Ramsey Numbers and Adiabatic Quantum Computing

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The graph-theoretic Ramsey numbers are notoriously difficult to calculate. In fact, for the two-color Ramsey numbers $R(m, n)$ with $m, n \geq 3$, only nine are currently known. We present a quantum algorithm for the computation of the Ramsey numbers $R(m, n)$. We show how the computation of $R(m, n)$ can be mapped to a combinatorial optimization problem whose solution can be found using adiabatic quantum evolution. We numerically simulate this adiabatic quantum algorithm and show that it correctly determines the Ramsey numbers $R(3, 3)$ and $R(2, s)$ for $5 \leq s \leq 7$. We then discuss the algorithm's experimental implementation, and close by showing that Ramsey number computation belongs to the quantum complexity class quantum Merlin Arthur.

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In an arbitrary party of N people, one might ask whether there is a group of m people who are all mutually acquainted, or a group of n people who are all mutual strangers. Using Ramsey theory [1,2], it can be shown that a threshold value $R(m, n)$ exists for the party size N so that when $N \geq R(m, n)$, all parties of N people will either contain m mutual acquaintances, or n mutual strangers. The threshold value $R(m, n)$ is an example of a two-color Ramsey number. Other types of Ramsey numbers exist, though we will focus on two-color Ramsey numbers in this Letter.

One can represent the N-person party problem by an N-vertex graph. Here each person is associated with a vertex, and an edge is drawn between a pair of vertices only when the corresponding people know each other. In the case where m people are mutual acquaintances, there will be an edge connecting any pair of the m corresponding vertices. Similarly, if n people are mutual strangers, there will be no edge between any of the n corresponding vertices. In the language of graph theory [3], the m vertices form an m clique, and the n vertices form an n-independent set. The party problem is now a statement in graph theory: if $N \geq R(m, n)$, every graph with N vertices will contain either an m clique, or an n-independent set. Ramsey numbers can also be introduced using colorings of complete graphs, and $R(m, n)$ corresponds to the case where only two colors are used.

Ramsey theory has found applications in mathematics, information theory, and theoretical computer science [4]. An application of fundamental significance appears in the Paris-Harrington (PH) theorem of mathematical logic [5] which established that a particular statement in Ramsey theory related to graph colorings and natural numbers is true, though unprovable within the axioms of Peano arithmetic. Such statements are known to exist as a consequence of Godel's incompleteness theorem, though the PH theorem provided the first natural example. Deep connections have also been shown to exist between Ramsey theory, topological dynamics, and ergodic theory [6].

Ramsey numbers grow extremely quickly and so are notoriously difficult to calculate. In fact, for two-color Ramsey numbers $R(m, n)$ with $m, n \geq 3$, only nine are presently known [3]. To check whether $N =^{?} R(m, n)$ requires examining *all* $2^{N(N-1)/2}$ N-vertex graphs. The number of graphs to be checked thus grows superexponentially with N, and so the task quickly becomes intractable. Ketonen and Solovay [7] have shown that this is the root cause for why the statement in the PH theorem cannot be proved within Peano arithmetic.

In this Letter, we (i) present a quantum algorithm for calculating Ramsey numbers based on adiabatic quantum evolution, (ii) numerically simulate the algorithm to verify that it correctly calculates small Ramsey numbers, (iii) discuss its experimental implementation, and (iv) show that Ramsey number computation belongs to the quantum complexity class quantum Merlin Arthur (QMA).

Optimization problem.—We begin by establishing a 1-1 correspondence between the set of N-vertex graphs and binary strings of length $L = N(N-1)/2$. To each N-vertex graph G there corresponds a unique adjacency matrix $A(G)$ which is an $N \times N$ symmetric matrix with vanishing diagonal matrix elements, and with the off-diagonal element $a_{i,j} = 1(0)$ when distinct vertices i and j are (are not) joined by an edge. It follows that $A(G)$ is determined by its lower triangular part. By concatenating columnwise the matrix elements $a_{i,j}$ appearing below the principal diagonal, we can construct a unique binary string $g(G)$ of length L for each graph G:

$$g(G) \equiv a_{2,1} \cdots a_{N,1} a_{3,2} \cdots a_{N,2} \cdots a_{N,N-1}. \quad (1)$$

Given the string $g(G)$, the following procedure determines the number of m cliques in G. Choose m vertices $S_\alpha = \{v_1, \ldots, v_m\}$ from the N vertices of G and form the

product $\mathcal{C}_\alpha = \prod_{(v_j, v_k \in S_\alpha)}^{(j \neq k)} a_{v_j, v_k}$. Note that $\mathcal{C}_\alpha = 1$ when S_α forms an m clique; otherwise $\mathcal{C}_\alpha = 0$. Now repeat this procedure for all $\rho = C(N, m)$ ways of choosing m vertices from N vertices, and form the sum $\mathcal{C}(G) = \sum_{\alpha=1}^{\rho} \mathcal{C}_\alpha$. By construction, $\mathcal{C}(G)$ equals the number of m cliques contained in G. A similar procedure determines the number of n-independent sets in G. Briefly, choose n vertices $T_\alpha = \{v_1, \ldots, v_n\}$ from the N vertices in G, and form the product $\mathcal{I}_\alpha = \prod_{(v_j, v_k \in T_\alpha)}^{(j \neq k)} \bar{a}_{v_j, v_k}$, where $\bar{a}_{v_j, v_k} = 1 - a_{v_j, v_k}$. If the vertex set T_α forms an n-independent set, then $\mathcal{I}_\alpha = 1$; otherwise $\mathcal{I}_\alpha = 0$. Repeat this for all $\nu = C(N, n)$ ways of choosing n vertices from N vertices, then form the sum $\mathcal{I}(G) = \sum_{\alpha=1}^{\nu} \mathcal{I}_\alpha$. By construction, $\mathcal{I}(G)$ gives the number of n-independent sets contained in G. Finally, define

$$h(G) = \mathcal{C}(G) + \mathcal{I}(G). \tag{2}$$

It follows from the above discussion that $h(G)$ is the total number of m cliques and n-independent sets in G. Thus $h(G) \geq 0$ for all graphs G; and $h(G) = 0$ if and only if G does not contain an m clique or n-independent set.

We can use $h(G)$ as the cost function for the following combinatorial optimization problem. For given integers (N, m, n), and with $h(G)$ defined as above, find an N-vertex graph G_* that yields the global minimum of $h(G)$. Notice that if $N < R(m, n)$, the (global) minimum will be $h(G_*) = 0$ since Ramsey theory guarantees that a graph exists which has no m clique or n-independent set. On the other hand, if $N \geq R(m, n)$, Ramsey theory guarantees $h(G_*) > 0$. If we begin with $N < R(m, n)$ and increment N by 1 until we first find $h(G_*) > 0$, then the corresponding N will be exactly $R(m, n)$. We now show how this combinatorial optimization problem can be solved using adiabatic quantum evolution.

Quantum algorithm.—The adiabatic quantum evolution (AQE) algorithm [8] exploits the adiabatic dynamics of a quantum system to solve combinatorial optimization problems. The AQE algorithm uses the optimization problem cost function to define a problem Hamiltonian H_P whose ground-state subspace encodes all problem solutions. The algorithm evolves the state of an L-qubit register from the ground-state of an initial Hamiltonian H_i to the ground-state of H_P with probability approaching 1 in the adiabatic limit. An appropriate measurement at the end of the adiabatic evolution yields a solution of the optimization problem almost certainly. The time-dependent Hamiltonian $H(t)$ for global AQE is

$$H(t) = \left(1 - \frac{t}{T}\right) H_i + \left(\frac{t}{T}\right) H_P, \tag{3}$$

where T is the algorithm runtime, and adiabatic dynamics corresponds to $T \to \infty$.

To map the optimization problem associated with computing $R(m, n)$ onto an adiabatic quantum computation, we begin with the 1-1 correspondence between N-vertex graphs G and length $L = N(N-1)/2$ binary strings $g(G)$. From Eq. (1) we see that position along the string is indexed by vertex pairs (i, j). We thus identify a qubit with each such pair (i, j), and will thus need L qubits. Defining the computational basis states (CBS) to be the eigenstates of $\sigma_z^0 \otimes \cdots \otimes \sigma_z^{L-1}$, we identify the 2^L graph strings $g(G)$ with the 2^L CBS: $g(G) \to |g(G)\rangle$. The problem Hamiltonian H_P is defined to be diagonal in the computational basis with eigenvalue $h(G)$ associated with eigenstate $|g(G)\rangle$:

$$H_P |g(G)\rangle = h(G) |g(G)\rangle. \tag{4}$$

Note that the ground-state energy of H_P will be zero iff there is a graph with no m cliques or n-independent sets. We give an operator expression for H_P below. The initial Hamiltonian H_i is chosen to be

$$H_i = \sum_{l=0}^{L-1} \frac{1}{2} (I^l - \sigma_x^l), \tag{5}$$

where I^l and σ_x^l are the identity and x-Pauli operator for qubit l, respectively. The ground-state of H_i is the easily constructed uniform superposition of CBS.

The quantum algorithm for computing $R(m, n)$ begins by setting N equal to a strict lower bound for $R(m, n)$ which can be found using the probabilistic method [9] or a table of two-color Ramsey numbers [3]. The AQE algorithm is run on $L_N = N(N-1)/2$ qubits, and the energy E is measured at the end of algorithm execution. In the *adiabatic limit* the result will be $E = 0$ since $N < R(m, n)$. The value of N is now incremented $N \to N + 1$, the AQE algorithm is rerun on L_{N+1} qubits, and the energy E measured at the end of algorithm execution. This process is repeated until $E > 0$ first occurs, at which point the associated N will be equal to $R(m, n)$. Note that any real application of AQE will only be approximately adiabatic. Thus, the probability that the measured energy E will be the ground-state energy will be $1 - \epsilon$. In this case, the algorithm must be run $k \sim \mathcal{O}(\ln[1 - \delta]/\ln\epsilon)$ times so that, with probability $\delta > 1 - \epsilon$, at least one of the measurement outcomes will be the true ground-state energy. We can make δ arbitrarily close to 1 by choosing k sufficiently large.

Simulation results.—To test the adiabatic quantum computation of $R(m, n)$, we numerically simulated the Schrödinger dynamics generated by the AQE Hamiltonian $H(t)$. Clearly, these simulations can only be run at finite values of T. As in Ref. [10], we chose T so that the algorithm success probability P_s is large compared to the probability that a randomly chosen CBS will belong to the D-degenerate ground-state eigenspace of H_P ($P_s \gg D/2^L$)). Here, P_s is the probability that an energy measurement done at the final time T will yield the ground-state energy E_{gs} of H_P. Since a classical computer cannot efficiently simulate the dynamics of a quantum system, we can only obtain small Ramsey numbers. In this case, H_P can be found by evaluating the cost function $h(G)$ using the procedure described above Eq. (2).

We simulated the AQE computation of $R(3, 3)$ and $R(2, s)$ for $5 \leq s \leq 7$. Straightforward arguments [3] give $R(3, 3) = 6$ and $R(2, s) = s$. We present our simulation results in Table I. We see that for all m, n considered, the threshold value N_t where $E_{gs} > 0$ first occurs is precisely at the Ramsey number: $N_t = R(m, n)$.

For $R(2, s)$ and $N = s$, Table I gives $E_{gs} = 1$. For these cases, graphs corresponding to ground-states of H_P will thus contain either a single s-independent set or a single 2 clique. There is only one s-vertex graph with an s-independent set, and there are $C(s, 2) = s(s-1)/2$ graphs with one 2 clique (viz. edge). Thus, the ground-state degeneracy $D = 1 + C(s, 2)$, in agreement with the $R(2, s)$ degeneracies in Table I for $N = s = 5, 6, 7$. For $R(3, 3)$ and $N = 6$, Table I gives $E_{gs} = 2$. Thus, graphs corresponding to ground-states are those with (i) two 3 cliques, (ii) two 3-independent sets, or (iii) one 3 clique and one 3-independent set. Ref. [11] derived the minimum number of 3 cliques and 3-independent sets that can be present in an N-vertex graph. This minimum is precisely our E_{gs} for $R(3, 3)$ and a given N. For $N = 6$, the minimum value is 2, in agreement with $E_{gs} = 2$ in Table I. We carried out both analytical [12] and numerical counts of the ground-state graphs for $R(3, 3)$ and $N = 6$. Both approaches found 1760 graphs giving a ground-state degeneracy $D = 1760$. In all cases appearing in Table I, the upward jump in D seen upon reaching the Ramsey threshold $N = R(m, n)$ (from below) is responsible for the jump in the success probability P_s also seen at this threshold.

Although we would like to have calculated larger Ramsey numbers, this was simply not practical. Note that the $N = 7$ simulations use $L = 21$ qubits. These simulations are at the upper limit of 20–22 qubits at which simulation of the full AQE Schrödinger dynamics is practical [10,13,14]. The next smallest Ramsey number is $R(2, 8) = 8$ which requires a 28 qubit simulation, well beyond what can be done practically.

Experimental implementation.—We begin by determining an operator expression for the problem Hamiltonian H_P which then fixes the AQE Hamiltonian $H(t)$ through Eqs. (3) and (5). Recall that the eigenvalue $h(G) = \mathcal{C}(G) + \mathcal{I}(G)$ counts the total number of m cliques and n-independent sets in a graph G. For an m-vertex set $S_\alpha = \{v_1, \ldots, v_m\}$, we define the edge set $E_\alpha = \{e_k^\alpha : k = 1, \ldots, C(m, 2)\}$ as the set of all edges connecting pairs of vertices $v_i, v_j \in S_\alpha$, and $C(m, 2)$ is the number of ways of choosing 2 vertices out of m. If S_α corresponds to an m clique in the graph G, the graph string $g(G)$ must have 1's at all bit positions associated with the edges of E_α. Let the states $|0\rangle$ and $|1\rangle$ satisfy $\sigma_z|a\rangle = (-1)^a|a\rangle$. Then the operator $H_\alpha = \prod_{e \in E_\alpha} P_1^e$ {where $P_1^e = (1/2)[I^e - \sigma_z^e]$, and e labels the qubit associated with edge e} will have $|g(G)\rangle$ as an eigenstate with eigenvalue 1 when S_α is an m clique, and zero otherwise. The operator that counts all m cliques in a graph G is then $H_{cl}^m = \sum_{\alpha=1}^{C(N,m)} H_\alpha$, and by construction, $H_{cl}^m|g(G)\rangle = \mathcal{C}(G)|g(G)\rangle$. A similar analysis can be carried out for n-independent sets. Let $T_\alpha = \{v_1, \ldots, v_n\}$ be an arbitrary n-vertex set, and $\bar{E}_\alpha$ its corresponding edge set. If T_α is an n-independent set in a graph G, then the graph string $g(G)$ must have 0's at all bit-positions associated with the edges of $\bar{E}_\alpha$. The operator $\bar{H}_\alpha = \prod_{e \in \bar{E}_\alpha} P_0^e$ {where $P_0^e = (1/2)[I^e + \sigma_z^e]$, and e labels the qubit associated with edge e} will have eigenstate $|g(G)\rangle$ with eigenvalue 1 (0) when T_α is (is not) an n-independent set. The operator that counts all n-independent sets in an arbitrary graph G is then $H_{is}^n = \sum_{\alpha=1}^{C(N,n)} \bar{H}_\alpha$, and by construction, $H_{is}^n|g(G)\rangle = \mathcal{I}(G)|g(G)\rangle$. For calculation of $R(m, n)$, the problem Hamiltonian H_P^{Nmn} is then

$$H_P^{Nmn} = H_{cl}^m + H_{is}^n. \tag{6}$$

Note that H_P^{Nmn} contains $\mathcal{O}(N^s)$ terms, where N is the number of vertices and $s = \max\{C(N, m), C(N, n)\}$. Since each H_α and $\bar{H}_\alpha$ is a projection operator, their operator norm will be unity and their matrix elements, being 0's and 1's, are specified with a single bit. Lastly, note that each term in H_P^{Nmn} is a product of at most $t = \max\{C(m, 2), C(n, 2)\}$ σ_z operators so that H_P^{Nmn} is a t-local Hamiltonian [15]. By using perturbative gadgets, it can be reduced to a 2-local Hamiltonian [16–18].

For a given Hamiltonian $H(t)$, two approaches have been demonstrated to experimentally implement AQE [19–21]. Refs. [19], [20] partitioned the full evolution into $\mathcal{N}$ subintervals of duration $\Delta t = T/\mathcal{N}$ which are sufficiently short that the propagator U_l for each subinterval l can be factored via a Trotter expansion. This approach was applied to three-qubit systems, though it can be used for arbitrary size qubit systems. Ref. [21] describes experiments using a quantum annealing device designed to implement adiabatic quantum optimization algorithms.

TABLE I. Simulation results for Ramsey numbers $R(3, 3)$ and $R(2, s)$ for $5 \leq s \leq 7$. Here N is the number of graph vertices, E_{gs} and D are the ground-state energy and degeneracy, respectively, for the problem Hamiltonian H_P, and T and P_s are, respectively, the algorithm runtime and success probability.

$R(2, 5)$					$R(2, 6)$					$R(3, 3)$					$R(2, 7)$				
N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s	N	E_{gs}	D	T	P_s
3	0.0	1	5.0	0.591	4	0.0	1	5.0	0.349	4	0.0	18	5.0	0.769	5	0.0	1	8.0	0.865
4	0.0	1	5.0	0.349	5	0.0	1	5.0	0.173	5	0.0	12	5.0	0.194	6	0.0	1	8.0	0.805
5	1.0	11	5.0	0.518	6	1.0	16	5.0	0.286	6	2.0	1760	5.0	0.693	7	1.0	22	8.0	0.938

Results are reported of AQE solutions for the groundstate of randomly generated instances of an 8-qubit quantum Ising spin glass. Work using perturbative gadgets is underway to convert H_P^{Nmn} into a 2-local form amenable to both AQE experimental approaches.

Ramsey numbers and QMA.—Quantum complexity theory formalizes the notion of efficient quantum algorithms. Our interest is in the quantum complexity class QMA which generalizes the randomized version of the classical complexity class *NP* [15,16].

QMA is a class of promise problems where each problem L is the union of two disjoint sets of binary strings L_y and L_n corresponding to Yes and No instances of the problem. For a string $x \in L_y \cup L_n$, the task is to determine whether $x \in L_y$ or $x \in L_n$ using polynomial resources. Let $\mathcal{H}$ denote a two-dimensional Hilbert space; and $|x\rangle$ the CBS labeled by the binary string x.

Definition 1.—(QMA) Let $x \in L = L_y \cup L_n$ and $\epsilon = 2^{-\Omega(|x|)}$. The promise problem L belongs to QMA if there exists a quantum polynomial-time verifier $V(|x\rangle, |y\rangle) \rightarrow \{0, 1\}$, and a polynomial $\pi(|x|)$ such that: (i) for all $x \in L_y$, there exists an $|\xi\rangle \in \mathcal{H}^{\pi(|x|)}$ such that $\Pr\{V(|x\rangle, |\xi\rangle) = 1\} \geq 1 - \epsilon$; and (ii) for all $x \in L_n$ and $|\xi\rangle \in \mathcal{H}^{\pi(|x|)}$, $Pr\{V(|x\rangle, |\xi\rangle) = 1\} \leq \epsilon$. Here $\Pr\{V(|x\rangle, |\xi\rangle) = 1\}$ is the probability that V concludes $x \in L_y$ when the quantum witness is $|\xi\rangle$. Informally, if x is a Yes (No) instance, there exists a (no) quantum witness $|\xi\rangle$ which causes V to correctly (mistakenly) conclude $x \in L_y$ with probability at least $1 - \epsilon$ (greater than ϵ).

A promise problem is QMA-complete if it belongs to QMA and all problems in QMA are polynomially reducible to it. It has been shown [15,16] that the k-local Hamiltonian is QMA-complete for $k \geq 2$.

Definition 2.— (k-local Hamiltonian) Consider an L-qubit Hamiltonian $H = \sum_{j=1}^{r} H_j$, where $r = \text{poly}(L)$; and each term H_j acts on at most k qubits (k-local); has operator norm $\|H_j\| \leq \text{poly}(L)$; and matrix elements specified by $\text{poly}(L)$ bits. Finally, two constants $a < b$ are specified. The Hamiltonian H is a Yes instance if its ground-state energy $E_{gs} < a$, and a No instance if $E_{gs} > b$. The problem is, given a k-local Hamiltonian H, determine whether H is a Yes or a No instance.

Our Ramsey number AQE algorithm leads naturally to an example of a t-local Hamiltonian which we call RAMSEY. We have seen that the Ramsey problem Hamiltonian H_P^{Nmn} is a t-local Hamiltonian; is a sum of a polynomial number of terms $H_j = H_\alpha$ or $\bar{H}_\alpha$; and each H_j satisfies the polynomial bounds specified in Definition 2. Suitable choices for the constants a and b are $0.01 < a < 0.1$ and $b = 1 - a$. Yes instances of RAMSEY then correspond to $N < R(m, n)$ since $E_{gs} = 0 < a$, and No instances to $N \geq R(m, n)$ where $E_{gs} \geq 1 > b$. It is possible to carry over the proof that the k-local Hamiltonian is in QMA [15] to show that RAMSEY is also in QMA.

For an AQE algorithm with *nondegenerate* ground-state, the runtime is largely determined [8] by the minimum energy gap $\Delta = \min_t\{E_1(t) - E_0(t)\}$. This connection fails for the Ramsey algorithm when $N = R(m, n)$ as the ground state becomes *degenerate* during its execution and so Δ vanishes. Determining how the runtime scales when $\Delta = 0$ (as with the Ramsey algorithm) is an open problem in adiabatic quantum computing.

In this Letter, we have presented a quantum algorithm that calculates two-color Ramsey numbers $R(m, n)$; numerically simulated the algorithm and shown it correctly determined small Ramsey numbers; discussed its experimental implementation; and shown that Ramsey number computation is in the quantum complexity class QMA.

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