MATHS340: Real and Complex Calculus
Tutorial 2
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1. Use the taylor command in MuPAD to

   nd the Taylor series expansion of $\tan(x)$ about $x = 0$ up to and including the x^7 term. (If you forgot how to use MuPAD, refer to the guide.mn file on Canvas)
   Solution:

   ```
   taylor(tan(x),x=0,7)
   ```
   $$x + \frac{x^3}{3} + \frac{2\,x^5}{15} + \frac{17\,x^7}{315} + O\!\left(x^9\right)$$

2. Let $f(x, y, z) = 3xy^2 + xe^z - \sin(zy)$.

   (a) Find the linear Taylor polynomial for f at the point $(2, 2, 2)$.

   (b) Check your answer using MuPAD with the command mtaylor(f,[x=2,y=2,z=2],2).

   Solution: Using the definition of a multi-variable Taylor series:

   $$P_1(x, y, z) = f(2, 2, 2) + \nabla f(2, 2, 2) \cdot (x - 2, y - 2, z - 2)$$
   $$= f(2, 2, 2) + f_x(2, 2, 2)(x - 2) + + f_y(2, 2, 2)(y - 2) + f_z(2, 2, 2)(z - 2)$$

   $$f_x = 3y^2 + e^z, \ f_y = 6xy - z\cos(zy), \ f_z = xe^z - y\cos(zy)$$
   $$f_x(2, 2, 2) = 12 + e^2, f_y(2, 2, 2) = 24 - 2\cos(4), \ f_y(2, 2, 2) = 2e^2 - 2\cos(4)$$

   Therefore,

   $$P_1(x, y, z) = (24 + 2e^2 - \sin(4)) + (12 + e^2)(x - 2) + (24 - 2\cos(4))(y - 2) + (2e^2 - 2\cos(4))(z - 2)$$
   $$= 39.53 + 19.39(x - 2) + 25.31(y - 2) + 16.09(z - 2)$$
   $$= 19.39x + 25.31y + 16.09z - 82.04$$

   Using MuPAD,

   ```
   mtaylor(3*x*y^2 + x*exp(z) - sin(z*y),[x=2,y=2,z=2],2)
   ```
   $$2\,e^2 - \sin(4) - (z - 2)\left(2\,\cos(4) - 2\,e^2\right) - (2\,\cos(4) - 24)\,(y - 2) + \left(e^2 + 12\right)(x - 2) + 24$$
   ```
   float(%)
   ```
   $$19.3890561\,x + 25.30728724\,y + 16.08539944\,z - 82.02857087$$

3. Find the tangent plane for the function $f(x, y) = xy^2 - x^3 + 10$ at the point $(1, \frac{3}{2})$.
   Solution:
   Let $z = f(x, y) = xy^2 - x^3 + 10$ and define $F(x, y, z) = z - xy^2 + x^3 - 10 = 0$.

   $$z(1, \frac{3}{2}) = \frac{9}{4} - 1 + 10 = \frac{45}{4}$$
   $$F_x = -y^2 + 3x^2, \ F_y = -2xy, \ F_z = 1$$
   $$F_x(1, \frac{3}{2}, \frac{45}{4}) = \frac{3}{4}, \ F_y(1, \frac{3}{2}, \frac{45}{4}) = -3, \ F_z(1, \frac{3}{2}, \frac{45}{4}) = 1$$

   The surface at $(1, \frac{3}{2}, \frac{45}{4})$ is perpendicular to the gradient.

   $$\nabla F(1, \frac{3}{2}, \frac{45}{4}) \cdot (x - 1, y - \frac{3}{2}, z - \frac{45}{4}) = 0$$
   $$\frac{3}{4}(x - 1) - 3(y - \frac{3}{2}) + (z - \frac{45}{4}) = 0$$
   $$z = -\frac{3}{4}x + 3y + \frac{15}{2}$$

4. Find $\|\mathbf{u} \times \mathbf{v}\|$, where $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{6}$.
   Solution:
   Using the formula: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \ \|\mathbf{v}\| \ \sin(\theta)$, where θ is the angle between $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

   $$\|\mathbf{u} \times \mathbf{v}\| = 2 \times 5 \times \sin(\frac{\pi}{6}) = 5$$

5. Let $\mathbf{u} = [4, 0, 0]$ and $\mathbf{v} = [0, 0, 3]$. Calculate the cross product $\mathbf{u} \times \mathbf{v}$ without using determinants.

    Solution: Using the formula: $\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \, sin(\theta)\hat{e}$, where $\hat{e}$ is the unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ while obeying the right hand rule.

    Notice that $\mathbf{u}$ and $\mathbf{v}$ are just scalar multiples of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ respectively. So $\hat{e}$ is $-\hat{\mathbf{j}}$. Trivially, $\|\mathbf{u}\| = 4$, $\|\mathbf{u}\| = 3$ and $\theta = \frac{\pi}{2}$.

    $$\mathbf{u} \times \mathbf{v} = 4 \times 3 sin(\frac{\pi}{2}) - \hat{\mathbf{j}} = -12\hat{\mathbf{j}} = [0, -12, 0]$$

6. Suppose $\mathbf{u} = [t, cos(2t), e^t]$. Use the diff command to find $d\mathbf{u}/dt$. You can find out about the diff command by typing ?diff.

    Solution:

    ```
    u := matrix([t,cos(2*t),exp(t)]):
    diff(u,t)
    ```
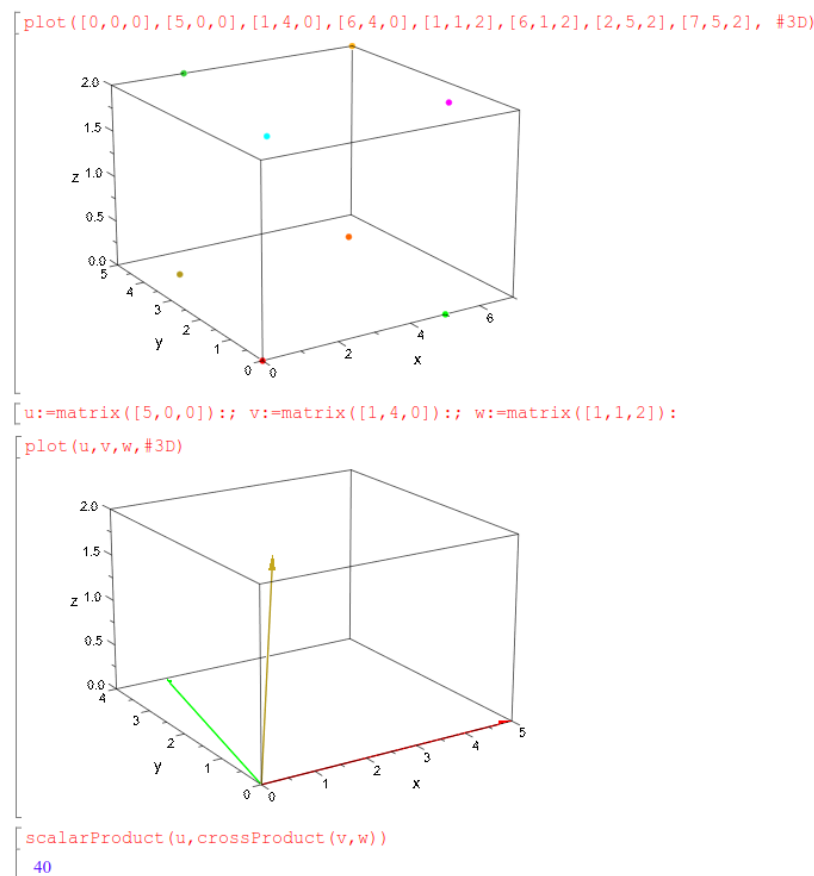 $$\begin{pmatrix} 1 \\ 2 \sin(2 \, t) \\ e^t \end{pmatrix}$$

7. If $\mathbf{u} = [t, cos(2t), e^t]$, $\mathbf{v} = [-t, 0, (1+t)^{-1}]$, what is $d(\mathbf{u} \times \mathbf{v})/dt$? **Solution:** Using the same $\mathbf{u}$ in the MuPAD code in Question 6;

    ```
    v := matrix([-t, 0, (1+t)^-1])
    ```
    $$\begin{pmatrix} -t \\ 0 \\ \frac{1}{t+1} \end{pmatrix}$$

    ```
    diff(crossProduct(u,v),t)
    ```
    $$\begin{pmatrix} -\frac{\cos(2\,t)}{(t+1)^2} - \frac{2\sin(2\,t)}{t+1} \\ \frac{t}{(t+1)^2} - e^t - \frac{1}{t+1} - t\,e^t \\ \cos(2\,t) - 2\,t\,\sin(2\,t) \end{pmatrix}$$

8. Plot the following points in $\mathbb{R}^3$ using MuPAD:

    $$(0, 0, 0), (5, 0, 0), (1, 4, 0), (6, 4, 0), (1, 1, 2), (6, 1, 2), (2, 5, 2), (7, 5, 2):$$

    These are the vertices of a parallelopiped. What three vectors can be used to define the paralellopiped? Find the volume of the parallelopiped using a scalar triple product. **Solution:**


    ```
    plot([0,0,0],[5,0,0],[1,4,0],[6,4,0],[1,1,2],[6,1,2],[2,5,2],[7,5,2], #3D)
    ```



    ```
    u:=matrix([5,0,0]):; v:=matrix([1,4,0]):; w:=matrix([1,1,2]):
    plot(u,v,w,#3D)
    ```

    ```
    scalarProduct(u,crossProduct(v,w))
    ```
    40

9. An ant is on a rotating disk whose rotation is accelerating at 0.01 revolutions per second-squared. The ant is walking radially outwards along the disk at 10 mm s^{-1}. What is the velocity and acceleration (in plane polar coordinates) of the ant when it is 7 centimetres from the centre of the disk and the disk is rotating at 0.1 revolutions per second?
Solution: Using polar co-ordinates and SI units,
We have $r' = 0.01$ m/s and $\theta'' = 0.01 \times 2\pi = 0.02\pi$ rad/s^2. We will assume that $r(0) = 0$. Therefore,
$$r(t) = 0.01t, \ \theta'(t) = 0.02\pi t + k,$$

At $t = 7$, the ant is at $r = 0.07$ m away from the centre of the disk and we also know that $\theta' = 0.1 \times 2\pi = 0.2\pi$ rad/s at this time.

$$\theta'(7) = 0.02\pi \times 7 + k = 0.2\pi \implies k = 0.06\pi$$

$$\theta'(t) = 0.02\pi t + 0.06\pi$$

Recall that velocity and acceleration in polar co-ordinates are given by:

$$\mathbf{v} = r'e_r + r\theta'e_\theta, \ \mathbf{a} = (r'' - r(\theta')^2)e_r + (r\theta'' + 2r'\theta')e_\theta$$

After substitution,
$$\mathbf{v} = 0.01e_r + 0.014\pi e_\theta$$

$$\mathbf{a} = -0.0276e_r + 0.017e_\theta$$

10. Express the position of the ant from the previous question after t seconds in a fixed Cartesian frame of reference. You may assume that the ant is at the centre of the disk when $t = 0$. In MuPAD, plot the ant's path for $0 \leq t \leq 20$ seconds. You may find the following commands useful:
curve:=plot::Curve2d([x(t),y(t)],t=t0..t1)
plot(curve)
Solution:
Carrying on from Question 9, we assume $\theta(0) = 0$.

$$\theta = 0.01\pi t^2 + 0.06\pi t$$

Hence,

$$R(t) = re_r = r[cos(\theta)\hat{\mathbf{i}} + sin(\theta)\hat{\mathbf{j}}]$$
$$= 0.01t cos(0.01\pi t^2 + 0.06\pi t)\hat{\mathbf{i}} + 0.01t sin(0.01\pi t^2 + 0.06\pi t)\hat{\mathbf{j}} =$$

```
theta1 = 0.01 * PI * t^2 + 0.06 * PI * t:; r = 0.01 * t:
x(t):= r * cos(theta1):
y(t):= r * sin(theta1):
curve:=plot::Curve2d([x(t),y(t)],t=0..20):
plot(curve)
```