# MATHS340: Real and Complex Calculus

**Tutorial 12**

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1. Determine all the singular points of the following functions and say whether they are poles or essential singularities. If a singularity is a pole, give its order.

   (a) $f(z) = \frac{e^z - 1}{z^3}$

   (b) $f(z) = \cos(^1/_z)$

   (c) $f(z) = \frac{1}{1+^1/(1+z)}$

   (d) $f(z) = \frac{z}{(\sin(z))^3}$

   **Solution:**

   (a) $z = 0$ is a singular point. Denote $P(z) = e^z - 1$ and $Q(z) = z^3$.
   $P(z)$ is of order 1 at $z = 0$. $Q(z)$ is of order 3 at $z = 0$.
   Thus, $f(z)$ is of order 2 at $z = 0$. $z = 0$ is a pole of order 2.

   (b) $z = 0$ is a singular point.
   Using a Taylor series expansion,

   $$\cos(^1/_z) = 1 - \frac{(^1/_z)^2}{2} + \frac{(^1/_z)^4}{4!} + ... = 1 - \frac{1}{2z^2} + \frac{1}{24z^4} + ...$$

   Thus, $f(z)$ is an essential singularity.

   (c)
   $$\frac{1}{1 + ^1/(1+z)} = \frac{1}{\frac{z+2}{z+1}} = \frac{z+1}{z+2} = \frac{P(z)}{Q(z)}$$

   $z = -2$ is a singular point.
   $P(z)$ is of order 0 at $z = -2$. $Q(z)$ is of order 1 at $z = -2$.
   Thus, $f(z)$ is of order 1 at $z = -2$. $z = -2$ is a simple pole.

   (d) $z = \pi k$, $k \in \mathbb{Z}$ are all singular points.
   Denote $P(z) = z$ and $Q(z) = \sin^3(z)$.
   $Q(z)$ is of order 3 for $z = \pi k$, $k \in \mathbb{Z}$.
   For $k = 0$, we have $z = 0$, so $P(z)$ is of order 1 at $z = 0$.
   For $k \neq 0$, $P(z)$ is of order 0 at $z = \pi k$, $k \in \mathbb{Z} \setminus \{0\}$.
   Thus, $z = 0$ is a pole of order 2 and $z = \pi k$, $k \in \mathbb{Z} \setminus \{0\}$ is a pole of order 3.

2. Use the residue theorem to evaluate

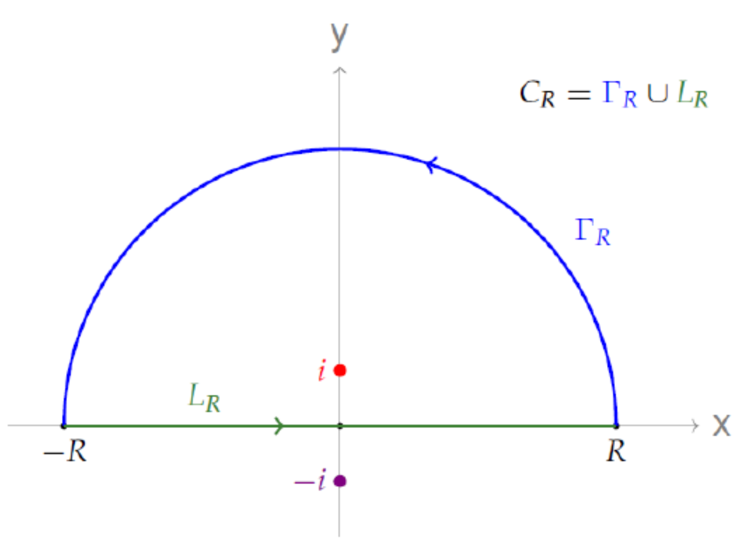
   $$\int_0^\infty \frac{x^2}{x^4 + 1}\, dx$$

   **Solution:**
   Notice that the integrand is an even function.

   $$\int_0^\infty \frac{x^2}{x^4+1}\, dx = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+1}\, dx$$

   Furthermore,

   $$\int_{-\infty}^\infty \frac{x^2}{x^4+1}\, dx = \lim_{R\to\infty}\int_{L_R} \frac{z^2}{z^4+1}\, dz$$




   Let $f(z) = \frac{z^2}{z^4+1}$,

   $$\oint_{C_R} f(z)\, dz = \int_{\Gamma_R} f(z)\, dz + \int_{L_R} f(z)\, dz$$

We need to find the singular points of $f(z)$, i.e: when $z^4+1=0$.

$$z^4 = -1 = e^{i(\pi+2\pi k)}$$

$$z = e^{\frac{1}{4}i(\pi+2\pi k)}, \text{ where we have 4 distinct roots for } k=0,1,2,3.$$

$$z_1 = e^{\frac{\pi}{4}i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,\ z_2 = e^{\frac{3\pi}{4}i} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

$$z_3 = e^{\frac{5\pi}{4}i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i,\ z_4 = e^{\frac{7\pi}{4}i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

The easiest way to convert from Euler form to Cartesian form in an exam is to draw a Argand diagram.
Only $z_1$ and $z_2$ lie inside $C$. $z_3$ and $z_4$ lie outside $C$. By Residue theorem, we obtain:

$$\oint_{C_R} f(z)\ dz = \oint_{C_R} \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\ dz = 2\pi i\left(\text{Res}_{z_1} f(z) + \text{Res}_{z_2} f(z)\right)$$

$$\text{Res}_{z_1} = \lim_{z\to z_1} f(z)(z-z_1) = \lim_{z\to z_1} \frac{z^2}{(z-z_2)(z-z_3)(z-z_4)} = \frac{1}{2\sqrt{2}(1+i)}$$

$$\text{Res}_{z_2} = \lim_{z\to z_2} f(z)(z-z_2) = \lim_{z\to z_1} \frac{z^2}{(z-z_1)(z-z_3)(z-z_4)} = \frac{1}{2\sqrt{2}(-1+i)}$$

Thus,

$$\oint_{C_R} f(z)\ dz = 2\pi i\left(\frac{1}{2\sqrt{2}(1+i)} + \frac{1}{2\sqrt{2}(-1+i)}\right) = \frac{\pi}{\sqrt{2}}$$

By the reverse triangle inequality, we have: $\left|z^4+1\right| \geq \left|z^4\right| - 1$.

$$\left|\int_{\Gamma_R} f(z)\ dz\right| \leq \int_{\Gamma_R} |f(z)|\ dz = \int_{\Gamma_R} \frac{\left|z^2\right|}{|z^4+1|}\ dz \leq \int_{\Gamma_R} \frac{\left|z^2\right|}{|z^4|-1}\ dz = \int_{\Gamma_R} \frac{R^2}{R^4-1}\ dz$$
$$= \frac{R^2}{R^4-1}\int_{\Gamma_R} dz = \frac{R^2}{R^4-1}\pi R = \frac{\pi R^3}{R^4-1} = \frac{\frac{\pi}{R}}{1-\frac{1}{R^4}}$$

By squeeze theorem,

$$0 \leq \lim_{R\to\infty}\left|\int_{\Gamma_R} f(z)\ dz\right| \leq \lim_{R\to\infty} \frac{\frac{\pi}{R}}{1-\frac{1}{R^4}} = 0$$

Hence,

$$\int_{\Gamma_R} f(z)\ dz = 0$$

So,

$$\frac{\pi}{\sqrt{2}} = \lim_{R\to\infty}\left|\int_{C_R} f(z)\ dz\right| = \lim_{R\to\infty}\left[\int_{\Gamma_R} f(z)\ dz + \int_{L_R} f(z)\ dz\right] = 0 + \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\ dx$$
$$= \int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\ dx$$

Finally,

$$\int_0^{\infty} \frac{x^2}{x^4+1}\ dx = \frac{\pi}{2\sqrt{2}}$$

3. Use the residue theorem to evaluate

$$\int_0^{2\pi} (\cos\theta)^6\ d\theta$$

**Solution:**

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} = \frac{z^2+1}{2z}, \text{ where } z = e^{i\theta}$$

$$dz = ie^{i\theta}\ d\theta = iz\ d\theta$$

Let C be the unit circle, i.e: $C : |z| = 1$.

$$\int_0^{2\pi} (\cos\theta)^6\ d\theta = \oint_C \left(\frac{z^2+1}{2z}\right)^6 \frac{1}{iz}\ dz = \frac{-i}{2^6}\oint_C \frac{(z^2+1)^6}{z^7}\ dz = \frac{-i}{2^6} 2\pi i\ \text{Res}_0\left(\frac{(z^2+1)^6}{z^7}\right)$$

$z=0$ is a pole of order 7. Therefore,

$$\text{Res}_0\left(\frac{(z^2+1)^6}{z^7}\right) = \frac{1}{6!}\lim_{z\to 0}\frac{d^6}{dz^6}\left((z^2+1)^6\right) = 20$$

Hence,

$$\int_0^{2\pi} (\cos\theta)^6\ d\theta = \frac{-i}{2^6} 2\pi i \times 20 = \frac{5\pi}{8}$$