Honours Thesis Research Proposal

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Provisional Thesis Title Quantum algorithms for determining Ramsey Numbers

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Declaration

I, Aceng Akii, declare that this proposal is my own, unaided work. It is being submitted for the degree of Bachelor of Science Honours in Computational and Applied Mathematics at the University of the Witwatersrand. It has not been submitted for any degree or examination at any other university.

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Abstract

The world often consists of large disorganized structures; upon closer examination, patterns and structures emerge from these complex arrangements. The study of Ramsey numbers is one such method to uncover these underlying patterns. Ramsey numbers come from the field of Graph Theory.In order to study Ramsey numbers, we must consider all possible structures that arise from a graph of a given size. This then informs the particular structures that will always arise from a complete graph of a given size. This is a Ramsey Number. Currently this search is an exhaustive process and one needs to search all possible graphs to find the Ramsey Number. As we increase the size of the graph we need to search exponentially more graphs and this is computationally expensive, eventually surpassing the capabilities of the classical computers. Therefore, the objective of this research is to explore an alternative approach to computing Ramsey Numbers using quantum computers.

Contents

1 Introduction

Graph theory is a field of study that can be leveraged to understand large and seemingly disorganised structures. Ramsey theory is a two colour, edge coloring method of drawing complete graphs. And it is a sub-topic of Graph theory that analyses the sub-graphs that exist in a graph. A Ramsey number is the smallest size a graph needs to be to ensure that, you will always find either, a complete sub-graph of size k in one colour - this is known as a k-clique. Or a compete sub-graph of size l in another colour - this is known as an l-independent set.

However, finding Ramsey numbers is a challenge because you need to search through all permutations of graph colorings for a graph of a given size to ensure that the properties for the Ramsey number is met. If not, you search through permutations of a graph of a larger size. This is where the challenge lies. Searching through graphs of a bigger size exponentially increases the number of graphs to consider. This poses a challenge to the limited computational capacity of a classical computer. And so we seek efficient algorithms, that computes higher order Ramsey numbers. In particular, I am investigating the use of quantum algorithms for computing Ramsey numbers.

2 Literature review

For a graph of size n , the number of graph permutations that emerges is $2^{n(n-1)/2}$. For graphs with a smaller size, analysing these permutations to determine the sub-graphs that emerge is not a problem, even if it is done manually. But as we increase n, running an algorithm on a classical computer to search through the different permutations is a task that quickly becomes time consuming and a huge amount of memory will be required to process these graphs. This type of problem is classified in mathematics as an intractable problem.

Intractable problems are characterised by having exponential run time that is , on classical computers. This is a problem for classical computers as the result may never be computed when the size of the graph becomes too big. So an alternative approach would be Quantum computers and quantum algorithms. A Quantum computer is a new tool that exploits the laws of quantum mechanics to reduce the time it takes to solve these types of problems.

2.1 Quantum computers versus Classical computers

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Both classical and quantum computers can perform logical operations and store information on bits of 0's and 1's. Classical bits are either a charged or uncharged atoms, and can only occupy a single state at any given moment. Quantum bits can be individual spins, photons or atom. Because these Q-bits obey the laws of quantum mechanics, they are allowed to interact and exist in superposition's of 0's and 1's. This means that you can perform a single computation on all possible states at once. Then, after any given computation on a quantum computer, you will have all possible states/outcomes with a given probability for that computation at once. This reduces computation time from exponential time on a classical computer to polynomial time on a quantum computer.

This leads to more precise and powerful computations, as the quantum inputs and outputs must also obey the laws of probability. Thus making quantum algorithms a more powerful approach for determining Ramsey numbers as with the algorithms that will be discussed in a later section.

In Lloyds discussion of quantum simulations, he contrasts classical and quantum computers [1]. He highlights that classical computers face limitations in run time and memory when computing intractable problems. In contrast, quantum systems can compress memory requirements per computation. For example, for the quantum simulation ran by Lloyd, he found that a classical computer will require 10^{12} numbers to capture the state and 10^{24} numbers to record the time evolution of 40 half spin particles. Whereas a quantum computer would require roughly a few hundreds or thousands of operations to simulate the system

Which means, the system is required to evolve according to a specific set of rules.

Schrodinger's equation (1) is an operator that links the system's energy to its time evolution and it dictates the behavior of quantum systems as they progress.

$$i\frac{d}{dt}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \tag{1}$$

with $|\Psi(t)\rangle = |z_1...z_k\rangle$ encodes all state information. For Ramsey Theory computations, the labels z_k are related to the structure of a graph. This is discussed further under the section of Quantum algorithms. $H(t)$ is the Hamiltonian of the system and it encodes the energy of the system and it is specific to each problem. It acts on a state $|\Psi(t)\rangle$, to produce a new state that corresponds to the time translation of the initial system. This is seen in the time derivative applied to the same state.

2.2 Background of quantum computing

There are decision problems that are solvable in polynomial time using QTM's, these are called Bounded error Polynomial Time Problems (BPP), and they are classified as tractable problems. The classical counterpart to this is Non-deterministic Polynomial (NP) time problems that are not solvable in Polynomial time. Bennet, Bernstein, Brassard, Vazirani establish that NP is not a subset of the quantum counterpart of the Bounded-error Polynomial time (BPP), BQP. As a consequence there is the separation of BPP from NP, with respect to a randomly chosen oracle[2]. This separation means that a quantum algorithm for computing deterministic problems exist.

Together this implies that not all NP problems are solvable on a quantum computer, because some are non-deterministic. This seemingly poses a contradiction to [3], since the initial phase of the tree must be defined. But, as will be seen below, quantum algorithms for determining Ramsey numbers have an initial phase (the initial Hamiltonian) defined. And the algorithm evolves according to logical structures. This allows for the quantum algorithm to be implemented.

Farhi, Goldstone, Gutmann, Lapan, Lundgren and Preda use adiabatic quantum computing to solve instances of NP complete cover problems, where no classical computing counter-part has been found [4]. A numerical simulation of Exact cover is run. It was found that for small data, the quantum algorithm only requires quadratic time, whereas on a classical computer, it would require exponential time.

The evolution of quantum systems is governed by equation 1. Choosing the Hamiltonian, $H(t)$ such that it encodes the solution required, then varying it slightly will result in the state vector $|\Psi(t)\rangle$ remaining near the ground state. Then specify T. Set $H(t) = H_P$ and $H(t) = (1-\frac{t}{T})H(0) + (\frac{t}{T})H_p$. This implies that $|\Psi(T)\rangle$ encodes the solution. Having corresponding ground state:

$$\frac{1}{\sqrt{2}}(|z_i = 0 + z_i = 1\rangle) \tag{2}$$

Adiabatic evolution ensures solution accuracy, but runtime poses a challenge. The algorithm numerically approximates the runtime complexity in a random experiment. The algorithm prioritizes bit specification over instances, aiming for a success probability of 1/8. This approach imposes constraints on the required iterations for enhanced success rates. In quantum simulation, complex problems, particularly around phase transitions, are challenging for classical computers but show promise for quantum solutions.

Qubits contain information about the system to be studied. Quantum networks determine how these qubits are transfered between physical systems. Direct operations between arbitrary qubits are hindered, requiring different interaction Hamiltonians for various two-qubit gates. A proposed way around this is by extending phase shift results to multiple qubits. Schuch and Siewert, developed a network that generates phase shift operations.

The network's parameters are determined by the rotation angles of single qubit operations, allowing the network to be programmable. This results in the realization of quantum algorithms.

Lastly, quantum computing's universality enables the development of general algorithms using one/two-qubit operations. However, these algorithms vary based on the system's nature, qubit count, and the function of choice to describe the system. This complicates the computation of Ramsey numbers because it makes generalizing algorithms challenging.

2.3 Quantum algorithms for computing Ramsey numbers

Ramsey number computation falls under the Quantum Merlin-Arthur class of functions. This is a generalisation of NP problems and therefor quantum computers are the approach to take when solving the problem. And, although [2] shows that not all NP problems can be solved, [5] suggests that the QMA class of functions can be solved and thus Ramsey numbers can be solved on a quantum computer.

The algorithm quantum adiabatic algorithm is an example of how an optimization problem can be used to compute two colour Ramsey numbers $R(m,n)$, by using an adiabatic quantum algorithm[6]. The algorithm is set up by first determining a string of length L of 0's and 1's, where: $L = N(N-1)/2$, the 0's represent no edge between two vertices and the 1's represent an edge. Then, all different combinations of these strings are collected into a matrix, call it G, the adjacency matrix.

From G, columns below the main diagonal are then put into a new string $g(G)$, which is set to represent the state $|g(G)\rangle$. Now to define the optimization problem that is to be solved. In this case it will be a minimization problem because we seek the smallest number of vertices (N) for which a graph containing m-cliques and n independent sets $(R(m,n))$ is formed.

From the set $S_\alpha = v_1 ... v_m$ choose m vertices. Then, to determine the number of cliques in G form the product:

$$C_a = \Pi^{j\neq i}_{v_j, v_{k\in S_\alpha}} a_{v_k, v_k} \tag{3}$$

where

$$a_{v,v'} = \begin{cases} 1, \text{if there is an edge} \\ 0, \text{ if there is no edge} \end{cases} \tag{4}$$

with the sum

$$C(G) = \Sigma^{\rho}_{\alpha=1} C_a \tag{5}$$

This determines how many m cliques there are in graph G , with a similar procedure for the independent sets $I(G)$.

$$I_\alpha = \Pi^{j\neq i}_{v_j, v_k \in T_\alpha} \overline{a}_{v_j, v_k} \; ; \; \overline{a} = 1 - a_{v_j, v_k} \tag{6}$$

Combining the two to find the cost function,

$$h(G) = C(G) + I(G) \tag{7}$$

Then, using $h(G)$, $g(G)$ to define the Hamiltonian, with eigenstate $g(G)$ and eigenvalue $h(G)$

$$H_P|g(G)\rangle = h(G)|g(G)\rangle \tag{8}$$

Then with H_p statisfying the above equation to find H_p, set the initial Hamiltonian:

$$H_i = \sum_{l=0}^{L-1} \frac{1}{2}(I^l - \sigma_x^l) \tag{9}$$

And then allow the equation:

$$H(t) = (1 - \frac{t}{T})H_i + (\frac{t}{T})H_p \tag{10}$$

To evolve overtime T, towards the ground state defined by: $E = 0$ of the system, that encodes the possibles solutions to the problem.

The algorithm developed by Gaitan and Clark is an example of an algorithm for computing Two-color Ramsey numbers using a resonance phenomenon and measuring changes in probe qubits states. The idea of measuring the phase changes of qubits helps us update the quantum Adiabatic algorithm for computing Ramsey numbers . Instead of computing the ground state of H_P as above, the algorithm detects energy levels by observing the state changes of the probe q-bits.

Gaitan and Clark update the quantum adiabatic algorithm by picking up from equation 10, and then defining a quantum register that encodes the Hamiltonian as follows :

$$H = -\frac{1}{2}\omega\sigma_z \oplus I_2^{\oplus(L+1)} + I_2 \oplus H_Q + c\sigma_x \oplus A \tag{11}$$

Where: I - identity matrix, σ_x and σ_z are pauli matrices . c , and ω - coupling co-efficients, $A = \sigma_x \bigotimes H_d^{\bigotimes L}$ and H_d if the Haddamard operation.

Set n equal to the lower bound of the Ramsey number and measure the resonance dynamics. Variation of the Hamiltonian does not exhibit resonance dynamics when approaching a solution, as in the case where Ramsey numbers are found. When it exhibits resonance dynamics, this directly relates to the ground state of H, suggesting a non-solution.

The updated algorithm determines the solution by measuring energy levels ,E a,s N increases until $E > 0$. Simulated on small Ramsey numbers up to $R(m,n) \leq 7$. It proved accurate and efficient. Governed by a t-local Hamiltonian, the algorithm falls within the QMA class. This study demonstrates the intractability of computing Ramsey numbers. But it aslo shows how the problem is solvable with a quantum computer due to its classification in the Quantum Merlin Arthur (QMA) , which is a complexity class, extending beyond NP.

Wangs' algorithm ensures that the simulated system evolves according to logical step, particularly equation 1, which is established as a requirement for the quantum simulation of a quantum system. . However, limitations on the number of qubits, meant that the algorithm can not be verified for larger problems, particularly those requiring 21 qubits or more.

3 Aim

To find a way of determining two colour Ramsey Numbers, using less computational time.

4 Hypothesis

The use of quantum algorithms will enable the determination of larger Ramsey Numbers with greater speed compared to classical algorithms.

5 Objectives

To first build a classical algorithm that works to calculate small Ramsey numbers. Then to build a quantum algorithm that calculates Ramsey numbers.

6 Methodology

6.1 Graph theory and Ramsey numbers

Given the literature on quantum algorithms, it is important to establish the classical algorithms, so in the following section I provide the background and a few lemmas and their proof that give some justification for the classical algorithm that determines Ramsey numbers.

6.2 Ramsey Numbers: N,k and l

The relationship between N, k and l, is as follows. N is the number of vertices of any given graph, k the set of vertices that form a clique and l is the set of mutually independent vertices. We can then postulate that the relationship between N, k and l is as follows:

$$N \leq |k| + |l| \tag{12}$$

This relationship can be checked by drawing all graphs of size $N = 1,2,3,4$. Then you define $\mathbb{H}_G$ to be the set of all graphs of size N that are homomorphic to each other. Next, you choose a representative from each $\mathbb{H}_G$. Then, if they exist, you count the number of k-cliques and l-independent sets. Noting that none of the vertices are connected to themselves. Do this for all sets of graphs of size N that are not homomorphic to each other.

Proof: Consider a proof by contradiction. If $N > |l| + |k|$ then for $N = 1$ there exists an l - independent set of size 1, and 1 k-cliques. However, if $l = 1$, then the we have that $1 > 1 + 1$ which is not possible. Therefore the relation $N \leq |k| + |l|$ must hold.

6.3 Homomorphic Graphs

Homomorphic graphs are graphs that have the same structure when you relabel the verticies of the graph, but the number of edges and the degree on the verticies is preserved. Thus, when a set of graphs are homomorphic, they give us the same information. Analysing homomorphic graphs helps us because it reduces the number of graphs we need to consider within our search.

Let G be a graph such that $G=(V,E)$ with $V=\{v_1, v_2, ..., v_N\}$ as the set that contains all the vertices in G and $E=\{e_1, e_2, ..., e_k\}$ as the set that contains all the edges in G.

Then we can define the set of graphs homomorphic to G by: $\mathbb{H}_G=\{H_1, H_2, ..., H_n\}$ whereby $n=|\mathbb{H}_G|$ the number of graphs homomorphic to G.

Then there exists a set of permutation mappings $S=\{\alpha^1, \alpha^2, ..., \alpha^n\}$ that map G to any element of $\mathbb{H}_G$, $\alpha : \mathbb{H}_G \to \mathbb{H}_G$. Where $\alpha^1 = e$, the identity map.
To say that this is a valid mapping, 3 properties must hold. They are, reflexivity, symmetry and transitivity.
Reflexivity
To show that $G \sim G$ Let $k \in \{1,2,...,n\}$. Assume $G \in \mathbb{H}_G$, then $\exists\ \alpha \in S$ Such that:

$$\alpha^k(G) = G \tag{13}$$

$$\alpha^k \alpha^k(G) = \alpha^k(G) \tag{14}$$

$$\alpha^{2k}(G) = \alpha^k(G) \tag{15}$$

but to ensure α^{2k} is a valid mapping, $2k \in \{1,2,...,n\}$, $2k \not> n \implies 2k = qn + r$, where q is the number of times you cycle through the set $S=\{\alpha^1, \alpha^2, ..., \alpha^n\}$ and r the remainder that tells you how to locate the specific mapping α you seek. Which then leaves us with

$$\alpha^r(G) = \alpha^k(G) \tag{16}$$

$$\alpha^r(G) = G \tag{17}$$

Therefore, $G \sim G$.

Symmetric
Similarly, suppose $G \sim H$, we must show that $H \sim G$.

$$\alpha^k(G) = H \tag{18}$$

Then,

$$\alpha^k(H) = \alpha^k(\alpha^k(G)) \tag{19}$$

$$\alpha^k(H) = \alpha^{2k}(G) \tag{20}$$

But since, $G \sim G$, and $\alpha^r(G) = G$

$$\alpha^k(H) = \alpha^r(G) \tag{21}$$

$$\alpha^k(H) = G \tag{22}$$

Transitivity
Suppose $G \sim H$ and $H \sim K$ then there exists k and s such that

$$\alpha^k(G) = H \tag{23}$$

and

$$\alpha^r(H) = K \tag{24}$$

Then,

$$\alpha^k(G) = H \implies \alpha^r(\alpha^k(G)) = K \tag{25}$$

$$\alpha^{r+k}(G) = K \tag{26}$$

Therefore, $G \sim K$.

6.4 Visualising Ramsey Numbers on Graphs

After establishing the properties of homomorphic graphs earlier, the number of graphs utilized for determining Ramsey numbers has been reduced. To find a Ramsey number from a graph we look for the occurrence of particular substructures in each permutation that is not homomorphic to another graph. And we begin the analysis for graphs of size $n = 2$ as the graph of $n = 1$ is trivial to this discussion.

In the section below are the complete graphs for graphs of size $n = 2, 3$ and 4. To find a two colour Ramsey number we look for all possible unique ways to assign the two colours to the edges of the graph. It is in the analysis of the coloured in graphs that we can determine whether or not we have a two colour Ramsey number.

For example, consider the graph of size $n = 4$, figure 3. There are 11 unique ways to colouring this graph, as shown in figures 10-20. Now, within this set of 11 graphs, notice that no matter how we colour the graph, we will alway be able to find a blue K_2 or a red K_3. This means a possible Ramsey number is $R(2,3) = 4$.

6.4.1 Complete graphs

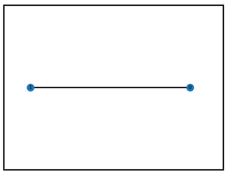

Figure 1: $n = 2$

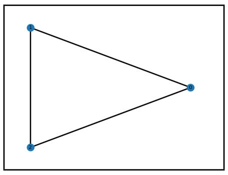

Figure 2: $n = 3$

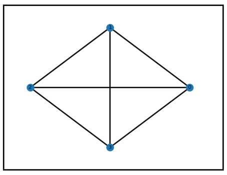

Figure 3: $n = 4$

6.4.2 Non-homomorphic graphs

For $N = 2$: The following are the set of graphs that are not homomorphic to each other. There are 2 in total.

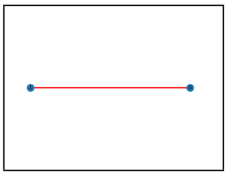

Figure 4:

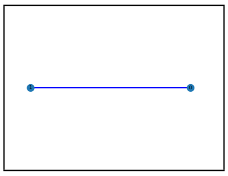

Figure 5:

For $N = 3$: The following are the set of graphs in that are not homomorphic to each other. There are 3 in total.

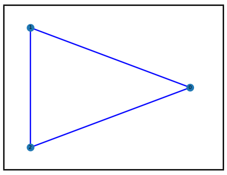

Figure 6:

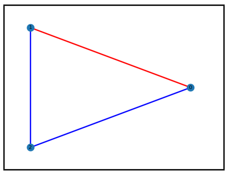

Figure 7:

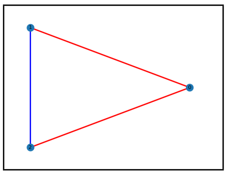

Figure 8:

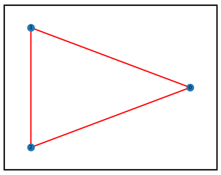

Figure 9:

For $N = 4$: The following are the set of all possible colourings for graphs of size 4. There are 11 in total.

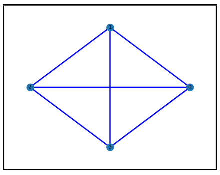

Figure 10:

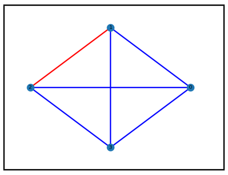

Figure 11:

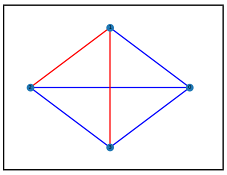

Figure 12:

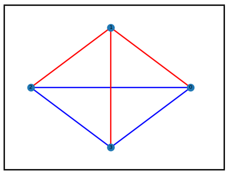

Figure 13:

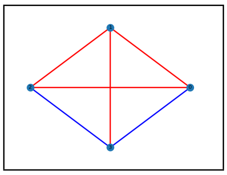

Figure 14:

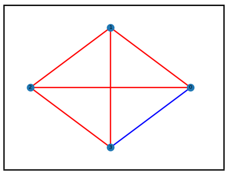

Figure 15:

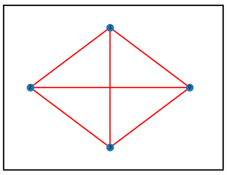

Figure 16:

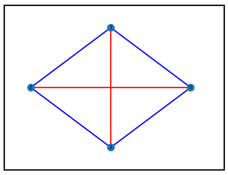

Figure 17:

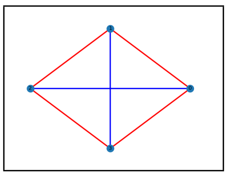

Figure 18:

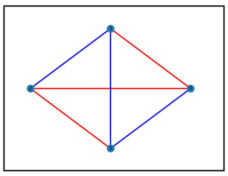

Figure 19:

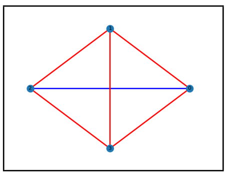

Figure 20:

A Ramsey number $R(k,l)$ can be found from graphs by looking for (in the set of graph permutations) the smallest possible graph, where you can find either k or l, showing up at least once.

Procedure: After forming all these graphs, one way to compute Ramsey numbers would be, to start with 1 vertex, and determine all the possible ways to arrange a graph of this size. Then find the subset of graphs that are not homomorphic to each other. Finally, count all possible k-cliques and l-independent sets that emerge from the graph. You then increase N by 1 and repeat the process of finding all possible graphs, removing homomorphics and analysing each graph. From the data the graphs provide you will be able to find the Ramsey number you seek.

7 Work Plan

My work plan will consist of two parts, where the first part will run over approximately 10 weeks. Within this time frame, each task will unfold over a two-week period, with an additional week allocated between tasks to scrutinise completed work and document my results..

The second part, will begin on the 1 August 2024, and extend through October. During this period, I will be developing my own quantum algorithm that will compute Ramsey numbers.

1. 20 May - 2 June - Qiskit: Understanding Quantum Information and Computation. Time to learn the theory and technical details required to produce a quantum algorithm. As well as complete the available tutorials.
2. 3-9 June - Physics, Quantum workshop.
3. 10-30 June - Classical algorithm : A further two weeks will be used to write and scrutinise my own classical algorithm.
4. 1 - 21 July - Quantum algorithm : This time will be used to recreate a existing quantum algorithm.
5. 1 August - 18 October: Building my own quantum algorithm that calculates Ramsey numbers.

8 Conclusion

The main reason it is so difficult to compute Ramsey numbers is because when you increase the size of the graph you need to consider, the number of graph colourings to consider increases exponentially. Checking all the graphs is highly time consuming and it uses a lot of memorythe Determining Ramsey numbers requires complex algorithms. However, there are still avenues, particularly in quantum computing, to be explored in order to find higher order Ramsey numbers. With this proposal, I intend to investigate classical algorithms and quantum algorithms in an attempt to develop my own quantum algorithm for determining Ramsey numbers.

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