MATHS340: Real and Complex Calculus

Tutorial 3

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1. Show the following partial derivatives of the spherical coordinate unit vectors are as follows:

$$\frac{\partial e_\theta}{\partial \theta} = -sin(\phi)e_\rho - cos(\phi)e_\phi,\ \frac{\partial e_\phi}{\partial \theta} = cos(\phi)e_\theta$$

Solution: Firstly, the unit vectors in spherical co-ordinates are:

$$e_\rho = cos(\theta)sin(\phi)\hat{\mathbf{i}} + sin(\theta)sin(\phi)\hat{\mathbf{j}} + cos(\phi)\hat{\mathbf{k}}$$

$$e_\phi = cos(\theta)cos(\phi)\hat{\mathbf{i}} + sin(\theta)cos(\phi)\hat{\mathbf{j}} - sin(\phi)\hat{\mathbf{k}}$$

$$e_\theta = -sin(\theta)\hat{\mathbf{i}} + cos(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Left hand side,

$$\frac{\partial e_\theta}{\partial \theta} = -cos(\theta)\hat{\mathbf{i}} - sin(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Right hand side,

$$\begin{aligned} -sin(\phi)e_\rho - cos(\phi)e_\phi &= -cos(\theta)sin^2(\phi)\hat{\mathbf{i}} - sin(\theta)sin^2(\phi)\hat{\mathbf{j}} + -cos(\phi)sin(\phi)\hat{\mathbf{k}} \\ &\quad - (cos(\theta)cos^2(\phi)\hat{\mathbf{i}} + sin(\theta)cos^2(\phi)\hat{\mathbf{j}} - sin(\phi)cos(\phi)\hat{\mathbf{k}}) \\ &= -cos(\theta)\hat{\mathbf{i}} - sin(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}} \end{aligned}$$

Since the left hand side equals the right hand side for all ρ, θ and ϕ; we have shown that: $\frac{\partial e_\theta}{\partial \theta} = -sin(\phi)e_\rho - cos(\phi)e_\phi$.
Similarly,
Left hand side,

$$\frac{\partial e_\phi}{\partial \theta} = -sin(\theta)cos(\phi)\hat{\mathbf{i}} + cos(\theta)cos(\phi)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Right hand side,

$$cos(\phi)e_\theta = -sin(\theta)cos(\phi)\hat{\mathbf{i}} + cos(\theta)cos(\phi)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

Since the left hand side equals the right hand side for all ρ, θ and ϕ; we have shown that: $\frac{\partial e_\phi}{\partial \theta} = cos(\phi)e_\theta$

2. A jet is flying directly north at 1000 kph in level flight 10 kilometres above Earth's surface. Find the acceleration of the jet when it is at latitude 45 degrees north. Assume that Earth is a sphere with radius 6371 kilometres and is not rotating.

Solution: Using spherical co-ordinates and SI units,
$\rho = 6.381 \times 10^6$m, $\rho' = 0$, $\rho'' = 0$ due to the "level flight".
$\theta' = 0$, $\theta'' = 0$, θ is some constant since the Earth is not rotating. We can set it to 0 for simplicity. $\theta = 0$.
The angular velocity is $\phi' = \frac{v}{\rho}$, where v is the linear velocity. $\phi' = \dfrac{\frac{10^6}{3600}}{6.381 \times 10^6}$ rad/s. $\phi'' = 0$ as ϕ' is constant.
The formula for acceleration in spherical co-ordinates is:

$$\begin{aligned} \mathbf{a}(t) &= (\rho'' - \rho(\phi')^2 - \rho(\theta')^2 sin^2(\phi))e_\rho \\ &\quad + (\rho\phi'' + 2\rho'\phi' - \rho(\theta')^2 sin(\phi)cos(\phi))e_\phi + (\rho\theta'' sin(\phi) + 2\rho'\theta' sin(\phi) + 2\rho\theta'\phi' cos(\phi))e_\theta \end{aligned}$$

After substitution with $\phi = \frac{\pi}{4}$ (45 degrees north),

$$\mathbf{a}(t) = -\rho(\phi')^2 e_\rho + 0e_\phi + 0e_\theta = -0.0120922e_\rho$$

This answer makes sense, since in circular motion, the centripetal acceleration is straight towards the centre of the circle.

3. Find the arc length of the curve $\mathbf{C}(\tau) = [2cos(3\tau + \frac{\pi}{4}), 2sin(3\tau + \frac{\pi}{4}), \tau^2]$ from $\tau = 0$ to $\tau = \tau_1$. If you have calculated the integral by hand, check your answer using MuPAD.

Solution: $\mathbf{C}'(\tau) = [-6sin(3\tau + \frac{\pi}{4}), 6cos(3\tau + \frac{\pi}{4}), 2\tau]$
$\|\mathbf{C}'(\tau)\| = \sqrt{36 + 4\tau^2} = 2\sqrt{9 + \tau^2}$
L $= \int_0^{\tau_1} \|\mathbf{C}\| \; d\tau = \int_0^{\tau_1} 2\sqrt{9 + \tau^2} \; d\tau$
This requires a trig substitution to solve.
Using MuPAD, where we have used a instead of τ_1.

```
C := matrix([2*cos(3*t+ PI * 4), 2*sin(3*t + PI*4), t^2]):
Cprime := diff(C,t):
assume(t, Type::Real)
norm(Cprime, 2):
simplify(%):
sqrt(scalarProduct(Cprime, Cprime)):
simplify(%)
```

$2\sqrt{t^2+9}$

```
int(%,t = 0..a)
```

$9\,\text{arcsinh}\left(\frac{a}{3}\right) + a\sqrt{a^2+9}$

4. Find the line integral of $f(x, y) = x^2 + y^2$ along the curve $\mathbf{C}(\tau) = [cos(\tau), sin(\tau), 0]$ from $\tau = 0$ to $\tau = \tau_1$. If you have calculated the integral by hand, check your answer using MuPAD.

Solution: $\mathbf{C}'(\tau) = [-sin(\tau), cos(\tau), 0], \;\; \|\mathbf{C}'(\tau)\| = 1$

$$\int_C f(x, y, z) \; ds = \int_0^{\tau_1} f(\mathbf{C}(\tau)) \; \|\mathbf{C}'(\tau)\| \; d\tau = \int_0^{\tau_1} (cos^2(\tau) + sin^2(\tau)) \times 1 \; d\tau$$
$$= \int_0^{\tau_1} \; d\tau = \tau_1$$

Using MuPAD, where we have used a instead of τ_1.

```
C := matrix([cos(t), sin(t), 0]):
Cprime := diff(C,t):
assume(t, Type::Real)
norm(Cprime, 2):
simplify(%):
sqrt(scalarProduct(Cprime, Cprime)):
simplify(%)
```

1

```
int(%,t = 0..a)
```

a

5. The mass M of a piece of wire that has the shape C is defined as

$$\int_C \sigma \; ds$$

where σ is the mass per unit length and ds is the differential arclength. Suppose the points R on C are given by $R = [2cos(\tau), 2sin(\tau), 2\tau]$, $0 \leq \tau \leq 10\pi$. Find M when $\sigma = 2\tau$.

Solution: $\mathbf{R}'(\tau) = [-2sin(\tau), 2cos(\tau), 2]$, $\|\mathbf{R}'(\tau)\| = \sqrt{4+4} = 2\sqrt{2}$

$$M = \int_C \sigma \; ds = \int_0^{10\pi} \sigma(\tau)\; \|\mathbf{R}'(\tau)\| \; d\tau = \int_0^{10\pi} 2\tau \times 2\sqrt{2} \; d\tau = 4\sqrt{2}\int_0^{10\pi} \tau \; d\tau$$
$$= 2\sqrt{2}\; [\tau^2]\Big|_0^{10\pi} = 200\sqrt{2}\pi^2$$

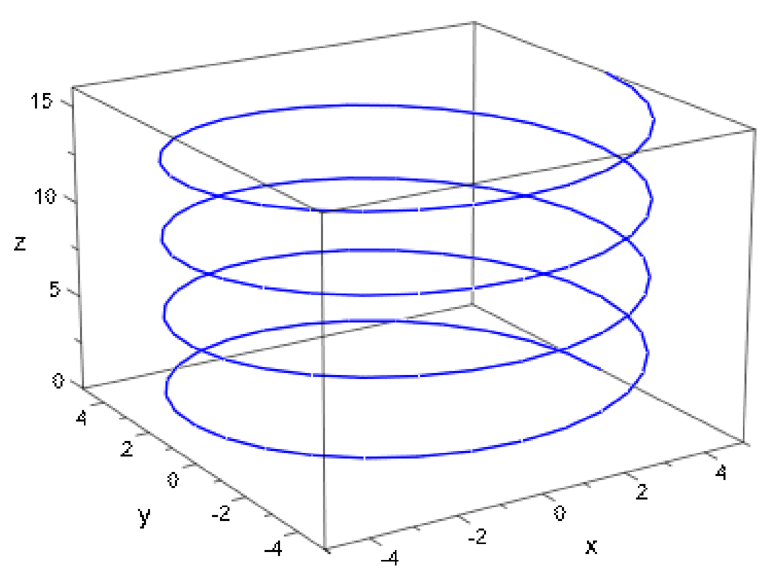
6. A spiral staircase in a building has the shape of a helix with a radius of 5 metres. Between two levels of the building, the stairs make one full revolution and climb by 4 meters.

   (a) Write down a parametrization for the staircase assuming the building contains four levels. Check your answer in MuPAD by plotting the parametric equations you have written down.

   (b) If a person walks up the stairs from the first floor to the second floor, how far have they travelled?

Solution: Using cylindrical co-ordinates, $R(t) = [rcos(\theta), rsin(\theta), z]$.
In this model, it takes 1 unit of time to walk up 1 floor; $r = 5$, $\theta = 2\pi t$, $z = 4t$.
So, $R(t) = [5cos(2\pi t), 5sin(2\pi t), 4t]$.

```
x(t) := 5 * cos(2*PI*t):; y(t) := 5 * sin(2*PI*t):; z(t) := 4*t:
curve:=plot::Curve3d([x(t),y(t),z(t)],t=0..4):
plot(curve)
```



$$\int_C ds = \int_0^1 \|\mathbf{R}'(t)\| \; dt$$

We need to find $\|\mathbf{R}'(t)\|$,

$$R'(t) = [-10\pi sin(2\pi t), 10\pi cos(2\pi t), 4],$$

$$\|\mathbf{R}'(t)\| = \sqrt{100\pi^2 + 16}$$

Therefore,

$$\int_C ds = \int_0^1 \|\mathbf{R}'(t)\| \; dt = \int_0^1 \sqrt{100\pi^2+16} \; dt = \sqrt{100\pi^2+16} = 31.7$$

The person will have travelled 31.7 metres.