Fuzzy logic

- Fuzzy thinking
- Fuzzy sets
- Linguistic variables and hedges
- Operations of fuzzy sets
- Fuzzy rules
- Summary

Fuzzy Thinking and Fuzzy Logic

- Experts rely on common sense.
- expert knowledge uses vague and ambiguous terms in a computer?
- not logic that is fuzzy, but logic that is used to describe fuzziness.
- the theory of fuzzy sets, sets that calibrate vagueness.
- based on the idea that all things admit of degrees.
- Temperature, height, speed, distance, beauty – all come on a sliding scale.

eg : The motor is running *really hot*.

Tom is a *very tall* guy.

- **Boolean logic** uses sharp distinctions.
  - draw lines between members of a class and non-members.
  - Tom is tall because his height is **181 cm**. If we drew a line at **180 cm,** we would find that David, who is **179 cm,** is small.
  - Is David really a small man or we have just drawn an arbitrary line in the sand?
- **Fuzzy logic** reflects how people think.
  - models our sense of words, our decision making and our common sense.
  - leads to new, more human, intelligent systems.

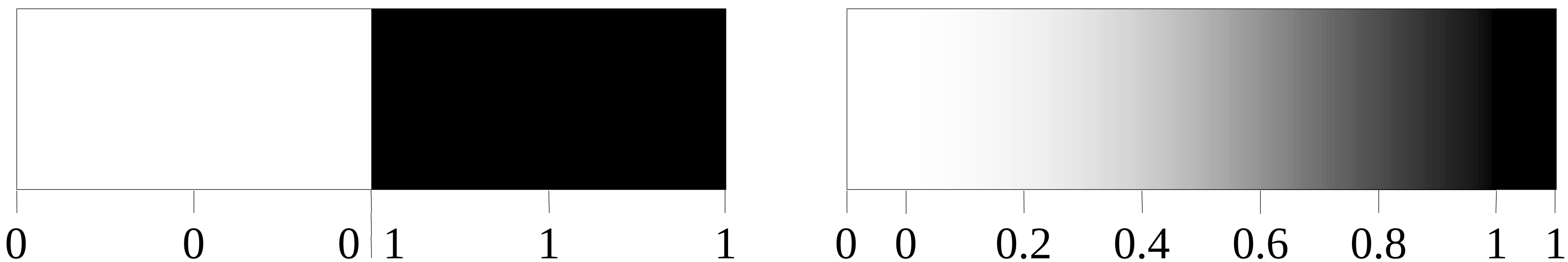
- **Fuzzy logic** was introduced in the 1930s by **Jan Lukasiewicz**, a Polish philosopher.
- While classical logic operates with only two values 1 (true) and 0 (false), Lukasiewicz introduced logic that extended the range of truth values to all real numbers in the interval between 0 and 1.
- the ***possibility*** that a given statement was true or false.
- For example, the possibility that a man 181 cm tall is “really tall” might be set to a value of 0.86.
- It is *likely* that the man is tall.
- This work led to an inexact reasoning technique often called **possibility theory**.

- In 1937, **Max Black** published a paper called "Vagueness: an exercise in logical analysis".
  - a continuum(a coherent whole characterized as a collection) implies degrees.
  - Imagine a line of countless "chairs". The first one is a Chippendale(various styles of furniture fashionable in the third quarter of the 18th century).
  
  - Next to it is a near-Chippendale, in fact indistinguishable from the first item.
  - Succeeding "chairs" are less and less chair-like, until the line ends with a log (bulky piece of a tree).
  
  - When does a ***chair*** become a ***log***?
- Max Black stated that if a continuum is discrete, **a number can be allocated to each element**. He accepted vagueness as a matter of probability.

- In 1965 **Lotfi Zadeh**, published his famous paper "**Fuzzy sets**".
- **L. A. Zadeh, "Fuzzy Sets,"** ***Information and Control*****, vol. 8, no. 3, pp. 338-353, June 1965.**
  - extended the work on possibility theory into a formal system of mathematical logic
  - introduced a new concept for applying natural language terms.
  - This new logic for representing and manipulating fuzzy terms was called **fuzzy logic**, and Zadeh became the *Master of fuzzy logic*.

- a set of mathematical principles for knowledge representation based on ***degrees of membership***.
- Unlike two-valued Boolean logic, fuzzy logic is multi-valued.
- deals with degrees of membership and degrees of truth.
- uses the continuum of logical values between 0 (completely false) and 1 (completely true).
- employs the spectrum of colours instead of just black and white, accepting that things can be partly true and partly false at the same time.

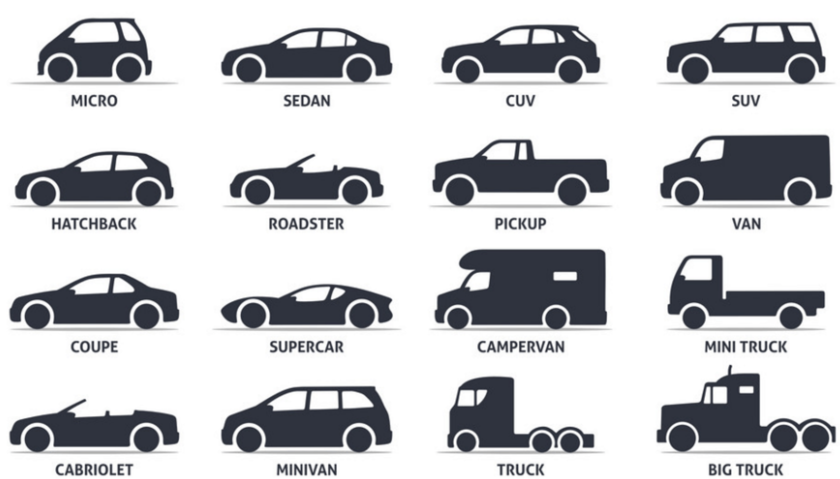
Range of logical values in Boolean and fuzzy logic



(*a*) Boolean Logic. (*b*) Multi-valued Logic.

Fuzzy sets

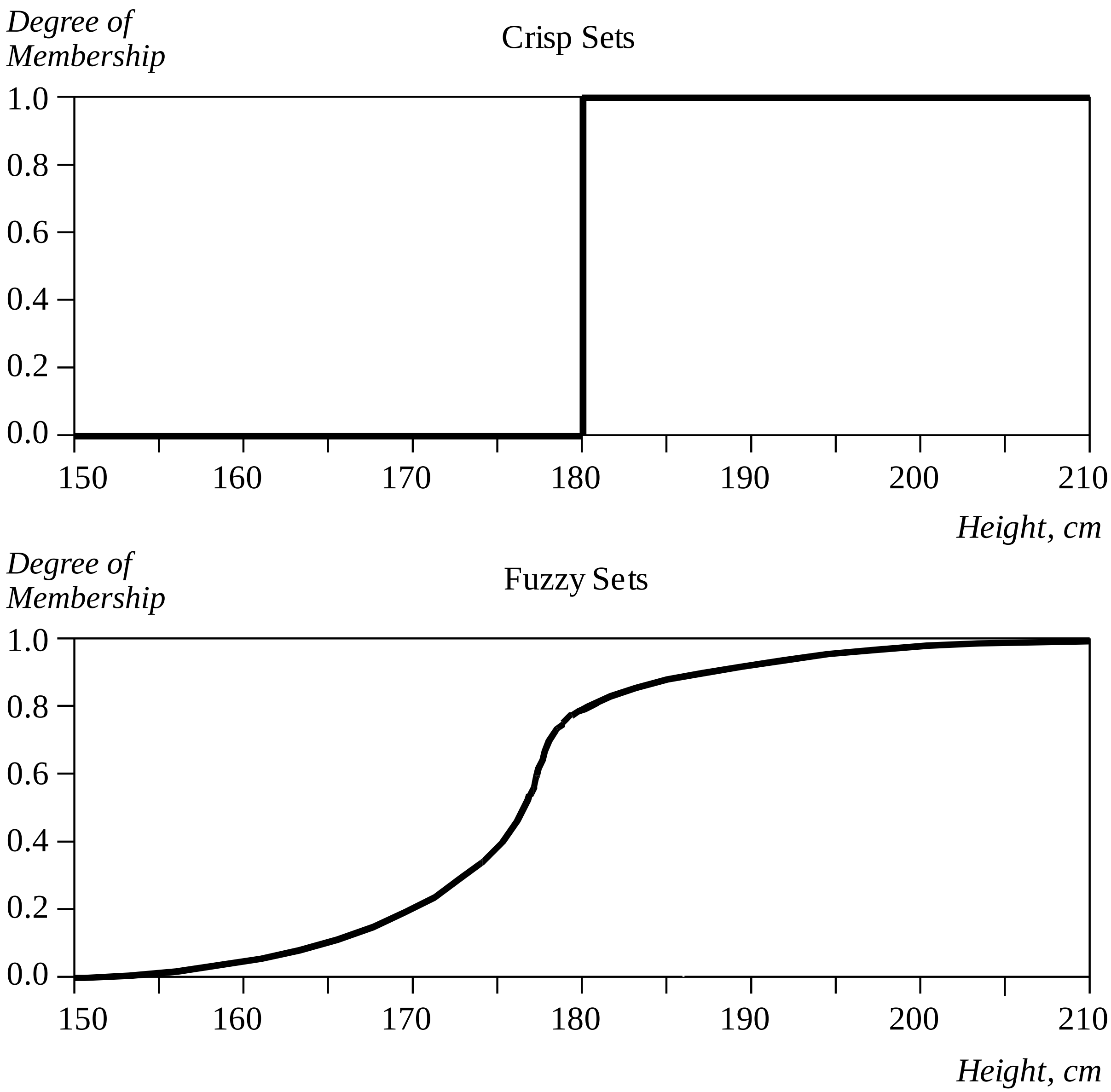
- The concept of a **set** is fundamental to mathematics.
- However, our own language is also the supreme expression of sets.
  - For example, ***car*** indicates the ***set of cars***. When we say *a car*, we mean one out of the set of cars.



- The classical example in fuzzy sets is *tall men*.
- The elements of the fuzzy set "tall men" are all men, but their degrees of membership depend on their height.

Name	Height, cm	Degree of Membership	
		Crisp	*Fuzzy*
Chris	208	1	1.00
Mark	205	1	1.00
John	198	1	0.98
Tom	181	1	0.82
David	179	0	0.78
Mike	172	0	0.24
Bob	167	0	0.15
Steven	158	0	0.06
Bill	155	0	0.01
Peter	152	0	0.00

Crisp and fuzzy sets of "*tall men*"



- The *x*-axis represents the universe of discourse – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.
- The *y*-axis represents the membership value of the fuzzy set. In our case, the fuzzy set of "*tall men*" maps height values into corresponding membership values.

Crisp set

- Let X be the universe of discourse and its elements be denoted as x.
- In the classical set theory, crisp set A of X is defined as function $f_A(x)$ called the characteristic function of A

$$f_A(x): X \to \{0, 1\}, \text{ where } \quad f_A(x) = \begin{cases} 1, \text{if } x \in A \\ 0, \text{if } x \notin A \end{cases}$$

- ➢ This set maps universe X to a set of two elements.
- ➢ For any element x of universe X, characteristic function $f_A(x)$ is equal to 1 if x is an element of set A, and is equal to 0 if x is not an element of A.

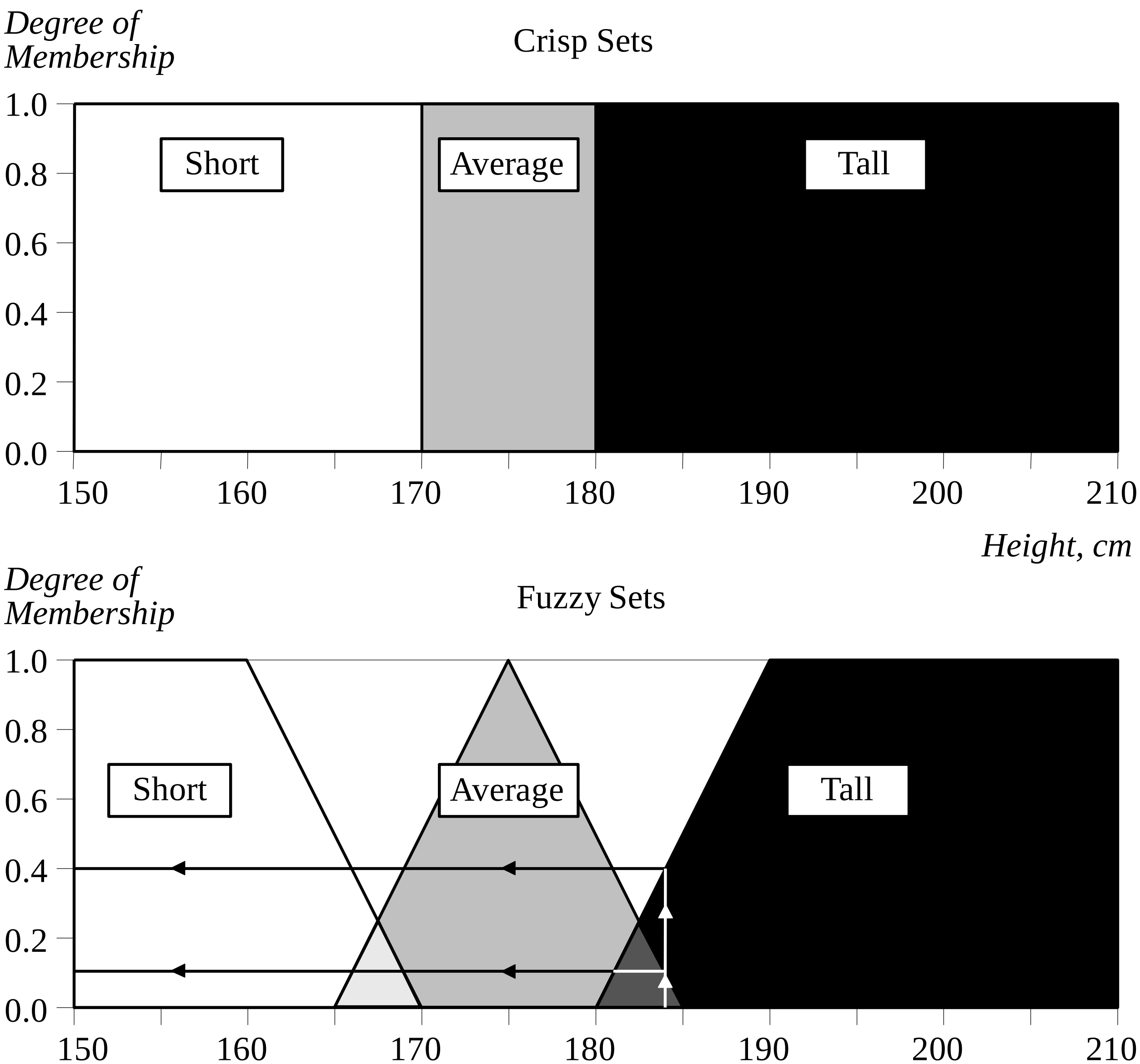
A fuzzy set is a set with fuzzy boundaries

- In the fuzzy theory, fuzzy set A of universe X is defined by function $\mu_A(x)$ called the ***membership function*** of set A

$\mu_A(x): X \rightarrow [0, 1]$, where $\mu_A(x) = 1$ if x is totally in A;
$\mu_A(x) = 0$ if x is not in A;
$0 < \mu_A(x) < 1$ if x is partly in A.

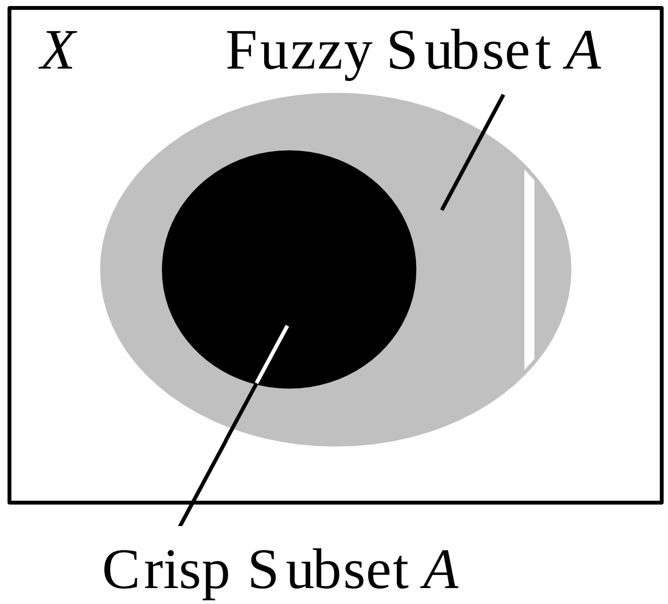
- For any element x of universe X, membership function $\mu_A(x)$ equals the degree to which x is an element of set A.
- This degree, a value between 0 and 1, represents the ***degree of membership***, also called membership value, of element x in set A.

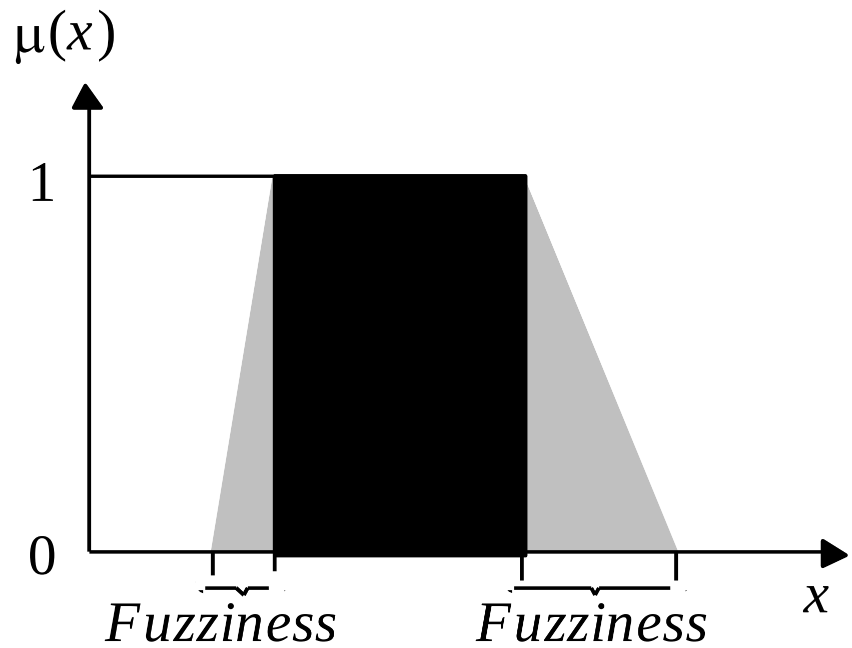
Crisp and fuzzy sets of short, average and tall men



- First, we determine the membership functions. In our "men's height" example, we can obtain fuzzy sets of *tall, short* and *average* men.
- The universe of discourse : the men's heights consist of three sets: *short*, *average* and *tall men*.
- A man who is 184 cm tall is a member of the *average men* set with a degree of membership of 0.1, and at the same time, he is also a member of the *tall men* set with a degree of 0.4.

Representation of crisp and fuzzy subsets





- Typical functions that can be used to represent a fuzzy set are sigmoid, gaussian and pi. However, these functions increase the time of computation.
- Therefore, in practice, most applications use linear fit functions.

Linguistic variables and hedges

- At the root of fuzzy set theory lies the idea of linguistic variables.
- A **linguistic variable** is a fuzzy variable.
  - For example, the statement "John is tall" implies that the linguistic variable *John* takes the linguistic value *tall*.

In fuzzy expert systems, linguistic variables are used in fuzzy rules. For example:

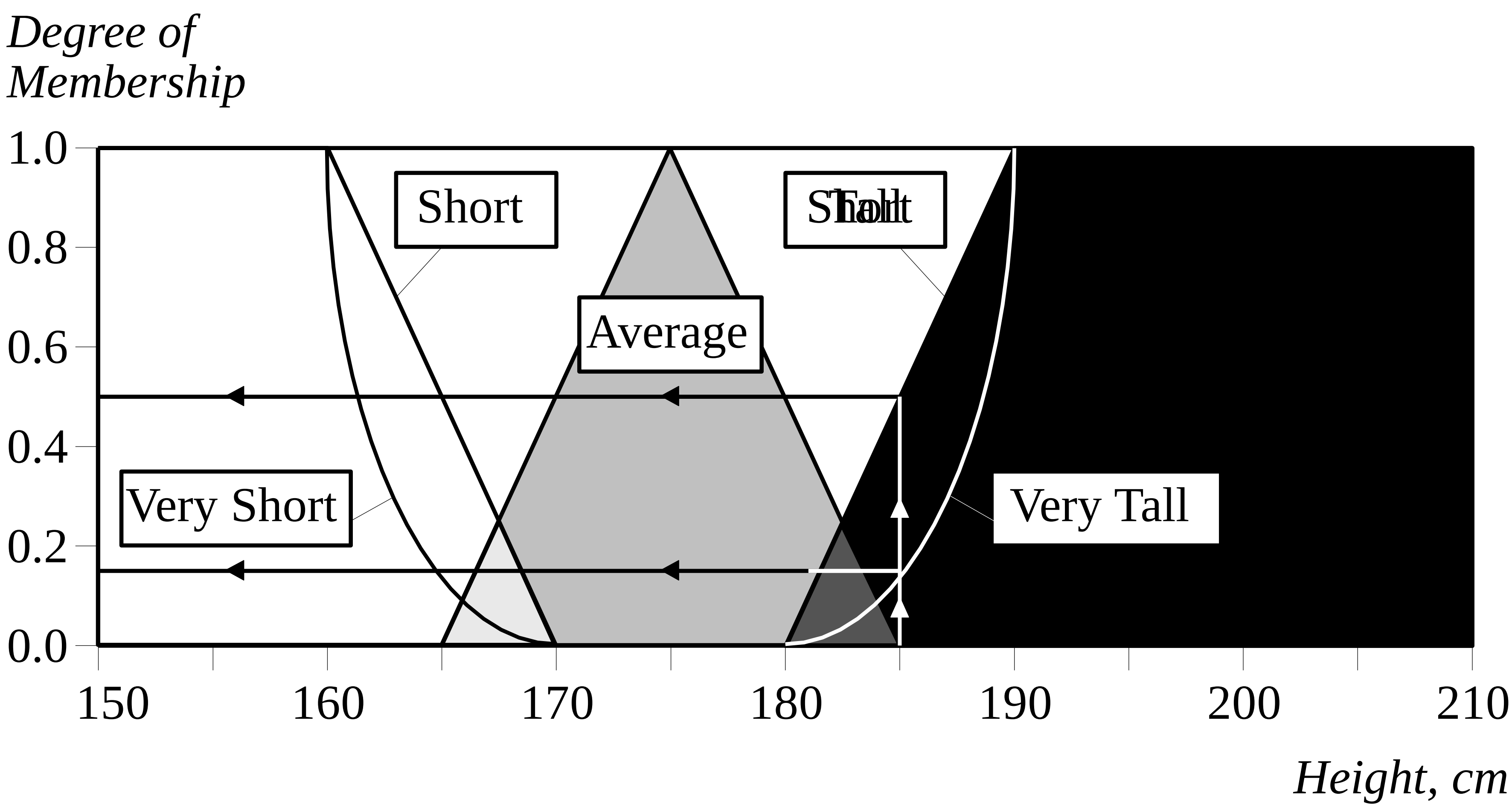
IF wind is strong
THEN sailing is good

IF project_duration is long
THEN completion_risk is high

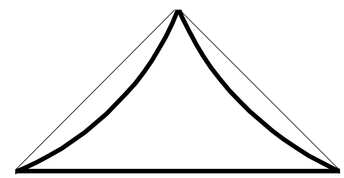
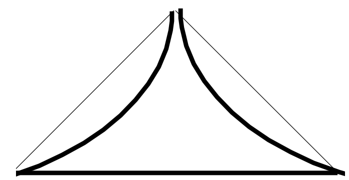
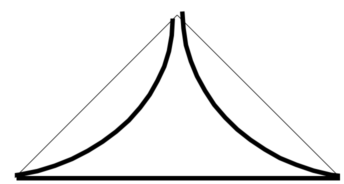
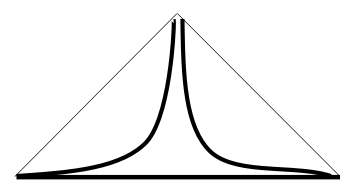
IF speed is slow
THEN stopping_distance is short

- The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.
- A linguistic variable carries with it the concept of fuzzy set qualifiers, called *hedges*.
- Hedges are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.

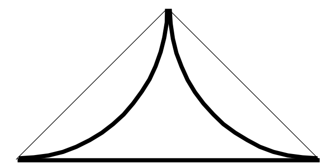
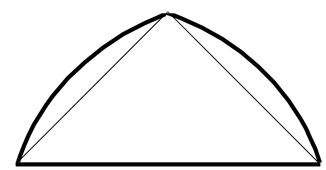
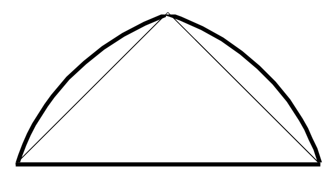
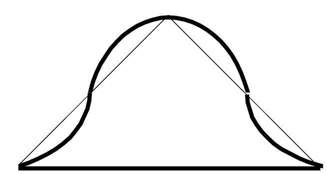
Fuzzy sets with the hedge *very*



Representation of hedges in fuzzy logic

Hedge	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

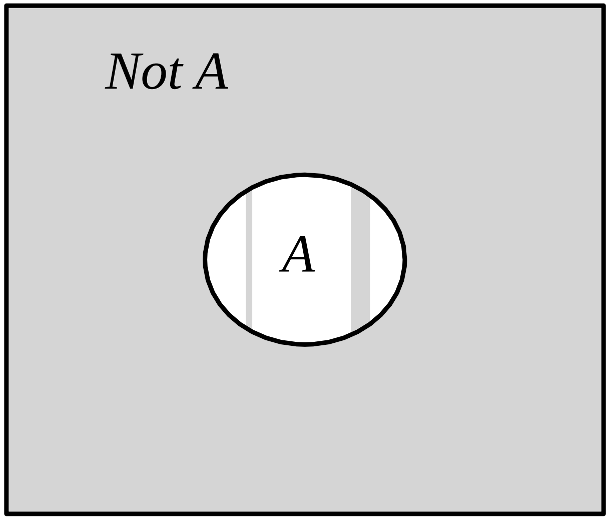
Representation of hedges in fuzzy logic (continued)

Hedge	*Mathematical Expression*	*Graphical Representation*
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2\,[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$ $1 - 2\,[1 - \mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

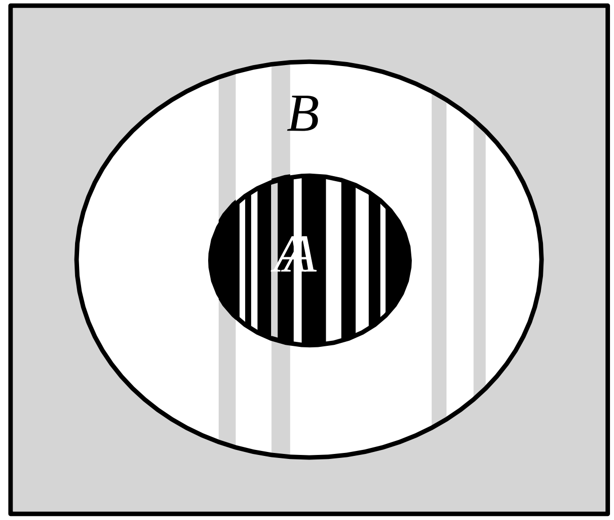
Operations of fuzzy sets

The classical set theory developed in the late 19th century by Georg Cantor describes how crisp sets can interact. These interactions are called operations.

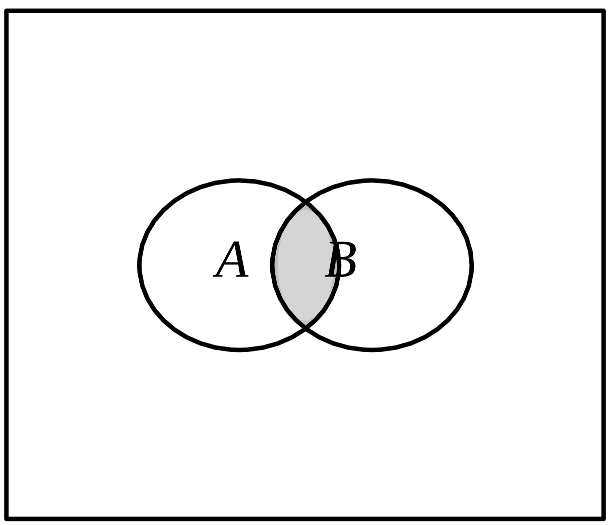
Cantor's sets



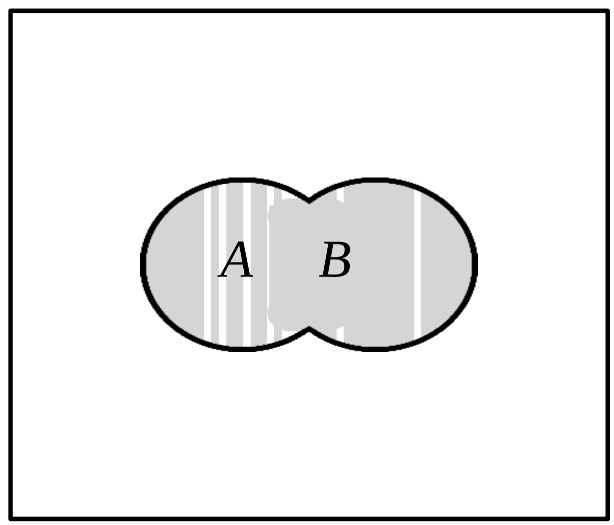
Complement



Containment

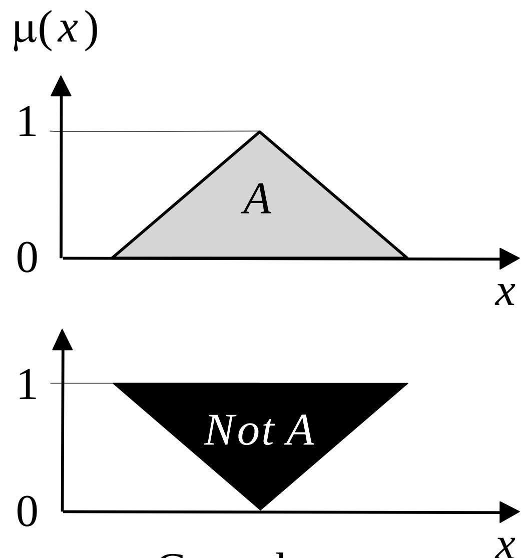


Intersection

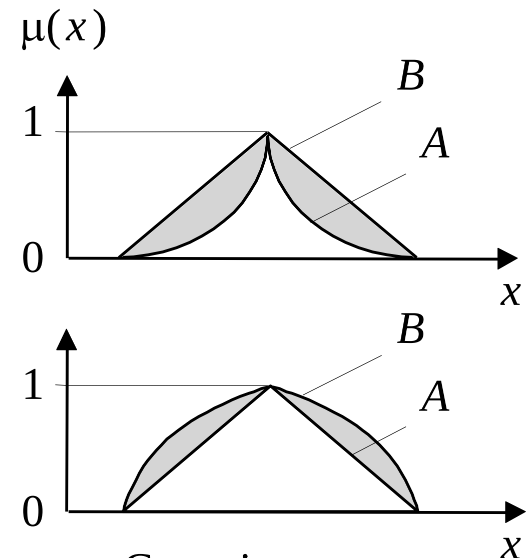


Union

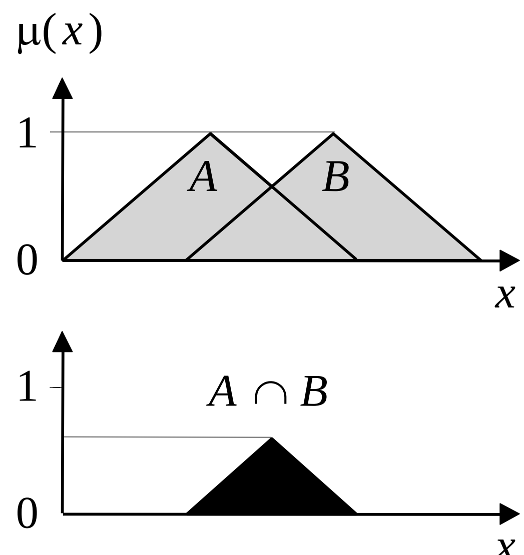
Operations of fuzzy sets



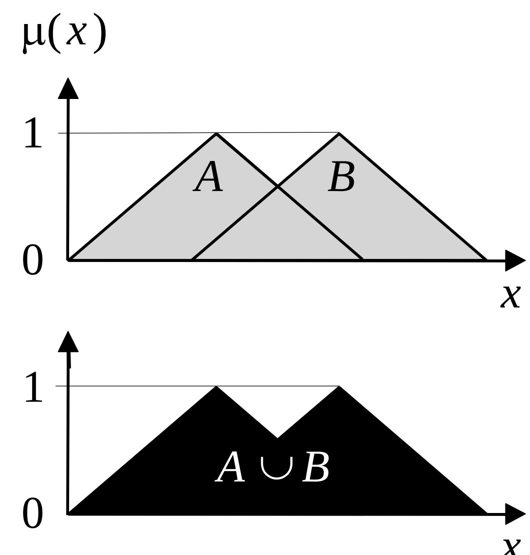
Complement



Containment



Intersection



Union

Complement

Crisp Sets: *Who does not belong to the set?*
Fuzzy Sets: *How much do elements not belong to the set?*
The complement of a set is an opposite of this set. For example, if we have the set of *tall men*, its complement is the set of *NOT tall men*. When we remove the tall men set from the universe of discourse, we obtain the complement. If A is the fuzzy set, its complement $\neg A$ can be found as follows:

$$\mu_{\neg A}(x) = 1 - \mu_A(x)$$

Containment

Crisp Sets: *Which sets belong to which other sets?*

Fuzzy Sets: *Which sets belong to other sets?*

Similar to a Chinese box, a set can contain other sets.The smaller set is called the subset.

For example, the set of *tall men* contains all tall men; *very tall men* is a subset of *tall men*. However, the *tall men* set is just a subset of the set of *men*. In crisp sets, all elements of a subset entirely belong to a larger set. In fuzzy sets, however, each element can belong less to the subset than to the larger set. Elements of the fuzzy subset have smaller memberships in it than in the larger set.

Intersection

Crisp Sets: *Which element belongs to both sets?*

Fuzzy Sets: *How much of the element is in both sets?*

In classical set theory, an intersection between two sets contains the elements shared by these sets. For example, the intersection of the set of *tall men* and the set of *fat men* is the area where these sets overlap.

In fuzzy sets, an element may partly belong to both sets with different memberships. A fuzzy intersection is the lower membership in both sets of each element. The fuzzy intersection of two fuzzy sets A and B on universe of discourse X:

$$\propto_{A \cap B}(x) = min\ [\propto_A(x), \propto_B(x)] = \propto_A(x) \cap \propto_B(x),$$

where $x \in X$

Union

Crisp Sets: *Which element belongs to either set?*

Fuzzy Sets: *How much of the element is in either set?*

The union of two crisp sets consists of every element that falls into either set.For example, the union of *tall men* and *fat men* contains all men who are tall OR fat.

In fuzzy sets, the union is the reverse of the intersection. That is, the union is the largest membership value of the element in either set. The fuzzy operation for forming the union of two fuzzy sets A and B on universe X can be given as:

$\propto_{A \cap B}(x) = max\ [\propto_A(x), \propto_B(x)] = \propto_A(x) \cap \propto_B(x)$,
where $x \in X$

Fuzzy rules

In 1973, Lotfi Zadeh published his second most influential paper.

This paper outlined a new approach to analysis of complex systems, in which Zadeh suggested capturing human knowledge in fuzzy rules.

What is a fuzzy rule?

A fuzzy rule can be defined as a conditional statement in the form:

IF *x* is *A*
THEN *y* is *B*

where *x* and *y* are linguistic variables; and *A* and *B* are linguistic values determined by fuzzy sets on the universe of discourses *X* and *Y*, respectively.

What is the difference between classical and fuzzy rules?

A classical IF-THEN rule uses binary logic, for example,

Rule: 1
IF speed is > 100
THEN stopping_distance is long

Rule: 2
IF speed is < 40
THEN stopping_distance is short

The variable *speed* can have any numerical value between 0 and 220 km/h, but the linguistic variable *stopping_distance* can take either value *long* or *short*. In other words, classical rules are expressed in the black-and-white language of Boolean logic.

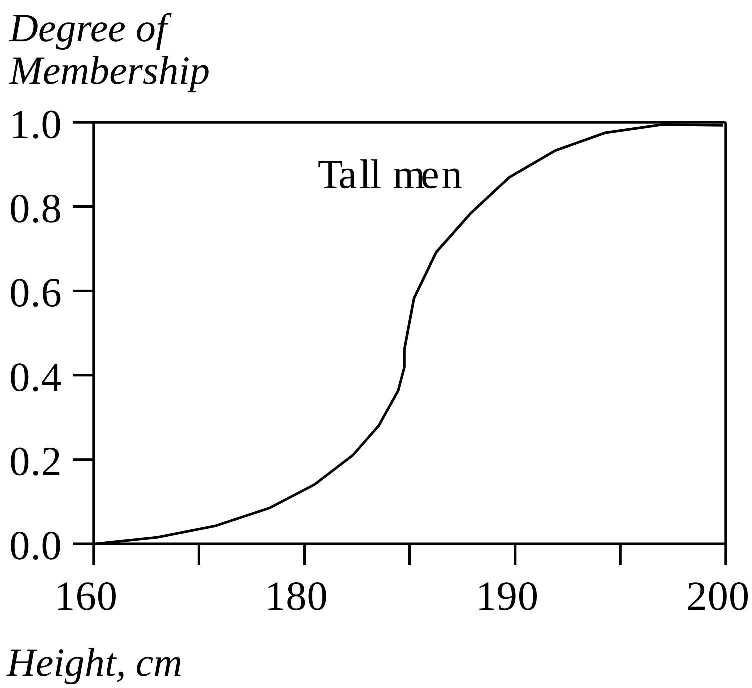
We can also represent the stopping distance rules in a **fuzzy form**:

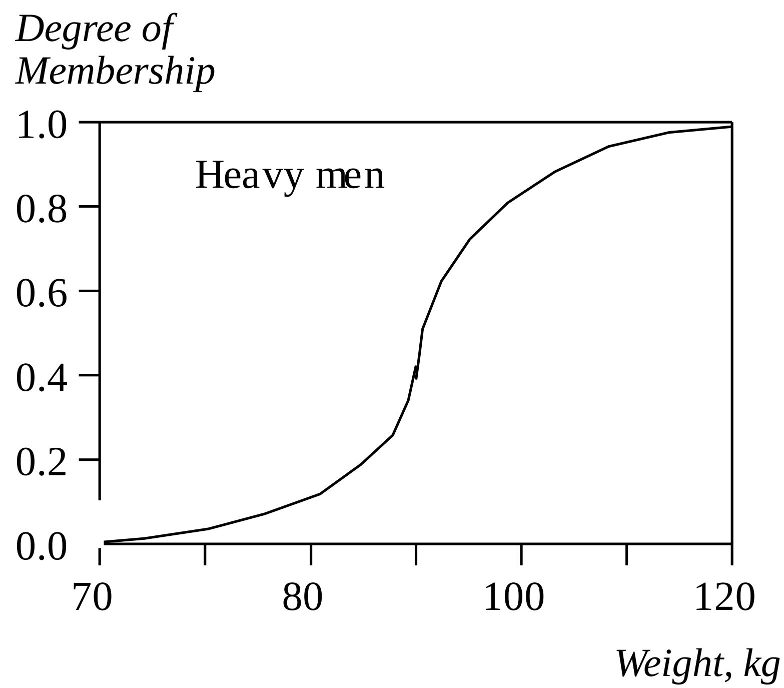
Rule: 1		Rule: 2	
IF	speed is fast	IF	speed is slow
THEN	stopping_distance is long	THEN	stopping_distance is short

In fuzzy rules, the linguistic variable ***speed*** also has the range (the universe of discourse) between 0 and 220 km/h, but this range includes fuzzy sets, such as *slow, medium* and *fast*. The universe of discourse of the linguistic variable ***stopping_distance*** can be between 0 and 300 m and may include such fuzzy sets as *short, medium* and *long*.

- Fuzzy rules relate fuzzy sets.
- In a fuzzy system, all rules fire to some extent, or in other words they fire partially.
- If the antecedent is true to some degree of membership, then the consequent is also true to that same degree.

Fuzzy sets of *tall* and *heavy* men

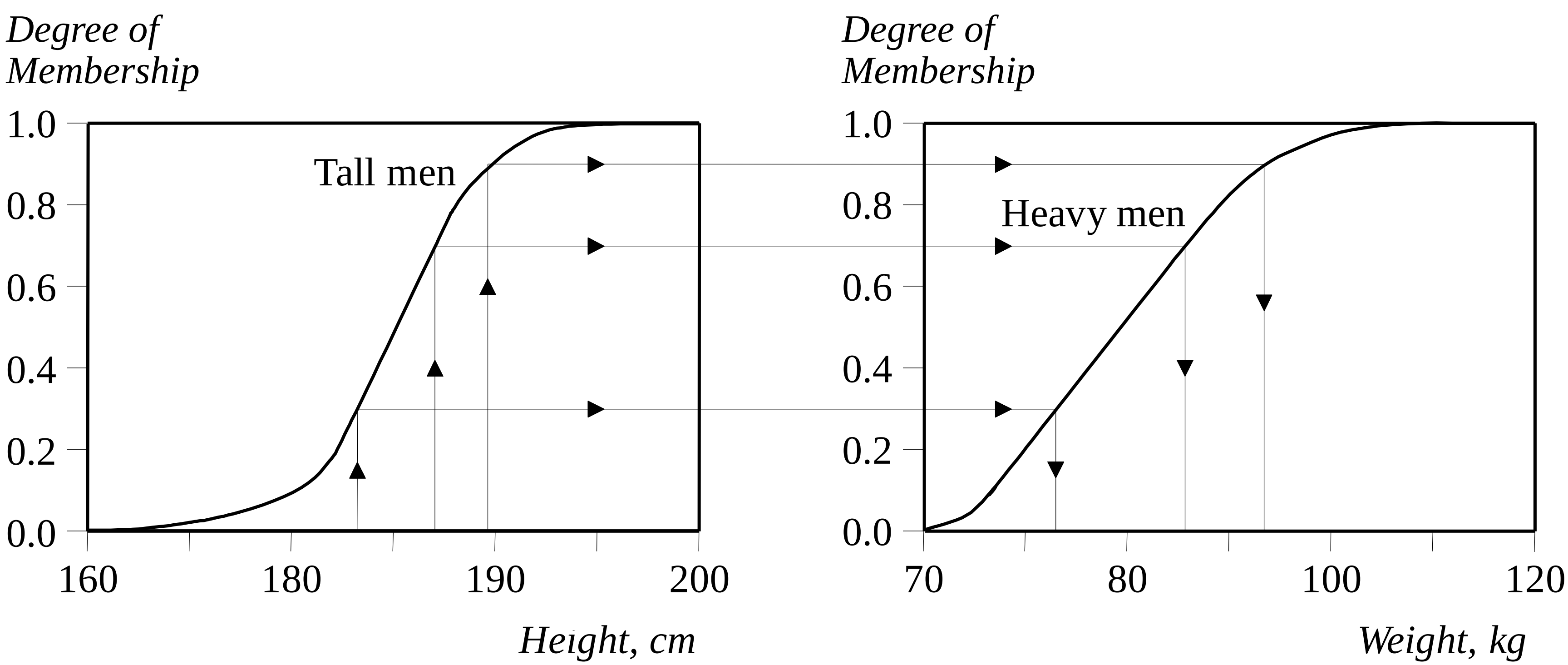




These fuzzy sets provide the basis for a weight estimation model. The model is based on a relationship between a man's height and his weight:

IF height is *tall*
THEN weight is *heavy*

The value of the output or a truth membership grade of the rule consequent can be estimated directly from a corresponding truth membership grade in the antecedent. This form of fuzzy inference uses a method called monotonic selection.



A fuzzy rule can have multiple antecedents, for example:

IF project_duration is long AND
project_staffing is large AND
project_funding is inadequate
THEN risk is high

IF service is excellent OR food is delicious
THEN tip is generous

The consequent of a fuzzy rule can also include multiple parts, for instance:

IF temperature is hot
THEN hot_water is reduced; cold_water is increased