MATHS340: Real and Complex Calculus

Tutorial 6

AceNighJohn

1. Calculate the integral $\iint_S \mathbf{v}\cdot\mathbf{n}\, dS$, where $v = e^x \sin(y)\hat{\mathbf{i}} + e^x \cos(y)\hat{\mathbf{j}} + yz^2\hat{\mathbf{k}}$, and S is the surface of the outward-oriented box bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 2$.

Solution: Using the Divergence Theorem, $\iint_S \mathbf{v}\cdot\mathbf{n}\ dS = \iiint_V div(v)\ dV$.
$div(\mathbf{v}) = e^x \sin(y) - e^x \sin(y) + 2yz = 2yz$
V $= \{(x, y, z)|\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 2\}$

$$\iint_S \mathbf{v}\cdot\mathbf{n}\ dS = \iiint_V div(\mathbf{v})\ dV = \int_0^1 \int_0^1 \int_0^2 2yz\ dzdydx = [1-0] \int_0^1 y\ [z^2]\Big|_0^2\ dy$$
$$= \int_0^1 4y\ dy = 2[y^2]\Big|_0^2 = 2$$

2. Let $\mathbf{F}(x, y, z) = -z\hat{\mathbf{i}} + x\hat{\mathbf{k}}$ and let S be the sphere of radius 3 centred at the origin with an outward-pointing normal vector.

   (a) Plot the vector field and the surface in MuPAD.

   (b) Find the value of the surface integral of F over S.

Solution: Using the following MuPAD commands;

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f1 := -z:; f2 := 0:; f3 := x:
x0:= -4:; x1 := 4:; y0:= -4:; y1 := 4:; z0:= -4:; z1 := 4:;
field:=plot::VectorField3d([f1, f2, f3], x=x0..x1,
y=y0..y1, z =z0..z1, Mesh=[5,5,5]):
rho:= 3:;
sphere:=plot::Spherical([rho, t, p],t = 0..2*PI, p = 0..PI):
plot(field, sphere)
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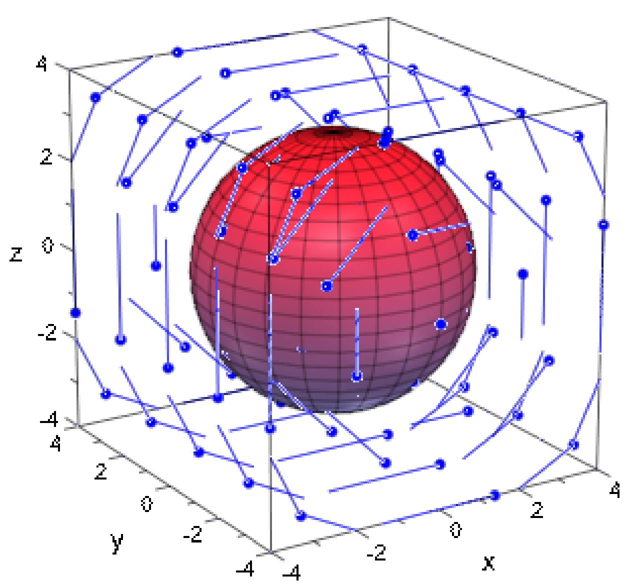


Using the Divergence Theorem, $\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iiint_V div(\mathbf{F})\ dV$.
Calculating the divergence of $div(\mathbf{F})$ yields: $div(\mathbf{F}) = 0 + 0 + 0 = 0$
Hence,

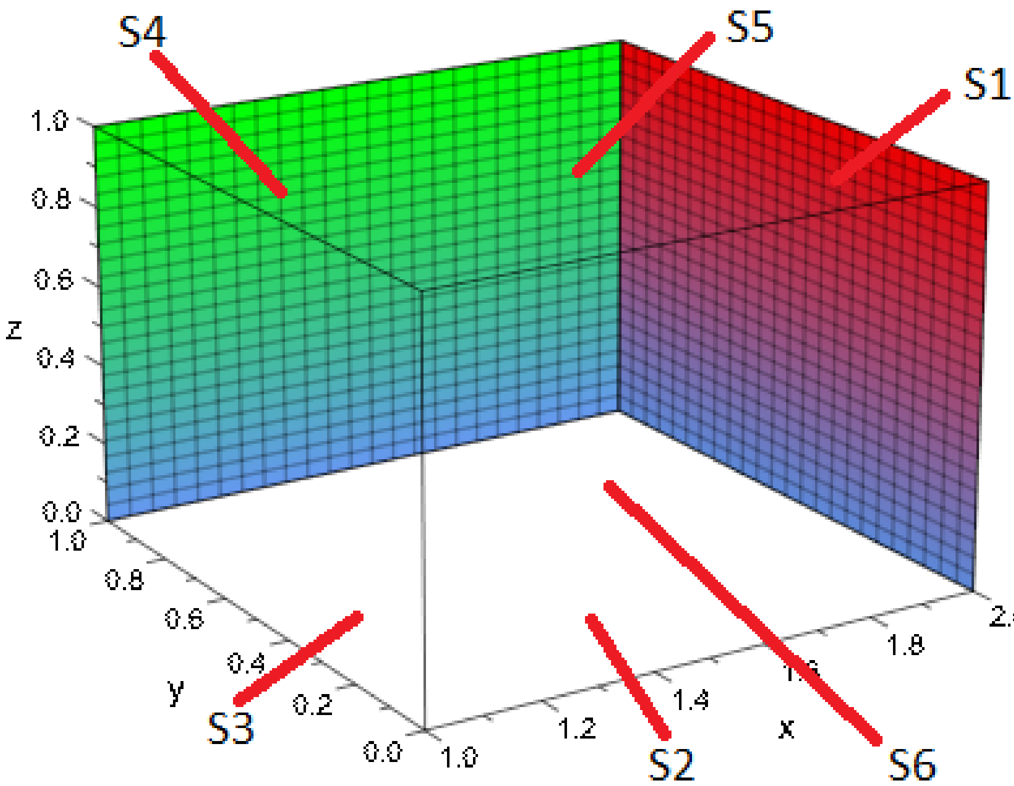
$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \iiint_V div(\mathbf{F})\ dV = \iiint_V 0\ dV = 0$$

3. Let $\mathbf{F}(x,y,z) = x^2\hat{\mathbf{i}} + 2y^2\hat{\mathbf{j}} + 3z^2\hat{\mathbf{k}}$ and let S be the surface of the box with faces $x = 1$, $x = 2, y = 0, y = 1, z = 0$ and $z = 1$. If S is oriented outwards, calculate the surface integral of $\mathbf{F}$ over S in two ways:

   (a) directly

   (b) using the Divergence Theorem.

   Solution: Computing the surface integral directly requires a picture.



S1 through to S4 are the sides of the cube, while S5 and S6 are the top and bottom of the cube respectively.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \sum_{i=1}^{6}\iint_{S_i}\mathbf{F}\cdot\mathbf{n}\,dS$$

Calculating each surface integral separately,

$$\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot\hat{\mathbf{i}}\,dS = \iint_{S_1}x^2\,dS = 4\iint_{S_1}dS = 4$$

Since $x = 2$ on S_1, and the final integrand is 1 so we yield the surface area.

$$\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot-\hat{\mathbf{j}}\,dS = \iint_{S_2}-2y^2\,dS = \iint_{S_2}0\,dS = 0$$

Since $y = 0$ on S_2.

$$\iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_3}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot-\hat{\mathbf{i}}\,dS = \iint_{S_3}-x^2\,dS = -\iint_{S_3}dS = -1$$

Since $x = 1$ on S_3, and the final integrand is 1 so we yield the surface area.

$$\iint_{S_4}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_4}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot\hat{\mathbf{j}}\,dS = \iint_{S_4}2y^2\,dS = 2\iint_{S_4}dS = 2$$

Since $y = 1$ on S_4, and the final integrand is 1 so we yield the surface area.

$$\iint_{S_5}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_5}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot\hat{\mathbf{k}}\,dS = \iint_{S_5}3z^2\,dS = 3\iint_{S_5}dS = 3$$

Since $z = 1$ on S_5, and the final integrand is 1 so we yield the surface area.

$$\iint_{S_6}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_6}\begin{bmatrix}x^2\\2y^2\\3z^2\end{bmatrix}\cdot-\hat{\mathbf{k}}\,dS = \iint_{S_6}-3z^2\,dS = \iint_{S_6}0\,dS = 0$$

Since $z = 0$ on S_6.
Hence,

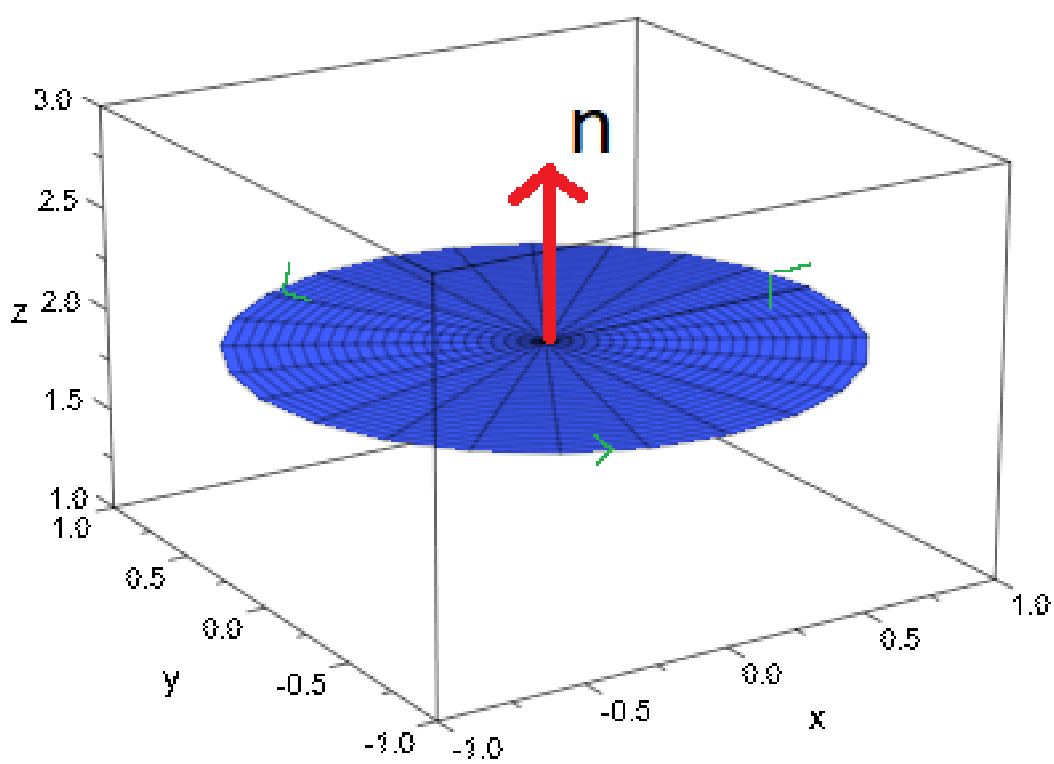
$$\iint_S \mathbf{F}\cdot\mathbf{n}\ dS = \sum_{i=1}^{6}\iint_{S_i}\mathbf{F}\cdot\mathbf{n}\ dS = 4+0+(-1)+2+3+0 = 8$$

Using the Divergence Theorem in this case is much easier.
V $= \{(x, y, z)|\ 1 \le x \le 2, 0 \le y \le 1, 0 \le z \le 1\}$
Calculating the divergence of $\mathbf{F}$ yields: $div(\mathbf{F}) = 2x + 4y + 6z$

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\ dS &= \iiint_V div(\mathbf{F})\ dV = \int_0^1\int_0^1\int_1^2 (2x+4y+6z)\ dxdydz \\ &= \int_0^1\int_0^1 [x^2+4xy+6xz]\Big|_1^2\ dydz = \int_0^1\int_0^1 (3+4y+6z)\ dydz \\ &= \int_0^1 [3y+2y^2+6yz]\Big|_0^1\ dz = \int_0^1 (5+6z)\ dz = [5z+3z^2]\Big|_0^1 = 8\end{aligned}$$

4. Let $\mathbf{F}(x, y, z) = xz\hat{\mathbf{i}} + (x + yz)\hat{\mathbf{j}} + x^2\hat{\mathbf{k}}$ and let C be the circle $x^2 + y^2 = 1, z = 2$ oriented counterclockwise when viewed from above. Use Stokes' Theorem to find the circulation of $\mathbf{F}$ around C.

Solution: Using Stokes Theorem, $\oint_C \mathbf{F}\cdot\ dr = \iint_S textcurl(\mathbf{F})\cdot\mathbf{n}\ dS$
Calculating the curl of $\mathbf{F}$ yields: $textcurl(\mathbf{F}) = -y\hat{\mathbf{i}} - x\hat{\mathbf{j}} + \hat{\mathbf{k}}$



The easiest surface to use is the plane, namely $z = 2$. And since the curve has anti-clockwise orientation, the right hand rule shows that the normal vector is upward facing. i.e: $\mathbf{n} = \hat{\mathbf{k}}$.
The projection of the surface onto the x-y plane is the unit circle $x^2 + y^2 = 1$.
It is natural to use polar co-ordinates, R $= \{(r, \theta)|\ 0 \le r \le\ 1, 0 \le \theta \le 2\pi\}$
Therefore,

$$\oint_C \mathbf{F}\cdot\ dr = \iint_S textcurl(\mathbf{F})\cdot\mathbf{n}\ dS = \iint_R \begin{bmatrix}-y\\-x\\1\end{bmatrix}\cdot\begin{bmatrix}0\\0\\1\end{bmatrix}\ dA = \iint_R dA = \pi$$

Since the integrand is 1, we are calculating the area of the unit circle.