = 2$.
The y-nullcline is $y = 2 - x$.

(c) Remember that your phase portrait doesn't have to look perfect.



(d) i. As $t \to \infty$, $(x, y) \to (2, 0)$.
As $t \to -\infty$, $(x, y) \to (0, \infty)$.

ii. As $t \to \infty$, $(x, y) \to (0, 2)$.
As $t \to -\infty$, $(x, y) \to (0, -\infty)$.