MATHS340: Real and Complex Calculus

Tutorial 4

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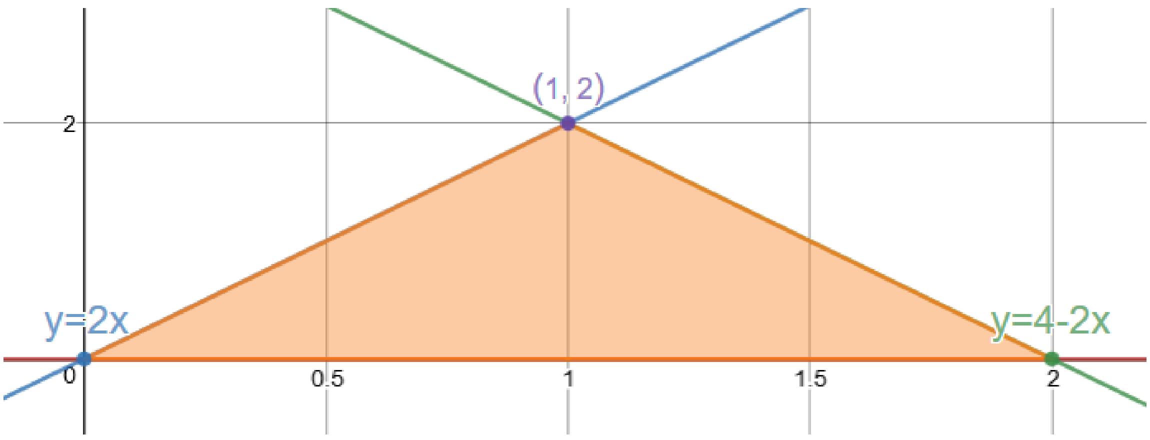
1. Evaluate

$$\iint_R x^2 y dy dx$$

where R is the triangle with vertices at (0,0), (2,0) and (1,2).

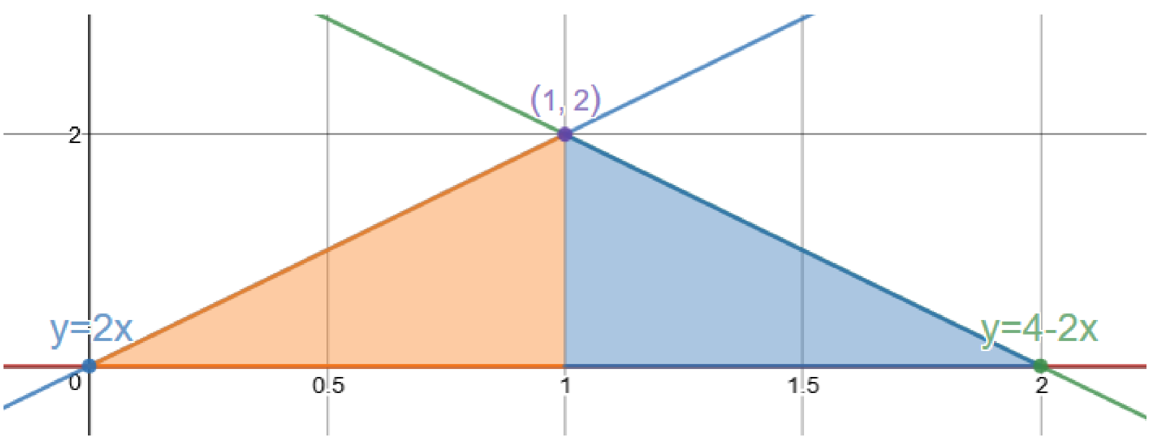
Solution: There are two approaches to the solution. The first approach is to let the region be of Type II. R = $\{(x,y)|\ 0 \le y \le 2, \frac{y}{2} \le x \le \frac{4-y}{2}\}$.



So the integral becomes:

$$\int_0^2 \int_{\frac{y}{2}}^{\frac{4-y}{2}} x^2 y\, dy\, dx = \frac{1}{3}\int_0^2 x^3\ [y]\Big|_{\frac{y}{2}}^{\frac{4-y}{2}} dy = \frac{1}{3}\int_0^2 y[(\frac{4-y}{2})^3 - (\frac{y}{2})^3]\, dy$$
$$= \frac{1}{3}\int_0^2 (-\frac{y^4}{4} + \frac{3y^3}{2} - 6y^2 + 8y)\, dy = \frac{1}{3}[-\frac{y^5}{20} + \frac{3y^4}{8} - 2y^3 + 4y^2)]\Big|_0^2$$
$$= \frac{1}{3} \times \frac{22}{5} = \frac{22}{15}$$

The second approach is to let the Region R be of Type I, to do this, we need to split up the region into two regions of Type I.



In this case,
R = $\{(x,y)|\ 0 \le x \le 1, 0 \le y \le 2x\} \cup \{(x,y)|1 \le x \le 2, 0 \le y \le 4-2x\}$

$$\iint_R x^2 y\ \ dx\, dy = \int_0^1 \int_0^{2x} x^2 y\, dx\, dy + \int_1^2 \int_0^{4-2x} x^2 y\, dx\, dy$$
$$= \frac{1}{2}[\int_0^1 x^2\ [y^2]\Big|_0^{2x} dx + \int_1^2 x^2\ [y^2]\Big|_0^{4-2x} dx]$$
$$= \frac{1}{2}[\int_0^1 x^2[(2x)^2]\, dx + \int_1^2 x^2(4-2x)^2\, dx]$$
$$= \frac{1}{2}[\int_0^1 4x^4\, dx + \int_1^2 (4x^4 - 16x^3 + 16x^2)\, dx\ ]$$
$$= \frac{1}{2}([\frac{4x^5}{5}]\Big|_0^1 + [\frac{4x^5}{5} - 4x^4 + \frac{16x^3}{3}]\Big|_1^2)$$
$$= \frac{1}{2}(\frac{4}{5} + [\frac{64}{15} - \frac{32}{15}]) = \frac{1}{2} \times \frac{44}{15} = \frac{22}{15}$$

2. Let S be the surface of the cylinder $x^2 + y^2 = 25$ bounded by $z = 0$ and $z = 3$. Evaluate the integral:

$$\iint_S xydS$$

Solution: Let's first parameterize the surface, it is natural to use cylindrical co-ordinates. Let $x = 5cos(\theta)$, $y = 5sin(\theta)$, $z = z$, $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 3$

Define $R = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = 5cos(\theta)\hat{\mathbf{i}} + 5sin(\theta)\hat{\mathbf{j}} + z\hat{\mathbf{k}}$
Taking derivatives,
$R_\theta = -5sin(\theta)\hat{\mathbf{i}} + 5cos(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$, $R_z = 0\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + k$
Then using the cross product (order doesn't matter since we'll take the norm of the cross product later),

$$R_\theta \times R_z = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -5sin(\theta) & 5cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = 5cos(\theta)\hat{\mathbf{i}} + 5sin(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

$\|\mathbf{R}_\theta \times \mathbf{R_z}\| = \sqrt{25cos^2(\theta) + 25sin^2(\theta) + 0^2} = 5$
Recall that the Jacobian when using cylindrical co-ordinates is r, which in this question is 5.

$$\iint_S xy\ dS = \iint_R 25sin(\theta)cos(\theta)\|\mathbf{R}_\theta \times \mathbf{R_z}\|\, dA$$

where R is the region in the $\theta - z$ plane; $R = \{(\theta, z)|0 \leq \theta \leq 2\pi, 0 \leq z \leq 3\}$.
Therefore,

$$\begin{aligned}\iint_S xy\ dS &= \int_0^3 \int_0^{2\pi} 25sin(\theta)cos(\theta) \times 5\ \ d\theta\, dz = 375 \int_0^{2\pi} sin(\theta)cos(\theta)\, d\theta \\ &= \frac{375}{2}[sin^2(\theta)]\Big|_0^{2\pi} = 0\end{aligned}$$

3. A city occupies a semicircular region on the coast. The radius of the city is 4km. Find the average distance from points in the city to the ocean.

Solution: Notice that the yellow lines in Figure 1 indicate the distance to the ocean for the 3 chosen points in the city. We are interested in finding the average value of y for every point in the city. Here, we will present two different approaches.
For the first approach, we will introduce another variable, z, where $z(x, y)$ represents the distance of a point (x, y). As discussed previously, $z(x, y) = y$.
Graphically, this introduces a third dimension, and we are interested in the average depth of the surface.

$$\bar{d} = \frac{V}{A}$$

where, $\bar{d}$ is the average depth, V is the volume of surface and A is the surface area of the base.

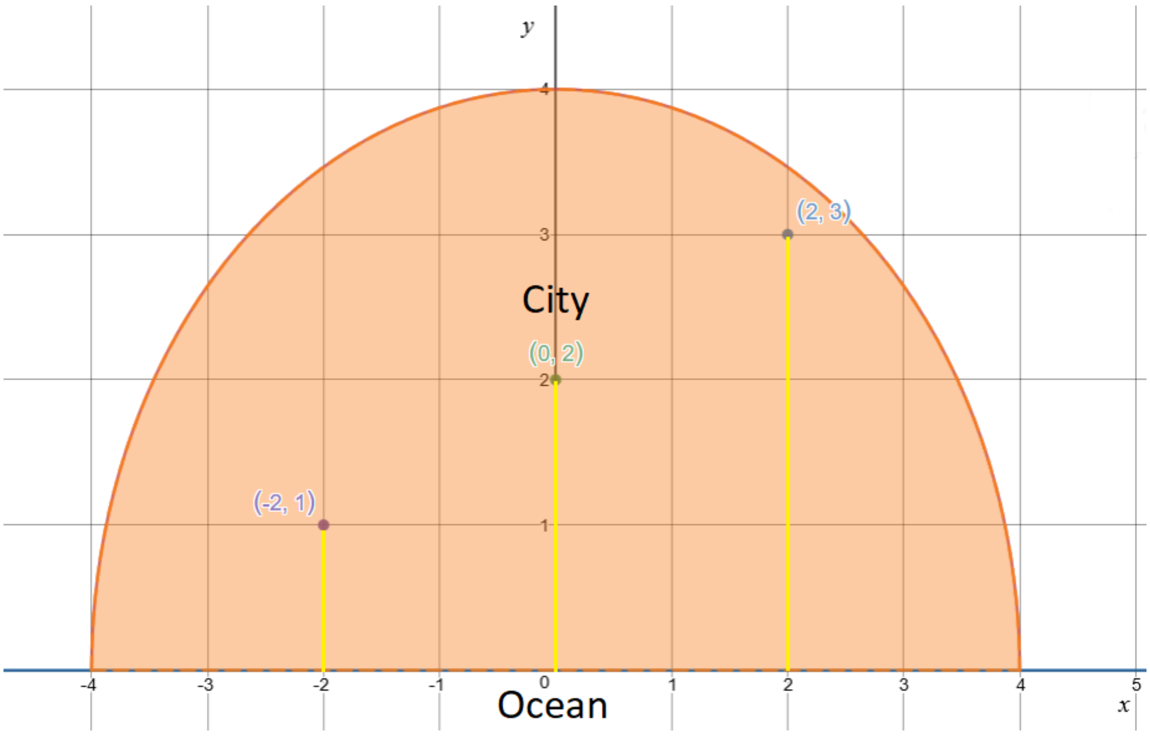


Figure 1: x-y plane

```
x:= u*cos(v):
y:= u*sin(v):
z1:= y:
z2:= 0:
s := plot::Surface([x(u,v), y(u,v), z1(u,v)], u = 0..4, v = 0..PI):
A := plot::Surface([x(u,v), y(u,v), z2(u,v)], u = 0..4, v = 0..PI):
plot(s, A)
```

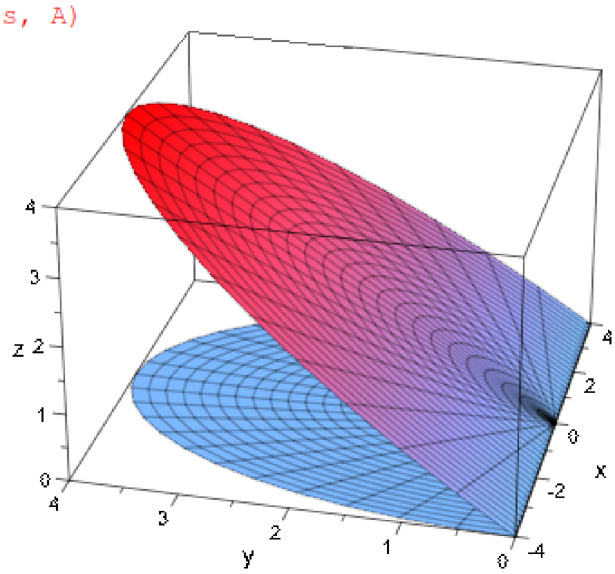


Figure 2: x-y-z plane

$V = \{(x,y,z)|\ 0 \leq x \leq 4, 0 \leq y \leq \sqrt{16-x^2}, 0 \leq z \leq y\}$
Clearly, we should use cylindrical co-ordinates.
$V = \{(r,\theta,z)|\ 0 \leq r \leq 4, 0 \leq \theta \leq \pi, 0 \leq z \leq rsin(\theta)\}$
So,

$$V = \iiint_V dV = \int_0^{\pi}\int_0^4\int_0^{rsin(\theta)} r\ dzdrd\theta = \int_0^{\pi}\int_0^4 r^2 sin(\theta)\ drd\theta$$
$$= \frac{1}{3}\int_0^{\pi} sin(\theta)[r^3]\Big|_0^4\ d\theta = \frac{-64}{3}[cos(\theta)]\Big|_0^{\pi} = \frac{128}{3}$$

Trivially, using the formula for the area of a semi-circle; $A = \frac{1}{2}\pi \times 4^2 = 8\pi$.
Hence, $\overline{d} = \frac{16\pi}{3}$.
The second method uses the concept of centre of mass. The average distance to the ocean is simply the total distance from the ocean of all points in the semi-circle/city divided by the number of points in the semi-circle/city. The number of points is simply the area of the semi-circle. Using this approach, we don't need to introduce a third variable z, but the calculations are identical: Let S be the total distance of all points, recall that y is the distance to the ocean of an individual point,

$$S = \iint_R y\ dA$$

where R is the semi-circle/city. Clearly, we should use polar co-ordinates to represent R. So,

$$S = \iint_R y\ dA = \int_0^{\pi}\int_0^4 rsin(\theta) \times r\ drd\theta = \int_0^{\pi}\int_0^4 r^2 sin(\theta)\ drd\theta$$

Define $\overline{d}$ to be the average distance for all points,

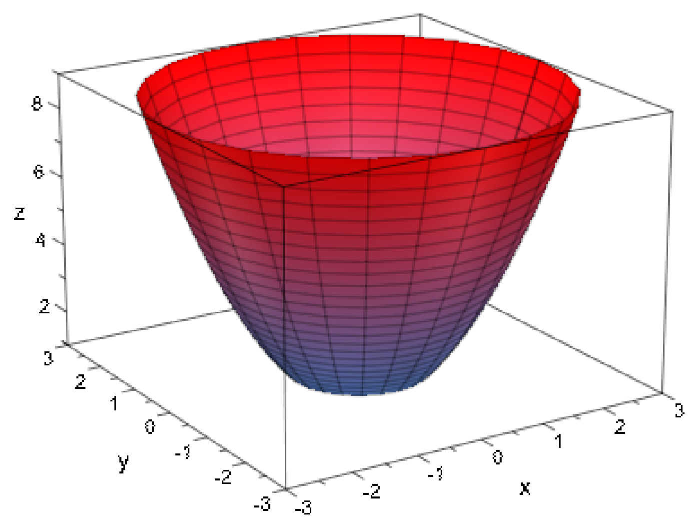
$$\overline{d} = \frac{S}{A}$$

A is the area of the semi-circle/city which is the same as in the first approach. Notice that S is also identical to V as they have identical integrals.
Hence, $\overline{d} = \frac{16\pi}{3}$.

4. The surface S is defined by $z = x^2 + y^2$, where $1 \leq z \leq 9$. Use an integral to find the surface area of S.

Solution: Start by drawing a picture,

```
x := u*cos(v):
y := u*sin(v):
z := x^2 + y^2:
s := plot::Surface([x(u,v), y(u,v), z(u,v)], u = 1..3, v = 0..2*PI):
plot(s)
```



Using cylindrical co-ordinates,
Let $R = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (x^2+y^2)\hat{\mathbf{k}} = rcos(\theta)\hat{\mathbf{i}} + rsin(\theta)\hat{\mathbf{j}} + r^2\hat{\mathbf{k}}$
Taking derivatives,
$R_\theta = -rsin(\theta)\hat{\mathbf{i}} + rcos(\theta)\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$, $R_r = cos(\theta)\hat{\mathbf{i}} + sin(\theta)\hat{\mathbf{j}} + 2r\hat{\mathbf{k}}$
Then using the cross product (order doesn't matter since we'll take the norm of the cross product later),

$$R_\theta \times R_r = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -rsin(\theta) & rcos(\theta) & 0 \\ cos(\theta) & sin(\theta) & 2r \end{bmatrix} = 2r^2cos(\theta)\hat{\mathbf{i}} + 2r^2sin(\theta)\hat{\mathbf{j}} - r\hat{\mathbf{k}}$$

$\|\mathbf{R}_\theta \times \mathbf{R}_\mathbf{r}\| = \sqrt{4r^4cos^2(\theta) + 4r^4sin^2(\theta) + r^2} = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$
The surface area is: $\iint_S dS$, where S is the surface.

$$\iint_S dS = \iint_R \|\mathbf{R}_\theta \times \mathbf{R}_\mathbf{r}\| dA$$

where R is the region on the x-y plane after the surface has been projected onto the x-y plane.
Using polar co-ordinates; R=$\{(r,\theta)|\ 1 \leq r \leq 3, 0 \leq \theta \leq 2\pi\}$.
Therefore,

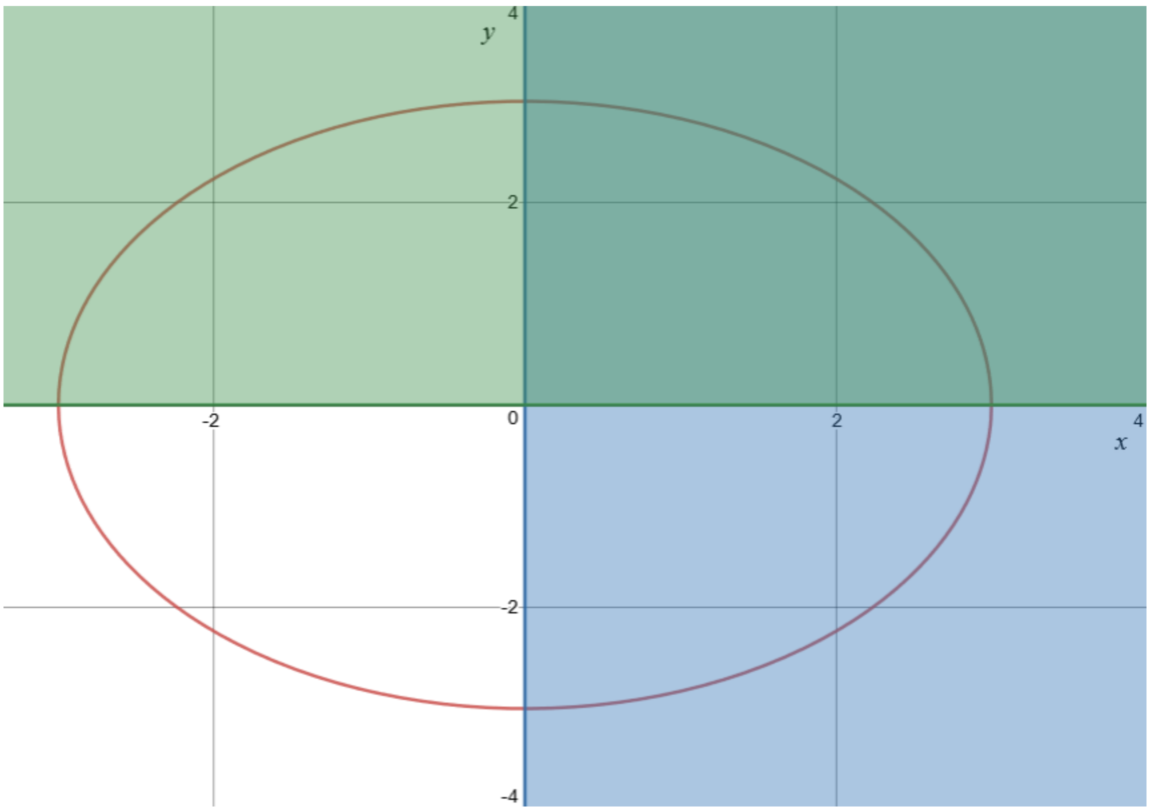
$$\iint_S dS = \iint_R \|\mathbf{R}_\theta \times \mathbf{R}_\mathbf{r}\| dA = \int_0^{2\pi}\int_1^3 r\sqrt{4r^2+1}\ drd\theta = 2\pi\int_1^3 r\sqrt{4r^2+1}\ dr$$
$$= 2\pi \times \frac{2}{3} \times \frac{1}{8}[(4r^2+1)^{\frac{3}{2}}]\Big|_1^3 = \frac{\pi}{6}[37^{\frac{3}{2}} - 5^{\frac{3}{2}}] = 111.988 = 112$$

5. Find the surface area of the plane $x + 2y + 4z = 16$ cut off by $x = 0$, $y = 0$ and $x^2 + y^2 = 9$, where $x, y \geq 0$.

Solution: Let $z = f(x, y)$, then $z_x = f_x$ and $z_y = f_y$
The surface is given by $x + 2y + 4z = 16$.
Differentiating with respect to x yields; $1 + 4z_x = 0$, $z_x = -\frac{1}{4}$
Differentiating with respect to y yields; $2 + 4z_y = 0$, $z_x = -\frac{1}{2}$
In Cartesian co-ordinates,

$$\iint_S dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

In this case, we will need to make a slight modification by using polar co-ordinates;



The region R is the projection of the surface down onto the x-y plane. Clearly, we are interested in the first quadrant of the circle $x^2 + y^2 = 9$.
R=$\{(r, \theta) |\ 0 \leq r \leq 3, 0 \leq \theta \leq \frac{\pi}{2}\}$.
Therefore,

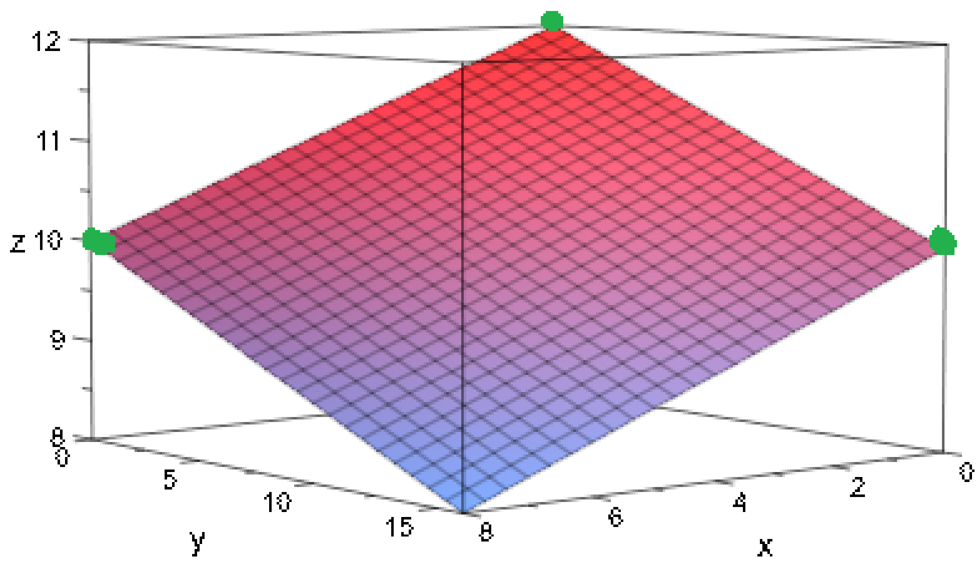
$$\iint_S dS = \iint_R \sqrt{f_x^2 + f_y^2 + 1} dA = \int_0^{\frac{\pi}{2}} \int_0^3 \sqrt{1 + \frac{1}{16} + \frac{1}{4}} \times r\ dr d\theta = \frac{\pi}{2} \int_0^3 \sqrt{\frac{21}{16}} r\ dr$$
$$= \frac{\pi}{2} \times \frac{\sqrt{21}}{4} r \times \frac{1}{2} [r^2] \Big|_0^3 = \frac{9}{16} \sqrt{21} \pi$$

6. A building is 8 metres wide and 16 metres long. It has a flat roof that is 12 metres high in one corner, and 10 metres high at each of the adjacent corners.
   - What is the surface area of the roof?
   - What is the volume of the building?

Solution: The roof is a section of a plane, we are given three points on this plane so we are able to determine the equation of the plane. We will choose the points $(0, 0, 12)$, $(8, 0, 10)$, $(0, 16, 10)$, and then define the vectors $\vec{P}$ and $\vec{Q}$, which lie on the plane and are linearly independent.
$\vec{P} = (8, 0, 10) - (0, 0, 12) = (8, 0, -2)$, $\vec{Q} = (0, 16, 10) - (0, 0, 12) = (0, 16, -2)$
The surface area is the norm of the cross product of $\vec{P}$ and $\vec{Q}$.

$$\vec{P} \times \vec{Q} = \begin{bmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 8 & 0 & -2 \\ 0 & 16 & -2 \end{bmatrix} = 32\hat{\mathbf{i}} + 16\hat{\mathbf{j}} + 128\hat{\mathbf{k}}$$

$$\left\| \vec{P} \times \vec{Q} \right\| = \sqrt{32^2 + 16^2 + 128^2} = 132.9$$



The equation of the plane is given by: $32x + 16y + 128z = d$, where the co-efficients are found from the cross product of $\vec{P}$ and $\vec{Q}$. To find, simply substitute any one of the three points. Let's substitute in (0,0,12), $d = 128 \times 12 = 1536$. Therefore, $32x + 16y + 128z = 1536$.
The volume is found using a triple integral.
V=$\{(x, y, z)|\ 0 \leq x \leq 8, 0 \leq y \leq 16, 0 \leq z \leq \frac{1}{128}(1536 - 32x - 16y)\}$

$$\begin{aligned}
\iiint_V dV &= \int_0^8 \int_0^{16} \int_0^{\frac{1}{128}(1536-32x-16y)} dzdydx = \int_0^8 \int_0^{16} \frac{1}{128}(1536 - 32x - 16y)\ dydx \\
&= \frac{1}{128} \int_0^8 [1536y - 32xy - 8y^2]\Big|_0^{16} dx = \frac{1}{128} \int_0^8 (22528 - 512x)\ dx \\
&= 4 \int_0^8 (44 - x)\ dx = 4[44x - \frac{1}{2}x^2]\Big|_0^8 = 1280
\end{aligned}$$

7. Consider the closed region bounded by the surfaces

$$z = e^{-x^2-y^2},\ z = 0,\ x^2 + y^2 = 1$$

Find the volume of the region.

Solution: V=$\{(x, y, z)|\ 0 \leq x^2 + y^2 \leq 1, 0 \leq z \leq e^{-x^2-y^2}\}$
It is natural to use cylindrical co-ordinates.
V=$\{(r, \theta, z)|\ 0 \leq r \leq 1, 0 \leq r \leq 2\pi, 0 \leq z \leq e^{-r^2}\}$

$$\begin{aligned}
\iiint_V dV &= \int_0^{2\pi} \int_0^1 \int_0^{e^{-r^2}} r\ dzdrd\theta = 2\pi \int_0^1 re^{-r^2} dr = 2\pi \times (-\frac{1}{2})[e^{-r^2}]\Big|_0^1 \\
&= -\pi(e^{-1} - 1) = \pi(1 - e^{-1})
\end{aligned}$$