MATHS340: Real and Complex Calculus

Tutorial 2

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1. Use the taylor command in MuPAD to

   nd the Taylor series expansion of tan(x) about x = 0 up to and including the x^7 term. (If you forgot how to use MuPAD, refer to the guide.mn file on Canvas)
   Solution:

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taylor(tan(x),x=0,7)
```

$$x+\frac{x^3}{3}+\frac{2\,x^5}{15}+\frac{17\,x^7}{315}+O(x^9)$$

2. Let $f(x, y, z) = 3xy^2 + xe^z - \sin(zy)$.

   (a) Find the linear Taylor polynomial for f at the point (2, 2, 2).

   (b) Check your answer using MuPAD with the command mtaylor(f,[x=2,y=2,z=2],2).

   Solution: Using the definition of a multi-variable Taylor series:

$$\begin{aligned}P_1(x, y, z) &= f(2, 2, 2) + \nabla f(2, 2, 2)\cdot(x - 2, y - 2, z - 2)\\ &= f(2, 2, 2) + f_x(2, 2, 2)(x - 2) + +f_y(2, 2, 2)(y - 2) + f_z(2, 2, 2)(z - 2)\end{aligned}$$

$$f_x = 3y^2 + e^z,\ f_y = 6xy - z\cos(zy),\ f_z = xe^z - y\cos(zy)$$
$$f_x(2, 2, 2) = 12 + e^2, f_y(2, 2, 2) = 24 - 2\cos(4),\ f_y(2, 2, 2) = 2e^2 - 2\cos(4)$$

Therefore,

$$\begin{aligned}P_1(x, y, z) &= (24 + 2e^2 - \sin(4)) + (12 + e^2)(x - 2) + (24 - 2\cos(4))(y - 2) + (2e^2 - 2\cos(4))(z - 2)\\ &= 39.53 + 19.39(x - 2) + 25.31(y - 2) + 16.09(z - 2)\\ &= 19.39x + 25.31y + 16.09z - 82.04\end{aligned}$$

Using MuPAD,

```
mtaylor(3*x*y^2 + x*exp(z) - sin(z*y),[x=2,y=2,z=2],2)
```

$$2\,\mathrm{e}^2-\sin(4)-(z-2)\left(2\cos(4)-2\,\mathrm{e}^2\right)-(2\cos(4)-24)\,(y-2)+\left(\mathrm{e}^2+12\right)(x-2)+24$$

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float(%)
```

$$19.3890561\,x+25.30728724\,y+16.08539944\,z-82.02857087$$

3. Find the tangent plane for the function $f(x, y) = xy^2 - x^3 + 10$ at the point $(1, \frac{3}{2})$.
   Solution:
   Let $z = f(x, y) = xy^2 - x^3 + 10$ and define $F(x, y, z) = z - xy^2 + x^3 - 10 = 0$.

$$z(1,\frac{3}{2}) = \frac{9}{4} - 1 + 10 = \frac{45}{4}$$

$$F_x = -y^2 + 3x^2,\ F_y = -2xy,\ F_z = 1$$
$$F_x(1,\frac{3}{2},\frac{45}{4}) = \frac{3}{4},\ F_y(1,\frac{3}{2},\frac{45}{4}) = -3,\ F_z(1,\frac{3}{2},\frac{45}{4}) = 1$$

The surface at $(1, \frac{3}{2}, \frac{45}{4})$ is perpendicular to the gradient.

$$\nabla F(1,\frac{3}{2},\frac{45}{4})\cdot(x - 1, y - \frac{3}{2}, z - \frac{45}{4}) = 0$$

$$\frac{3}{4}(x - 1) - 3(y - \frac{3}{2}) + (z - \frac{45}{4}) = 0$$

$$z = -\frac{3}{4}x + 3y + \frac{15}{2}$$

4. Find $\|\mathbf{u}\times\mathbf{v}\|$, where $\|\mathbf{u}\| = 2$ and $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\frac{\pi}{6}$.
   Solution:
   Using the formula: $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta)$, where θ is the angle between $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$.

$$\|\mathbf{u}\times\mathbf{v}\| = 2\times 5\times\sin(\frac{\pi}{6}) = 5$$

5. Let $\mathbf{u} = [4,\ 0,\ 0]$ and $\mathbf{v} = [0,\ 0,\ 3]$. Calculate the cross product $\mathbf{u} \times \mathbf{v}$ without using determinants.
   Solution: Using the formula: $\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\| \sin(\theta)\hat{e}$, where $\hat{e}$ is the unit vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$ while obeying the right hand rule.
   Notice that $\mathbf{u}$ and $\mathbf{v}$ are just scalar multiples of $\hat{\mathbf{i}}$ and $\hat{\mathbf{k}}$ respectively. So $\hat{e}$ is $-\hat{\mathbf{j}}$. Trivially, $\|\mathbf{u}\| = 4$, $\|\mathbf{u}\| = 3$ and $\theta = \frac{\pi}{2}$.
   $$\mathbf{u} \times \mathbf{v} = 4 \times 3\sin(\frac{\pi}{2}) - \hat{\mathbf{j}} = -12\hat{\mathbf{j}} = [0, -12, 0]$$

6. Suppose $\mathbf{u} = [t, \cos(2t), e^t]$. Use the diff command to find d$\mathbf{u}$/dt. You can find out about the diff command by typing ?diff.
   Solution:

```
u := matrix([t,cos(2*t),exp(t)]):
diff(u,t)
```

$$\begin{pmatrix} 1 \\ 2\ \sin(2\ t) \\ e^t \end{pmatrix}$$

7. If $\mathbf{u} = [t, \cos(2t), e^t]$, $\mathbf{v} = [-t, 0, (1+t)^-1]$, what is d$(\mathbf{u} \times \mathbf{v})$/dt? **Solution:** Using the same $\mathbf{u}$ in the MuPAD code in Question 6;

```
v := matrix([-t, 0, (1+t)^-1])
```

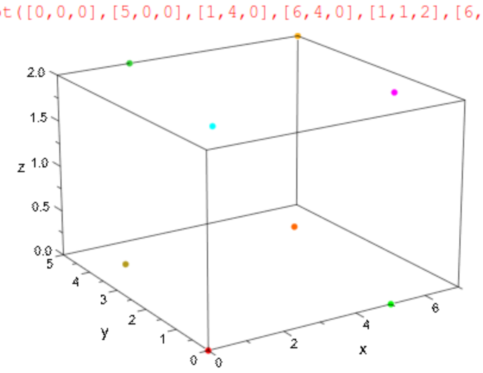
$$\begin{pmatrix} -t \\ 0 \\ \frac{1}{t+1} \end{pmatrix}$$

```
diff(crossProduct(u,v),t)
```

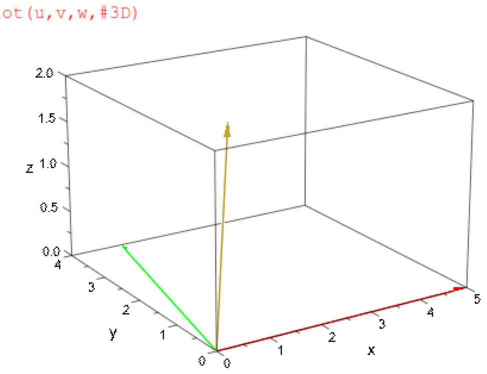
$$\begin{pmatrix} -\frac{\cos(2\ t)}{(t+1)^2} - \frac{2\ \sin(2\ t)}{t+1} \\ \frac{t}{(t+1)^2} - e^t - \frac{1}{t+1} - t\ e^t \\ \cos(2\ t) - 2\ t\ \sin(2\ t) \end{pmatrix}$$

8. Plot the following points in $\mathbb{R}^3$ using MuPAD:
   $$(0,0,0), (5,0,0), (1,4,0), (6,4,0), (1,1,2), (6,1,2), (2,5,2), (7,5,2):$$
   These are the vertices of a parallelopiped. What three vectors can be used to define the paralellopiped? Find the volume of the parallelopiped using a scalar triple product. **Solution:**

```
plot([0,0,0],[5,0,0],[1,4,0],[6,4,0],[1,1,2],[6,1,2],[2,5,2],[7,5,2], #3D)
```




```
u:=matrix([5,0,0]):; v:=matrix([1,4,0]):; w:=matrix([1,1,2]):
plot(u,v,w,#3D)
```




```
scalarProduct(u,crossProduct(v,w))
```

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9. An ant is on a rotating disk whose rotation is accelerating at 0.01 revolutions per second-squared. The ant is walking radially outwards along the disk at 10 mm s^{-1}. What is the velocity and acceleration (in plane polar coordinates) of the ant when it is 7 centimetres from the centre of the disk and the disk is rotating at 0.1 revolutions per second?
Solution: Using polar co-ordinates and SI units,
We have $r' = 0.01$ m/s and $\theta'' = 0.01 \times 2\pi = 0.02\pi$ rad/s^2. We will assume that $r(0) = 0$. Therefore,

$$r(t) = 0.01t,\ \ \theta'(t) = 0.02\pi t + k,$$

At $t = 7$, the ant is at $r = 0.07$ m away from the centre of the disk and we also know that $\theta' = 0.1 \times 2\pi = 0.2\pi$ rad/s at this time.

$$\theta'(7) = 0.02\pi \times 7 + k = 0.2\pi \implies k = 0.06\pi$$

$$\theta'(t) = 0.02\pi t + 0.06\pi$$

Recall that velocity and acceleration in polar co-ordinates are given by:

$$\mathbf{v} = r'e_r + r\theta' e_\theta,\ \ \mathbf{a} = (r'' - r(\theta')^2)e_r + (r\theta'' + 2r'\theta')e_\theta$$

After substitution,

$$\mathbf{v} = 0.01e_r + 0.014\pi e_\theta$$

$$\mathbf{a} = -0.0276e_r + 0.017e_\theta$$

10. Express the position of the ant from the previous question after t seconds in a fixed Cartesian frame of reference. You may assume that the ant is at the centre of the disk when $t = 0$. In MuPAD, plot the ant's path for $0 \le t \le 20$ seconds. You may find the following commands useful:
curve:=plot::Curve2d([x(t),y(t)],t=t0..t1)
plot(curve)
Solution:
Carrying on from Question 9, we assume $\theta(0) = 0$.

$$\theta = 0.01\pi t^2 + 0.06\pi t$$

Hence,

$$\begin{aligned} R(t) &= re_r = r[\cos(\theta)\hat{\mathbf{i}} + \sin(\theta)\hat{\mathbf{j}}] \\ &= 0.01t\cos(0.01\pi t^2 + 0.06\pi t)\hat{\mathbf{i}} + 0.01t\sin(0.01\pi t^2 + 0.06\pi t)\hat{\mathbf{j}} = \end{aligned}$$

```
theta1 = 0.01 * PI * t^2 + 0.06 * PI * t:; r = 0.01 * t:
x(t):= r * cos(theta1):
y(t):= r * sin(theta1):
curve:=plot::Curve2d([x(t),y(t)],t=0..20):
plot(curve)
```