MA 202: MATHEMATICS IV

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Lighthill-Whitham-Richards (LWR) Traffic Flow Model and Its Numerical Analysis

Problem Statement

The movement from one place to another, whether they be kilometres or miles apart, is a primary necessity of the population. Traffic on the road can be understood as all the road users, including pedestrians, animals, and automobiles. The exercised movement of traffic on the roads is known as ‘traffic flow.’ Any activity related to transportation can be defined by the origin, destination, departure time, arrival time, and path. A multitude of these activities brings in a vast network of trails and produces complex traffic flow patterns.

Since the globe is facing escalating population growth, the traffic flow is constantly becoming less manageable. In addition, with a majority of the population migrating to urban and suburban areas, the road networks are getting highly congested. It is mainly affecting the functioning and efficiency of the traffic networks in addition to consuming plenty of each user’s time.

Traffic flow models are a class of scientific models that study the flow of traffic based on some significant parameters. These models are classified as:

a. Microscopic traffic flow models
b. Macroscopic traffic flow models

While the microscopic models explain vehicular traffic dynamics by studying each vehicle-driver unit based on its position and velocity, the macroscopic models formulate the interdependence of the traffic flow characteristics, including density, flow, and the mean speed of a traffic stream, etc.

Our group aims to understand the LWR (Lighthill-Whitham-Richards) traffic flow model, study its mathematical aspects, and build a numerical model to solve it. LWR models represent the behaviour of traffic streams by using a continuity equation and an assumed equilibrium speed-density relationship. It assumes that the traffic stream will remain in equilibrium. In simpler words, it assumes that the values of speed and density at any point in the stream at any time remains consistent with the equilibrium relation. Apart from developing the defined LWR model, our group seeks to add some additional features to the model, which are listed below:

1. Understand all the factors governing the traffic flow of the LWR model in detail, and derive a relationship between them
2. Assume a set of initial and boundary conditions to start with solving the developed flow problem
3. Study the solutions that obey the flow conservation law
4. Decide on particular numerical methods to proceed with and develop a programming code to solve the final equation

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1. Background

The LWR model can be described as a system of hyperbolic PDEs. It assumes that an equilibrium speed-density relationship in the form of $v = v'(\rho)$ exists. It is based on the principle of conservation of mass or traffic conservation, and its dynamics can be described by the following first-order, nonlinear PDE:

$$\rho_t + f(\rho)_x = 0 \qquad \textbf{(Eq 1)}$$

Here, the $f(\rho) = \rho v'(\rho)$ subscript variable signifies the partial derivative with respect to it. In equation 1, the function is known as the fundamental diagram of traffic flow. We have represented in its conservative form. The function f(ρ) is assumed to be concave in nature, without any bounds on the sign of characteristic wave speed, λ(ρ). The characteristic wave speed can be defined as the derivative of f(ρ) with respect to ρ. The weak solutions for equation 1 satisfy the integral form of the conservation law:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2} \rho(x,t)dx = f(x_1, t) - f(x_2, t) \qquad \textbf{(Eq 2)}$$

We can solve the LWR model by several methods. Two of the most popular approaches to tackle this problem are:

1. Solve the Riemann problem and apply a Godunov method. The obtained solutions will remain consistent with hyperbolic conservation laws.
2. Usage of Demand and supply functions (a variant of Godunov's method).

Even though the LWR model was developed for homogenous roadways, we can still use it for the paths with inhomogeneities. The only significant change will be the dependence of the equilibrium speed-density relationship on the location.

Let a(x) be our inhomogeneity function, representing a roadway profile at the location x. Then, the inhomogeneous LWR model can be represented by:

$$\rho_t + f(a, \rho)_x = 0 \qquad \textbf{(Eq 3)}$$

where, $f(a,\rho)$ is the traffic flow rate as a function of inhomogeneity and density.

Notations

Following notations are followed throughout the text to be consistent.

v : Velocity as a function of density

v' : Velocity

ρ : Density

ρ_0 : Initial Density

ρ_l : Upstream traffic density

ρ_r : Downstream traffic density

$\rho(\xi)$: Rarefaction wave

$f(\rho)$: Fundamental Equation

q : Traffic flow

λ : Characteristic wave speed

$a(x)$: Inhomogeneity function

s : Shock wave speed

2. Assumptions

1. The primary assumption of the LWR model is that the velocity depends only on the main traffic density.
2. A macroscopic traffic flow model's underlying assumption is that traffic can be represented to be as continuous, similar to a fluid or a gas.
3. The values of speed and density at any point in the stream at any time remains consistent with the equilibrium relation.
4. The traffic flow is conserved, which means that no new vehicles can be produced or demolished. Therefore, It is possible to use the conservation or continuity equation.
5. The equilibrium flow-density relationship can only be written by assuming that both time and space scales are sufficiently large.

Further assumptions will be made when needed and will be depicted in italicised and under-lined text.

3. Governing Equations

As we have already assumed that the traffic flow will remain conserved, we can use either the continuity equation or the conservation equation and the state equation to derive a mathematical model for our problem.

3.1 Conservation Equation

Let us assume vehicles to be flowing from left to right. Then, we can write the conservation equation as

$$\frac{d\rho(x,t)}{dt} + \frac{dq(x,t)}{dx} = 0 \qquad \textbf{(Eq 4)}$$

Here, t denotes time, and x represents the spatial coordinates along the direction of traffic flow.

3.2 CFL Condition

Courant–Friedrichs–Lewy Condition, popularly known as CFL condition is used to ensure the stability of unstable numerical methods. When we solve certain partial differential equations, it is a necessary condition for convergence. For a one-dimensional continuous-time model, CFL condition can be written as:

$$C = \frac{u\Delta t}{\Delta x} \leq C_{max} \qquad \textbf{(Eq 5)}$$

Here, the dimensionless number C is called the Courant number, u is the magnitude of the velocity, Δt is the time step and Δx is the spatial interval. The courant number can have a maximum value (C_{max}) of 1.

4. Boundary Conditions

To proceed with solving the final equation derived in equation 7 later, we need to define a set of boundary and initial conditions that the solution could build upon.

1) The problem need not necessarily begin with an empty road (initial density = 0). To calculate the densities at each point on the road, the initial density at the point needs to be known. Therefore:

   The initial density(ρ_0) is formulated as a function of space parameter(x) at time t=0.

$$\rho_0 = 20,\ x \leq 0.5$$

$$\rho_0 = 100, x \geq 0.5$$

2) At any point of time, an ideal road has a fixed flux that is permitted to enter the first section and exit the last section of the road. These parameters are termed as Upstream demand and Downstream supply respectively.

$$\text{Upstream demand} = 1800 \text{ vehicle/hr}, \ \forall \ t$$

$$\text{Downstream supply} = 3000 \text{ vehicle/hr}, \ \forall \ t$$

5. Parameters

5.1 Speed-Density Relation

The primary speed-density relation is :

$$\text{Flow rate} = \text{Density} \times \text{speed}$$

$$q = \rho \times v$$

where flow rate implies the number of vehicles passing through a particular area per unit time and density implies the number of vehicles passing by a particular distance.

We know that as speed increases, density will decrease and vice versa. Therefore, we can say that speed is a function of density:

$$v = v(\rho)$$

and,

$$\frac{dv}{d\rho} < 0$$

i.e. v should be a decreasing function of ρ because, for a higher value of density, the average speed of the vehicle should be lower.

Therefore, when the value of ρis zero, $v = v_{max}$and when the value of ρis ρ_{max}(Jam Density), the value of v will become zero.

5.2 Interval Parameter

$$x_i = a + i\Delta x$$

With $$\Delta x = \frac{(b-a)}{N}$$

where the value of i varies from 1 to N, the space domain is [a,b] and N is no. of grid intervals. Δt is also defined in a similar way.

6. Numerical solution to the LWR model

The conservation equation given in equation 4 can be approximated using the Finite Difference Method as follows:

$$\frac{\rho_i^{j+1} - \rho_i^j}{\Delta t} + \frac{q(\rho_{i-1/2}^j) - q(\rho_{i+1/2}^j)}{\Delta x} = 0 \qquad \textbf{(Eq 6)}$$

Here, ρ_i^j is the average of the density ρ(x,t) in the space grid $(x_{i-1/2}, x_{i+1/2})$ at the time step t^j.

Also, $\rho_{i-1/2}^j$ is the average of density ρ(x,t) during the time interval (t_j, t_{j+1}) at the space grid $x_{i-1/2}$

The above equation can be adjusted as:

$$\rho_i^{j+1} = \rho_i^j + \frac{\Delta t}{\Delta x}(q(\rho_{i+1/2}^j) - q(\rho_{i-1/2}^j)) \qquad \textbf{(Eq 7)}$$

7. Algorithm Used

To solve the problem via a computer program, a code was written in MATLAB ®. The following steps depict the algorithm followed in the code

1) First of all, specify the characteristics of the path used in the problem.
2) Define initial conditions, demand function, supply function, spatial step and time horizon.
3) Create two empty matrices for storing the midpoint and length of every section of the path.
4) In the case of more than one links, choose the maximum out of specified two max velocities for different links.
5) Store the length of each link in the previously generated matrix and sum them to obtain the total length of the path.
6) Store the location of every section in the other generated matrix.
7) Specify the value of the security factor (inverse of courant number) and calculate the value of the time step consistent with the Courant–Friedrichs–Lewy Condition.
8) Create a nested time-space for the loop and calculate the values of density using the Godunov scheme.
9) Required density matrix and time step consistent with CFL condition are obtained.
10) View colourmap to find out the behaviour of density with space and time.

8. Results and Discussion

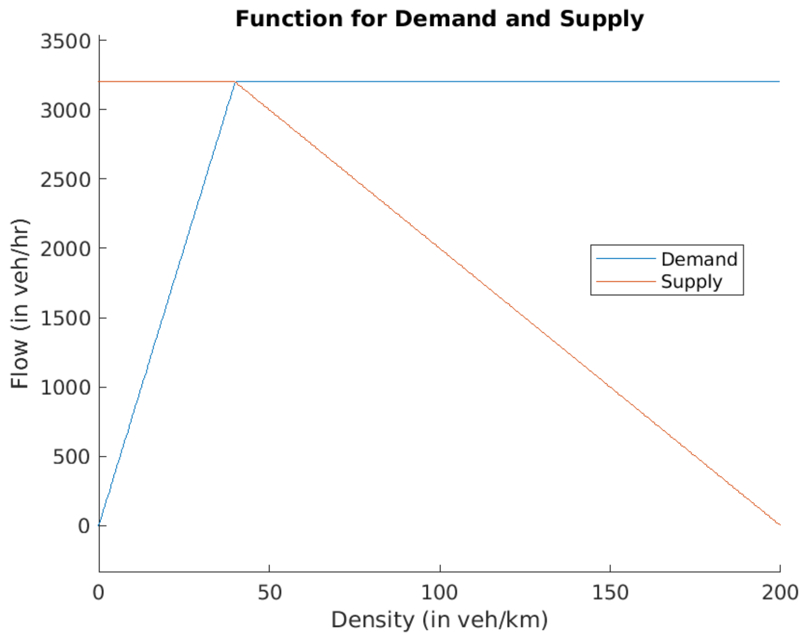

Figure 1: Variation of Demand and Supply Functions

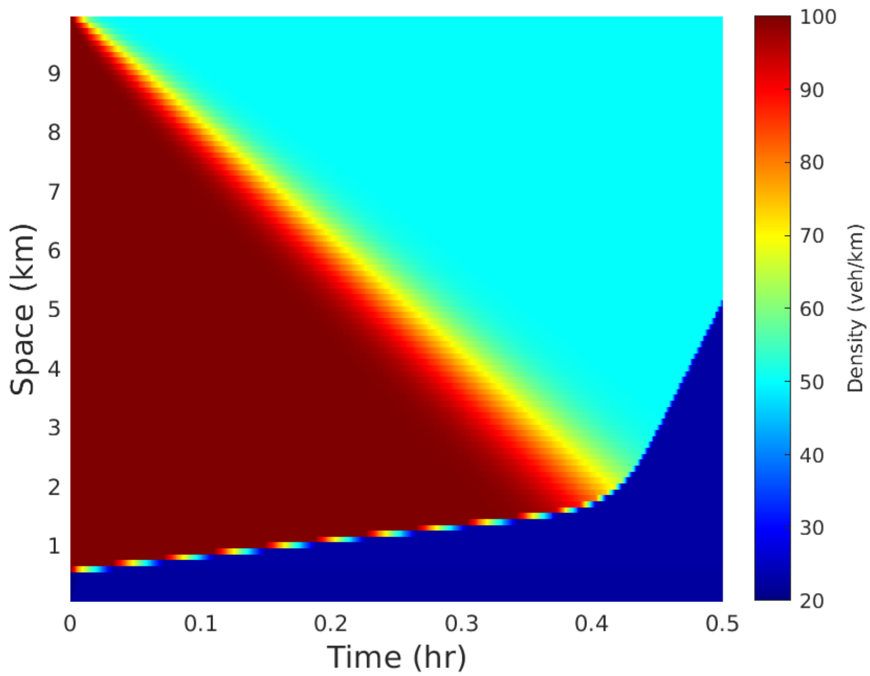

Figure 2: Density Distribution at Each Point in Time and Space

($\Delta x = 0.1$ and $\Delta t = 7.4074 \times 10^{-4}$)

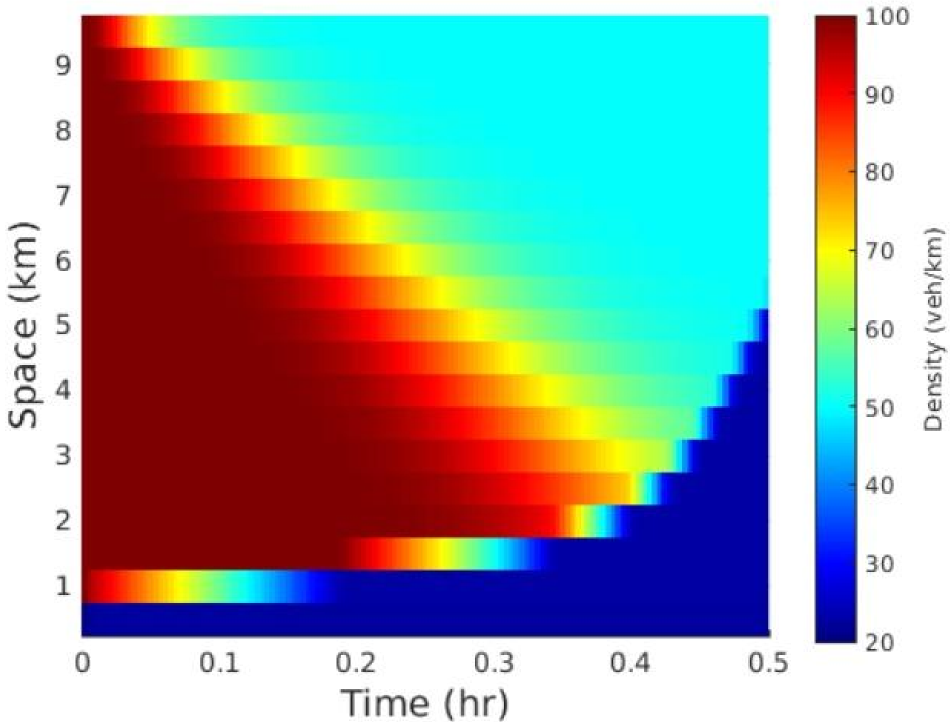

Figure 3: Density Distribution at Each Point in Time and Space

(Δx = 0.5 and Δt = 0.0037)

9. References

[1] Jin, Wenlong. (2004). Traffic Flow Models and Their Numerical Solutions.

[2] Li, J., Chen, Q., Wang, H., & Ni, D. (2012). Analysis of LWR model with fundamental diagram subject to uncertainties. Transportmetrica, 8(6), 387-405. https://doi.org/10.1080/18128602.2010.521532

[3] Marušić, S. (2012). A Note on Boundary Conditions for the LWR Model. PROMET - Traffic&Transportation, 21(6), 379-386. https://doi.org/10.7307/ptt.v21i6.253

[4] Vikram, D., Chakroborty, P., & Mittal, S. (2013). Exploring the Behavior of LWR Continuum Models of Traffic Flow in Presence of Shock Waves. Procedia - Social And Behavioral Sciences, 104, 412-421. https://doi.org/10.1016/j.sbspro.2013.11.134