MATHS260: Differential Equations

2018 Semester 1 Exam

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1. (12 marks)

   (a) Find a solution to the initial value problem

   $$\frac{dy}{dt} = ty + 2t,\ \ y(0) = 3$$

   (b) Could there be any other solutions to this initial value problem? Explain your answer carefully.

   Solution:

   (a) Using separation of variables, remind yourself that we assume $y \neq -2$.

   $$\int \frac{1}{y+2}\ dy = \int t\ dt$$

   $$y = -2 + Ae^{\frac{1}{2}t^2}$$

   Using the initial condition, we find that $A = 5$.

   $$y = -2 + 5e^{\frac{1}{2}t^2}$$

   (b) Let $f(t, y) = \frac{dy}{dt} = ty + 2t$,

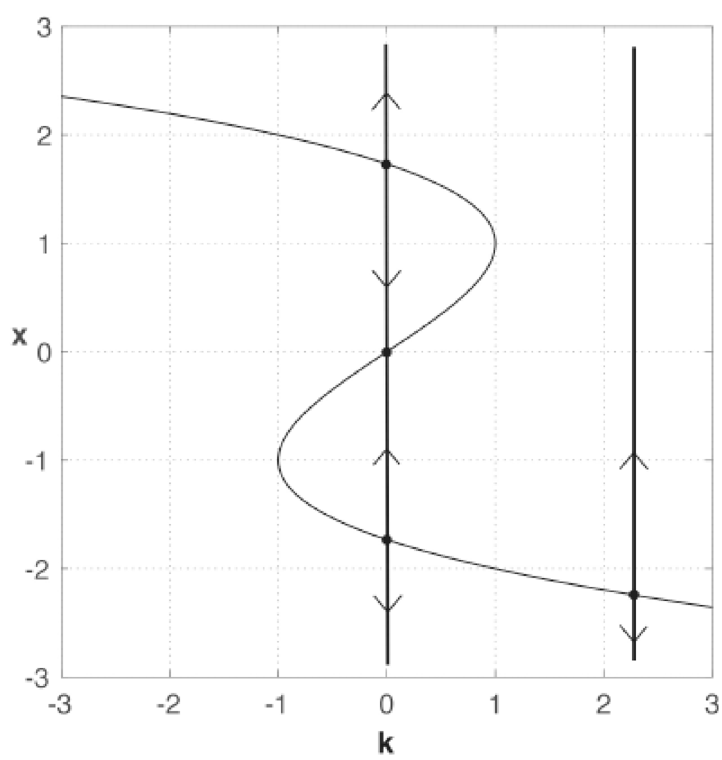
   $$\frac{\partial f}{\partial y} = t$$

   $f(t, y)$ and $\frac{\partial f}{\partial y}$ are continuous for all $t, y \in \mathbf{R}$. Hence, by the existence and uniqueness theorem, the solution we found in (a) is the only solution to the IVP.

2. (10 marks) The figure below shows a bifurcation diagram for a differential equation

   $$\frac{dx}{dt} = f_K(x)$$

   where k is the bifurcation parameter.



   (a) Find all values of k for which there is a bifurcation.

   (b) Describe the long term behaviour of the solution to the differential equation if: (i) $k = 0$ and $x(0) = 1$; (ii) $k = 0$ and $x(2) = 0$; (iii) $k = 2$ and $x(0) = -1$.

   (c) Sketch the phase lines for the cases: (i) $k = -2$; (ii) $k = 1$.

   Solution:

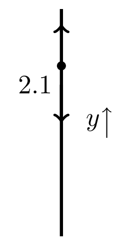
   (a) Bifurcations occur at $k = -1$ and $k = 1$.

   (b) i. Assume that the equilibrium are $x = -1.75,\ 0,\ 1.75$.
   As $t \to \infty$, $x \to 0$.
   As $t \to -\infty$, $x \to 1.75$

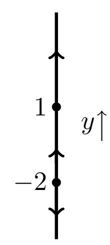
   ii. As $t \to \infty$, $x \to 0$.
   As $t \to \infty$, $x \to ?$. We don't know what direction we came from since $x = 0$ is a sink.

iii. Assume that the equilibrium is at $x = -2.1$.
As $t \to \infty$, $x \to \infty$.
As $t \to -\infty$, $x \to -2.1$

(c) i. Assume that the equilibrium is at $x = 2.1$. We need our equilibrium to match the top equilibrium for when $k = 0$ which is provided. Thus, $x = 2.1$ is a source.

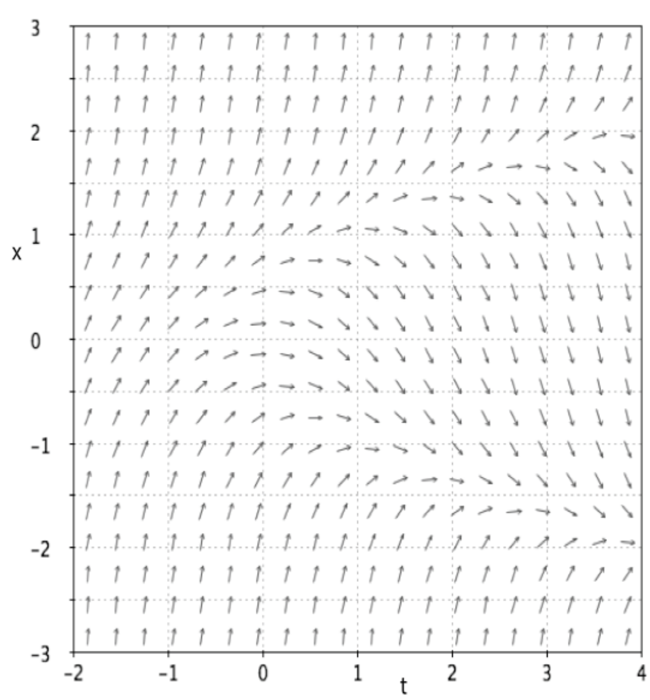


ii. The equilibrium are at $x = -2$ and $x = 1$.
The bottom equilibria are sources for both $k = 0$ and $k = 2$. Thus, $x = -2$ is a source.
For $x = 1$, recall that this is a bifurcation point, two equilibria "squash" into a single equilibria which results in two upward arrows.



3. (12 marks) The figure below shows the direction field for a differential equation

$$\frac{dx}{dt} = (t, x)$$

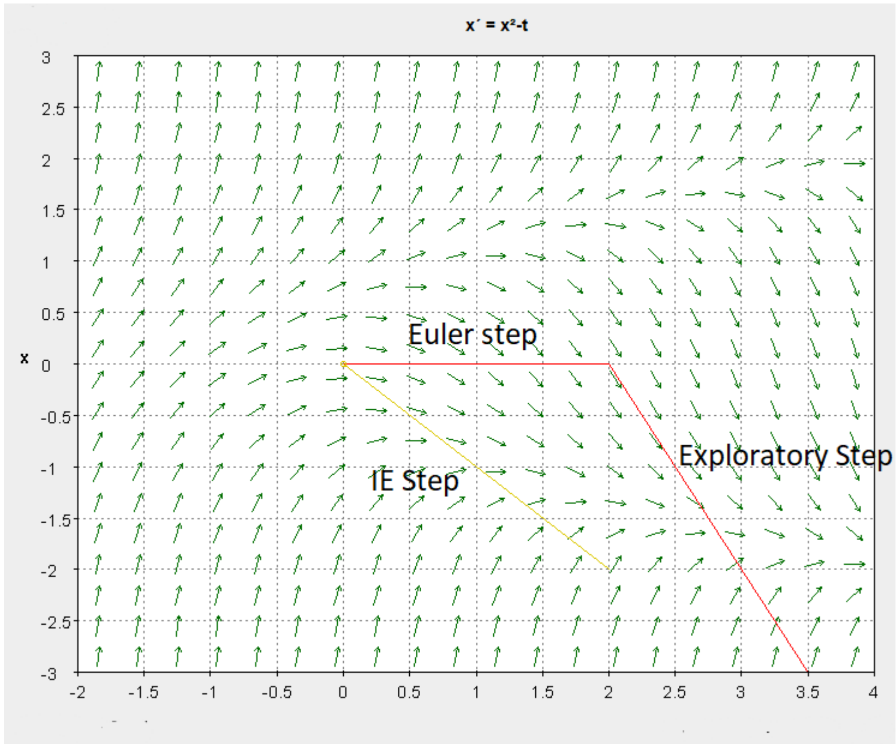


(a) On the answer sheet at the end of the question paper carefully draw the approximate solution that you would obtain if you used one step of Improved Euler's method with step length $h = 2$ to calculate the solution to this differential equation with initial condition $x(0) = 0$. Note: You do not need to do any computations to answer this part of the question, but you should explain how you obtained your answer.

(b) On the second copy of the answer sheet at the end of the question paper, find the points in the (t, x)-plane where the direction field has zero slope, and write down a formula for a curve that goes through these points. Hence, write down a differential equation that might have been used to produce the direction field shown in the figure.

Solution:

(a) The slope at (0,0) looks to be approximately 0, therefore the Euler step with $h = 2$ takes us to (2,0). The slope at (2,0) looks to be approximately -2. The IE method requires the average of these two slopes; -1. Now we go back to (0,0) and take a step with the slope -1 with step size $h = 2$. We should end up around (2, -2).



(b) Notice that the slope is 0 when $x = \pm\sqrt{t}$. This prompts us to guess:

$$\frac{dx}{dt} = x^2 - t.$$

4. (22 marks) This question is about the system of differential equations

$$\frac{d\mathbb{Y}}{dt} = \begin{bmatrix} 3 & 1 \\ k & 1 \end{bmatrix} \mathbb{Y}$$

(a) Consider the case $k = 0$.

i. Determine the type of equilibrium at (0,0) (e.g. sink, spiral source).

ii. Write down the general solution.

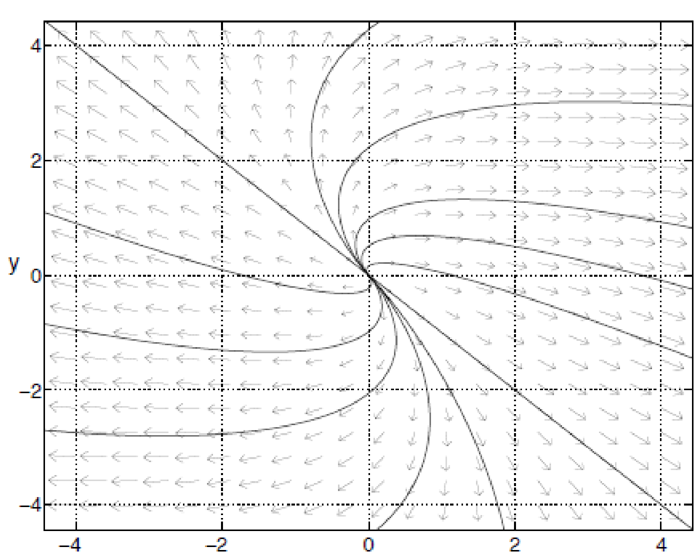
iii. Sketch a phase portrait for the system.

(b) Now consider the case $k = -3$.

i. In this case, the matrix has an eigenvalue $2 + i\sqrt{2}$ with eigenvector $\begin{pmatrix} 1 \\ -1 + i\sqrt{2} \end{pmatrix}$ and an eigenvalue $2 - i\sqrt{2}$ with eigenvector $\begin{pmatrix} 1 \\ -1 - i\sqrt{2} \end{pmatrix}$. Determine the type of equilibrium at (0,0). (e.g. sink, spiral source).

ii. Write down the general solution in terms of real-valued functions.

(c) For a given value of k the following phase portrait was produced. What value of k do you think was used? Justify your answer.



Solution:

(a) i.

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}$$

The eigenvalues of A are $\lambda = 1$ and $\lambda = 3$.

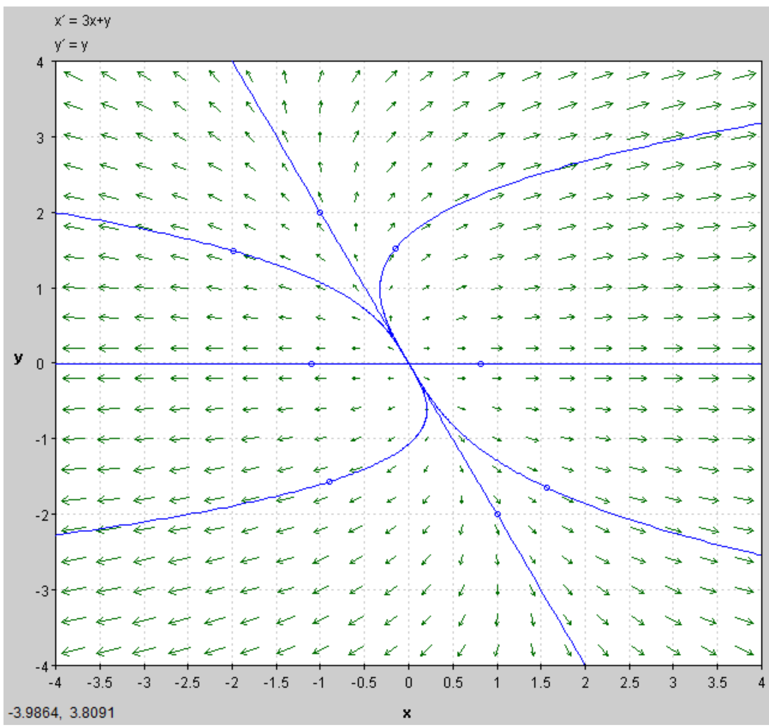
The corresponding eigenvectors are $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. respectively.

(0,0) is a nodal source as both eigenvalues are positive.

ii. Denote $Y(t)$ as the general solution,

$$Y(t) = k_1 e^t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } k_1, k_2 \in \mathbf{R}$$

iii. As long as the straight line solutions are drawn correctly, any nodal source will do.



(b) i. (0,0) is a spiral source since the real part of the eigenvalues are positive.

ii.

$$e^{(2+i\sqrt{2})t} \begin{pmatrix} 1 \\ -1 + i\sqrt{2} \end{pmatrix} = e^{2t} \begin{pmatrix} \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + ie^{2t} \begin{pmatrix} \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}$$

$$= Y_R + iY_I$$

Therefore, the general solution is:

$$Y(t) = k_1 Y_R + k_2 Y_I = k_1 e^{2t} \begin{pmatrix} \cos(\sqrt{2}t) \\ -\cos(\sqrt{2}t) - \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + k_2 e^{2t} \begin{pmatrix} \sin(\sqrt{2}t) \\ -\sin(\sqrt{2}t) + \sqrt{2}\cos(\sqrt{2}t) \end{pmatrix}$$

(c) The phase portrait shows that (0,0) is an improper source. This occurs when we have an eigenvalue with algebraic multiplicity 2 but geometric multiplicity 1. i.e: we have a repeated eigenvalue but only one distinct eigenvector (looks like only one straight line solution).

Let $A = \begin{bmatrix} 3 & 1 \\ k & 1 \end{bmatrix}$, the characteristic polynomial is: $\lambda^2 - 4\lambda + (-k+3) = 0$.

$$\lambda = \frac{4 \pm \sqrt{16 - 4(-k+3)}}{2}$$

In order to have a single eigenvalue, we need the discriminant to equal to 0.

$$16 - 4(-k+3) = 0 \Longrightarrow k = -1$$

Thus, when $k = -1$, the eigenvalue is $\lambda = 2$ which explains the source behaviour in the diagram.

5. (26 marks) Consider the following system of equations:

$$\frac{dx}{dt} = -2y$$

$$\frac{dy}{dt} = x - x^2 - y$$

A grid is provided on the answer sheet attached to the back of the question paper. Use the grid for your answers to parts (b), (c), and (d) of this question.

(a) Find all equilibrium solutions and determine their types (e.g., saddle, spiral source). For each equilibrium that you find, draw a phase portrait showing the behaviour of solutions near that equilibrium.

(b) Find the nullclines for the system and sketch them on the answer sheet provided. Show the direction of the vector field in the regions between the nullclines and on the nullclines themselves.

(c) Sketch the phase portrait for the system on the answer sheet provided. Your phase portrait should show the behaviour of solutions near the equilibria, and should show some solutions in each different part of the phase plane. Include the solution that satisfies the initial condition $(x(0), y(0)) = (0.5, 0)$.

(d) Describe the long-term behaviour (as t increases) of the solution with initial condition $(x(0), y(0)) = (0.5, 0)$.

Solution:

(a) Setting $\dot{x} = 0$ yields: $y = 0$.
Setting $\dot{y} = 0$ yields: $y = x(x-1)$.
For equilibria, we require both $\dot{x} = 0$ and $\dot{y} = 0$. Thus, we have two equilibria: $(x, y) = (0, 0)$ and (1,0).

$$J(x,y) = \begin{bmatrix} 0 & -2 \\ 1-2x & -1 \end{bmatrix}$$

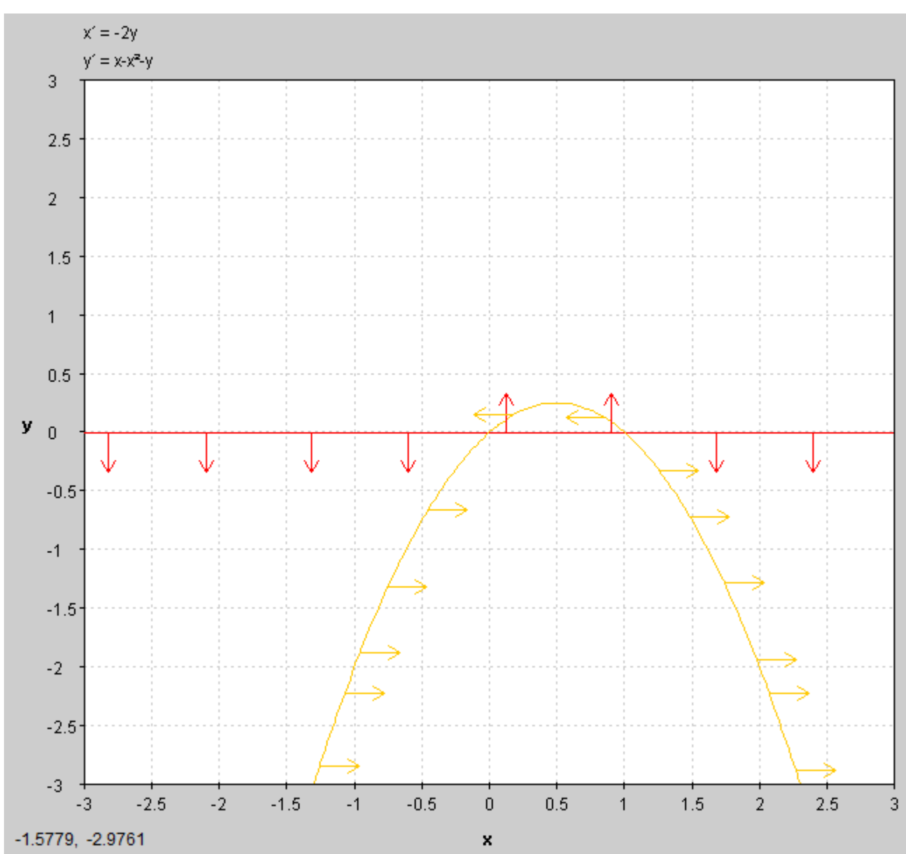
$$J(0,0) = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix}$$

for which has eigenvalues $\lambda = \frac{-1 \pm \sqrt{7}i}{2}$, thus (0,0) is a spiral sink.

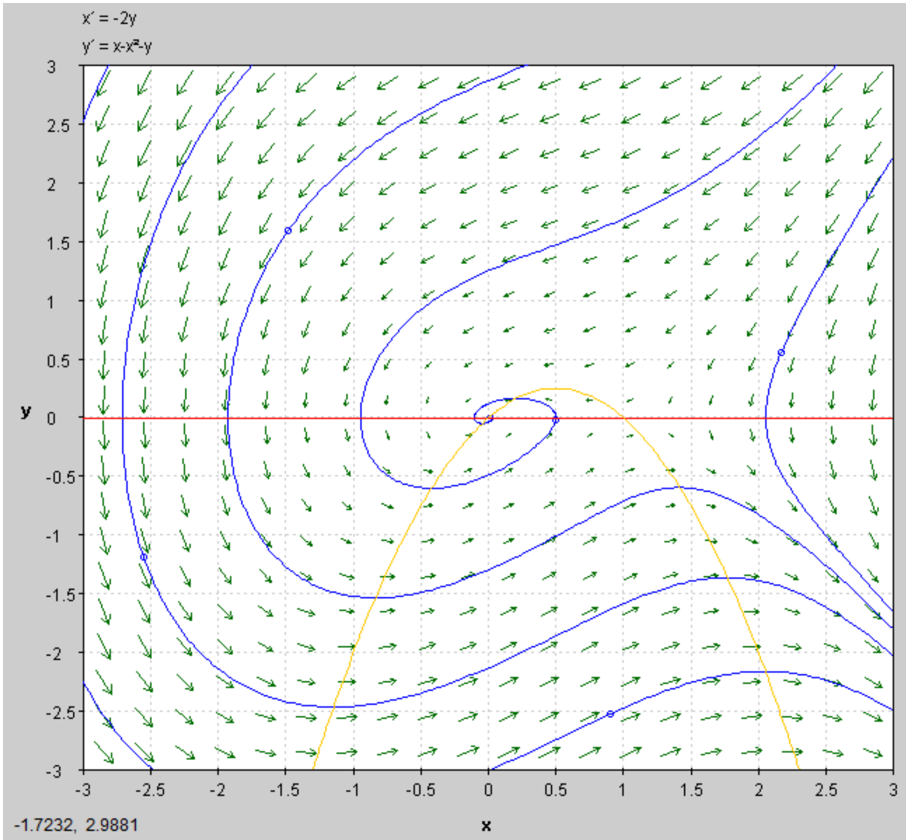
$$J(1,0) = \begin{bmatrix} 0 & -2 \\ -1 & -1 \end{bmatrix}$$

for which has eigenvalues $\lambda = -2$ and $\lambda = 1$, thus (1,0) is a saddle.

(b) x-nullcline: $y = 0$.
y-nullcline: $y = x(1-x)$.
$\dot{x} > 0$ when $-2y > 0 \implies y < 0$
$\dot{y} > 0$ when $x - x^2 - y > 0 \implies y < x(1-x)$



(c) Remember that your phase portrait doesn't have to look perfect.



(d) As $t \to \infty$, $(x, y) \to \infty$.

6. (18 marks)

(a) A sailing boat is floating on the harbour. The following equation is used to measure the vertical displacement, x, of the boat from its equilibrium position.

$$5\frac{d^2x}{dt^2} + x = 0$$

i. Find the general solution to this differential equation.

ii. Suppose that a storm results in a forcing function of the form $f(t) = te^{-t}$, i.e. consider the model

$$5\frac{d^2x}{dt^2} + x = te^{-t}$$

Find the general solution to this differential equation.

(b) A more general and realistic model for the motion of the boat is:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + c\left(\frac{dx}{dt}\right)^2 + kx = 0$$

where m, b, c and k are positive constants.

i. Briefly explain what each term in the equation might mean physically.

ii. Rewrite this equation as a two-dimensional system of first-order differential equations.

Solution:

(a) i. The characteristic polynomial is: $5\lambda^2 + 1 = 0$.
The solutions to this quadratic are: $\lambda = \pm\frac{1}{\sqrt{5}}i$.

$$e^{\lambda t} = e^{\frac{1}{\sqrt{5}}it} = \cos\left(\frac{1}{\sqrt{5}}t\right) + i\sin\left(\frac{1}{\sqrt{5}}t\right) = x_R + ix_I$$

The general solution is:

$$x(t) = k_1x_R + k_2x_I = k_1\cos\left(\frac{1}{\sqrt{5}}t\right) + k_2\sin\left(\frac{1}{\sqrt{5}}t\right)$$

ii. We need to find a particular solution. UC Set $= \{e^{-t}, te^{-t}\}$.
Guess a solution of the form:

$$x_p = x = ate^{-t} + be^{-t}$$

$$x' = -ae^{-t} + b\left(e^{-t} - te^{-t}\right)$$

$$x'' = e^{-t}(a-2b) + bte^{-t}$$

LHS: $5x'' + x = e^{-t}(6a - 10b) + 6bte^{-t}$
RHS: te^{-t}
In order for LHS=RHS, we must have $a = \frac{1}{6}$ and $b = \frac{5}{18}$. Thus,

$$x_p = \frac{1}{6}te^{-t} + \frac{5}{18}e^{-t}$$

The general solution is:

$$x(t) = x_p + x_h = \frac{1}{6}te^{-t} + \frac{5}{18}e^{-t} + k_1\cos\left(\frac{1}{\sqrt{5}}t\right) + k_2\sin\left(\frac{1}{\sqrt{5}}t\right)$$

(b) i. m is the mass of the boat.
$\frac{d^2x}{dt^2}$ is the vertical acceleration of the boat.
b is the damping co-efficient due to the water.
$\frac{dx}{dt}$ is the vertical velocity of the boat.
c is the damping co-efficient due to the air.
k is a constant due to the buoyancy of the boat.
x is the vertical displacement of the boat.

ii. Let $u = \frac{dx}{dt}$, then $\frac{du}{dt} = \frac{d^2x}{dt^2}$.

$$m\frac{du}{dt} + bu + cu^2 + kx = 0$$

$$\frac{du}{dt} = \frac{-bu - cu^2 - kx}{m}$$

Our system of differential equations is:

$$\begin{cases} \frac{du}{dt} = \frac{-bu-cu^2-kx}{m} \\ \frac{dx}{dt} = u \end{cases}$$