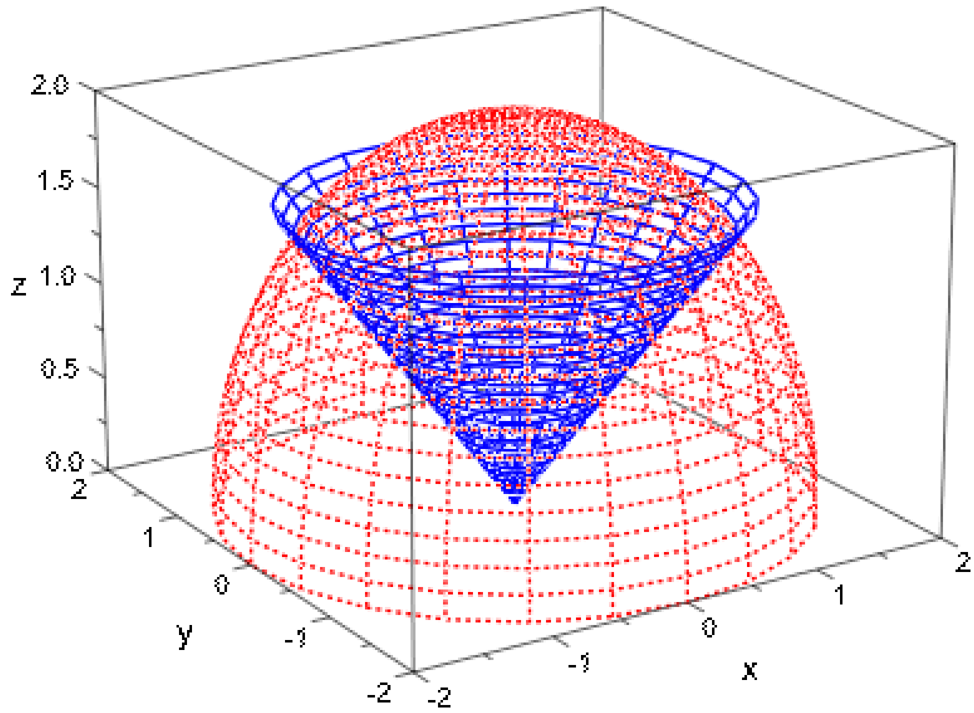
MATHS340: Real and Complex Calculus

Tutorial 5

AceNighJohn

1. Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$ above the $x - y$ plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Solution: Start by drawing a picture;



It is natural to use spherical co-ordinates, the challenging part is to find the range of values for ϕ. To do this, we need to use the equation of the cone.

$$z = \sqrt{x^2 + y^2} \Rightarrow \rho\cos(\phi) = \sqrt{\rho^2\sin^2(\phi)\cos^2(\theta) + \rho^2\sin^2(\phi)sin^2(\theta)} \Rightarrow \cos(\phi) = \sin(\phi)$$

Therefore, $\phi = \frac{\pi}{4}$ and $\frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}$.
Furthermore, V=$\{(\rho, \theta, \phi) |\ 0 \leq \rho \leq 2, 0 \leq \theta \leq 2\pi, \frac{\pi}{4} \leq \phi \leq \frac{\pi}{2}\}$

$$\begin{aligned}\iiint_V dV &= \int_0^{2\pi} \int_0^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \rho^2\sin(\phi)\ d\phi d\rho d\theta = 2\pi \int_0^2 \rho^2\ [-\cos(\phi)]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ d\rho \\ &= \frac{2}{\sqrt{2}}\pi \int_0^2 \rho^2 d\rho = \frac{\sqrt{2}}{3}\pi\ [\rho^3]\Big|_0^2 = \frac{8\sqrt{2}}{3}\pi\end{aligned}$$

2. Evaluate the integral $\iiint_V (x^2+y^2)^{\frac{3}{2}} dV$ where V is the solid region bounded by the paraboloid $z = x^2 + y^2$ and the paraboloid $z = 2 - x^2 - y^2$.

Solution: We are going to use cylindrical co-ordinates. We need to find where the two paraboloids intersect. Equating the two yields:

$$x^2 + y^2 = 2 - x^2 - y^2 \Rightarrow x^2 + y^2 = 1$$

Hence, the intersection is the unit circle.
V=$\{(r, \theta, z) |\ 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, r^2 \leq z \leq 2 - r^2\}$

$$\begin{aligned}\iiint_V (x^2 + y^2)^{\frac{3}{2}} dV &= \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} (r^2)^{\frac{3}{2}} \times r\ dz dr d\theta = 2\pi \int_0^1 r^4\ [2 - r^2 - r^2]\ dr \\ &= 4\pi \int_0^1 r^4(1 - r^2)\ dr = 4\pi\ [\frac{1}{5}r^5 - \frac{1}{7}r^7]\Big|_0^1 = \frac{8}{35}\pi\end{aligned}$$

3. Let $v = x^2y\hat{\mathbf{i}} + (3x - yz)\hat{\mathbf{j}} + z^3\hat{\mathbf{k}}$. Calculate div$(v)$ and curl(v).

Solution: Standard question
div$(v) = 2xy - z + 3z^2$
curl$(v) = y\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + (3 - x^2)\hat{\mathbf{k}}$

4. Consider the vector field $\mathbf{F}(x, y) = y\cos(x)\hat{\mathbf{i}} + \sin(x)\hat{\mathbf{j}}$.

   (a) Show that $\mathbf{F}$ is a conservative vector field.

   (b) Calculate a potential function for $\mathbf{F}$.

   (c) Evaluate the line integral $\int_C \mathbf{F} \cdot dr$, where C is the upper half of the unit circle $x^2+y^2 = 1$ starting at the point $(1, 0)$ and ending at (-1, 0).

Solution: Define P$(x, y) = y\cos(x)$ and Q$(x, y) = \sin(x)$ for convenience.
We require $P_y = Q_x$ in order for $\mathbf{F}$ to be conservative.
Since, $P_y = Q_x = \cos(x)$. Therefore, $\mathbf{F}$ is conservative.

Define ϕ as a potential function for $\mathbf{F}$.

$$\phi = \int P(x, y)\ dx = \int y(x)\ dx = y\sin(x) + C(y)$$

$$\phi_y = \sin(x) + C'(y) = Q(x, y) = \sin(x) \Rightarrow\ C'(y) = 0 \Rightarrow C(y) = c$$

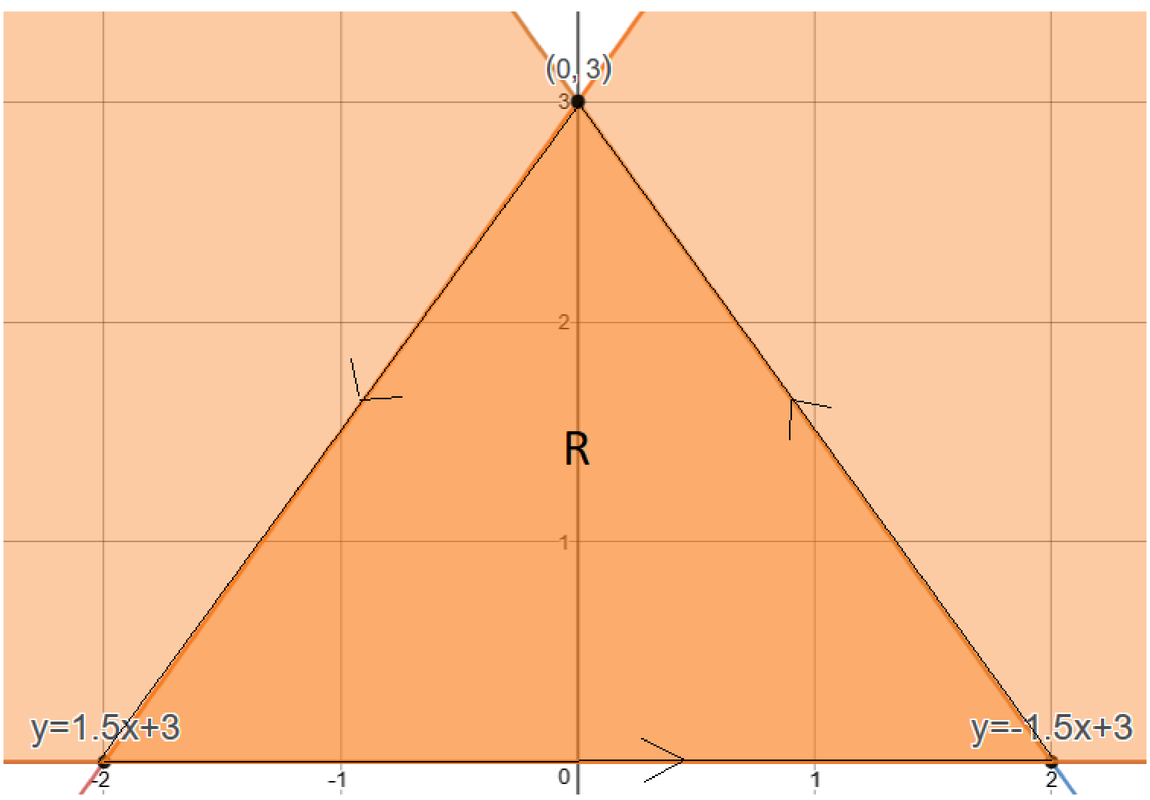
Since we only need a single potential function, we can just choose $c = 0$, and $\phi = y\sin(x)$.

Since $\mathbf{F}$ is conservative, we can use the Fundamental Theorem of Line Integrals.

$$\int_C \mathbf{F} \cdot\ dr = \phi(-1, 0) - \phi(1, 0) = 0$$

5. Use Green's Theorem to calculate the circulation of $\mathbf{F} = (2x^2 + 3y)\hat{\mathbf{i}} + (2x + 3y^2)\hat{\mathbf{j}}$ around the triangle with vertices (2, 0), (0, 3) and (-2, 0), oriented counterclockwise.

Solution: Define P$(x, y) = 2x^2 + 3y$ and Q$(x, y) = 2x + 3y^2$,
Using Green's Theorem, $\oint_C \mathbf{F} \cdot dr = \iint_R (Q_x - P_y)\ dA$, R can be easily found using a diagram.



Letting R be a Type II Region: R $= \{(x, y)|\ 0 \le y \le 3, \frac{2}{3}(y - 3) \le x \le -\frac{2}{3}(y - 3)\}$

$$\oint_C \mathbf{F} \cdot\ dr = \iint_R (Q_x - P_y)\ dA = \iint_R (2 - 3)\ dA = -\iint_R\ dA = -(\frac{1}{2} \times 4 \times 3) = -6$$

Notice that the final integral has an integrand of 1, this means we are computing the area of the region. All we need to is use the formula for the area of a triangle.