

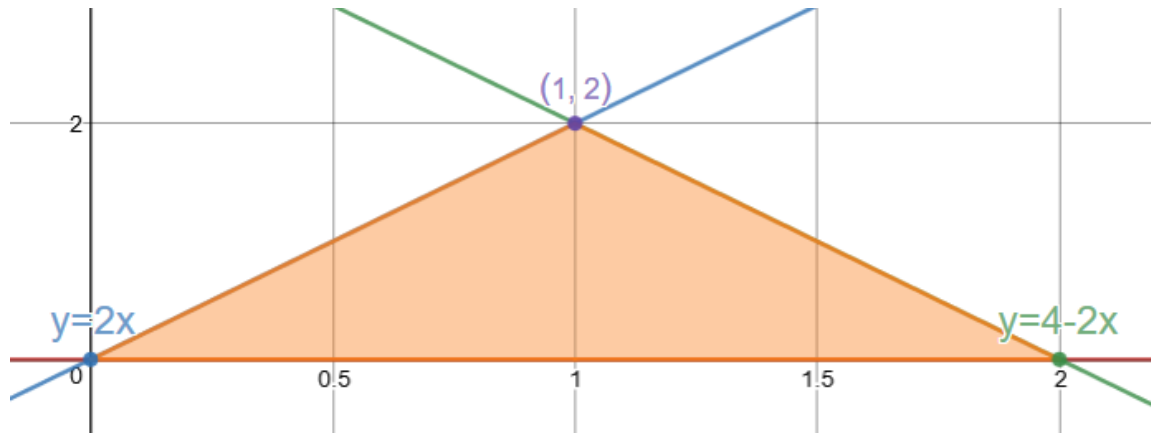
MATHS340: Real and Complex Calculus
Tutorial 4
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1. Evaluate

$$\iint_R x^2 y dy dx$$

where R is the triangle with vertices at $(0,0)$, $(2,0)$ and $(1,2)$.

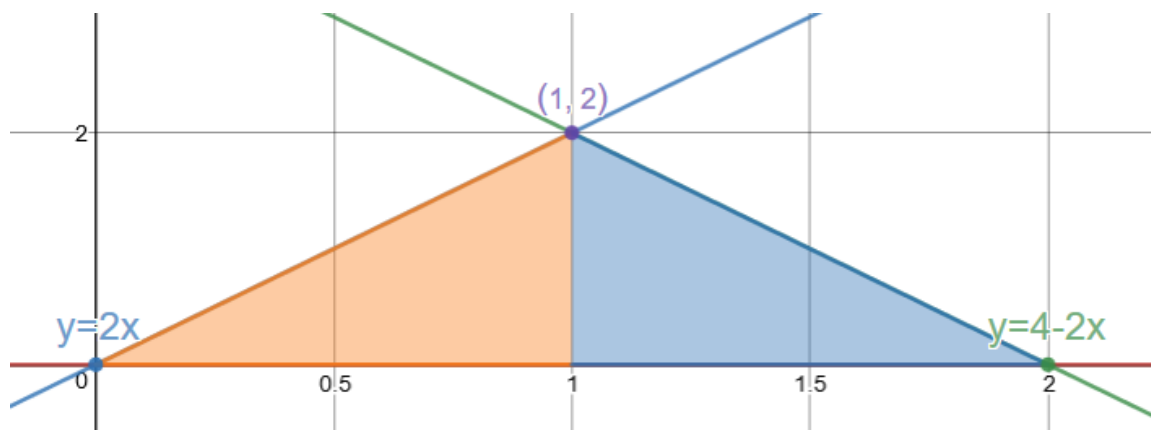
Solution: There are two approaches to the solution. The first approach is to let the region be of Type II. $R = \{(x, y) \mid 0 \leq y \leq 2, \frac{y}{2} \leq x \leq \frac{4-y}{2}\}$.



So the integral becomes:

$$\begin{aligned} \int_0^2 \int_{\frac{y}{2}}^{\frac{4-y}{2}} x^2 y dy dx &= \frac{1}{3} \int_0^2 x^3 [y] \Big|_{\frac{y}{2}}^{\frac{4-y}{2}} dy = \frac{1}{3} \int_0^2 y \left[\left(\frac{4-y}{2} \right)^3 - \left(\frac{y}{2} \right)^3 \right] dy \\ &= \frac{1}{3} \int_0^2 \left(-\frac{y^4}{4} + \frac{3y^3}{2} - 6y^2 + 8y \right) dy = \frac{1}{3} \left[-\frac{y^5}{20} + \frac{3y^4}{8} - 2y^3 + 4y^2 \right] \Big|_0^2 \\ &= \frac{1}{3} \times \frac{22}{5} = \frac{22}{15} \end{aligned}$$

The second approach is to let the Region R be of Type I, to do this, we need to split up the region into two regions of Type I.



In this case,

$$R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2x\} \cup \{(x, y) \mid 1 \leq x \leq 2, 0 \leq y \leq 4 - 2x\}$$

$$\begin{aligned} \iint_R x^2 y dx dy &= \int_0^1 \int_0^{2x} x^2 y dx dy + \int_1^2 \int_0^{4-2x} x^2 y dx dy \\ &= \frac{1}{2} \left[\int_0^1 x^2 [y^2] \Big|_0^{2x} dx + \int_1^2 x^2 [y^2] \Big|_0^{4-2x} dx \right] \\ &= \frac{1}{2} \left[\int_0^1 x^2 [(2x)^2] dx + \int_1^2 x^2 (4 - 2x)^2 dx \right] \\ &= \frac{1}{2} \left[\int_0^1 4x^4 dx + \int_1^2 (4x^4 - 16x^3 + 16x^2) dx \right] \\ &= \frac{1}{2} \left(\left[\frac{4x^5}{5} \right] \Big|_0^1 + \left[\frac{4x^5}{5} - 4x^4 + \frac{16x^3}{3} \right] \Big|_1^2 \right) \\ &= \frac{1}{2} \left(\frac{4}{5} + \left[\frac{64}{15} - \frac{32}{15} \right] \right) = \frac{1}{2} \times \frac{44}{15} = \frac{22}{15} \end{aligned}$$

2. Let S be the surface of the cylinder $x^2 + y^2 = 25$ bounded by $z = 0$ and $z = 3$. Evaluate the integral:

$$\iint_S xy dS$$

Solution: Let's first parameterize the surface, it is natural to use cylindrical co-ordinates. Let $x = 5\cos(\theta)$, $y = 5\sin(\theta)$, $z = z$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 3$

Define $R = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = 5\cos(\theta)\hat{\mathbf{i}} + 5\sin(\theta)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$

Taking derivatives,

$$R_\theta = -5\sin(\theta)\hat{\mathbf{i}} + 5\cos(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}, R_z = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Then using the cross product (order doesn't matter since we'll take the norm of the cross product later),

$$R_\theta \times R_z = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5\sin(\theta) & 5\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5\cos(\theta)\hat{\mathbf{i}} + 5\sin(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$$\|R_\theta \times R_z\| = \sqrt{25\cos^2(\theta) + 25\sin^2(\theta) + 0^2} = 5$$

Recall that the Jacobian when using cylindrical co-ordinates is r , which in this question is 5.

$$\iint_S xy dS = \iint_R 25\sin(\theta)\cos(\theta)\|R_\theta \times R_z\| dA$$

where R is the region in the $\theta - z$ plane; $R = \{(\theta, z) | 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3\}$.

Therefore,

$$\begin{aligned} \iint_S xy dS &= \int_0^3 \int_0^{2\pi} 25\sin(\theta)\cos(\theta) \times 5 d\theta dz = 375 \int_0^{2\pi} \sin(\theta)\cos(\theta) d\theta \\ &= \frac{375}{2} [\sin^2(\theta)]_0^{2\pi} = 0 \end{aligned}$$

3. A city occupies a semicircular region on the coast. The radius of the city is 4km. Find the average distance from points in the city to the ocean.

Solution: Notice that the yellow lines in Figure 1 indicate the distance to the ocean for the 3 chosen points in the city. We are interested in finding the average value of y for every point in the city. Here, we will present two different approaches.

For the first approach, we will introduce another variable, z , where $z(x, y)$ represents the distance of a point (x, y) . As discussed previously, $z(x, y) = y$.

Graphically, this introduces a third dimension, and we are interested in the average depth of the surface.

$$\bar{d} = \frac{V}{A}$$

where, \bar{d} is the average depth, V is the volume of surface and A is the surface area of the base.

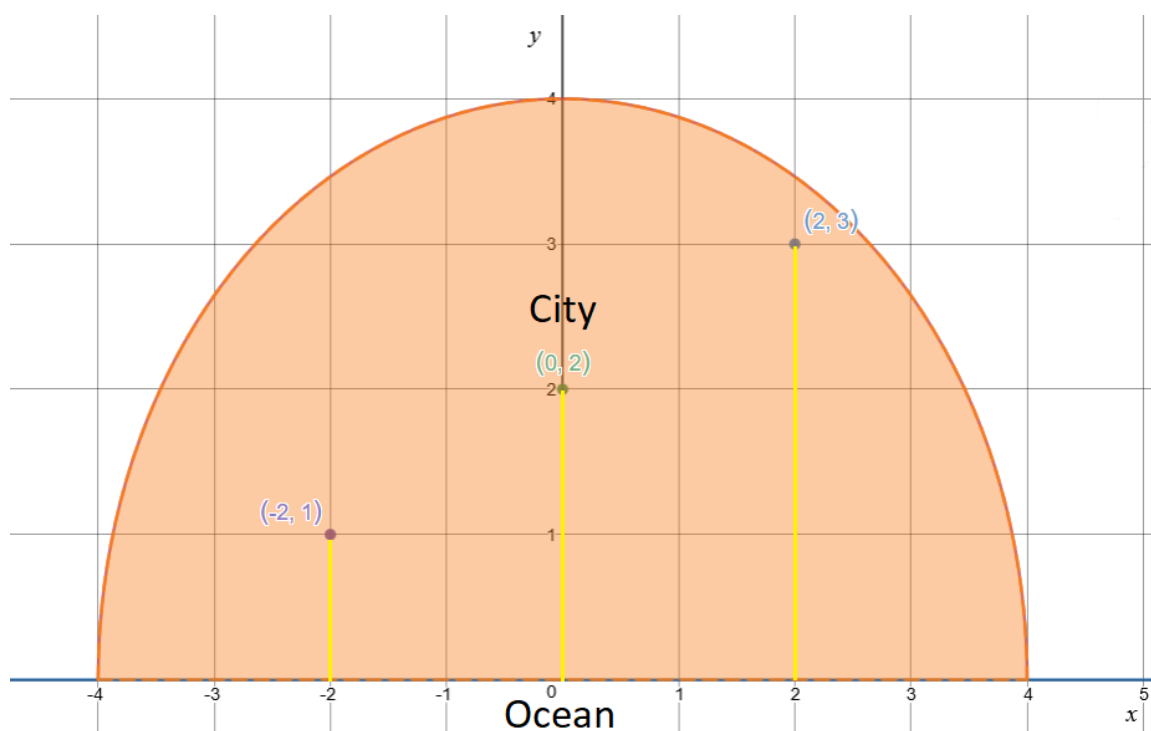


Figure 1: x-y plane

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[x:= u*cos(v):
[y:= u*sin(v):
[z1:= y:
[z2:= 0:
[s := plot::Surface([x(u,v), y(u,v), z1(u,v)], u = 0..4, v = 0..PI):
[A := plot::Surface([x(u,v), y(u,v), z2(u,v)], u = 0..4, v = 0..PI):
plot(s, A)

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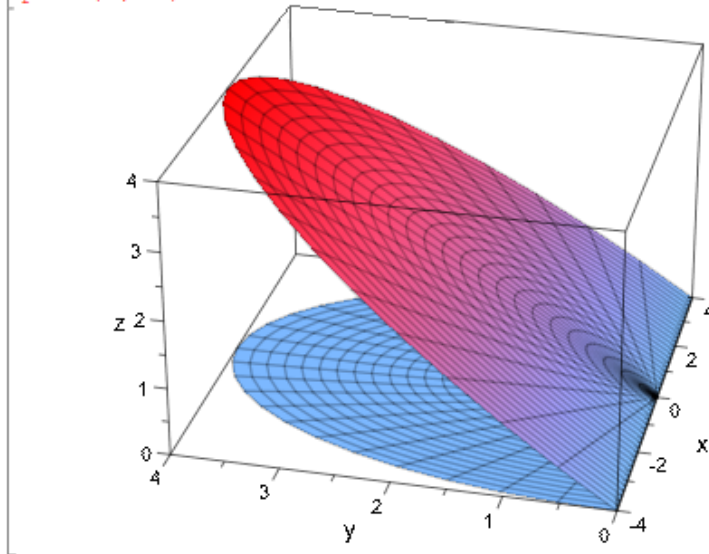


Figure 2: x-y-z plane

$$V = \{(x, y, z) \mid 0 \leq x \leq 4, 0 \leq y \leq \sqrt{16 - x^2}, 0 \leq z \leq y\}$$

Clearly, we should use cylindrical co-ordinates.

$$V = \{(r, \theta, z) \mid 0 \leq r \leq 4, 0 \leq \theta \leq \pi, 0 \leq z \leq r \sin(\theta)\}$$

So,

$$\begin{aligned}
 V &= \iiint_V dV = \int_0^\pi \int_0^4 \int_0^{r \sin(\theta)} r \, dz \, dr \, d\theta = \int_0^\pi \int_0^4 r^2 \sin(\theta) \, dr \, d\theta \\
 &= \frac{1}{3} \int_0^\pi \sin(\theta) [r^3]_0^4 \, d\theta = \frac{-64}{3} [\cos(\theta)]_0^\pi = \frac{128}{3}
 \end{aligned}$$

Trivially, using the formula for the area of a semi-circle; $A = \frac{1}{2}\pi \times 4^2 = 8\pi$.

Hence, $\bar{d} = \frac{16\pi}{3}$.

The second method uses the concept of centre of mass. The average distance to the ocean is simply the total distance from the ocean of all points in the semi-circle/city divided by the number of points in the semi-circle/city. The number of points is simply the area of the semi-circle. Using this approach, we don't need to introduce a third variable z , but the calculations are identical: Let S be the total distance of all points, recall that y is the distance to the ocean of an individual point,

$$S = \iint_R y \, dA$$

where R is the semi-circle/city. Clearly, we should use polar co-ordinates to represent R . So,

$$S = \iint_R y \, dA = \int_0^\pi \int_0^4 r \sin(\theta) \times r \, dr \, d\theta = \int_0^\pi \int_0^4 r^2 \sin(\theta) \, dr \, d\theta$$

Define \bar{d} to be the average distance for all points,

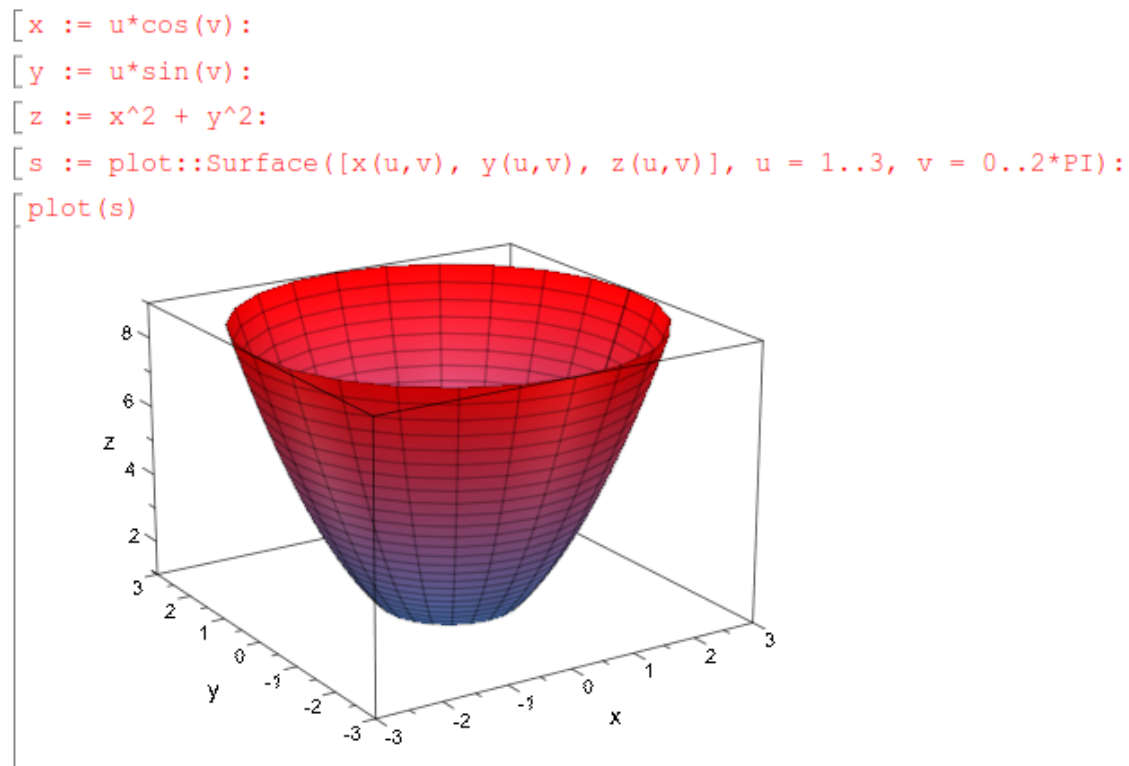
$$\bar{d} = \frac{S}{A}$$

A is the area of the semi-circle/city which is the same as in the first approach. Notice that S is also identical to V as they have identical integrals.

Hence, $\bar{d} = \frac{16\pi}{3}$.

4. The surface S is defined by $z = x^2 + y^2$, where $1 \leq z \leq 9$. Use an integral to find the surface area of S .

Solution: Start by drawing a picture,



Using cylindrical co-ordinates,

Let $R = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (x^2 + y^2)\hat{\mathbf{k}} = r\cos(\theta)\hat{\mathbf{i}} + r\sin(\theta)\hat{\mathbf{j}} + r^2\hat{\mathbf{k}}$

Taking derivatives,

$R_\theta = -r\sin(\theta)\hat{\mathbf{i}} + r\cos(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$, $R_r = \cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}} + 2r\hat{\mathbf{k}}$

Then using the cross product (order doesn't matter since we'll take the norm of the cross product later),

$$R_\theta \times R_r = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 2r \end{bmatrix} = 2r^2\cos(\theta)\hat{\mathbf{i}} + 2r^2\sin(\theta)\hat{\mathbf{j}} - r\hat{\mathbf{k}}$$

$$\|R_\theta \times R_r\| = \sqrt{4r^4\cos^2(\theta) + 4r^4\sin^2(\theta) + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2 + 1}$$

The surface area is: $\iint_S dS$, where S is the surface.

$$\iint_S dS = \iint_R \|R_\theta \times R_r\| dA$$

where R is the region on the x-y plane after the surface has been projected onto the x-y plane.

Using polar co-ordinates; $R = \{(r, \theta) \mid 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.

Therefore,

$$\begin{aligned} \iint_S dS &= \iint_R \|R_\theta \times R_r\| dA = \int_0^{2\pi} \int_1^3 r\sqrt{4r^2 + 1} dr d\theta = 2\pi \int_1^3 r\sqrt{4r^2 + 1} dr \\ &= 2\pi \times \frac{2}{3} \times \frac{1}{8} [(4r^2 + 1)^{\frac{3}{2}}]_1^3 = \frac{\pi}{6} [37^{\frac{3}{2}} - 5^{\frac{3}{2}}] = 111.988 \approx 112 \end{aligned}$$

5. Find the surface area of the plane $x + 2y + 4z = 16$ cut off by $x = 0$, $y = 0$ and $x^2 + y^2 = 9$, where $x, y \geq 0$.

Solution: Let $z = f(x, y)$, then $z_x = f_x$ and $z_y = f_y$

The surface is given by $x + 2y + 4z = 16$.

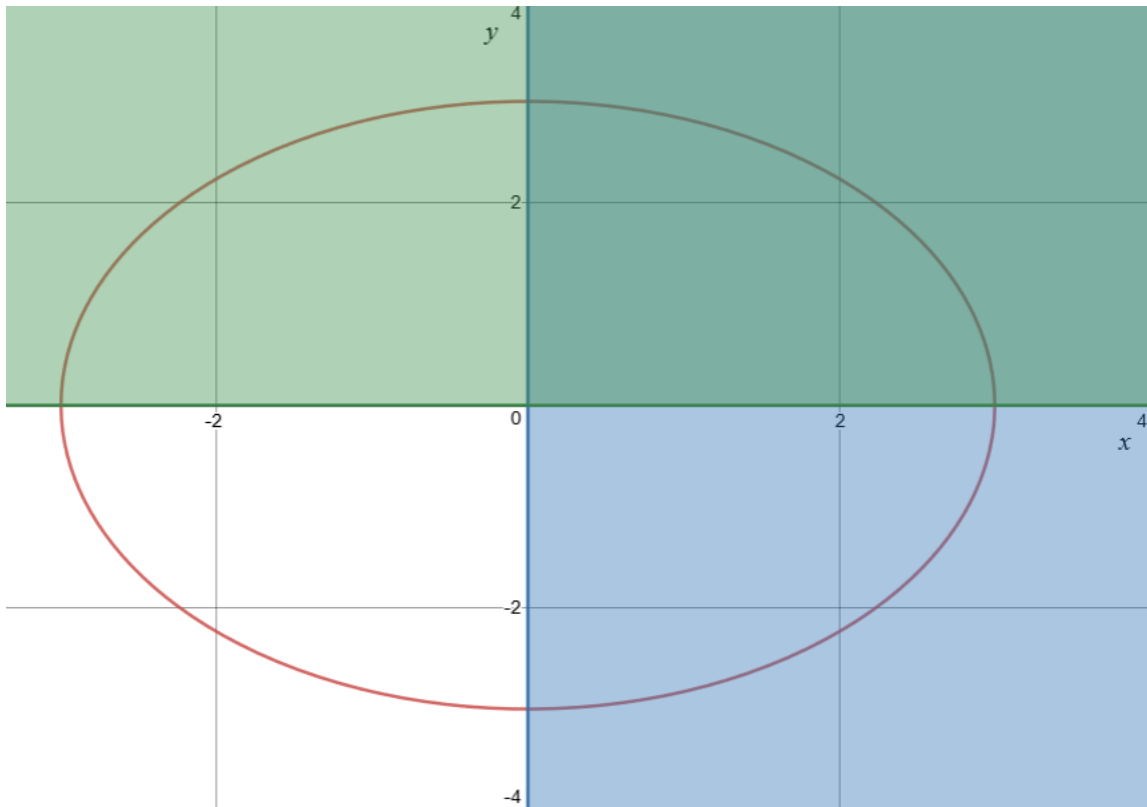
Differentiating with respect to x yields; $1 + 4z_x = 0$, $z_x = -\frac{1}{4}$

Differentiating with respect to y yields; $2 + 4z_y = 0$, $z_y = -\frac{1}{2}$

In Cartesian co-ordinates,

$$\iint_S dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA$$

In this case, we will need to make a slight modification by using polar co-ordinates;



The region R is the projection of the surface down onto the x - y plane. Clearly, we are interested in the first quadrant of the circle $x^2 + y^2 = 9$.

$R = \{(r, \theta) \mid 0 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\}$.

Therefore,

$$\begin{aligned} \iint_S dS &= \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA = \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{1 + \frac{1}{16} + \frac{1}{4}} \times r dr d\theta = \frac{\pi}{2} \int_0^3 \sqrt{\frac{21}{16}} r dr \\ &= \frac{\pi}{2} \times \frac{\sqrt{21}}{4} r \times \frac{1}{2} [r^2] \Big|_0^3 = \frac{9}{16} \sqrt{21} \pi \end{aligned}$$

6. A building is 8 metres wide and 16 metres long. It has a flat roof that is 12 metres high in one corner, and 10 metres high at each of the adjacent corners.

- What is the surface area of the roof?
- What is the volume of the building?

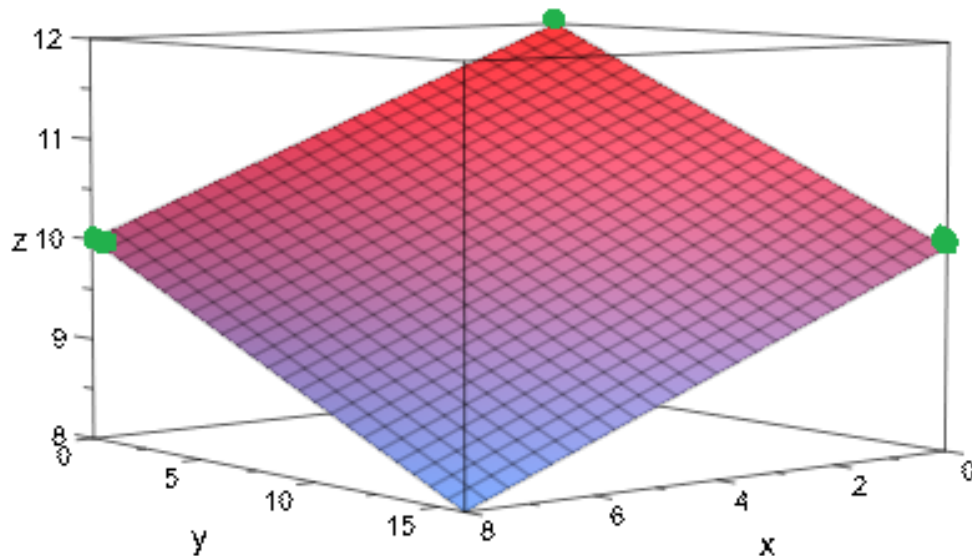
Solution: The roof is a section of a plane, we are given three points on this plane so we are able to determine the equation of the plane. We will choose the points $(0, 0, 12)$, $(8, 0, 10)$, $(0, 16, 10)$, and then define the vectors \vec{P} and \vec{Q} , which lie on the plane and are linearly independent.

$\vec{P} = (8, 0, 10) - (0, 0, 12) = (8, 0, -2)$, $\vec{Q} = (0, 16, 10) - (0, 0, 12) = (0, 16, -2)$

The surface area is the norm of the cross product of \vec{P} and \vec{Q} .

$$\vec{P} \times \vec{Q} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 8 & 0 & -2 \\ 0 & 16 & -2 \end{bmatrix} = 32\hat{i} + 16\hat{j} + 128\hat{k}$$

$$\|\vec{P} \times \vec{Q}\| = \sqrt{32^2 + 16^2 + 128^2} = 132.9$$



The equation of the plane is given by: $32x + 16y + 128z = d$, where the co-efficients are found from the cross product of \vec{P} and \vec{Q} . To find, simply substitute any one of the three points. Let's substitute in $(0,0,12)$, $d = 128 \times 12 = 1536$. Therefore, $32x + 16y + 128z = 1536$. The volume is found using a triple integral.

$$V = \{(x, y, z) \mid 0 \leq x \leq 8, 0 \leq y \leq 16, 0 \leq z \leq \frac{1}{128}(1536 - 32x - 16y)\}$$

$$\begin{aligned} \iiint_V dV &= \int_0^8 \int_0^{16} \int_0^{\frac{1}{128}(1536-32x-16y)} dz dy dx = \int_0^8 \int_0^{16} \frac{1}{128}(1536 - 32x - 16y) dy dx \\ &= \frac{1}{128} \int_0^8 [1536y - 32xy - 8y^2] \Big|_0^{16} dx = \frac{1}{128} \int_0^8 (22528 - 512x) dx \\ &= 4 \int_0^8 (44 - x) dx = 4[44x - \frac{1}{2}x^2] \Big|_0^8 = 1280 \end{aligned}$$

7. Consider the closed region bounded by the surfaces

$$z = e^{-x^2-y^2}, \quad z = 0, \quad x^2 + y^2 = 1$$

Find the volume of the region.

Solution: $V = \{(x, y, z) \mid 0 \leq x^2 + y^2 \leq 1, 0 \leq z \leq e^{-x^2-y^2}\}$

It is natural to use cylindrical co-ordinates.

$$V = \{(r, \theta, z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq e^{-r^2}\}$$

$$\begin{aligned} \iiint_V dV &= \int_0^{2\pi} \int_0^1 \int_0^{e^{-r^2}} r dz dr d\theta = 2\pi \int_0^1 r e^{-r^2} dr = 2\pi \times \left(-\frac{1}{2}\right)[e^{-r^2}] \Big|_0^1 \\ &= -\pi(e^{-1} - 1) = \pi(1 - e^{-1}) \end{aligned}$$