Linear elastic stress analysis of shear-flexible axisymmetric cylindrical shells under uniform meridional edge load and normal pressure

Achilleas Filippidis and Adam J. Sadowski
Department of Civil and Environmental Engineering
Imperial College London, UK

Introduction

This manuscript serves as a supporting document to the article 'Selected closed-form algebraic solutions for linear stress and buckling analyses of elastic shear-flexible axisymmetric cylindrical shells', as written by the present authors. A step by step derivation of the displacement fields associated with uniform meridional edge loading and normal pressure is first presented, to be followed by illustrative examples that compare these solutions to thin shell theory and FE results. Sections relating to kinematics of the finite-thickness cylinder that have already been presented in the parent article, are included here for completeness.

The following assumptions are essential to define the basis of the derivations found in the next section.

Assumption 1 *Through thickness coordinate ζ over radius r ratio is ignored during integration of stress resultants.*
While the scope of the derivations presented in what follows relates to cylinders of finite-thickness, inclusion of the ζ/r terms when computing the stress resultant integrals [1, 2] would introduce an additional layer of complexity by not allowing for the assumption of $N_{\phi\theta} \approx N_{\theta\phi}$ for the in-plane shear membrane stress resultants and $M_{\phi\theta} \approx M_{\theta\phi}$ for the twisting stress resultants. This simplification is adopted in almost all previously mentioned analytical efforts, with the exception of Flügge's detailed thin-shell theory [2].

Assumption 2 *Through thickness variations of strains and stresses are of linear order.*
The effect of changing radii at different levels through the shell-wall thickness on the strains, in particular the circumferential strain ϵ_θ [2], is omitted. This allows for linear variations of strains with the exception of transverse shear strains, assumed to be uniform through the thickness. Their parabolic variation is amended with the aid of a shear correction factor k [3].

1 Bending theory solution for shear-flexible cylinders

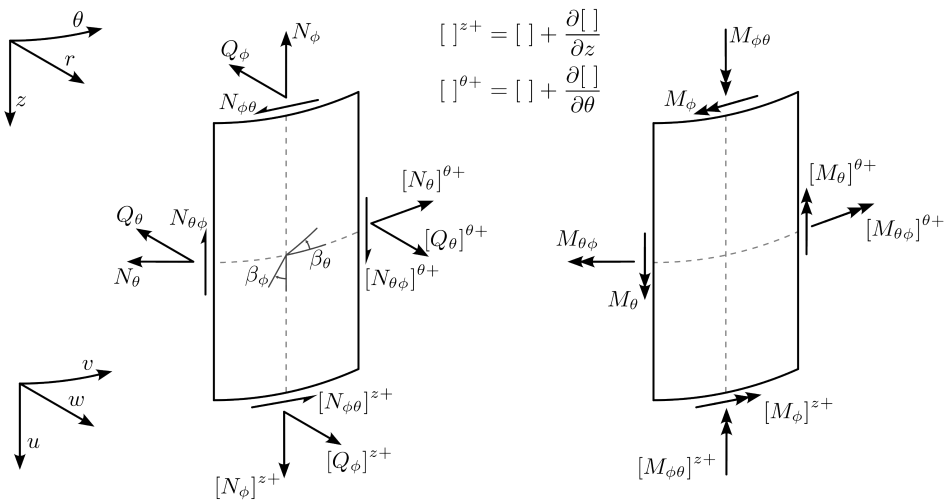


Figure 1: Cylindrical shell system

1.1 Kinematic relations

The following kinematic relations are based on Green's strain tensor for shells [4] and have been specialised for the case of a cylinder [5, 6] (meridional angle set to $\phi = \pi/2$, constant radius of circumferential curvature $r = r_\theta$ and radius of meridional curvature tending to infinity $r_\phi \to \infty$). With reference to the conventions presented in Fig. 1, the full set of midsurface strains and curvatures for a finite-thickness cylindrical shell are given as:

Membrane strains

$$\epsilon_\phi = \frac{\partial u}{\partial z} \tag{1a}$$

$$\epsilon_\theta = \frac{w}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \tag{1b}$$

Shear strains

$$\gamma_{\phi\zeta} = \frac{\partial w}{\partial z} + \beta_\phi \tag{2a}$$

$$\gamma_{\theta\zeta} = \frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v}{r} + \beta_\theta \tag{2b}$$

$$\gamma_{\phi\theta} = \gamma_{\theta\phi} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial z} \qquad \textit{(in-plane)} \tag{2c}$$

Curvatures

$$\kappa_\phi = \frac{\partial \beta_\phi}{\partial z} \tag{3a}$$

$$\kappa_\theta = \frac{1}{r}\frac{\partial \beta_\theta}{\partial \theta} \tag{3b}$$

$$\kappa_{\phi\theta} = \kappa_{\theta\phi} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial \beta_\phi}{\partial \theta} + \frac{\partial \beta_\theta}{\partial z}\right) \tag{3c}$$

The strains and curvatures of Eqs. 1-3 are identical to those presented in reference shell theory textbooks [7, 1], with the exception of the arguably more complete and complex theory derived from first principles by Flügge's 'Stresses in Shells' [2], when the assumption of thin shell theory is applied:

$$\beta_\phi \approx -\frac{\partial w}{\partial z} \tag{4a}$$

$$\beta_\theta \approx -\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{v}{r} \tag{4b}$$

The meridional and circumferential transverse shear strains of Eqs. 2 are essential to the derivation of a thick cylinder bending theory solution, where the positive sign of the rotations β_ϕ and β_θ stems from the convention corresponding to their positive direction, as illustrated in Fig. 1.

1.2 Constitutive relations

According to classical bending shell theory [7, 2, 8], with additional considerations for transverse shear effects and the simplifications of Assumption 1, the full set of membrane, bending and shear stress resultants for a cylinder may be expressed as:

Membrane stress resultants:

$$N_\phi = C(\epsilon_\phi + \nu\epsilon_\theta) \tag{5a}$$

$$N_\theta = C(\epsilon_\theta + \nu\epsilon_\phi \tag{5b}$$

$$N_{\phi\theta} = N_{\theta\phi} = C\frac{1-\nu}{2}\gamma_{\phi\theta} \tag{5c}$$

Bending stress resultants:

$$M_\phi = D(\kappa_\phi + \nu\kappa_\theta) \tag{6a}$$

$$M_\theta = D(\kappa_\theta + \nu\kappa_\phi) \tag{6b}$$

$$M_{\phi\theta} = M_{\theta\phi} = D(1-\nu)\kappa_{\phi\theta} \tag{6c}$$

Transverse shear stress resultants:

$$Q_\phi = kGt\gamma_\phi \tag{7a}$$

$$Q_\theta = kGt\gamma_\theta \tag{7b}$$

where the extensional stiffness C, flexural stiffness D and shear modulus G are defined as:

$$C = \frac{Et}{1-\nu^2} \tag{8a}$$

$$D = \frac{Et^3}{12(1-\nu^2)} \tag{8b}$$

$$G = \frac{E}{2(1+\nu)} \tag{8c}$$

for an elastic isotropic material. The shear correction factor is set to $k = 5/6$ [3], appropriate for rectangular cross-sections, an assumption that is valid for both sides of an infinitesimal element $dz\, rd\theta$ which is used for the derivations of the cylindrical shell theory [9, 10].

1.3 Equilibrium equations for axisymmetric cylinders

Only a handful of the force terms and their increments displayed in Fig. 1 will participate in the equilibrium equations for a (pre-buckling) stress state described by circumferential axisymmetry which characterises many lower slenderness reference shell systems of industrial importance. Any derivatives with respect to the circumferential angle θ may be ignored, as they would violate the axisymmetric stress state, together with transverse shear forces Q_θ and in-plane shear terms. The circumferential degrees of freedom v and β_θ are also assumed to be zero.

Summation of forces along the radial r and axial z axes, as well as moments about the circumferential axis θ, referring to the undeformed configuration of Fig. 1, yields:

$$\frac{\partial N_\phi}{\partial z} = 0 \tag{9a}$$

$$r\frac{\partial Q_\phi}{\partial z} - N_\theta + rp = 0 \tag{9b}$$

$$\frac{\partial M_\phi}{\partial z} - Q_\phi = 0 \tag{9c}$$

after application of the reductions on the equilibrium equations outlined above due to the axisymmetric nature of the problem. Force equilibrium along the circumferential axis θ leads to an identity for the case of axisymmetry, as does moment equilibrium about the r and z axes.

1.4 System of equations

Substitution of the kinematic relations of Eqs. 1-3 into the constitutive relations of Eqs. 5-7, after application of reductions arising from the condition of circumferential stress uniformity, produces the following generalised stress resultants:

$$N_\phi = C(\epsilon_\phi + \nu\epsilon_\theta) = C\left(\frac{\partial u}{\partial z} + \nu\frac{w}{r}\right) \tag{10a}$$

$$N_\theta = C(\epsilon_\theta + \nu\epsilon_\phi) = C\left(\frac{w}{r} + \nu\frac{\partial u}{\partial z}\right) \tag{10b}$$

$$M_\phi = D\kappa_\phi = D\frac{\partial \beta_\phi}{\partial z} \tag{10c}$$

$$M_\theta = \nu D\kappa_\phi = \nu D\frac{\partial \beta_\phi}{\partial z} \tag{10d}$$

$$Q_\phi = kGt\gamma_\phi = kGt\left(\frac{\partial w}{\partial z} + \beta_\phi\right) \tag{10e}$$

while the $N_{\phi\theta}$, $M_{\phi\theta}$, Q_θ stress resultants are equal to zero due to the condition of axisymmetry. A further substitution of Eqs. 10 into the reduced equilibrium equations of Eqs. 9 leads to:

$$C\left(\frac{\partial^2 u}{\partial z^2} + \frac{\nu}{r}\frac{\partial w}{\partial z}\right) = 0 \tag{11a}$$

$$rkGt\left(\frac{\partial^2 w}{\partial z^2} + \frac{\partial \beta_\phi}{\partial z}\right) - C\left(\frac{w}{r} + \nu\frac{\partial u}{\partial z}\right) + rp = 0 \tag{11b}$$

$$D\frac{\partial^2 \beta_\phi}{\partial z^2} - kGt\left(\frac{\partial w}{\partial z} + \beta_\phi\right) = 0 \tag{11c}$$

Integration of Eq. 11a with respect to the z coordinate introduces a constant to the equation for meridional force equilibrium, representing a natural boundary condition (BC) N_z for the meridional membrane stress resultant N_ϕ to be set equal to at the loaded edge of the cylinder:

$$C\left(\frac{\partial u}{\partial z} + \frac{\nu}{r}w\right) = N_z \tag{12}$$

Solving for the first derivative of u from Eq. 12 and substituting into Eq. 11b allows for partial decoupling of the system of Eqs. 11, which now only includes terms in the normal displacement w and meridional rotation β_ϕ. The partially decoupled set of equations may be written as:

$$rkGt\frac{\partial^2 w}{\partial z^2} + rkGt\frac{\partial \beta_\phi}{\partial z} - \frac{C(1-\nu^2)}{r}w - \nu N_z + rp = 0 \tag{13a}$$

$$D\frac{\partial^2 \beta_\phi}{\partial z^2} - kGt\left(\frac{\partial w}{\partial z} + \beta_\phi\right) = 0 \tag{13b}$$

1.5 Solution of the homogeneous problem for a thick cylinder

The system of second order linear ordinary differential equations (ODEs) in Eqs. 13 governs the bending behaviour of a cylinder under the combined action of a meridional force N_z and normal pressure p. The solution of this system allows for the computation of stress patterns along the meridian of the cylinder. In compliance with the general mathematical treatment for ODE systems of this nature, the homogeneous part of these equations will be isolated and solved for, neglecting any terms that do not include the degrees of freedom (DOFs) w and β_ϕ. The computation of a particular solution will follow and complete the expressions for the displacement field along the meridian of the cylinder.

1.5.1 Reduction into first order linear system of ODEs

Due to the nature of the problem, a substitution of variables is required in order to transform the system of second order linear ODEs into a first order one. Let:

$$w_a = w \tag{14a}$$

$$w_b = \frac{\partial w}{\partial z} \tag{14b}$$

$$\beta_{\phi,a} = \beta_\phi \tag{14c}$$

$$\beta_{\phi,b} = \frac{\partial \beta_\phi}{\partial z} \tag{14d}$$

Hence:

$$w_b = w_a' \tag{15a}$$

$$\beta_{\phi,b} = \beta_{\phi,a}' \tag{15b}$$

The homogeneous part of the original system can now be rewritten as:

$$rkGtw_b' + rkGt\beta_{\phi,b} - \frac{C(1-\nu^2)}{r} w_a = 0 \tag{16a}$$

$$D\beta_{\phi,b}' - kGtw_b - kGt\beta_{\phi,a} = 0 \tag{16b}$$

Solving for the now first order derivatives of the system w_b' and $\beta_{\phi,b}'$ leads to:

$$w_b' = \frac{C}{kGt}\frac{1-\nu^2}{r^2} w_a - \beta_{\phi,b} \tag{17a}$$

$$\beta_{\phi,b}' = \frac{kGt}{D} w_b + \frac{kGt}{D}\beta_{\phi,a} \tag{17b}$$

The homogeneous part of the system may now be expressed in matrix form:

$$\begin{Bmatrix} w'_a \\ w'_b \\ \beta'_{\phi,a} \\ \beta'_{\phi,b} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{C}{kGt}\frac{1-\nu^2}{r^2} & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{kGt}{D} & \frac{kGt}{D} & 0 \end{bmatrix} \begin{Bmatrix} w_a \\ w_b \\ \beta_{\phi,a} \\ \beta_{\phi,b} \end{Bmatrix} \tag{18}$$

1.5.2 Characteristic equation for the homogeneous problem

The matrix form of the system allows for the extraction of the characteristic equation through the solution of the eigenvalue problem:

$$|A - \Lambda I| = \begin{vmatrix} -\Lambda & 1 & 0 & 0 \\ \frac{C}{kGt}\frac{1-\nu^2}{r^2} & -\Lambda & 0 & -1 \\ 0 & 0 & -\Lambda & 1 \\ 0 & \frac{kGt}{D} & \frac{kGt}{D} & -\Lambda \end{vmatrix} = 0 \tag{19}$$

Computation of the above determinant, allows for the characteristic equation to be found as:

$$r^2\Lambda^4 - \frac{2(1+\nu)}{k}\Lambda^2 + \frac{12(1-\nu^2)}{t^2} = 0 \tag{20}$$

where the membrane and bending stiffnesses, as well as the shear modulus, have been substituted for according to Eqs. 8. The quartic characteristic equation is found to be quadratic in Λ^2, allowing for the root finding problem to be approached in a classical quadratic equation fashion. The discriminant of the 'quadratic' form can be evaluated to be:

$$\Delta = \frac{4(1+\nu)^2}{k^2} - 4r^2\frac{12(1-\nu^2)}{t^2} \tag{21}$$

For reasonable values of the above parameters and for $r/t > 1$ (but not necessarily $r/t \gg 1$) the discriminant Δ is negative, therefore the two Λ^2 roots will be:

$$\Lambda^2 = \frac{1+\nu}{kr^2} \pm i\frac{1}{r^2}\sqrt{\frac{(1+\nu)^2}{k^2} - 12(1-\nu^2)\left(\frac{r}{t}\right)^2} \tag{22}$$

In turn, the four Λ roots of the quartic characteristic equation can be evaluated as:

$$\Lambda = \pm\sqrt{\frac{1+\nu}{kr^2} \pm i\frac{1}{r^2}\sqrt{\frac{(1+\nu)^2}{k^2} - 12(1-\nu^2)\left(\frac{r}{t}\right)^2}} \tag{23}$$

The real and imaginary parts of these roots can be isolated as follows:

$$\begin{aligned} \Lambda_1 &= \quad a + ib \\ \Lambda_2 &= \quad a - ib \\ \Lambda_3 &= -a + ib \\ \Lambda_4 &= -a + ib \end{aligned} \tag{24}$$

where the positive real parameters a and b are defined as:

$$a = \frac{1}{\sqrt{2}r}\sqrt{\sqrt{12(1-\nu^2)\left(\frac{r}{t}\right)^2} + \frac{1+\nu}{k}} \tag{25a}$$

$$b = \frac{1}{\sqrt{2}r}\sqrt{\sqrt{12(1-\nu^2)\left(\frac{r}{t}\right)^2} - \frac{1+\nu}{k}} \tag{25b}$$

1.5.3 Eigenvectors for the matrix system of the homogeneous problem

The matrix system $[A - \Lambda_j I]$ can be readily shown to be singular, as the Λ_j parameters with $j = 1, 4$ correspond to the roots of the Eq. 19 eigenvalue problem. Due to the singularity of the matrix system, there is an infinite number of eigenvector solutions $\{\phi_j\}$ such that:

$$[A - \Lambda_j I]\{\phi_j\} = \{0\} \tag{26}$$

where the $\{\phi_j\}$ eigenvector is given as:

$$\{\phi_j\} = \begin{Bmatrix} w_{a,j} \\ w_{b,j} \\ \beta_{a,j} \\ \beta_{b,j} \end{Bmatrix} \tag{27}$$

To allow for a sensible solution, let $w_{a,j} = 1$ for all Λ_j roots. The j^{th} eigenvector of Eq. 26 then becomes:

$$\{\phi_j\} = \begin{Bmatrix} w_{a,j} \\ w_{b,j} \\ \beta_{a,j} \\ \beta_{b,j} \end{Bmatrix} = \begin{Bmatrix} 1 \\ \Lambda \\ \frac{kGt\Lambda}{kGt - D\Lambda^2} \\ \frac{kGt\Lambda^2}{kGt - D\Lambda^2} \end{Bmatrix} \tag{28}$$

1.5.4 Displacement field solution for the homogeneous problem

The normal displacement w and meridional rotation β_ϕ fields that satisfy the homogeneous problem of Eq. 13:

$$w = C_1 e^{\Lambda_1 z} + C_2 e^{\Lambda_2 z} + C_3 e^{\Lambda_3 z} + C_4 e^{\Lambda_4 z} \tag{29a}$$

$$\beta = \beta_{a,1} C_1 e^{\Lambda_1 z} + \beta_{a,2} C_2 e^{\Lambda_2 z} + \beta_{a,3} C_3 e^{\Lambda_3 z} + \beta_{a,4} C_4 e^{\Lambda_4 z} \tag{29b}$$

Expansion of the Λ_j roots in the above expression can lead to:

$$\begin{aligned} w = {} & C_1 e^{az}(\cos(bz) + i\ \sin(bz)) + C_2 e^{az}(\cos(bz) - i\ \sin(bz)) \\ & + C_3 e^{-az}(\cos(bz) + i\ \sin(bz)) + C_4 e^{-az}(\cos(bz) - i\ \sin(bz)) \end{aligned} \tag{30a}$$

$$\beta = \beta_{a,1} C_1 e^{az} (\cos(bz) + i\ \sin(bz)) + \beta_{a,2} C_2 e^{az} (\cos(bz) - i\ \sin(bz))$$
$$+ \beta_{a,3} C_3 e^{-az} (\cos(bz) + i\ \sin(bz)) + \beta_{a,4} C_4 e^{-az} (\cos(bz) - i\ \sin(bz)) \tag{30b}$$

The half-wavelength for the shear-flexible bending theory solution may now be found as:

$$\lambda_{thick} = \frac{\pi}{b} = \frac{\pi\sqrt{2}r}{\sqrt{\sqrt{12(1-\nu^2)\left(\frac{r}{t}\right)^2} - \frac{1+\nu}{k}}} \tag{31}$$

It can be readily shown that the relationship for the bending half-wavelength of Eq. 31 quickly reduces to the corresponding thin shell theory result as the r/t ratio in the denominator increases and overshadows the shear correction factor quantity:

$$\frac{\lambda_{thick}}{\lambda_{thin}} = \frac{\dfrac{\pi\sqrt{2}r}{\sqrt{\sqrt{12(1-\nu^2)\left(\frac{r}{t}\right)^2} - \frac{1+\nu}{k}}}}{\dfrac{\pi\sqrt{rt}}{[3(1-\nu^2)]^{\frac{1}{4}}}} = \frac{1}{\sqrt{1 - \frac{t}{r}\frac{1+\nu}{k\sqrt{12(1-\nu^2)}}}} > 1 \tag{32}$$

An equivalent reduction may also be achieved for high values of the k factor, essentially 'enforcing' a thin shell theory to the finite-thickness approach adopted here. Closer examination of Eq. 32 additionally indicates that the inclusion of transverse shear effects on the cylinders deformation permits the bending boundary layers to cover a larger portion of the shell's meridian, owing to their increased flexibility.

1.6 Particular solution to the inhomogeneous problem for a thick cylinder

A particular solution for the normal displacement w and meridional rotation β can be found through membrane theory [11] as:

$$w_m = \frac{pr^2}{tE} - \frac{N_z \nu r}{tE} \tag{33}$$

and

$$\beta_m = 0 \tag{34}$$

As is always the case with displacement field solutions produced through membrane theory, they cannot satisfy essential boundary conditions relating to restraints of the displacements or rotations on the cylinder's edges.

1.7 Bending theory solution to cylinder under meridional edge load and normal pressure

The general solution to the ODE system may now be found to be:

$$\begin{aligned} w = {} & C_1 e^{az}(\cos(bz) + i\ \sin(bz)) + C_2 e^{az}(\cos(bz) - i\ \sin(bz)) \\ & + C_3 e^{-az}(\cos(bz) + i\ \sin(bz)) + C_4 e^{-az}(\cos(bz) - i\ \sin(bz)) + \frac{pr^2}{tE} - \frac{N_z \nu r}{tE} \end{aligned} \tag{35a}$$

$$\begin{aligned} \beta = {} & \beta_{a,1} C_1 e^{az}(\cos(bz) + i\ \sin(bz)) + \beta_{a,2} C_2 e^{az}(\cos(bz) - i\ \sin(bz)) \\ & + \beta_{a,3} C_3 e^{-az}(\cos(bz) + i\ \sin(bz)) + \beta_{a,4} C_4 e^{-az}(\cos(bz) - i\ \sin(bz)) \end{aligned} \tag{35b}$$

The last step in producing a thick shell bending theory solution is the evaluation of the unknown constants C_j. These four unknowns require four essential boundary conditions for their calculation, two at each end of the cylinder. Arrangement into a matrix system for the calculation of the C_j unknowns has been formally outlined elsewhere [12], with the process being indifferent to the shell theory utilised for the derivation of said unknowns.

Computation of the normal displacement w and meridional rotation β_ϕ allows in turn for the evaluation of the meridional displacement u at any point along the cylinder. Integration with respect to the meridional coordinate z on Eq. 12, which was used for the elimination of the $\partial u/\partial z$ derivative, can lead to the following solution of the meridional displacement u:

$$\begin{aligned} u = {} & u_c + z\frac{N_z}{C} - \frac{\nu}{r}\left(\frac{C_1}{\Lambda_1} e^{az}(\cos(bz) + i\ \sin(bz)) + \frac{C_2}{\Lambda_2} e^{az}(\cos(bz) - i\ \sin(bz))\right. \\ & \left. + \frac{C_3}{\Lambda_3} e^{-az}(\cos(bz) + i\ \sin(bz)) + \frac{C_4}{\Lambda_4} e^{-az}(\cos(bz) - i\ \sin(bz)) + z\frac{pr^2}{tE} - z\frac{N_z \nu r}{tE}\right) \end{aligned} \tag{36}$$

with the integration constant u_c chosen such that it satisfies a boundary condition at either end of the cylinder. Strains and stresses can now be readily calculated at any point using the displacement and rotation fields of Eqs. 35-36 and their derivatives, providing a full stress resultant profile for shear-flexible cylinders.

Unlike the displacement field expressions for thin shell bending theory, in Eqs. 35-36 the exponent az does not match the bz quantity in the sine and cosine terms. Consequently, the finite-thickness expressions for u, w and β cannot be rearranged in a way that would allow for the imaginary portions to be eliminated. However, numerical results illustrate that the imaginary part of these seemingly complex solutions can be found to be negligible, within reasonable tolerance levels for machine precision. This finding can be reproduced even with a common spreadsheet implementation of the present section's finite-thickness solution, as is shown in the accompanying GitHub repository files.

2 Linear elastic stress analysis examples for shear-flexible axisymmetric cylinders

Owning to the principles of superposition that linear elasticity allows for, the individual study of a cylindrical system under either uniform meridional compression or uniform external pressure illustrates the theory's correctness for combinations of these two load cases while showcasing the discrepancies between thin and thick analytical shell theories and results obtained computationally from the well-established ABAQUS finite element software [13].

In all that follows, the material parameters of the cylinders are kept constant, with a Poisson ratio of $\nu = 0.3$ and a Young's modulus of $E = 2e5\ N/mm^2$. The boundary conditions classification scheme of EN 1993-1-6 [14] is adopted for the definition of edge restraints, with Yamaki's equivalents additionally used as a reference [15]. Characterisation of the cylinder length is made in a dimensionless manner in terms of a group ω, defined as:

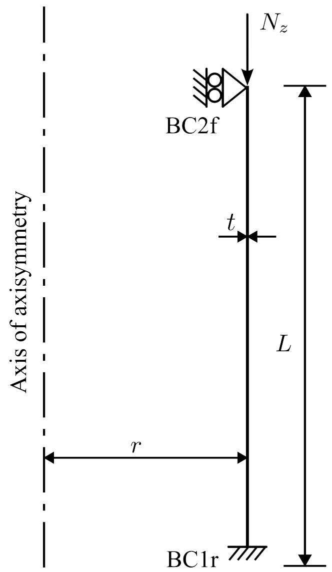
$$\omega = \frac{L}{\sqrt{rt}} \tag{37}$$

For the stress analysis comparisons to be meaningful and any discrepancies between thin and thick cylindrical shell theories to be made apparent, a very low $r/t = 5$ ratio is chosen for both loading scenarios accompanied by a unit thickness. Geometric definition of the following models is concluded by a dimensionless group $\omega = 5$. Even according to thin shell theory, the material and geometric parameters used for the stress analysis examples here lead to a normalised length ratio of $L/\lambda_{thin} = 2.05$. The thick theory bending half-wavelength λ_{thick} of Eq. 31 has been found to be always higher than its thin theory equivalent, thus ensuring interaction of the $2\lambda_{thick}$ bending boundary layers.

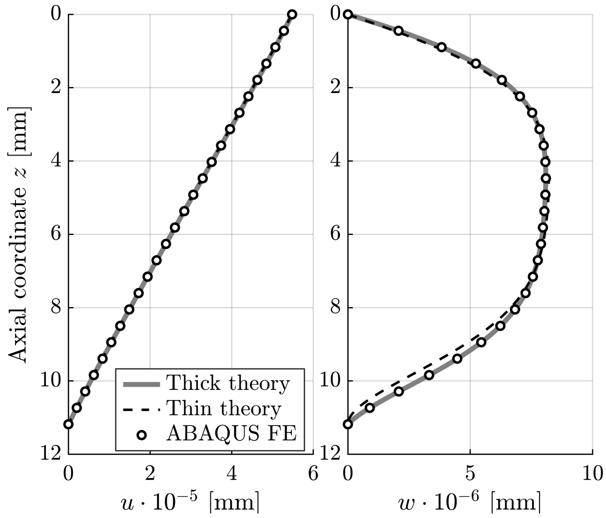
2.1 Cylinder under uniform meridional compression

A cylinder under uniform meridional compression (introduced via a compressive meridional edge load) constitutes the first example presented here. The generating meridian with its loading and BCs is illustrated in its axisymmetric space in Fig. 2a, with the corresponding analytical and computational displacement fields in Fig 2b. A unit compressive edge load $N_z = -1\ N/mm$ is applied on its top end, with the negative sign owning to the convention for membrane stress resultants being positive under external loading.

The present derivations offer a more flexible cylinder deformation pattern due to the inclusion of shear strain effects in the formulation, as displayed in Fig. 2b. More pronounced deflection profiles are commonly associated with higher strain energy outputs and hence closer estimates of the ideal continuum behaviour of a mechanical body. A very close agreement between the thick shell solution with reference finite element results is presented in



(a) System geometry.



(b) Meridional u and normal w displacement profiles.

Figure 2: Cylinder under uniform meridional compression.

Fig. 2b for the u and w displacement profiles and in Fig. 3 for the meridional and circumferential stress patterns.

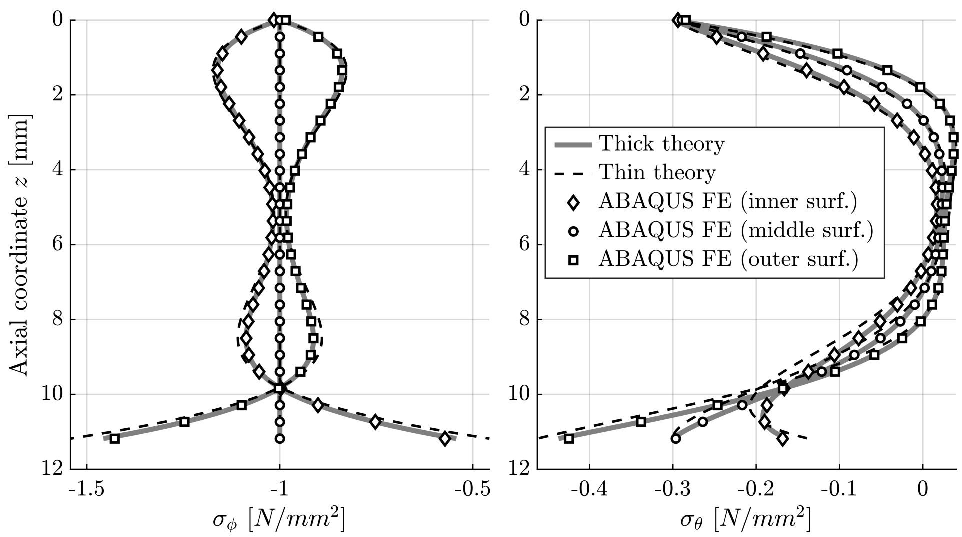
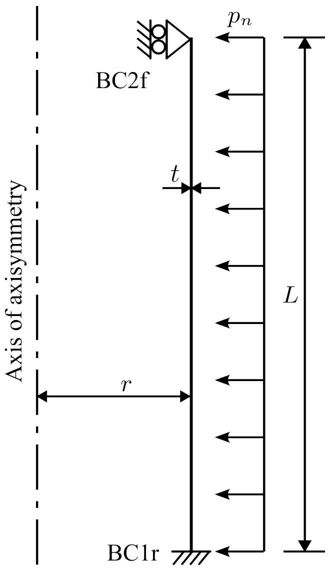


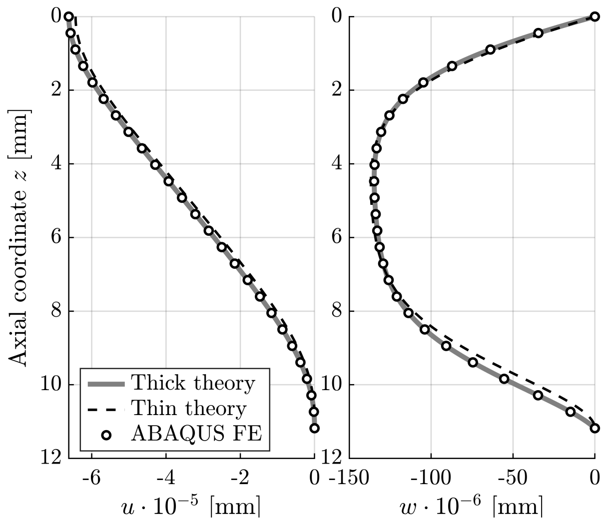
Figure 3: Meridional (left) and circumferential (right) axisymmetric stress patterns for a cylinder under uniform meridional compression.

2.1.1 Cylinder under uniform external pressure

The displacement field of Eqs. 35-36 additionally allows for a finite-thickness solution to the problem of a cylinder under normal pressure. A unit external pressure $p = -1\ N/mm^2$ is applied on the cylinder of Fig. 4a.



(a) System geometry.



(b) Meridional u and normal w displacement profiles.

Figure 4: Cylinder under uniform external pressure.

The more flexible solution of the thick cylindrical theory is in agreement with the ABAQUS FE displacements (Fig. 4b) and stresses (Fig. 5). As was the case with the bending half-wavelength λ of Eq. 31, the solutions for both loadcases considered here quickly reduced to their thin shell theory equivalents, either naturally by increasing the r/t aspect ratio of the models or by enforcing a thin theory through values of $k \to \infty$.

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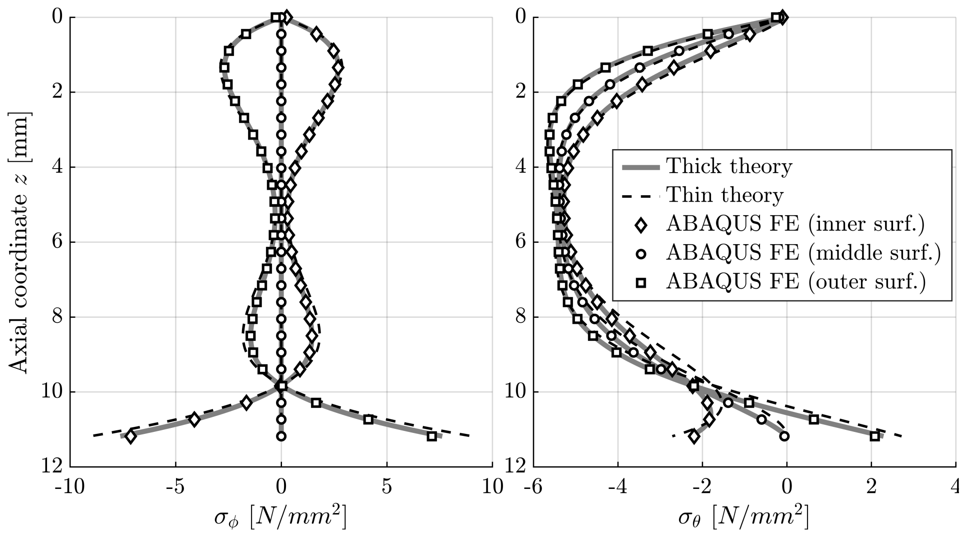

Figure 5: Meridional (left) and circumferential (right) axisymmetric stress patterns for a cylinder under uniform external pressure.

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